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Students' conceptual understanding of functions at upper secondary level

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This thesis deals with danish secondary school students' conceptual understanding of functions. The function concept is chosen as a case study because of its importance to mathematics as a discipline, and because of its central role in the danish educational system. The aim of the thesis is to characterise students' conceptual understanding of functions through a combination of theoretical work and empirical studies. The theoretical framework serves as a guidance for the development of the methodology, as well as an analytical tool for describing, understanding, and communicating observations.

The main theory is Anna Sfard's theory of reification – a theory that describes fundamental aspects of the development of mathematical concepts, which can be found within the historical development of concepts in mathematics as a discipline, as well as within the learning process of individuals. Based on theoretical arguments and observations from the literature, the theory is revised and extended. The essence of the extension is, that during the learning process, an individual's conceptual understanding can develop differently across different representations of the same concept. The extension is referred to as disjoint-reification-of-representations hypothesis. The hypothesis has significant consequences for the characterisation of students' conceptual understanding and therefore potentially for how teachers can support and challenge students' conceptual understanding.

Three empirical studies, containing a total of 17 pairs of students, spread across two danish secondary schools, are conducted. The studies consist of video/audio recordings of pairs of students who are solving a set of mathematical problems, which were designed according to the theory of reification, with the aim of characterising the students' understanding of the function concept.

The data analysis consists of a close scrutiny of the video recording and the students' written answers to the problems. Observations are presented in the form of transcribed dialogues between the students. These dialogues are then analysed within the theoretical framework of the thesis. Based on the analysis a characterisation of the students' understanding of the function concept is given.

The overall conclusion of the thesis is, that important aspects of secondary students' conceptual understanding can be probed through video recording of problem solving sessions, and that it can be characterised by stating the level of reification which the student has attained of the concept in its different representations.

Master's Thesis

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Students' conceptual understanding of functions at upper secondary level

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Abstract in danish

Dette speciale omhandler danske gymnasieelevers forståelse af funktionsbegrebet. Funktionsbegrebet er valgt på baggrund af dets vigtige rolle i matematik som videnskab og især på grund af dets centrale rolle i det danske uddannelsesystem. Formålet med specialet er at karakterisere individers forståelse af funktionsbegrebet gennem en kombination af teoretisk arbejde og empiriske studier. Den teoretiske ramme lægger fundamentet for udviklingen af de empiriske studier, og fungerer som et analyseværktøj til at beskrive, forstå og kommunikere observationer.

Hovedteorien er Anna Sfards reifikationsteori – en teori, der beskriver de grundlæggende træk ved matematisk begrebsdannelse, hvilket kan genfindes både i den historiske udvikling af begrber i matematik, som videnskab, og i begrebsdannelsen hos individer.

Baseret på teoretiske argumenter samt observationer fra litteraturen revideres og udvides teorien. Udvidelsen består i at åbne op for muligheden for, at begrebsdannelsen, i læreprocesser, kan forkomme forskelligt inden for et begrebs forskellige repræsentationer. Udvidelsen kaldes for hypotesen om disjunkt-reifikation-af-repræsentationer. Hypotesen har afgørende betydning for karakterisering af elevers begrebsforståelse og dermed også potentielt for, hvordan man, som lærer, kan støtte og udfordre elevers begrebsforståelse.

Der er udført tre empiriske studier, der inkluderer 17 par af studerende spredt over to danske gymnasier. Studierne består af video/audio optagelser af studerende, der løser matematiske problemer, som er designet jf. reifikationsteorien med henblik på at karakterisere de studerendes forståelse af funktionsbegrebet.

Dataanalysen består af en grundig gennemgang af video-optagelserne og de skrevne opgavebesvarelser. Observationerne præsenteres i form af transskriberede dialoger fra video-optagelserne. Dialogerne er derefter analyseret i lyset af reifikationsteorien. Baseret på disse analyser gives en karakterisering af de studerendes forståelse af funktionsbegrebet.

Den overordnede konklusion på specialet er, at vigtige aspekter af gymnasieelevers forståelse af funktionsbegrebet kan belyses gennem problemløsningssessioner, og at deres begrebsforståelse kan karakteriseres ved at udpege det stadie af reifikation som de studerende har opnået af begrebet i dets forskellige repræsentationer.

And so is this one.

Acknowledgements

Even though I have written this thesis alone, I have not done so without significant help from others. I would like to take this opportunity to thank the people who have helped me along the way.

I would like to thank Martin Niss for introducing me to Alan Schoenfelds work on problem solving; Kasper Bjering for getting me in touch with the teacher at Roskilde Katedral Skole, and for useful discussions and suggestions; Jens Højgaard, Sif Skjoldager, and Steen Grode for listening to a presentation of my thesis, and for the follow discussion.

Lastly I would like to thank my supervisor for guidance throughout the process and especially for agreeing to initiating the supervision process a full year before officially starting the thesis.

Thank you.

Contents

Part I

Introduction

1 Introduction

Functions play an important role in mathematics education, but it turns out that a thorough comprehension of the function concept is remarkably complex and difficult for students to attain. [Harel and Dubinsky, 1992]

Historically the evolution of the function concept can be traced back to Bernoulli's definition from around 1720: *"One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants"* [Kleiner, 1989, p. 3]. This is to be compared to Bourbaki's definition from around 1940, where it is defined as a subset of ordered paris (x, y) in a product set $A \times B$, in which there are no different paris with the same first component. The subset is then said to define a function $f : A \to B$ [Kleiner, 1989, p. 18]. Thus, during the approximately 200 year development, the function concept has undergone extensive revision; e.g. in the first definition an analytic expression is required; this is not so in the latter definition.

As an example we can consider the parsing of barcodes into prices; each barcode determines a unique price. This parsing from barcodes to prices can be considered a function in the Bourbaki definition, but not in the Bernoulli definition.

Having evolved with mathematics, the function concept now plays a central and unifying role; functions occur throughout mathematics and are used in very diverse ways. Consequently, functions can be represented in many different ways: as a set of ordered pairs, a correspondence, a graph, a dependent variable, or a formula, just to name a few points of view. Students often hold only some of these views incompletely, and are having considerable difficulty changing from one point of view to another. [Harel and Dubinsky, 1992]

It is important to understand students' conceptual understanding for at least two reasons: First of all, mathematics is, for the most parts, built from the bottom up; new concepts rely on old ones – in history as well as for the individual's learning process. Therefore, making sure that students have a sound understanding of core concepts is vital for the success of the students. In the Danish educational system, for example, the function concept is introduced during elementary school and serves as a gateway to calculus during secondary school. Thus, the function concept is central to mathematics in the danish educational system, and any student who wish to succeed in this subject had better have a sound understanding of this concept.

However, when the function concept is taught in school, only a subset of the possible representations of the function concept is introduced, and only few standardised connections between these representations are focused on. This way of teaching has some unfortunate consequences.

For example, it has been observed by Tall and Bakar in [Tall and Bakar, 1992], that students' understanding of a given concept is restricted to the number of examples and non-examples they have encountered. There is reason to expect that if students see a limited numbers examples, presented in a limited number of representations, the students understanding will be limited.

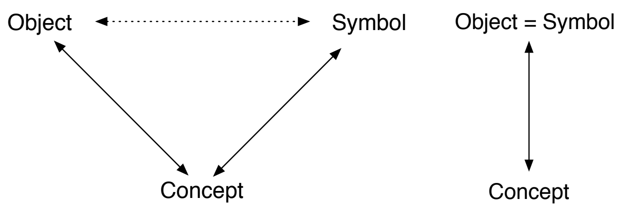
Furthermore, according to Steinbring [Steinbring, 1989, 1997], the meaning of mathematical concepts emerge in the interplay between symbols – the notation used to represent an instances of a concept – and object domains – the concrete instances of the concept we wish to describe. Steinbring describes this relationship between the symbols and objects with the epistemological triangle: object – symbol – concept, as shown in figure 1.1.

As an example, let us consider the function concept: The symbols we use are, e.g., graphs, algebraic expressions or table of values. The specific objects we wish to describe are, e.g. polynomials.

Steinbring points out, that the connection between mathematical objects and the symbols used to represent these object is indirect. It is important to distinguish between the objects one wishes to describe, and the symbol one uses to do so; only by relating these two sides to each other *without* identifying them directly, can the development of knowledge advance, since the objects and symbols are the students only means of accessing the concept [Steinbring, 1989].

However, this is precisely what happens for many students. They equate the objects with the signs and symbols used to represent them, and as a consequence the epistemological collapses (figure 1.1b), and becomes a linear connection between object and concept; the representation of an object *becomes* the object, and the concepts are taken out of the focus – as a consequence the students' conceptual understanding suffers.

One interpretation of this is be that a student might think that the expression $f(x) = x^2$ and the graph of this function is in fact two different mathematical objects, not two different representations of the same object.



(a) Epistemological triangle **(b)** Collapse

Figure 1.1 – Steinbring's epistemological triangle describes

To summarise: Due to the way in which mathematical concepts are taught, students can obtain an unsatisfactory conceptual understanding – students can only access the mathematical concepts thought its different representations and the meaning they attached to different situations, but some students equate the representations with the objects, and this removes the focus from the concept itself.

Thus, there is a need to be able to properly understand and characterise students' conceptual understanding of functions such that we can improve the way the concept is taught.

The second reason for studying students' conceptual understanding of functions is that problem solving skill can be divided into four domains: resources, heuristic, control, and beliefs. Simply put: in order to be proficient at solving mathematical problems, one needs some basic mathematical knowledge (resources); a set of procedures, or rules of thumb, for making progress on a difficult problem (heuristics); and the mental oversight to monitor ones actions, and plan ahead (control). Furthermore ones ideas (beliefs) about what mathematics is, has an influence on the problem solving process [Schoenfeld, 1985]. Thus, conceptual understanding, being a part of the students resources, has an influence on students problem solving performance.

It is well known that students' conceptual understanding of many core concepts is less than ideal. The mathematics education literature is filled with reports about students difficulties with concepts such as function, variable, limit, and compactness, to name just a few; thus, it is reasonable to ask:

1.1 Research question

What characterises students' conceptual understanding of functions?

That is, how can we adequately describe students' conceptual understanding? By adequately describe, I mean be able to understand, explain, and – not in the least – communicate students' behaviour, when they are dealing with the function concept. An answer to this question will be sought partly through studying the relevant literature, and partly through an empirical study.

The relevant literature is immense; thus, the real challenge does not lie in finding relevant literature, as much as choosing parts of the literature, which is representative for the multitude of theories and observations made by researcher scattered across the globe.

Many have attempted to understand and model students' conceptual understanding: Vinner with his concept image/concept definition theory[Tall and Vinner, 1981, Vinner, 1983, Vinner and Dreyfus, 1989]; Dubinsky with his Action-Process-Object-Schema approach (APOS)[Asiala et al., 1997]; and Sfard with her theory of reification[Sfard, 1988, 1991, 1994, Sfard and Linchevski, 1994a,b], just to name a few of the most prominent examples of general theories of the development of conceptual understanding in mathematics. These theories have different origins and, to some extent, a different take on how to explain students' conceptual understanding. I will briefly outline the three theories as examples of what the literature has to offer.

Vinner observed that some students were unable to figure out whether or not a graph, or an algebraic expression, represented a function, despite the fact that the students were able to give a correct, set theoretical, definition of a function; that is, the students were unable to use their definition of the concept. This lead Vinner to introduce the notion of concept image and concept definition. The concept definition is a linguistic definition of the concept that accurately explains the concept in a non-circular way, while the concept image is the set of mental images, properties, and examples of the given concept. Vinner noticed that the students concept image and concept definition can disagree but coexist, apparently without giving rise to cognitive conflicts for the student. [Vinner, 1983, Vinner and Dreyfus, 1989]

For instance, the concept definition of a mathematical function might be taken to be *"a relation between two sets A and B in which each element A is related to precisely one element in B"*, but individuals who have studied functions may, or may not, remember the concept definition, and the concept image may include many other aspects, such as the idea that a function is given by a rule or a formula, or perhaps that several different formulae may be used on different parts of the domain. There may be other notions. For instance, the function may be thought of as an action which maps a in A to $f(a)$ in B, or as a graph, or as a table of values. All or none of these aspects may be in an individual's concept image. [Tall and Vinner, 1981]

Sfard's theory of reification is inspired by historical accounts of how mathematical concepts have evolved through time. Based on these anecdotes, as well as observation of students misconception about a set of mathematical concepts, she constructs a model of how mathematical concepts evolve, in mathematics as a discipline, as well as for the individual learner. The foundation of the theory is the notion that mathematical concepts can be conceived in two fundamentally different ways; as a process, or as an object. She goes on to specify the way in which students' conception of a concept can move from the process conception to the object conception, though three consecutive stages: interiorization, condensation, and reification. [Sfard, 1988, 1991, 1994, Sfard and Linchevski, 1994a,b]

An entire chapter of the thesis will be devoted to this theory, so I will say no more about it now.

Dubinsky et. al. have developed a theory with the basic hypothesis that an individual's mathematical knowledge can be described through that individual's mental ***actions***, ***processes***, and ***objects*** organised in ***schemas*** to make sense of a given situation and help solve problems. The theory is often referred to as the APOS theory. An action is a transformation of an object carried out by an individual through some algorithm. When an individual constructs an internal operation that preforms the same transformation as the actions, the action is said to be *interiorized* to a processes. When an individual is capable of carrying out an action on the process, it is said to be *encapsulated* to an object. A coherent collection of processes and objects is called a schema. Schemas are invoked by the individual in respond to a specific situation or problem. They can be *thematised* to become another kind of objects. At times it might be necessary for the individual to de-encapsulate a given object or de-thematise a schema, in order to deal with a given situation or problem, thus the encapsulation and thematisation are no irreversible. [Asiala et al., 1997, p. 2]

The APOS theory is based on the same fundamental observation as Sfard; the fact that most mathematical concepts can be conceived as both objects and processes; however, the focus of the theory is on how a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts, and by suggesting directions for pedagogy that can help in this learning process [Asiala et al., 1997, p. 3]. The theory has a special emphasis on application.

As I see it, the three theories differ in interesting ways. The APOS theory is normative – it attempts to explain *how* teachers should go about introducing the mathematical concepts to the students – while both the concept image/definition and the theory of reification are descriptive – they strive to *characterise and explain* problems and phenomena which occur in students learning of mathematical concepts, but mostly without inferring how the teachers should go about solving these problems. Another difference lies in the origins of the theories; firstly we have the theory of reification which is, to a large extent, bases on historical accounts of how the concepts have developed, with the occasional empirical *verification*. Secondly we have the APOS theory, which is developed in intimate collaboration with practicing teachers and with an explicit interest in how to improve educational programs. Thirdly we have the concept image/ concept definition theory, which has its origins in observations about students misconceptions of different mathematical concepts. I think that these theories are representative for the different types of theories the literature has to offer.

One of the main points of this thesis is to establish a two-way contact between theory and experiment: On the one hand, the theory should be able to help you classify and understand observations; that is, the theory should be *useful*. On the other hand the observations should be the foundation of further revision and development of the theory; that is, the theory should not be set in stone. This is one of the reasons why this thesis contains both a theoretical *and* an empirical part.

For the empirical part I have chosen to use problem solving as a way of probing the students' conceptual understanding of functions. I have done so because I think that in order to comprehend students' understanding of the function concept, it is essential to see how the students use the concept when they actually *do* mathematics.

The term *problem solving* is used abundantly in the mathematics education literature, but there is no consensus among researchers on how to use the term. It it therefore necessary to clarify exactly how I use the term *problem solving*. The term is generally used in two very different ways: Either as a synonym for any activity involving mathematical thought and arguments, or in a more rigid and strict way, in which *problem* is to be understood as a specific kind of mathematical question; one which can not be answered by simply imploring a set of routine activities. When problem is used in that context, it is clear that the classification of problem depends on the person who needs to answer the specific question; thus, the classification of "problem" is subjective.

I will use the term in an intermediate way. The assignments the students solve would most likely not be problems in the strict sense, so it might make sense to call them non-problematic. However, most of them are non-standard in their phrasing and general feel. Let me give some examples. A standard problem might be: $f(x) = x$, $g(x) = -x + 2$ for which x is $f(x) = g(x)$? A question like this could very well appear in a classic A-level mathematics text book in the danish secondary school. A non-standard, non-problematic example could be: $x + y = 10$, what can you say about x in relation to y? It is unlikely that you will find a problem like this in any danish mathematics textbook; thus, it is not standard for the students, but the secondary school students will most likely not find it problematic in the strict sense. An example of a mathematical question that would probably be problematic in the strict sense is: Can you construct a triangle from three arbitrary side lengths? An answer to this question would require a proof, probably by contradiction, or a counter example. Again the subjectivity of the categorisation of a mathematical question as problematic should be stressed: what is problematic to some is standard to others.

There are several reasons why I have decided not to use problems in the strict sense. First of all, I wanted to be able to conduct the empirical studies within the time limit of a normal secondary school mathematics class. This was prioritised to make it more likely that the secondary school teacher would agree to let me use their students for the studies and, perhaps more importantly, that the students would not find it too time consuming and tedious. Secondly, I wished to attain a broad sense of the students conceptual understanding which requires them to work on several different kinds of problems, containing different representation of the concept. Having a lot of problems in the strict sense would, most likely, yield the information I need, but it would take longer than the 45-60 minutes at my disposal. Asking a lot of standard questions would make it possible to include a lot of problems and fish in time; however, since such problems can be solved by imploring routine procedures, I might not get the necessary amount of information. A compromise was necessary. Non-standard, non-problematic "problems" seemed to fit the bill. For the reminder of the thesis I will simply use the term "problem", but keep in mind, it is, most likely, not problems in the strict sense.

Another note about the problems: The problems have been designed with a focus on the different representations of the function concept, and great care have been taken cover many representations of the function concept, as well as connections between them. I have chosen to focus on different representations and their connections for two reasons: Firstly, the literature suggest that students understanding of the different representations is more superficial than one could hope, see e.g. [Knuth, 2000]. Secondly, the problems have been designed with the theory of reification in mind. The ability to skilfully change between different representations of the same concept indicates that a person has reified (made an object of) the concept. Thus, studying students understanding of different representations of the function concept should indicate how solid the students' understanding of the function concept is.

1.2 Outline of the report

The thesis is divided into four parts. The first part was the introduction.

The second part of the thesis presents the theoretical framework on which the thesis is based. The framework serves as a guide for designing the empirical studies as well as an analytical tool for presenting and discussing the data. It includes a chapter about how $f(x)$ is used as a symbol in mathematics; a chapter on Schoenfeld's work on problem solving; and a chapter on Anna Sfard's the theory of reification. The chapter on the theory of reification culminates with the development of a hypothesis that extends the theory. Part one is concluded with a summary in which theories within the didactics of mathematics are discussed in general.

The third part of the thesis reports on three studies carried out in the period 2012-2013. It starts off with a chapter on the methodological considerations regarding the thesis as a whole, and the empirical studies in particular. These considerations are delayed to the second part of the thesis because they are so deeply rooted in the theoretical framework, that it would not make sense to present them prior to the presentation of the theoretical framework itself. The three studies consists of video recordings of pairs of students solving a set of problems designed specifically to probe the students' conceptual understanding of functions. Each study is presented as a independent whole with its own results and discussion. Since the purpose of the thesis is to characterise students' conceptual understanding of the function concept an attempt at this is made at the end of each empirical study.

The fourth part of the thesis presents a coherent discusses of the main results and observations from the theoretical work, as well as the empirical studies. This discussion also includes suggestions for further work. Lastly, the part includes a conclusion on the thesis as a whole.

The thesis is written under the assumption that the reader has an understanding of mathematics equivalent to a bachelors degree in mathematics. The reader is also assumed to have no prior knowledge of the didactics of mathematics.

Part II

Theoretical background

2 Schoenfeld's model of mathematical problem solving

In his pioneering book about mathematical problem solving from 1985 Alan Schoenfeld attempts to categorise mathematical problem solving [Schoenfeld, 1985]. In his own words his goal is

> "... to explain, as accurately as possible, what takes place during the solution attempt. What mathematical knowledge is accessible to the problem solver? How is it chosen? How is it used? Why does the solution evolve the way it does? In what ways do the approaches taken to solve the problem reflect the individual's understanding of this area of mathematics, and what is the relationship between the understanding and the individual's problem solving performance? And finally what accounts for the success or failure of the problem solving attempt."

In order to answer these many questions he identifies four categories in which the problem solver can have different abilities. The four categories are: *Resources*, *heuristics*, *control* and *belief system*. A section will be devoted to each category.

Schoenfeld used the term problem in the strict sense, but even though we are not dealing with problems in the strict sense, the general characterisation that Schoenfeld has developed has proven relevant even for non-problematic problems. Thus, I will introduce his work, and it will serve as a tool for communicating parts of the analysis in the empirical part of the thesis.

2.1 Resources

A persons resources is the mathematical knowledge the individual might bring to use during the problem solving process. Resources cover: Intuition and informal knowledge about the mathematical domain; facts; algorithmic procedures; routine and non-algorithmic procedures; and understanding of the appropriate norms of the mathematical domain in which the problem takes place. A complete inventory of resources will characterise each of the skills in these categories that the individual might be able to use. It is important to note that resources may be more than shaky, they may be wrong. Therefore, a complete inventory of an individual's knowledge should also describe how solid each of these resources is.

It is obvious that resources has an influence on the success or failure of problem solving attempts. One might naively think that an individual's resources influence the problem solving process in a trivial way; either you have to resources, or you don't. However, since resources cover the students standard procedure (not heuristics), informal knowledge about the mathematical domain, and knowledge of mathematical concepts, the ways in which resources might influence problem solving attempts are quite diverse.

2.2 Heuristics

Heuristics are defined as strategies and techniques for making progress on unfamiliar or nonstandard problems; rules of thumb for effective problem solving including, but not limited to: Drawing figures; introducing suitable notation; exploiting related problems; reformulating problems; working backwards; and testing and verification procedures. Thus, heuristics is, per definition, not covered under resources.

One might think that resources and heuristics alone should be enough to get through even the most demanding mathematical problems. The resources describe the students archive of mathematical knowledge, and the heuristics lets you make progress on unfamiliar ground. This, however, is not the case.

Schoenfeld has studied the influence of heuristics on problem solving and found that heuristics alone was not enough to guarantee successful problem solving. Furthermore he found that even though students *can* master a particular problem solving technique, there is no guarantee that they *will* use it [Schoenfeld, 1979]. This brings us the to next category: Control.

2.3 Control

Control covers the global decisions regarding the selection and implementation of resources and strategies covering: planning; monitoring and assessment; decision-making; and conscious and metacognitive acts.

It should come as no surprise, that an individual's ability to select the relevant resources, make a solution plan, and monitor and asses their progress along the way, has an influence on the overall success of the problem solving process.

The effect of control on problem solving was largely studied in [Schoenfeld, 1985]. In a study, Schoenfeld investigates the possibility of directly teaching control techniques to students. The study yielded two results: (i) Even in a simple and straightforward domain (integration), students do not, by themselves, develop efficient control strategies, and their problem solving performance suffers because of it; and (ii) a prescriptive control strategy can result in significant improvements in students' performance.

Thus, one should not expect secondary school students to have great control, and consequently, control issues might have a great impact on their overall success.

2.4 Belief system

Ones mathematical "world view" is termed belief; it is defined by Schoenfeld to be the set of, not necessarily conscious, determinants of an individuals behaviour. It covers their beliefs about: themselves; the environment; the mathematical topic; and mathematics in general. Schoenfeld found that students' mathematical beliefs shape the ways that they behave in mathematical situations; thus, the influence of students' beliefs on their problem solving performance is important, but difficult to study, and for many years it has been considered a *hidden variable* in educational studies. [Törner, 2013]

To get an idea of to what extent beliefs have an influence on the problem solving performance, we can consider an example from Schoenfeld's book: Two students have worked on a geometrical problem in which they are to make a specific construction. The approach the students take is entirely empirical; the students make conjectures and then test them by construction, with the sole standard for accepting or rejecting a potential solution to the problems being the accuracy of the construction. From this behaviour it appears that that the two students are completely unfamiliar with deductive geometry; however, it is later shown that they are perfectly capable of making the deductive arguments that provide the answer the the problem. They simply had not though to approach the problem that way.

Belief also covers students thoughts about the norms of a given mathematical subject. As an example of this behaviour we can consider two students, from study A in this thesis, who are working on a standard mathematics assignment; they are given the graph of two functions, $f(x)$ and $g(x)$, which intersect in $x = 2$, and are asked to find the x value that solves $f(x) = g(x)$. Rather than reading off the x values from the graph, the students find the expression for the two function, and continue to solve for x in the equation $f(x) = g(x)$. You might think that the students were simply not able to recognise the intersection of the graphs as the point where $f(x) = g(x)$, but later the students attempt to solve a similar problem by finding the intersection of two graphs; thus, this is not the case.

These examples goes to show that students' success or failure to solve mathematical problems can not be adequately describe by their resources, heuristic, and control alone. More recently the influence of beliefs on students' learning of mathematics in general has been studied by several authors, see [Törner, 2013] for a review.

2.5 Summary

We have seen how problem solving ability can be characterised through the four categories resources, heuristics, control and belief. The most important thing to take away from this short presentation of Shoenfeld's work it this: Problem solving processes are complex, and the success or failure of a given solution attempt can be due to *any* of the categories; thus, when analysing problem solving sessions it should be done with great care.

3 | $f(x)$ as a symbol

This chapter grew out of several discussions about students' difficulties with the notation most frequently used for functions in the danish educational system: $f(x)$. The discussions were part of the analysis of the data from the empirical studies also presented in this thesis, and I felt that a discussion of the symbol $f(x)$ was necessary in its own right. The different uses of the symbol $f(x)$ is sometimes mentioned in the literature, but I am not aware of any literature devoted solely to the discussion of this symbol; that is the aim of this chapter. The chapter stands alone in the sense that it transcends the theories otherwise presented.

The symbols used to represent instances of the function concept are more numerous and diverse than the three most used representation: graph, algebraic expression, and table of function values. For example, the symbol $f(x)$ is used across the other representation; we write $f(x) = ...$ when we represent a function as an algebraic expression; we write $f(x)$ over the graph of the function; and we write $f(x)$ in the column of the table containing the function values. $f(x)$ is use both as a symbol for representing the function values, e.g., $f(2) = 3$, and as a symbol referring to the function as an object.

This way of using the symbol has a lot of advantages; we can express ideas effectively across different representation. For example, we can talk about $f(f(x))$, $f'(x)$, $f(x)+g(x)$, and $f(x+y) = f(x) + f(y)$, without specifying the representation in which we want to carry out the operation or which specific function that should be used. $f(x)$ is a very potent symbols; it enriches to function concept substantially. Furthermore, the notation $f(a)$ also serves as a reminder of the uniqueness of function values; something which students are known to forget.

However, we run the risk that the students, according to Steinbringe, identify the symbol $f(x)$ with the concrete examples which the symbol is referring to. This is no trivial risk, if the students identify the symbol $f(x)$ with the objects they wish to describe, they will miss out on the richness and flexibility that the symbol brings to the concept. For example, this would pose a difficulty for an individual learning about the concept of differentiation; if one does not have a flexible understanding of the $f(x)$ symbol it is conceivable that $f(x+\Delta x)$ will not make a lot of sense to the individual, and consequently the difference quotient might seem like a strange concept.

4 The theory of reification

In this chapter Anna Sfard's theory of reification is presented. The problems that are used in the empirical studies were designed with this theory in mind, and the theory will be used as the theoretical basis for the data analysis.

The theory, as it is presented here, is my interpretation of the what have been stated in the following publications: [Sfard, 1988, 1991, 1992, 1994, Sfard and Linchevski, 1994a,b]. Thus, the theory is not in itself original; however, this specific presentation of it is. The presentation will include many examples, some of these will by my own interpretation of how the theory might be used on a specific concept – some will not be. For each example I have attempted to make it clear which one is the case.

4.1 The general theory

In [Sfard, 1991] Anna Sfard presents a theoretical framework for describing the development of mathematical concepts within mathematics as a discipline, as well as the development of individuals' conceptual understanding during the learning process. She states that mathematical concepts, such as number or function, can be conceived in two fundamentally different ways: *structurally* - as objects, and *operationally* - as processes. The model is general in the sense that it seeks to capture the essential aspects of the development of mathematical concepts in mathematics as a scientific discipline as well as for in the individual.

A distinction is made between the mathematical *concept*, an abstract mathematical notion which can be represented in different ways, and a persons *conception* of a given concept which is *"the whole cluster of internal representations and associations evoked by the concept"* for that particular individual.

She conjectures that the operational conception is, for most people, the first step in acquisition of new mathematical concept. Furthermore she argues that the transition from a operational conception to a structural conception is a long and inherently difficult process, accomplished in three steps: *interiorization*, *condensation*, and *reification*.

The relationship between structural and operational conception is discussed at length in the article. The main point is, that the two views are not a dichotomy, but rather a duality; they are complementary in the same way as the wave and particle description of light is complementary – in order to explain the whole range of phenomena involving light, it is necessary to adopt both points of view. Thus, having a structural conception of a concept does not exclude ones ability to carry out *"the process"*. In fact, the ability to flexibly switch between these two conceptions in a sign of a healthy conceptual understanding.

