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Puzzles and Siegel disks

by

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Abstract

Using as examples quadratic polynomials with a fixed Siegel disk, whose boundary is a Jordan curve containing the critical point, we give an abstract definition of puzzles and use it to prove that the Julia set of any quadratic polynomial with a constant type Siegel disk is locally connected.

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Abstract

Using as examples quadratic polynomials with a fixed Siegel disk, whose boundary is a Jordan curve containing the critical point, we give an abstract definition of puzzles and use it to prove that the Julia set of any quadratic polynomial with a constant type Siegel disk is locally connected.

1 Introduction.

Suppose $\theta \in [0,1]$ is an irrational and write it as a continued fraction:

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$.

Definition 1.1 *An irrational $\theta \in [0,1]$ is said to be of* constant type *if and only there exist $N \geq 1$ such that*

$$a_n \leq N \qquad \textit{for all} \quad n \in \mathbb{N}.$$

Suppose $z_0 \in \mathbb{C}$ is a say k-periodic point for some holomorphic map f and that the multiplier $\lambda = f^{k'}(z_0)$ equals $e^{i2\pi\theta}$ for some irrational θ. If f^k is linearizeable in a neighbourhood of z_0 we let Δ_0 denote the maximal domain of linearization and call it a Siegel-disk after C. L. Siegel, who was the first to study such domains and prove their existence. In fact Siegel proved that when θ is of constant type then

f^k always has a Siegel disk around z_0, we call such a Siegel disk a constant type Siegel disk. We call k the period of the Siegel disk. As everywhere in this paper, k-periodic and k-cycle means period exactly k, if not stated explicitly otherwise. The number θ is called the rotation number both of the Siegel disk Δ_0 and its periodic center z_0.

Main Theorem 1 *Suppose the quadratic polynomial Q has a constant type Siegel disk. Then the Julia set J_Q for Q is locally connected.*

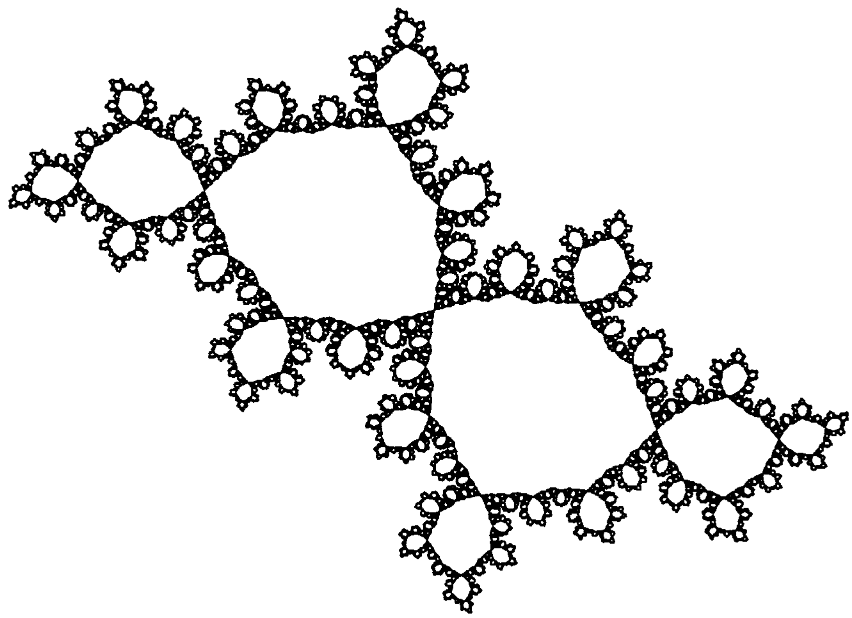

Figure 1: The Julia set of a quadratic polynomial with a fixed Siegel disk with rotation number the "golden mean" $\theta = \frac{1}{2}(\sqrt{5}-1)$.

One may show that these Julia sets also have planar Lebesgue measure 0, using a similar technique as the one presented here. However we shall desist from doing so, in part for simplicity, and in part because conjecturally their Hausdorff dimension is strictly less than 2, which would be stronger. In fact in the case of a fixed (period 1) constant type Siegel disk this conjecture is even a Theorem due to McMullen, see [McM1].

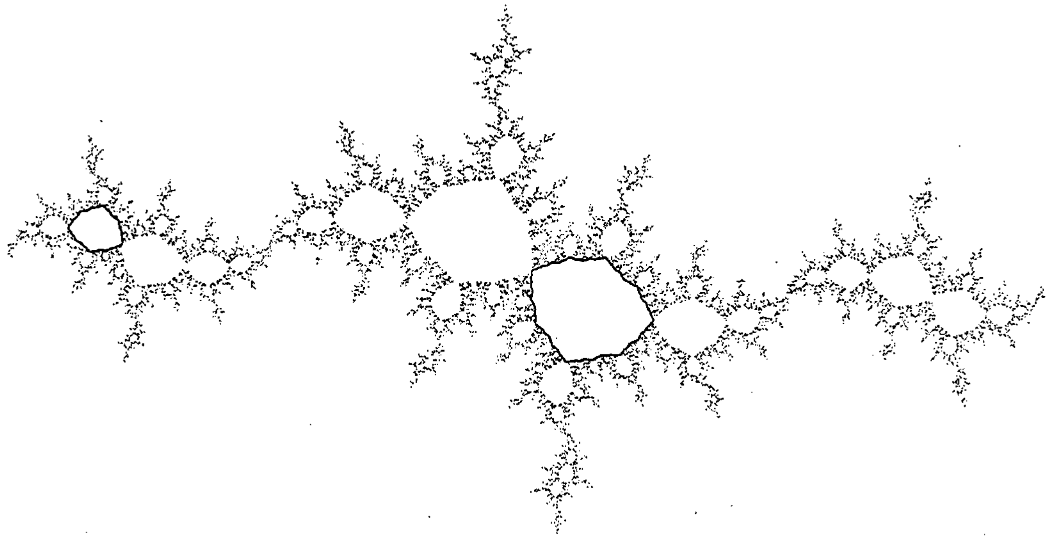

Figure 2: The Julia set (in grey) of a quadratic polynomial with a 2-cycle of Siegel disks (boundaries in black) with rotation number the "golden mean" $\theta = \frac{1}{2}(\sqrt{5}-1)$.

For a polynomial P the point ∞ is always a superattractive fixed point. We define the attracted basin of infinity $\Lambda_P(\infty)$ by

$$\Lambda_P(\infty) = \left\{ z \mid P^n(z) \underset{n\to\infty}{\longrightarrow} \infty \right\}.$$

It is non-empty and connected by the maximum principle. The compact complement $K_P = \mathbb{C} \setminus \Lambda_P(\infty)$ is called the *filled Julia set*, because it is full (the complement is connected and unbounded in $\mathbb{C}$) and $\partial K_P = J_P$, the *Julia set* for P. The filled-Julia set K_P is connected if and only if $\Lambda_P(\infty) \simeq \mathbb{C} \setminus \overline{\mathbb{D}}$. Suppose K_P is connected and let $\phi_P : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \Lambda_P(\infty)$ denote the unique biholomorphic map with

$$\frac{\phi_P(z)}{z} \underset{z\to\infty}{\longrightarrow} r \in \mathbb{R}_+.$$

If $P(z) = z^d + a_{n-1}z^{d-1} + \ldots$, i.e. P is monic of degree $d \geq 1$, then $r = 1$ and

$$\begin{array}{ccc} \mathbb{C} \setminus \overline{\mathbb{D}} & \xrightarrow{z \mapsto z^d} & \mathbb{C} \setminus \overline{\mathbb{D}} \\ {\scriptstyle \phi_P}\downarrow & & \downarrow{\scriptstyle \phi_P} \\ \Lambda_P(\infty) & \xrightarrow{P} & \Lambda_P(\infty) \end{array}$$

By a classical theorem of Caratheodory, [Car] we have :

Theorem 1.2 *If the Julia set J_P is connected then it is locally connected if and only if the map $\phi_P : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \Lambda_P(\infty)$ extends continuously to the boundary, i.e. to a continuous map*

$$\phi_P : \mathbb{C} \setminus \mathbb{D} \longrightarrow \overline{\Lambda_P(\infty)}$$

If the Julia set J_P is connected and locally connected we define an equivalence relation $\sim_P$ on $\mathbb{S}^1$ by $z_1 \sim_P z_2 \Leftrightarrow \phi_P(z_1) = \phi_P(z_2)$. Then $z_1 \sim_P z_2 \Rightarrow z_1^d \sim_P z_2^d$. Let Σ_P denote the quotient space $\mathbb{S}^1/\sim_P$ and let $\widetilde{P} : \Sigma_P \longrightarrow \Sigma_P$ denote the quotient dynamics induced by $z \mapsto z^d$ on $\mathbb{S}^1$ (for an enlargement of this discussion see [D3]).

Corollary 1.3 *If the Julia set J_P is connected and locally connected, then J_P is homeomorphic to the quotient Σ_P of the circle. More precisely there exists a homeomorphism $\chi_P : \Sigma_P \longrightarrow J_P$, which conjugates dynamics: $\chi_P \circ \widetilde{P} = P \circ \chi_P$.*

The quotients of quadratic polynomials (and their quotient dynamics) have been studied among others by [K-B-1], [K-B-2], [K].

We shall discuss the proof of the Main Theorem first in the case of a fixed Siegel disk, ie. a constant type Siegel disk of period 1.

As a representative example consider the Julia set for the quadratic polynomial $Q_c(z) = z^2 + c$, where $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, $\lambda = e^{i2\pi\theta}$, $\theta = \frac{1}{2}(\sqrt{5} - 1)$ as pictured above in Figure 1. The number θ here is the golden number, which have all 1-s in the continued fraction. Figure 3 shows the basic dynamics of this quadratic polynomial. The indifferent fixed point $\alpha = \frac{\lambda}{2}$ is surrounded by a Siegel disk on which the dynamics is conjugate to the rigid rotation $z \mapsto \lambda z$.

By a theorem of Herman and Światec the boundary of this Siegel disk is a Jordan curve, in fact a quasi-circle containing the critical point 0. The Siegel disk and its other preimage thus have 0 as their unique common boundary point. Moreover there is a sequence of iterated preimages of the Siegel disk one attached to the next at an iterated preimage of the critical point 0, just like pearls on a string. This sequence converge to the other and repelling fixed point $\beta = 1 - \frac{\lambda}{2}$.

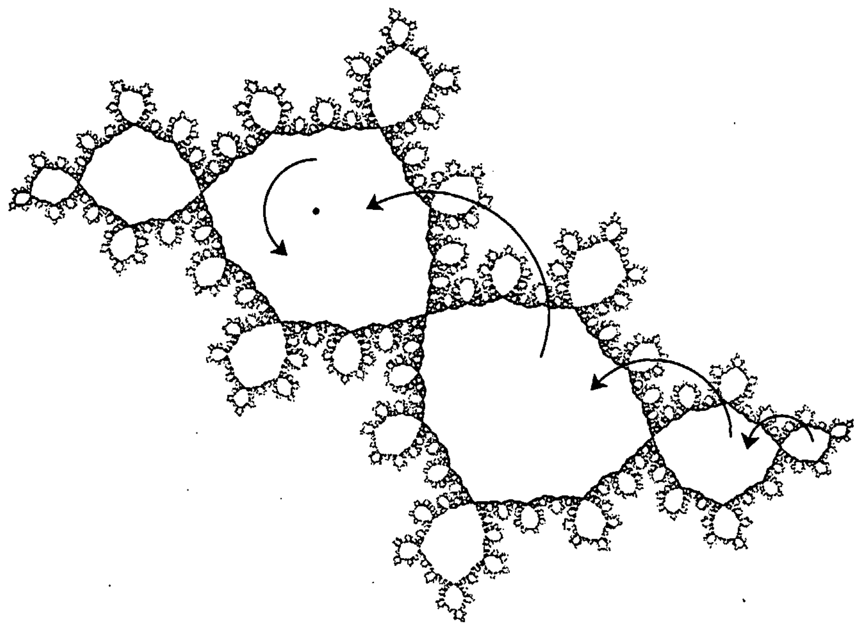

Figure 3: The basic dynamics of the quadratic polynomial with a fixed Siegel disk with rotation number the "golden mean" $\theta = \frac{1}{2}(\sqrt{5}-1)$. (Julia set in grey and the string of pearls in black.)

2 Puzzles

2.1 A puzzle

A principal tool for proving the Main Theorem is a puzzle, which we are about to define. This puzzle can be constructed and we shall do so under the assumption, that the boundary of the Siegel disk is a Jordan curve, containing the critical point 0. This condition is actually much weaker than the above constant type condition on the rotation number. In fact M. Herman has proven that this condition has full measure in the unit circle.

General Assumption 2.1 *Let $\theta \in [0,1]$ be a(ny) irrational such that the quadratic polynomial $Q_c(z) = z^2 + c$, where $c = c(\theta) = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, and $\lambda = e^{i2\pi\theta}$, has a Siegel disk Δ, whose boundary δ is a Jordan curve containing the critical point. (Q_c has a fixed Siegel disk with* *rotation number θ and the boundary of the disk is nice.)*

Let $\kappa_0^{\pm}$ denote the two closed but complementary subarcs of δ bounded by the critical point 0 and the critical value c. Define $\kappa_1^{\pm} \subset -\delta$ to be the lifts of $\kappa_0^{\pm}$ to Q_c not contained in δ but in $-\delta$. Define inductively $\kappa_n^{\pm}$ to be the lifts of $\kappa_{n-1}^{\pm}$ to Q_c starting at the common endpoint of $\kappa_{n-1}^{\pm}$. Define

$$\gamma_0^{\pm} = \kappa_1^{\pm} \cdot \kappa_2^{\pm} \cdot \ldots \cdot \kappa_n^{\pm} \cdot \ldots,$$

where $\cdot$ means curve product. Then the curves $\gamma_0^{\pm}$ converge to the unique repelling fixed point β for Q_c, which is belanded by the fixed external ray $R_c(0)$ of argument 0. We shall thus add β to the arcs $\gamma_0^{\pm}$, to make them closed. Let $\gamma_1^{\pm}$ denote the lifts of $\gamma_0^{\pm}$ to Q_c, starting at $x_1 = Q_c^{-1}(0) \cap \delta$. Define

$$\gamma^{\pm} = (\gamma_0^{\pm})^{-1} \cdot (-\kappa_1^{\mp}) \cdot \gamma_1^{\mp}$$

so that $\gamma^{\pm}$ are two arcs in J_c from β to $-\beta$, and which intersects only at the endpoints and at 0 and iterated preimages there of.

We shall extend the two arcs above to form two Jordan curves bounding disjoint open disks as follows. Define $\Gamma^{\pm}$ to consist of first the segment of the external ray $R_c(0)$ from potential level 1 into β, secondly the arc $\gamma^{\pm}$, thirdly the segment of the external ray $R_c(\frac{1}{2}) = -R_c(0)$ from $-\beta$ to potential level 1 and fourth and last the appropriate segments (one for each) of the level-1 equipotential to close up the arcs.

Let $\breve{P}_1^0$ be the open disk, (simply connected domain) bounded by the Jordan curve $\Gamma_1^0 = \Gamma^+$ and let $\breve{P}_1^1$ be the open disk, bounded by the Jordan curve $\Gamma_1^1 = \Gamma^-$. Moreover let $P_1^0 = \overline{\breve{P}_1^0}$, $P_1^1 = \overline{\breve{P}_1^1}$.

Let Γ_0 denote the common image curve $\Gamma_0 = Q_c(\Gamma_1^0) = Q_c(\Gamma_1^1)$. Moreover define $\breve{P}_0$ and P_0 similarly as for Γ_1^0, Γ_1^1 above. Then the restrictions $Q_c : P_1^i \longrightarrow P_0$ are homeomorphisms, holomorphic in the interior for each $i = 0, 1$.

We note for later use that the complements of both $\breve{P}_0$ and P_0 are forward invariant, i.e. $P_c(\mathbb{C} \setminus P_0) \subset \mathbb{C} \setminus P_0$. Moreover the closure of the critical orbit is contained in the complement of the disk $\breve{P}_0$. And finally $J_c \subset P_1^0 \cup P_1^1 \subset P_0$ and both sets $J_c \cap P_1^0, J_c \cap P_1^1$ are connected.

The two sets P_1^i, $i = 0, 1$ shall be called *level* 1 *puzzle pieces* and the set $\mathcal{P}_1 = \{P_1^0, P_1^1\}$ consisting of the level 1 puzzle pieces shall be called the *level* 1 *puzzle.* We define *puzzle pieces* and *puzzles* at all levels $n \geq 1$ as follows. Let $n \geq 1$ be given. A level n prepuzzle piece $\check{P}_n$ is any connected component of $P_c^{-n}(\check{P}_0)$. A level n puzzle piece P_n is the closure of a level n prepuzzle piece. And the level n puzzle is the set or collection $\mathcal{P}_n$ of all level n puzzle pieces. It follows from the construction, that each puzzle piece at level $n+1$ maps homeomorphicly onto a puzzle piece at level n. In particular each puzzle piece is homeomorphic to a closed disk. Moreover each level n puzzle piece is the image of precisely two level $n+1$ puzzle pieces so that by induction there are precisely 2^n level n puzzle pieces. In fact this kind of puzzle is special kind of puzzle. We call it the dyadic puzzle and shall come back to this point later.

