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- a geometric point of view -

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The fine structure of prime ends of the third and fourth kind are analysed by decomposing the impressions into intrinsicly defined subcontinua ordered by inclusion. This is accomplished by associating two (possibly identical) intrinsicly defined impressions (subcontinua of the impression $I(P)$) to every pair consisting of a prime end P and a continuous and increasing function F on $\mathbb{R}_+$. Further more a notion of equivalence of monotone continuous functions is introduced, so that two equivalent functions always yield identical impressions for all prime ends P. Moreover it is shown that many inequivalent functions give distinguishable impressions (a full 1 real parameter family is given as example). It follows that the Caratheodory classification scheme can be substantially extended.

1 Introduction

The study of boundaries of simply connected domains $U \subsetneq \mathbb{C}$ was initiated by Caratheodory, [Car]. He founded the theory of prime ends, which provide a convenient language for discussing the boundary behavior of univalent maps on the unit disc $\mathbb{D}$. Caratheodory classified prime ends into four kinds using purely topological considerations. In this paper hyperbolic geometry is used to further classify prime ends and to study the fine structure of prime ends. This naturally leads to distinguish real infinitely many different types/kinds of prime-ends. Thus substantially extending the classification scheme of Caratheodory.

Prime ends was defined by Caratheodory as follows. Let $U \subsetneq \mathbb{C} \subset \overline{\mathbb{C}}$ be an arbitrary simply connected proper sub domain. A *cross cut* in U is an arc $c : [0,1] \longrightarrow \overline{U}$ with $c(0), c(1) \in \partial U$ and $c(]0,1[) \subset U$.

A *chain* in U is a sequence $\mathcal{C} = \{c_n\}_{n\geq 0}$ of cross cuts in U, such that :

1. $c_n \cap c_{n+1} = \emptyset$ for each $n \geq 0$.
2. Each arc c_n, $n \geq 0$ separates in U the arcs $\{c_k \cap U\}_{k\geq 0}^{n-1}$ from the arcs $\{c_k \cap U\}_{k\geq n+1}$.
3. The spherical diameter diameter of c_n converge to 0 as $n \to \infty$.

Two chains $\mathcal{C} = \{c_n\}_{n\geq 0}$ and $\mathcal{C}' = \{c'_n\}_{n\geq 0}$ are termed *equivalent* provided there exists a chain consisting of alternately cross cuts from $\mathcal{C}$ and $\mathcal{C}'$. A *prime end* P is an equivalence class of chains.

Let $\mathcal{P}(U)$ denote the set of prime ends of U. Caratheodory proved that a and hence any Riemann map $\phi : \mathbb{D} \longrightarrow U$ defines a $1:1$ correspondence $s \mapsto P_\phi(s)$ between $\mathbb{S}^1 = \partial\mathbb{D}$ and $\mathcal{P}(U)$.

To each prime end $P \in \mathcal{P}(U)$ Caratheodory associated a set, the *impression* $I(P) \subseteq \partial U$ as follows : Let $\mathcal{C} = \{c_n\}_{n\geq 0}$ be a chain representing P. For each $n \geq 0$ let ω_n denote the connected component of $U \backslash c_n$ containing $c_{n+1} \cap U$. Then $\omega_n \subset \omega_{n+1}$ for each $n \geq 0$ and :

$$I(P) := \bigcap_{n\geq 0} \overline{\omega}_n \subset \partial U.$$

(For $\omega \subset \overline{\mathbb{C}}$ we let $\overline{\omega}$ denote the closure of ω.) Clearly $I(P)$ does not depend on the choice of representing chain $\mathcal{C}$; it is an intrinsic characteristic of the domain ω. One also says that the prime end P contains the points of its impression. But note that a point may belong to several prime ends. Further more a point $p \in I(P)$ is called *principal* in P if there exists a chain $\mathcal{C} = \{c_n\}_{n\geq 0}$ representing P and with $c_n \to p$ as $n \to \infty$. The set of principal points in P is called the *principal impression* of P and is denoted $\Pi(P)$. The points of $I(P)\backslash\Pi(P)$ are called *subsidiary points* of P. Caratheodory classified prime ends according to the following schema :

-	$I_\phi(P) = \Pi_\phi(P)$	$I_\phi(P) \neq \Pi_\phi(P)$
$\#(\Pi_\phi(P)) = 1$	first kind	second kind
$\#(\Pi_\phi(P)) > 1$	third kind	fourth kind

If $\phi : \mathbb{D} \longrightarrow U$ is a Riemann map then we also write $I_\phi(s) = I(P_\phi(s))$ and $\Pi_\phi(s) = \Pi(P_\phi(s))$.

For prime ends of any other than the first kind, the impression is some continuum with a topological structure and a topological/geometric relation to U, the fine structure of the prime end. We shall say that a subset of (the impression of) a prime end $P \in P(U)$ is defined *intrinsicly* to U, if it only depends on the shape of U, i.e. it does not depend on the choice of a Riemann map from U to $\mathbb{D}$ (see also Definition 2.3 – Remark 2.5). In this paper we shall analyse the fine structure of prime ends of the second and fourth kind by decomposing the impression into an increasing sequence of subcontinua defined intrinsicly to U. For doing so hyperbolic geometry will be an essential tool. Similar techniques may be applied to study also the fine structure of the principal impression and in particular prime ends of the third kind. However this is beyond the scope of the present paper.

The idea is that as soon as the impression $I(P)$ of some impression is non trivial it should be possible to find some kind of stratification of this impression. Indeed this is possible in particular we shall define one-parameter families of subsets of $I(P)$. Moreover we shall show by way of example that these subsets can all be different for some prime ends P. We need some terminology.

Ursell and Young [U-Y] gave a first qualitative discussion of the fine structure of prime ends by means of curves. In particular they showed that one may at least write the impression of a given prime-end as an increasing union of subcontinua. But they did not attempt to find intrinsicly defined continua. Later also Collingwood and Piranian, [C-P] and Hamilton, [Ha] have contributed to our understanding of the fine structure.

The classical discussion of prime ends initiated by Caratheodory concerns the possible distributions of the different kinds of prime ends for domains U. Urysohn noted that $\mathcal{P}(U)$ carries a natural topology induced by the Caratheodory $1:1$ correspondence between $\mathbb{S}^1$ and $\mathcal{P}(U)$. There are numerous results on the distribution in terms of this topology of the different kinds of prime ends, see e.g. [U], [Pi1], [C-P], [C-L] and [L-P]. But this is a completely different discussion.

2 Impressions

For $U \subset \overline{\mathbb{C}}$ a domain (an open connected subset) let $CL(U)$ denote the set of non empty relatively closed subsets of U, i.e. $\omega \in CL(U)$ if and only if $\emptyset \neq \omega = \overline{\omega} \cap U$.

Definition 2.1 *Let $\phi : \mathbb{D} \longrightarrow \overline{\mathbb{C}}$ be a univalent map. For $\omega \in CL(\mathbb{D})$ define the impression $I_\phi(\omega)$ of ω by ϕ as*

$$I_\phi(\omega) = \overline{\phi(\omega)} \backslash \phi(\omega) \subset \partial U,$$

where $U = \phi(\mathbb{D})$. Moreover if $s \in \overline{\omega} \cap \mathbb{S}^1$ we define the pointed impression $I_\phi(\omega, s)$ as

$$I_\phi(\omega, s) = \left\{ \zeta \in \partial U | \exists \{z_n\}_{n \in \mathbb{N}} \subset \omega, \text{ with } z \underset{n \to \infty}{\longrightarrow} s, \phi(z) \underset{n \to \infty}{\longrightarrow} \zeta \right\}$$

For any connected set $\omega \in CL(\mathbb{D})$ with $\overline{\omega} \cap \mathbb{S}^1$ non empty and connected, and for any $s \in \overline{\omega} \cap \mathbb{S}^1$, the impressions $I_\phi(\omega), I_\phi(\omega, s)$ are non empty connected sets for all univalent maps ϕ of $\mathbb{D}$. Moreover when $\overline{\omega} \cap \mathbb{S}^1 = \{s\}$ a singleton we shall say that $s \in \mathbb{S}^1$ is the *root* of $\omega \in CL(\mathbb{D})$. Remark that when ω has a root s then $I_\phi(\omega) = I_\phi(\omega, s)$. We shall focus our attention on sets with a root.