The three stage model – *interiorization*, *condensation* and *reification* – for the development of mathematical concepts is guided by historical anecdotes, as well as observations about students ability to use mathematical concepts. A description of the stages is given below:

Interiorization. At the stage of interiorization the learner gets acquainted with the process which will eventually give rise to a new concept. A process has been interiorized if it can be carried out through mental representations, and in order to be considered, analysed, and compared it needs no longer to be actually performed.

Condensation. The stage of condensation is a period of squeezing lengthy sequences of operations into more manageable units. The condensation phase lasts as long as the new entity remains tightly connected to a certain process.

Reification. The stage of reification is defined as an ontological shift – a sudden qualitative jump in the way of looking at things. A process solidifies into an object, into a static structure. The individual will be able to consider different representations of the concept and skilfully alternate between different representations. The stage of reification is the point where an interiorization of higher level concepts begins.

It should be stressed that the fact that a process has been interiorized and condensed into a compact self-sustained entity, does not mean, by itself, that a person has acquired the ability to think about it in the structural way. Without reification the individual's approach will remain purely operational.

Once the stage of reification is reached, a now object is born. This new object can then be used in other processes and the development continues. A schematic representation of this development is shown in figure 4.1. It should be clear that that the three stages are hierarchical.

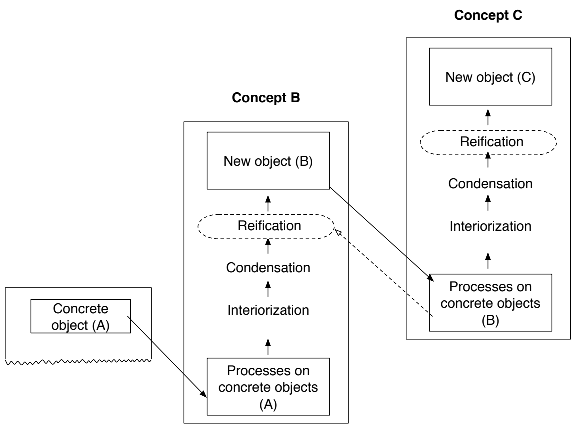


Figure 4.1 – Visual representation of Anna Sfard's model of concept formation. A mathematical concept can be conceived as a process or as an object. To be able to conceive it as an object the learner must pass through the three stages interiorization, condensation, and reification. When the stage of reification is reached a new mathematical object is born, and the concept can be used as inputs for other processes.

The schematic representation of the three stage model illustrates an unfortunate fact about the development of new mathematical concepts. The reification of a concept takes place simultaneously with the interioiaztion of concepts involving the newly reified concept. This forms the *"vicious circle"*. The problem is that, on the one hand, without an attempt at the higher-level interiorization, the reification will not occur – without the need to preform some process on the concept, it hardly seems necessary to consider the concept as an object. On the other hand, the existence of objects on which the higher-level processes are preformed on seems indispensable for interiorization – without such objects the processes must appear quite meaningless. The problem with the vicious circle can be stated more compactly: *"the lower-level reification and the higher-level interiorization are prerequisites for each other!"* [Sfard, 1991, p 31].

To get a feel for the inherent difficulty one can imagine trying to differentiate a function or solve a differential equation, when ones conception of a function is still in the process stage. The vicious circle thesis explains some of the inherent difficulty in acquiring a satisfactory understanding of new concepts.

But is this progress from operational to structural conception necessary at all? Why should we strive to reach it? The short answer is that without the structural conception, some problems are simply too complex to be handled with the rather limited processing power of our human minds. Furthermore, the formation of structural conception seem essential for further learning – for acquisition of more advanced concepts. [Sfard, 1992]

This is the essence of the theory of reification. The theory is very general, but as Sfard herself points out:

> "Our model should not be expected to fit in with every possible concrete example, because like any formal structure imposed on natural phenomena, it is not much more than a rough approximation of a prevailing tendency." [Sfard, 1992, p 64]

Before the theory can be utilised as an analytical tool, it is necessary to consider how the model relates to a series of concrete examples. This is the purpose of the next couple of sections. The theory will be used to describe the development of the mathematical concepts: Variable, function, and derivative of a function.

4.2 The case of variable

In [Sfard and Linchevski, 1994a] Sfard and Linchevski apply the theory of reification to algebra. They give an historical account of how algebra has changed from being process orientated to object orientated throughout history, in accordance with the theory of reification. They identify several different stages in students' conception of algebra, but focus on two main transitions: The transition from conceiving algebra as a general arithmetic process, to algebra of an unknown; and the transition from the algebra of an unknown to a functional algebra.

It has been shown by Usiskin in [Zalman, 1999] that students' different uses of variable reflects their conception of algebra, so even though Sfard and Linchevski's focus is on the development of students conception of algebra, it is not far fetched to use their analysis of algebra as the basis for my analysis of the variable concept.

In [Zalman, 1999] Usiskin identifies four different conceptions of algebra each related with a specific use of the variable concept. They are: Algebra as generalised arithmetic; algebra as a study of procedures for solving certain kinds of problems; algebra as a study of relationships; and algebra as the study of structure. This characterisation of the different conceptions of algebra, and the use of the variable concept is extended by Trigueros and Ursini in [Trigueros and Ursini, 2003]. They identify three conceptions of variable: *as a specific unknown*, *as general number*, and *in functional relationships*. For each of the conceptions they have identified several indications, that might suggest that an individual has the given conception. Bases on these bodies of work, I have reached the following description of the stages of reification for the variable concept:

Interiozation: (Variable as unknown). The individual learns how to solve for a specific unknown in an an equation, e.g. solving for x in $x + 7 = 2x + 12$. It is at this stage that the individual becomes familiar with carrying out operations on symbols representing unknown numbers. It could be argued that an unknown is not, in fact, a manifestation of a variable because it represents a fixed value; nevertheless, like Trigueros and Ursini in [Trigueros and Ursini, 2003], I consider it the first step in acquiring the variable concept.

Condensation: (Algebra with general numbers). The individual starts to understand that symbols can represent general numbers, not just specific unknowns. They are able to deduce general methods by distinguishing the constant aspects from the variable ones. They can manipulate the symbols and symbolise general statements, rules or methods. An example would be the ability to express the commutative and distributive property of addition and multiplication of scalars as $a \cdot b = b \cdot a$ and $a(b + c) = a \cdot b + a \cdot c$.

Reification: (Symbols as functional relations) This conception involves interpreting the symbols as representing a correspondence and joint variation in analytical representations, tables, and graphs. The individual will recognise the correspondence between related variables independent of the representation used. The individual becomes confident in changing between the different views of a variable and becomes familiar with the functional relation. It is necessary to be able to change between different views of symbols when learning about functional algebra since functions like $y = ax + b$ contains symbols representing general numbers (a and b) and variables (x and y).

Thus, the process which eventually evolves into the concept of variable, is the ability to let symbols represent numbers and solving for an unknown, even though the symbol represents a static unknown initially. The reification consists in the ability to interpret the symbols as representing a correspondence and joint variation.

4.3 The case of function

The concept of function is closely related to that of variable. A lot of researchers have focused on students' misconception about functions. In this section, however, I will outline how a student's (healthy) transition from an operational conception to a structural conception might be described using the theory of reification. Sfard is not the only one who have been working on students conceptual development. Therefore I will present some of the literature regarding students conceptual understanding of the function concept, as inspiration for the application of the theory of reification.

4.3.1 Some historical examples

It is worthwhile to briefly consider how the concept of function has developed historically. This is done because the theory of reification should describe the development of mathematical concepts in history as well as in the individual. Thus, the historical development of the function concept should serve as an excellent inspiration for how the theory of reification can be applied to the function concept. Now, there is no real reason to believe that the development of an individual's conceptual understanding should mirror that of history, but it can still be used for inspiration. I will show that the definition has changes over a period of time, from a process orientated definition to a object orientated definition. The brief historical survey is based on [Kleiner, 1989], and is by no means an attempt at making a complete and comprehensive presentation of all the historical curiosities that lead to the function concept we have today.

We start with Bernoulli's definition of a function from around 1720:

> "One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants" [Kleiner, 1989, p. 3]

This was the first formal definition of a function, but it was not explained exactly how one should interpret *"composed in any manner whatever"*. In this definition the function concept is closely tied up with the variable concept.

Another definition along the same line is the one by Eulers' from around 1750:

> "A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities"[Kleiner, 1989, p. 3]

Both Euler and Bernoulli's definition leans heavily on the variable concept. Euler stresses that fact that a function should be an analytical expression. Fourier gave the following definition of a function:

> "In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there are an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given if it were a single quantity." [Kleiner, 1989, p. 8]

This definition no longer focuses on that fact that the relation should be describable by an analytical expression, but functions are still restricted to numbers. Cauchy gave a similar definition of a function:

> "When the variable quantities are linked together in such a way that, when the value of one of them is given, we can infer the value of all the others, we ordinarily conceive that these various quantities are expressed by means of one of them which then takes the name of *independent variable*; and the remaining quantities, expressed by means of the independent variable, are those which one calls the *functions* of this variable" [Kleiner, 1989, p. 10]

Although Cauchy's definition is rather general, evidence suggests that he had something more limited in mind [Kleiner, 1989, p. 10]. Dirichlet takes the definition one step further in 1830:

> "y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established." [Kleiner, 1989, p. 10]

Dirichlet was the first to take the notion of function as an arbitrary correspondence seriously. However, his definition is still a correspondence between *numbers*. Dedekind gave a fairly modern definition of function around 1890:

> "By a mapping of a system S a law is understood, in accordance with which is called the image of s and is denoted $\phi(s)$; we say too, that $\phi(s)$ corresponds to the element s, that $\phi(s)$ is caused or generated by the mapping ϕ out of s, that s is transformed by the mapping ϕ into $\phi(s)$." [Kleiner, 1989, p. 18]

Here functions are considered to be mappings between sets, not necessarily between numbers. Bourbaki elaborated further on this idea. The following definition is from around 1940:

> "Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E, there exists a unique y in F which is in the given relation with x.
>
> We give the name of function to the *operation* which in this way associates with every element x in E the element y in F which is in the given relation with x; y is said to be the value of the function at the element x, and the function is said to be determined by the given functional relation. Two equivalent functional relations determine the same function" [Kleiner, 1989, p. 18](my emphasis)

This is the definition of a function as a set or ordered pairs. Even though this definition mentions an operation, it is purely structural.

Thus, we have seen examples of how the definition of a function has changes over a period of little over 200 years. The evolution of the concept goes from variable, to correspondence, to mapping, to ordered pairs, with each successive change being more structural than the preceding.

4.3.2 Sierpinska's epistemological obstacles and acts of understanding

Now that we have used the historical development for inspiration it is time to turn to the mathematics education literature for further inspiration. As was explained in the introduction, there are quite a few different theories of how conceptual understanding develops for the individual. I focused on the APOS theory, the concept image/definition theory, and the theory of reification. In this section I will turn to another theory which I will refer to as the theory of epistemological obstacles.

In [Sierpinska, 1992] Sierpinska attempts to clarify what it means to understand the function concept. She concentrates on *"the jumps in understanding i.e. the qualitative changes related to mathematical knowledge in the human mind, jumps from old ways of knowing to new ways of knowing"* [Sierpinska, 1992, p. 27].

There are two complementary ways of looking at these jumps – as *epistemological obstacles* and as *acts of understanding*. Things that prevent one from a new way of knowing are called epistemological obstacles, while changes in new ways of knowing are called acts of understanding. The acts of understanding are separated into four different categories: *identification*, *discrimination*, *generalisation* and *synthesis*. The two views are complementary because neither alone is enough to fully describe the jumps in understanding. Sierpinska has devised 19 levels of understanding, and 16 epistemological obstacles to overcome, when dealing with the function concept, all of which I have given in table 4.1.

As should be clear from table 4.1, Sierpinska's analysis is more fine-grained than the description we aim at; she has 19 acts of understanding and 16 epistemological obstacles to overcome, while the theory of reification has a meagre three stages. Therefore, in order to use Sierpinska's analysis we will have to single out the most important acts of understanding, and epistemological obstacles, or group them together. Not all of Sierpinska's points are relevant for our analysis, in fact, many of them belong with the development of the concept of variable. Furthermore, her epistemological obstacles have a focus which is not quite covered by the theory of reification; namely belief.

Act of understanding number 5 fits nicely with the stage of interiorization; in order to calculate the function value, you should know which variable is dependent, and which is independent. Acts of understanding 11 and 15-16 are all covered by the stage of reification. It is the point where the general notion of function is synthesised as an object, and an individual will be able to discriminate between different means of representing functions and the functions themselves. It seems that Sierpinska's analysis does not cover the stage of condensation. This is not very surprising because the stage of condensation is a period of becoming more efficient at carrying out processes, or grouping processes together – there is no real act of understanding connected to the stage of condensation.

Table 4.1 – Summary of the different levels of understanding of the function concept (U) and the epistemological obstacles to be overcome (E).

	Understanding		Epistemological obstacle
U1	Identification of changes observed in the surrounding world as a practical problem to solve.	E1	**(A philosophy of mathematics)** Mathematics is not converned with practical problems
U2	Identification of regularities in relationship between changes as a way to deal with the changes.	E2	**(A philosophy of mathematics)** Computational techniques used in producing tables of numerical relationships are not worthy of being an object of study in mathematics.
U3	Identification of the subjects of change in studying changes	E3	**(Unconscious scheme of thought)** Regarding changes as phenomena; focusing on how things change, ignoring what changes.
U4	Discrimination between two modes of mathematical thought: one in terms of known and unknown quantities, the other - in terms of variable and constant quantities.	E4	**(Unconscious scheme of thought)** Thinking in terms of equations and unknowns to be extracted from them.
U5	Discrimination between the dependent and independent variables.	E5	**(Unconscious scheme of thought)** Regarding the order of variables as irrelevant.
U6	Generalisation and synthesis of the notion of number	E6	**(An attitude towards the concept of number)** A heterogeneous conception of number.
U7	Discrimination between number and quantity	E7	**(An attitude towards the notion of number)** A pythagorean philosophy of number: everything is number.
U8	Synthesis of the concepts of law and the concept of function; in particular , awareness of the possible use of functions in modelling relationships between physical or other magnitudes.	E8	**(An unconscious scheme of thought)** Laws in physics and functions in mathematics have nothing in common; they belong to different domains (compartments) of thought.
U9	Discrimination between a function and the analytic tool sometimes used to describe its law.	E9	**(An unconscious scheme of thought)** Proportion is a privileged kind of relationship.
U10	Discrimination between mathematical definitions and description of objects.	E10	**(A belief concerning mathematical methods)** Strong belief in the power of formal operations on algebraic expressions.
U11	Synthesis of the general conception of function as an object.	E11	**(A conception of function)** Only relationships described by analytic formulae are worthy of being given the name of functions.
U12	Discrimination between the concept of function and relation.	E12	**(A conception of definition)** Definition is a description of an object otherwise known by senses or insight. The definition does not determine the object: rather the object determines the definition. A definition is not binding logically.
U13	Discrimination between the notions of function and sequence.	E13	**(Conception of functions)** Functions are sequences.
U14	Discrimination between coordinates of a point of a curve and the line segments fulfilling some function for the curve.	E15	**(Conception of co-ordinates)** Coordinates of a point are line segments (not numbers)
U15	Discrimination between different means of representing functions and the functions themselves.	E15	**(Conception of graph of function)** The graph of a function is a geometrical model of the functional relationship. It need not be faithful, it may contain points (x,y) such that the function is not defined in x.
U16	Synthesis of the different ways of giving functions, representing functions and speaking about functions.	E16	**(A conception of variable)** The changes of a variable are changes in time
U17	Generalisation of the notion of variable		
U18	Synthesis of the role of notions of function and causes in the history of science: Awareness of the fact that search for functional and causal relationships are both expressions of the human endeavour to understand and explain changes in the world.		
U19	Discrimination between the notions of functional and causal relationships.		

4.3.3 Application of the theory of reification to the function concept

Based on the historical account, and Sierpinska's analysis, we are ready to apply the theory of reification to the concept of function. Here are the stages:

Interiozation: The operational conception of a function is a processes which, given an input, yields an output. The stage of interiorization is reached when the ability to find function values is acquired.

Condensation: At the stage of condensation, the learner will become capable of playing with a function as a whole, without actually looking into its specific values. Eventually the learner will be able to investigate functions, combine couples of functions, and even find the inverse of a given function.

Reification: Functions are conceived as objects. The student becomes capable of discrimination between different means of representing functions and the functions themselves. The individual will begin to be able carry out processes in which a function serves as an input, and be able to solving equations in which the "unknowns" are functions, that is, e.g. differential equations.

The process which will eventually become the concept of function, is the process of finding function values. When the concept of function is eventually reified, the student will be able to treat functions as objects.

4.4 The case of derivative of a function

As an example of a process which is carried out on functions, we will consider the concept of derivative of a function. For inspiration I turn to Asiala et. al.'s study of students' understanding of the concept of derivative, which is presented in [Asiala et al., 1997]. Their analyses is based on the APOS theory; however, the theories are similar enough, that it serves as a good inspiration.

They make a *generic decomposition* of the concept in both its algebraic and graphical representation. A generic decomposition, or model of cognition, is a description of specific mental constructions that a learner might make in order to develop his or her understanding of the concept. These mental constructions are called actions, processes, objects, and schemas, hence the name: the APOS Theory. Their generic decomposition consists of eleven stages of development. Three stages for the graphical representation, three for the analytical (algebraic) representation, and five stages that describe different stages of encapsulation, interiorization, coordination, and reconstruction of the relation between the two representations. The stages are as follows:

1a. Graphical: The action of connecting two point on a curve to form a chord which is a portion of the secant line through he two points together with the action of computing the slope of the secant line through the two points

1b. Analytical: The action of computing the average rate of change by computing the difference quotient at a point

2a. Graphical: Interiorization of the actions in point 1a to a single process as the two points on the graph get "closer and closer" together.

2b. Analytical: Interiorization of the actions in point 1b to a single process as the difference in the time intervals get "smaller and smaller", i.e., as the length of the time interval get "closer and closer" to zero.

3a. Graphical: Encapsulation of the process in point 2a to produce the tangent line as the limiting position of the secant lines and also produce the slope of the tangent line at a point on the graph of a function.

3b. Analytical: Encapsulation of the process in point 2b to produce the instantaneous rate of change of one variable with respect to another.

4. Encapsulation: Encapsulation of the processes in point 2a and 2b, in general, to produce the definition of the derivative of a function at a point as a limit of a difference quotient at a point.

5. Coordination: Coordination of the processes in points 2a and 2b in various situations to relate the definition of the derivative to several other interpretations.

6. Interiorization: Interiorization of the action of producing the derivative at a point into the process of a function f' which takes as input a point x and produces the output value $f'(x)$ for any x in the domain of f'.

7. Encapsulation: Encapsulation of the process in point 6 to produce the function f' as an object.

8.Reconstruction: Reconstruction of the schema for the graphical interpretation of a function using the relation between properties of functions and derivatives.

It should be quite obvious from their "generic decomposition" that the APOS theory and the theory of reification are similar in essence. Asiala et. al. have chosen to describe the initial development within two different representations. It is an interesting point, and we will spend more time on it in section 4.7.

Based on this analysis, I have constructed the three stages of reification, which are presented below.

Interiozation: The students understand the process of finding the differential quotient at a point.

Condensation: An individual will be able to find the derivative of a function, f', sketch the graph of the f', and use f' to e.g. find maxima and minima of f.

Reification: f' is conceived as a object. The individual will be able to find f'' and to think of $f'(x)$ as an unknown in other problems.

This conclude the application of the theory of reification to the concepts variable, function, and derivative of function.

4.5 Summary of the general theory

We have now considered the development from process to object of the three concepts: variable, function and derivative of a function. Hopefully this has given the reader an impression of the applicability of the theory of reification. The development of the concepts are summarised in figure 4.2.

Notice the *visiouse circle* which was mentioned in the beginning of the chapter in section 4.1. Before the a new concept can be interiorized it is necessary to have reified the previous concept, but a concept will not be reified before the interiorization of a new concept seems necessary. For example: Before the concept of function can be interiorized is is necessary to see variables as functional relations, but before the concept of variable can be reified into a functional relation, it is necessary to think about calculating function values; before the concept of derivative of a function can be reified it is necessary to consider $f(x)$ to be an object on which processes can be carried out, but before the concept of function can be reified, it is necessary have some idea of a process which would use $f(x)$ is input.

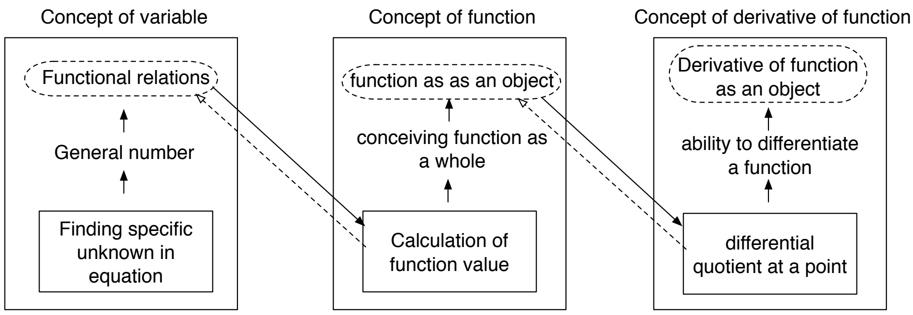


Figure 4.2 – Schematic representation of the development of the concepts variable, function and derivative of function.

4.6 Pseudo-structural conception

Until now we have considered the development of an individual's healthy conceptual understanding. It is well known that a students' conceptual understanding can be limited at times, and often filled with misconceptions [Harel and Dubinsky, 1992]. So in order for the theory of reification to serve as a tool for analysing students conceptual understanding it is useful to see how the theory might explain some of students common misconceptions. This is the aim of this section.

In [Sfard and Linchevski, 1994b] Sfard and Linchevsky carry out an analysis based on the theory of reification. They noticed that the majority of mathematical notions draw their meaning from two kinds of processes: *the primary process*, that is, the processes from which the given concept originated, and *secondary processes*, which are the processes that can be preformed on the given concept. They state that abstract objects act as a link between these two kinds of processes, and therefore seem to be crucial for our understanding of the corresponding concepts. They use the notion of Pseudo-structural conceptions for the conceptions which develop when the students, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result remains unaware of the relations between the secondary and primary processes.

When dealing with the concept of algebraic equations they state that the primary process is when one refers to the arithmetical procedures hiding behind the formulae, whereas the algebraic manipulations themselves will be called secondary processes. The abstract objects behind the concept are the truth sets. One of the main conclusions of the empirical study presented in [Sfard and Linchevski, 1994b] is that, among secondary school students the pseudo-structural conception of algebra may be more widespread than suspected.

For the function concept, they consider the sequence of numerical operations necessary to compute the values of a function as primary processes, whereas procedures which may be applied to a function as a whole, e.g. adding, composing, deriving, or integrating, are secondary processes.[Sfard and Linchevski, 1994b, p. 283] The abstract object linking the two, I would say, are the ordered pairs.

Sfard and Linchevsky have collected a number of phenomena which indicate that an individual have a pseudo-structural conception. They are [Sfard and Linchevski, 1994b]:

1. When dealing with equations and functions the specific form of the expression becomes the sole basis for judgements and decisions. This is a result of algebraic symbols being treated as things in their own right, not standing for anything else.

2. The secondary operation seems arbitrary and unjustified. This is the case for individuals who cannot see beyond the symbols.
3. Inability to see different representations of a concept as equivalent. If a sign serves also as its own referent, there is little hope that the student will be able to see different representations of the same mathematical concept as equivalent.
4. Inability to handle non-routine problems, even if the individual already has learned the relevant facts and the appropriate methods of solution. This is because mathematical objects are vitally important for our mathematical thinking. Mathematical objects tie together facts, concepts, and rules which would otherwise be stored in separate compartments of out memory. In the absence of such abstract mathematical objects, it is difficult to see the connection between new facts, when dealing with a new problem. As a consequence, students will choose algorithms of solutions based only on certain external features. Sometimes, even the most obvious discrepancies and absurdities will not make the students realise the inadequacy of the method they are using.
5. Confusion related to use of mathematical notation and terms. Students' confusion expresses itself as messy statements, in which different kinds of entities are mentioned at random and mistaken for each other. This kind of confusion will be called out-of-focus phenomenon, OOF for short.

I think that the notion of pseudo-structural conception serves well to encompass many of the most common observations regarding students misconceptions about mathematical concepts. Sfard argues that manny of these difficulties – common to the attentive teacher – may be regarded as an indication of a students inability to think structurally.

In [Sfard, 1992] Sfard considers the possibility of stimulating a structural way of thinking in students, with the concept of function as a case study. The way she evaluates the students ability of thinking structurally is, to a large extent, based on how well the development of a pseudo-structural conception of the function concept is avoided. She is motivated by the observation that despite the object-orientated way of teaching, the full-blown structural conception of function is rather rare in secondary-school. She supports this statement by presenting the findings gathered by many other researches, as well as the answers to a small questionary which she herself have designed. The papers presents results from a teaching experiment with the aim of teaching the function concept in an operational manner. This is done because the theory of reification suggests that the operational way of thinking precede the structural one, even if the concept is taught structurally initially. The results suggest that the students do not fully develop a structural conception, but that there are good reasons to believe that the danger of pseudo-structural conception were considerably diminished.

It seems to me, that when analysing the data from my own empirical studies, it would be a good idea to keep an eye out for signs of a pseudo-structural conception of functions.

4.7 An extension of the theory of reification

The idea that some students consider representations of concepts to be objects in themselves, rather than means of signifying the mathematical concept, have been mentioned twice in this thesis; once in the introduction as a part of Steinbringe's epistemological triangle, and in the previous section as an explanation of the a pseudo-structural conception.

It has been observed that students' conceptual understanding can differ across different representations of the same concept. For example, in the case of the function concept, it has been shown that many students prefer to use the algebraic representation, even when the graphical representation would yield an answer to the problems more efficiently [Knuth, 2000]. Thus, it seems logical to extend the theory of reification to accommodate for the observation that different representations are not necessarily considered equal by the students.

In this section I will entertain the idea that some students do in fact consider different representations of the same concept to be different objects, rather than different representations of the same concept, and see how this can be incorporated into the theory of reification. This will be done by describing how the different stages of reification can be reached across the different representations separately. The hypothesis will be referred to as the *disjoint-reification-of-representations hypothesis* (DRR hypothesis for short).

Since the extension should to be consistent with they general theory of reification, the initial stage of conceptual development should be the interiozation of a process, and the final stage should be the reification of the process into a mathematical object. In addition to this, I will introduce the possibility of different representations of the same concept evolving seperately, but still following the stages of interiozation, condensation and reification.

The interioization of a processes require a specific representation in which to carry out the process. Once this representation has been chosen, the student will gradually become more efficient in carrying out the process in the specific representation. This is the interioization of a representation of a process. The stage of condensation of a representation is a rather technical change, in accordance with the general theory of reification. The reification of a representation means that a student considers the representation to be an object in itself. The students will refer to the representation as if it was an object, and be able to carry out processes on the representation. However, the concept itself will not be reified before the student realises that the representation it not an object in itself, but a means of representing a mathematical concept.

Let us consider some consequences that should follow naturally from this hypothesis. It follows that a student might have reached the stage of condensation in one representation, and not in another. This means that the student will feel more comfortable, and be more efficient, in one representation than in another. This behaviour have been observed in e.g. [Knuth, 2000].

The hypothesis gives another dimension to the difficulty of reification of a concept. Earlier, the difficulty was mostly justified with the vicious circle – the observation that reification of a concept, and the interiorization of a higher level concept, are prerequisites for each other. Here, we see that reification of a concept require the reification of at least one, and quite possibly several, representations.

Notice also, that reification of a single representation is not necessarily a good thing. Having reified a single representation, the student might use it as an object in itself, rather than a means of representing a mathematical object. In the previous section, when we were discussing the pseudo-structural conception, this was one of the indication of having a pseudo-structural conception. Thus, it does not seem unreasonable that a pseudo-structural conception can be obtained as a consequence of having reified one or several representation, without having achieved reification of the concept itself. Once again, making reification of a concept very difficult to achieve.

A schematic representation of the disjoint-reification-of-representations hypothesis is shown in figure 4.3. I would like to point out that the reification of a representation is not equivalent to the reification of the concept. The reification of a representation is an ontological shift in the way the specific representation of the concept is conceived – the representation is considered an object, and not the actual concept. The lines going from the representations to the reification of the concept are dotted because the reification of the representation do not guarantee the reification of the concept. An ontological shift is still required for the concept. As long as the ontological shift has not taken place, the student will not have reified the concept.