The forward invariance of $\mathbb{C} \setminus \check{P}_0$ under P_c implies that any two puzzle pieces either are interiorly disjoint or one is nested inside the other. Thus we shall define

Definition 2.2 *A* **nest** $\mathcal{N} = \{P_n\}_{n \geq 1}$, $P_n \in \mathcal{P}_n$ *for each* $n \geq 1$ *is a (any) nested sequence of puzzle pieces*

$$P_{n+1} \subset P_n, \qquad \forall n \geq 1.$$

Definition 2.3 *The* **End** *of a nest* $\mathcal{N} = \{P_n\}_{n \geq 1}$ *is the set*

$$\mathrm{End}(\mathcal{N}) = \bigcap_{n \geq 1} P_n \subset J_c.$$

The nest $\mathcal{N}$ *is called* **convergent** *if the set* $\mathrm{End}(\mathcal{N})$ *is a singleton, a one point set, and the nest is called divergent otherwise.*

For $x \neq x' \in \delta$ let $\lceil x, x' \rceil$ and $\rceil x, x' \lceil$ denote respectively a closed or open subarc of $\delta = \partial\Delta$ bounded by x and x'. It will be clear from the context, which of the two possible arcs it refers to, but as a rule with exceptions, it will be the smaller.

Recall that the restriction $Q_c : \delta \longrightarrow \delta$ is a homeomorphism. For each $n \in \mathbb{Z}$ let $x_n = Q_c^{-n}(0) \cap \delta$, and thus in particular $x_0 = 0$ and $x_{-1} = c$. Let $U_0 = -\Delta$, the preimage of Δ different from Δ, so that Δ and U_0 has the critical point x_0 as unique common boundary point. For $n \geq 1$ let U_n denote the unique connected component of $Q_c^{-n}(U_0)$ having a common boundary point (in fact the point x_n) with Δ.

Below are listed some basic properties of our puzzle pieces, which are easily proved by induction.

1. The boundary ∂P_n is a Jordan curve for every level $n \geq 1$ puzzle piece P_n.
2. If $P_n \cap \delta \neq \emptyset$, then $P_n \cap \delta = \partial P_n \cap \delta = [x_k, x_m]$ for some $k, m \geq 0$, $k \neq m$. Moreover in this case the sets $P_n \cap \partial U_k$ and $P_n \cap \partial U_m$ are non trivial arcs (i.e. not points) and $P_n \cap U_k = P_n \cap U_m = \emptyset$ (see also Figure 4).

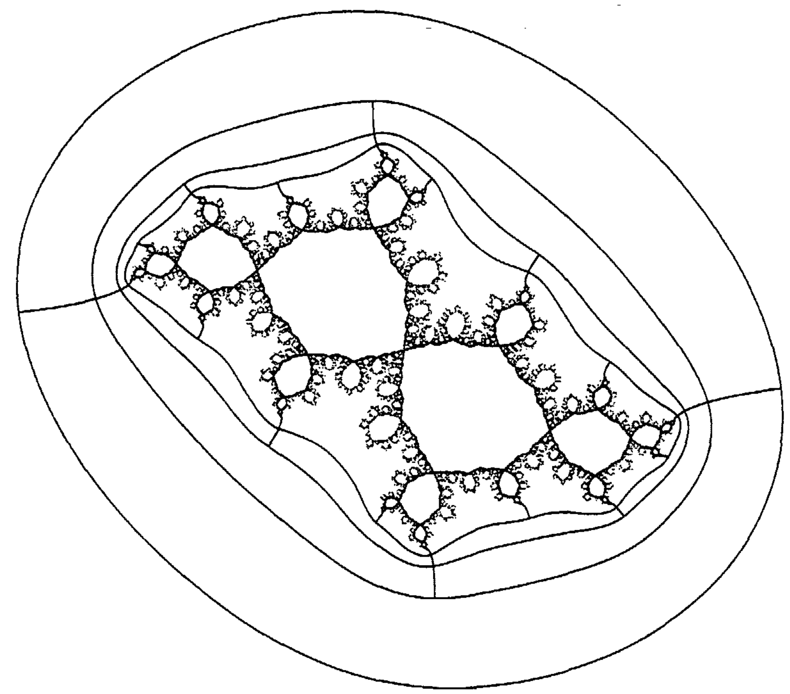

Figure 4: Julia set in grey and the first four puzzles in black.

Proposition 2.4 *If every nest is convergent, then the Julia set J_c is locally connected.*

Proof : By construction and induction we have the following properties

1. For any $n \geq 1$, any point in J_c belongs at least one puzzle piece at level n and to at most two such puzzle pieces.
2. If a point z in J_c belongs to a level n puzzle piece P_n, then z also belongs to a level $n+1$ puzzle piece $P_{n+1} \subset P_n$.

3. The intersection $P_n \cap J_c$ is connected for any puzzle piece P_n. Moreover let $z \in J_c$. If z belongs to only one level n puzzle piece P_n, then $J_c \cap P_n$ is a connected neighbourhood of z in J_c. And if z belongs to two level n puzzle pieces P_n, P'_n, then $J_c \cap (P_n \cup P'_n)$ is a connected neighbourhood of z in J_c.

It follows that every $z \in J_c$ belongs to at least one and at most two nests. Moreover a nest $\mathcal{N} = \{P_n\}_{n \geq 1}$ is convergent if and only if $\operatorname{diam}(P_n) \longrightarrow 0$, as $n \to \infty$. Thus local connectivity follows.

q.e.d.

Local connectivity, given convergence of all dyadic nests could also be deduced indirectly from the following which also explains the term dyadic puzzle. Let $\phi : \overline{\mathbb{C}} \setminus K_c \longrightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ denote the Riemann map, which is tangent to the identity at infinity. Then ϕ conjugates Q_c to $Q_0(z) = z^2$. Define the dyadic puzzle for Q_0 similarly as for Q_c. That is (see Figure 5) let the level 0 prepuzzle piece be the set bounded by the unit circle, the circle of radius e and the line segment $[1, \mathrm{e}]$.

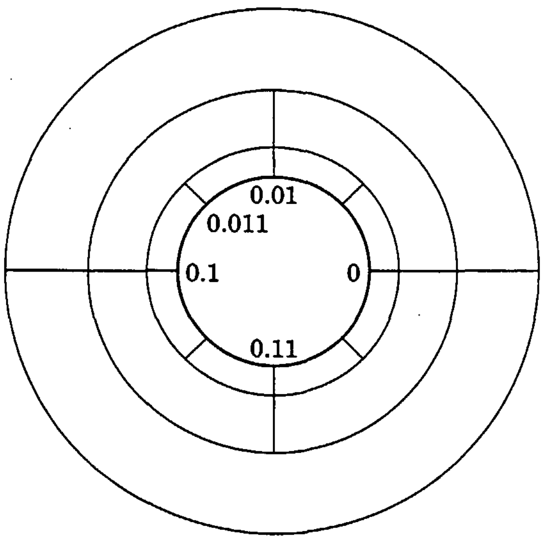

Figure 5: The dyadic puzzle (first three levels) for the quadratic polynomial $Q_0(z) = z^2$.

Define prepuzzle pieces at all levels by pulling back by Q_0 and define puzzle pieces as closures of prepuzzle pieces etc. Then ϕ defines a bijection $\widetilde{\phi}$ between the set of all puzzle pieces for Q_c and the set of all puzzle pieces for Q_0, given by $\widetilde{\phi}(P_n) = \overline{\phi(P_n \setminus K_c)}$. Thus ϕ

defines a $1:1$ correspondence between the (dyadic) nests for Q_c and the (dyadic) nests for Q_0. This proves the following Proposition

Proposition 2.5 *The map $\phi^{-1} : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \mathbb{C} \setminus K_c$ extends continuously to the unit circle $\mathbb{S}^1$, if and only if every nest is convergent.*

And as a Corollary of the Caratheodory Loop Theorem we obtain

Corollary 2.6 *The Julia set J_c is locally connected, if and only if every nest is convergent.*

There is a natural bijection between $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ and the set of all (dyadic) nests for Q_0 and hence a bijection $\eta : \Sigma_2 \longrightarrow \{\mathcal{N} | \mathcal{N} \text{ a nest for } Q_c\}$ given by

$$\eta(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) = \mathcal{N} = \{P_n\}_{n \geq 1}$$

if and only if for all $n \geq 1$

$$\exp(i2\pi([0, 2^{-n}] + 0, \epsilon_1\epsilon_2 \ldots \epsilon_n)) = \mathbb{S}^1 \cap \overline{\phi(P_n \setminus K_c)}$$

where $0, \epsilon_1\epsilon_2 \ldots \epsilon_n$ is the binary decimal number with digits $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.

This explicitly exhibits J_c as a quotient of Σ_2, in case J_c is locally connected.

2.2 Puzzles in general

The idea of puzzles was invented by Branner and Hubbard, in order to understand the structure of certain cubic polynomials, where one of the two finite critical points escapes to infinity. In particular they described precisely, which cubic polynomials has a Cantor Julia set see [B-H]. Puzzles were subsequently used by Yoccoz to prove local connectivity of the Julia set for any quadratic polynomial Q_c, which is not infinitely renormalizable and for which all periodic orbits are repelling (see [Hu] and the beautifull exposition [Mi] by Milnor). Infinitely renormalizable polynomials have also been studied using (Yoccoz) puzzles. See for instance the papers [L] by Lyubich, [J] by Jiang and [L-vS] by Levin and vanStrien. The puzzles used by Branner and Hubbard was in a way simpler, than those employed by Yoccoz, because their boundary was fully contained in the attracted basin of ∞, while the puzzles of Yoccoz intersected the Julia set in preperiodic repelling points. The puzzles presented here represents yet a generalization, in that full boundary arcs of the puzzle pieces are contained in the Julia set, and these boundaries may even contain the critical and or post critical points. Ignoring the risk of being too restrictive for future uses, we shall try to give an abstract definition of puzzles, which at least includes the puzzles used so far:

Definition and Construction 2.7 *Given a rational map $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$, a puzzle is given by a finite collection of open, connected and non empty sets:*

$$\breve{\mathcal{P}}_0 = \left\{ \breve{P}_0^1, \breve{P}_0^2 \ldots, \breve{P}_0^N \right\}$$

such that

1. $J_R \subseteq \bigcup_{\breve{P} \in \breve{\mathcal{P}}_0} \overline{\breve{P}}$.
2. $\breve{P} \cap \breve{P}' = \emptyset$, *for all* $\breve{P} \neq \breve{P}' \in \breve{\mathcal{P}}_0$.
3. *The complement* $\mathcal{C} = \overline{\mathbb{C}} \setminus \bigcup_{\breve{P} \in \breve{\mathcal{P}}_0} \breve{P}$ *is forward invariant, i.e. satisfies* $R(\mathcal{C}) \subset \mathcal{C}$.

Define $\breve{\mathcal{P}}_n$ for all $n \geq 1$ by pulling back, i.e. $\breve{P}_n \in \breve{\mathcal{P}}_n$ if and only if $\breve{P}_n$ is a connected component of $R^{-n}(\breve{P})$ for some $\breve{P} \in \breve{\mathcal{P}}_0$.

Finally the puzzles $\mathcal{P}_n$, $n \geq 0$ are obtained by taking closures. That is

$$P \in \mathcal{P}_n \Leftrightarrow \exists \breve{P} \in \breve{\mathcal{P}}_n : P = \overline{\breve{P}}$$

When used in local connectivity arguments, we also require

4. $\overline{\breve{P}_0} \cap J_R$ *is connected for all* $\breve{P}_0 \in \breve{\mathcal{P}}_0$.
5. *The number of postcritical points belonging to at least two different level-1 puzzle pieces is finite.*

It is easily checked that the properties 1.–3. are inherited by all levels $n \geq 1$. Moreover properties 2. and 3. implies that puzzle pieces are either nested or "interiorly disjoint", i.e. if $P_n \in \mathcal{P}_n$, $P_m \in \mathcal{P}_m$, $m \geq n$ then either $P_m \subset P_n$ or $\breve{P}_n \cap \breve{P}_m = \emptyset$. Concerning property 4. one has to put in extra assumptions in order for it to be inherited to subsequent levels. However it easily follows that if the puzzle piece P_{n+1} is mapped properly by degree d' onto the puzzle piece P_n, then the number of connected components of $P_{n+1} \cap J_R$ equals d' times the number of connected components of $P_n \cap J_R$.

Nests, the End of a nest and convergence/divergence of a nest is defined as in the previous definitions.

The objective for defining puzzles and for the further study of puzzles is to decide when nests (a given nest) is convergent and if not describe why not.

In the case of cubic polynomials as studied by Branner and Hubbard the convergence of all nests for a given polynomial proves that the Julia set in question is a Cantor set. Moreover they prove that when this is not the case, so that at least one nest is not convergent, then the cubic polynomial has a quadratic like renormalization (this term shall be defined in Section 5) and the End of the nest is homeomorphic to the filled Julia set of some quadratic polynomial with connected Julia set. In the case of Yoccoz puzzles again either all nests for a given quadratic polynomial are convergent and the Julia set is locally connected, or there is at least one non convergent nest, the quadratic polynomial is quadratic like renormalizable and the end of the non convergent (in fact all non convergent nests) is homeomorphic to the filled Julia set of some quadratic polynomial with connected Julia set. Moreover both the results of Branner-Hubbard and Yoccoz about the dynamical spaces admit conclusions about the corresponding parameter spaces.

The results of Branner-Hubbard and Yoccoz, not only share the idea of puzzles. Also when it comes to proving that nests are convergent they share the main idea:

Given a nest $\mathcal{N} = \{P_n\}_{n\geq 1}$ define $A_n = \overset{\circ}{P_n} \setminus P_{n+1}$. If A_n does not separate the plane we say A_n is degenerate and define $\mathrm{mod}(A_n) = 0$ and if it does define $\mathrm{mod}(A_n)$ to be the conformal modulus of A_n. That is if A_n separates the plane then it is biholomorphic to a circular annulus $A(r, R) = \{z | r < |z| < R\}$. The number $\mathrm{mod}(A(r, R)) = \frac{1}{2\pi} \log R/r$ is a conformal invariant called the conformal modulus. The central part in proving convergence of nests is then to prove that

$$\sum_{n\geq 1} \mathrm{mod}(A_n) = \infty$$

using dynamics. Convergence of the nest $\mathcal{N}$ then follows by a basic Grötzsch inequality.

2.3 General Puzzle Tools.

Going back to our puzzles and nests for quadratic polynomials with a fixed Siegel disk, we see that the above method fails fatally. For every nest $\mathcal{N} = \{P_n\}_{n\geq 1}$ every $A_n = \overset{\circ}{P_n} \setminus P_{n+1}$ is degenerate. Thus we need a completely different approach.

One such was given by the author in the paper [P]. There the idea is to look at a quasi conformal model of the Julia set described by Douady in [D2]. In this model the Siegel disk is replaced by the unit circle and its iterated preimages are analytic arcs. For this model it is proved directly that every nest contains puzzle pieces, which are arbitrarily small, simply because their circumference are small. One then obtains local connectivity or convergence of all nests because the model (with its puzzles) are homeomorphic to the quadratic Julia set (with its puzzles).

The proof we present here is a simplified version of the proof used in [P]. This simplification was suggested by Lyubich and Yampolsky, (see also [Ya]). The simplified approach combines a central result from [P] on control of the nests containing the critical point with a general idea for spreading this control to all nests. This general spreading idea has its roots in what Lyubich have named the Köbe principle in holomorphic dynamics. It has been used succesfully in several places and by many authors e.g. Lyubich, Shishikura, Jiang, Hu, Levin, van Strien, Yampolsky, ...

Lemma 2.8 *Suppose $U \subsetneq \mathbb{C}$ is a simply connected domain, $z \in U$ and $r > 0$. Let $d = \operatorname{diam}(B_U(z,r))$ denote the Euclidean diameter of the closed hyperbolic U-ball $B_U(z,r)$. Then*

$$\mathbb{D}(z, \frac{d}{2}\mathrm{e}^{-2r}) \subset B_U(z,r).$$

Proof : The Lemma is an immediate consequence of the distortion estimates for univalent maps of $\mathbb{D}$ applied to an inverse Riemann map $\phi : \mathbb{D} \longrightarrow U$, with $\phi(0) = z$.

q.e.d.

Proposition 2.9 *Let $z \in \operatorname{End}(\mathcal{N}) \subseteq J_c$ for some nest $\mathcal{N} = \{P_n\}_{n\geq 1}$. Suppose there exist $K > 0$, two sequences of integers $\{m_k\}_{k\in\mathbb{N}}$, $\{n_k\}_{k\in\mathbb{N}}$ and a sequence of simply connected domains $\{U_k\}_{k\in\mathbb{N}}$ such that*

1. $m_k \longrightarrow \infty$ as $k \to \infty$.

2. $P_{n_k} \subset U_k$ and $\mathrm{diam}_{U_k}(P_{n_k}) \leq K$, *for all* $k \in \mathbb{N}$.

3. $Q_c^{m_k} : U_k \longrightarrow \mathbb{C}$ *is univalent, for all* $k \in \mathbb{N}$.

Then $\mathcal{N}$ *is convergent, and* $\mathrm{End}(\mathcal{N}) = \{z\}$.

Proof : Under the hypothesis of the Proposition suppose to the contrary that

$$E = \mathrm{End}(\mathcal{N}) = \bigcap_{n\geq 1} P_n = \bigcap_{k\in\mathbb{N}} P_{n_k} \neq \{z\}.$$

and let $d = \mathrm{diam}(E) > 0$. As $E \subset P_{n_k} \subset B_{U_k}(z, K)$ for all $k \in \mathbb{N}$ the above Lemma implies

$$\mathbb{D}(z, \tfrac{d}{2}\,\mathrm{e}^{-2K}) \subseteq B_{U_k}(z, K) \subset U_k.$$

Hence the restrictions $Q_c^{m_k} : \mathbb{D}(z, \frac{d}{2}\,\mathrm{e}^{-2K}) \longrightarrow \mathbb{C}$ are univalent for all $k \in \mathbb{N}$. But then the family of iterates $\{Q_c^{m_k}\}_{k\in\mathbb{N}}$ is a normal family. This contradicts that $z \in J_c$ and as such is accumulated by repelling periodic points.

q.e.d.