The classical terminology for the generalized impressions as defined above is cluster sets (se e.g. [C-L]). I find however the term impression more suggestive and shall use it here.

Example 2.2 Let $\phi : \mathbb{D} \longrightarrow U \subset \overline{\mathbb{C}}$ be a univalent map and let $s \in \mathbb{S}^1$ be any point. Then the (Caratheodory) impression $I_\phi(s)$ equals $I_\phi(\mathbb{D}, s)$.

The radial impression $I_\phi(R_s)$ corresponding to s is the impression of the ray $[0, s[$, $I_\phi(R_s) = I_\phi([0, s[)$. Lindelöf proved that $\Pi_\phi(s) = I_\phi(R_s)$, i.e. the principal impression equals the radial impression. A version of the Lindelöf Theorem can be found in the monograph by Pommerenke [Po, Th. 2.16]. The radial impression $I_\phi(R_s)$ is sometimes called the oblique impression or cone impression, because any small cone in $\mathbb{D}$ of angle strictly less than π, centered on R_s and with top point s have the same impression, the principal impression by Lindelöfs theorem.

Using topology alone the Caratheodory classification is almost as far as one can distinguish different types of prime ends. One may in fact also distinguish left and right wing impressions, as is done in the monograph [C-L]. This gives in total 8 different kinds in contrast to the full 1 (real)-parameter families of different types of prime ends being presented in the following.

Definition 2.3 *Two non empty and relatively closed subsets $\omega_1, \omega_2 \in CL(\mathbb{D})$ are termed (impression) equivalent relative to $\mathbb{D}$, written $\omega_1 \equiv_{\mathbb{D}} \omega_2$ if for all univalent maps $\phi : \mathbb{D} \longrightarrow \overline{\mathbb{C}}$*

$$I_\phi(\omega_1) = I_\phi(\omega_2).$$

Definition 2.4 *Suppose $\omega \in CL(\mathbb{D})$ has root $s \in \mathbb{S}^1$. We shall say that ω is* conformally natural *if $A(\omega) \equiv_{\mathbb{D}} \omega$ for any automorphism A of $\mathbb{D}$ fixing s.*

Remark 2.5 *Suppose $\omega \in CL(\mathbb{D})$ with root $s \in \mathbb{S}^1$ is conformally natural. Then for any biholomorphic map $\phi : \mathbb{D} \longrightarrow U$ the impression $I_\phi(\omega) \subset I_\phi(s)$ is intrinsic to U, that is it does not depend on the choice of biholomorphic map $\phi : \mathbb{D} \longrightarrow U$.*

One may of course suspiciously ask : Does this really yield new information? Does there exist any conformally natural set $\omega \in CL(\mathbb{D})$ say with root 1 and a univalent map ϕ, for which the impression $I_\phi(\omega)$ is different from both the principal impression $\Pi_\phi(1)$ and the prime end impression $I_\phi(1)$?

A first positive indication of this was found by Gaier and Pommerenke in response to a question raised by Piranian :

Theorem 2.6 (Gaier and Pommerenke) *Let $\{z_p\}_{p\geq 1} \subset \mathbb{D}$ be any sequence with $z_p \to 1$ and $Arg(z_p - 1) \to \frac{\pi}{2}$ as $p \to \infty$. There exists a bounded univalent map $\phi : \mathbb{D} \longrightarrow \mathbb{C}$ with*

$$I_\phi([0,1[) = \Pi_\phi(1) = \{0\} \quad \textit{and} \quad \{0\} \subsetneq I_\phi(\{z_p\}_{p\geq 1}).$$

This Theorem at least hints that the Caratheodory classification of prime ends can be refined substantially.

For $U \subset \overline{\mathbb{C}}$ a hyperbolic domain we let $\mathrm{d}_U(\cdot,\cdot)$ denote the corresponding hyperbolic distance. As a main tool in our analysis we shall use the hyperbolic Hausdorff (pseudo) distance:

Let $U \subset \overline{\mathbb{C}}$ be any hyperbolic subset. Define the hyperbolic Hausdorff semi-distance $D_U^* : CL(U)^2 \longrightarrow [0,\infty]$ by

$$D_U^*(\omega_1,\omega_2) = \sup_{z_1\in\omega_1} \inf_{z_2\in\omega_2} d_U(z_1,z_2) \qquad \omega_1,\omega_2 \in CL(U).$$

The hyperbolic Hausdorff (pseudo) distance $D_U : CL(U)^2 \longrightarrow [0,\infty]$ on $CL(U)$ is the symmetrized function defined by

$$D_U(\omega_1,\omega_2) = \max\{D_U^*(\omega_1,\omega_2), D_U^*(\omega_2,\omega_1)\} \qquad \forall \omega_1,\omega_2 \in CL(U).$$

The function D_U has all the properties of a metric except that it takes the value ∞. Moreover the restriction of D_U to the set of compact subsets $Comp(U) \subset CL(U)$ is a complete metric on $Comp(U)$.

Proposition 2.7 *Let $\omega_1, \omega_2 \in CL(\mathbb{D})$. If $D^*_{\mathbb{D}}(\omega_1, \omega_2) < \infty$ then*

$$I_\phi(\omega_1) \subset I_\phi(\omega_2)$$

for all univalent maps $\phi : \mathbb{D} \longrightarrow \overline{\mathbb{C}}$.

Proof : Suppose $D^*_{\mathbb{D}}(\omega_1, \omega_2) = R < \infty$ and let $\phi : \mathbb{D} \longrightarrow U \subset \overline{\mathbb{C}}$ be an arbitrary, but fixed univalent map. We can suppose $I_\phi(\omega_1) \neq \emptyset$, as $\emptyset$ is a subset of any set. Let $y \in I_\phi(\omega_1)$ be arbitrary and let $\{z_n\}_{n\geq 1} \subset \omega_1$ be a sequence with $\phi(z_n) \to y$ as $n \to \infty$. Choose any sequence $\{w_n\}_{n\geq 1} \subset \omega_2$ with $d_{\mathbb{D}}(z_n, w_n) \leq R$ for all $n \geq 1$ so that

$$\phi(w_n) \in B_U(\phi(z_n), R) \qquad \forall n \geq 1.$$

The spherical diameter of the closed hyperbolic balls $B_U(\phi(z_n), R)$ converges to 0 as $n \to \infty$, because the centers converge to $y \in \partial U$. It follows that $\phi(w_n) \to y$ as $n \to \infty$. Thus $y \in I_\phi(\omega_2)$ and $I_\phi(\omega_1) \subset I_\phi(\omega_2)$ as $y \in I(\omega_1)$ was arbitrary. **q.e.d.**

Corollary 2.8 *Let $\omega_1, \omega_2 \in CL(\mathbb{D})$. If $D_{\mathbb{D}}(\omega_1, \omega_2) < \infty$ then $\omega_1 \equiv_{\mathbb{D}} \omega_2$, i.e.*

$$I_\phi(\omega_1) = I_\phi(\omega_2)$$

for all univalent maps $\phi : \mathbb{D} \longrightarrow U \subset \overline{\mathbb{C}}$

Define $\omega_1, \omega_2 \in CL(\mathbb{D})$ to be *Hausdorff equivalent in* $\mathbb{D}$ if the condition $D_{\mathbb{D}}(\omega_1, \omega_2) < \infty$ holds. We may then formulate the above Corollary as Hausdorff equivalence implies impression equivalence. Most of the results exposed in the following will be derived from this observation. However there is more to the story than Hausdorff distance. There are plenty of impression equivalent sets, which are not Hausdorff equivalent (see page 9).