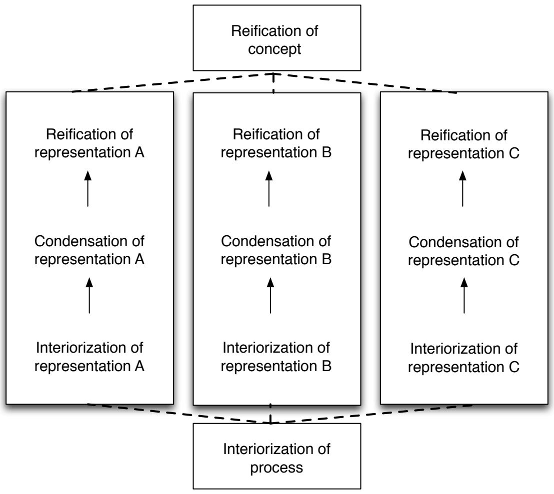


Figure 4.3 – Schematic representation of the disjoint-reification-of-representations hypothesis. Conceptual understanding can evolve separately across different representations. The concept will not be reified before the student realise that the different representation do in fact represent the same mathematical object.

I would like to discuss the relation between pseudo-structural conception and the DRR hypothesis in further detail. In the previous section the term pseudo-structural conception was used *"for the conceptions which develop when the students, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result remains unaware of the relations between the secondary and primary processes"*. Thus, pseudo-structural conception develops when students use symbols as objects in themselves. Now, the DRR hypothesis states that students will use symbols (representations) as things in themselves *because* their conceptual understanding have developed separately across different representations – the reification of a single representation can result in students using symbols as things in themselves. Therefore, the DRR hypothesis explains how a pseudo-structural conception may evolve.

As an example I will apply the hypothesis to the concept of function. I have chosen to focus on the three representations that are most commonly used: graphical, algebraic and table of values. In accordance with section 4.3.3, the process that must be interiozed is the process of finding the function value corresponding to a value of the dependent variable. The stage of condensation is a technical change which require that the student can play with the function as a whole, without actually looking into the specific values. This includes being able to investigate function, combine function and finding inverse of a given function. At the stage of reification functions are conceived as objects, and the students should before capable of changing between different means of representing functions. The result is presented in table 4.2.

Table 4.2 – The theory of reification applied to the different representation of the function concept.

	Graph	Algebraic	Table value
Interiozation	The student will be able to find values of the dependent variable by reading them off the graph. The process is following from the x-axis to the graph, to the y-axis.	The student is able to calculate values of the dependent variable by inserting values of the independent variable.	The student will be able to use the table to determine the dependent value.
Condensation	The student will be able to superimpose the graphs of two function and even draw the composite of two graphs.	The student is capable of adding function and calculating composite function.	It will be possible for the student to use the table of two graphs to find their sum or the composite of the two.
Reification	The student will consider the graph of a function as an object	The student will consider the algebraic expression as an object	The student will consider the table of values as an object.

Let us consider some specific consequences of this analysis. First of all, it is not strange that students ability can vary across different representations of the function concept. For example, it seems reasonable to assume that some students will be able to add the algebraic expressions for two functions, but be unable to superimpose the graphs of two function. Secondly, if a student has reified the algebraic representation, but not the graph representation, he may refer to the algebraic expression as an object, but not the graphical. Even if both representations are reified, he may still not realise that they are connected. Furthermore, secondary processes, like finding the derivative of a function, may seem justified in one representation, but not in another, if the latter representation of the function concept has not been reified.

5 | Summary and discussion of the theoretical framework

This chapter gives a very brief summary of the theoretical framework to be used in this thesis and it discusses the nature and present state of theories within this particular branch of the didactics of mathematics.

The theoretical framework includes Schoenfeld's characterisation of problem solving and Sfard's theory of the development of mathematical concepts. Furthermore there has been a short discussion of how $f(x)$ is used as a symbol in mathematics and the challenges for understanding the concept of function that this complex symbolic representation causes. The primary points of this part of the thesis are:

1. That problem solving can be divided into four categories, and that conceptual understanding falls within the category of resources. This has several implications that are of the upmost importance for the development of the methodology which is presented in the next part of the thesis.
2. That concepts can be conceived as both processes and objects, and that the development from a process to a object conception is achieved through three stages: interioziation, condensation, and reification. The theory of reification has been applied to the function concept, such that it can serve as an analytical tool during the data analysis.
3. That an individual's conceptual understanding might develop disjointly across different representations of the same concept. This is referred to as the disjoint-reification-of-representations hypothesis.
4. That the symbol $f(x)$ is very diverse. It can be used both to refer to a function as an object, or as an instruction to e.g. find the function value for some specific number as, $f(a)$. Thus, the way in which students use this symbol might tell something about their conception of functions. In addition awareness of the epistemological obstacles caused by the complexity of $f(x)$ as a symbolic representation of a function or of a particular function in a given context, could and should influence the practice of mathematics teaching.

These point form the basis on which the methodology will be developed in the next part of the thesis. But before we move on to the next part, I would like to discuss shortly the nature and present state of theories within this particular branch of the didactics of mathematics – it is hard to work within a field without forming some opinion about the field itself.

It seems to me that this branch has reached a point at which there are many different theories attempting to explain the same fundamental observation, and that they do so with strikingly similar thoughts and ideas. Thus, there seem to be some convergence of ideas towards a consensus.

However, these theories seem to flourish quite independently of each other at the moment. It seems to me, that no real attempt is made at comparing the different theories with the aim of determining which theory explains the most observations in a consistent manner. Thus, it seems that the theories at not so much competing as they are coexisting.

This might have something to do with the nature of the theories, and the quality of the observations they attempt to explain. The predictions that the theories make are often of a very qualitative nature. This makes it hard to compare the predictions of the theories to each other, because the predictions are qualitatively very similar. The situation is not improved by the fact that the data is often not detailed enough, or convincing enough, to distinguish between the two predictions.

As an example we can consider the APOS theory and the theory of reification. Both theories focus on the duality of mathematical concepts: they can be considered as being process or objects. Both theories are concerned with the development from the one conception to the other. However, the nomenclature is slightly different: Sfard talk about interiozation, condensation, and reification; Dubinsky talk about action, process, and object. But as we have seen the interiozation stresses the *action* aspect of the concept, while Condensation is the stage of automatisation of different complicated *processes*, e.g. adding function. Reification is the stage at which the concept is considered by the individual to be an *object*. Clearly the two theories are quite similar. Thus, a potential difference between the two is to be found in the details and not in the general ideas.

Now, in order for two theories to be different, they should have different predictions. However, specific predictions do not fall out of these theories easily. As will hopefully be apparent from the preceding sections, a great deal of effort has gone into actually applying the theory of reification to the function concept. This can be considered an example of a prediction based on the theory of reification. When I was applying the theory of reification to the concept of derivative of function, however, I was greatly inspired by Asiala's application of the APOS theory to the very same concept. Thus, even the details of the two theories seem to be quite compatible.

Another example of a prediction could be the disjoint-reification-of-representation hypothesis, which was put forward in one of the preceding sections. This hypothesis was cast within the framework of the theory of reification, but the very essence of the hypothesis is very simple: an individuals understanding of a given mathematical concept can develop differently, and even disjointly, across different representations of the concept. Here there is no reference to the theory of reification, and I do not think that it is far fetched to say that this hypothesis could just as easily have been implemented within the framework of the APOS theory.

I think that it would be interesting to investigate this further. After all, if one of the theories turns out to be more more precise and correct than the other, there is no reason that it should not be prioritised. As it is now, different groups seem to be inventing the wheel separately.

Part III

Empirical studies

6 Methodological considerations

The aim of this chapter is to explain how the research question, the theory, and the empirical studies all come together.

The chapter is divided into two parts; one concerning methodological considerations regarding the emperical studies; another regarding the data analysis. Before we get to those considerations, let me explain the time scope of the master's thesis. Officially, the thesis is written between February of 2013 and August of 2013. However, prior to that, during Marts of 2012, I conducted a pilot study which is referred to as Study A in the thesis. The two studies, B and C, were conducted in april of 2013. A timeline is presented in figure 6.1.

Study A was conducted at Mulernes Legat Skole in Odense and involved eight students separated into four pairs. Study B was conducted on the same class one year after, and also involved eight students, four of which also participated in Study A. Study C was conducted at Roskilde Katedral Skole. It involved an entire class of 18 students which were divided into 9 pairs. Since I do not have nine recording devices at my disposal only four of the problem solving sessions were recorded; the rest of the pairs simply handed in their written answers.

Between the study A and B the problems were revised. Since the choice of problems is an enormous part of the preparation for the empirical studies an entire sections will be devoted to this – it takes time and effort to come of with problems that should neither be too difficult, or too trivial; thus, prior to study A, and between study A and B, time have been spend on evaluating and coming up with relevant problems.

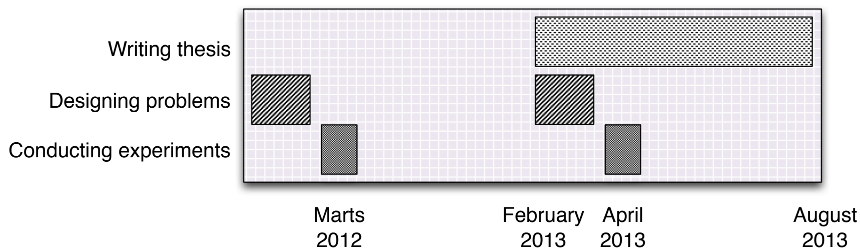


Figure 6.1 – Overview of events in chronological order

Now it should be clear roughly *what* I have done and *when* it was done, so let us discuss *why* and *how*.

6.1 Methodological considerations concerning the empirical studies

The purpose of the empirical studies is to get a broad sense of the possible conceptions that danish secondary school students can have of the function concept; thus, what I wish to gain from the empirical studies is of qualitative nature rather than quantitative; I do not wish to carry out a statistical analysis. This, of course, has implications for the method that needs to be developed, and since didactics, being an interdisciplinary field, utilities quite a broad range of methods to acquire data – qualitative as well as quantitative – developing the right method is not a trivial task.

The most common methods seems to be: Reports from the class room; multiple choice tests; concept maps; and problem solving. All these methods are applied in slightly different variants. The problem solving activities usually fall within three categories: clinical interviews, analysis of written answers, and video recordings of the problem solving process. Researchers can, and do, use one or several of these methods to illuminate their particular research interests.

I have chosen to use video recording of students solving problems in pairs – problems which have been designed specifically to probe their conceptual understanding of functions.

This method was chosen because I believe, that if you want to know something about students' understanding of mathematical concepts, you have to observe how they actually use those concepts while *doing* mathematics. Also, since I wish to be able to characterise the students conceptual understanding, potential discussions between the students could come in handy. The method is deeply inspired by Schoenfeld's book [Schoenfeld, 1985]. Even though I do not use the protocol analysis which he developed, the book did leave me with the impression that recordings of the problem solving processes can yield very interesting information about the students, and that transcriptions of the students' dialogues is an excellent way of presenting and discussing the observations.

As I briefly explained in the introduction, the term "problem" is a slippery one, and the decision to use what I call non-standard, non-problematic problems was no taken lightly. As I also explained in the introduction, it was, however, made mostly for pragmatic reasons; I wanted to be able to carry out the empirical studies during a single mathematics lesson, and I did not want the students' possible lack of control or heuristics to overshadow their ability to use their mathematical concepts.

But is it really necessary to film the students while they solve the problems? Isn't it enough to collect their written answers, and see how they answered the assignment? Well, often the answers to the assignments will be along the lines of "$x < y$" or "*Yes, because of the equation*". These answers do not give a very satisfactory insight into the students' thinking; thus, more information is needed.

As an example, consider one of the answers given above. A pair students were given the assignment: "$y = x + 5$, *what can you say about x in relation to y*", and their written answer was "$x < y$". This following dialogue took place during the problem solving process:

CAIN: $y = x + 5$. What can you say about x in relation to y?
ABEL: That x is less than y, because you have to add five to x in order to obtain y. So what can we write?
CAIN: We can write $x < y$. Okay.

The students correctly state the reasoning behind their written answer, but this information would not be available if I had not recorded the problem solving process. I would not consider $x < y$ to be a satisfactory answer to the question, while the answer *"x is less than y, because you have to add five to x to obtain y"* would be considered to be very close to a satisfactory answer, e.g. *"x is five less than y"* would be considered to be correct. Thus, the video recoding shows that the students are on the right track, more so than would would guess by simply looking at their written answer.

The method of filming students while they solve problems has both strengths and weaknesses. One strength is that you can obtain very detailed information on the students' actions, information which would be available to you otherwise; this information is what allows us to really probe the students' understanding of the mathematical concepts. One weakness is the fact that choosing the right problems can be difficult since we know from Schoenfeld's work on mathematical problem solving, that a student's success or failure in solving a mathematical problem is a complex mixture of that students resources, heuristics, control, and beliefs about the mathematical domain in which the problem takes place, but this is, of course, always the case when dealing with problem solving.

Since we are chiefly interested in students' conceptual understanding, this leaves us with quite a lot of experimental variables that we need to get under control, and one way of controlling this is designing problems that focus more on resources than, say, control and heuristics.

Another weakness is the fact that working with human subjects is not easy. People do not always act rationally or consistently, and you can never know exactly what they are thinking. This poses a challenge when you want to conduct empirical studies involving humans, a challenge which is expressed by Dubinsky and Heral:

> " ... the same individual will behave very differently at different times, in both constant an varying contexts. ... This leads to a mosaic that for some individuals allows us to judge their process conception as relatively strong or relatively weak. But for many subjects we have only a contradictory collection of conclusions and so we must refer to the totality of episodes for an individual." [Dubinsky and Harel, 1992]

This is, of course, a problem shard by all the methods, but it is important to remember none the less.

The quote has several implication. First of all, one should not read too much into isolated episodes, since students apparently behave inconsistently; what is the point of attempting to give a complete and comprehensive description of a student's every though and move, if the student behaves differently under similar circumstances ten minutes later? Secondly, reproducibility, in its strictest sense, is out of the question; you a not guaranteed to get the same results from consecutive empirical studies. Then, where does that leave us? Well, on a larger scale, the same observations have been made by different people across different countries. Thus, not all is lost for reproducibility.

Yet another weakness is the fact that I make the students solve the problems in pairs. It has been shown that the meta-cognitive processes (i.e. control in the schoenfeld terminology) that take place during problem solving in pairs differ from the ones that take place during individual problem solving process [Iiskala et al., 2011]. I do so anyway, because I am not interested in studying control, but their resources, and I believe that the pros outweigh the cons. The fact that they solve the problems in pairs will force the students to explicitly state the reasoning behind their answers in a way that is more natural to the students.

As an example of this we can consider Mads and Michael's work on the problem: "is $x = 2$ the solution to $3x - 2 = 2x$":

MADS: Yes. It is simple. We just insert $x = 2$. *[he continues to reads aloud while he inserts $x = 2$, and concludes that $4 = 4$.]*

MICHAEL: I didn't get that at all. You said $3 \cdot 2$. Why did you do that?

MADS: Because it says $x = 2$. Is $x = 2$ the solution?

MICHAEL: Ah, ok. Well, you could just isolate x.

MADS: Yeah, sure.

Here Michael's question makes Mads elaborate on the reasoning behind his solution method.

One alternative would be to tell the students to "think aloud"; however, without extensive instruction it is most unlikely that the student will say what they think unconditionally. The students are told that they should attempt to solve as many problems as possible; that they are not allowed to erase anything; and that they should state explicitly when they move on to a new assignment. I have done this in an attempt to keep the environment as natural as possible.

6.1.1 Designing the right problems

The choice of problems is perhaps to most important aspect of planning these empirical studies. I have argued for the general genre of the problems: they are non-standard and non-problematic, but this statement does not get us much closer to actually having ten problems which are likely to tell us anything about the students' conceptual understanding regarding the function concept.

I wanted the problems to allow me to get information about the students ability to use and reason with the function concept in a broad sense; I wanted to test the limit of their ability to use the different representations in ways that might be unfamiliar, or at least uncommon to them.

The representations most commonly used in the danish secondary school (in fact, I think this is so for any secondary school) is algebraic, table of values, and graphical. The problems I have designed are supposed to cover these representations, but I have also covered an additional representation: the linguistic representation. The reason that I have chosen to include the linguistic representation is that students are having a hard time expressing statements mathematically. For example, when a group of 150 engineering students at the University of Massachusetts were asked to write an equation using the variable S for the number of students, and P for the number of professors, to represent the following statement: *"At this university there are six times as many students as professors"*, 37% were unable to write up the correct equation $S = 6P$ in any form. In fact, the most common error was that they wrote $6S = P$. When the ratio was changed to 4:5 from 6:1 the error rate increased to 75%. When another group of students were given the problem in reverse, that is, they were given the equation $S = 6P$ and were asked to choose from a list what S and P stood for, over 40% were unable to pick "number of professors" for P and "number of students" for S. In fact, over 22% chose "S stands for professor" as their answer [Rosnick, 1981]. I wanted to see if the students in the danish secondary school had similar difficulties with the linguistic representation.

Choosing the right problems is hard, and it would be very time consuming to develop sets of problems for different grades, or possibly even different parts of the danish educational system. Thus, it made sense to focus on a small part of the educational system: First year of secondary school. The study include two secondary school. The two schools were chosen mostly for pragmatic reasons. The contact to Mulernes Legat Skole was established during a meeting on cognition and infinity in mathematics at a secondary school in Ringsted. One of the teachers said that she would be happy to let me use her students, and I took her up on the offer. The contact to Roskilde Katedral Skole was established through Kasper Bjering Søby Jensen at Roskilde University. Thus, both secondary school were chosen out of convenience. I wanted to have two different classes because I think the teachers' influence on the students' conceptual understanding is paramount, so I wanted to have classes with different teachers. The fact that the two secondary school are separated by some 100 kilometres is fine, but it is of no real importance to my study.

In the two following subsection I will present and discuss the problems that I have designed. Since the problems were revised between study A and B, a section has been devoted to each.

The problems for study A

In this section the problems used in study A presented and discussed. The problems are all included on the page following the next one. It should be noted that when these problems were designed, it was still my ambition to investigate the relation between conceptual understanding and problem solving; therefore, the problems are divided into assignments – which were supposed to probe their conceptual understanding – and problems – which were supposed to test their problem solving ability.

Assignment 1 tests if the students know what it means for an unknown to be the solution to a given equation. In the literature there are examples of students who are able to solve the equation $7x - 3 = 13x + 15$ to find $x = -3$, but that same individual will be unable to say if $x = 10$ solves $7x - 3 = 13x + 15$.

Assignment 2 tests whether or not the students know the difference between a variable and an unknown. This assignment was included because some studies suggest that students have a weak understanding of the variable concept [Trigueros and Ursini, 2003]. Since the variable and function concepts are closely related I wanted to investigate the students understanding of the variable concept.

Assignment 3 tests the students ability to give linguistic descriptions of equations. This is not something that the danish secondary school curriculum focuses on, so I wanted to see how the students reacted to such assignments. The assignment "$y = x + 5$, what can you say about x in relation to y" has been used in the literature already. Blomhøj used it in [Blomhøj, 1997], and by Orjan Hansson in [Hansson and Grevholm, 2003].

In [Blomhøj, 1997] Blomhøj discusses the difficulties related to the formulation of the assignment. The formulation suggests that the students should give a linguistic formulation of the connection that the equation describes. The fact that the assignment asks for x in relation to y, and not the other way around, is no coincidence, and it probably makes the assignment harder. If the student sees the equation as an expression for a function them the assignment consists in giving a linguistic description of a function. Before one can consider the expression as a function is it necessary to see x and y as variables. However, the assignment does not specify the domains of x and y, so in that regard the assignment requires that the students interpret the assignment. The formulation of the assignment also require an interpretation: Are we talking about the pairs of $x-$ and $y-$values that obey the expression? How should one understand the phrasing: "x in relation to y"?

Assignment 4 also tests the students ability to give linguistic descriptions of equations. The assignment is very similar to assignment 3, but the equation is not in intercept-slope form.

Assignment 5 focuses on the students ability to go from the graphical representation of a function to the algebraic one. Studies suggest that students prefer to use the algebraic representation when dealing with functions. Since reification of a concept involves the ability utilise different representations and being able to change between them, this is something that I need to study. The assignment is very standard, and its sole purpose is to check if the students are in fact able to carry out the procedures necessary to change from the graph representation to the algebraic.

Assignment 6 is meant to test if the students can use the graphical representation of a function to reason about a function – do the students know what it means for a point to be part of the graph of a function? The assignment is similar to assignment 1 in essence, since the students simply need to evaluate an expression – in assignment

1 they are asked if $x = 2$ solves $x + 3 = 5$ and in this assignment they are asked if $x = 1$ solves $f(x) = 1$. I have chosen a point which does not lie on the graph of the function because I do not want the students beliefs about the graph of a function to interfere too much. If the point had been on the graph of the function you run the chance that students will not use the point because, in principle, they can not know if the point lies on the graph, or if it just looks like it.

In [Knuth, 2000] a set of similar, but not identical, assignments were given to 284 students. He found that students hardly ever used the graphical representation, even though the assignments were specifically designed to be solved most easily with the graphical representation. Many students even found the graph irrelevant. His final conclusion was that many students had a limited understanding of the connection between equations and their graphs.

Assignment 7 is meant, like assignment 6, to test their ability to reason with the graphical representation of functions. Do they students know what it means for the graph of functions to intersect?

Assignment 8 is essentially the same as assignment 7, but given in a different representation. Assignment 7 and 8 are standard in the sense that they could easily appear in the textbooks used in the danish secondary school. I have included them because it may show if students ability can vary across different representations.

These assignments were all supposed to assess the students' conceptual understanding of functions. The last three assignments (termed 'problems' on the hand-out.) were meant the test the students problem solving ability with problems involving the function concept.

Problem 1 is similar to assignment 7 and 8, but the students need to write up the correct expressions for the functions themselves. The students are given a text describing a scenario where two cars are driving along a road with different velocities, and an initial offset of five kilometres. Their assignment is to figure out when the two cars meet. This problem is included to see if the students are able to carry out the analysis of the written text, and if they would use graphs or algebraic expressions to solve the problem. That fact that the students have also solved problems 7 and 8 tells me if they are able to carry out the relevant reasoning in a more standardised context.

Problem 2 gives Martin and Maria's heights as measured from january to may, represented in a table, and asks when Martin will be taller than Maria. Again the students need to construct the functions describing the heights and figure out when the functions give the same function value. The purpose is similar to problem one; I want to see if they prefer to use graphs or expressions for solving such problems.

Problem 3 was included to have something more abstract than the other two problems. Here the students need to equate the expressions for the area of a circle and a square, and figure out, under which conditions, the two areas are similar. This assignment requires the students to give an "open" answer to the question.

The students were asked to try and solve at least one of the three problems.

1 Opgaver

Opgave 1 *Er $x = 2$ løsningen til ligningen $x + 3 = 5$?*

Opgave 2 *Er der forskel på x'et i de følgende ligninger?*

$$f(x) = 3 \cdot x + 5 \tag{1}$$
$$3 \cdot x + 5 = 11 \tag{2}$$
$$x = 5 \tag{3}$$

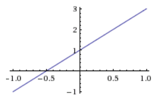
Opgave 3

$$y = x + 5 \tag{4}$$

Hvad kan du sige om x i forhold til y?

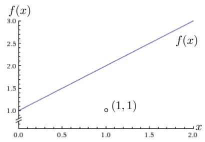
Opgave 4 *Hvad kan du sige om de x og y der opfylder ligningen $x+y = 10$.*

Opgave 5 *Forneden er grafen for funktionen $f(x)$. Hvad er liniens ligning?*



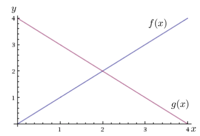
Figur 1 – grafen af funktionen $f(x)$.

Opgave 6 *Kan du fra grafen for funktionen $f(x)$ og punktet $(x, f(x)) = (1, 1)$ sige noget om hvorvidt $x = 1$ løser ligningen $f(x) = 1$?*



Figur 2 – Grafen for funktionen $f(x)$ og punktet $(1, 1)$.

Opgave 7 *Graferne for funktionerne $f(x)$ og $g(x)$ ses forneden. For hvilke(n) x værdier er $f(x) = g(x)$?*



Figur 3 – Grafen for funktionerne $f(x)$ og $g(x)$.

Opgave 8 *Du er givet to funktioner $f(x) = x$ og $g(x) = -x + 4$. For hvilke x værdier er $f(x) = g(x)$?*

2 Problemer

Problem 1 *To biler A og B holder ved den samme vej. De sætter begge igang samtidig og kører i samme retning, A med* 80 `km/time` *og B med* 60 `km/time`. *Hvornår overhaler A B, hvis A fra starten holdte 5 kilometer længere nede ad vejen?*

Problem 2 *Martin og Maria er søskende med et år imellem. Forneden er en tabel af deres højder, målt hvert måned. Hvis vi antager at de bliver ved med at vokse med samme fart, hvornår er Martin så højere end Maria?*

Tabel 1 – Martin og Marias højder målt hvert måned over en periode af 5 måneder.

	Martin (højde i cm)	Maria (højde i cm)
Januar	160	164
Feburar	161	164.5
Marts	162	165
April	163	165.5
Maj	164	166

Problem 3 *Under hvilken betingelse er arealet af en cirkel og et kvadrat ens?*

The problems for study B

One of the main motivations for conducting study A already in 2012, was to assess and revise the problems. It is the purpose of this section to explain the changes that were made to the problems.

Some of the coordinate systems I used were made up by 0.5×1 cells rather than the 1×1 cells normally used. This caused confusion for the students. Even though it is interesting that this is enough to cause a considerably amount of confusion, it is unnecessary. All the coordinate systems have been changed to standard 1×1 cells.

It appeared that all the students had problems with assignments two and six. The major difficulty seemed to arise from my phrasing of the assignments. Assignment two was meant to lure the students into considering the difference between x as a variable, and x as an unknown. All of the students misunderstood the purpose of the assignment, and simply checked if one value of x could satisfy all the equations. This could be interpreted as an indication of the students preference for x as an unknown, but it could just as well indicate that the problem was poorly phrased. One could attempt to fix the problem by giving the students a set of equations, and asking them to tell which one include x as an unknown, and which one include x as a variable. This would definitely clear things up. However, the problem has been removed from the set in favour of another problem.

In assignment seven and eight they are asked to find the x value for which two function $f(x)$ and $g(x)$ are equal. In assignment seven they are given the graphs of the functions, and in assignment eight they are given the algebraic expressions. The functions happen to be the same in both assignments. This was changed such that they have to carry out the procedure in assignment eight. Assignment one and six have both been changed slightly. Two new assignments have been introduced:

New assignment 1 asks for a definition of a function. They are invited to both draw and write, as they see fit.

New assignment 2 Focuses on the students ability to change between different representations; The students are given six cards, each with a function represented either as a graph, an analytical expression, or as a table. The students are then told to figure out which cards include the same function; there are three pairs.

Since the focus of the thesis has changed from the relation between conceptual understanding and problem solving, to focusing purely on conceptual understanding, the "problems" (assignment 9,10 and 11) have been removed.

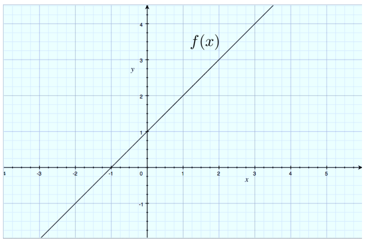
New assignment 3 Another problem similar to – but slighter more difficult than – the old assignment 9 has been introduced. Two cars are now separated by 520 km, and they are driving towards each other. One car sets off at 13 o'clock with a constant velocity of 80 km/hr, and the other car sets off at 15 o'clock with 100 km/hr, when do they meet? This problem is included to see how the students prefer to solve it; which representation do they use?

The assignments are presented on the following page.

Opgave 1 *Kan du give en definition af en funktion? Du må meget gerne både skrive og tegne.*

Opgave 2 *Er $x = 2$ løsningen til ligningen $3x - 2 = 2x$?*

Opgave 3 *Forneden er grafen for funktionen $f(x)$. Hvad er funktionens forskrift?*



Figur 1 – grafen af funktionen $f(x)$.

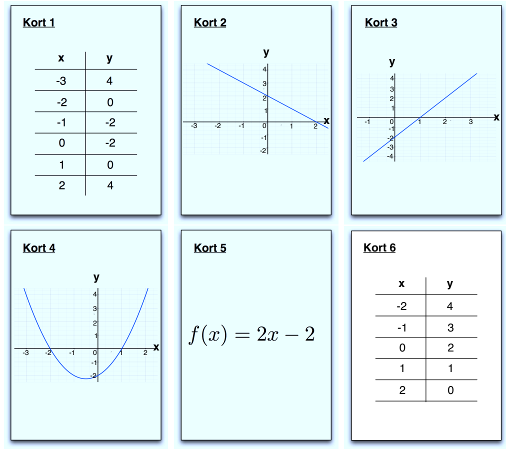
Opgave 4

$$y = x + 5 \tag{1}$$

Hvad kan du sige om x i forhold til y?