Note that the univalence of the restrictions $Q_c^{m_k}$ to U_k was only used to ensure normality of the family $\{Q_c^{m_k}\}_{k\in\mathbb{N}}$ on the disk $\mathbb{D}(z, \frac{d}{2}\,\mathrm{e}^{-2k})$. Hence many other properties such as omitting 3 distinct points in $\overline{\mathbb{C}}$ could be used instead of 3.

A puzzle piece P is called *critical* if it contains the critical point. A nest $\mathcal{N} = \{P_n\}_{n\in\mathbb{N}}$ is called *critical* if every puzzle piece P_n contains a critical point so that $\mathrm{End}(\mathcal{N})$ also contains a critical point. In our case with a Siegel disk there are precisely two critical puzzle pieces at every level and precisely two critical nests.

A puzzle piece P is called *postcritical,* if and only if $P \cap \delta \neq \emptyset$. (Recall that the postcritical set $\overline{\{Q_c^n(0)\}_{n\geq 1}}$ equals δ.) Moreover we let $\mathcal{P}'_n$ denote the set of postcritical level-n puzzle pieces.

Corollary 2.10 *Let* $z \in J_c$ *and suppose there exists a level* N *and a sequence of integers* $\{m_k\}_{k\in\mathbb{N}}$ *diverging to* ∞ *such that*

$$Q_c^{m_k}(z) \notin \bigcup_{P\in\mathcal{P}'_N} P.$$

Then every nest $\mathcal{N}$ *with* $z \in \mathrm{End}(\mathcal{N})$ *is convergent with* $\mathrm{End}(\mathcal{N}) = \{z\}$.

Proof : For each non postcritical level-N puzzle piece $P \in \mathcal{P}_N \backslash \mathcal{P}'_N$ let $U_P \subset \mathbb{C}$ be an open simply connected neighbourhood of P with $U_P \cap \delta = \emptyset$. (To construct such a neighbourhood note that $P\cap = \emptyset$ and that P is compact and connected. Let $\phi_P : \mathbb{D} \longrightarrow \overline{\mathbb{C}} \backslash P$ denote an inverse Riemann map. Choose $0 < r < 1$ such that $\delta \cup \{\infty\} \subseteq \phi_P(\overline{\mathbb{D}(r)})$ and define $U_P = \overline{\mathbb{C}} \backslash \phi_P(\overline{\mathbb{D}(r)})$.)

Let $K = \max\{\mathrm{diam}_{U_P}(P) | P \in \mathcal{P}_N \backslash \mathcal{P}'_N\}$ and let $n_k = N + m_k$ for $k \in \mathbb{N}$. Given a nest $\mathcal{N} = \{P_n\}_{n \geq 1}$ with $z \in \mathrm{End}(\mathcal{N})$ we shall define a sequence of simply connected domains $\{U_k\}_{k \in \mathbb{N}}$ such that K, $\{m_k\}_{k \in \mathbb{N}}$, $\{n_k\}_{k \in \mathbb{N}}$ and $\{U_k\}_{k \in \mathbb{N}}$ full fills the hypotheses of Proposition 2.9.

For $k \in \mathbb{N}$ let $P = Q_c^{m_k}(P_{n_k}) \in \mathcal{P}_N \backslash \mathcal{P}'_N$ and let U_k denote the unique connected component of $Q_c^{-m_k}(U_P)$ containing P_{n_k}. Then the restriction $Q_c^{m_k} : U_k \longrightarrow U_P$ is biholomorphic because $U_P \cap \overline{\{Q_c^m(0)\}}_{m \geq 0} = \emptyset$. In particular $Q_c^{m_k}$ is univalent on U_k, U_k is simply connected and $\mathrm{diam}_{U_k}(P_{n_k}) = \mathrm{diam}_{U_P}(P) \leq K$. Hence the hypotheses of Proposition 2.9 are satisfied and this Proposition furnishes the conclusion $\mathrm{End}(\mathcal{N}) = \{z\}$.

q.e.d.

3 Controlling the dyadic Siegel puzzle

In order to use the general tools developed so far, we need to have some initial control over the (two) critical nests. In this section we shall state one such type of control over the critical nests and prove that it suffices to prove that all nests are convergent and hence that the Julia set J_c is locally connected. In the subsequent section we shall show that this particular control over the critical nests can be obtained provided the rotation number for the Siegel disk is of constant type.

Definition 3.1 *Given $k \geq 1$ let $I_k = I_{k,0} = [0, x_{q_k}]$ denote the closed interval containing $x_{-q_{k+1}}$ and let $J_k = J_{k,0} =]x_{-q_k}, x_{-q_{k+1}+q_k}[$ denote the open interval containing I_k. Moreover for $0 \leq j < q_{k+1}$ let the intervals $I_{k,j}$ and $J_{k,j}$ be the unique connected components in δ of $Q_c^{-j}(I_{k,0})$ and $Q_c^{-j}(J_{k,0})$ respectively.*

Note that $I_{k,j} \subset J_{k,j}$ and that the restrictions $Q_c : J_{k,j} \longrightarrow J_{k,j-1}$ are diffeomorphisms for each $0 < j < q_{k+1}$, because $x_{-q_{k+1}}$ is the first return (iterate) of the critical point $0 = x_0$ into $J_{k,0}$.

Given an open interval $]x, x'[\subseteq \delta$ in δ (notion defined on page 7) define a hyperbolic domain $\mathbb{C}_{]x,x'[} = (\mathbb{C} \setminus \delta) \cup]x, x'[$. To simplify notation we shall use the shorthand $\mathbb{C}_{k,j} = \mathbb{C}_{J_{k,j}}$ and similarly we shall simplify the notation for other objects such as distance related to the hyperbolic domain $\mathbb{C}_{k,j}$.

Proposition and Definition 3.2 *Suppose the (critical) level-n puzzle piece P satisfies $P \cap \delta = I_k$. For $0 \leq j < q_{k+1}$ there is a unique level $n + j$ puzzle piece P^j (connected component of $Q_c^{-j}(P)$) with $P^j \cap \delta = I_{k,j}$. Moreover $c \in I_{k,q_{k+1}-1} \subset P^{q_{k+1}-1}$ and the two level-$(n + q_{k+1})$ critical puzzle pieces map homeomorphicly onto $P^{q_{k+1}-1}$ by Q_c and hence map homeomorphicly onto $P = P^0$ by $Q_c^{q_{k+1}}$. One of these critical level-$(n + q_{k+1})$ puzzle pieces, call it P' is nested inside P and the other, call it P'' satisfies $P'' \cap \delta = I_{k+1}$. The later puzzle piece is called the Swap of P and is denoted* $\mathrm{Swap}(P)$.

Proof : Note that $Q_c^j : Q_c^{-j}(\mathbb{C}_{k,0}) \longrightarrow \mathbb{C}_{k,0}$ is a covering map because it is unbranched. Hence any connected component of $Q_c^{-j}(P)$ is mapped homeomorphicly onto $P \subset \mathbb{C}_{k,0}$ by Q_c^j and is thus a level $(n + j)$ puzzle piece. Moreover $Q_c^j(I_{k,j}) = I_{k,0} \subsetneq P \cap \delta$ and thus $I_{k,j}$ is contained in a unique connected component of $Q_c^{-j}(P)$. This provides for the existence and uniqueness of P^j. Clearly $c \in I_{k,q_{k+1}-1}$ and thus the mapping properties of P' and P'' follows. Hence we need only check that $P'' \cap \delta = I_{k+1}$. To this end we have

$$Q_c^{-q_{k+1}}(I_k) \cap \delta = \lceil x_{q_k+q_{k+1}}, x_{q_{k+1}} \rceil = \lceil x_{q_k+q_{k+1}}, 0 \rceil \cup \lceil 0, x_{q_{k+1}} \rceil$$

and $\lceil x_{q_k+q_{k+1}}, 0 \rceil \subset I_k \subset P$.

q.e.d.

Hypothesis 3.3 *There exist a constant $K > 0$ and a sequence of critical puzzle pieces $\{\mathcal{CP}_{n_k}\}_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$*

1. $\mathcal{CP}_{n_k} \cap \delta = I_k$.
2. $\mathrm{diam}_{k,0}(\mathcal{CP}_{n_k}) \leq K$ *and* $\mathrm{diam}_{k+1,0}(\mathrm{Swap}(\mathcal{CP}_{n_k})) \leq K$.

In order to transform this hypothesis into something useful we shall introduce a little more notation and some properties of hyperbolic metrics.

For $U \subsetneq \mathbb{C}$ a hyperbolic subset we let $\lambda_U : U \longrightarrow \mathbb{R}_+$ denote the coefficient function of the hyperbolic metric on U relative to the Euclidean. We let $d_U(\cdot, \cdot)$ denote the hyperbolic distance, further more for $z \in U$ and $r \geq 0$ we let $D_U(z, r)$ and $B_U(z, r)$ denote the open and closed respectively hyperbolic U-ball of radius r around z. Finally we define the outer (Euclidean) radius of $B_U(z,r)$ as

$$\mathrm{Out}(B_U(z,r)) = \sup\{|w-z|\,|w \in B_U(z,r)\}$$

the minimal radius of a Euclidean ball centered at z and containing $B_U(z,r)$. It is well known that $\mathrm{Out}(B_U(z,r))$ for r fixed, converge to 0 as z approaches a finite boundary point of U. We shall in the following quantify this statement.

Assume first that U is simply connected. The famous Köbe $\frac{1}{4}$-Theorem is equivalent to

$$\frac{1}{2\,\mathrm{d}(z,\partial U)} \le \lambda_U(z),$$

where $\mathrm{d}(z,\partial U)$ denotes the Euclidean distance from z to the boundary of U. Let $z_0 \in U$ be arbitrary but fixed and let $\alpha \in \partial U$ be a point with $|z_0-\alpha| = \mathrm{d}(z_0,\partial U) = d$. Then by the triangle inequality

$$\lambda_U(z) \ge \frac{1}{2\,\mathrm{d}(z,\partial U)} \ge \frac{1}{2|z-\alpha|} \ge \frac{1}{2(d+|z-z_0|)} = \kappa(z),$$

where $\kappa : \mathbb{C} \longrightarrow \mathbb{R}_+$ is defined by the last equal sign. We let also κ denote the metric with coefficient function κ relative to the Euclidean metric. Given any $r \ge 0$ we have $B_U(z_0,r) \subseteq B_\kappa(z_0,r)$, where the later denotes the closed κ ball. The later is also a Euclidean ball centered at z_0, by rotational symmetry of κ around z_0 and its Euclidean radius R can be calculated from the formula

$$r = \int_{[z_0,z_0+R]} \kappa(z)|dz| = \frac{1}{2}\int_0^R \frac{1}{d+t}dt = \log\frac{d+R}{d}$$

and hence $R = d(\mathrm{e}^{2r}-1)$. We have thus deduced that for any simply connected domain $U \subsetneq \mathbb{C}$, for any point $z_0 \in U$ and for any $r \ge 0$

$$\frac{\mathrm{Out}(B_U(z_0,r))}{\mathrm{d}(z_0,\partial U)} \le (\mathrm{e}^{2r}-1).$$

Assymptoticly this also holds in the general case, where U is not simply connected, but the center z_0 approaches a non singleton boundary component of U. This is the content of the following Lemma

Lemma 3.4 *Let $U \subsetneq \mathbb{C}$ be a hyperbolic subset and let K be a connected component of $\mathbb{C}\backslash U$. For all $r \geq 0$ and for all $z_0 \in U$ with $\mathrm{d}(z_0, K) \leq \frac{1}{2}\,\mathrm{e}^{-2r}\,\mathrm{diam}(K)$:*

$$\frac{\mathrm{Out}(B_U(z_0, r))}{\mathrm{d}(z_0, K)} \leq \frac{\mathrm{e}^{2r} - 1}{1 - 2\frac{\mathrm{d}(z_0,K)}{\mathrm{diam}(K)}\,\mathrm{e}^{2r}}.$$

Proof : The set $V = \overline{\mathbb{C}}\backslash K$ contains U and thus it suffices to prove the bound for V. If K is unbounded then V is simply connected and $\mathrm{diam}(K) = \infty$. The bound in the Lemma coincides with the bound in the simply connected case, found above. Thus we shall suppose K is compact. Moreover both the inequality in the statement of the Lemma and the inequality in the hypothesis of the Lemma are invariant under affine maps. We shall use this liberty however only to assume $z_0 = 0$ and that $-d \in K$, where $d = \mathrm{d}(z_0, K)$. Let $\beta \in K$ be a point with

$$|\beta + d| = |\beta - (-d)| = D \geq \frac{\mathrm{diam}(K)}{2} \geq d\,\mathrm{e}^{2r}.$$

Let $H(z) = \frac{(\beta+d)z}{\beta - z}$. The Möbius transformation H fixes 0 and $-d$, moreover it maps β to ∞ and ∞ to $-(\beta + d)$. Let $W = H(V) \subsetneq \mathbb{C}$, then W is simply connected with $\mathbb{C}\backslash W = \mathbb{C} \cap H(K)$. Moreover H maps the hyperbolic ball $B_W(0, r)$ onto the hyperbolic ball $H(B_U(0, r))$ and $\mathrm{d}(0, H(K)) \leq d$, hence

$$B_W(0, r) \subseteq \mathbb{D}(d(\mathrm{e}^{2r} - 1))$$

To complete the proof let us show that

$$B_U(z_0, r) \subseteq B_V(z_0, r) \subseteq H^{-1}(\mathbb{D}(d(\mathrm{e}^{2r} - 1))) \subseteq \mathbb{D}\left(\frac{d(\mathrm{e}^{2r} - 1)}{1 - 2\frac{\mathrm{d}(z_0,K)}{\mathrm{diam}(K)}\,\mathrm{e}^{2r}}\right)$$

where $H^{-1}(z) = \frac{\beta z}{\beta + d + z}$ denotes the inverse of H. Here the only non trivial inclusion is the last one: For $|z| \leq d(\mathrm{e}^{2r} - 1)$ we have

$$\begin{aligned} |H^{-1}(z)| &\leq \frac{|\beta||z|}{|\beta + d| - |z|} \leq \frac{(D + d)|z|}{D - d(\mathrm{e}^{2r} - 1)} \\ &= \frac{(D + d)|z|}{D + d - d\,\mathrm{e}^{2r}} \leq \frac{|z|}{1 - \mathrm{e}^{2r}\,\frac{d}{D+d}} \\ &\leq \frac{d(\mathrm{e}^{2r} - 1)}{1 - 2\,\mathrm{e}^{2r}\,\frac{\mathrm{d}(z_0,K)}{\mathrm{diam}(K)}}. \end{aligned}$$

q.e.d.

Lemma 3.5 *The Euclidean diameter* $\mathrm{diam}(J_{k,j})$ *converges to zero uniformly in* $0 \le j < q_{k+1}$ *as* $k \to \infty$.

Proof : The Lemma holds because the restriction $Q_c : \delta \longrightarrow \delta$ is homeomorphicly conjugate to the rigid circle-rotation, with the same rotation number. The Lemma holds for the images under the conjugacy (in fact the images are for fixed k all of equal length and interiorly disjoint) and the inverse of the conjugacy is uniformly continuous.

q.e.d.

Remark that the Lemma implies

$$\frac{\mathrm{diam}(\delta \backslash J_{k,j})}{\mathrm{diam}(J_{k,j})} \longrightarrow \infty \qquad \text{as} \qquad k \to \infty.$$

Lemma 3.6 *Given* $R > 0$ *and* $\epsilon > 0$, *there exists* $k_0 \in \mathbb{N}$ *such that for* $\forall k \ge k_0$, $\forall 0 \le j < q_{k+1}$ *and for all* $z \in J_{k,j}$:

$$\mathrm{Out}(B_{k,j}(z,R)) \le \epsilon$$

and the disk $D_{k,j}(z,R)$ *is simply connected.*

Proof : Let $\epsilon' = \min\{\mathrm{diam}(\delta)/2, \epsilon\}$ and choose $k_0 \in \mathbb{N}$ according to Lemma 3.5 such that $\mathrm{diam}(J_{k,j}) \le \frac{\epsilon'}{2(e^{2R}-1)}$. We can suppose that $\mathrm{diam}(\delta \backslash J_{k,j}) > \mathrm{diam}(\delta)/2$ and $2\,\mathrm{diam}(J_{k,j}) \le \frac{1}{2}\, e^{-2R}\, \mathrm{diam}(\delta \backslash J_{k,j})$ for $k \ge k_0$ and $0 \le j < q_{k+1}$, increasing k_0 if necessary. Combining the above with Lemma 3.4 we obtain the estimate of the Lemma. Moreover any loop in $D_{k,j}(z,R)$, based at z is homotopicly trivial in $\mathbb{C}_{k,j}$ and hence also in $D_{k,j}(z,R)$, because we have proved the estimate with $\epsilon' = \min\{\mathrm{diam}(\delta)/2, \epsilon\}$ and $\mathrm{diam}(\delta \backslash J_{k,j}) > \mathrm{diam}(\delta)/2$. Thus $D_{k,j}(z,R)$ is simply connected. This proves the Lemma.

q.e.d.

For $k \in \mathbb{N}$ and $0 < j < q_{k+1}$ let $A_{k,j} = Q_c^{-1}(\mathbb{C}_{k,j-1}) \subsetneq \mathbb{C}_{k,j}$. Then the restriction $Q_c : A_{k,j} \longrightarrow \mathbb{C}_{k,j-1}$ is a covering and hence a local hyperbolic isometry and $J_{k,j-1} = Q_c(J_{k,j}) = Q_c(-J_{k,j})$.