For $s \in \mathbb{S}^1$ and $0 < r < 1$ the horo disc in $\mathbb{D}$ with root s and of (Euclidean) radius r is the disc $\mathbb{D}(s(1-r), r)$. Let $Horo(s, r) = \overline{\mathbb{D}(s(1-r), r)} \cap \mathbb{D}$. For any $s \in \mathbb{S}^1$ and any two numbers $0 < r_1, r_2 < 1$

$$D_{\mathbb{D}}(Horo(s, r_1), Horo(s, r_2)) = |\log \frac{(1-r_1)r_2}{r_1(1-r_2)}| < \infty$$

as can be seen in various ways. Thus any two horo discs in $\mathbb{D}$ with the same root are Hausdorff and thus impression equivalent. This justifies the following definition.

Definition 2.9 *For $\phi : \mathbb{D} \longrightarrow U \subset \overline{\mathbb{C}}$ a univalent map define the horo cyclic impression $h_\phi(s)$ corresponding to $s \in \mathbb{S}^1$ as*

$$h_\phi(s) = I_\phi(Horo(s, \tfrac{1}{2})).$$

Note that the horo cyclic impression $h_\phi(s)$ is conformally natural, and thus it is intrinsicly defined.

3 More impressions

3.1 Impressions of functions

Let $\mathbb{H} = \{z = x + iy \mid y > 0\}$ and $S = \{z = x + iy | 0 < y < \pi\}$. Moreover define $\overline{S} = \{z = x + iy | 0 \leq y \leq \pi\} \cup \{-\infty, \infty\}$ and let $\overline{\mathbb{D}}$ and $\overline{\mathbb{H}}$ denote the respective closures in $\overline{\mathbb{C}}$. So that $\exp : S \longrightarrow \mathbb{H}$ extends to a homeomorphism between $\overline{S}$ and $\overline{\mathbb{H}}$, and $i\frac{z-i}{z+i} : \mathbb{H} \longrightarrow \mathbb{D}$ extends to a homeomorphism of the closures.

Remark 3.1 *A biholomorphic map $\psi : V \longrightarrow W$ preserves relatively closed subsets. Hence the notions of impression and impression equivalence together with the notion of Hausdorff equivalence and Corollary 2.8 generalize to arbitrary hyperbolic domains V in $\overline{\mathbb{C}}$.*

In brief say in $\mathbb{H}$, any two *Hausdorff equivalent* (relatively closed) subsets $\omega_1, \omega_2 \in CL(\mathbb{H})$ are *impression equivalent*. That is $D_H(\omega_1, \omega_2) < \infty$ implies that : $I_\phi(\omega_1) = \overline{\phi(\omega_1)} \backslash \phi(\omega_1) = \overline{\phi(\omega_2)} \backslash \phi(\omega_2) = I_\phi(\omega_2)$ for any univalent map $\phi : \mathbb{H} \longrightarrow \mathbb{C}$. Moreover a relatively closed subset $\omega \in CL(\mathbb{H})$ with root s, $\{s\} = \overline{\omega} \cap \partial\mathbb{H}$ is (by definition) conformally natural, if and only if it is impression equivalent to all of its images under automorphisms of $\mathbb{H}$ fixing s. Finally the *impression* $I_\phi(\omega)$ of a conformally natural ω is defined intrinsicly to $U = \phi(H)$ for any univalent map $\phi : \mathbb{H} \longrightarrow \mathbb{C}$.

Inspired by the horocyclic impression one may try to consider impressions of sets with different orders of tangency. As examples define for each $\delta \geq 0$ the graph

$$\Gamma_\delta = \{z = x + iy | 0 < x \leq 1, y = x^{1+\delta}\}.$$

The following Theorem shows that this idea is quite useful.

Theorem 3.2 *For every $\delta \geq 0$ the graphs Γ_δ are conformally natural. And hence for any univalent map $\phi : \mathbb{H} \longrightarrow \mathbb{C}$ the impression $I_\phi(\Gamma_\delta)$ is intrinsic to $U = \phi(\mathbb{H})$. Moreover there exists a domain U with a prime end P, for which any of the above impressions are different.*

The proof of the Theorem is postponed until we have the right machinery. The above Theorem shows that we may enlarge the Caratheodory classification scheme by at least a 1-real parameter family.

Note that the graphs $\Gamma_{\delta,c} = \{z = x + iy | 0 < x \leq 1, y = cx^{1+\delta}\}$, $c > 0$ are Hausdorff equivalent to Γ_δ, in fact $D_{\mathbb{H}}(\Gamma_\delta, \Gamma_{\delta,c}) \leq |\operatorname{Log} c|$. Thus the graphs $\Gamma_{\delta,c}$ do not produce any finer classifications than does Γ_δ.

The arcs Γ_δ are not conveniently parametrized. Instead we introduce another parametrization, which is better suited for hyperbolic geometry. Fix $\delta \geq 0$. We associate a continuous function $F_\delta : [0, \infty[\longrightarrow [0, \infty[$ to Γ_δ as follows: For $s \in [0, \infty[\; = \mathbb{R}_+$ let $\gamma_s = \{z | |z| = \mathrm{e}^{-s}\} \cap \mathbb{H}$ denote the unique (hyperbolic) geodesic orthogonal to the geodesic $i\mathbb{R}_+$ at $i\,\mathrm{e}^{-s}$. The arcs γ_s and Γ_δ have a unique point of intersection z_s. Define $F_\delta(s) = \mathrm{d}_{\mathbb{H}}(z_s, i\mathbb{R}_+)$, the hyperbolic length of the sub arc of γ_s between $i\,\mathrm{e}^{-s}$ and z_s.

Thus the graph Γ_δ defines a new continuous function F_δ. On the other hand a continuous function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defines an arc $\Gamma(F)$ by reversing the construction above. For F_δ the arc $\Gamma(F_\delta)$ coincides with all but a (in $\mathbb{H}$) relatively compact subset of Γ_δ. In general the arc $\Gamma(F)$ need not bee the graph of a function $x \mapsto y(x)$. Never the less we shall refer to $\Gamma(F)$ as the graph of F.

Using signed distance along the geodesic γ_s we may even define the "graph" (arc) of an arbitrary continuous function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$. Then the impressions of graphs of positive functions are subsets of the left wing of the (prime end) impression and of negative functions are subsets of the right wing of the impression. We shall concentrate on positive functions and thus study the left wing of the impression. The right hand study being similar.

Let $C^+(\mathbb{R}_+) \subset C(\mathbb{R}_+)$ denote the spaces of continuous functions from $\mathbb{R}_+$ to respectively $\mathbb{R}_+$ and $\mathbb{R}$. Two functions $F_1, F_2 \in C(\mathbb{R}_+)$ are defined to be *equivalent*, if $||F_1 - F_2||_\infty < \infty$. Evidently graphs of equivalent functions are Hausdorff equivalent and hence impression equivalent. The reader shall easily supply a proof of the following Lemma.

Lemma 3.3 *For every* $0 < \delta$: $F_\delta(s) - \delta(s) \underset{s\to\infty}{\longrightarrow} \log 2$. *Hence for every* $0 \leq \delta$ *the function* F_δ *is equivalent to the linear function* $s \mapsto \delta s$.

Given $F \in C^+(\mathbb{R}_+)$ we define also the set $\omega(F) \in CL(\mathbb{H})$ as the set, whose boundary consists of $]0, i]$, $\Gamma(F)$ and the circular arc $\delta_0([0, F(0)])$.

Lemma 3.4 *For every* $F \in C^+(\mathbb{R}_+)$: $\Gamma(F) \equiv_{\mathbb{H}} \omega(F)$, *i.e. for all univalent maps* $\phi : \mathbb{H} \longrightarrow \overline{\mathbb{C}}$: $I_\phi(\Gamma(F)) = I_\phi(\omega(F))$.

Proof : Clearly $I_\phi(\Gamma(F)) \subset I_\phi(\omega(F))$ as $\Gamma(F) \subset \omega(F)$. For the other inclusion fix a univalent map $\phi : \mathbb{H} \longrightarrow \overline{\mathbb{C}}$. Choose a sequence $\{s_n\}_{n\geq 0} \subset \mathbb{R}_+$ strictly increasing to ∞, such that $\{\overline{\phi(C(0, \mathrm{e}^{-s_n}) \cap \mathbb{H})}\}_{n\geq 0}$ is a chain converging to some point $z_0 \in \Pi_\phi(0) = I_\phi(]0, i])$. The existence of the sequence $\{s_n\}_{n\geq 0}$ is a standard application of length-area inequalities (see [Po]). Let $U_0 = \omega(F)\backslash\mathbb{D}_{\mathrm{e}^{-s_0}}$ and $U_n = \omega(F) \cap \{z | \, \mathrm{e}^{-s_n} \leq |z| \leq \mathrm{e}^{-s_{n-1}}\}$ for each $n \geq 1$.