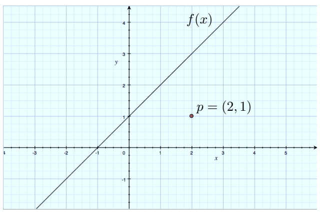
Opgave 5 *Hvad kan du sige om de x og y der opfylder ligningen $x + y = 10$.*

Opgave 6 *Her er 6 "kort". Hvert kort viser en funktion enten som* tabel, *graf eller* regneforskrift. *Hvilke kort viser den samme funktion? Der er 3 par i alt. Angiv de 3 par.*



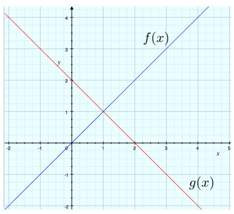
Figur 2 – De 6 kort.

Opgave 7 *Kan du fra grafen for funktionen $f(x)$ og punktet $(x, y) = (2, 1)$ sige noget om, hvorvidt $x = 2$ løser ligningen $f(x) = 1$?*



Figur 3 – Grafen for funktionen $f(x)$ og punktet $(2, 1)$.

Opgave 8 *Graferne for funktionerne $f(x)$ og $g(x)$ ses forneden. For hvilke x værdier er $f(x) = g(x)$?*



Figur 4 – Grafen for funktionerne $f(x)$ og $g(x)$.

Opgave 9 *Du har to funktioner $f(x) = x$ og $g(x) = -x + 4$. For hvilke x værdier er $f(x) = g(x)$?*

Opgave 10 *Vi betragter to biler, bil 1 og bil 2. Der er to punkt er A og B 520 km fra hinanden. Bil 1 starter ved A klokken 1 og kører mod B med 80 km/timen. To timer efter starter bil 2 fra B og kører mod A med 100 km/timen. Hvor mange timer går der før de to biler mødes?*

6.2 Methodological considerations concerning the data analysis

The main methodological considerations concerning the data analysis is this: It is theory driven.

Some researchers have developed comprehensive coding algorithms to make the data analysis as objective as possible; they even let different people (who are trained in using the algorithm, of course!) preform the coding and compare the results.

My take on the data analysis is different. I do not aspire to eliminate every grain of subjectivity from the data analysis, but rather I wish for my conclusions to be based on observations, and that the reader will be able to follow the logic of my analysis; the reader need not agree with my analysis, but it should be possible for the reader to judge if it is a *possible* interpretation of the events. In a way, you can say that subjectivity is at the heart of the data analysis: *I* have chosen parts of the dialogues shared by students while they solve problem, that can be *interpreted* using theories that *I* have hand picked from a multitude of different theories. A person with other theories in mind might have focused on other parts of the dialogues. I do not consider this to be a weakness in the methodology, but it is something that one should be aware of.

In all three studies the data will be processed in the same fashion. The process of analysing the data involves several steps:

1. First the written answers are analysed. I evaluate all the written answers and present the success ratio of each assignment in a plot. This gives an overview of the difficulty of the different assignments. All written answers are presented and the reasoning behind their evaluation is given.
2. I then turn to the recorded problem solving processes and scrutinise the recording, to better understand the reasoning behind the students written answers. Parts of the dialogues are chosen and transcribed for presentation and further analysis.
3. The findings are then analysed in the light of the theoretical framework.
4. Based on the analysis of their written answers and their dialogues, a characterisation of the students conceptual understanding of functions is put forward.

The actual analysis have not been quite as structured as this – I have gone over the written answers and the recording several times, often seeing episodes differently the second or third time round, but this is the essence of it.

When the data is presented, it will be with a section for each problem. The section will include all the students' written answers to the assignment together with the chosen transcribed dialogues followed by the analysis of the dialogues. A discussion of the observations is given at the end of each study.

7 | Study A

Study A took place during the spring semester of 2012. It was conducted on four pairs of students from Mulernes Legat Skole in Odense, Denmark. The student were going through their 2. semester of A-level mathematics. The aims of this study were:

1. To get acquainted with the empirical method; to detect pitfalls and disadvantages of the method early on, with a special focus on assessing the chosen problems.
2. To investigate students understanding of the function concepts and their ability to solve mathematical problems involving this concept.

The majority of the motivation for doing this study came from a need to assess the quality of the assignments.

Based on the literature and my knowledge of the danish secondary school curriculum, I constructed a set of 11 assignments, which I hoped would neither be too difficult, nor to hard to handle for the students, as explained in section 6.1.1.

Before we move on to the presentation of the the data analysis it is important to note that the focus of the thesis has changed from the spring of 2012 to the spring of 2013. Initially, the aim was to study the relation between conceptual understanding and problem solving competency, as is reflected in the second aim of this study. The focus has since then changed to conceptual understanding alone, but studied by analysing the problem solving process of students. This decision was based on the observation that it is difficult to study both conceptual understanding and problem solving ability – let alone investigate the relation between the two – within the limited time at my disposal. Furthermore, conceptual understanding is not at all well understood.

The results of this study are still of relevance to the thesis and are therefore included in the report.

7.1 Data analysis

Figure 7.1 shows both the number of attempts to solve the problems, and the number of successful attempts. From the graph it is quite clear that the students must have found the assignments rather difficult; five of the assignments, roughly 45%, were not answered correctly by any of the pairs.

I will present the written answers of all the pairs in the following subsections – one assignment at a time. The pairs have all been given pretend names, which do not necessarily reflect the sex of the students.

7.1.1 Assignment 1

All of the students were able to answer this assignment correctly; the students either solve for x or insert $x = 2$, to see if it is true.

Cain & Abel: $x + 3 = 5$; $x + 3 - 3 = 5 - 2$; $x = 2$

Tegan & Sara: Yes, $2 + 3 = 5$.

Hans & Grete: $x = 2$, true. $x + 3 = 5$; $5 - 3 = 2$; $x = 2$

Seth & Rogan: $x = 2$

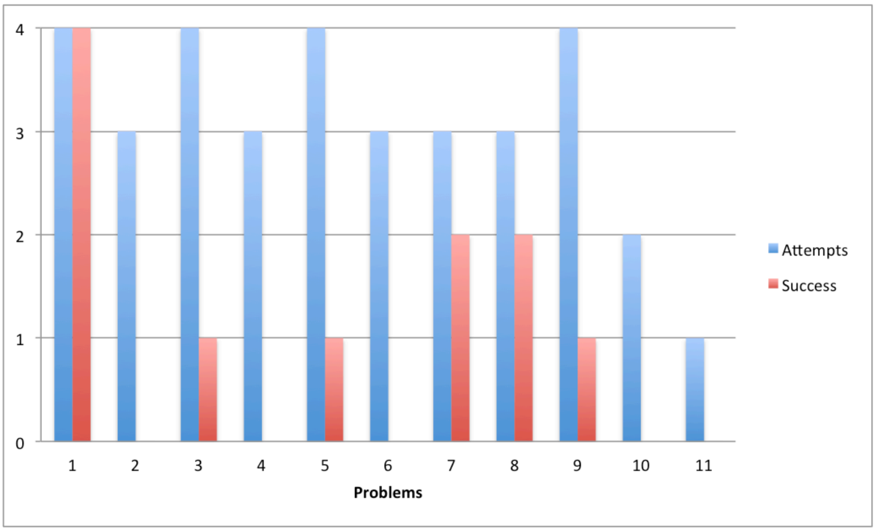


Figure 7.1

We can look at the video recording of Cain and Abel's problem solving process to get an idea of their reasoning. Their immediate respond is: *"yes"*, followed by the following dialogue:

CAIN: Well, we need to show it. We can show it by solving for x. If we subtract 3 from both sides we obtain $x = 2$.

ABEL: We could also just insert $x = 2$.

CAIN: Well, we need to show it, so...

ABEL: yes, you are right.

From the dialogue it seems that Cain thinks it is necessary so solve the equation, in order to show that $x = 2$ is the solution to the equation. Abel knows that it is enough to simply insert x to see if it is true, but Cain does not think that this "shows it".

7.1.2 Assignment 2

None of the students were able to answer this assignment correctly, and one of the four paris neglected to write anything at all. I wanted the students to consider the difference between x used as an unknown, and x used as a variable. Here are their written answers:

Cain & Abel: Yes, they do not have the same value.

Tegan & Sara: The last one is not the same.

Hans & Grete: Difference is present.

It seems quite clear that the students misunderstood the purpose of this assignment. To get an idea of what the students did, let us consider how Cain and Abel approached this assignment. They wanted to check if the value of x was the same in all the three equations. They realised that $x = 5$ is stated in the last equation, compared it with $3x + 5 = 11$ and concludes that $20 \neq 11$. Thus, the x could not be the same. They briefly considered the function but they state: *"well this is a function, so this will end up with a graph"*, which confirms their suspicion that x is *"not the same"*. Their final answer to the question is *"yes, they do not have the same value"*. They do not explicitly state that the difference lies in x being an "unknown" in $3x + 5 = 11$ and a variable in $f(x) = 3x + 5$, but they seem to have an idea that the x's are at least qualitatively different.

As another example, we can consider the dialogue shared by Tegan & Sara:

TEGAN: Well, this is just a function. This actually means y. You know $y = ax + b$ *[referring to* $f(x) = 3 \cdot x + 5$ *I assume]*.

SARAH: But down here x will end up being a single number *[referring to* $3 \cdot x + 5 = 11$ *i assume]*

TEGAN: Ah, but in both cases $x = 2$ right?

SARAH: Oh we only have to say if there is a difference.

TEGAN: But we insert $x = 2$ it gives 11.

SARAH: Yes, yes it does.

TEGAN: But we insert this *[pointing to* $x = 5$ *i assume]* it doesn't make sense. So no x is not the same. That must be the answer

SARAH: Yes.

It seems that Tegan and Sara have a good idea about the difference between the x as a variable and x as an unknown. However, when they notice the equation $x = 5$ does not agree with $3x + 5 = 11$, their focuses changes; they think that I ask them to check if there is a single x, which satisfies all the equations.

Tegan's comment: *"Ah, but in both cases x=2, right?"*, when referring to $f(x) = 3x+5$ and $3x + 5 = 11$, does blur the picture a little, though. Sure, $f(2) = 11$ and $x = 2$ does solve $3x + 5 = 11$ but the fact that she talks about x *being* a specific value for the function suggests that her conception of variable might be sketchy.

Hans and Grete's written answer suggests, that they also think that I ask them if the same value for x applies to all the equations. A brief inspection of the video recording reveals the following dialogue:

HANS: Are the x's in the following different? *[he continues to read aloud the equations]*. Yes there is, isn't there?

GRETE: 5... 15... 20, yes there is a difference.

Again it seems they insert $x = 5$ into $3x + 5 = 11$ and notice that $20 \neq 11$.

Seth and Rogan does not write anything, but the video recording reveals that they insert $x = 5$ into $3 \cdot x + 5 = 11$ and notice that it is not the same. After this observation they simply move on to the next problem.

7.1.3 Assignment 3

The students are asked: *"y = x+5, what can you say about x compared to y"*. Judging wether or not an answer to assignments three and four is correct is not trivial; it requires convention. The phrasing of the question is intentionally left open; I did not ask the students specifically *not* to use mathematical notation, because I wanted to see how the students responded to such questions. Since I want to give a graphical presentation of the number of successful answers, however, it is necessary to decide wether or not the answers are correct. To be consistent, I apply the following convention: correct answer must have two properties: It should not rely solely on mathematical notation; and it should convey enough information for a third party to reconstruct the relations, based on the written answer – there has to be no ambiguity. Since I do not ask the students explicitly to give answers which have these properties, it is to be expected that the many of the students' written answers will be considered to be incorrect, but this is not so important; the important thing is how the students go about answering such open questions.

Only one of the four pairs were able to answer the assignment correctly. The written answers are given below.

Cain & Abel: $x < y$

Tegan & Sara: y is always 5 larger than x

Hans & Grete: Proportional

Seth & Rogan: That when $y = 5$, x is 0.

These are three very different answers. Cain and Abel's answer is considered to be wrong since it does not have any of the two properties. First, they relying solely on mathematical notation. Secondly, even if they had written: *"x is less than y"*, it still would not have contained sufficient information for someone to reconstruct the exact relation. Seth and Rogan only consider a single case, which is not sufficient. Hans & Grete's answer is considered to be incorrect because x and y are in fact not proportional. Tegan & Sara's answer is considered to be correct.

Let us look at the video recording to better understand the reasoning behind their answers. We will begin by considering Tegan and Sara's dialogue:

TEGAN: I see it as a graph. Which i guess is just horizontal then.

SARAH: This one?

TEGAN: Yes, with intersection 5

SARAH: But what can we say? what is meant with this assignment? ... If it were like this, they would be the same.

TEGAN: We can say that y is always x plus this other term. I don't know what to say. It must be explained in a better way.

SARAH: Yes for example y is five bigger than x. But we must not say 5 times bigger, just 5 bigger.

TEGAN: But what about if we look at it as a graph?

SARAH: ... mmhm... Then it goes like... We have... x and y is just the same. no its not.

TEGAN: Never mind that graph

With that final comment they move on to the next assignment.

It is quite interesting that their first respond is *"I see it as a graph"*. This is indicative of a structural conception; however, they seem to think that their final answer is somehow not compatible withe the graphical representation of the function.

When they consider the graph for the second time, Sarah states that "x and y is just the same". Even though she corrects herself straight after, it seem that they would need to convince themselves that x is five less than y in the graphical representation as well.

They also think $y = x + 5$ is a horizontal line (they are able to correctly give the algebraic expression for the graph in assignment five). From the dialogue we can not say why they make this mistake, but a reason might be, that it does not say $1x$, and since there is no a present, a must be 0, and everyone knows, that if a line has no slope, then it is horizontal, but this is speculation.

Cain & Abel dialogue is as follows:

CAIN: $y = x + 5$. What can you say about x in relation to y?

ABEL: That x is less than y, because you have to add five to x in order to obtain y. So what can we write?

CAIN: We can write $x < y$. Okay

It seems that they do not have any trouble going from an algebraic representation to a linguistic one; however, they write down their answer as $x < y$. They make the correct consideration, but choose to write $x < y$.

One reason for this could be that they think that answers to mathematical questions should be given in symbolic notation. Their answer suggests a process conception of function; this is based on Abel's statement: *"That x is less than y, because you have to add five to x in order to obtain y"*, which focuses on the act of adding 5 to x, rather than simply stating that x is 5 less than y.

Hans & Grete's answer seems to be due to a confusion of concepts; x and y are proportional if there is a number, a, such that $y = ax$. The students probably also know that the equation for a straight line is $y = ax + b$. They probably notice the similarities, and in an attempt to give a linguistic description of $y = x + 5$, proportional might seem more correct than e.g. "straight line". Let us inspect the problem solving process.

GRETE: That y depends on x. So what does one say? Proportional?

HANS: Yes, one talks about proportional. Constant. It must be like that.

GRETE: This is a constant. *[probably pointing at 5]*, so it is proportional.

HANS: Yes, so it is proportional.

It seems that they have mixed the concept of proportionality with that of straight line. This is an example of an out-of-focus phenomena.

Seth and Rogan's dialogue is all in a foreign language, so I can not tell anything useful from their dialogue. Based on their written answer alone, one might speculate, that they have a process conception of the function, since they simply insert values for y and find the corresponding x-values.

7.1.4 Assignment 4

Assignment four is along the same line as assignment three, but the phrasing is different. None of the four pairs were able to answer the assignment correctly.

Cain & Abel: The sum of x and y gives 10. $x + y = 10$.

Tegan & Sara: x and y are two different number which give 10

Seth & Rogan: $5 + 5 = 10$,$x + y = 10$

Again the answers are quite different. Let us start by considering Cain and Abel's answer. I consider their answer to be wrong, but it is a close call. Their answer is considered to be wrong, because it closely resembles what you would get from simply reading aloud the mathematical notation: x plus y equals 10. Their problem solving process stats with the following dialogue:

ABEL: That the sum of x and y gives 10. Hehe. Then we can say... I guess... Well I don't know, it can really be a lot of numbers. You can have a negative number plus y, for example, If x is negative.

CAIN: So we can say..

ABEL: Well, it sort of whats already there. x and y put together should give 10

Cain and Abel are not oblivious to the fact, that their answer closely resembles what you would get by simply reading aloud the symbolic notation. They do allow for the x to be negative, which, as we will see later, students do not always do.

Tegan and Sara's answer is considered to be wrong because they specifically state the x and y should be different. Their dialogue shows that this is not just an unfortunate choice of words:

TEGAN: That both are 5. Haha

SARA: No they have to be different, otherwise it would just be $x + x$, you know.

TEGAN: Well, I guess you can say that the one is always larger than the other. That would be a start. We can probably say all kinds of things.

SARA: Yes, we can say that they are two different number, which give ten.

It is interesting, that they do not "see" the graph in this case, as they did in assignment 3. It seems that the expression has to be on intercept-slope form before they recognise it as a function.

It would seem it is no coincidence that they use the word "different" in their answer; they do not think that $x = y = 5$ is valid. This suggest that they do not completely see x and y as variables, and that they do not have a good understanding of the variable concept, otherwise it should have been clear to them that when x is 5 then so is y.

Hans & Grete did not give a written answer to the assignment; however, if we look at Hans' monologue during the problem solving process we see the following:

HANS: If one of them is one, then the other one should be 9. So lets say if x is one, we can pull it over to the other side and subtract, then if x is one, then y is 9. If x is 2 then y is 8 and so on until you reach the opposite, or.. 5 plus 5.

Hans basically rewrote the equation to intercept-slope form and calculated the y values for a few x values. He sees the pattern, but chooses to stop when $x = y = 5$.

Seth and Rogan simply look at $x = y = 5$. They did the same with problem three. The video recording does not reveal much; there is a silent mumbling for 30 seconds followed by the utterance of "Yes, of course! 5+5=10!". It seems quite clear from assignment 3 and 4 that Hans and Grete do not see functional relations when they look at $y = x + 5$ and $x + y = 10$.

7.1.5 Assignment 5

In this assignment the students are given a graph of a function, and are asked to give the equation describing it. Below their written answers are given.

Cain & Abel: $a = 1$, $b = 0.5$, $(f(x) = 1x + 0.5)$.

Tegan & Sara: $y = 2x + 1$

Hans & Grete: $y = x \cdot 3 + 1$

Seth & Rogan: $f(x) = 1, 3$.

Only one of the pairs were able to find the algebraic expression for the graph of the function $f(x) = 2x + 1$. This is rather surprising, since you learn the procedure much earlier in the danish educational system. You would think that students who choose to study A-level mathematics would know how to carry out this procedure.

Let us consider Cain & Abel's attempt to solve this assignment. They generally spend a long time trying to figure out if a is the intersection or the slope. I have transcribed their dialogue in quite some detail, because it shows where they go wrong and how fragile their mathematical knowledge is. The dialogue includes some of their work on assignment 6, because it is a continuation of their discussion. Their solution to assignment 5 starts with the following lines:

ABEL: b must be one.
CAIN: Then we need to find the slope.
ABEL: We can say that for each time it goes 1/2 along, we go one up.
CAIN: Seems correct. So what is it?
ABEL: So x is... haha...
CAIN: How is it?
ABEL: It is a number you multiply x with, such that for each time we go one along the x-axis. Okay, then we can say it is one.
CAIN: We have to write it up as an equation. Ugh, what is the formula for linear regression?
ABEL: It is ax... $f(x)$... the function of x
CAIN: mx times b or a times bx.

The confusion continues along these lines for several minutes. They constantly interchange a, b, and x when they talk about linear functions which indicates that they do not have a very robust understanding of the concept of a variable. It is a clear example an OOF phenomena, and it continues.

They start to interchange the dependent variable y with the independent variable x, as is clear from the following lines of dialogue

CAIN: We know every time y in increased by 1, x is increased by 1/2. Isn't it just $1 \cdot 1/2$?
ABEL: Well you need an x in there somewhere. One times x?
CAIN: What is the x a symbol for?
ABEL: Isn't it a symbol for all the numbers on the x-axis?
CAIN: I don't remember
ABEL: Yes, it has to be.
CAIN: Ok, let's return to it if we have the time.

With this comment they continue to problem six. They read the text several times. When they try to make sense of the assignment Cain states that *"f(x) must the the result of $a \cdot bx$"*, which indicates a process conception, since he talks about $f(x)$ being the result of a calculation. They go on to find the formula for the slope. They find b to be one and since, as Cain mentions:

CAIN: every time y increases 1/2, x increases with 1/2, so the slope must be 1. That is one times one which is one. And for x the point is at one, so one times one times one is one so $f(x) = 1$.
[This is an example of the fact that Cain seems to think that the algebraic expression for a linear function is $f(x) = axb$.]
ABEL: Well this is exactly what it says there, but it is follow by a question mark?
CAIN: Well the intersection is one. and the slope is –
ABEL: Its 1/2, because every time you go 1/2 out you go 1/2 up.
CAIN: ... should we then add the two numbers?
ABEL: Yes i think so. Normally with a linear regression. I think that is what it is called. Then you have a number b, which is the intersection, and a slope which is a number times x.
[Abel starts to look at assignment two]

ABEL: You see this? This is a function. Here I would say that if I had to draw this is would intersect in –

CAIN: Okay! Now I see. This is the starting point, a, and +5 five is... *[It seems that Cain interchange the intersection and the slope.]*

ABEL: this is b

CAIN: no

ABEL: yes

CAIN: no

ABEL: yes

CAIN: no it must be 3

ABEL: no

CAIN: Yes because +5 must be the slope!
[They are looking at the function $f(x) = 3x + 5$]

ABEL: I disagree. I would say that b is the intersection.

CAIN: Oh. So if we go one up, we have to multiply 2 with three to obtain x. Wouldn't it make more sense if 3 was the intersection and 5 was the slope? Because if you go 1/2 up then you have to add 1/2 on the x-axis.

ABEL: No I think that the number 3 means that you have to go 3 along the y-axis and x-number along the x-axis.

CAIN: No, because we are at 1. Lets assume it said 1 there and 1/2 there. We start at 1 and add 1/2, so every time we go 1/2 up we add 1/2... Or every time we go one up we have to add 1.

ABEL: I don't think it is correct.

They finally agree to continue to the next problem.

I think that this dialogue shows several things. First, their understanding seems to have a very fragile foundation; they make one mistake, and after just a few minutes, they start to wonder that the x in $f(x)$ actually is. At some point they even attempt to find the slope by varying y by one and finding the corresponding change in x; that is, they mix up the dependent and independent variable.

They do not seem to have a solid understanding of what the symbols they use actually mean. During the approximately 10 minutes of this dialogue Cain and Abel give two different expressions for a linear function. The fact that they constantly interchange the slope and intersection shows that they do not have a solid understanding of the mathematical notation they use, e.g. Cain states *"What is the x a symbol for?"*

The dialogue also shows an unfortunate side effect of the methodology: Abel seems to have a clear idea about how to answer the assignment, but he is unable to convince Cain, and consequently end up agreeing to answers, which he not fully comfortable with. Let us move on to the remaining teams.

Tegan and Sara quickly find the expression for the graph by using the rise over run method.

Hans and Grete's answer is incorrect. Generally Hans and Grete do not communicate a lot when they attempt to solve the problems. This makes the video recording of limited use, but let us inspect their dialogue anyhow.

HANS: Below the graph of the function $f(x)$ is given. What is the equation describing the line? $y = ax + b$, and it intersects in 1.
[Hans continues to carry out the rise over run procedure. He marks off a line from (0,1) to (1,3) and notice that if you go vertically from (1,3) to the y-axis, then:]

GRETE: It crosses in 3

GRETE: So its $x \cdot 3 + 1$

Their mistake lies in carrying out the rise over run procedure incorrectly.

Seth and Rogan give a very strange answer to this problem: "$f(x) = 1, 3$". Let us see if we can better understand this by looking at the video recordings of their problem solving process. They realise that the intersection is 1, and make the same mistake and Hans and Grete:

SETH: So 3,1 right?
ROGAN: You have to write up the equation.
SETH: 1,3
ROGAN: I think you have to say $f(x) = 1, 3$.
SETH: There we go.

They seem to have some idea that a straight line is defined by intersection with the y-axis and its slope; however, they do not know how to convey this information in a mathematical manner. This is quite surprising, they are following A-level mathematics after all.

7.1.6 Assignment 6

None of the pairs were able to answer the assignment correctly. Their written answers are given below.

Cain & Abel: $f(x) = 1$, $f(x) = 1 \cdot 1 \cdot 0.5 = 0.5$. $1 > 0.5$ ergo, $f(x) = 1$ does not solve the equation.

Hans & Grete: $1 \cdot x + 1$, $f(x) = x$

Seth & Rogan: $f(x) = 1$, because you have to go one along and one up, and the graph is a linear function.

We have already seen Cain and Abel's attempt to solve the problem. Their idea is correct, but they fail to give the correct expression for the graph, because of their confusion about the expression for a straight line.

Hans & Grete spend almost five minutes on this assignment, but without much dialogue; they do not have a clear idea about how to solve this assignment. It would seem that they think that $f(x) = 1$ is true, and that their assignment is to show it. They write up the expression for the graph, that is the $1 \cdot x + 1$ part of their written answer, but they don't really know what to do with it. Then they begin to focus on the point, and since $f(x) = 1$ should be true, the equation for the graph should be $f(x) = x$. Then the following dialogue takes place:

GRETE: So $f(x) = x$?
HANS: Yeah, it must be something like that, but it does not go through the point though?

After this they move on to the next problem.

Hans & Grete's failure is due to a mixture of lack of control and resources. They do not have a plan and spend their five minutes without really knowing what to do.

The dialogue shows that Hans seems to be worried about the fact that the graph does not pass through the point. It shows that he has some idea about the fact that the algebraic and the graphical representation have to be consistent, but the fact that he does not follow up on it shows that he is not complete sure about what to make of it.

It seems that Seth & Rogan try to say that $f(x) = x$, but write $f(x) = 1$ instead. Nothing can be gained from watching the video recordings.

Tegan and Sara did not write anything. Let us look at their dialogue to figure out what went wrong.

TEGAN: We should justify that x is one... And then we should solve the equation

SARA: No we have to check if $x = 1$ solves the equation. We should use the information to tell if $x = 1$ solves the equation

TEGAN: Ohh, I thought we had to show that $x = 1$ first.

SARA: No I don't think so.

TEGAN: I guess I just misunderstood the text.

SARA: Okay. So this is the point $(x, 1)$ and this is $(1, 1)$, but then $x = 1$, as it says there, and y is also 1.

TEGAN: So it is (1,1).

SARA: I don't understand it.

TEGAN: No it's strange. I don't see what he wants us to do.

SARA: He wants us to state if $x = 1$ solves the equation. I don't think it is difficult, I just cant figure it out.

TEGAN: It's just a strange explanation.

Tegan explicitly states that she does not understand the phrasing of the question. Sara seems to understand that they have to figure out if $x = 1$ solves $f(x) = 1$, but she does not have a good idea about how to go about doing just that. It is not completely clear to me what lies behind the sentence *"Okay. So this is the point* $(x, 1)$ *and this is* $(1, 1)$*, but then x=1, as it says there, and y is also one."*.

I think that it is safe to say that the students had a hard time understanding the assignment. As I see it, this assignment can be solved in one of two ways. Either you find the expression for $f(x)$ check if $f(1) = 1$, or you realise that the point $(1, 1)$ does not lie on the graph of the function $f(x)$, consequently $f(1) \neq 1$. In order to use the latter approach it is helpful to realise that the graph of a function are the points $(x, f(x))$. Thus, one reason for the students failure might be that they do not have a clear definition of the graph of a function. Another reason might be that the assignment is deceivingly simple. If I had written "is $f(1) = 1$?", rather than "does x=1 solve the equation f(x)=1", then most likely, the students would all be able to answer. This only requires the students to understand that the notation $f(a)$ specifies the function value of $f(x)$ for $x = a$.

I would argue that the first approach is indicative of a process conception of the function concept, while the latter is indicative of a structural conception.

7.1.7 Assignment 7

In this problem the students are given the graphs of two functions, $f(x)$ and $g(x)$, and are asked for which x $f(x) = g(x)$. Two pairs answered the assignment correctly.

Cain & Abel: $f(x) = 1$, $g(x) = -1$.

Tegan & Sara: We need to figure out where $g(x)$ and $f(x)$ intersect. $x = 2$.

Hans & Grete: $\underline{\underline{2}}$

Cain and Abel seem to have misunderstood the assignment. Let us take a look at the video recording to figure out what went wrong.

ABEL: I have no clue.

CAIN: The graphs of the function f(x) and g(x)...

ABEL: Well, I guess they are the same, one of them are just negative, right?

CAIN: For which n values are $f(x) = g(x)$. I don't understand. *[the confusion about "n" is caused because i write "for hvilke(n) værdier..." where then (n) is meant to keep open the possibility of there being several values of x.]*

ABEL: Ahh! n is...

CAIN: What did n mean? What was it?

ABEL: I have no clue

CAIN: Was it rational number? No it was...

ABEL: Natural numbers!

CAIN: Ah yes natural number. 1, 2, 3, 4...

ABEL: For which natural number of the x value are f(x)=g(x)?

CAIN: Oh, then we have to calculate $f(x)$ and $g(x)$.