Lemma 3.7 *The hyperbolic distance* $\mathrm{d}_{A_{k,j}}(J_{k,j}, -J_{k,j})$ *converges to* ∞ *uniformly in* $0 < j < q_{k+1}$, *as* $k \to \infty$.

Proof : Let $\gamma = \gamma_{k,j} : [0,1] \longrightarrow A_{k,j}$ be a (geodesic) arc realizing the hyperbolic distance $\mathrm{d}_{A_{k,j}}(J_{k,j}, -J_{k,j})$ and say with $\gamma(0) \in J_{k,j}$, $\gamma(1) \in -J_{k,j}$. Let $\kappa = \kappa_{k,j-1} = Q_c(\gamma) : [0,1] \longrightarrow \mathbb{C}_{k,j-1}$ and define a Jordan arc $\mu_{k,j-1} = \kappa \cdot \lceil \kappa(1), \kappa(0) \rceil$, where the later arc in the curve product is contained in the interval $J_{k,j-1}$. The arc μ has non zero index (± 1) around the critical value c and hence around any $z \in \delta \backslash J_{k,j-1}$. Combining Lemma 3.4, Lemma 3.5 and the remark following it with the above yields the Lemma.

q.e.d.

Proposition 3.8 *Hypothesis 3.3 implies the two critical nests are convergent. And hence that any nest $\mathcal{N}$ containing a precritical point z, i.e. $Q_c^m(z) = 0$ for some $m \geq 0$, is convergent with* $\text{End}(\mathcal{N}) = \{z\}$.

Proof : The convergence of the two critical nests follows by combining Lemma 3.5 and Lemma 3.4 with Hypothesis 3.3. Suppose next that $Q_c^m(z) = 0$ for some $m \geq 0$ and let $\mathcal{N} = \{P_n\}_{n \geq 1}$ be a nest with $z \in \text{End}(\mathcal{N})$. Then $\mathcal{N}_m = Q_c^m(\mathcal{N}) = \{Q_c^m(P_n)\}_{n>m}$ is a critical nest and hence convergent. But $Q_c^m(\text{End}(\mathcal{N})) = \text{End}(\mathcal{N}') = \{0\}$ and hence $\text{End}(\mathcal{N}) = \{z\}$ by analytic continuation.

q.e.d.

Given $k \geq 1$ let $I_{k,j+q_{k+1}} = I_{k+1,j}$ and $J_{k,j+q_{k+1}} = J_{k+1,j}$ for $0 \leq j < q_k$.

Definition 3.9 *Let P be a puzzle piece with $P \cap \delta = I_{k,0}$ for some $k \in \mathbb{N}$ as in Proposition and Definition 3.2 and let P^j, $0 \leq j < q_{k+1}$ be as defined there. Extend this sequence of puzzle pieces by defining $P^{q_{k+1}} = \text{Swap}(P)$ and $P^{q_{k+1}+j} = \text{Swap}(P)^j$ for $0 \leq j < q_k$. Where $\text{Swap}(P)^j$ is as in Proposition and Definition 3.2, but with P replaced by $\text{Swap}(P)$. Then $Q_c^j(P^j) = P^0 = P$ for all $0 \leq j < q_{k+1} + q_k$. This finite sequence $\{P^j\}_{j=0}^{q_{k+1}+q_k-1}$ will be called the Yampolsky sequence based on P.*

The following Lemma is due to Yampolsky.

Lemma 3.10 *For every critical puzzle piece P with $P \cap \delta = I_{k,0}$ for some $k \in \mathbb{N}$ the union of the puzzle pieces in the Yampolsky sequence based on P contains a neighbourhood of δ in J_c. That is the set*

$$\bigcup_{j=0}^{q_{k+1}+q_k-1} P^j \cap J_c$$

is a neighbourhood of δ in J_c. Moreover the puzzle pieces of the Yampolsky sequence are mutually interiorly disjoint.

Proof : We have

$$P^j \cap \delta = I_{k,j} \qquad \text{for} \qquad 0 \le j < q_{k+1}$$
$$P^j \cap \delta = I_{k,j} = I_{k+1,j-q_{k+1}} \quad \text{for} \quad q_{k+1} \le j < q_{k+1} + q_k.$$

The intervals $I_{k,j}$, $0 \le j < q_{k+1} + q_k$ are mutually interiorly disjoint and their union covers δ, because the similar property holds for the corresponding rigid rotation. Thus the puzzle pieces in the Yampolsky sequence are also mutually interiorly disjoint. Moreover

$$\Delta \cup \bigcup_{j=0}^{q_{k+1}+q_k-1} (P^j \cup U_j)$$

is a neighbourhood of δ in $\mathbb{C}$ by the above and the general properties of puzzle pieces (see page 7). This completes the proof.

q.e.d.

A point $z \in J_c$ is said to be *critically recurrent* (recurrent to the critical point) if the forward orbit of z, $\{Q_c^n(z)\}_{n \ge 0}$ intersects every neighbourhood ω of the critical point 0.

Proposition 3.11 *Assume Hypothesis 3.3 and suppose $z \in J_c$ is not critically recurrent. Then every nest $\mathcal{N}$ with $z \in \mathrm{End}(\mathcal{N})$ is convergent, i.e. $\mathrm{End}(\mathcal{N}) = \{z\}$.*

Proof : Let ω be a neighbourhood of the critical point 0 with $\omega \cap \{Q_c^m(z) | m \ge 1\} = \emptyset$. Choose k with $(\mathcal{CP}_{n_k} \cup \mathrm{Swap}(\mathcal{CP}_{n_k})) \subseteq \omega$ and let $\{P^j\}_{j=0}^{q_{k+1}+q_k-1}$ denote the Yampolsky sequence based on $\mathcal{CP}_{n_k}$. The orbit of z also stays outside the union of the puzzle pieces P^j, because every puzzle piece in the Yampolsky sequence is an iterated preimage of $\mathcal{CP}_{n_k}$. Moreover by Lemma 3.10 the union the puzzle pieces from the Yampolsky sequence forms a neighbourhood of δ. Thus by the nestedness property of puzzle pieces the orbit of z does not intersect any of the postcritical level $N = n_k + q_{k+1} + q_k$ puzzle pieces. Hence we can take N above and $m_k = k$ in Corollary 2.10. This Corollary then furnishes the conclusion.

q.e.d.

Lemma 3.12 *Given $K > 0$ there exists $\widehat{K} = \widehat{K}(K) > 0$ such that for any hyperbolic domain $U \subsetneq \mathbb{C}$, for any $z_0 \in U$ with $D_U(z_0, 2K)$ simply connected and any compact set $z_0 \in L \subset U$ with $\mathrm{diam}_U(L) \le K$:*

$$\mathrm{diam}_{D_U(z_0,2K)}(L) \le \widehat{K}.$$

Proof : As $D_U(z_0, 2K)$ is simply connected we can assume, lifting to a universal covering if necessary, that $U = \mathbb{D}$ and $z_0 = 0$. We have $D_{\mathbb{D}}(0, 2K) = \mathbb{D}(\tanh(K))$ and $L \subset D_{\mathbb{D}}(0, K) = \mathbb{D}(\tanh(K/2))$. Thus we can even calculate a bound for $\widehat{K}$:

$$\widehat{K}(K) \leq 2 \log \frac{1+R}{1-R} \quad \text{where} \quad R = \frac{\tanh(K/2)}{\tanh(K)}.$$

q.e.d.

Proposition 3.13 *Assume Hypothesis 3.3 holds and let $z \in J_c$ be critically recurrent, but not precritical. Then every nest $\mathcal{N}$ with $z \in \operatorname{End}(\mathcal{N})$ is convergent, i.e. $\operatorname{End}(\mathcal{N}) = \{z\}$.*

Proof : Let K be as in the hypothesis and let $\widehat{K}(K)$ be as in Lemma 3.12. Choose k_0 according to Lemma 3.6 such that $D_{k,0}(0, 2K)$ is simply connected for all $k \geq k_0$ and according to Lemma 3.7 such that

$$\mathrm{d}_{A_{k,j}}(J_{k,j}, -J_{k,j}) > 2K \tag{1}$$

for all $k \geq k_0$ and all $0 < j < q_{k+1}$.

Let $\mathcal{N} = \{P_n\}_{n \geq 1}$ be a nest with $z \in \operatorname{End}(\mathcal{N})$ and let $z_m = Q_c^m(z)$. For $k \geq k_0$ arbitrary let $\{P_k^j\}_{j=0}^{q_{k+1}+q_k-1}$ denote the Yampolsky sequence based on $\mathcal{CP}_{n_k}$.

Claim 3.13.1 *There exists $m \geq 0$ (in fact infinitely many) such that*

$$Q_c^m(P_{n_k+m}) = \mathcal{CP}_{n_k}.$$

Proof of Claim: The union of $P_k^{q_{k+1}} = \operatorname{Swap}(\mathcal{CP}_{n_k})$ and $-P_k^{q_{k+1}}$ contains a neighbourhood of 0 and are the only level $(n_k + q_{k+1})$ puzzle pieces which intersects $\mathbb{D}(r)$ for r sufficiently small. There exists $m \geq 0$ (in fact infinitely many) with $z_m \in \mathbb{D}(r)$, because $z_0 = z$ is critically recurrent. But for such m $Q_c^m(P_{n_k+q_{k+1}+m})$ equals either $P_k^{q_{k+1}}$ or $-P_k^{q_{k+1}}$ and hence $Q_c^{m+q_{k+1}}(P_{n_k+q_{k+1}+m}) = \mathcal{CP}_{n_k}$, because

$$Q_c^{q_{k+1}}(P_k^{q_{k+1}}) = Q_c^{q_{k+1}}(-P_k^{q_{k+1}}) = P_k^0 = \mathcal{CP}_{n_k}.$$

q.e.d.

Let $m'_k \geq 0$ denote the minimal $m \geq 0$ with $Q_c^m(P_{n_k+m}) = \mathbb{C}P_{n_k}$ and define $n'_k = n_k + m'_k$. Moreover let $0 \leq j_k < q_{k+1} + q_k$ be maximal with $Q_c^{m_k - j_k}(P_{n'_k}) = P_k^{j_k}$, so that $P_k^{j_k}$ is the first puzzle piece from the Yampolsky sequence based on $\mathbb{C}P_{n_k}$ in the forward orbit of the puzzle piece $P_{n'_k}$. If $j_k < q_{k+1}$ define $m_k = m'_k$ and let U_k denote the connected component of $Q_c^{-m_k}(D_{k,0}(0, 2K))$ containing z, call this **Case a)**. If $j_k \geq q_{k+1}$ define $m_k = m'_k - q_{k+1}$ and let U_k denote the connected component of $Q_c^{-m_k}(D_{k+1,0}(0, 2K))$ containing z, call this **Case b)**.

Claim 3.13.2 *For each $k \geq k_0$ the set U_k is simply connected and $\mathrm{diam}_{U_k}(P_{n'_k}) \leq \widehat{K}$. Moreover the restrictions*

$$\textbf{Case a):} \qquad Q_c^{m_k} : U_k \longrightarrow D_{k,0}(0, 2K) \qquad (2)$$

$$\textbf{Case b):} \qquad Q_c^{m_k} : U_k \longrightarrow D_{k+1,0}(0, 2K) \qquad (3)$$

are biholomorphic.

Note first that $m_k \to \infty$ as $k \to \infty$. Hence the Claim combined with Proposition 2.9 yields the desired conclusion $\mathrm{End}(\mathcal{N}) = \{z\}$.
Proof of Claim: Consider first **Case a)**. The hyperbolic estimate follows from the biholomorphicnes of the restriction (2) as follows: We remark at first that $P_{n'_k} \subset U_k$ because $Q_c^{m'_k}(P_{n'_k}) = \mathbb{C}P_{n_k} \subset D_{k,0}(0, 2K)$. As $D_{k,0}(0, 2K)$ is simply connected we obtain from Lemma 3.12

$$\mathrm{diam}_{U_k}(P_{n'_k}) = \mathrm{diam}_{D_{k,0}(0,2K)}(\mathbb{C}P_{n_k}) \leq \widehat{K}.$$

We proceede to prove the restriction (2) is biholomorphic. Note at first that $Q_c^{-1}(P_k^{q_{k+1}-1}) = P_k^{q_{k+1}} \cup (-P_k^{q_{k+1}})$ and $-P_k^{q_{k+1}} \subset P_k^0$. Thus either $m_k = j_k = q_{k+1} - 1$ or $j_k < q_{k+1} - 1$ by the maximality of $j_k < q_{k+1}$.

For each $0 \leq j < q_{k+1}$ the restriction

$$Q_c^j : Q_c^{-j}(\mathbb{C}_{k,0}) \longrightarrow \mathbb{C}_{k,0} \supset D_{k,0}(0, 2K) \qquad (4)$$

is a covering map, because the first return (iterate) of the critical point into $\mathbb{C}_{k,0}$ is $Q_c^{q_{k+1}}(0) = x_{-q_{k+1}} \in J_{k,0}$. Thus for any connected component ω of $Q_c^{-j}(D_{k,0}(0, 2K))$, the restriction $Q_c^j : \omega \longrightarrow D_{k,0}(0, 2K)$ is biholomorphic, because $D_{k,0}(0, 2K)$ is simply connected. Hence if $m_k = j_k$ or $m_k = j_k + 1 < q_{k+1}$ we are through. If $m_k > j_k + 1$ then $Q_c^{m_k-(j_k+1)}(P_{n'_k}) = -P_k^{j_k+1}$ by maximality of j_k. Let ω_k denote the connected component of $Q_c^{-(j_k+1)}(D_{k,0}(0, 2K))$ containing

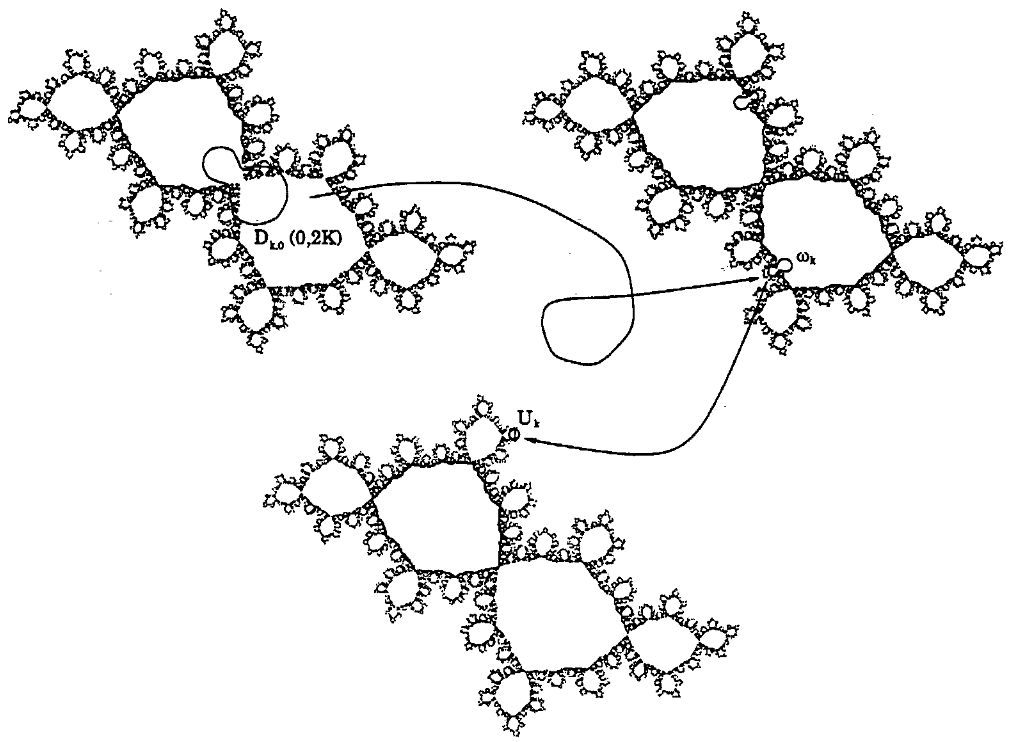

Figure 6: The central steps in pulling back the hyperbolic disk around 0 to a disk U_k around z.

$z_{m_k-(j_k+1)} \in -P_k^{j_k+1}$ and hence $-P_k^{j_k+1} \subset \omega_k$. Then U_k is a connected component of $Q_c^{-m'_k+(j_k+1)}(\omega_k)$. Moreover $y_{j_k+1} = -x_{j_k+1} \in -P_k^{j_k+1}$ and $Q_c^{-(j_k+1)}(\mathbb{C}_{k,0}) \subseteq A_{k,j_k+1} = Q_c^{-1}(\mathbb{C}_{k,j_k})$. Hence

$$\omega_k \subseteq D_{A_{k,j_k+1}}(y_{j_k+1}, 2K) \subseteq \mathbb{C}\backslash\overline{\Delta},$$

where the first inclusion holds, because the restriction (4) is a local isometry, and the second holds, because of the lower bound (1). But then the restriction $Q_c^{m'_k-(j_k+1)} : U_k \longrightarrow \omega_k$ is biholomorphic because $Q_c : \mathbb{C}\backslash(\overline{\Delta} \cup -\overline{\Delta}) \longrightarrow \overline{\Delta}$ is a (holomorphic) covering map. This proves the Claim in the **Case a)**. The proof of **Case b)** is similar and is left to the reader. Note however that if $j_k = q_{k+1} + q_k - 1$ and $m'_k > j_k$ then

$$y_{q_{k+1}+q_k} = -x_{q_{k+1}+q_k} \in Q_c^{m'_k-(j_k+1)}(P_{n'_k}) = P'.$$

The reason is the minimality of m'_k and the fact that the other preimage $-P'$ of $P_k^{q_{k+1}+q_k-1}$ is contained in P_k^0.

q.e.d.
q.e.d.