Let $y \in I_\phi(\omega(F))\backslash I_\phi(]0, 1])$ be arbitrary. We shall prove that $y \in I_\phi(\Gamma(F))$ (by Lindelöfs theorem $\Pi_\phi(0) = I_\phi(]0, 1]) \subset I_\phi(\Gamma(F)))$. Choose a sequence $\{z_k\}_{k\geq 1} \subset \omega(F)$ with $\phi(z_k) \to y$ as $k \to \infty$. Choose $n(k)$ such that $z_k \in U_{n(k)}$ for each k. By the maximum principle there exists $x_k \in \partial U_{n(k)}$, $k \in \mathbb{N}$ with $d_\sigma(\phi(x_k), y) \leq d_\sigma(\phi(z_k), y)$, where $d_\sigma(\cdot, \cdot)$ denotes the spherical distance. Then also $\phi(x_k) \longrightarrow y$ as $k \to \infty$. There exists N such that $x_k \in \Gamma(F)$ for $k \geq N$, because $\phi(\partial U_n \backslash \Gamma(F))$ converge to $I_\phi(]0, 1])$ and $\phi(x_k) \to y \notin I_\phi(]0, 1])$. Thus $y \in I_\phi(\Gamma(F))$. **q.e.d.**

Thus we may use which ever is the most convenient of the sets $\Gamma(F)$ and $\omega(F)$. Remark that when $\Gamma(F)$ is tangent to the real axis at 0, the sets $\Gamma(F)$ and $\omega(F)$ are not Hausdorff equivalent. And thus (impression) equivalence does not imply Hausdorff equivalence.

One of the advantages of the graphs $\Gamma(F)$ is that they behave well under automorphisms of $\mathbb{H}$ fixing 0. For instance let $L \in \mathbb{R}$ and $F \in C(\mathbb{R}_+)$. Then $\mathrm{e}^L \cdot \Gamma(F) = \Gamma(F(\cdot + L))$, where $\Gamma(F(\cdot + L))$ is the "graph" of the continuous function $\widehat{F} : [-L, \infty[\longrightarrow \mathbb{R}$, $\widehat{F}(s) = F(s + L)$.

Proposition 3.5 *There exists* $0 < r \leq 1$ *and* $K > 0$ *such that any local change of coordinates in* $\mathbb{H}$ *around* 0, *i.e. univalent map* $\psi : \mathbb{D}_R \longrightarrow \mathbb{C}$ *with* $\psi(\overline{z}) = \psi(z)$, $\psi(0) = 0$ *and* $\psi'(0) > 0$ *satisfies:*

$$\forall z \in \mathbb{H} \cap \mathbb{D}_{r\cdot R} \; : \; |\, \mathrm{d}_{\mathbb{H}}(\psi(z), i\mathbb{R}_+) - \mathrm{d}_{\mathbb{H}}(z, i\mathbb{R}_+)| < K|z|/R.$$

Proof : Elementary calculations using de Branges Theorem proves, that for any univalent map $\psi : \mathbb{D}_R \longrightarrow \mathbb{C}$

$$\forall\,|z| \le R \ : \qquad \left|\frac{\psi(z)-\psi(0)}{\psi'(0)z} - 1\right| \le \frac{1}{(1-|z|/R)^2} - 1.$$

And hence

$$\forall\,|z| \le \frac{1}{4}R \ : \qquad \left|\frac{\psi(z)-\psi(0)}{\psi'(0)z} - 1\right| \le 4|z|/R. \tag{3.1}$$

Similarly

$$\left|\frac{\psi'(z)}{\psi'(0)} - 1\right| \le \frac{1+|z|/R}{(1-|z|/R)^3} - 1.$$

And hence

$$\forall\,|z| \le \frac{1}{8}R \ : \qquad \left|\frac{\psi'(z)}{\psi'(0)} - 1\right| \le 7|z|/R. \tag{3.2}$$

Suppose that $\psi(0) = 0$, $\psi'(0) > 0$ and $\overline{\psi(\overline{z})} = \psi(z)$. As dilations are isometries of $\mathbb{H}$ preserving the geodesic $i\mathbb{R}_+$ we can suppose $\psi'(0) = 1$.

Write $z = x + iy$, we distinguish two cases $y \ge x$ and $y \le x$. The first case follows immediately from (3.1). For the second case one may procede as follows. Note that $d_{\mathbb{H}}(i|z|, z) = \log \frac{1+\sqrt{1-(y/|z|)^2}}{y/|z|}$, which for $|z|$ fixed is asymptotic in the C^1 topology to $\log \frac{2|z|}{y}$ as $y \to 0$. Hence writing $\psi(z) = z' = x' + iy'$ it suffices to prove the following inequality

$$\forall |z| \le \frac{1}{8}R \ : \qquad \left|\frac{y'}{|z'|}\frac{z}{y} - 1\right| \le \exp(17|z|/R) - 1. \tag{3.3}$$

To obtain this we apply (3.1) first to ψ and obtain :

$$\forall\,|z| \le \frac{1}{4}R \ : \qquad \left|\frac{\psi(z)}{z} - 1\right| \le 4|z|/R \tag{3.4}$$

Secondly we apply it to the univalent restriction $\psi| : \mathbb{D}_{\frac{7}{8}R}(x) \longrightarrow \mathbb{C}$ and obtain :

$$\forall |z| \le \frac{1}{8}R \ : \qquad \left|\frac{\psi(z)-\psi(x)}{\psi'(x)(z-x)} - 1\right| \le 4|z-x|/(\frac{7}{8}R) \le 5|z|/R.$$

Which, as $z - x = iy$, implies

$$\forall |z| \leq \frac{1}{8}R \;: \qquad \left| \frac{y'}{\psi'(x)y} - 1 \right| \leq 4|y|/(\frac{7}{8}R) \leq 5|z|/R. \tag{3.5}$$

Thirdly we apply (3.1) to ψ and obtain for $|z| \leq \frac{1}{8}R$

$$\forall |z| \leq \frac{1}{8}R \;: \qquad |\psi'(x) - 1| \leq 7|y|/(\frac{7}{8}R) \leq 8|z|/R. \tag{3.6}$$

Combining the three estimates (3.4), (3.5) and (3.6) we obtain (3.3) completing the proof. **q.e.d.**

Let $F \in C(\mathbb{R}_+)$ be arbitrary and define for each $k \in \mathbb{R}$ a new function $F_k \in C(\mathbb{R}_+)$ by translation in the domain as follows

$$F_k(s) = F(s+k) \qquad \text{for } k \geq 0$$

$$F_k(s) = \begin{cases} F(s+k) & \text{if } s \geq -k \\ F(0) & \text{if } 0 \leq s \leq -k. \end{cases} \qquad \text{for } k \leq 0.$$

Then for every $k \leq 0 : F = (F_k)_{-k}$ and for every $k \geq 0$ the two functions agree for $s \geq k$.

Proposition 3.6 *Let $\psi : \mathbb{D}_R \longrightarrow \mathbb{C}$ be any local change of coordinates in $\mathbb{H}$ around 0. For any Lipschitz function $F \in C(\mathbb{R}_+)$ the set $\Gamma(F) \in CL(\mathbb{H})$ is Hausdorff equivalent and hence impression equivalent to $\psi(\Gamma(F) \cap \overline{\mathbb{D}}_r) \in CL(\mathbb{H})$, $0 < r < R$. Moreover write $\psi'(0) = \mathrm{e}^L$. Then for any $k < L < K$ and any non-decreasing function $F \in C^+(\mathbb{R}_+)$*

$$D^*_{\mathbb{H}}(\omega(F_k), \psi(\omega(F) \cap \overline{\mathbb{D}}_r)), D^*_{\mathbb{H}}(\psi(\omega(F) \cap \overline{\mathbb{D}}_r), \omega(F_K)) < \infty \tag{3.7}$$

where $0 < r < R$ is arbitrary.