ABEL: Yes, and they must be the same, *just negative*.

CAIN: Okay, so f(x), what is that?

ABEL: You go one along and one up, so $f(x) = 1x$

CAIN: Yes $f(x) = 1$, and then $g(x) = -1$

ABEL: $f(x)$ is equal one *times* x.

CAIN: yes, and $g(x) = -1$, don't you see?

ABEL: Yeah. sure. Write that. It's out best offer.

CAIN: So if we calculate everything $f(x) = 1$ and $g(x) = -1$. *[He reads aloud what he is writing]*.

ABEL: Well, now you are answering problem 8, because there we have to find the x-value. I don't know, fuck it.

After this they move on to problem 8.

I Believe that this dialogue shows two things. First, they students can get confused by the smallest things. In this case it was the "hvilke(n)" in the phrasing of the assignment text. If the students met this notation in a danish class, they probably would not have thought twice about it, but because they memt it as part of a mathematical question, they immediately thought about the set of natural numbers. This is seen clearly when Abel rephrases the question to *"For which natural number of the x value are f(x)=g(x)"*. Secondly, Abel seems to have a clear idea about how to describe a straight line mathematically, while Cain does not. One example is when Abel specifies *"$f(x)$ is equal one times x."* and Cain responds *"yes, and $g(x) = -1$, don't you see?"*. We will get back to Cain's $f(x) = -1$ notation in the next subsection.

Somehow the focus changes from the x value they are supposed to find, to the equations for the lines. As a consequence they never get around to finding the actual x value. Abel's last statement *"Well, now you are answering problem 8, because there we have to find the x-value."* shows that he has forgotten what problem 7 was about.

Tegan and Sara, and Hans and Grete solved the problem correctly. It is quite clear from their written answer that Tegan and Sara knew what they were doing when they arrived at $x = 2$. For Hans and Grete the reasoning is less clear. Let us take a look at the video recordings.

HANS: It must be here, right? Where they intersect. At $(2, 2)$.

It seems they do indeed look at the intersection.

Seth and Rogan did not write anything and nothing meaning full can be recovered from the video recordings. It is a mixture of foreign language and silence.

7.1.8 Assignment 8

The two paris which solved assignment 7 correctly also succeeded in solving this assignment.

Cain & Abel: $f(x) = 2$, $g(x) = -2$.

Tegan & Sara: Its the same as in assignment 7. $x = 2$.

Hans & Grete:

$$\begin{aligned} x &= -x + 4 \\ x + x &= 4 \\ 2x &= 4 \\ x/2 &= 4/2 \\ x &= 2 \end{aligned} \tag{7.1}$$

Tegan and Sara realise that the functions are actually the same as in assignment seven, and conclude that the function are equal for $x = 2$.

Hans and Grete correctly equate $f(x)$ and $g(x)$, and solve for x to find their intersection.

Seth and Rogan did not write anything. Again, nothing can be gained from looking at the video recording.

Cain and Abel attempt to solve the assignment in a way similar to what they did in assignment seven; however, from their written answer it is not clear what went wrong. In assignment seven they attempted to give the expressions for the two functions, which is alright, since they were given graphs, but here they are given two expressions for the function, and yet, they end up with $f(x) = 2$ and $g(x) - 2$. What is going on? Let us inspect the problem solving process.

ABEL: You are given two function $f(x) = x$ and $g(x) = -x+4$. Because it intersects in 4! You see? This is our b, and this is the slope, a, because +4 is where it intersects the y-axis.. you see... otherwise it should be the intersection with the x-axis. However, I don't think so. It a little unfortunate that it also intersects the x-axis at 4.

CAIN: So what should we write? For which values? I mean $f(x)$ must be $x + 0$.

ABEL: Sure write that. I don't think its completely true, but I don't have anything better to offer.

CAIN: It's the same as before. $-1 + 4 = 3$, and if you insert 0 and x... then if you insert 1, then $x+1$ gives 1 plus 2 it gives... So what should it be $f(x) = 0+x$? or $f(x) = x + 0$?

ABEL: I don't think thats what we have to find at all. We should just find out which x value it is, you know, what slope. Or what? *[x is confused with the slope again]*

CAIN: Ohh

ABEL: What should x be for those two to make sense. In $g(x)$ we're told that is it $-x$ because it goes towards minus, and here it is x, it goes toward plus.

CAIN: Ohh! I think it should be 2 and -2!

ABEL: Yes, Okay, lets write that.

CAIN: Because then they hit the same place.

ABEL: Yes, sure, try that.

CAIN: $f(x) = 2$ and $g(x) = -2$

Once again they are having some difficulties with the notation, i.e. Abel confuses x with the slope of the line. It seems that they do, in fact, understand what the assignments asks for, since Abel states: *"What should x be for those two to make sense."* The way in which they express their final answer is a little strange, though; let us see if we can make sense of their notation.

As I see it the notation $f(x) = 2$ can be taken to mean one of two things: Either it means that the slope of $f(x)$ is 2, or it is an instruction to put $x = 2$ in $f(x)$, similar to – but not identical to – the notation $f(2)$. Let me explain why.

In assignment seven it seemed that Cain talked about the function $f(x)$ and $g(x)$ as though they were only specified by the slopes, e.g. they wrote $f(x) = 1$, because the slope of $f(x)$ was 1. This suggests that $f(x) = 1$ is their way of specifying the slope of the function.

However, at the end of the dialogue from problem seven Cain says *"so if we calculate everything $f(x) = 1$ and $g(x) = -1$"*. The fact that he says: *"if we calculate everything"*, might suggest that he is thinking about the function values.

In assignment eight Cain seems to think that he is on to something when he says *"It's the same as before. $-1 + 4 = 3$, and if you insert 0 and x... then if you insert 1, then $x + 1$ it gives 1, then x plus 2 it gives... So what should it be $f(x) = 0 + x$? or $f(x) = x + 0$"*. It is not quite clear to me exactly what is going on in Cain's head, but if we ignore, for a second, the fact that he seems to be varying "0" rather than x, we can attempt to make sense of this. I think he is attempting to insert values for x, which would make $f(x)$ and $g(x)$ be equal; that what he attempts to say is $f(2) = 2$, and that he is ignoring the minus in front of the x in $-x + 4$, thinking about $-x$ as a whole, such that $-x = -2 \Rightarrow -2 + 4 = 2$. This could be why he writes $g(x) = -2$. To him the notation $f(a) =$ number, does not mean the function value for $x = a$, but rather something along the lines: $f(x) =$ "what you should put in the place of the term containing x". If this is truly what his notation means, he would write $f(x) = 25$ as an instruction for someone to calculate the function value of $f(x) = x^2 + 8$, for $x = 5$. Which is different that the usual notation $f(5) = 33$.

If my analysis is correct, their answer should be read as an instruction: insert 2 in place of x in $f(x)$ to obtain 2; insert -2 in place of $-x$ in $g(x)$ to obtain 2; see that they are equal.

I realise that it a bit of a stretch, but it is hard to make sense of their final answer otherwise.

This is a good time to summarise the notation that we have seen Cain utilise so far. He has given $f(x) = axb$ and $f(x) = mxb$ as expressions for a straight line and now he uses the notation $f(x) =$ number, not to mean an horizontal line, but to mean something along the line of an instruction to calculate the function value, when the x term is replaced by the number.

7.1.9 Assignment 9

This is the first of the problems which were meant to be problematic for the students. It can be solved using some of the same method as the previous assignments. Only Tegan and Sara were able to solve the problem correctly. The written answers are given below.

Cain & Abel: *[They have drawn a graph]*

$$\begin{aligned} f(A) &= 80x + 0 \\ f(B) &= 60x + 5 \\ 80x &= 60x + 5 \\ 75x &= 60x \\ \underline{\underline{1.25}} &\underline{\underline{= x}} \end{aligned} \tag{7.2}$$

Tegan & Sara: *[They have drawn a graph]*

$$\begin{aligned} \text{Bil a} : y &= 60x \\ \text{Bil b} : y &= 80x - 5 \\ 60x &= 80x - 5 \\ 5 &= 20x \\ \frac{5}{20} &= x \\ 0.25 &= x \end{aligned} \tag{7.3}$$

Hans & Grete: *[They have drawn a graph]*

$$\begin{aligned} y &= ax + b \\ A &= a \cdot 80 - 5 \\ B &= a \cdot 60 + 0 \\ a \cdot 80 - 5 &= a \cdot 60 \\ a \cdot 75 &= a \cdot 60 \\ a^2 &= 75/60 \\ a^2 &= 1,25. \end{aligned} \tag{7.4}$$

Seth & Rogan: $80 - 60 = 20$

Cain and Abel spend approximately 9 minutes working on the assignment. In their first attempt to solve the problem they try to graph the position of the cars as a function of time, and then finding the intersection between the two functions. This solution, if carried out correctly, will yield the correct answer. After about four minutes they give up on the plan, because they are having trouble drawing the graph rigorously enough. After a brief period of planning ($\approx$ 1 min.) they try to implement their next plan: Equating the algebraic expression of the positions of the cars. This plan would also yield the correct answer, if carried out correctly. They are able to correctly write up the equations describing the positions of the cars as a function of time, but they make a mistake when they try so solve for x, and end up with a wrong answer. Briefly after they write down their answer they run out of time. Considering their previous difficulties with describing a straight line algebraically, it is quite remarkable that they get it right this time.

They way in which they express the functions describing the positions of the cars is interesting. We have already seen them use a quite strange notation regarding functions. This time they use $f(A)$ and $f(B)$ to denote the function describing respectively car A and B. The fact that they use this notation suggests that to them, what you write in the parentheses of the $f(...)$ has no mathematical consequence; once again showing that their understanding of standard mathematical notation is less than ideal.

Let us look at Tegan & Sara's attempt to solve the problem. Assignment nine is where they spend the majority of their time (35% of the entire session). Their main difficulty lies with translating the problem text into mathematical notation; almost immediately after reading the problem text aloud Sara states that *"we just need to find the intersection of their graphs"*. Thus, their problem does not consist in being unable to find a solution to the problem, but rather in their implementation of their chosen solution method. Figure 7.2 shows their graph. Their attempt fails because they fail to scale the axis of the figure conveniently.

They realise that they can equate the expressions for the positions of the cars, and they correctly give the algebraic expressions and solve for x without making mistakes; however, they are unsure about their answer. They explicitly state that they do not know what $x = 0.25$ means. This is an example of students difficulties with "reading meaning" into the mathematical notation they use.

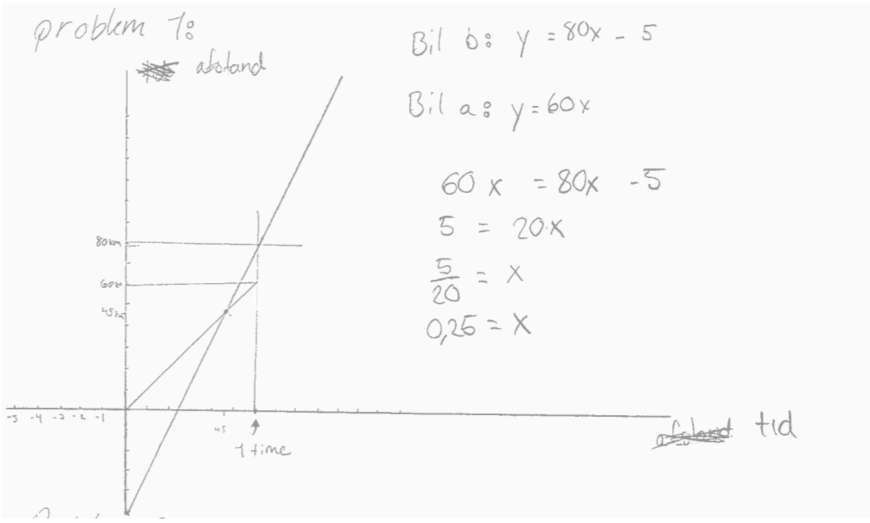


Figure 7.2 – Tegan & Sara's solution to problem 1

Hans and Grete correctly write up formula for a straight line as $y = ax + b$. Immediately blow, however, they give the equations describing the motion of the cars as $A = a \cdot 80 - 5$ and $B = a \cdot 60 + 0$. It is a little bit confusing that they choose a to be the variable rather than x, but since they are consistent it is not a problem. They equate A and B but run into trouble because they can not solve for a correctly.

From the video recording it becomes clear that Hans and Grete's initial attempt to solve the problem was through graphing the positions of the cars as a function of time. They spend five minutes attempting the graph the positions, but they realise that they have scaled the x-axis completely wrong (it goes from 0 to 10 hours, with very small separation between the ticks on the axis). Rather than redrawing the coordinate system, they decide to equate the two functions; This is their dialogue:

HANS: a must be -5 and b must be 0. x must be 80. *[he has written $y = ax + b$ followed by $A = a \cdot 80 - 5$ and $B = a \cdot 60 + 0$]*

GRETE: $+5$ you mean.

HANS: No, because it starts further down the road. So after one hour it will be at 75. I'm talking about A. And B will be at 60.

GRETE: Well, then they have already crossed.
HANS: Hah, yes, then we have to make the graph a lot smaller.
GRETE: Well, I guess we can equate them.
HANS: Yes, of course. Then we say $a \cdot 80 - 5 = a \cdot 60$

They continue to solve for a, but make a mistake along the way. When they arrive at their final answer, Grete notes: *"no, it cannot be true. It has to be less."* After this Hans suggests that *"we can try again. This time we call it x so we won't get confused"*. Before they can finish they run out of time.

It is not clear why they choose to use a as the variable and x as the slope initially; they do not use this convention when solving any of the previous problems, and their first axis is marked as "Time". I is hard to tell, if they change notation by choice or by accident. Hans's statement *"we can try again. This time we call it x so we won't get confused"* makes it clear, that he is aware of the change, however.

Based on Seth and Rogans written answer it is difficult to say anything about their thoughts. It looks like they simply subtracted the two velocities of the cars. Yet again, the video recording reveals nothing useful.

7.1.10 Assignment 10

Only Tegan & Sara had the time to work on assignment 10 and 11.

Tegan & Sara:

$$\begin{aligned} y &= 164x + 0.5 \quad \text{Marias} \\ y &= 160x + 1 \quad \text{Martin} \end{aligned} \tag{7.5}$$

At some point they will intersect.

After reading the problem text aloud the following dialogue takes place:

TEGAN: Every time he grows one centimetre she grows half a centimetre.
SARAH: Yes, she only grows a half.
TEGAN: So we can just count? Or we can make an equation.
SARAH: I see it as a graph. So we have January, February and so one here, and then we have the height here.
[Since this is not a video recording it hard to tell what she is pointing at, but it might very well be the x-axis and y-axis of one the graphs on the assignment set.]
TEGAN: The months are x, its the independent variable. If we can say that the equation for martin is.. January is 0 and there it's 160, so the b value is 160 and the a value is x.
SARAH: And for her b is 164 and a is $1/2$.
TEGAN: So the equation is $y = 164x + 0.5$ and for martin it is $160x + 1$.
SARAH: So if we had Inspire *[Inspire is the mathematical software most danish schools use]* we could simply graph them and find the intersection and we would be done.

They start to draw the graph but realise they they have made the scale on the y axis way to small and give up. It is interesting, that they talk about increasing 1 for each month, even though they give the formula for the height as $f(x) = 164x + 0.5$. In this case they exchange the intersection and the slope, when they write up the algebraic expression for the function, but when they attempt to graph the function, they do it correctly ($f(x) = 0.5x + 164$). They do not notice this inconsistency at any point. In assignment five and nine they do not make this kind of mistake.

It is interesting that they do not equate the two function. They have shown earlier that they are clearly able to do so. It might be because they already have their mind set on using Inspire.

7.1.11 Assignment 11

The students are asked under which conditions the area of a circle and a square are equal. The students written answers are given below.

Tegan & Sara: Area of circle: πr^2
Areal of square: $l \cdot b$

Tegan and Sara correctly write up the expression for the area of a circle. Instead of writing up the area of a square, however, they use the more general formula for an arbitrary rectangle. On paper, they fail to state any conditions for the equality of the area. Let us consider what Tegan and Sara did during the problem solving process.

They quickly state that they want to equate the formula for the area of a square with that of a circle. This gives them the following expression:

$$\pi \cdot r^2 = l \cdot b, \tag{7.6}$$

and they state *"if $l = \pi$ and $b = r^2$"* it is the same, but they don't find the answer satisfactory, and as a consequence they do not write it down. Maybe if they had considered a square instead of a rectangle, they would have felt more comfortable, but this is speculation.

7.2 Characterisation of the students understanding of the function concept

Since these 11 assignment were meant to be the basis on which the students conceptual understanding of function were to be evaluated, let us attempt to draw some conclusions regarding the students conceptual understanding of functions.

I have grouped the students together, mostly out of convenience, since an individual assessment would require a more fine grained analysis of the data.

Cain & Abel are at the stage of interiorization of the function concept or very early stage of condensation, since they have difficult time changing between different representations of the function concept. There were many instances of out-of-focus phenomena during their problem solving session. They used no less than four different expressions for a straight line, and they generally seemed to have a poor understanding of the mathematical notation they used. Their understanding of the graphical representation seems to be limited to its potential of being translated into an algebraic expression, and even this is hard for them to do.

Tegan & Sara seem to be at the stage of condensation of the function concept. They are capable of changing between different representations; they solved assignment seven and eight, and they even realised that the functions in the two assignments were in fact the same. There have been episodes where they seem to "see" a graph when to look at algebraic expressions, and they solve assignment seven by inspection, but their failure to solve assignment six suggests that their understanding of the graphical representation is rather limited.

Hans & Grete seem to be able to reason with the different representations of the function concept, and they solved assignment seven and eight. The fact that they failed to solve assignment five suggests that they are capable of reasoning with the different representations of the function concept, but unable to alternate between them. This indicates that they are at a late stage of interiorization, or very early condensation, of the function concept. There is at least one instance of out-of-focus phenomena when they use the term proportional to describe the relation $y = x + 5$.

Seth & Rogan seem to have a very week understanding, not only of the function concept, but of the variable concept as well. I would argue, that they are at the earliest stage of interiorization of the function concept, and that their variable concept has yet to be reified. In both the assignment that require them to describe a functional relation they simply examples of values that would make the statement true, which suggests that, to them, the symbols x and y represent unknowns rather than variables.

7.3 Discussion

In this section I will attempt to summarise and discuss the main results of this study.

7.3.1 Pseudo-structural conception

In section 4.6 we discussed five indication of what is referred to as a pseudo-structural conception.

As the reader may have noticed, there were quite a lot of episodes that indicate a pseudo-structural conception. Most of the examples were instances of out-of-focus phenomenon, which basically is the imprecise use of otherwise precise mathematical concepts and notions. One example of this is Hans and Grete's incorrect use of the notion of proportionality. Most of these out-of-focus phenomenons occurred disguised as notational difficulties; several of the students ran in to problems with their notation, during their work with the assignments. For example, we can consider Cain and Abel's ideas about the algebraic representation of a function. Their final answer to assignment 5, written in a small parentheses, to show that they are not completely sure about the answer, is $f(x) = 1x{+}0.5$ (it is wrong, the answer is $f(x) = 2x + 1$), while they write $f(x) = 1 \cdot 1 \cdot 0.5 = 0.5 < 1$ as their solution to assignment 6. In assignment 7 they give the formula for $f(x)$ and $g(x)$ as $f(x) = 1$ and $g(x) = -1$, and again in assignment 8 as $f(x) = 2$ and $g(x) = -2$, even though they are talking about the slopes of the graphs as being respectively 2 and -2. During the thirty minutes they express that the algebraic expression for a linear function is:

$$f(x) = a \cdot x + b \quad , \quad f(x) = mxb \quad , \quad f(x) = a \cdot x \cdot b \quad \text{and} \quad f(x) = \text{number} \tag{7.7}$$

It seems that they do not have a clear cut idea how to give the algebraic expression for a linear function.

This supports the observation of Heral and Dubinsky, in [Dubinsky and Harel, 1992], that students can react differently in similar context, at different times. In this case, the students have three different idea about the algebraic expression of a linear function over the period of just 30 minutes.

Similarly, Tegan and Sara think that $y = x + 5$ is a horizontal line, when they discuss assignment three. However, they are able to correctly give the expression for the line in assignment five, but when they get to assignment ten, they confuse the intersection with the slope, and as a consequence, they write up equations describing a person growing 164 cm each month.

While we are at Tegan and Sara's answer to assignment ten: It is an excellent example of students inability to "read meaning" into the mathematical notation they use, as is seen from their comment: *"$x = 0.25$, 0.25 what?"*. They do not know what they use the symbol x to represent.

Seth and Rogan use the notation $f(x) = 1,3$ to describe, what they think is a function whose graph intersects the y-axis at 1, and has a slope of 3. These are mostly examples of out-of-focus phenomena.

Another indication of a pseudo-structural conception is the inability to see different representations of a concept as equivalent. Many of the students were not able to change from the graphical representation to the graphical representation. This does not, in it self, mean that they do not see the two representations as equivalent. They might simply be unable to carry out the rise over run procedure. However, it does seem that the different representation are not considered equal by the students. For example, all the students that attempt to give an answer to assignment six did it by changing to the algebraic representation, rather than trying to solve it in the graphic representation. Another example is Cain and Abel's attempt to solve assignment seven – they attempt to give the algebraic representation of the two function, rather than just checking where the graphs intersect. This is not because they are unaware of the possibility to use the graphs to find when $f(x) = g(x)$, because they attempt to utilise exactly this solution method on assignment nine.

Inability to solve "non-routine" problems is also indicative of a pseudo-structural conception. Seeing as only one of the pairs solved any of the assignments which were meant to be "non-routine", it is safe to say that this is the case.

7.3.2 Difficulties with linguistic descriptions

It is quite clear that the students are having a difficult time with the assignments that require them to give a description of an equation. Initially, most of the students are completely dumbstruck by the assignments; they simply do not know what to do with it.

This is not a strange result. The linguistic representation is not really treated in the danish curriculum, so there is no reason to expect that the students should be able to use it on equal footing with the other three representations.

7.3.3 Importance of context

In assignment seven I ask the students "for hvilke(n) værdier..." which translates into "for which value(s)...". The (n) or (s) are there to indicate that there may be several values for x. Mathematically it is pretty obvious that there can only be one x-value for which $f(x) = g(x)$, but the student do not necessarily know this, so I wanted to keep the question open. When the students encounter this notations, because they encounter it as part of a mathematics assignment, they take the "(n)" to mean natural numbers and immediately start looking for which natural numbers value of x that make $f(x) = g(x)$. As luck would have it, it is in fact a natural number, namely $x = 2$. It goes to show just how important context is.

8 Study B

Study B took place approximately one year after study A. It included four paris of students from the same class as study A. Tegen, Sara, Hans, and Grete from study A participated both studies. The revised version of the problems were used.

It should be noted that the students only spend about 20 minutes going through all of the 10 problems, even though they were given an hour to do so. This might indicate that they found the problems too easy, but it might also indicate that they were in a hurry to get out. The students were not forced to participate, but the teacher did ask a select few in a rather suggestive manner. As a consequence, not all the students were eager to solve the problems in a nice manner, but rather get through them quickly. This is most obvious from Tegan and Sara's recording. After reading the text of some of the assignment aloud a few times, they simply move on; after reading the text of problem ten, and making a small diagram, they simply say *"Thank you for today"* and switch off the recording device. Clearly they did not wish to stay long. This is unfortunate, but it is one of the difficulties when working with people as test-subjects.

I did not have enough recording equipment for all of the teams, so I do not have any recording of Hans and Grete's work. Their answers are included anyway because their written answers can be compared to the remaining teams.

8.1 Data analysis

The number of attempts to solve each problems, together with the successful attempts are shown in figure 8.1.

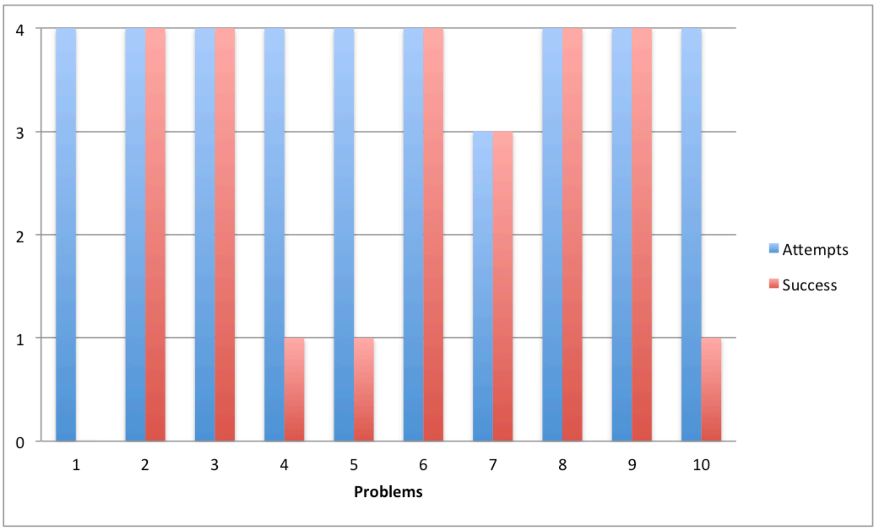


Figure 8.1

A few things are clear from the graph. First of all, the students generally fared better than the previous year. Secondly, none of the students were able to give a satisfactory definition of a function, and the students are having a difficult time giving linguistic descriptions of functional relations. The students did very well on problem seven, in which they are given a point $(2, 1)$, a graph of $f(x) = x + 1$ and are asked if $x = 2$ solves $f(x) = 1$. In the previous year none were able to solve a very similar problem.

Let us dive into the analysis of the their written answers and problem solving processes.

8.1.1 Assignment 1

None of the students gave a satisfactory definition of a function. They all fail to mention uniqueness of y for a given x. Some of the students simply give examples of functions they have encountered during their mathematics classes, which suggests that they have a different understanding of the word definition than I do.

Hans & Grete: $f(x) = ax + b$, $f_2(x) = ax^2 + bx + c$, $f_3(x) = ax^3 + bx^2 + cx + d$.

Mads & Michael: y depends on x

Nora & Helmut: $f(x) = x^2$

Tegan & Sara: A function is defined by $f(x)$.

It is interesting that none of the students are able to give a satisfactory definition of a function. The students were on their second year of A-level mathematics and were currently learning integration. It would seem that the students simply forget the definition as time goes by.

Let us begin by looking at Tegan and Sara's attempt at answering the assignment:

TEGAN: A definition? It's something with $f(x)$.
SARA: Yes, I think so too.
TEGAN: Look at assignment three, haha.
SARA: Yes.

Assignment three contains the graph of a function $f(x)$ and asks "what is the algebraic expression for the function". The fact that they are referring to this assignment might suggest that they are simply thinking of an example of a function, and that *"a function is defined by $f(x)$"* should be understood as "a function is defined through the equation relating x and y". This is, of course, speculation.

Mads and Michael gave a rather general answer to the assignment. Let us see if this generality is also present in the reasoning that lead them to give this answer.

MADS: Sure. $f(x) = x^2$
MICHAEL: Yes or $f(x) = ax + b$ for example
MADS: Yes, but now it is just x^2. Then $f(x)$ is described by x
MICHAEL: Yes, the y value is described through the x value
MADS: Exactly. And x is a placeholder for every number.
MICHAEL: Yes, if it is continues
MADS: Yes, if it is continues
MICHAEL: Or actually, I guess that doesn't matter.
MADS: No no.
MICHAEL: Yes, because it would just jump up higher.
MADS: Well, if there is a large whole here, and then it jumps further up.
MICHAEL: Yes, ok. If it skips along *[he is gesturing a whole in the x-axis, with his arms, it seems]*. Thats right.
MADS: Thank you

We can see that they start off by giving an example of a specific parabola, followed by a more general example for a straight line. Based one these examples they arrive at a more general statement about functions: *"the y value is described through the x value"*. The statement *"and x is a placeholder for every number"* sets off a discussion about the requirement of continuity in the definition of a function. The requirement is discussed through examples about, what is popularly called, a "staircase function" and a split domain function. It seems that Michael is thinking of a staircase function when he says *"or actually, that doesn't matter, it would just jump up higher"*, and when Mads comments *"well, if there is a large whole"*, it seems that he is thinking about a split domain function, something along the lines of $f(x) = 1$ for $x \in \,]-\infty, 1\,] \;\wedge\; f(x) = 2$ for $x \in [2, \infty\,[$. It would seem that their conclusion is that a function must be continues, at least if x should be able to be a place holder for *"every number"*.

My interpretation of this is based partly on their words, and partly on their gesticulation. When Michael says *"or actually, that doesn't matter, it would just jump up higher."* he is gesturing, what seems to be, a jump in the vertical direction. When Mads is saying *"Well, if there is a large whole here, and then it jumps further up."* he is moving his arm both horizontally and vertically. I think my interpretation is reasonable, but may be it is not the only interpretation. This is a good example of why it is important to have video recording of the problem solving process.

I think this is a good example of an out-of-focus phenomenon, since they use the word continues in a different manner, than what is usually meant in a mathematical context; they use it to mean x can be varied continuously. It is interesting that continuity has crept its way into the students' concept definition of a function.