4 Fulfilling the Hypothesis

Recall that for $\theta \in [0,1]$ we let $\lambda = \lambda(\theta) = e^{i2\pi\theta}$ and $c = c(\theta) = \frac{\lambda}{2} + \frac{\lambda^2}{4}$. The following Theorem can be found in [D2]. It relies on a priori estimates for critical circle mapping by Herman and Światec see also [S],[He] and [Yo].

Theorem 4.1 *Suppose the irrational $\theta \in [0,1]$ is of constant type. and let $c = c(\theta)$ be as above. Then there exists a quasiconformal homeomorphism $\phi_\theta : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that:*

1. *The map ϕ_θ is conformal on the attracted basin of ∞.*
2. *$\phi_\theta(\overline{\Delta_\theta}) = \overline{\mathbb{D}}$ and $\phi_\theta(0) = 1$.*
3. *There exists $\mu = \mu(\theta) \in \mathbb{S}^1$ such that for all $z \in \overline{\mathbb{C}} \setminus \mathbb{D}$*

$$\phi_\theta \circ Q_c \circ \phi_\theta^{-1}(z) = f_\theta(z) = \mu z^2 \frac{z-3}{1-3z}.$$

Remark that the number μ is unique, but the map ϕ_θ is not. In fact the number $\mu = \mu(\theta)$ is the unique unimodular constant such that the analytic circle homeomorphism $f_\theta : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ has rotation number θ. Technically one starts with the Blaschke function f_θ. Then one replaces the dynamics of f_θ in $\overline{\mathbb{D}}$ by a quasiconformal conjugate of the rigid rotation with rotation number θ and the complex structure on $\mathbb{D}$ is replaced by the structure pulled back by the quasiconformal conjugacy. The map F_θ thus obtained has topological degree 2. By further pulling back by F_θ one obtains an invariant complex structure for F_θ. The Ahlfors-Bers Theorem, [A-B] yields the existence of a quasiconformal homeomorphism ϕ_θ integrating the invariant structure. Finally topological considerations shows that the conjugate map $\phi_\theta \circ F_\theta \circ \phi_\theta^{-1}$ is Möbius conjugate to the quadratic polynomial Q_c. Thus we have synthetically constructed the map Q_c by constructing a topological degree 2 map with an invariant complex structure and the right topological data.

The condition θ of constant type enters, because this is the precise condition under which analytic circle maps with a critical point (the point 1 is a double critical point for f_θ) are quasi-symmetrically conjugate to the rigid rotation see [He] and [Yo]. And moreover an orientation preserving circle homeomorphism extends to a quasiconformal selfmap of the disk if and only if it is quasi symmetric.

Let the map $f_\theta = \mu(\theta) z^2 \frac{z-3}{1-3z}$, $\mu(\theta) \in \mathbb{S}^1$ have rotation number θ on the unit circle. Let F_θ be a continuous degree 2 map obtained from f_θ by gluing into $\mathbb{D}$ a conjugate of the rigid rotation $R_\theta(z) = e^{i2\pi\theta} z$. (Note that if θ is not of constant type we can not obtain an invariant complex structure, but this is not of concern to us right now.) Moreover let J_{F_θ} denote the boundary of the (immediate) attracted basin of ∞ for F_θ. Then J_{F_θ} has the same structure and dynamics as was supposed for J_c in constructing the dyadic puzzle. Thus one may define a dyadic puzzle for F_θ completely analogous to those for the quadratic polynomials with a fixed Siegel disk whose boundary is a Jordan curve containing the critical point. Such puzzles were constructed and studied in great detail in the paper [P] by the author.

We shall use almost the same notation for F_θ as for Q_θ, for instance $x_j = F_\theta^{-j}(1) \cap \mathbb{S}^1$, $I_k = [x_{q_k}, 1] \subset \mathbb{S}^1$, $J_k =]x_{-q_{k+1}+q_k}, x_{-q_k}[\subset \mathbb{S}^1$.

The following analogue of Hypothesis 3.3 for F_θ can be deduced from [P] [Proposition and Definition 3.1, Lemma 3.3 and their proofs]

Theorem 4.2 *For every irrational rotation number $\theta \in [0,1]$, there exist $K > 0$ and a sequence of critical puzzle pieces $\{\mathcal{CP}_{n_k}\}_{k\in\mathbb{N}}$ with $\mathcal{CP}_{n_k} \cap \mathbb{S}^1 = I_k$ and with*

$$l_{J_k}(\partial\, \mathcal{CP}_{n_k}) \leq K, \tag{5}$$

$$l_{J_{k+1}}(\partial\, \mathrm{Swap}(\mathcal{CP}_{n_k})) \leq K. \tag{6}$$

Note that the boundaries of puzzle pieces for F_θ turns out to be rectifiable and that bounds are in fact given in terms of hyperbolic lengths of boundaries of puzzle pieces ($l_{J_k}(\cdot,\cdot)$ denotes hyperbolic arc length in $\mathbb{C}_{J_k} = J_k \cup (\mathbb{C}\backslash\mathbb{S}^1)$). Such a property is impossible in the polynomial Siegel case, where presumably any arc in the Julia set is non rectifiable (quasiconformal maps can and often do map lots of rectifiable curves to non rectifiable curves).

Proof : *Sketch* The candidate sequence of arcs is found in [P] [Proposition and Definition 3.1]. Secondly we note that it suffices to obtain the desired bounds in $\mathbb{C}^*_{J_k} = \mathbb{C}_{J_k}\backslash\{0\}$, where J_k is a hyperbolic geodesic by symmetry. This is done in [P][Lemma 3.3]. To provide also bounds for $l_{J_{k+1}}(\partial\, \mathrm{Swap}(\mathcal{CP}_{n_k}))$ one needs the proof of [P][Lemma 3.3]. Technically (in the terminology of the paper [P]) the bound

$l_\lambda(G_n) \leq L_{G,\theta} + 3L_{R,\theta}$ is replaced by the bound $l_\lambda(G_n) \leq L_{G,\theta} + 4L_{R,\theta}$, because $\mathrm{Swap}(\mathbb{CP}_{n_k})$ may at worst be a fourth instead of a third Swap of a controlled puzzle piece $P \in \mathcal{F}_\theta$.

q.e.d.

To yield results for quadratic polynomials we shall use Theorem 4.1 and the fact that quasiconformal maps distorts quasiconformal distances by a bounded amount. To this end consider two arbitrary points $z_1, z_2 \in \mathbb{D}$ and define

$$\mathrm{mod}_{\mathbb{D}}(z_1, z_2) = \sup\{\mathrm{mod}(A) \mid A \subsetneq \mathbb{D} \text{ an annulus separating } \mathbb{S}^1 \text{ from } \{z_1, z_2\}\}$$

For $0 < r < 1$ we have $\mathrm{mod}_{\mathbb{D}}(0, r) = \mathrm{mod}(\mathbb{D}\backslash[0, r])$ and further more the function $r \mapsto \mathrm{mod}(\mathbb{D}\backslash[0, r])$ is a decreasing homeomorphism from $]0, 1[$ onto $]0, \infty[$(see e.g. the monograph [L-V]). Define a decreasing homeomorpism $\mu :]0, \infty[\longrightarrow]0, \infty[$ by $\mu(\mathrm{d}_{\mathbb{D}}(0, r)) = \mathrm{mod}_{\mathbb{D}}(0, r)$. Then $\mu(\mathrm{d}_{\mathbb{D}}(z_1, z_2)) = \mathrm{mod}_{\mathbb{D}}(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{D}$ because both the hyperbolic distance $\mathrm{d}_{\mathbb{D}}(z_1, z_2)$ and $\mathrm{mod}\,\mathbb{D}(z_1, z_2)$ are invariant under automorphisms of $\mathbb{D}$.

Lemma 4.3 *Suppose $\phi : U \longrightarrow V$ is a K-quasiconformal homeomorphism between hyperbolic subsets $U, V \subset \mathbb{C}$. Then for any pair of points $z_1, z_2 \in U$*

$$\mu^{-1}(K\mu(\mathrm{d}_{\mathbb{D}}(z_1, z_2))) \leq \mathrm{d}_{\mathbb{D}}(\phi(z_1), \phi(z_2)) \leq \mu^{-1}(\frac{1}{K}\mu(\mathrm{d}_{\mathbb{D}}(z_1, z_2)))$$

Proof : Lifting to universal coverings if necessary we can suppose that $U = V = \mathbb{D}$. Then $\forall z_1, z_2 \in \mathbb{D}$:

$$\frac{1}{K}\mathrm{mod}_{\mathbb{D}}(z_1, z_2) \leq \mathrm{mod}_{\mathbb{D}}(\phi(z_1), \phi(z_2)) \leq K\,\mathrm{mod}_{\mathbb{D}}(z_1, z_2)$$

as K-quasiconformal maps distorts moduli of annuli by at most a factor K. Hence the Lemma follows

q.e.d.

Thus we obtain as Corollary of the above Theorem 4.1 and Theorem 4.2:

Corollary 4.4 *Suppose $\theta \in [0, 1]$ is of constant type, then Hypothesis 3.3 holds for the quadratic polynomial $Q_{c(\theta)}$. In particular all nests are convergent and the Julia set is locally connected.*

5 Siegel disk of period $p > 1$.

We shall turn to the study of quadratic polynomials with a $p > 1$-cycle of Siegel disks whose boundaries are Jordan curves, one of which contains the critical point 0. For this we need the notions of polynomial like map, renormalizeability, and some basic knowledge of Julia sets of quadratic polynomials.

A *polynomial like map of degree* $d \geq 2$ is a triple (f, U', U), where $U' \subset\subset U \subset\subset \mathbb{C}$ are open sets with closures homeomorphic to $\overline{\mathbb{D}}$ and $f : U' \longrightarrow U$ is a proper holomorphic mapping of degree d. We shall usually write $f : U' \longrightarrow U$ instead of (f, U', U). Moreover we shall use the synonym *quadratic like map* for a polynomial like map of degree two. The *filled Julia set of a polynomial like map* is the set of non escaping points $K_f = \{z \in U' | f^n(z) \in U', \forall n \geq 0\}$. The *Julia set* J_f is the boundary ∂K_f of the filled Julia set for f (note the abbreviation f for the polynomial like map).

Two degree d polynomial like maps $f : U' \longrightarrow U$ and $g : V' \longrightarrow V$ are said to be conformally equivalent if there exist disk neighbourhoods, ω_f of K_f and ω_g of K_g respectively and a biholomorphic map $\phi : \omega_f \longrightarrow \omega_g$ conjugating f to g, ie $g \circ \phi = \phi \circ f$. In the situation and notation above, if ϕ is only quasiconformal on ω_f, but conformal on K_f (in the sense of distributions) we say that f and g are *hybrid equivalent* and call ϕ a *hybrid equivalence*. Clearly both a conformal and a hybrid equivalence maps filled Julia set onto filled Julia set.

Note that any polynomial like map $f : U' \longrightarrow U$ is conformally equivalent to a polynomial like restriction $f : V' \longrightarrow V$ of f, where both $V \subseteq U$ and $V' \subseteq U'$ have real-analytic boundary. (Let V be a sufficiently large hyperbolic disk in U containing both U' and all critical values for f and let $V' = f^{-1}(V)$). Polynomials are included in the class of polynomial like maps, taking suitable restrictions, when necessary. The following is a principal tool in the theory of polynomial like maps developed by Douady and Hubbard and presented in [D-H].

Theorem 5.1 (Straightening) *Let $f : U' \longrightarrow U$ be a degree d polynomial like map, for which both U and U' have at least C^2 two boundary. Then there exists a degree-d polynomial P and a quasiconformal homeomorphism, $\phi : U \longrightarrow V \supset K_P$ conjugating f to P on U' and* *which is conformal on K_f (in the sense of distributions). i.e.*

$$\begin{array}{ccc} U' & \xrightarrow{f} & U \\ \phi\downarrow & & \downarrow\phi \\ V' & \xrightarrow{g} & V \end{array}$$

In particular any polynomial like map is hybrid equivalent to a polynomial. Moreover if K_f is connected then the polynomial P is unique up to affine conjugation.

Though the polynomial P of the above Theorem is uniquely determined up to affine conjugation by f (if K_f is connected), the straightening map (hybrid equivalence) is not. It is determined only on K_f. (If P has (affine) symmetries, it is even only defined up to postcomposition by such a symmetry.)

A quadratic polynomial $Q_c(z) = z^2 + c$ is said to be ($p \geq 2$)-renormalizeable if there exists a quadratic like restriction $Q_c^p : U' \longrightarrow U$ of Q_c.

The following Theorem is a particular case of a Theorem by Douady and Hubbard. We have however for completeness included a proof in the Appendix, Theorem A.6 (See also [McM2]).

Theorem 5.2 *Suppose the quadratic polynomial Q_c has a $p \geq 2$-cycle with multiplier $\lambda \in \overline{\mathbb{D}}\backslash\{1\}$. Then there exists a p-renormalization $f = Q_c^p : U' \longrightarrow U$ with $0 \in U'$ and which is hybrid equivalent to the unique quadratic polynomial $Q_{\widehat{c}}$, $\widehat{c} = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, which has a fixed point of multiplier λ.*

Corollary 5.3 *In the terminology of the above Theorem. The quadratic polynomial Q_c has a cycle of Siegel disks, whose boundaries are Jordan curves, one of which contains the critical point, if and only if the quadratic polynomial $Q_{\widehat{c}}$ has a fixed Siegel disk, whose boundary is a Jordan curve containing the critical point.*

Suppose the quadratic polynomial Q_c has a $p \geq 2$-cycle of Siegel disks with rotation number $\theta \in [0,1]$ (and multiplier $\lambda = e^{i2\pi\theta}$). Let f denote both $f = Q_c^p$ and a p-renormalization $f = Q_c^p : U' \longrightarrow U$ of Q_c with $0 \in U'$. Moreover let $\phi : U \longrightarrow V$ be a straightening homeomorphism as in Theorem 5.1 with

$$\begin{array}{ccc} U' & \xrightarrow{f} & U \\ \phi\big\downarrow & & \big\downarrow\phi \\ V' & \xrightarrow{Q_{\widehat{c}}} & V \end{array}$$

where $V' = \phi(U')$ and $\widehat{c} = \frac{\lambda}{2} - \frac{\lambda^2}{4}$. Suppose furthermore the boundary of the Siegel fixed disk for $Q_{\widehat{c}}$ is a Jordan curve containing the critical point 0.

Define $K_{f,0} = K_f = \phi^{-1}(K_{Q_{\widehat{c}}})$ and moreover $K_{f,j} = Q_c^j(K_{f,0})$ for $0 \leq j < p$. Then the restrictions $Q_c^{p-j} : K_{f,j} \longrightarrow K_{f,0}$ are homeomorphisms (restrictions of holomorphic diffeomorphisms) for each $0 < j < p$. Let β_0 denote the β-fixed point for f, i.e. $\beta_0 = \phi^{-1}(\widehat{\beta})$, where $\widehat{\beta}$ is the unique repelling fixed point for $Q_{\widehat{c}}$, belanded by the external ray of external argument 0 for $K_{\widehat{c}}$. For $p \geq 2$ there are at least two p-periodic external rays landing at β_0 (see also the Appendix). Let R_0^+, R_0^- denote the two such external rays of K_c, which are closest to $K_f = K_{f,0}$. That is let S be the open sector bounded by the arc $R_0^+ \cup \{\beta_0\} \cup R_0^-$ and disjoint from K_f. Then any other external ray of K_c landing at β_0 is contained in S. Moreover let $\widetilde{\beta}_0 = -\beta_0$ and $\widetilde{R}_0^\pm = -R_0^\pm$, so that

$$f(\widetilde{\beta}_0) = f(\beta_0) = \beta_0 \qquad \text{and} \qquad f(\widetilde{R}_0^\pm) = f(R_0^\pm) = R_0^\pm.$$

We let W_p denote the strip containing $K_{f,0}$ and bounded by the two arcs $R_0^+ \cup \{\beta_0\} \cup R_0^-$ and $\widetilde{R}_0^+ \cup \{\widetilde{\beta}_0\} \cup \widetilde{R}_0^-$. Moreover let $\gamma_0^\pm = \phi^{-1}(\widehat{\gamma}^\pm)$, where $\widehat{\gamma}^\pm$ are the Jordan arcs in $J_{\widehat{c}}$ constructed on page 6. Pull back these points, rays and arcs homeomorphicly to $K_{f,j}$ by Q_c^{j-p} to define points $\beta_j, \widetilde{\beta}_j \in K_{f,j}$ belanded by rays $R_j^\pm$ and $\widetilde{R}_j^\pm$ and finally arcs $\gamma_j^\pm$. To complete the picture add the equipotential Γ at say level 1.

Then any level-1 puzzle piece P, (the closure of a level-1 prepuzzle piece,) is bounded by a Jordan curve. This Jordan curve consists of piece(s) of the equipotential Γ, two or more pieces of external ray with landing point β_j or $\widetilde{\beta}_j$ for some j and either nothing more or one of the arcs $\gamma_j^\pm$, $0 \leq j < p$.