Here as in the rest of this paper the Lipschitz requirement is far too strong. Rather a term like *Lipschitz in the large*, meaning $|F(s) - F(s+k)| \leq C|k|$ say $\forall\, |k| > 1$ and some $C > 0$ is needed. Henceforth we shall use the term Lipschitz in the sence Lipschitz in the large.

Proof : If the statements holds for one value of $0 < r < R$, it holds for all. Let $\psi : \mathbb{D}_R \longrightarrow \mathbb{C}$ be as in the statement, with $\psi'(0) = \mathrm{e}^L > 0$ and let

$k < L < K$. By the standard growth estimate for univalent maps, there exists $0 < r < R$ such that

$$\forall\, |z| \leq r \ : \qquad \mathrm{e}^k < \left|\frac{\psi(z)}{z}\right| < \mathrm{e}^K .$$

Decreasing r if necessary we can suppose $r < \mathrm{e}^{-K}$ and by Proposition 3.5

$$|\mathrm{d}_{\mathbb{H}}(i|z|, z) - \mathrm{d}_{\mathbb{H}}(i|\psi(z)|, \psi(z))| < 1.$$

Suppose first that $F \in C(\mathbb{R}_+))$ is $C > 0$ Lipschitz and define:

$$F_-(s) = F(s) - 1 - C \cdot \max\{|k|, |K|\}$$
$$F_+(s) = F(s) + 1 + C \cdot \max\{|k|, |K|\}$$

Then $\psi(\Gamma(F) \cap \overline{\mathbb{D}}_r)$ is an arc leading to 0 with in the set bounded by the graphs $\Gamma(F_-), \Gamma(F_+)$ and the circular arc $\delta_0([F_-(0), F_+(0)])$. Hence the first statement follows. For the second note that $\psi([0, iR[)$ is tangent to $i\mathbb{R}_+$ at 0. Suppose $F \in C^+(\mathbb{R}_+)$ is non decreasing and define

$$F_- = F_k - 1 \qquad F_+ = F_K + 1$$

Then again $\psi(\Gamma(F) \cap \overline{\mathbb{D}}_r)$ is contained in the set bounded by the graphs $\Gamma(F_-), \Gamma(F_+)$ and the circular arc $\delta_0([F_k(0), F_K(0)])$. Thus the second statement follows. **q.e.d.**

Corollary 3.7 *For any Lipschitz function $F \in C(\mathbb{R}_+)$ the arc $\Gamma(F)$ is conformally natural and hence for any univalent map $\phi : \mathbb{H} \longrightarrow \mathbb{C}$ the impression $I_\phi(\Gamma(F))$ is intrinsic to $\phi(\mathbb{H})$. In particular the impressions $I_\phi(\Gamma_\delta) = I_\phi(\delta\cdot)$ are intrinsic for every $0 \leq \delta$.*

3.2 Upper and Lower impressions of functions.

Not all functions are Lipschitz. In order to produce intrinsicly defined impressions also from non Lipschitzs functions the following definitions are introduced.

Definition 3.8 *For every function $F \in C(\mathbb{R}_+)$ and every univalent map $\phi : \mathbb{H} \longrightarrow \overline{\mathbb{C}}$ define upper and lower impressions*

$$\overline{I_\phi(\Gamma(F))} = \bigcup_{k\in\mathbb{N}} I_\phi(\Gamma(F_k)) \qquad \text{and}$$
$$\underline{I_\phi(\Gamma(F))} = \bigcap_{k\in\mathbb{N}} I_\phi(\Gamma(F_{-k})).$$

Note that when F is increasing (in the weak sense, i.e. non-decreasing), then obviously

$$\Pi_\phi(0) \subseteq \underline{I_\phi(\Gamma(F))} \subseteq I_\phi(\Gamma(F)) \subseteq \overline{I_\phi(\Gamma(F))} \subseteq I_\phi(0).$$

Proposition 3.9 *For every Lipschitz or increasing $F \in C(\mathbb{R}_+)$, and every univalent map $\phi : \mathbb{H} \longrightarrow \overline{\mathbb{C}}$ the subsets*

$$\underline{I_\phi(\Gamma(F))}, \overline{I_\phi(\Gamma(F))} \subseteq I_\phi(0)$$

are intrinsic to $U = \phi(\mathbb{H})$. Moreover for every Lipschitz $F \in C(\mathbb{R}_+)$

$$\underline{I_\phi(\Gamma(F))} = I_\phi(\Gamma(F)) = \overline{I_\phi(\Gamma(F))}$$

Proof : This is an immediate Corollary of Proposition 3.6. **q.e.d.**

Let $U \subset \overline{\mathbb{C}}$ denote a simply connected hyperbolic domain and recall that $\mathcal{P}(U)$ denotes the space of prime ends of U. Moreover the Urisohn topology on $\mathcal{P}(U)$ is the topology induced by the unique topology on $U \cup \mathcal{P}(U)$ for which any biholomorphic map $\phi : \mathbb{D}\ (\mathbb{H}) \longrightarrow U$ extends to a homeomorphism $\widetilde{\phi} : \overline{\mathbb{D}}\ (\overline{\mathbb{H}}) \longrightarrow U \cup \mathcal{P}(U)$. In view of Proposition 3.9 we make the following definition:

Definition 3.10 *Let $U \subset \overline{\mathbb{C}}$ denote a simply connected hyperbolic domain. Let $P \in \mathcal{P}(U)$ be arbitrary and let $F \in C(\mathbb{R}_+)$ be any Lipschitz or increasing function. Then*

$$\overline{I_P(F)} := \overline{I_\phi(\Gamma(F))},$$
$$\underline{I_P(F)} := \underline{I_\phi(\Gamma(F))}.$$

Where $\phi : \mathbb{H} \longrightarrow U$ is any biholomorphic map with $\widetilde{\phi}(0) = P$

We extend the notion of equivalence of functions in $C(\mathbb{R}_+)$ as follows : Two functions $F, G \in C(\mathbb{R}_+)$ are termed *equivalent*, if there exists $k \leq K \in \mathbb{R}$ such that both functions $F_K - G, G - F_k \in C(\mathbb{R}_+)$ are bounded from above, i.e.

$$\limsup_{s \to \infty} F_K(s) - G(s), \limsup_{s \to \infty} G(s) - F_k(s) < \infty$$

Then for any simply connected hyperbolic domain $U \subset \overline{\mathbb{C}}$, for any prime end $P \in \mathcal{P}(U)$ and for any pair of equivalent increasing or Lipschitz functions $F, G \in C(\mathbb{R}_+)$ the upper and lower impressions of F and G are equal, that is

$$\underline{I_P(F)} = \underline{I_P(G)} \qquad \text{and} \qquad \overline{I_P(F)} = \overline{I_P(G)}$$

Upper and lower impressions are also invariant under holomorphic dynamics. More precisely let $f : \Omega \longrightarrow \overline{\mathbb{C}}$ denote any holomorphic function defined on some domain $\Omega \subseteq \overline{\mathbb{C}}$. Suppose $\Lambda_1 \subseteq \Omega$ and and $\Lambda_2 \subset \overline{\mathbb{C}}$ are simply connected hyperbolic domains for which the restriction $f| : \Lambda_1 \longrightarrow \Lambda_2$ is proper of degree $d \geq 1$. Then it is well known that f induces a degree d covering $\widetilde{f} : P(\Lambda_1) \longrightarrow P(\Lambda_2)$.