The video recording of Helmut and Nora's answer does not show similar considerations:

HELMUT: So should we just show some functions? We have a linear one: $f(x) = ax + b$

NORA: Then we can continue and make a 2. degree one. $f_2(x) = ax^2 + bx + c$.

HELMUT: And then we can take a 3. degree one and so on.

Apparently, they think that it is enough to simply give some examples of functions.

8.1.2 Assignment 2

All the students answered this assignment correctly; they either solve for x, or insert $x = 2$ and check it the resulting statement is true.

Tegan, Sara, Helmut, and Nora simply solve the equation straight away. Hans and Grete have written *"yes!"*, and since we do not have any recording of their attempt, we can not know how they arrived at the conclusion. Mads and Michael inset $x = 2$; lets have a look at the video recording:

MADS: Yes. It is simple. We just insert $x = 2$. *[he continues to reads aloud while he inserts $x = 2$, and concludes that $4 = 4$.]*

MICHAEL: I didn't get that at all. You said $3 \cdot 2$. Why did you do that?

MADS: Because it says $x = 2$. Is $x = 2$ the solution?

MICHAEL: Ah, ok. Well, you could just isolate x.

MADS: Yeah, sure.

Initially Michael did not understand what Mads did, when he inserted $x = 2$. This might be due to the fact that he had an idea about how to solve the problem himself, and as a consequence, he was confused about Mads's deviation from this method. The written answers, together with the dialogue between Mads and Michael, goes to show that isolating for x is by far the most popular way of showing that some value of x solves an equation; even though Mads carries out the method of inserting $x = 2$ while reading aloud every step, Michael is not quite sure why you would want to do that.

8.1.3 Assignment 3

All of the students were able to correctly give the algebraic expression for the graph of the function. This is quite an improvement from the previous year, where only one pair of students were able to do so (Tegan and Sara).

8.1.4 Assignment 4

Only one of the pairs were able to give a completely satisfactory answer to this problem. Their written answers are given below.

Hans & Grete: x controls y; 2+5=7; 3+5=8 etc.

Mads & Michael: x is always 5 larger than y

Nora & Helmut: y will always be 5 larger than x

Tegan & Sara: y is larger than x

Nora and Helmut are the only pair which answers the problem in a satisfactory manner. Based on their description it would be possible to reconstruct the equation. Let us have a look at the video recording:

NORA: That x is the slope?

HELMUT: Well, I think that you have to say something. I mean if x is a number, then y will be five larger. I think that is what they mean. I don't think that it has to be a graph, I think that we should just describe it.

NORA: Are you sure?

HELMUT: Well, if it is a function then we normally say $f(x)$.

NORA: Yeah, and it does say "what can you say"

It is interesting that Helmut doesn't think that $y = x + 5$ should be a graph based solely on the fact that it doesn't say $f(x)$. This could be interpreted as the following: To him, the graph of $y = x + 5$ and the expression $y = x + 5$ are two completely different things – Helmut equates symbols with the objects they are supposed to represent.

However, since Helmut says *"if it is a function then we normally say $f(x)$"* it might suggest that it is the fact that it says $y = x + 5$, rather than $f(x) = x + 5$; to him $f(x)$ invokes the function concept and its different representation. Maybe he would be able to see $f(x) = x + 5$ and a graph with slope 1 and intersection 5 as two different symbols for the same object.

It seems that they agree that a graph would be inappropriate because the assignment: "y=x+5, what can you say about x in relation to y" include the word "say".

Mads and Michael make the same mistake that Blomhøj observed in [Blomhøj, 1997] for Danish 9. grade students. They think that $y = x + 5$, means that x is larger than y. They spend 35 seconds on the assignment, and Michael states almost instantly *"yes. x is five larger than y"*, which is what they write down. It is interesting that misconceptions like this are still present at the second year of secondary school.

Hans and Grete's answer is considered to be wrong because they do not adequately convey the necessary information to be able to reconstruct the relation. While it is true that the two points (2,7) and (3,8), which they have essentially given, would be enough to reconstruct the relation $y = x + 5$, it is not certain that this is what caused Hans and Grete to give two examples. It is just as likely that they simply give two examples to say *something* about the the relation.

Tegan and Sara's answer does not contain sufficient information to reconstruct the relation. It is interesting that they fail to answer this problem correctly since they were able to do so the previous year. Let us take a look at the video recording:

TEGAN: x is five times larger. Right?

SARA: Well sure.

TEGAN: Well, not 5 times larger. Just five larger. Can you say that? Five larger?

SARA: Yeah sure. Or what does one say about x.

TEGAN: You can say that it is less than y. Its y that is larger than x.

SARA: That is only if you add the five to it. They could be the same in principle.

TEGAN: No they can not. Not when it is stated like that. Then it would say $y = x$. y must be larger then x if you have to add something to x in order to get y.

SARA: Well, sure, when you add the five to x, then it gets larger. Right?

TEGAN: Well, not that they are equal. It is an equation.

SARA: Yes of course.

TEGAN: So you can say that y is larger than x. It is a simple equation so you can not really say it more precisely without saying the equation directly.

This last statement from Tegan suggest that their written answer is vague because they do not wish to *"to say the equation directly"*.

Initially they make the same mistake as Mads and Michael; they think that x is five larger than y. Tegan notices the mistakes and corrects it, but it takes some time to convince Sara. Sara seems to think that x and y could be equal in principle, but that once you add the five, then they are different. Notice that she still seems to think that x is the larger one, even though Tegan has stated that y must be the larger one, at least that is how I interpret her statement: *"when you add the five to x, then it gets larger"*. It seems that she is really parsing the mathematical notation from left to right: y equals x plus 5, tells her that y and x are equal until you add the five to x, after which x will obviously be the bigger one. The statement *"it is an equation"*, seems to convince her that she had been using faulty reasoning.

It seems that only Nora "sees" a graph when he looks at the expression $y = x + 5$, and he it convinced that this is in fact the wrong way to look at it. There is reason to believe that the students would see a graph if it had said $f(x) = x + 5$; to the students, it seems, $y = x + 5$ and $f(x) = x + 5$ are not the same.

8.1.5 Assignment 5

The written answers are given below:

Hans & Grete: They are opposite *[They have given a table with the x-y values corresponding to x going from 9 to 2. At the end of the table they have written "etc."]*

Mads & Michael: Positive numbers = interval between [0;10]. Negatives numbers can in principle be ANYTHING, e.g. $-21 + 31 = 10$, $x = -21$ $y = 31$.

Nora & Helmut: Together they must give 10, and they can be both positive and negative, at least one of them must be positive.

Tegan & Sara: They give 10. They are good friends.

Nora and Helmut gave a satisfactory answer. They convey both the fact that the sum of x and y must give 10, and that x and y can be any number, positive and negative, as long as $x + y = 10$. Let's have a look at their dialogue:

NORA: Uhm. $8 + 2$, $6 + 4$

HELMUT: So what can you say. Together they should give 10.

NORA: We could also have -10 and 20.

HELMUT: Yeah, they just have to give 10.

NORA: Yeah, so if one is negative, the other one must be positive.

They start with two examples and conclude with a general statement based on the examples. It is interesting that Nora does not "see" the graph in this assignment.

Hans and Grete give table values corresponding to x going from two to nine, and conclude that they are *"opposite"*. While the table values are indeed enough information to reconstruct the relation $x + y = 10$; the statement *"they are opposite"* is definitely not sufficient.

Mads and Michael give two separate descriptions of the relation, one for x and y being positive and one with the possibility of them being negative. Then fail to mention the fact that the sum of x and y should be 10 in their written answer, and as a consequence their descriptions are too vague to enable reconstruction of the relation. This is their dialogue:

MICHAEL: They are within an interval of 0 to 10. Both variable.

MADS: No, it can be infinitely many. In the place of x we can have -21.

MICHAEL: Are you sure?

MADS: Yes, and $+11$ for y.

MICHAEL: That would give -10

MADS: Ah, yes $+31$ then.

MICHAEL: Well, sure. If you look at it like that. If they are allowed to go negative, it can be infinitely many. But it we only look at positive numbers, it goes from 0 to 10.

It seems that their written answer is an attempt to satisfy both Michael and Mads's considerations.

Tegan and Sara give an intentionally silly answer. Let us have a look at their dialogue:

TEGAN: I was thinking 5 and 5, but it doesn't have to be.

SARA: or it could be 7 and 3

TEGAN: So what can you say. They are good pals. haha.

SARA: Well, they are good friends. They give.

TEGAN: Together they give 10. Their sum should be 10.

It seems that the students are having difficulties given linguistic descriptions of equations. If one compares the answers to this assignment with those to assignment four, one will notice that they are not very similar, even though the essence of the assignments are the same. This may be due to, at least, two factors; first of all, the assignment in phrased differently, we ask them to describe the x and y that obey $x + y = 10$ in assignment five, and in assignment four we ask them what they can say about x in *relation* to y.

Maybe if the assignment was rephrased as *"what can you say about the x and y that obey $y = -x + 10$"* the students would be more inclined to think about the points that lie on the graph of the function, but then again, we have already seen that the students will most likely not think of a graph unless the notation $f(x)$ is used explicitly.

8.1.6 Assignment 6

All the students gave correct answers to this problem, and there is nothing of interest to report. It goes to show that the students can easily change between the different representations as long as the assignment is phrased in a familiar way.

8.1.7 Assignment 7

Three of four students successfully solved this problem. This is quite an improvement from the previous year, where none of the four pairs of students were able to do so. The students who solved it all did it in the exact same way; they find the expression for $f(x)$ and check if $f(2) = 1$. Their written answers are given below:

Hans & Grete: No, $y = ax + b$. $y = x + 1$. $y = 2 + 1 = 3$.
Mads & Michael: $f(x) = x + 1$. $f(2) = 1 =$ False.
Nora & Helmut: $f(x) = x + 1$. $x + 1 = 1$. $2 + 1 \neq 1$.

Let us start by looking at Michael and Mads's recording:

MICHAEL: Yes. If you write $f(2) = 1$.

MADS: We can see what the equation describing the graph is.

MICHAEL: Yes

MADS: It is $x+1$, so $f(x) = x+1$ and $f(2)$ should be 1, which it is not. So it does not solve the equation. Then Inspire would say "false". haha.

Their reasoning is sound and clear. I think that it is interesting that they do not simply inspect if $f(2) = 1$ is true by looking at the graph, rather than finding the expression, and *then* checking.

Nora and Helmut share the following dialogue.

HELMUT: Well, what is our function? It is the same as before. Right.

NORA: Yes.

HELMUT: So that should be equals one. So $2 + 1$, that does not give 1.

Once again, the assignment seems to cause no problems at all.

Tegan and Sara solve it in a similar way:

TEGAN: Solves the equation. So we should insert 2 there?

SARA: I don't know if we have to use the point for anything?

TEGAN: Well, x is here.

SARA: Ok, so if we use the two values, x and y, then we could put this in the place of y, or $f(x)$, and then we can find the equation for the graph, it is $f(x) = x + 1$ so $f(2) = 3$. So it is not true at all.

Apparently they also solve the problem without great difficulty.

The students fared much better on this problem, than they did the year before. They all solved the problem in the same way, and this is a fine way to solve the problem. I would like to note, however, that no one use the more structural approach of inspecting if the point lies on the graph. If the equation for the function was not immediately accessible to them, their solution strategy would not have worked.

8.1.8 Assignment 8

Everyone solved this assignment by inspection.

Mads and Michael answer it almost without thinking. So does Tegan and Sara and Helmut and Nora; however, Helmut and Nora continue to write up the equation for the two graphs, equate them, and solve for x, just in case.

8.1.9 Assignment 9

All of the pairs were able to solve this problem correctly as well. They all solve it by equating $f(x)$ and $g(x)$. Once again showing that the students are quite efficient as long as the assignment takes place in a familiar setting.

8.1.10 Assignment 10

Only Mads and Michael manage to solve this problem correctly. The students approach the problem in a mixture of drawing diagrams, and manipulating the the functions describing the positions of the cars. Their written answers are given below.

Hans & Grete:

$$\begin{aligned} 80x - 360 &= 100x \\ 20x &= 360 \\ x &= 18 \end{aligned} \tag{8.1}$$

18 hours.

Mads & Michael:

$$\begin{aligned} 80x + 100x &= 360 \\ 180x &= 360 \\ 1x &= 2 \end{aligned} \tag{8.2}$$

4 Hours after car 1 starts and 2 hours after car 2 starts.

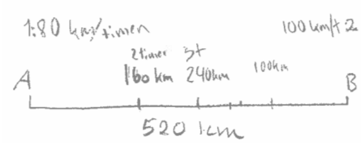
Nora & Helmut: We can conclude that they meet after about 3.5 hours. *[they have drawn a diagram.]*

Tegan & Sara: ... *[they have drawn a diagram]*

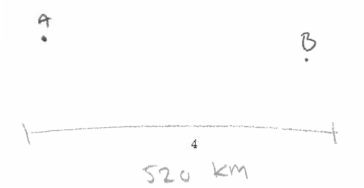
It would seem that Hans and Grete's problem lies in taking into account that the cars are driving towards each other. They have correctly accounted for the fact that, once both cars are moving, they only have to travel the remaining 360 km. The fact that their result is an astonishing 18 hours does not seem to bother them in the least.

Nora and Helmut attempt to solve the problem purely by drawing a diagram, see figure 8.2a. This should work, but is seems that they do not draw the diagram rigorously enough. The video recording shows what you would expect; they spend about four minutes on the problem and they start by drawing the diagram (figure 8.2) almost immediately after they have read the assignment.

Tegan and Sara have only drawn a diagram, see figure 8.2b. After reading the problem text slowly and carefully they decide that they don't feel like doing any more math and turn off the camera.



(a) Nora and Helmut



(b) Tegan and Sara

Figure 8.2 – 8.2a: Nora and Helmut's diagram to assignment 10. 8.2b Tegan and Sars' diagram.

Mads and Michael solve it correctly. They take into account that once both cars are moving they are only separated by 360 km and the fact that the cars are moving towards each other. The last two lines of their written answer seems a little strange though. Let us consider their problem solving process.

They start by writing up the two expression for the positions of the cars as $80x + 160$ for the car starting at A and $100x$ for the car starting at B, and they equate these two terms as $80x + 160 = 100x$. They notice that they have been making a mistake, and the dialogue continues as follows:

MADS: Oh, we're making a mistake. 2 hours after, a cars starts from B and drives *toward* A. So they are driving towards each other. B starts at 520, and A stats at 0, and it has driven 160, when B begins, so B must be at 360.

MICHAEL: Yes, so one drives with 100 km/hr and the other one with 80 km/hr

MADS: So we can say $80x + 100x = 360$, because for each hour this one drives 100 km and the other one drives 80 km. Then $x = 2$.

Which is correct.

8.2 Characterisation of the students understanding of the function concept

Hans & Grete are able to solve all the standard problem (2, 3, 6, 8, and 9) as well as problem 7, suggesting that they are at the stage of condensation. They solve assignment 7 by changing to the algebraic representation rather than solving it in the graphic representation, indicating that they are not as confident with the graphic representation. They fail to solve both assignments that require a linguistic description, showing that they are unfamiliar with this representation.

Mads & Michael solve all the standard problems correctly and efficiently, and they are able to solve assignment 10, leading me to believe that they are at the stage of condensation. They were unable to give a satisfactory definition of a function, and they failed to solve both the assignments that requires a linguistic description of a functional relation, even giving a wrong description, rather one that is just insufficient.

Nora & Helmut seem to be at the early stage of reification. They are able to change between the different representations quite efficiently – they are even able to solve both the assignments that require linguistic descriptions, suggesting that they have a sound understanding of the variable concept as well. However, they shown sign of a having a pseudo-structural conception, since they seem to think that the algebraic and graphical representation are different mathematical objects. Furthermore, they are unable to give a proper definition of a function.

Tegan & Sara seem to be at the early stage of condensation, since they solve all the standard problems efficiently, but they are unable to give a proper definition of a function. They are unable to solve the assignments that require them to give a linguistic description. They skip assignment 7 all together, suggesting that they have a weak understanding of the graphical representation. Furthermore, they were unable to solve assignment ten.

8.3 Discussion

Since I have been fortunate enough to study some of the same students with a year in between each study, it seems appropriate to comment on the signs of change in conceptual understanding, and attitude towards mathematics in general, that have taken place during that year.

The students generally fared better than the previous year – the most obvious change being their success on assignment seven, an assignment that no-one was able to solve the previous year. However, when solving the assignment, none of the students used the graphic representation. It seems that even during the second year of A-level mathematics the fundamental challenges have not changed much from the ones experienced by first year students.

One thing that does seem to have changed is their use of $f(x)$ as a symbol for a function. The video recording shows signs which are both positive and negative. On the one hand, the students seem to be confident with using $f(a)$ to denote the function value at a. This was something that caused quite some difficulty the previous year. I would argue that this improved understanding of $f(x)$ as a symbol, is in fact, the main reason that they are able to solve assignment 7 this year, and were unable to do so the previous year.

On the other hand, some of the students showed signs of using $f(x)$ as the *only* sign for representing a function. This tendency is present in the assignments that require a linguistic description of functional relations. They students generally don't think of the expressions as functions because it does not say $f(x)$. We saw this with Helmut and Nora when they noted that *"if it is a function then we normally say $f(x)$"*.

This is not so for all the students, however. Tegan and Sara noted that *"we could put this in the place of y, or $f(x)$"*, which suggests that y and $f(x)$ is the same to them, when they are sure that the are dealing with functions, at least – they did not talk about functions when they solve assignment four and five. Similarly, Mads and Michael defined a function as a relation between x and y, not x and $f(x)$.

There are generally fewer out-of-focus phenomena than the previous year. But some still occur, e.g. when Mads and Michael's use of the concept of continuity.

9 Study C

Study C was conducted at Roskilde Katedralskole on a Bio-tech class with A-level mathematics. The math teacher was using "Vejen til matematik" as the text book. The class were in the middle of their first year of mathematics. A set of 18 students were divided into 9 pairs, 4 of which were recorded. The pairs which were recorded are Laura & Lotte, Louise & Henriette, Matilde & Hilda and Lars & Erik. The students were given 50 minutes to solve the 10 revised problems.

In this study, all the students were placed in the same room. They were placed as far from each other as possible, but there is the change that the students may have overheard each others solutions, which is of course unfortunate. However, as long as the students state the reasoning behind their answers, it should not be a problem.

9.1 Data analysis

Figure 9.1 shows both the number of attempts to solve the assignment and the number of correct answers to the assignments.

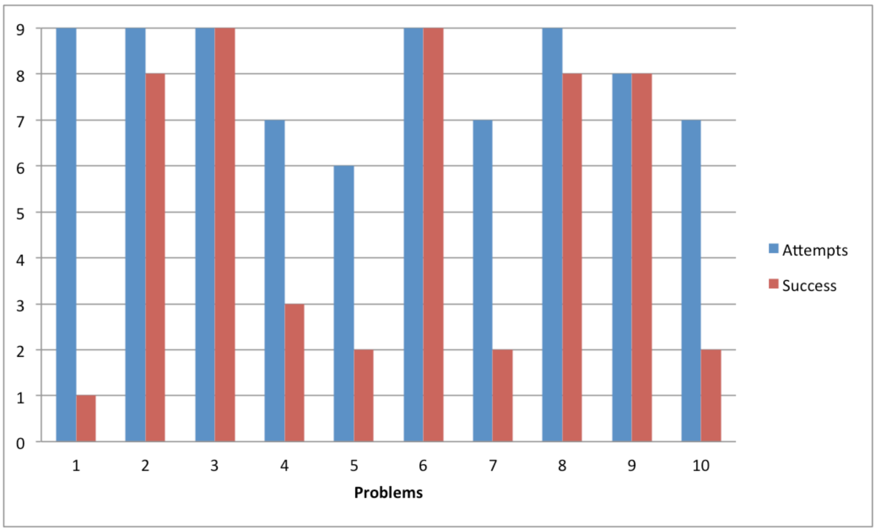


Figure 9.1 – Overview of the number of correct answers to the assignments.

Looking at the plot, we see a pattern similar to the results from the previous studies. Only one pair were able to give a correct definition of a function; assignments four and five, in which the students are asked to give a description of functional relations both have comparably low success rate; problem seven and ten also gave the students a hard time.

9.1.1 Assignment 1

The correctness of assignment one is based on the students ability to articulate what they consider to be the essence of a function. I do not expect the students to give a correct set theoretical definition of a function, since this is not taught to upper secondary students. Before we start to analyse the definitions the students give, I will give a definition of a function, as it is given in the textbook "Vejen til matematik".

A function $y = f(x)$ is a relation between two variables x and y for which it holds that there exists a unique y for every x in the domain.

All the written answers are given below. I will comment on the definitions and state which ones I considered to be correct. The following definition is considered to be correct:

Jon & Troels For a function the following holds: There can only be 1 y-value to 1 x-value. A function describes the connection between two variables.

Jon and Troels both convey the fact that a function is a connection between two variables, and that there exists a unique y for every x. All the really need is to specify that the x's need to bee in the domain of the function as well.

The rest of the definitions of the students are wrong for various reasons.

Laura & Lotte A function describes a connection between a series of numbers. A linear function is $f(x) = ax + b$. An exponential function is $f(x) = b \cdot a^x$.

They fail to mention the uniqueness of y. Laura and Lotte are one of the pair I have recorded, so let us dive into the recordings:

LAURA: A function is when there is a connection between two quantities.
LOTTE: Yes, or a long series of them, several quantities, and that they can be described graphically.
LAURA: Yes. A function can be shown graphically, but yes, a connection between different quantities.
LOTTE: Should we make a model of a linear relation and an exponential relation?
LAURA: Yes sure, I guess it would be fine to do it like that. We can draw it.

It would seem that Laura and Lotte think that a function is a connection between two quantities, and that his connection can be represented in different ways, e.g. as a graph. After this, a minute or two passes while one of the girls draw the graph of a function on the paper. Suddenly Lotte stats the following dialogue:

LOTTE: I guess that it is through variables that it describes a connection between a series of numbers, right? Through variables.
LAURA: Why variables?
LOTTE: No, just forget it.
LAURA: No, but why? why variables?
LOTTE: Just forget it.
LAURA: No, I mean what was the reason that you said it? there must have been a reason.
LOTTE: I was just thinking that there are variables in... hmm... in the function. I was thinking about the fact that you can just interchange the numbers and it would still be the... same...
LAURA: Yes, I can see what you mean.
LOTTE: But what can you say. I mean, it is not a specific series of number, you can insert all kinds of stuff, and it would still be a linear relation.
LAURA: I see what you mean. You would write a series of numbers. Variables. Is that what you were thinking?

During the dialogue Laura states that *"A function is, when there is a connection between two quantities"*, and Lotte elaborates that *"is through variables that it describes a connection between a series of numbers"*. Put together, it brings them close to stating that a function is a connection between variables. However, Laura is not convinced by Lotte's logic, and Lotte has to elaborate further. She states that *"there are variables in the function"* and that *"it is not a specific series of number, you can insert all kinds of stuff"*. Suggesting that she thinks that the statement *"a connection between variables"* is more general than *"a connection between numbers"*. They end up writing down the latter, though.

Matilde & Hilda A function is a connection between x and y

Louise & Henriette A connection between two values. E.g. height dependent of time.

Dan & Anders A function is a unambiguous connection between x and y. For 1 y-value there exists 1 x-value.

Laura, Lotte, Matilde, Hilda, Louise and Henriette all fail to mention the uniqueness of the y corresponding to a given x. Dan and Anders remember that uniqueness is important, but the get the uniqueness wrong. All these definitions focus on the fact that a function is a relation between x and y values.

Let us have a look at Matilde and Hilda's dialogue.

MATILDE: A function is when there is a connection between they way...
HILDA: x and y
MATILDE: There is a connection between the way x and y grow. Right?
HILDA: A function... is when there is a connection between...
MATILDE: Is *the* connection between x and y. Or the connection between two variables actually.
HILDA: Yeah. But a function can also be exponential. Then there is more variable than two.
MATILDE: A connection between a minimum of two variables, then.
HILDA: No, because we talk about the connection between x and y. It is a connection that can be described with several variables, so there are only two. So should we just call them x and y?
MATILDE: Yeah. That is the question. We can say the way a number.. or a graph.. how do we describe this in a simply way? A function is the connection between x and y, described by variables
HILDA: Described by an equation.
MATILDE: Described with variables.

It seems that Matilde knows that a function is a connection between variables, but they never get around to the uniqueness of the function value for a given x.

There is some confusion about what role the variables play. It is unclear why Hilda thinks that an exponential function requires several variables or what she means that a function is a connection that can be described with several variables, I do not think that she is function of a function of several variables. Maybe she is thinking about the fact that an exponential function $f(x) = b \cdot a^x$ contain more symbols than just x and y, since there is both a and b, and that the function is then described through the specification these symbols, but this is speculation.

Matilde states that a function is *the* connection between x and y, which suggests that she knows that the function describes the connection between variables, and not the other way around, as Hilda seems to think. Over all there is quite a lot of confusion regarding the definition of a function, and they are not complete certain what role the variables play.

Let us have a look at Louise and Henriette's dialogue

LOUISE: I guess we can draw a coordinate system, if we should make a drawing.

HENRIETTE: Yeah.

LOUISE: *[she draws a coordinate system containing a straight line and starts to label the axes]* so we can put time here *[the fist-axis]* and... height here *[the second-axis]*.

HENRIETTE: Should we give an algebraic expression as well, do you think?

LOUISE: ...Can you give a definition? What is a function a connection between. Is it two values? Proportional values?

HENRIETTE: I don't think that they have to be proportional

LOUISE: ... a connection between two values.

HENRIETTE: Should we give an example?

LOUISE: Well, we already have with the graph.

The concept of proportionality has somehow crept it way into Louise's concept definition of a function.

Here are some definitions which have some of the right ingredients, but lack a crucial component of the definition and are therefore considered to be incorrect.

Julie & Esben There can only be one y-value to one x-value.

Julie and Esben seem to understand the a function consists of something ordered, but they fail to state that there must be some connection between domain and codomain.

Jens & Martin A function is connection between x and y values which can be describes by a mathematical model.

It is not clear exactly what the students mean by mathematical model. It might be that there should be an mathematical formula for the connection between the x and y-values. Either way, they do not mention the uniqueness of y.

Lars & Erik A function is when an independent variable is put into an equations which yields a product.

This definition is wrong because it focus on the algebraic expression and it does not mention the uniqueness of function values. Lets have a look at their dialogue.

LARS: A function is when you get an dependent variable from an input

ERIK: Yes, you insert a number which has to be random, and then you can calculate the other.

LARS: You get a product from a constant.

ERIK: Yes, basically yes.

LARS: Ok, what do we write?

ERIK: It is important about the constants, where you have a variable which should be put into an equation which then yields a result.
LARS: Not a variable. Well, yes, an independent variable can be put in and then you get an dependent... you know $f(x) =$...
ERIK: Yes, exactly. But, how do we explain it. Maybe we should give an example. It says that we can both write and draw, so i guess we can give examples too.
LARS: It is when an independent variable is *[he reads aloud while he writes, and pauses]* should we said "is being used" or?
ERIK: It is more like "put into". They always try to explain it as a machine or something stupid like that.
LARS: ... is put into a *[he reads aloud while he writes, and pauses]*
ERIK: say formula
LARS: ... ok formula, which then yields an *[he reads aloud while he writes, and pauses]* independent variable or product?
ERIK: Just write product.

It is clear that there is a strong focus on the algebraic representation. Lars state that you get a product from a constant, and Erik states that it is important about the constants. It is not entirely clear to me what they are thinking about. It is quite clear that they have a heavy process conception of the function concept, since they talk about an input being put into an expression which then yields and output, and they make a specific reference to the "function machine".

Fenja & Ronja A function can both be linear and exponential. A Function has both a dependent and an independent variable. From this, you can write up an equation describing the function. The equation describing a linear function is $f(x) = ax \cdot b$ and the equations describing the exponential function is: $f(x) = b \cdot a^x$.

Fenja and Ronja simply give examples of functions they can remember from their classes. Furthermore they give an incorrect definition of a linear function.

9.1.2 Assignment 2

Eight of the nine pairs of students answered this assignment correctly. Basically the students apply one of two methods; either they solve for x, or they insert $x = 2$ into the equation and checks if the statement is correct. It is worth analysing what went wrong for the pair that answered incorrectly.

This is what they wrote:

Fenja & Ronja

$$\begin{aligned} 3x - 2 &= 2x \\ 3 \cdot 2 - 2 &= 2 \cdot 2 \\ 4 &= 4 \\ x &= 1 \end{aligned} \tag{9.1}$$

It is interesting that the students would have solved the assignment correctly if they had just left out that last line. I think they added it because solving equations usually required you to give the answer in the form of $x =$ something. When they got to the last line and realised that there were no longer any x in play, the simply added one on the lefthand side as $4x = 4$ which lets you conclude that $x = 1$.