We are now in position to define a puzzle for Q_c. A level-1 prepuzzle piece is any bounded connected component of the set (see also Figure 7).

$$\mathbb{C} \setminus \left(\Gamma \cup \bigcup_{j=0}^{p-1} \left(R_j^+ \cup \gamma^+ \cup \widetilde{R}_j^- \cup R_j^- \cup \gamma^- \cup \widetilde{R}_j^+ \right) \right).$$

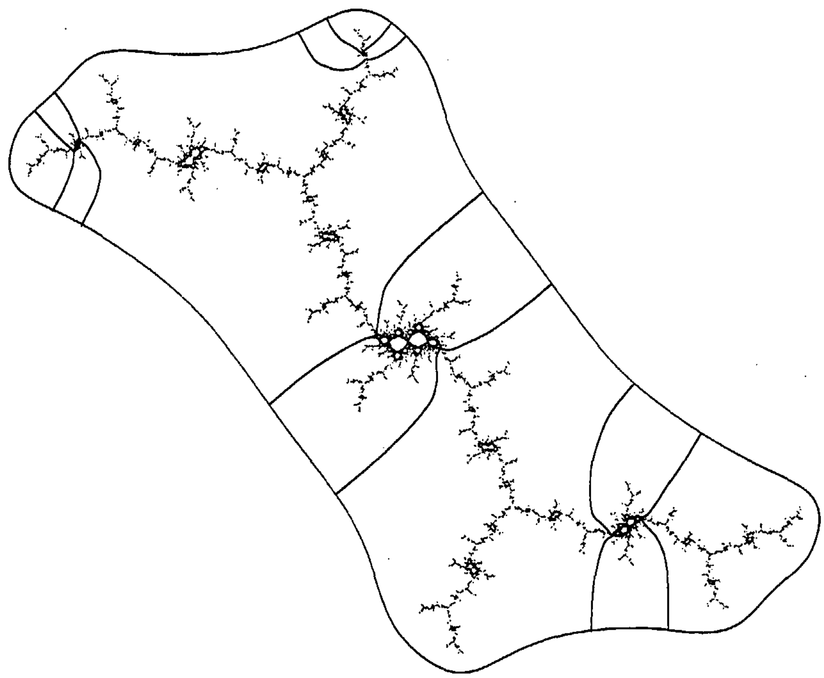

Figure 7: Boundaries of the level-1 puzzle pieces with Julia set in grey.

It is easy to verify that the above prepuzzle satisfies the puzzle properties (see page 11) to define a puzzle and that this puzzle also satisfies the two additional puzzle properties 4. and 5. (see page 11. Moreover since the postcritical set is contained in the complement of the level-1 prepuzzle, any level $n \geq 2$ puzzle piece P is mapped homeomorphicly onto a level-$n-1$ puzzle piece. Hence the connectedness condition 4. is inherrited to all subsequent levels.

Proposition 5.4 *If every nest is convergent, then the Julia set J_c is locally connected.*

The proof is a direct copy of Proposition 2.4 and is left to the reader. The only difference is that a point $z \in J_c$ can belong to more than two puzzle pieces at any level. This occurs precisely for the periodic point β_0 and all its iterated preimages: If the period p' of β_0 equals the period p (of the Siegel disk and) of the rays $R_0^{\pm}$ landing on β_0, then there are precisely three puzzle pieces at each level containing β_0, and if $k = \frac{p}{p'} > 1$ then β_0 belongs to exactly $2k$ puzzle pieces at each level.

Proposition 5.5 *In the terminology above. Let P^+ and P^- be the two level-1 critical puzzle pieces containing γ_0^+ and γ_0^- respectively. There exists a p-renormalization $f = Q_c^p : U' \longrightarrow U$ with $P^+, P^- \in U$, a quadratic like restriction $Q_{\widehat{c}} : V \longrightarrow V'$ and a straightening map $\phi : U \longrightarrow V$ satisfying*

$$\phi(P^+) \subset \widehat{P}^+, \quad \phi(P^-) \subset \widehat{P}^- \quad \text{and} \quad \phi(P^\pm \cap K_f) = (\widehat{P}^\pm \cap K_{Q_{\widehat{c}}}).$$

Proof : For the quadratic like restriction take V to be the open disk bounded by the level-2 equipotential, and let $V' = Q_{\widehat{c}}^{-1}(V)$ (the disk bounded bounded by the level-1 equipotential). And thus $Q_{\widehat{c}} : V' \longrightarrow V$. For the p-renormalization of Q_c let W denote the the intersection of the closed disk bounded by the level-2 equipotential and the strip W_p (recall W_p from page 30). Moreover let $W' = f^{-1}(W) \cap W$ so that $f = Q_c^p : W' \longrightarrow W$ is a proper degree two map with $K_{f,0} \subset W'$. However $\partial W' \cap \partial W \neq \emptyset$. This difficulty can be overcome by fattening W and W' along the ray parts of the boundary, because β_0 is repelling and f is expanding in the Bötcher coordinate (see also the Appendix for a more precise description of fattening). Thus fattening W and restricting again to a sufficiently large hyperbolic disc U centered at 0 we obtain a p-renormalization $f : U' \longrightarrow U$ with real analytic boundaries, with $P^\pm \subset U$ and with $K_f = K_{f,0}$. Choose $z_1 \in S \cap \partial U'$ and let $z_0 = f(z_1) \in S \cap \partial U$ (recall S from page 30). Moreover let κ denote the closure of the hyperbolic geodesic of $S \cap (U \backslash \overline{U'})$ from z_0 to z_1.

Choose a diffeomorphism $\phi_1 : \overline{U} \backslash U' \longrightarrow \overline{V} \backslash V'$ with $Q_{\widehat{c}} \circ \phi_1 = \phi_1 \circ f$ on $\partial U'$ and which maps the arcs κ and $-\kappa$ onto $\overline{V} \backslash V' \cap R_{\widehat{c}}(0)$ and $\overline{V} \backslash V' \cap R_{\widehat{c}}(\frac{1}{2})$ respectively. Then ϕ_1 is K quasiconformal for some $K > 1$. Define inductively $\phi_n : \overline{U} \backslash f^{-n}(U) \longrightarrow \overline{V} \backslash Q_{\widehat{c}}^{-n}(V)$ by $\phi_n = \phi_1$ on $\overline{U} \backslash U'$ and $Q_{\widehat{c}} \circ \phi_n = \phi_{n-1} \circ f$ on $\overline{U'} \backslash f^{-n}(U)$. Then ϕ_n converges locally uniformly to $\phi_0 : U \backslash K_f \longrightarrow V \backslash K_{Q_{\widehat{c}}}$ a K quasiconformal homeomorphism. Applying the proof of [D-H, Proposition 5, last 8 lines] to the f invariant complex structure on U obtained by keeping the standard structure on K_f, and pulling back the standard structure on $V \backslash K_{Q_{\widehat{c}}}$ to $U \backslash K_f$ by ϕ_0, we obtain a hybrid equivalence $\phi : U \longrightarrow V$ between f and $Q_{\widehat{c}}$. Moreover by construction $\phi = \phi_0$ on $U \backslash K_f$ (see e.g. [D-H, Proposition 6]) and hence

$$\phi(P^\pm) \subsetneq \widehat{P}^\pm \qquad \text{and} \qquad \phi(P^\pm \cap K_f) = (\widehat{P}^\pm \cap K_{Q_{\widehat{c}}}).$$

q.e.d.

Corollary 5.6 *Let $\phi : U \longrightarrow V$ be the hybrid equivalence of the above Proposition 5.5 and let W be the initial disk constructed in the proof of this Proposition. Then ϕ defines a $1:1$ correspondence between nests $\mathcal{N} = \{P_n\}_{n\in\mathbb{N}}$ for Q_c with $Q_c^{mp}(P_{mp+1}) \subset W_p$, $\forall m \geq 0$ and nests $\mathcal{N}' = \{P_n\}_{n\in\mathbb{N}}$ of $Q_{\widehat{c}}$ given by : P_{mp} is the unique level-mp, Q_c puzzle piece with $\phi(P_{mp}) \subseteq P'_m$. In particular $\phi(\mathrm{End}(\mathcal{N})) = \mathrm{End}(\mathcal{N}')$ and $\mathcal{N}$ is convergent if and only if $\mathcal{N}'$ is convergent.*

Note that $Q_c^{mp}(P_{mp+1}) \subset W_p$ is equivalent to $Q_c^{mp}(P_{mp+1}) = P^+$ or $Q_c^{mp}(P_{mp+1}) = P^-$. Because $P^\pm$ are the only level-1 puzzle pieces entirely contained in W.

Let Δ_0 denote the Siegel disk for f and let $\delta_0 = \partial\Delta_0$ denote the boundary of Δ_0, so that δ_0 by hypothesis is a Jordan arc containing the critical point 0. Moreover let $\Delta_j = Q_c^j(\Delta_0)$ and $\delta_j = Q^j(\delta_0)$.

The definitions and Propositions of the previous sections have been carefully designed so as to carry over with almost only substitutional changes. The reader is encouraged to do so. However we shall present here a different more general approach due to Lyubich.

The fundamental observation of Lyubich is that the local expansion of the renormalization implies that if a Q_c-nest $\mathcal{N} = \{P_n\}_{n\in\mathbb{N}}$ does not iterate to one of the nest of the above Corollary, then there are infinitely many puzzle pieces $P_n \in \mathcal{N}$, which under iteration eventually maps on to a non postcritical level-1 puzzle piece.

Theorem 5.7 *In the terminology above. Every nest for $Q_{\widehat{c}}$ is convergent if and only if every nest for Q_c is convergent. Thus if the Julia set $J_{\widehat{c}}$ for $Q_{\widehat{c}}$ is locally connected then the Julia J_c for Q_c is locally connected.*

Note that by an effective version of Proposition 5.4, J_c is locally connected if and only if $J_{\widehat{c}}$ is locally connected.

Proof : If every nest for Q_c is convergent then every nest for $Q_{\widehat{c}}$ is convergent by the above Corollary 5.6.

Hence let $\mathcal{N}_0 = \{P_n\}_{n\in\mathbb{N}}$ be a nest for Q_c and let $z_0 \in \mathrm{End}(\mathcal{N})$. Define $z_k = Q_c^k(z_0)$ and $\mathcal{N}_k = Q_c^k(\mathcal{N}_0) = \{Q_c^k(P_{n+k})\}_{n\in\mathbb{N}}$, so that $z_k \in \mathrm{End}(\mathcal{N}_k)$.

Consider first the case where $z_{k_0} \in K_f = K_{f,0}$ for some $k_0 \geq 0$. Then either $Q_c^{k_0+mp}(P_{k_0+mp+1}) \subset W$ for all $m \geq 0$, $\mathcal{N}_k$ is convergent by the above Corollary 5.6 and $\mathcal{N}_0$ is convergent by analytic continuation. Or there exists a possibly higher k_1 with $z_{k_1} = \beta_0$ and $Q_c^{k_1}(P_{k_1+1}) \subset \overline{S}$. In this case there are two possibilities. If the period p' of β_0 properly divides the period p of the rays $R_0^\pm$, then there exists

$0 < j < p$ such that also $z_{k_1+j} = \beta_0$ and $Q_c^{k_1+j+mp}(P_{k_1+j+mp+1}) \subset W_p$ for all $m \geq 0$. This case then falls under the above already treated case. Finally if $p' = p$ then $P = Q_c^{k_1}(P_{k_1+1})$ is the then unique non critical level-1 puzzle piece containing β_0 and $Q_c^{k_1+mp}(P_{k_1+mp+1}) = P$ for all $m \geq 0$. The puzzle piece P is also non postcritical and hence in this case convergence follows directly from Corollary 2.10 and analytic continuation.

Secondly if $z_m \notin K_{f,0}$ for all $m \geq 0$, then there are infinitely many $n \in \mathbb{N}$ for which $Q_c^{n-1}(P_n)$ is a non postcritical level-1 puzzle piece. Hence convergence follows in this case directly from Corollary 2.10.

q.e.d.

A A proof of renormalizeability

In this Appendix we shall portrait the proof by Douady and Hubbard, that a quadratic polynomial with a period $p \geq 2$ Siegel disc, is p-renormalizable. Another proof can be found the monograph [McM2]. We first need a result of Douady and Hubbard based on ideas of Sullivan.

Theorem A.1 *Suppose the quadratic polynomial $Q_{c'}$ has a non repelling p-periodic point α_0 with multiplier $\lambda \in \overline{\mathbb{D}}$. Then there exist continuous maps $C, \alpha : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ such that $C(\lambda_0) = c'$, $\alpha(\lambda_0) = \alpha_0$ and for each $\lambda \in \overline{\mathbb{D}}$ the point $\alpha(\lambda)$ is p-periodic for $Q_{C(\lambda)}$ with multiplier λ. Moreover both functions C, α are holomorphic in $\mathbb{D}$ and C is a homeomorphism onto its image $\overline{H} = C(\overline{\mathbb{D}})$*

The reader shall find proofs in [D1, Theoreme 4,] and in [ON2, Exposé XIV and Exposé XIX]. The open set $H = C(\mathbb{D})$ of the above Theorem is called a hyperbolic period-p component of the Mandelbrot set. The point $c_0 = C(0)$ is called the center of the hyperbolic component H and the boundary point $c_1 = C(1)$ is called the root of H.

In light of the above Theorem A.1 we shall use the natural parameter c on H rather than λ. Thus we shall consider the inverse function $\lambda(c) = C^{-1}(c)$ and the function also denoted by α, $\alpha(c) = \alpha(\lambda(c))$. Both functions $\alpha, \lambda : \overline{H} \longrightarrow \mathbb{C}$ depend on H and are restrictions of holomorphic functions defined on a neighbourhood of $\overline{H} \backslash \{c_1\}$. Obviously there is a one to one correspondence between the set of hyperbolic components H and the set of c_0, for which c_0 is periodic for Q_{c_0}, given by c_0 is the center of a unique hyperbolic component H. Less obvious but true is it that, a point c for which Q_c has a parabolic cycle is the root of precisely one hyperbolic component.

Note that a quadratic polynomial can have at most one non repelling cycle see e.g. [D1].

Suppose the critical point 0 is $p \geq 2$ periodic for Q_{c_0} and let Λ_1 denote the immediate attracted basin for c_0. Then $\partial\Lambda_1$ (and J_{c_0}) is locally connected. Hence $\partial\Lambda_1$ contains a unique repelling periodic point β_1 of period p' dividing p. (Let $\phi : \mathbb{D} \longrightarrow \Lambda_1$ denote the inverse Riemann map conjugating $Q_{c_0} : \Lambda_1 \longrightarrow \Lambda_1$ to $z \mapsto z^2$ on $\mathbb{D}$. Then ϕ extends continuously to $\overline{\mathbb{D}}$ and $\beta_1 = \phi(1)$. The point β_1 is repelling, because there can only be one non repelling cycle).

Proposition A.2 *Suppose the critical point* 0 *is* $p \geq 2$ *periodic for* Q_{c_0}. *Let* Λ_1 *denote the immediate attracted basin for* c_0 *and let* $\beta_1 \in \partial\Lambda_1$ *denote the unique repelling periodic point of period* p' *dividing* p. *Let* $\alpha, \lambda : \overline{H} \longrightarrow \mathbb{C}$ *be as above. Then there exists a continuous function* $\beta : \overline{H} \longrightarrow \mathbb{C}$, *holomorphic in* H *with* $\beta(c_0) = \beta_1$ *and with* $\beta(c)$ *a* p'*-periodic point for* Q_c, *repelling for each* $c \in \overline{H}\backslash\{c_1\}$. *Moreover* $Q_c^m(c) \neq \beta(c)$, $m \geq 0$ *for each* $c \in \overline{H}$ *and the (periodic) orbits of* $\alpha(c)$ *and* $\beta(c)$ *are disjoint for for each* $c \in \overline{H}\backslash\{c_1\}$.

Proof : Note that any periodic point not in the orbit of $\alpha(c)$ is repelling for Q_c, for $c \in \overline{H}\backslash\{c_1\}$. Moreover such a periodic point is simple, i.e. it is not a multiple periodic point. In particular any such periodic point can be followed holomorphicly in c for c in a neighbourhood of any $c' \in \overline{H}\backslash\{c_1\}$. The existence of the function β then follows by analytic continuation. If $Q_c^m(c) = \beta(c)$, then either c and hence 0 belong to the orbit of $\beta(c)$, which is then (super) attracting, a contradiction, or the critical point 0 is strictly preperiodic, but then all periodic orbits of Q_c are repelling in contradiction with $c \in \overline{H}$. Moreover the orbits of $\alpha(c)$ and $\beta(c)$ can coalesce only at the root c_1, and hence $\beta(c)$ is repelling for $c \in \overline{H}\backslash\{c_1\}$.

q.e.d.

Note that one can prove that in fact $\alpha(c_1) = \beta(c_1)$. Moreover the hyperbolic component is called *primitive*, if the period p' of $\beta(c)$ equals the period p of $\alpha(c)$ and it is called a *satelite* if p' properly divides p.

Theorem A.3 *Suppose the critical point* 0 *is periodic for* Q_{c_0} *of period* $p \geq 2$. *Let* Λ_1 *and* $\beta_1 \in \partial\Lambda_1$ *be as above. Then there are at* *least two p-periodic external rays of K_{c_0}, which lands on β_1. Moreover if $R_{c_0}(\theta_-), R_{c_0}(\theta_+)$, $0 < \theta_- < \theta_+ < 1$ are the two such rays with $\theta_+ - \theta_-$ minimal, i.e. they are the two rays adjacent to Λ_1. Then the* arc $R_{c_0}(\theta_-) \cup \{\beta_1\} \cup R_{c_0}(\theta_+)$ separates c_0 from 0 *and the other points in the orbit of* 0.

This Theorem is often referred to, but I have not found it stated together with an actual proof. One can however read a proof between the lines in [ON1]: Combine the local connectivity of J_{c_0}, [ON1, Exposé III, Prop. 4] with the extremality of c_0 in the Hubbard tree $H_{c_0} \subset K_{c_0}$ for Q_{c_0}, [ON1, Exposé IV, Prop. 4] and the fact that every access to $\beta_1 \in H_{c_0} \cap J_{c_0}$ relative to H_{c_0} contains at least one external ray of K_{c_0}, landing at β_1, [ON1, Exposé VI, Prop. 1].