Proposition 3.11 *In the notation above let $P_1 \in \mathcal{P}(\Lambda_1)$ be arbitrary and let $P_2 = \widetilde{f}(P_1) \in \mathcal{P}(\Lambda_2)$. Then for every Lipschitz or increasing function $F \in C(\mathbb{R}_+)$*

$$f(\overline{I_{P_1}(F)}) = \overline{I_{P_2}(F)}$$
$$f(\underline{I_{P_1}(F)}) = \underline{I_{P_2}(F)}.$$

Proof : Let $\phi_i : \Lambda_i \longrightarrow \mathbb{D}$ denote Riemann maps for $i = 1, 2$. Then the composite map $\phi_2 \circ f \circ \phi_1^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ is a degree d Blaschke product. In particular it has univalent restrictions to sufficiently small neighbourhoods of any point on the circle. Hence the Proposition is a Corollary of Proposition 3.6. **q.e.d.**

Example 3.12 For every degree d polynomial $Q \in C(\mathbb{R}_+)$ with leading coefficient $a > 0$ and for every $k \in \mathbb{R}$ the function $Q_k \in C(\mathbb{R}_+)$ is again a degree d polynomial with leading coefficient a, but with $a \cdot d \cdot k$ added to the coefficient of z^{d-1}. Hence the upper and lower impressions for Q are independent of the lower order terms of Q. Thus fixing $d \geq 1$ the upper and lower (left wing) impressions, obtainable by degree d polynomials, are parametrized by the monomials $a \cdot t^d$, $a > 0$. Moreover obviously for any $0 < a_1 < a_2$ and any univalent map $\phi : \mathbb{H} \longrightarrow \mathbb{C}$

$$\overline{I_\phi(\Gamma(a_1 t^d))} \subseteq \underline{I_\phi(\Gamma(a_2 t^d))}.$$

However fixing the growth order at^d, $d \in [1, \infty[$ at infinity, there appears to be still lots of room for structure: Suppose $f \in C^+(\mathbb{R}_+)$ satisfies

$$f(t), \frac{t}{f(t)} \longrightarrow \infty \qquad \text{as} \qquad t \longrightarrow \infty.$$

Let $F_-, F_+ \in C(\mathbb{R}_+)$ be any two functions of the form

$$\begin{aligned} F_-(t) &= at^d - f(t)t^{d-1} + \mathcal{O}(t^{d-1}) \\ F_+(t) &= at^d + f(t)t^{d-1} + \mathcal{O}(t^{d-1}), \end{aligned}$$

then for $0 < a_1 < a < a_2$:

$$\begin{aligned} \overline{I_\phi(\Gamma(a_1 t^d))} \subseteq \underline{I_\phi(\Gamma(F_-))} \subseteq \overline{I_\phi(\Gamma(F_-))} \subseteq \underline{I_\phi(\Gamma(at^d))} \subseteq \\ \overline{I_\phi(\Gamma(at^d))} \subseteq \underline{I_\phi(\Gamma(F_+))} \subseteq \overline{I_\phi(\Gamma(F_+))} \subseteq \underline{I_\phi(\Gamma(a_2 t^d))}. \end{aligned}$$

And for $a = 0 < a_2$

$$\overline{I_\phi(\Gamma(F_+))} \subseteq \underline{I_\phi(\Gamma(a_2 t^d))}.$$

4 Proof of distinguishability

Proposition 4.1 *There exists a univalent map $\phi : \mathbb{H} \longrightarrow U \subset \mathbb{H}$ such that $\forall\, 0 \le \delta \le \infty$:*

$$I_\phi(\Gamma(\delta \cdot)) = I_\phi(\Gamma_\delta) = [0, \delta]. \tag{4.1}$$

Note that combining Corollary 3.7 with Proposition 4.1 we obtain Theorem 3.2.

Proof : For $n \in \mathbb{N}$ define $y_n = 2^{-n}$ and $x_n = \frac{y_n}{2} e^{-\frac{\pi}{2y_n}}$. Define an open subset $U \subset \mathbb{H}$ by

$$U = \mathbb{H} \setminus \bigcup_{n \in \mathbb{N}} \{z = x + iy \,|\, y = y_n, |x| \ge x_n\}$$

and let $\phi : \mathbb{H} \longrightarrow U$ denote the Riemann map with $\phi(i) = i$ and $\phi(ri) \to 0$ as $r \to 0$. We shall prove that ϕ satisfies the equalities (4.1). The proof is based on three Claims, which are proved independently. Note that $i\mathbb{R}_+ = \phi(i\mathbb{R}_+)$ a hyperbolic geodesic in U. Moreover write $P = P_\phi(0)$, then clearly

$$\Pi(P) = I_\phi([i, 0]) = \{0\}, \qquad I(P) = \mathbb{R}$$

and the left wing of $I(P)$ equals $[0, \infty]$.

Let $z_0 = i$ and for $n \geq 1$ let $z_n = \frac{i}{2}(y_n + y_{n+1}) = i\frac{3}{4}y_n \in i\mathbb{R}_+$.

Claim 1 *There exists a constant $K_1 > \log 2$ such that $\forall n \geq 1$:*

$$\pi 2^n + \log 2 \leq \mathrm{d}_U(z_{n-1}, z_n) \leq \pi 2^n + K_1 \tag{4.2}$$

Claim 2 *There exists a constant $K_2 > 0$ such that $\forall\, 0 \leq \delta$ and $\forall\, n \geq 1$:*

$$\delta\pi 2^{n+1} - K_2 \leq \mathrm{d}_U(i\mathbb{R}_+, \delta + i]y_{n+1}, y_n[) \leq \mathrm{d}_U(z_n, z_n + \delta) \leq \delta\pi 2^{n+1} \tag{4.3}$$

For $w \in i\mathbb{R}_+$ let G_w denote the hyperbolic geodesic in U orthogonal to $i\mathbb{R}_+$ at w.

Claim 3 *Given $\epsilon > 0$ there exists $M_\epsilon > 0$ such that for all $n \geq 1$ and all $w \in [iy_{n+1}, iy_n]$ with $\mathrm{d}_U(w, z_n) > \epsilon$:*

$$\mathrm{diam}(G_w) \leq 2M_\epsilon(y_n - y_{n+1}),$$

where $\mathrm{diam}(\cdot)$ *denotes Euclidean diameter.*

Summing up (4.2) we obtain $\forall n \geq 1$:

$$\pi(2^{n+1} - 2) + n \log 2 \leq \mathrm{d}_U(z_n, z_0) \leq \pi(2^{n+1} - 2) + n(K_1 + \log 2) \tag{4.4}$$

Fix $0 < \delta_0 < \infty$ and consider $I_\phi(\Gamma(\delta_0 \cdot)) = I_\phi(\omega(\delta_0 \cdot))$. For each $n \in \mathbb{N}$ let G_{w_n}, denote the unique geodesic through $\delta_0 + z_n$ and orthogonal to $i\mathbb{R}_+$ (at w_n). We shall prove that the geodesic segment $G_{w_n}([w_n, z_n + \delta_0])$ between w_n and $z_n + \delta_0$ is contained in $\phi(\omega(\delta_0 \cdot))$ for n sufficiently big. To do so we just need to prove that its length $\mathrm{d}_U(w_n, z_n + \delta_0)$ is bounded by δ_0 times the distance $s_n = \mathrm{d}_U(z_0, w_n)$. Choose $N \in \mathbb{N}$ such that $n \log 2 > 2\pi + 1$ and (invoking Claim 3) such that $\mathrm{d}_U(z_n, w_n) \leq 1$. Then

$$s_n \geq \mathrm{d}_U(z_0, z_n) - \mathrm{d}_U(z_n, w_n) \geq \pi(2^{n+1} - 2) + n \log 2 - 1 \geq \pi 2^{n+1}$$

by the left hand inequality of (4.4). Hence invoking the right hand inequality of (4.3)

$$\delta_0 s_n \geq \delta_0 \pi 2^{n+1} \geq \mathrm{d}_U(i\mathbb{R}_+, z_n + \delta_0) = \mathrm{d}_U(w_n, z_n + \delta_0).$$

Thus $G_{w_n}([w_n, \delta_0 + z_n]) \subset \phi(\omega(\delta_0\cdot))$ for $n \geq N$. As $\phi(G_{w_n}([w_n, \delta_0 + z_n]))$ converges (Hausdorff) to $[0, \delta_0]$ we have proved that $[0, \delta_0] \subseteq I_\phi(\Gamma(\delta_0\cdot))$.