9.1.3 Assignment 3

All of the students were able to correctly state the equation describing the function. Lets have a look at Louise and Henriette dialogue:

LOUISE: It is linear, so we have...
HENRIETTE: $f(x) = 1$...
LOUISE: Normally it is b, right?
HENRIETTE: Yes, plus a. No. $ax + b$.
LOUISE: our b value is equal 1, right?
HENRIETTE: And a is... when the x-value rises one, the y-value rises one.
LOUISE: So we have a slope of one
HENRIETTE: Is this not always *a [she is carrying out the rise-over-run method and is pointing at the vertical line, which shows how much you have to "rise" to reach the graph of the function after having moved unity along the x-axis.]*
LOUISE: Yes, that is the slope. Then it must be one.
HENRIETTE: yes.
LOUISE: So we can just say x.

They carry out the rise-over-run method with only a little confusion as to how to give the algebraic expression for a straight line.

9.1.4 Assignment 4

Only three of the nine pairs of students were able to correctly answer assignment four and five. Both assignments required the students to give a linguistic description of an equation. The two assignments were; "$y = x + 5$, what can you say y compared to x", and "what can you say about the x and y that obey the equation $x + y = 10$?"

Generally the student either wrote up the equations with words, or they focused on only a part of the property of the equations. Here are some of the written solutions:

Lars & Erik y is always 5 higher than x
Matilde & Hilda y is 5 higher than x
Laura & Lotte The value y is 5 larger than x

These are the three answers I have considered to be correct. They all convey enough information for someone to express up the relation between x and y in another representation.

Let us look at Matilde and Hilda's dialogue

MATILDE: That x is five larger than y.
HILDA: Hmm... Yes.
MATILDE: Yes, that is all. Is there any more we can say?
HILDA: No, not really.
MATILDE: x starts off being five larger than y. Then, every time y... then there is just a linear growth thereafter.
HILDA: Yes, because it wont be more than five. More or less than five.

MATILDE: No. Do we have to write it? x is five larger than y. No y is five larger than x. Yes.

HILDA: y is five larger, no not times, larger than x. It that just it?

MATILDE: It almost seems too easy.

Initially they think that x is the larger one, because it says +5 next to the x. Matilde corrects the mistake without much fuss, and they go on to write down their final answer.

Let us look at Lars and Erik's dialogue.

ERIK: That one is always bigger than the other?

LARS: Yes. Yes.

ERIK: x will always be five higher than y *[Erik notices the mistake immediately and says "no" as the following line takes place]*.

LARS: y will always be five higher x.

So initially Erik makes the mistake of thinking that x is the larger one because it says "+5" next to the x, but he spots his mistake immediately.

Let us consider Laura and Lotte's dialogue:

LAURA: We can say that the value x always will be five larger than y. Right?

LOTTE: No, because when you have the value x, then you need to add five to obtain y

LAURA: Ah, so x will always be 5 less than y, so y is five larger than x.

It seems that they make the mistake of thinking that x is the larger one as well, but they are also able to correct the mistake.

Let us look at the remaining answers:

Louise & Henriette $y > x$

while this is true, it is not enough to reconstruct the relation between x and y; some information is lost. Let us have a look at their dialogue

LOUISE: $x + 5$. It must be that y is larger than x

HENRIETTE: Yes

LOUISE: Then it must be $y > x$

HENRIETTE: So for every y, we add 5 and then we have x

LOUISE: No you subtract five. If you have y and you want x, you subtract five. Because $x + 5$ gives y.

HENRIETTE: Oh, yes. Of course. It's a good think that I've got you, Louise.

LOUISE: But the only thing we can say is that y is larger than x. Or five larger.

HENRIETTE: But it is not five times larger

LOUISE: No you are right.

HENRIETTE: So you can't say that.

LOUISE: And y will always be larger than five. x can be equal zero, but not matter what y must be five.

HENRIETTE: But x can be negative

LOUISE: Yes, you are right. Well, then we can only say that y should be larger than x.

Louise actually gets it right, but she is discouraged by Henriette's misunderstanding. Initially Louise thinks that x is restricted to positive values or zero, but Henriette corrects her. Very briefly Henriette thinks that one should add five to y to obtain x, even though they have just agreed that y is larger than x, but this is corrected quickly by Louise.

Jens & Martin y depends on a given x-value (+5)

Julie & Esben x is the independent variable, y is the dependent variable. The value of y will be the x-value +5

These answers are considered to be incorrect because they use the mathematical notation +5.

Jon & Troels x is a constant and y is a variable

This is simply not true, but it is interesting that the students can see one of them as being constant while the other one is able to vary. Please note, that Jon and Troels were actually the only students who were able to give a correct definition of a function; never the less, they give am answer like this.

Fenja and Ronja wrote nothing at all. The same goes for Dan and Anders.

Based on the number of students who have neglected to even attempt to answer the assignment, I think it is safe to say that the students felt that the phrasing of the assignment was unusual, if not difficult to understand.

9.1.5 Assignment 5

And here are the two correct answers to assignment 5

Jon & Troels Their total value should be 10

Laura & Lotte The values for x and y can be infinitely big/small (positive/negative) – when x is negative y must equally positive in order for them to obey $x + y = 10$.

Let us consider Laura and Lotte's dialogue:

LAURA: Well, there are infinitely many x and y values that obey the equation.
LOTTE: Should we just write that?
LAURA: Yes, we can not say any more, I mean, they can even be negative. So if x is negative y will just be correspondingly larger
LOTTE: All number?
LAURA: I guess that they can.
LOTTE: I don't know. Okay. what about this: can x be a million?
LAURA: Yeah, sure then y has to be... correspondingly negative
LOTTE: Then y has to be 999990.
LAURA: So they can be infinitely many numbers?
LOTTE: Lets write that.

It is interesting that Lotte state that *"y has to be 999990"* even though they have just decided that y has to be negative, but she probably just forgot to say "minus".

The rest are considered to be incorrect.

Lars & Erik They are inversely proportional, since their sum must be 10.

Jens & Martin $x, y \leq 10$

Louise & Henriette $x \leq 10$ and $y \leq 10$

Matilde & Hilda $x =]0;10[$ and $y =]0;10[$. It is linear *[they have drawn a liniear function $f(x) = -x + 10$ shown in the interval $x = [0;10]$.]*

Let us have a look at Matilde and Hilda's dialogue

MATILDE: Ok. So how do we do this?

HILDA: What are you trying to do?

MATILDE: The domain of definition

HILDA: Why?

MATILDE: To describe what values x and y can take. They can both be as close to zero as possible and as close to ten as possible. So we can do like this.

HILDA: Well, we need to describe that they can not both be 10 simultaneously.

MATILDE: Ok, so how to we do that in an easy way. $x + y = 10$ is what i would usually say.

HILDA: Yes, it is a little difficult. We can almost do it with... But are we allowed to draw, because then we could just draw the graph.

They go on to draw the graph, but they do not allow x to run negative. Matilde and Hilda apparently understand that $x + y = 10$ is a function, and the the points that obey this relation defines the graph of the function.

Let us have a look at Louise and Henriette's attempt:

LOUISE: That neither can be larger than 10. They should be less than 10.

HENRIETTE: yes.

LOUISE: I guess we can not say anything about x and y in relation to each other.

HENRIETTE: No.

Again Louise restricts the variables to be positive or zero, but this time Henriette does not correct her. Apparently the expression $x + y = 10$ does not convey any information to them about relation between the values that x and y can take, even though they were able to say that $y = x + 5$ meant that y was 5 larger than x in the previous problem.

Let us have a look at Erik and Lars' dialogue

ERIK: Well, we can say that they are inversely proportional.

LARS: What can you say about the x and y...

ERIK: ...that solve the equation. They must be inversely proportional if they have to give the same always.

LARS: Yes. I don't know. Yes

ERIK: If one is increased by one the other one has to decrease by one.

From this dialogue it is quite clear that Erik takes "inversely proportional" to mean that if one increases then the other decreases by the same amount, rather than the usual definition where $y = k/x$ with k being the constant of proportionality. I take this to be an example of a out-of-focus phenomena.

Fenja and Ronja wrote nothing at all. The same goes for Dan, Anders, Esben, and Julie.

9.1.6 Assignment 6

All the students were able to correctly list the three pairs of functions.

9.1.7 Assignment 7

The following two answers were considered to be correct:

Jens & Martin $b = 1$, $a = 1$, $f(x) = ax + 1$, $1 \cdot 2 + 1 = 3$, thus $x = 2$ does not satisfy the equation.

Louise & Henriette $f(x) = 1$, $f(2) = 1 = (2, 1)$. The point $(2, 1)$ can be read off the coordinate system and does not lie anywhere near $f(x)$. $f(x) = 1$ can be read off the coordinate system and $f(x) = 1$ gives $x = 0$.

Louise and Henriette initially have a hard time understanding the question. They seem to be confused about the fact that the point does not lie on the graph. In an attempt to do *something*, they find the expression for the graph, and this lets them get on with the problem. It does not seem that they find the expression for the graph with the sole purpose of evaluating $f(2)$, but rather as an automatic response. Once they have the equation, however, they are able to think about the assignment more clearly; they start to consider what they are supposed to use the point for:

LOUISE: Oh, we are not supposed to find the equation for the graph.

HENRIETTE: No, we just have to decide if $x = 2$ solves the equation. Which it does not.

LOUISE: Which equation?

HENRIETTE: The equation $f(x) = 1$. This is what were asked about. Does $x = 2$ solve that equation. It does not.

They do not write anything down at this point, but continue to discuss the problem. They have actually solved the problem, but it seems that they are not quite satisfied with the answer.

HENRIETTE: This point says the same as what they said there *[pointing at $f(x) = 1$ in the problem text]*, that if $x = 2$ then we should also have... So we can say it does not lie on the line *[referring to the graph]*

LOUISE: No wait, no I'm confused again.

Shortly after this they decide to move on to the next assignment and leave room for the assignment, if they have time to get back to it. When they return to the assignment they assume that $f(2) = 1$ is correct.

HENRIETTE: If we assume that this is correct, then we would have $x = 2$. This we have found. And if that should fit the equation, then $f(x) = 1$, and the function always denotes the y-corrdinates right?

LOUISE: Yes

HENRIETTE: Then this will be be the point, if this is correct, and if our function is here, then we can just see that it is not true.

LOUISE: Well. It is this equation *[pointing at $f(x) = 1$]*, that we have to check for $x = 2$?

HENRIETTE: Yes

LOUISE: Then...

HENRIETTE: Well, I'm thinking this point shows if this equation should be correct for $x = 2$. That is what the point tells us.

LOUISE: So we have to insert $x = 2$?

HENRIETTE: Well, I don't think that we have to calculate anything. I just think the this tells us if it is correct or not.

LOUISE: It is correct for the point. But it is not correct for that one *[pointing at the graph]*

HENRIETTE: This point point tells us that this *[points at the equation $f(x) = 1$]* should be correct. But this point does not line of the graph of the function. Can that not be what we are supposed to answer?

LOUISE: I guess so.

HENRIETTE: Because we can not calculate anything, because we know it is wrong.

They go on to write down the final answer. Their language is imprecise at times, but they get to the essence of the problem.

Henriette talks about the point showing that $f(2) = 1$ *should* be true; about the point $(2, 1)$ *"telling her"* the same as the equation $f(2) = 1$; if the point lay on the graph then $f(2) = 1$ *would* be true. The fact that she says *"we can just see that it is not true"* shows that she has understood the essence of the problem – she is giving a structural argument.

The rest are considered to be wrong:

Jon & Troels If you insert $x = 2$ in $f(x) = 1$, you get the point P. (we had trouble understanding the phrasing of the assignment).

Dan & Anders We assume that $f(x)$ is the same as y. Thus $x = 2$ and $f(x) = 1$ are the same as the point $P = (2, 1)$

Matilde & Hilda Correct, based on the expression of the graph. $f(x) = x + 1$

It is interesting that Dan and Anders feel that they have to *assume* that y and $f(x)$ are the same.

It is hard to understand how Matilde and Hilda can conclude that $f(2) = 1$ based on $f(x) = x + 1$. Let us have a look at their dialogue:

HILDA: Ok, so we can see that the function is $f(x) = x + 1$. So if we insert $x = 2$, does that give 1?

MATILDE: Why do we have that point?

HILDA: I don't know.

MATILDE: So from the given information we should... We can just see it. It is much easier. We agree that the function is $f(x) = x + 1$.

HILDA: Yes

MATILDE: Like the one we had before

HILDA: Yes, so let us set 1 in place of x. $1 + 1 = 2$, yes it does.

MATILDE: Therefore it is correct.

Hilda actually solves the essence of the problem with her first comment. Unfortunately Hilda's comment about the point seems to throw her off. They end up checking if $f(1) = 2$ rather than if $f(2) = 1$. They seem to completely ignore the point.

Let us look at Laura and Lotte's written answer

Laura & Lotte $f(x) = x + 1$.

Laura and Lotte simply give the equation for the graph, illustrating once again that the students first instinct when they see a graph, is to give the algebraic expression for it. Let us look at their problem solving process:

LAURA: What are you supposed to do?
LOTTE: I don't understand the formulation either...
LAURA: If the equation $f(x) = 1$ –
LOTTE: – we can determine the formula for the function and see if that helps us at all.
LAURA: It intersects in the point 1 and the slope is 1. So $f(x) = x + 1$ which was what we had before... mmhm...
LOTTE: ... If we insert x...
LAURA: I don't understand how we should interpret this...
LOTTE: But $f(x) = 1$... Then x is 0...
LAURA: Yes...
LOTTE: Therefore, $x = 2$ does not solve the equation $f(2) = 1$... Then what should we use the point for.
LAURA: I don't get it either.

After this they decide to move on to the next problem. It is interesting that they actually solve the problem, but that they are unable to accept their answer as correct. The dialogue also shows that they do in fact find the expression for the function just to do *something*.

The last of the written answers is the following:

Lars & Erik If the point does not lie on the curve, it can not be used to conclude anything about the curve

Lars and Erik give what I would call a structural argument, but sadly, it is wrong. The fact that the point does *not* lie on the curve is enough to conclude that $f(2) \neq 1$. Let us have a look at their dialogue

LARS: Can you from the graph... and the point... Well, it does because it is something with x, which gives 1
ERIK: Yes, then it pretty much has to be two here. My first though was: "is this a trap".
LARS: Yes, when the point (2,1) *[he starts to write]*
ERIK: Lars, are you sure about that?
LARS: Well it can not be anything but true.
ERIK: Well, on the graph. It [the graph of the function] is not a part of the dot. This point does not lie on the line. It is not a part of the function. I don't think that we interpreted the question correctly. He asks if we can say anything, should we just say no?
LARS: Well, yes because we can not say anything because the point is not a part of the graph. That was a tricky one. *[he starts to write up the answer]*. If the point does not lie on the curve or graph?
ERIK: The curve, because the graph is the entire thing.
LARS: If the point does not lie on the curve, the function of the curve can not be used.

ERIK: No no no. The function can still be used. There is nothing wrong with the function. We just can not conclude anything about it.

LARS: So if the point does not lie on the curve, we can not use the curve to say anything about it

ERIK: The point. We can not use the *point* to say anything about the curve.

LARS: Well, we should say if $x = 2$ solve the equation $f(x) = 1$.

ERIK: The problem is that, when we are not informed that the point should lie on the...

LARS: Yes. well. It does not state that the function... $x = 2$ does not give us anything. If the point does not lie on the curve we can not use it to say anything.

It would seem that they think that the graph is the entire coordinate system, and that the curve is what we would normally refer to as the graph of the function. The fact that Erik states that *"the problem is that, when we are not informed that the point lie on the [curve]"* suggests that he might have been able to solve the assignment if it had asked about a point on the graph of the function. The problem may be that they are not used to equation being wrong. To him the expression $f(x) = 1$ may not be something that can possibly be wrong.

Fenja and Ronja wrote nothing at all. Neither did Julie and Esben.

9.1.8 Assignment 8

Fenja and Ronja were the only pair of students who answered the assignment incorrectly. Their answer was "$f(x) = 1$ and $g(x) = -1$". It is interesting that we see this notation used here as well – remember Cain and Abel's notation from Study A.

Laura and Lotte spend 10 minutes on this problem, which is a lot longer than you would expect. It turns out, that they actually solve the assignment within the first minute where the following dialogue takes place.

LAURA: they intersect at (1,1)

LOTTE: yes

LAURA: they must be. $f(x) = g(x)$ must be the point where they intersect

LOTTE: yes at (1,1)... For which x-values?

LAURA: Then it must be for the x-value 1.

LOTTE: yes

LAURA: no... yes, it must be

LOTTE: yes. It just sounds as if there should be several points in that formulation.

LAURA: for which...

LOTTE: But if.. yes...

LAURA: I don't know how else to interpret it

LOTTE: Neither can I. If you found the algebraic expression for both functions then you could equate them, then would would probably get the same. Or, I mean, you really should get the same.

LAURA: For the x-value 1 $f(x) = g(x)$ *[she reads aloud while she writes]*

The actually solve the assignment by inspection, and the only reason that they are not completely sure about their answer seems to be that my formulation of the assignment is sloppy, since it leads them to believe that there might be several intersection points.

To make sure that they got the right answer they decide to solve the assignment in the algebraic representation as well. The remaining 8-9 minutes are spend on attempting to find the expressions of the to function and equating them. This is very difficult for them

– more so than you would expect from their rapid solution of assignment 3. It turn out, that their problem is that they are unable to write up the expression for the function with the negative slope; they talk about $g(x) = x - 1$. As a consequence they do not get $x = 1$ when they equate $f(x)$ and $g(x)$. At the end of the 10 minutes they decide to get back to it later. The fact that they decide to solve the assignment in another representation to double check their answer shows that they understand that the two representations are equivalent. After having worked at problem 10 they come back to this assignment and write it up correctly.

Lars and Erik solve the assignment by inspection.

Matilde and Hilda solve the assignment correctly, but their road to the correct answer is interesting.

MATILDE: It guess it is where they intersect

HILDA: Yes, at 1. On the other hand, it is only x-values. They are also equal... Here it is both x and y.

MATILDE: They are equal everywhere. Because there is 90 degrees between them. Or actually it is 180.

HILDA: But they intersect at one place.

MATILDE: So everywhere. Don't we agree? ... or wait. Not down here. When they become negative.

HILDA: Well, the x-values are not the same here, it is the y values.

MATILDE: Well, if you look here, these two have the same x-value, but $f(x)$ is not the same. This is what we need to consider, and that only counts there *[pointing at (1,1)]*.

HILDA: Yes, there they're equal.

MATILDE: And that is the only place where $f(x) = g(x)$.

HILDA: Yeah, but it does not mention the y values. It asks for which x values.

MATILDE: Yeah. The formulation is difficult. I mean, either it is only at the point $(1, 1)$ or it is for all x-values.

HILDA: Yeah, but why?

MATILDE: Ok. Description. Im good at that. Since there are 180 degrees between... Can we not describe it through the algebraic expressions for the graphs? Let's write them up $f(x) =$...

HILDA: No, $f(x)$ intersects in 0, so it should be $0 \cdot x$, so $f(x) =$...

MATILDE: $f(x) = x$. $g(x) = -x$. Heh, no it intersects in 2, so $g(x) = 2x$.

HILDA: $2x$ plus. no wait. It is decreasing.

MATILDE: $2x$.

HILDA: No, it is b that is the intersection with the $y-$axis, a is the slope. We forgot.

MATILDE: Oh, yes.

HILDA: So it will be $-x$, right? $+2$. Do we agree.

MATILDE: Yes.

HILDA: I cant describe this. Don't you think that it is enough that we have said it out loud?

MATILDE: I don't know.

HILDA: For all $x-$values, $f(x) = g(x)$... No, I think that we have to say for which points is counts, with the help of the $x-$value... I think we have to find the point at which they are equal, and then we should just give the $x-$value the function has for the point where $f(x) = g(x)$.

MATILDE: Ohh. That makes sense. But the algebraic expressions are still right. So we should just equate them, say $f(x) = g(x)$.

HILDA: Can't we just inspect it.

MATILDE: No, that is too easy.

HILDA: For the $x-$value 1, $f(x) = g(x)$ since this is where they intersect. Or should we take the fun route and say that because we can not be 100% certain that what we see it correct?

MATILDE: Yes. Well, on the other hand we used the graphs to find the algebraic expressions.

HILDA: Hah. Damn, the other way is more entertaining.

MATILDE: Well, the phrasing is quite difficult.

HILDA: At the $x-$value one, $f(x) = g(x)$.

They solve the assignment within the first line of dialogue. From that point, however, a lang string of confusion arguments follow. They talk about the functions being equal for all x values and the argument is, that there is a angle of $180°$ between the graph of the two function.

One way of interpreting this is, that they are in fact think about for which $x-$values the functions are defined. It is, however, a bit confusion that Matilde says that they are not equal when "they become negative". I don't know what to make of it.

The fact that Hilda says that the assignment doesn't mention the y-values suggests that she does not fully understand that the function values are $y-$values, and that the functions are equal for some x if they take on the same y-value.

It is interesting that they bring up the fact that you can not truly trust the graph representation because of the limited resolution of the graph. I have wondered if this is the reason that the students sometimes do not like relying on the graph representation, because they think it is not as "exact" as the algebraic representation.

9.1.9 Assignment 9

Only Dan and Anders answered this assignment incorrectly. They attempted to graph the two function $f(x) = x$ and $g(x) = -x + 4$, but graphed $f(x) = 0$ and $g(x) = -x + 4$ and arrived at $f(x) = g(x)$ for $x = 4$. For some reason they could not graph $f(x) = x$ correctly. The same pair answered assignment 2 and 8 correctly.

Lars and Erik solve the assignment correctly. They do not do it by equating the algebraic expression for the two functions initially. They "draw" the graphs of the function in their mind and find the intersection. Erik says *"This increases one for one. And this decreases. They must intersect in two. If one starts at 0 and increases and the other decreases and starts at four. Then they meet at two."* After this they go on to solve the assignment by equating the algebraic expression for the two functions.

Let us have a look at Matilde and Hilda's approach to the problem.

MATILDE: So we should just solve for x. We can just put $f(x) = 4$ and then -4. No wait, what am I doing? What am i thinking. No, what I am thinking is, we can put these two together to one, and we know that $f(x) = x$, so we can insert $f(x)$ in place for x in the $g(x)$ function. Right?

HILDA: Okey. Ehm. I'm thinking. No thats how you find y. Can't we just put them together?

MATILDE: Well, thats what I'm doing.

HILDA: Yeah, but you want it in place of x, I want it to say $x = -x + 4$

MATILDE: Yeah sure we can do that. It should give the same.

HILDA: Ok. So we can write it up as $x = -x + 4$, can we not do that?

MATILDE: I'm not sure you can. $f(x) = x$, $g(x) = -x + 4$. If we isolate x as positive [in $g(x)$] and then insert it up here [in $f(x)$], the we get $f(x)+$... No. Maybe we are going at this all wrong. If we just choose two points then we can determine...

HILDA: We could also just draw it.

MATILDE: Should we just draw it.

HILDA: But the other way is much more fun.

MATILDE: We can draw it, and then we can calculate backwards afterwards.

HILDA: Ok.

They go on to draw the graph of the two functions. After having drawn the graph of the two function and found the intersection to be at $x = 2$ they go on to solve the assignment by equating the expression for the two functions which they found earlier. This time they solve for x and find that $x = 2$.

9.1.10 Assignment 10

Two pairs were able to solve this problem. One of them with startling clarity. Their written answer is the following:

Jens & Martin During the first two hours car 1 drives alone. It drives with 80 km/hr and at 3 o'clock the two cars will be separated by 360 km. After this the 2. cars also starts driving and their combined velocity is 180 km/hr. $360/180 = 2$ hours. So the total time before the cars meet is 4 hours.

Another group used the same approach:

Erik & Jens The distance between the two cars after two hours is given by: $520 - 2 \cdot 80 = 360$. After this the cars will be closer by 180 km for each hour. Thus another two hours pass before the cars meets, since $360/(100 + 80) = 2$. $2 + 2 = 4$, since car one had been driving for 2 hours.

Initially, Erik and Jens attempt to just add the velocities of the two cars and calculate how long it takes to travel the 520 kilometers. Here they do not take into account that one car starts before the other. This solution attempt is interrupted because Lars misunderstands the assignment, and thinks that both cars starts at point A and travel to B. They spend some time on this approach. Initially they attempt to equate the algebraic expression for the functions describing the positions of the cars. They give the position of the cars respectively as $f(x) = 80x$ and $g(x) = 100x - 160$ and equate them. This is the correct solution to the assignment they are trying to solve, but unfortunately it is not the assignment I asked them to solve. Erik notices that the car are in fact travel towards each other and not in the same direction. From here they go on to solve the assignment within minutes.

During their solution attempt they use all the representation: graph, table of values and algebraic expression.

Another pair of students give an answer which is almost correct. They takes a slightly differ approach; their solutions goes like this:

Matilde & Hilda .

1. hour: $520 - 80 = x$
2. hour: $520 - 80 - 80 = x$
3. hour: $360 - (100 + 8) = x$
4. hour: $180 - (100 + 80) = x$.

Since the result $x \leq 0$ it is the number of hours.

Hilda starts off by noting that she has solved a similar assignment before. They discusses if they want to solve in the algebraic or graph representation and they decide on using the algebraic representation because, as Hilda notes *"if we had Inspire we could use the graph because it could be very precise. But we cant use Inspire."* Once again they decide not to use the graphic representation because they see it as being less precise than the algebraic representation.

Initially they think that the two cars start at the same point, and drives in the same direction. They are having a hard time taking into account that one car has a head start. At a point they decide to graph the position of the cars anyhow, but they get confused about when the cars should intersection.

After about 12 minutes they figure out that the cars are in fact traveling towards each other, and they start to rework the problem. They still do not take into account that one car has a head start. At the end they decide to calculate the distance between the two cars for each hour after the start of the first car. This method only works because the number of hours is a natural number.

They let x denote the distance between the two cars, and calculate the value of x for each hour until they meet; however, The fact that they write *"Since the result $x \leq 0$ it is the number of hours"* suggest that they do not think that x denote the number of hours before they meet. This is strange, because they write *"4. hour:* $180 - (100 + 80) = x$*"*, but if x was the number of hours, it should be four, and here $x = 0$. Apparently, they do not notice this inconsistency.

Laura & Lotte .

Car 1: $a = 80$, $b = 0$, $f(x) = 80x$
Car 2: $a = -100$, $b = 2$, $g(x) = -100x + 2$

$$80x = -100x + 2 \tag{9.2}$$
$$\frac{180x}{180} = \frac{2}{180} \tag{9.3}$$
$$x = \frac{1}{60} \tag{9.4}$$

It takes them less than 3 minutes to get the idea of how to solve the assignment. This is their dialogue:

LAURA: If we drew a graph car one would be driving with the slope of 80 km. Where on the x-axis we have time. No kilometers along the x-axis and time up the y-axis. And then we have...
LOTTE: They are quite far from each other, I mean, it's 520 kilometers.
LAURA: Yes, but if we find the algebraic expression for both we can equate them

This method will yield the correct answer if carried out correctly. The dialogue shows that Laura thinks in terms of the graph representation, but that she wants to use the algebraic representation to solve the problem.

LOTTE: Don't they intersect at 520? I mean, they are both 520 kilometers from each other.

LAURA: yes? ...

LOTTE: Is that not the y-value, and then the x-value is the slope – the velovity which they drive. They both start at 520, they are 520 away from each other. B is 520 from A, A is 520 from B. It is the same value, and then their slopes are just different. Couldn't you say that their initial position is y-value. Or don't you understand it?

LAURA: Yes i understand it

This is quite a long sequence of out-of-focus phenomena. They had just agreed that kilometers should be along the "x"-axis, and time along the "y"-axis. Lotte starts talking about their position being the y-value which is good, but then she goes on to talk about the cars velocities being the x-value. I think that Lotte though is the following: Since the cars are separated by 520 kilometers, then the graphs representing the positions of the cars (as a function of *something*) should intersect when both cars have traveled the total of 520 kilometres. She realises that the slope of the graphs should be the velocities of the cars, but somehow she thinks that this should be on the "x" -axis.

LAURA: Ok, I just need to finish drawing this... No it doesn't make any sense what I am drawing.

LOTTE: Could we set it up as some kind of exponential. *[Laura keeps drawing]*

LAURA: So time is here, and distance is here.

LOTTE: Why did you change it?

LAURA: It is the only way it made sense to me... I really don't know what I am doing right now, I just know where I want to go.

LOTTE: I just keep thinking that they have the same initial point, I all depends on their velocities.

LAURA: Write down what you are thinking. Write it down.

LOTTE: Im not sure it makes any sense. *[she starts to draw a coordinate system with "kilometers" along the y-axis and "time/hours" along the x-axis]*

They give up on the graph representation and start working on the algebraic representation. It seems that their main problem is that the don't know how to take into account that the cars do not start off at the same time – they write $g(x) = -100x + 2$ rather than $g(x) = 100(x - 2)$ to compensate for car B starting two hours after car A. They have accounted for the fact that the cars are indeed approaching each other, but they fail to take into account that the cars are separated by 520 kilometres initially. Also, they solve for x incorrectly. Thus, their problem lies with the technicality and not with getting the ideas for solving the problem. A part of their dialogue sums it up perfectly

LAURA: I wan't two graphs that intersect or two algebraic expressions which I can equate somehow

LOTTE: But we don't know how to set it up.