Douady and Hubbard announced in [D-H, page 332] that given c_0 as in the above Theorem A.3, there exists $\chi_{c_0} : M \longrightarrow M$, a continuous map with $\chi(0) = c_0$ and $\chi_{c_0}(\partial M) \subseteq \partial M$, moreover χ_{c_0} can be chosen to satisfy $\overline{\partial}\chi_{c_0} = 0$ almost everywhere on M.

We shall not need the full power of this Theorem. Actually we only need what I believe was motivating this main Theorem of [D-H].

Proposition A.4 *Suppose the critical point* 0 *is periodic for Q_{c_0} of period $p \geq 2$ and let $\alpha, \beta, \lambda : \overline{H} \longrightarrow \mathbb{C}$ be as in Proposition A.2. For $c \in H$ let $\Lambda_1(c)$ be the immediate attracted basin of $\alpha(c)$. Then $\beta(c)$ is the unique p' periodic point in the boundary of $\Lambda_1(c)$.*

Let $0 < \theta_- < \theta_+ < 1$ be the external arguments of $\beta_1 = \beta(0)$ in Theorem A.3. Then the external rays $R_c(\theta_-), R_c(\theta_+)$ lands on $\beta(c)$ for all $c \in \overline{H}$.

Moreover let $S_1(c)$ denote the open sector bounded by the Jordan arc by $R_c(\theta_-) \cup \{\beta(c)\} \cup R_c(\theta_+)$ and not containing c. Then $S_1(c)$ contains any other external ray landing at $\beta(c)$ and contains all points of the orbit of $\alpha(c)$ except $\alpha(c)$, which is contained in the complementary closed sector $W_{p+1}(c)$.

Proof : We shall prove here only the case $c \in \overline{H} \backslash \{c_1\}$. The interested reader shall find a proof for $c = c_1$ in [ON2, Exposé XVIII]. The periodic point $\beta(c)$ stays repelling for $c \in \overline{H} \backslash \{c_1\}$. For $c \in H$ the boundary of the p periodic basin $\Lambda_1(c)$ contains a unique periodic point $\widehat{\beta}(c)$ of period dividing p, and which can be accessed from $\Lambda_1(c)$ via a p periodic arc, a generalized ray. By a generalized version of the stability of landing of (pre)periodic rays [ON1, Exposé VIII, II, Prop. 3] the point $\widehat{\beta}(c)$ moves continuously, even holomorphicly with the parameter c. Since $\widehat{\beta}(c_0) = \beta(c_0)$ the two functions agree on all of H. This proves the first statement. Similarly the second statement of the Proposition is a particular case of the stability of landing of (pre)periodic external rays, [ON1, Exposé VIII, II, Prop. 3]. Finally the statement about the orbit of $\alpha(c)$ follows by continuity, because it holds for $c = c_0$, by Theorem A.3 and the orbit of $\alpha(c)$ does not intersect (the orbit of) $R_c(\theta_-) \cup \{\beta(c)\} \cup R_c(\theta_+)$.

q.e.d.

Let 0 be $p \geq 2$ periodic for Q_{c_0} and let $\alpha, \beta, \lambda : \overline{H} \longrightarrow \mathbb{C}$ be as in Proposition A.2. Moreover let $W_{p+1}(c)$ be the closed sector defined in Proposition A.4. Define

$$\alpha_0(c) = Q_c^{(p-1)}(\alpha(c)), \quad \alpha_j(c) = Q_c^{(j-1)}(\alpha(c)), \quad \text{for} \quad 0 < j < p \quad (7)$$

$$\beta_0(c) = Q_c^{(p-1)}(\beta(c)), \quad \beta_j(c) = Q_c^{(j-1)}(\beta(c)), \quad \text{for} \quad 0 < j < p \quad (8)$$

Moreover let $W_p(c) = Q_c^{-1}(W_{p+1}(c))$ and let $W_j(c)$ denote the connected component of $Q_c^{j-p}(W_p(c))$ containing $\alpha_j(c)$, for $0 \leq j < p$. As $Q_c^{p-j}(\alpha_j(c)) = \alpha_0(c)$, the above is well defined.

Proposition A.5 *For every $c \in \overline{H}$:*

1. $Q_c^{mp+j}(0) \in \mathring{W}_j(c)$, *$m \geq 0$ and $0 \leq j < p$.*
2. *The restrictions $Q_c : W_j(c) \longrightarrow W_{j+1}(c)$ are biholomorphic for every $0 < j < p$ and are degree two coverings branched at the critical point 0 for $j = 0$ and $j = p$.*
3. $\mathring{W}_j(c) \cap \mathring{W}_{j'}(c) = \emptyset$ *for $j \neq j' \mod p$.*

Proof : To prove 2. of the Propostion note that the restrictions $Q_c : W_j(c) \longrightarrow W_{j+1}(c)$ are proper, because $W_j(c)$ is a connected component of $Q_c^{-1}(W_{j+1}(c))$ for each j, thus we need only calculate the degree. As $W_{p+1}(c)$ is simply connected each $W_j(c)$ is simply connected. Thus the degree is 2 if $c \in W_j(c)$ and 1 if not.

We shall prove the Proposition for the point $c_0 \in H$, then it follows for all other $c \in \overline{H}$ by continuity, since $Q_c^m(c) \neq \beta(c)$ for all $m \geq 0$, $c \in K_c$ and the boundary of each $W_j(c)$ moves continuously with c. We shall henceforth omit for the rest of this proof the dependence on $c = c_0$.

Let Λ_j denote the immediate attracted basin for the attracting p-periodic point α_j. Then $\Lambda_j \subset W_j$, because Λ_j is a connected open subset (component) of the interior of K_{c_0} containing α_j. And moreover the Λ_j are mutually disjoint. We have $c_0 = \alpha = \alpha_1 \in \Lambda_1$ and thus the first statement of the Proposition holds.

As $Q^p_{c_0}(c_0) = c_0 = \alpha_1 \in W_{p+1}$ the degree is 2 for $j = 0$ and p and moreover $W_1 \subsetneq W_{p+1}$, $W_0 \subsetneq W_p$.

Define $\Lambda_{j+p} = \Lambda_j$ and $\alpha_{j+p} = \alpha_j$. Then $\alpha_{j+1} \in S_1 = \mathbb{C} \backslash W_{p+1}$ for $0 < j < p$ and hence $\Lambda_{j+1} \subset S_1$ for $0 < j < p$. Moreover $\overline{\Lambda}_{j+1} \cup \partial W_{j+1}$ is connected and hence $W_j \subset \overline{S}_1$ for each $0 < j < p$, since external rays do not cross and $R(\theta_\pm)$ are adjacent to Λ_1. Thus if $c_0 \in W_{j+1}$ for some $0 < j < p$, then $W_{p+1} \subseteq W_{j+1}$. We shall show by induction on $p - j$, that this is not possible. For the same price we get that $\mathring{W}_{p+1} \cap \mathring{W}_{j+1} = \emptyset$ for $0 < j < p$. Note that W_{p+1} contains all rays with argument in the interval $[\theta_-, \theta_+]$. And W_p contains only rays with arguments in two disjoint intervals of length $(\theta_+ - \theta_-)/2$. Thus $W_{p+1} \not\subseteq W_p$ and $c_0 \notin W_p$. Hence the restriction $Q_{c_0} : W_{p-1} \longrightarrow W_p$ is a homeomorphism and the arguments of external rays in W_{p-1} are contained in two intervals each of length $(\theta_+ - \theta_-)/4$. And by induction the arguments of external rays contained in W_{j+1} are contained in two intervals each of length $(\theta_+ - \theta_-)/2^{p-j}$, $W_{p+1} \not\subseteq W_{j+1}$ and the restriction $Q_{c_0} : W_j \longrightarrow W_{j+1}$ is a homeomorphism. This completes the proof that the restrictions $Q_{c_0} : W_j \longrightarrow W_{j+1}$ are homeomorphisms for each $0 < j < p$ and that

$$\mathring{W}_{p+1} \cap \mathring{W}_{j+1} = \emptyset. \tag{9}$$

To complete the proof recall that $W_1 \subsetneq W_{p+1}$, $W_0 \subsetneq W_p$ and suppose that $\mathring{W}_{j+1} \cap \mathring{W}_{i+1} \neq \emptyset$ for some $0 < i < j < p$. Then

$$Q_c^{p-j}(\mathring{W}_{j+1}) \cap Q_c^{p-j}(\mathring{W}_{i+1}) = \mathring{W}_{p+1} \cap \mathring{W}_{p+1+i-j} \neq \emptyset$$

contradicting equation (9).

q.e.d.

Proposition A.6 (Fattening) *Suppose the quadratic polynomial Q_c has a $p \geq 2$-cycle with multiplier $c \in \overline{\mathbb{D}} \backslash \{1\}$. Then there exists a p-renormalization $f = Q_c^p : U' \longrightarrow U$ with $0 \in U'$ and hybrid equivalent to the unique quadratic polynomial $Q_{\widehat{c}}$, $\widehat{c} = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, which has a fixed point of multiplier c.*

Proof : (See also [B-F].) We shall use freely the notation introduced previously in this Appendix. Moreover we shall suppress the functionality of c for typographical reasons. Let $W = B(\Gamma, 2) \cap W_p$, where

$B(\Gamma, 2)$ denotes the closed disk bounded by the level-2 equipotential and W_p is the 'strip' defined above. Moreover let $W' = f^{-1}(W) \cap W$. Then the restriction $f : W' \longrightarrow W$ is proper of degree 2 by the above Proposition A.5. We shall *fattend* W and W' along the the nonequipotential part of the boundaries to obtain a quadratic-like restriction $f : U' \longrightarrow U$ with $W' \subset \overline{U}'$ and $W \subset \overline{U}$.

Let S denote the connected component of $Q_c^{-1}(S_1)$ with $\beta_0 \in \overline{S}$ (for S_1 see Proposition A.4). Choose $r > 0$ such that f is injective on the disk $D = \mathbb{D}(\beta_0, r)$ and $f(D) \supset \overline{D}$. Note that $-\beta_0 = \widetilde{\beta}_0 \notin W_j$ for $0 < j < p$, because $\widetilde{\beta}_j \in \overset{\circ}{W}_{p+1}$. Hence decreasing r if necessary we can assume furthermore, that $-(D \cap S)$ is disjoint from all the sets W_j, $0 \leq j \leq p$ and in particular contains no critical value for f. Let T denote the torus obtained as the quotient of the punctured disk $D^* = D \backslash \{\beta_0\}$ by the dynamics of f and let $\pi : D^* \longrightarrow T$ denote the natural projection, i.e. $\pi \circ f = \pi$ on $f^{-1}(D)$. Consider the cylinder $A = \pi(D \cap S) \subset T$ and let $\eta \subset f^{-1}(D) \cap S$ be a lift to π of a geodesic in A connecting the two boundary arcs of A. We extend η to a closed arc by adding to it its two endpoints $w_\pm \in R_0^\pm$. Let z_+, z_- be interior points of η such that the two subarcs $[w_+, z_+]_\eta$ and $[w_-, z_-]_\eta$ of η are contained in $\Lambda_c(\infty)$ and the external rays $R_\pm$ through $z_\pm$ intersects η for the first time at $z_\pm$, when looking from ∞ and intersects $f(\eta)$ only once on the way to $z_\pm$ from ∞. The existence of two such points $z_\pm$ follows from the fact, that the similar property holds for the endpoints $w_\pm \in R_0^\pm$ of η and that η is transverse (orthogonal) to $R_0^\pm$ at $w_\pm$.

We construct the open disk U as the Jordan domain containing 0, whose boundary consists of the segments of the rays $R_\pm$ and $-R_\pm$ from potential level 2 to $z_\pm$ and $-z_\pm$ respectively together with the subarcs $[z_+, z_-]_\eta$, $-[z_+, z_-]_\eta$ of η and $-\eta$ respectively and the connecting segments of the level-2 equipotential (see Figure 8). Then $W \subset \overline{U}$, because $\partial U \cap \partial W$ is a subset of the level-2 equipotential and the rest of the boundary of W is contained in U. Let U' denote the connected component of $f^{-1}(U)$ containing say the interior of W', then U' is a Jordan domain and $f : U' \longrightarrow U$ is proper of degree 2, because $\overline{U}$ contains only the single critical value $f(0) \in U$ for f. Finally $\overline{U'} \subset U$ because each preimage of $R_\pm \cap \partial U$ on $\partial U'$ intersects η or $-\eta$ only once and thus is contained in U. Hence $f : U' \longrightarrow U$ is quadratic-like and a p-renormalization of Q_c. Let $Q_{\widehat{c}}$ be a quadratic polynomial hybrid equivalent to f. Then $Q_{\widehat{c}}$ has a fixed point of multiplier $c \in \overline{\mathbb{D}} \backslash \{1\}$, because f has such a fixed point and hybrid equivalences preserve the multiplier of any non repelling periodic point. Hence it follows by direct calculation that $\widehat{c} = \frac{\lambda}{2} - \frac{\lambda^2}{4}$.

q.e.d.

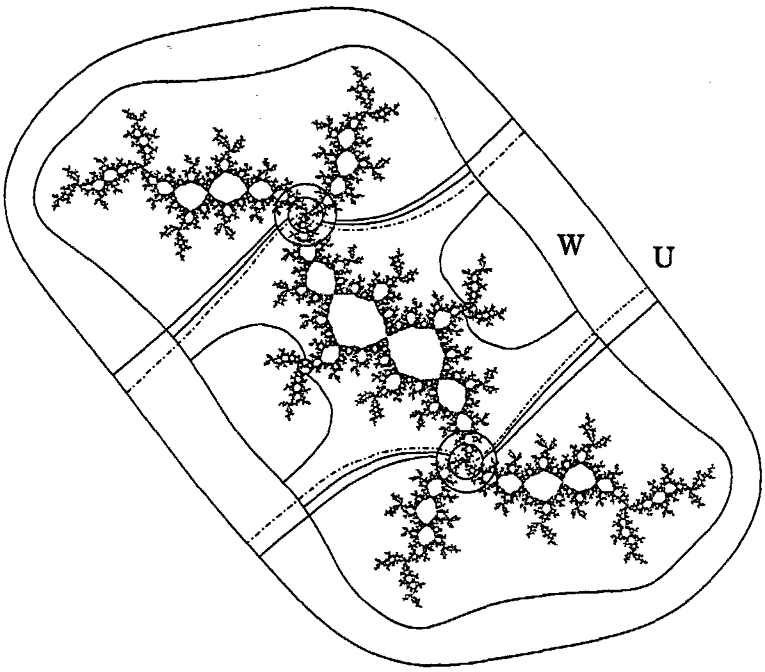

Figure 8: Fattening of W and W'. The dot–dash curves are the f-invariant rays $R_0^{\pm}$ and their preimages $-R_0^{\pm}$ on the boundary of W and $W' \subset W$.

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by: Thomas P. Branson

224/92 UNIVERSAL AC CONDUCTIVITY OF NON-METALLIC SOLIDS AT LOW TEMPERATURES

by: Jeppe C. Dyre

225/92 "HATMODELLEN" Impedansspektroskopi i ultrarent en-krystallinsk silicium

af: Anja Boisen, Anders Gorm Larsen, Jesper Varmer, Johannes K. Nielsen, Kit R. Hansen, Peter Bøggild og Thomas Hougaard

Vejleder: Petr Viscor

226/92 "METHODS AND MODELS FOR ESTIMATING THE GLOBAL CIRCULATION OF SELECTED EMISSIONS FROM ENERGY CONVERSION"

by: Bent Sørensen

227/92 "Computersimulering og fysik"

af: Per M.Hansen, Steffen Holm, Peter Maibom, Mads K. Dall Petersen, Pernille Postgaard, Thomas B.Schrøder, Ivar P. Zeck

Vejleder: Peder Voetmann Christiansen

228/92 "Teknologi og historie"

Fire artikler af:

Mogens Niss, Jens Høyrup, Ib Thiersen, Hans Hedal

229/92 "Masser af information uden betydning"

En diskussion af informationsteorien i Tor Nørretranders' "Mærk Verden" og en skitse til et alternativ basseret på andenordens kybernetik og semiotik.

af: Søren Brier

230/92 "Vinklens tredeling - et klassisk problem"

et matematisk projekt af

Karen Birkelund, Bjørn Christensen

Vejleder: Johnny Ottesen

231A/92 "Elektrondiffusion i silicium - en matematisk model"

af: Jesper Voetmann, Karen Birkelund, Mette Olufsen, Ole Møller Nielsen

Vejledere: Johnny Ottesen, H.B.Hansen

231B/92 "Elektrondiffusion i silicium - en matematisk model" Kildetekster

af: Jesper Voetmann, Karen Birkelund, Mette Olufsen, Ole Møller Nielsen

Vejledere: Johnny Ottesen, H.B.Hansen

232/92 "Undersøgelse om den simultane opdagelse af energiens bevarelse og isærdeles om de af Mayer, Colding, Joule og Helmholtz udførte arbejder"

af: L.Arleth, G.I.Dybkjær, M.T.Østergård

Vejleder: Dorthe Posselt

233/92 "The effect of age-dependent host mortality on the dynamics of an endemic disease and Instability in an SIR-model with age-dependent susceptibility

by: Viggo Andreasen

234/92 "THE FUNCTIONAL DETERMINANT OF A FOUR-DIMENSIONAL BOUNDARY VALUE PROBLEM"

by: Thomas P. Branson and Peter B. Gilkey

235/92 OVERFLADESTRUKTUR OG POREUDVIKLING AF KOKS

- Modul 3 fysik projekt -

af: Thomas Jessen

236a/93 INTRODUKTION TIL KVANTE HALL EFFEKTEN

af: Anja Boisen, Peter Bøggild

Vejleder: Peder Voetmann Christiansen
Erland Brun Hansen

236b/93 STRØMSSAMMENBRUD AF KVANTE HALL EFFEKTEN

af: Anja Boisen, Peter Bøggild

Vejleder: Peder Voetmann Christiansen
Erland Brun Hansen

237/93 The Wedderburn principal theorem and Shukla cohomology

af: Lars Kadison

238/93 SEMIOTIK OG SYSTEMEGENSKABER (2)