Let $\delta > \delta_0$ be arbitrary. We shall prove that there exists $N \in \mathbb{N}$ such that

$$\phi(\omega(\delta_0)) \cap \{z = x + iy | x \geq \delta, y \leq y_N\} = \emptyset \tag{4.5}$$

from which follows the inclusion $I_\phi(\Gamma(\delta_0\cdot)) \subseteq [0, \delta]$ and hence by arbitrariness of $\delta > \delta_0$ the inclusion $I_\phi(\Gamma(\delta_0\cdot)) \subseteq [0, \delta_0]$. This will complete the proof of (4.1), modulo the three Claims.

Choose $N \in \mathbb{N}$ such that for all $n \geq N$

$$\delta_0(\pi 2^{n+1} + n(K_1 + \log 2)) < \delta\pi 2^{n+1} - K_2 \quad \text{and} \quad M_1(y_n - y_{n+1}) < \delta. \tag{4.6}$$

Given $0 < y < y_N$ with $\delta + iy \in U$, we shall prove that $\delta + iy \notin \phi(\omega(\delta\cdot))$. Choose $n \in \mathbb{N}$ with $y_{n+1} < y < y_n$ and let $w_y \in i\mathbb{R}_+$ denote the unique point for which $\delta + iy \in G_{w_y}$. Then $\mathrm{d}_U(w_y, z_n) \leq 1$ by the right hand of (4.6) and Claim 3. Let $s_y = \mathrm{d}_U(z_0, w_y)$, then

$$s_y \leq \mathrm{d}_U(i, z_n) + \mathrm{d}_U(z_n, w_y) \leq \pi 2^n + n(K_1 + \log 2)$$

by the right hand inequality of (4.4). On the other hand the left hand inequality of (4.3) and the choice of N yields for $n \geq N$

$$\mathrm{d}_U(w_y, \delta + iy) \geq \delta\pi 2^{n+1} - K_2 > \delta_0(\pi 2^{n+1} + n(K_1 + \log 2)) \geq \delta_0 s_y$$

Hence $\delta + iy \notin \phi(\omega(\delta_0\cdot))$.

To complete the proof of the Proposition we shall prove each of the three Claims above. But first a Lemma used in the proofs of the Claims.

Lemma 4.2 *Let $U \subset V \subset \mathbb{C}$ be hyperbolic subsets and let λ denote the coefficient function of the hyperbolic metric on U relative to the hyperbolic metric on V. Then for all $z \in U$*

$$1 \leq \lambda(z) \leq \coth(\frac{1}{2}\,\mathrm{d}_V(z, \partial U)),$$

where $d_V(\cdot,\cdot)$ *denotes hyperbolic distance in* V. *Moreover for any piecewise differentiable curve* $\gamma : [0, L] \longrightarrow U$, *which is parametrized by hyperbolic arclength in* V *and which satisfies* $d_V(\gamma(t), \partial U) \geq mt + l$ *for some positive constants* $m, l > 0$:

$$L \leq l_U(\gamma) \leq L + \frac{2}{m} \log \frac{1}{1 - e^{-l}},$$

where $l_U(\cdot)$ *denotes hyperbolic arclength in* U.

Proof : The first estimate is standard and can be found by considering a universal cover of V. For the second we obtain by integration, as γ is parametrized by hyperbolic arclength in V :

$$\begin{aligned} l_U(\gamma) &= \int_0^L \lambda(t)dt \leq \int_0^L \coth(\frac{1}{2}(mt + l))dt \\ &= \left[1 + \frac{2}{m} \log(1 - e^{-(mt+l)})\right]_0^L . \end{aligned}$$

Thus the result follows. **q.e.d.**

Proof of Claim 1: The case $n = 1$ needs a little extra care, which is left to the reader. Define

$$\begin{aligned} U_n &= \mathbb{C}\backslash\{z = x + iy | y = y_n, |x| \geq x_n\} \\ V_n &= \{z = x + iy | y_{n+1} < y < y_{n-1}\} \cap U_n \end{aligned}$$

so that $[z_{n-1}, z_n] \subset V_n \subset U \subset U_n$ and hence

$$d_{U_n}(z_{n-1}, z_n) \leq d_U(z_{n-1}, z_n) \leq d_{V_n}(z_{n-1}, z_n) \tag{4.7}$$

Note that the map $z \mapsto \psi(z) = \frac{x_n}{2}(z + 1/z) + iy_n : \mathbb{H} \longrightarrow U_n$ is biholomorphic. Thus by elementary calculations we find that

$$\begin{aligned} \pi 2^{n-1} &= \log 2 \frac{|z_n - iy_n|}{x_n} \leq d_{U_n}(z_n, iy_n) \\ &\leq \log(2 \frac{|z_n - iy_n|}{x_n} + 1) \leq \pi 2^{n-1} + e^{-(\pi 2^{(n-1)})} . \end{aligned}$$

Similarly we find that

$$\log 2+\pi 2^{n-1} = \log 2\frac{|z_{n-1}-iy_n|}{x_n} \le \mathrm{d}_{U_n}(z_{n-1}, iy_n)$$
$$\le \log(2\frac{|z_{n-1}-iy_n|}{x_n}+1) \le \log 2+\pi 2^{n-1}+\frac{1}{2}\mathrm{e}^{-(\pi 2^{(n-1)})}.$$

As $i\mathbb{R}_+$ is a geodesic we obtain by adding the two pairs of inequalities

$$\log 2+\pi 2^n \le \mathrm{d}_{U_n}(z_{n-1}, z_n) \le \log 2+\pi 2^n+\frac{3}{2}\mathrm{e}^{-(\pi 2^{(n-1)})}.$$

This provides the left hand inequality of (4.2). Moreover to complete the proof of Claim 1 we need only prove that there exists $K>0$ such that

$$l_{V_n}([z_{n-1}, z_n]) \le l_{U_n}([z_{n-1}, z_n])+K,$$

because of the inequalities (4.7). To this end note that $]iy_{n-1}, iy_{n+1}[$ is a geodesic arc in both U_n and V_n and let w_n denote its hyperbolic midpoint. Let $l_n^o=\mathrm{d}_{U_n}(z_{n-1}, iy_{n-1})$ and $l_n^u=\mathrm{d}_{U_n}(z_n, iy_{n+1})$ denote the U_n distances from the extreme points to the boundary of $V_n \subset U_n$ and parametrize the two geodesic arcs $[z_{n-1}, w_n]$, $[z_n, w_n]$ by U_n-hyperbolic arc length. Then Lemma 4.2 yields

$$l_{V_n}([z_{n-1}, z_n]) \le l_{U_n}([z_{n-1}, z_n])+2(\log\frac{1}{1-\mathrm{e}^{-l_n^o}}+\log\frac{1}{1-\mathrm{e}^{-l_n^u}})$$
$$\underset{n\to\infty}{\longrightarrow} l_{U_n}([z_{n-1}, z_n])+4\log 2,$$

where we have used that $l_n^o, l_n^u \underset{n\to\infty}{\longrightarrow} \log 2$, as

$$\frac{iy_n-iy_{n+1}}{iy_n-z_n}=\frac{iy_{n-1}-iy_n}{z_{n-1}-iy_n}=2.$$

This completes the proof of the Claim. **q.e.d. (of Claim 1)**

Proof of Claim 2: The upper bound, the right hand side of (4.3) follows from the fact that U contains the strip $\{z=x+iy|y_{n+1}<y<y_n\}$ of width $2^{-(n+1)}$. The middle inequality is trivial. Finally for the left hand inequality one may procede as follows. Define simply connected domains

$$U_n=\mathbb{C}\backslash\{z=x+iy||y|=\pi/2, |x|\ge \pi\,\mathrm{e}^{-\frac{\pi}{2y_n}}\}.$$