LAURA: No.

After this Lotte notes that they need to take into account the 520 kilometers.

LAURA: If car B had to travel the 520 kilometers it would take 5.2 hours, and if car A had to travel the 520 kilometers it would take 6.5 hours.

LOTTE: But it starts two hours later...

As is seen from their written answer they end up with subtracting these two numbers, even though they are aware of the fact that they do not take into the account that the cars do not start off at the same time.

Louise & Henriette .
Car 1: $\frac{520}{80} = 6.5$ hours
Car 2: $\frac{520}{100} = 5.2$ hours
$\frac{420}{80} - \frac{520}{100} = x$

Louise and Henriette start off by calculating the time it would take each car to travel the 520 kilometers between them. Once this is calculated, they subtract these two times from each other, and conclude that this must be x. This is part of their dialogue

LOUISE: So 520 divided by 80 is some number of hours. 520 divided by 100 is some number of hours. And this number of hours should be the same. We should be able to equate them.

HENRIETTE: They shouldn't be the same.

LOUISE: We want to know when they meet.

HENRIETTE: Well, they don't have to travel the entire distance.

It shows that their plan was to subtract the time it would take from each car to travel the 520 kilometers, but Henriette realises that they have failed to take into account that the cars are traveling towards each other.

LOUISE: If we equate the two, we get a number of hours. Then we can calculate how long they drive. Car 1 gets a head start of two hours, so it would take it 4.5 hours to travel the entire distance. Car 1 will get the furthest.

HENRIETTE: Why?

LOUISE: Because it has a head start of 2 hours, which means that we can remove two hours before it will meet the other car at any point.

HENRIETTE: Well, that still does not give us anything.

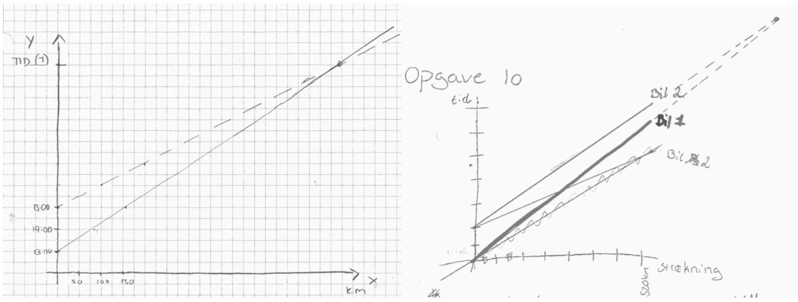
The follow this approach for a while but decide to take a break and go back to assignment 7. When they return to assignment 10. They spend an additional 15 minutes, but they are unable to give a better answer to the problem. They are aware that the cars are driving towards each other, and the one of the cars have a head start of two hours, but they are unable to use this information to solve the problem correctly.

Dan & Anders $T \cdot S = \frac{KM}{T}$; Isolating T; $T = \frac{KMT}{S}$; $\frac{80}{520} = 6.5$ hours. They meet when they have driven more than 520 KM.

Their final answer, that *"they meet when they have driven more than 520 km"* is a bit cheeky.

Jon & Troels [They have drawn a graph of of the positions of the cars]
2. car: $f(x) = 100x + 3$
1. car: $f(x) = 80x + 1$
They meet after 8.5 hours

Their idea of equating the position of the two cars is good, but they fail to take account for the two hour delay.



(a) Jon & Troels **(b)** Dan & Anders

Figure 9.2 – Jon, Troels, Dan, and Anders attempted to solve assignment 10 partly by graphing the positions of the cars.

Ronja and Fenja did not write anything.

9.2 Characterisation of the students understanding of the function concept

Laura & Lotte seem to be at the early stage of condensation. They seem to have a somewhat sound understanding of the variable concept, based on their answer to assignment four and five, together with their work on assignment 1. In assignment seven they are able to give the algebraic expression for the function, but they are unable to use this to check if $f(2) = 1$, showing that they do not fully understand how $f(x)$ is used as a symbol.

Louise & Henriette fail to give a proper definition of a function. They have problems with the linguistic description of functional relations. This is based on the fact that they do not allow for x and y to run negative in assignment five, furthermore they state that $x + y = 10$ can not be used to say anything about the relation between x and y. Their answer to assignment seven is very confusing and it suggests that they do not have a good understanding of the graph representation of functions. They are able to solve assignment three, eight and nine, showing that they are able to reason with both the algebraic and graph representation as long as the assignments are fairly standard, which suggests that they might be at the stage of condensation.

Matilde & Hilda fail to give a proper definition of a function, and it seems that they do not see how the variable concept ties in with the function concept. They are able to give a correct linguistic description in assignment four, but in assignment five they only consider positive values, suggesting that their understanding of the variable concept is somewhat limited. In assignment seven they conclude that $x = 2$ solves $f(x) = 1$, even though $f(x) = x + 1$, which suggests that both their understanding of the graph representation and the use of $f(x)$ as a symbol is limited.

Lars & Erik. Their definition of a function shows that they have a strong process-conception of the function concept. They are able to correctly answer assignment four showing that they can translate the mathematical notation, but they fail to solve assignment five because they use the notion of proportionality incorrectly. Their answer to assignment seven shows that they are willing to use the graph representation to support arguments and reason. They solve assignment eight and nine which suggests that they are at the stage of condensation.

9.3 Discussion

The students are obviously having a difficulty with the assignments that require them to give a linguistic description of functional relations. It seems that their main difficulties lies in figuring out when they have given "enough of a description". They attempt to avoid simply translating the mathematical notation, as e.g., y is x plus five. In assignment five, three of nine of the paris restrict the equation $x + y = 10$ to positive x and y values.

The students generally do well on the assignments in which algebraic and graphical representations are needed. The fact that assignment seven, which focuses on the graphical representation, is the assignment with the lowest success rate might seem surprising. What is it about the assignment that makes it so difficult for the students to solve? It is very likely that the students have not encountered an assignment phrased similarly before. If the assignment was rephrased as *"does the point (2,1) lie on the graph of f(x)"*, there is no doubt that anyone could answer the assignment correctly, it is trivial. Thus, the difficulty lies in realising that this is actually what is being asked. The student must first realised that the graph consists of the points $(x, f(x))$ so checking if $(2, 1)$ lies on the curve is equivalent to checking if it is in fact true that $f(2) = 1$.

There are a couple of examples of episode that are indicative of a pseudo-structural conception.

One example is the difference in the answers to assignment four and five. The assignments are similar in essence, but the students generally give very different answers – the specific algebraic expression becomes the sole basis on which the function is evaluated. They are given the expression $y = x + 5$ and $x + y = 10$. For the first expression some are able to conclude that y is five larger than x, but no one conclude that, in the second expression, y is the difference between 10 and x. This may be because the assignments are phrased differently; in the first I ask for the relation between x and y, and in the second I ask for what they can say about the x and y which obey the expression. For a specific example we can consider Louise and Henriette's answer to the two problems. In the first of the two, they state that *"... the only thing we can say is that y is larger than x. Or five larger"*, in the second of the two, they state that *"I guess we can not say anything about x and y in relation to each other"*, even though they write down that $x \leq 10$ and $y \leq 10$.

An example of an out-of-focus phenomena is Matilde and Hilda's work on assignment eight. Matilde states that *"they [the functions] are equal everywhere. Because there is 90 degrees between them. Or actually it is 180."*, which suggests that she does not have clear understanding of what it means for functions to be equal.

Part IV

Results, discussion, perspective, and conclusion

10 Results, Discussion, perspective

As the title suggests, it is the aim of this chapter to present and discuss the main results of the thesis, and to offer some perspective on the results in the form of suggestions for further work.

Section 10.1 presents an overview of the main results and observation of the thesis. This includes the result of the theoretical work presented in part II, and the empirical observation presented in part III. Section 10.2 discusses the empirical observation in the light of the theoretical results, and offers suggesting for further work.

10.1 Theoretical results and empirical observations

I consider the main results from the theoretical part of the thesis to be:

Problem solving as a way of probing students conceptual understanding: Because conceptual understanding is a part of students' resources when they do problem solving, there is reason to believe that problem solving can be used to study students' conceptual understanding. Problem solving activities shows the students' conceptual understanding in action.

The disjoint-reification-of-representations (DRR) hypothesis: Inspired by the observation that some students use symbols as objects in themselves, rather than means of representing an abstract mathematical object, an extension to the theory of reification is proposed. The extension is presented in the form of the hypothesis, that it is possible for students' conceptual understanding to develop separately in the different representations of the same concept. The observation that students use symbols as objects in themselves is explained by saying that the students have reified a representation, without having reified the concept itself. The extension is referred to as the disjoint-reification-of-representations hypothesis.

These results were obtained through theoretical work, which was inspired by observations presented in the literature.

Based on Schoenfeld's work on problem solving and the theory of reification, a methodology for studying students' understanding of the function concept was developed. In all, a total of 15 pairs of students have participated in the empirical studies, and 17[1] characterisation have been given. I consider the main observations from the empirical part to be:

Limited concept definition of function: Only 1 of 13 pairs of students were able to give a satisfactory definition of the function concept.

Limited mastery of linguistic representation: The assignments which required the students to give a linguistic description of functional relations caused quite a lot of difficulty for the students – of the 17[2] pairs of students 5 answered the assignment: "$y = x + 5$, what can you say about x in relation to y", correctly, and 3 of the 17 answered the assignment: "What can you say about the x and y that satisfy $x + y = 10$?" correctly.

[1] Tegan, Sara, Hans and Grete participated in two of the studies, so they their understanding have been characterised twice.

[2] Counting both the answers from Tegan & Sara and Hans & Grete

Limited understanding of the graphic representation: Many of the students are able to solve the problems that involve graphs, as long as the graphs are used in a familiar way, e.g., when they are asked to find the algebraic expression corresponding to a certain graph, or when they are asked for intersections of graphs. However, when the assignments are not familiar to the students, they seem to be unable to reason with the graph representation. This is based on the observations that very few were able to solve the assignment in which they were given the graph of a function, $f(x)$, and a point $(2, 1)$, which clearly did not lie on the graph of $f(x)$, and were asked if $x = 2$ solves $f(x) = 1$. Those who solved it, did so by changing to the algebraic representation first. Furthermore, in the assignment where the students were given the graphs of two function $f(x)$ and $g(x)$, and were asked which value of x solves the equation $f(x) = g(x)$, some even choose to change to the algebraic representation rather than looking at the intersection.

Limited understanding of the algebraic representation: The students have a hard time handling the algebraic representation; carrying out even the most simple of algebraic manipulations, e.g. solving simple linear equations, can cause considerable amounts of trouble.

Notational difficulties: Almost all of the students show signs of having difficulties with notation at one point or another during the problem solving session. Most of the notational difficulties were related to the use of $f(x)$ as a symbol – e.g. using $f(A)$ to describe the position of car A, even though it was a function of time, and x was used to signify time – and how to express a straight line algebraically – e.g. thinking that $f(x) = axb$ describes an arbitrary straight line.

Limited ability to solve "real life" problems: Some of the problems I posed required the students to interpret a situation and formulate the information mathematically. These are the problem 1-3 in the set used in Study A, and assignment 10 from study B and C. Only one of the four pairs solve any of those problems in study A. In study B and C 3 our of 13 pairs were able to solve assignment 10.

Based on these observations, one can attempt to give a general characterisation of danish secondary school students' understanding of the function concept. The characterisation will, of course, be a rather rough generalisation, but it may be useful none the less.

Generalised 1. year student named Bob: Bob is at the early stage of condensation of the function concept. He can carry out some standard procedure, but generally, he does not have a very sound understanding of the mathematical notation he uses – the notation $f(x)$ is especially difficult for him to understand. Bob is aware that different representation can be in play, when dealing with the function concept, but the different representation are mostly used to obtain an algebraic expression, which then can be used to solve the assignment. Bob finds it difficult to give linguistic descriptions of functional relations and can not give a definition of the function concept.

10.2 Discussion

This section discusses the empirical observations in the light of the theoretical results.

10.2.1 About the disjoint-reification-of-representations hypothesis

The main point of the disjoint-reification-of-representations hypothesis is, as the name suggests, that conceptual understanding can develop separately across different representations of the same concept. This thesis was developed rather late in the process of writing the thesis, and consequently, the problem I have designed were not designed to test this hypothesis. It is the aim of this section to discuss some preliminary observations regarding the hypothesis and how the hypothesis might be used to improve the teaching of the function concept.

Preliminary results

First of all, the students did not fare equally well in assignments with different representations. This might be because some of the problems are simply more difficult in some representation than other, but this is not always the case. For example, almost all of the students were able to solve the problems in which they are asked if a given value of x solved an equation – they simply inserted the value and inspected, or isolated for x – while very few of the students were able solve the problem in which the students were given a graph of the function $f(x) = x + 1$ and a point, $(2, 1)$, and were asked if $x = 2$ solves $f(x) = 1$. Now, the essence of the two problems are strikingly similar, yet almost none of the students were able to solve the problem in the graphical representation, and the ones who managed to solve it, did so by switching to the algebraic representation.

Another concrete example comes from Fenja and Ronja from study C. They are unable to solve the assignment in which they are given the graph of two function, $f(x)$ and $g(x)$, and are asked when $f(x) = g(x)$. However, they are able to solve the assignment in which they are given the expression for two functions, $f(x)$ and $g(x)$, and are asked what value of x solves the equation $f(x) = g(x)$. The same assignment, posed in different representation, can lead to different outcomes.

Secondly, there have also been episodes suggesting that some students think that different representations are in fact different mathematical objects. For example, when Helmut and Nora were working on problem four of the revised problems, they shared the following dialogue:

HELMUT: Well, I think that you have to say something. I mean if x is a number, then y will be five larger. I think that is what they mean. I don't think that it has to be a graph, I think that we should just describe it.
NORA: Are you sure?
HELMUT: Well, if it is a function then we normally say $f(x)$.

Helmut does not think that $y = x + 5$ should be a graph based solely on the fact that it doesn't say $f(x)$. This suggests that, to him, the graph of $y = x + 5$ and the expression $y = x + 5$ are two different things.

These observations are all in agreement with the disjoint-reification-of-representations hypothesis, but they are suggestive at best. To test the hypothesis one would need to design a different set op problems, which brings us to the next section.

Developing problems for further studies

I would design a series of problems posed in different representations, making sure that each problem is given in each representation. Some of the problems should focus on students ability to carry out procedures like finding function values or adding function, and some should focus on changing between the different representation. Care should be taken to cover all usual representations and the links between them. In table 10.1 I have given suggesting for two of such sets of problems.

One could also ask students to draw the graph of the function $h(x) = f(x) + g(x)$ with $f(x)$ and $g(x)$ given in the algebraic representation. It would be interesting to see if the students would add the algebraic representation and then draw the graph, or if they would draw the graphs of both functions and superimpose them.

Table 10.1 – Example of the kind of pairs of assignments that may test the DRR hypothesis in more detail.

Procedure	algebraic	graph	table
finding function value	$f(x) = 2x + 1$ what is $f(2)$	Given the graph of $f(x)$, what is $f(2)$?	Given a table for $f(x)$, what is $f(2)$
addition of function	$f(x) = 2x + 1$ and $g(x) = 5$ what is $(f + g)(x)$?	given the graphs of $f(x)$ and $g(x)$, draw $(f + g)(x)$	Given the tables of $f(x)$ and $g(x)$, create the table of $(f + g)(x)$

A study containing many problems like this could show, more clearly, if the students had different skills across the different representations – can they find function values across all the different representation, what about adding functions?

One should try to design problems that are likely to reveal if a student has reified a representation or not. I think that assignment seven, in the revised version of the problems, is an example of this kind of problems. If one has reified the graph representation, it should be no problem to realise that $f(a) \neq b$ if the point (a, b) does not lie on the graph of the function $f(x)$.

I think that this is definitely something that could be promising, and it would be worth while to look in to.

Avoiding pseudo-structural conception when teaching

So what does the DRR hypothesis tell us about how the function concept should be taught? Well, the aim is to have the student reify the concept whilst avoiding the development of a pseudo-structural conception. One should make sure that the student is at the same stage of concept formation for each representation, since you do not want the student to reify one representation before the others. This could be done by keeping the students active across the different representations whilst focusing on how different representation can be used to solve a given assignment. It is clear that different representations might not be equally efficient for solving a given task, but this can be a point in itself – there is a reason why you should know how to change between representations.

During the teaching of the function concept the teacher should check continually that the students can carry out the same processes in different representations. If this is done, one could hope that by the time reification is reached in the different representations, it might be clear to the students that they do in fact represent the same concept, and reification of the function concept may be reached.

One way to stress this relation even further may be to focus on the symbol $f(x)$ to make to connection between different representations explicit. From this study it is quite obvious that the students have a difficult time using $f(x)$ as a symbol – it is the source of much confusion for Cain an Abel when they discuss functions, and it is probably also behind some of the confusion that the students experience in assignment seven of the revised set of problems. I think that some of the difficulties arise from the fact that the symbol $f(x)$ transcends the different representations of the function concept. Now, it is quite clear from the empirical studies that none of the students have reified the function concept. Any symbol that refers to a function as an object must therefore be quite difficult to handle. For the students it might be hard to coincide the different uses of the symbol, because they do not see the object that $f(x)$ refers to. When teaching the function concept, the use of $f(x)$ in the different representation should be made explicit. Taught correctly, it may very well help the students see the connection between different representation by offering a symbol that transcends the representations, and help them see function as objects that can be represented by different means.

When teaching, one should be aware of that difficulties connected to the concept formation in the students may differ across the different representations. Sierpinska epistemological obstacles/acts-of-understanding analysis of the function concept does touch upon different representations of the function concept, but the graph representation is only mentioned in one act of understanding and two epistemological obstacles. With the DRR hypothesis in mind, one could make a similar analysis, but with equal focus on the different representation.

Sfard has put forward some principles on how to teach mathematical concepts. The principles are based on the theory of reification, and with the aim of avoiding pseudo-structural conception in mind. The principles are:

Principle I New concepts should not be introduced in structural terms.

Principle II A structural conception should not be required as long as a student can do without it.

Base on the discussion above, I would add:

Principle III One should be aware that the students' conceptual understanding can vary across the different representation of a concept. If the goal is for students to students attain the same level of conceptual understanding across the different representations, one needs to actively work towards this goal.

10.2.2 Regarding the theory of reification

Having used the theory of reification throughout the thesis, I would like to take a step back and discuss the theory.

The hierarchical structure of the stages of reification

The theory of reification is hierarchical in nature. A concept is reified by going though the three stages: interiorization, condensation, and reification, in that order. This makes sense because at the stage of interiorization the learner gets acquainted with the process which will eventually lead to a new concept. The stage of condensation is a stage of technical improvements, and the stage of reification is defined as an ontological shift; the process solidifies into an object. Thus, it seems reasonable that the stage of interiozation precedes that of condensation, after all, how can you condense a processes you have not yet gotten acquainted with? The hierarchical ordering of the stages make sense, logically.

From the empirical studies it seems, however, that some students are at the stage of condensation of the function concept even though they have not interiorized the process of finding function values. This may be because of the fact, that the acts which we take to indicate that an individual has reached the stage of condensation of the function concept, can be learned by heart – adding functions, finding intersection between graphs of function or finding inverse function. All of these procedures can be carried out without referring to the process nature of the functions.

The empirical studies suggest that none of the students have reified the function concept. This is despite the fact that many of the students were able to change between the different representations of the function concept. Thus, it seems that being able to change between different representations is not, in it self, a sign that a person has reified a concept.

The vicious circle states that reification of a concept A, and the interiorization of a higher level process B – a process that takes concept A as input – are requisites for each other. We have seen that students at the second year of secondary school, students who know how to differentiate and integrate, have not yet reified the concept of function. Thus, the ability to carry out higher level processes does not, in it itself, guarantee reification of the lower level concept. They are not, strictly speaking, requisites for each other.

The point I'm trying to make is not that they theory of reification is wrong, but rather that the observables we use are too crude.

How can one know if a student is at the stage of condensation? In stead of simply looking at the students ability to carry out standardised procedures we must focus on weather or not the student sees the connection between standardised procedures and the underlying (primary) process. The stage of condensation was defined as technical improvement, a period of squeezing lengthy procedures into smaller manageable units, but it also requires a process conception of the concept. Thus, if students can carry out these standardised procedures without knowing about the underlying process, I do not think that it is fair to say that they are at the stage of condensation.

How can one know if a student has reified a concept or not? We must look for it in the details. The stage of reification was defined as an ontological shift – the student must think of instances of the concept as if they are actual objects. Being able to skilfully alternate between different representation, and being able to carry out processes on these supposed objects both indicate that a student has reified the concept, but it does not prove it. We must catch people in referring to the mathematical notions as if they were objects.

This does make the characterisation more difficult and it is discussed in the following section.

10.2.3 About the characterisations

Now that I have given 12 characterisation of students understanding of the function concept, I would like to discuss their quality and their potential.

Improving the characterisations and the methodology

The characterisation I have giving are rather limited in their detail of description and many of the conclusion are based on the students work on just a few problems. Having more problems and consistency checks would be nice.

An obvious improvement of the characterisation would be to individualise them, rather than having them in pairs. I would not change the actual set up of the experiment – I think that having them solve the problems in pairs is an over all good solution – but with more time and possibly more problems, it should be possible to do the characterisation for the individuals rather than for the pairs.

One way of improving the detail of the characterisations would be to use the disjoint-reification-of-representation hypothesis. With the right amount of problems designed as described in section 10.2.1, it should be possibly to specify the stage of reification in each of the representation, that are used in the problems.

Another way of improving the characterisations may be to make a double study. That is, to use the first characterisation to design new problems that dig deeper into an individual's conceptual understanding. For example, we can consider Seth and Rogan's characterisation. I am not entirely sure that the students have properly interiorized the function concept; do they actually know what it means to find function values, and can they do this in different representation? Is their problems that they don't understand the diversity of the $f(x)$ notation? Some light could be shed on these questions by giving them additional assignments. For example, one could give them the problems "f(x)=2x, what is the function value for $x = 2$", "Given the graph of the function $f(x)$, what is $f(2)$?", "Given a table of values of x and f(x), what is f(8)", or "$f(x) = x + 2$ and $g(x) = -x + 5$, what is $f(2) - g(3)$?" Seeing how they go about solving these problems should shed some light on how well they have interiorized the process of finding function values.

An alternative to designing an additional set of problems would be to follow up the problem solving sessions with clinical interviews. Because of the fact that the students' solution attempts are recorded, it would be possible to play them back to the students and ask clarifying questions. The advantage of using clinical interviews as a method is that it lets you ask several consecutive clarifying equations. Often, when analysing the dialogues between students I have had to make educated guesses as to what the students are thinking and why, doing clinical interviews should improve the quality of the guesses.

For example, when Nora and Helmut were working the the problem "$y = x + 5$, what can you say about x in relation to y", Helmut talk about not using the graph because if it were a function, it would say $f(x)$. It would be interesting to ask him what he could say about the relation between $f(x)$ and x, when $f(x) = x+5$. Would he then use a graph to describe this relation? If so, what has changed? Having his answers to questions like these would definitely improve the quality of the analysis, and the following characterisation.

The influence of beliefs

I have not focused on students' beliefs. Students beliefs can have a great influence on their problem solving performance, and should not be ignored. For example, an alternative explanation for students preference for the algebraic representation might be that student are often asked to show the "intermediate result". Writing down that $x = 5$ because the graphs of the the functions intersect at $x = 5$ may not feel like intermediate results. Similarly, stating that $f(2) \neq 1$, because the point $(2, 1)$ does not lie on the graph of $f(x)$, may not seem like a proper mathematical explanation – there are no intermediate calculations.

The students' focus on the algebraic representation may be explained by the fact that CAS tool are used extensively in the danish educational system. CAS tools does allow you to switch between different representations of the function concept, but most of time it is done by typing in an algebraic expression, and then inspecting other representations. This may lead students to believe that it is it fact the algebraic expression that is being represented as a graph or a table of values, and not a function per say.

I do not think that beliefs like this can sufficiently account for all of the empirical observations, but I am quite sure that they do play a role in how the students give their written answers, and further work should not ignore it to the same extent that I have done in this work.

One example of just how strong an influence beliefs have on the students' thinking is Hilda and Matilde's dialogue during problem eight:

HILDA: Can't we just inspect it.

MATILDE: No, that is too easy.

HILDA: For the $x-$value 1, $f(x) = g(x)$ since this is where they intersect. Or should we take the fun route and say that because we can not be 100% certain that what we see it correct?

MATILDE: Yes. Well, on the other hand we used the graphs to find the algebraic expressions.

HILDA: Hah. Damn, the other way is more entertaining.

First Matilde refuses to simply inspect where the function intersect because it is too easy – where is the fun in that? Thus, the students' preference for the algebraic representation might be that is cooler and more math-like to do calculations. A second explanation is revealed by Hilda comment about the vagueness of the graph representation; they can never be 100% sure that what they see is correct. Matilde's answer is a valid point. In this instance Matilde and Hilda were very explicit about their reasoning, but I have no way of know how normal these beliefs are – they are probably not uncommon.

Following up the problem solving session with clinical interview may be a way to better understand how strong the influence of beliefs is.

The relation between conceptual understanding and problem solving

Now we have a way of characterising students' conceptual understanding of functions. What then? For me, the ultimate goal has been to study the relation between conceptual understanding and different mathematical competencies[3]. A competency, that I think would be particularly interesting to study, is the problem solving competency.

We have seen that the theory of reification touches on problem solving shortly. Sfard states that reification of a concept makes problem solving a lot easier. This is because conceiving concepts as objects makes it possible to store, and recall, information in a way, that a person who have not reified the concept would not be able to. In her own words:

> "It is the static object-like representation which squeezes the operational information into a compact whole and turns the cognitive schema into a more convenient structure." [Sfard, 1991, p. 26]
>
> ...
>
> "Thus, almost any mathematical activity may be seen as an intricate inter play between the operational and the structural versions of the same mathematical ideas: when a complex problem is being tackled, the solver would repeatedly switch from one approach to the other in order to use his knowledge as proficiently as possible." [Sfard, 1991, p. 28]

It would be interesting to study this relation in detail. One could, for example, focus on the problem solving process. In his book from 1985 Schoenfeld briefly presents a model for the problem solving process. This model has been extended by Yimer [Yimer and Ellerton, 2006, 2009]. It would be interesting to better understand *where* in the problem solving process conceptual understanding plays a role: is it during planning? Is it while implementing the chosen solution strategy? Does it have an effect at all?

Some work has already been done in this direction by Elia, see [Elia et al., 2007]. They focus on a relation between students' constructed definitions of the concept of function in relation to their abilities in dealing with tasks involving different representations of functions and problem solving tasks. I would focus on relating conceptual understanding to the different parts of the problem solving process.

How should one go about studying the relation between conceptual understanding and problem solving process? I would start by characterising the students' conceptual

[3] I realise that competencies is possibly a loaded word, so to be clear I use it as described in [Niss and Jensen, 2002]

understanding by using the method developed in this thesis, with the revisions suggested in section 10.2.3. This should be followed by a session of problem solving, where the assignment would be problems in the “strict” sense.

The data from the (strict) problem solving session should be analysed with the aid of protocol analysis as presented in Schoenfeld’s book. The protocol will give an overview of the problem solving process, and help monitor the students’ control abilities.

Having characterised the students’ conceptual understanding and analysed their problem solving protocols, with a focus on the understanding the problem solving process, one might be able to understand the relation between conceptual understanding and the problem solving process.

11 Conclusion

The goal of this thesis has been to find a way to adequately describe students' conceptual understanding of the functions. This includes being able to understand, explain, and communicate students' behaviour when they are dealing with the function concept. The research question was: *"What characterises students conceptual understanding of functions?"*, and the short answer is: The level of reification which the student has attained of the concept in its different representations and as a whole.

As for the description and communication of students' conceptual understanding, several examples have been given, two of them are:

Cain & Abel are at the stage of interiorization of the function concept or very early stage of condensation, since they have difficult time changing between different representations of the function concept. There were many instances of out-of-focus phenomena during their problem solving session. They used no less than four different expressions for a straight line, and they generally seemed to have a poor understanding of the mathematical notation they used. Their understanding of the graphical representation seems to be limited to its potential of being translated into an algebraic expression, and even this is hard for them to do.

Nora & Helmut seem to be at the early stage of reification. They are able to change between the different representations quite efficiently – they are even able to solve both the assignments that require linguistic descriptions, suggesting that they have a sound understanding of the variable concept as well. However, they shown sign of a having a pseudo-structural conception, since they seem to think that the algebraic and graphical representation are different mathematical objects. Furthermore, they are unable to give a proper definition of a function.

It has been demonstrated that problem solving – in the restricted sense, as non-problematic, non-standard assignments – can be used as a probe to investigate students' conceptual understanding.

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