Vektorbånd og tensorer

af: Peder Voetmann Christiansen

239/93 Valgsystemer - Modelbygning og analyse

Matematik 2. modul

af: Charlotte Gjerrild, Jane Hansen, Maria Hermannsson, Allan Jørgensen, Ragna Clauson-Kaas, Poul Lützen

Vejleder: Mogens Niss

240/93 Patologiske eksempler. Om sære matematiske fisks betydning for den matematiske udvikling

af: Claus Dræby, Jørn Skov Hansen, Runa Ulsøe Johansen, Peter Meibom, Johannes Kristoffer Nielsen

Vejleder: Mogens Niss

241/93 FOTOVOLTAISK STATUSNOTAT 1

af: Bent Sørensen

242/93 Brovedligeholdelse - bevar mig vel

Analyse af Vejdirektoratets model for optimering af broreparationer

af: Linda Kyndlev, Kare Fundal, Kamma Tulinius, Ivar Zeck

Vejleder: Jesper Larsen

243/93 TANKEEKSPERIMENTER I FYSIKKEN

Et 1.modul fysikprojekt

af: Karen Birkelund, Stine Sofia Korremann

Vejleder: Dorthe Posselt

244/93 RADONTRANSFORMATIONEN og dens anvendelse i CT-scanning

Projektrapport

af: Trine Andreasen, Tine Guldager Christiansen, Nina Skov Hansen og Christine Iversen

Vejledere: Gestur Olafsson og Jesper Larsen

245a+b /93 Time-Of-Flight målinger på krystallinske halvledere
Specialerapport

af: Linda Szkotak Jensen og Lise Odgaard Gade

Vejledere: Petr Viscor og Niels Boye Olsen

246/93 HVERDAGSVIDEN OG MATEMATIK - LÆREPROCESSER I SKOLEN

af: Lena Lindenskov, Statens Humanistiske Forskningsråd, RUC, IMFUFA

247/93 UNIVERSAL LOW TEMPERATURE AC CONDUCTIVITY OF MACROSCOPICALLY DISORDERED NON-METALS

by: Jeppe C. Dyre

248/93 DIRAC OPERATORS AND MANIFOLDS WITH BOUNDARY

by: B. Booss-Bavnbek, K.P.Wojciechowski

249/93 Perspectives on Teichmüller and the Jahresbericht Addendum to Schappacher, Scholz, et al.

by: B. Booss-Bavnbek

With comments by W.Abikoff, L.Ahlfors, J.Cerf, P.J.Davis, W.Fuchs, F.P.Gardiner, J.Jost, J.-P.Kahane, R.Lohan, L.Lorch, J.Radkau and T.Söderqvist

250/93 EULER OG BOLZANO - MATEMATISK ANALYSE SET I ET VIDENSKABSTEORETISK PERSPEKTIV

Projektrapport af: Anja Juul, Lone Michelsen, Tomas Højgård Jensen

Vejleder: Stig Andur Pedersen

251|93 Genotypic Proportions in Hybrid Zones

by: Freddy Bugge Christiansen, Viggo Andreasen and Ebbe Thue Poulsen

252|93 MODELLERING AF TILFÆLDIGE FÆNOMENER

Projektrapport af: Birthe Friis, Lisbeth Helmgaard, Kristina Charlotte Jakobsen, Marina Mosbæk Johannessen, Lotte Ludvigsen, Mette Hass Nielsen

253/93 Kuglepakning

Teori og model

af: Lise Arleth, Kåre Fundal, Nils Kruse

Vejleder: Mogens Niss

254/93 Regressionsanalyse

Materiale til et statistikkursus

af: Jørgen Larsen

255/93 TID & BETINGET UAFHÆNGIGHED

af: Peter Harremoës

256/93 Determination of the Frequency Dependent Bulk Modulus of Liquids Using a Piezoelectric Spherical Shell (Preprint)

by: T. Christensen and N.B.Olsen

257/93 Modellering af dispersion i piezoelektriske keramikker

af: Pernille Postgaard, Jannik Rasmussen, Christina Specht, Mikko Østergård

Vejleder: Tage Christensen

258/93 Supplerende kursusmateriale til "Lineære strukturer fra algebra og analyse"

af: Mogens Brun Heefelt

259/93 STUDIES OF AC HOPPING CONDUCTION AT LOW TEMPERATURES

by: Jeppe C. Dyre

260/93 PARTITIONED MANIFOLDS AND INVARIANTS IN DIMENSIONS 2, 3, AND 4

by: B. Booss-Bavnbek, K.P.Wojciechowski

261/93 OPGAVESAMLING
Bredde-kursus i Fysik
Eksamensopgaver fra 1976-93

262/93 Separability and the Jones Polynomial
by: Lars Kadison

263/93 Supplerende kursusmateriale til "Lineære strukturer fra algebra og analyse" II
af: Mogens Brun Heefelt

264/93 FOTOVOLTAISK STATUSNOTAT 2
af: Bent Sørensen

265/94 SPHERICAL FUNCTIONS ON ORDERED SYMMETRIC SPACES
To Sigurdur Helgason on his sixtyfifth birthday
by: Jacques Faraut, Joachim Hilgert and Gestur Olafsson

266/94 Kommensurabilitets-oscillationer i laterale supergitre
Fysikspeciale af: Anja Boisen, Peter Bøggild, Karen Birkelund
Vejledere: Rafael Taboryski, Poul Erik Lindelof, Peder Voetmann Christiansen

267/94 Kom til kort med matematik på Eksperimentarium - Et forslag til en opstilling
af: Charlotte Gjerrild, Jane Hansen
Vejleder: Bernhelm Booss-Bavnbek

268/94 Life is like a sewer ...
Et projekt om modellering af aorta via en model for strømning i kloakrør
af: Anders Marcussen, Anne C. Nilsson, Lone Michelsen, Per M. Hansen
Vejleder: Jesper Larsen

269/94 Dimensionsanalyse en introduktion metaprojekt, fysik
af: Tine Guldager Christiansen, Ken Andersen, Nikolaj Hermann, Jannik Rasmussen
Vejleder: Jens Højgaard Jensen

270/94 THE IMAGE OF THE ENVELOPING ALGEBRA AND IRREDUCIBILITY OF INDUCED REPRESENTATIONS OF EXPONENTIAL LIE GROUPS
by: Jacob Jacobsen

271/94 Matematikken i Fysikken.
Opdaget eller opfundet
NAT-BAS-projekt
vejleder: Jens Højgaard Jensen

272/94 Tradition og fornyelse
Det praktiske elevarbejde i gymnasiets fysikundervisning, 1907-1988
af: Kristian Hoppe og Jeppe Guldager
Vejledning: Karin Beyer og Nils Hybel

273/94 Model for kort- og mellemdistanceløb
Verifikation af model
af: Lise Fabricius Christensen, Helle Pilemann, Bettina Sørensen
Vejleder: Mette Olufsen

274/94 MODEL 10 - en matematisk model af intravenøse anæstetikas farmakokinetik
3. modul matematik, forår 1994
af: Trine Andreasen, Bjørn Christensen, Christine Green, Anja Skjoldborg Hansen. Lisbeth Helmgaard
Vejledere: Viggo Andreasen & Jesper Larsen

275/94 Perspectives on Teichmüller and the Jahresbericht
2nd Edition
by: Bernhelm Booss-Bavnbek

276/94 Dispersionsmodellering
Projektrapport 1. modul
af: Gitte Andersen, Rehannah Borup, Lisbeth Friis, Per Gregersen, Kristina Vejrø
Vejleder: Bernhelm Booss-Bavnbek

277/94 PROJEKTARBEJDSPÆDAGOGIK - Om tre tolkninger af problemorienteret projektarbejde
af: Claus Flensted Behrens, Frederik Voetmann Christiansen, Jørn Skov Hansen, Thomas Thingstrup
Vejleder: Jens Højgaard Jensen

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by: Mette Olufsen(Math-Tech), Finn Nielsen (RISØ National Laboratory), Per Føge Jensen (Herlev University Hospital), Stig Andur Pedersen (Roskilde University)

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af: Tage Christensen

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by Lars Kadison and Matthias T. Kromann

281/94 Modellering af Det Cardiovaskulære System med Neural Pulskontrol
Projektrapport udarbejdet af:
Stefan Frello, Runa Ulsøe Johansen, Michael Poul Curt Hansen, Klaus Dahl Jensen
Vejleder: Viggo Andreasen

282/94 Parallelle algoritmer
af: Erwin Dan Nielsen, Jan Danielsen, Niels Bo Johansen

283/94 Grænser for tilfældighed
(en kaotisk talgenerator)

af: Erwin Dan Nielsen og Niels Bo Johansen

284/94 Det er ikke til at se det, hvis man ikke lige ve' det!
Gymnasiematematikkens begrundelsesproblem

En specialerapport af Peter Hauge Jensen og Linda Kyndlev

Vejleder: Mogens Niss

285/94 Slow coevolution of a viral pathogen and its diploid host

by: Viggo Andreasen and Freddy B. Christiansen

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by: Bent Sørensen

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by: Jeppe C. Dyre and Jacob Jacobsen

290/95 Modellering af uregelmæssige bølger
Et 3.modul matematik projekt

af: Anders Marcussen, Anne Charlotte Nilsson, Lone Michelsen, Per Mørkegaard Hansen

Vejleder: Jesper Larsen

291/95 1st Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

an example of using methods developed for the OECD/IEA and the US/EU fuel cycle externality study

by: Bent Sørensen

292/95 Fotovoltaisk Statusnotat 3

af: Bent Sørensen

293/95 Geometridiskussionen - hvor blev den af?

af: Lotte Ludvigsen & Jens Frandsen

Vejleder: Anders Madsen

294/95 Universets udvidelse - et metaprojekt

Af: Jesper Duelund og Birthe Friis

Vejleder: Ib Lundgaard Rasmussen

295/95 A Review of Mathematical Modeling of the Controled Cardiovascular System

By: Johnny T. Ottesen

296/95 RETIKULER den klassiske mekanik

af: Peder Voetmann Christiansen

297/95 A fluid-dynamical model of the aorta with bifurcations

by: Mette Olufsen and Johnny Ottesen

298/95 Mordet på Schrödingers kat - et metaprojekt om to fortolkninger af kvantemekanikken

af: Maria Hermannsson, Sebastian Horst, Christina Specht

Vejledere: Jeppe Dyre og Peder Voetmann Christiansen

299/95 ADAM under figenbladet - et kig på en samfunds-videnskabelig matematisk model

Et matematisk modelprojekt

af: Claus Dræby, Michael Hansen, Tomas Højgård Jensen

Vejleder: Jørgen Larsen

300/95 Scenarios for Greenhouse Warming Mitigation

by: Bent Sørensen

301/95 TOK Modellering af træers vækst under påvirkning af ozon

af: Glenn Møller-Holst, Marina Johannessen, Birthe Nielsen og Bettina Sørensen

Vejleder: Jesper Larsen

302/95 KOMPRESSORER - Analyse af en matematisk model for aksialkompressorer

Projektrapport sf: Stine Bøggild, Jakob Hilmer, Pernille Postgaard

Vejleder: Viggo Andreasen

303/95 Masterlignings-modeller af Glasovergangen
Termisk-Mekanisk Relaksation
Specialerapport udarbejdet af:

Johannes K. Nielsen, Klaus Dahl Jensen

Vejledere: Jeppe C. Dyre, Jørgen Larsen

304a/95 STATISTIKNOTER Simple binomialfordelingsmodeller

af: Jørgen Larsen

304b/95 STATISTIKNOTER Simple normalfordelingsmodeller

af: Jørgen Larsen

304c/95 STATISTIKNOTER Simple Poissonfordelingsmodeller

af: Jørgen Larsen

304d/95 STATISTIKNOTER Simple multinomialfordelingsmodeller

af: Jørgen Larsen

304e/95 STATISTIKNOTER Mindre matematisk-statistisk opslagsværk indeholdende bl.a. ordforklaringer, resuméer og tabeller

af: Jørgen Larsen

305/95 The Maslov Index:
A Functional Analytical Definition
And The Spectral Flow Formula

By: B. Booss-Bavnbek, K. Furutani

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Preprint of a chapter for the forthcomming International Handbook of Mathematics Education (Alan J.Bishop, ed)

By: Mogens Niss

307/95 Habit Formation and the Thirdness of Signs

Presented at the semiotic symposium

The Emergence of Codes and Intensions as a Basis of Sign Processes

By: Peder Voetmann Christiansen

308/95 Metaforer i Fysikken

af: Marianne Wilcken Bjerregaard, Frederik Voetmann Christiansen, Jørn Skov Hansen, Klaus Dahl Jensen Ole Schmidt

Vejledere: Peder Voetmann Christiansen og Petr Viscor

309/95 Tiden og Tanken
En undersøgelse af begrebsverdenen Matematik udført ved hjælp af en analogi med tid

af: Anita Stark og Randi Petersen

Vejleder: Bernhelm Booss-Bavnbek

310/96 Kursusmateriale til "Lineære strukturer fra algebra og analyse" (E1)

af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Hélène Connor-Lajambe, Bernd Kuemmel, Stefan Krüger Nielsen, Bent Sørensen

312/96 Grassmannian and Chiral Anomaly

by: B. Booss-Bavnbek, K.P.Wojciechowski

313/96 THE IRREDUCIBILITY OF CHANCE AND THE OPENNESS OF THE FUTURE
The Logical Function of Idealism in Peirce's Philosophy of Nature

By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian Cardiovascular System

By: Johnny T. Ottesen

315/96 a + b "Rejsen til tidens indre" - Udarbejdelse af et manuskript til en fjernsynsudsendelse + manuskript

af: Gunhild Hune og Karina Goyle

Vejledere: Peder Voetmann Christiansen og Bruno Ingemann

316/96 Plasmaoscillation i natriumklynger

Specialerapport af: Peter Meibom, Mikko Østergård

Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer

af: Ulla Rasmussen

Vejleder: Anders Madsen

318/96 Modelling the Respiratory System

by: Tine Guldager Christiansen, Claus Dræby

Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming Impacts

by: Bent Sørensen

320/96 Grassmannian and Boundary Contribution to the -Determinant

by: K.P.Wojciechowski et al.

321/96 Modelkompetencer - udvikling og afprøvning af et begrebsapparat

Specialerapport af: Nina Skov Hansen, Christine Iversen, Kristin Troels-Smith

Vejleder: Morten Blomhøj

322/96 OPGAVESAMLING
Bredde-Kursus i Fysik 1976 - 1996

323/96 Structure and Dynamics of Symmetric Diblock Copolymers
PhD Thesis

by: Christine Maria Papadakis

324/96 Non-linearity of Baroreceptor Nerves

by: Johnny T. Ottesen

325/96 Retorik eller realitet ?
Anvendelser af matematik i det danske Gymnasiums matematikundervisning i perioden 1903 - 88

Specialerapport af Helle Pilemann

Vejleder: Mogens Niss

326/96 Bevisteori
Eksemplificeret ved Gentzens bevis for konsistensen af teorien om de naturlige tal

af: Gitte Andersen, Lise Mariane Jeppesen, Klaus Frovin Jørgensen, Ivar Peter Zeck

Vejledere: Bernhelm Booss-Bavnbek og Stig Andur Pedersen

327/96 NON-LINEAR MODELLING OF INTEGRATED ENERGY SUPPLY AND DEMAND MATCHING SYSTEMS

by: Bent Sørensen

328/96 Calculating Fuel Transport Emissions

by: Bernd Kuemmel

329/96 The dynamics of cocirculating influenza strains conferring partial cross-immunity

and

A model of influenza A drift evolution

by: Viggo Andreasen, Juan Lin and Simon Levin

330/96 LONG-TERM INTEGRATION OF PHOTOVOLTAICS INTO THE GLOBAL ENERGY SYSTEM

by: Bent Sørensen

331/96 Viskøse fingre

Specialerapport af:

Vibeke Orlien og Christina Specht

Vejledere: Jacob M. Jacobsen og Jesper Larsen

332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG

Specialerapport af:

Stine Sofia Korremann

Vejleder: Dorthe Posselt

333/97 Biodiversity Matters

an extension of methods found in the literature on monetisation of biodiversity

by: Bernd Kuemmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Bernd Kuemmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids

by: Jeppe C. Dyre

336/97 PROBLEM-ORIENTATED GROUP PROJECT WORK AT ROSKILDE UNIVERSITY

by: Kathrine Legge

337/97 Verdensbankens globale befolkningsprognose - et projekt om matematisk modellering

af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen

Vejleder: Jørgen Larsen

338/97 Kvantisering af nanolederes elektriske ledningsevne

Første modul fysikprojekt

af: Søren Dam, Esben Danielsen, Martin Niss, Esben Friis Pedersen, Frederik Resen Steenstrup

Vejleder: Tage Christensen

339/97 Defining Discipline

by: Wolfgang Coy

340/97 Prime ends revisited - a geometric point of view -

by: Carsten Lunde Petersen

341/97 Two chapters on the teaching, learning and assessment of geometry

by Mogens Niss

342/97 LONG-TERM SCENARIOS FOR GLOBAL ENERGY DEMAND AND SUPPLY

A global clean fossil scenario discussion paper

prepared by Bernd Kuemmel

Project leader: Bent Sørensen

343/97 IMPORT/EKSPORT-POLITIK SOM REDSKAB TIL OPTIMERET UDNYTTELSE AF EL PRODUCERET PÅ VE-ANLÆG

af: Peter Meibom, Torben Svendsen, Bent Sørensen

344/97 Puzzles and Siegel disks

by Carsten Lunde Petersen