Given $n \geq 1$ the image of U under the affine map

$$A_n(z) = \frac{2\pi}{y_n}(z - z_n) = \pi 2^{n+1}(z - z_n) \tag{4.8}$$

is contained in $U_n \subset U_1$. Moreover a point $z = \delta + iy$, $y_{n+1} < y < y_n$ is mapped by A_n to a point $z' = \delta\pi 2^{n+1} + y'$, with $|y'| < \pi/2$. Hence it suffices to find $K_2 > 0$ such that for all $L \geq 0$ and for all $|y| < \pi/2$:

$$\mathrm{d}_{U_1}(i\mathbb{R}, L + iy) \geq L - K_2. \tag{4.9}$$

Let $\psi : U_1 \longrightarrow S = \{z = x + iy \| y| < \pi/2\}$ denote the biholomorphic map with $\psi(0) = 0$ and $\psi'(0) > 0$. So that $\psi(\mathbb{R}) = \mathbb{R}$ and $\psi(i\mathbb{R}) = i\mathbb{R}$. By symmetry and the Schwartz reflection principle, the restriction of ψ to $S \subset U_1$ extends to a $i\pi$ periodic univalent map

$$\widetilde{\psi} : \mathbb{C} \backslash \{z = x + iy \| x| \leq \pi \, \mathrm{e}^{-\frac{\pi}{2y_1}}, y \in (\pi/2 + \pi\mathbb{Z})\} \longrightarrow \mathbb{C}.$$

Hence there exists $\alpha \in \mathbb{R}$ such that

$$|\psi(x + iy) - (x + iy) - \alpha| = \mathcal{O}(\mathrm{e}^{-x})$$

for $x \geq 0, |y| < \pi/2$. As ψ is a hyperbolic isometry the lower bound (4.9) easily follows. Completing the proof of the Claim **q.e.d. (of Claim 2)**

Proof of Claim 3: We modify slightly the construction above. Let A_n be given by (4.8) above, but define $U_n = A_n(U)$. Moreover let $\psi_n : U_n \longrightarrow S$ denote the biholomorphic map with $\psi_n(i\mathbb{R} \cap U_n) = i[-\pi/2, \pi/2]$ and with $\psi(\mathbb{R})$ tangent to $\mathbb{R}$ at both ends. Similarly to above the restriction of ψ_n to $S \subset U_n$ extends to (this time) a $i2\pi$ periodic univalent map

$$\begin{aligned}\widetilde{\psi_n} : \mathbb{C} \backslash \big\{ z = x + iy \,\big|\, |x| \leq \pi \, \mathrm{e}^{-\frac{\pi}{2y_n}}, y \in (\pi/2 + 2\pi\mathbb{Z}), \\ |x| \leq \tfrac{\pi}{2} \mathrm{e}^{-\frac{\pi}{2y_{n+1}}}, y \in (-\pi/2 + 2\pi\mathbb{Z}) \big\} \longrightarrow \mathbb{C}.\end{aligned}$$

Since the cuts converge (Hausdorff) to the points $i(\pi/2 + \pi\mathbb{Z})$. The maps $\widetilde{\psi_n}$ converge to the identity uniformly on $\mathbb{C} \backslash (\cup_{n \in \mathbb{Z}} \overline{\mathbb{D}(i(\pi/2 + 2\pi n), r)})$ and in particular on $S \backslash (\overline{\mathbb{D}(i\pi/2, r)} \cup \overline{\mathbb{D}(-i\pi/2, r)})$ for any $r > 0$. Hence the Claim easily follows. **q.e.d. (of Claim 3)**

q.e.d.

Conjecture 1 (weak) *Let $F_0 \leq F_1 \in C(\mathbb{R}_+)$ be any pair of increasing and strongly inequivalent Lipschitz functions, i.e. $F_1(s) - F_0(s) \longrightarrow \infty$ as $s \to \infty$. Then there exists a simply connected hyperbolic domain $U \subset \overline{\mathbb{C}}$ and a prime end $P \in \mathcal{P}(U)$ for which*

$$\{0\} = \Pi(P) \subsetneqq I_P(F_0) \subsetneqq I_P(F_1) \subsetneqq I(P).$$

Conjecture 2 (strong) *Let $F_0 \leq F_1 \in C(\mathbb{R}_+)$ be any pair of increasing and strongly inequivalent Lipschitz functions.*

For $t \geq 0$ let $F_t = tF_1 + (1-t)F_0 \in C^+(\mathbb{R}_+)$ Then there exists a simply connected hyperbolic domain $U \subset \overline{\mathbb{C}}$ and a prime end $P \in \mathcal{P}(U)$ for which

$$\{0\} = \Pi(P) \subsetneqq I_P(F_{t_1}) \subsetneqq I_P(F_{t_2}) \subsetneqq I(P)$$

for any $0 \leq t_1 < t_2$.

5 Results in the Non Lipschitz range

This final section proves some further results and concludes by an open question for non Lipschitz functions $F \in C^+(\mathbb{R}_+)$. In order to facilitate this discussion we shall move the scene from the upper half plane $\mathbb{H}$ to the strip $S = \{z = x + iy | 0 < y < \pi\}$ via Log. Let $g = \{z = x + iy | y = \pi/2\}$ denote the central translation invariant geodesic in S corresponding to $i\mathbb{R}_+$ under Log. Note also that the geodesics $]\log r, \log r + i\pi[$ orthogonal to g corresponds to the geodesics $C_r = \{z || z| = r\} \subset \mathbb{H}$ orthogonal to $i\mathbb{R}_+$. We shall use frequently the following elementary estimate of hyperbolic distances:

$$\log \pi - \log 2 - \log y \leq \mathrm{d}_S(x + iy, g) = -\log \tan \frac{y}{2} \leq \log 2 - \log y, \qquad (5.1)$$

for any $0 < y \leq \pi/2$ and any $x \in \mathbb{R}$.

For $F \in C(\mathbb{R}_+)$ define $\Upsilon(F) = \{z = x + iy | x \leq 0; y = \mathrm{e}^{-F(-x)}\}$ and for $F \in C^+(\mathbb{R}_+)$ define $\varpi(F) = \{z = x + iy | x \leq 0; \mathrm{e}^{-F(-x)} \leq y \leq \pi/2\}$. (We remark that $\Upsilon(F)$ is an actual graph of a function.)

Lemma 5.1 *For any function $F \in C^+(\mathbb{R})$ the two sets $\Upsilon(F)$ and $\mathrm{Log}(\Gamma(F))$ are Hausdorff equivalent in S. Moreover so are the two sets $\varpi(F)$ and $\mathrm{Log}(\omega(F))$*

Proof : It follows from (5.1) above that

$$D_S(\Upsilon(F), \operatorname{Log}\Gamma(F)) \leq \log 2.$$

This proves the first statement from which also the second follows. **q.e.d.**

To study Hausdorff equivalence of sets of the form $\Gamma(F), \omega(F) \subset \mathbb{H}$ and Hausdorff equivalence of the corresponding sets $\Upsilon(F), \varpi(F) \subset S$ is equivalent by the above Lemma 5.1. We shall henceforth do the later.

In the following we shall use freely the notions and results of extremal lengths, as can be found in e.g. [L-V]. We shall study simply connected domains $U \subset \mathbb{C}$, which are $l > 0$ periodic, that is invariant under translation by l, and which do not contain either an upper or a lower half plane. For such a domain there exists a unique l periodic hyperbolic geodesic g_U, which converge to ∞ at one end and $-\infty$ at the other end. We denote by P_∞ and $P_{-\infty}$ the corresponding two prime ends. Note that the quotient $U/(l \cdot \mathbb{Z})$ is a cylinder with core geodesic $g_U/(l \cdot \mathbb{Z})$ and a natural Euclidean infinitessimal metric induced from U. Let $\operatorname{Area}(U/l \cdot \mathbb{Z}) = A < \infty$ then by Rengels inequality

$$\operatorname{mod}(U/l \cdot \mathbb{Z}) := \frac{\pi}{l'} \leq \frac{\operatorname{Area}(U/l \cdot \mathbb{Z})}{l^2}$$

where $l' > 0$ and the equality sign holds, if and only if U is a straight strip of height A/l.

Theorem 5.2 *Let $U \subset \mathbb{C}$ be a simply connected and $l > 0$ periodic domain. If $\operatorname{Area}(U/l \cdot \mathbb{Z}) = A < \infty$ then for every $c > 0$*

$$\overline{I_{P_{-\infty}}(c \cdot s^2)} = \Pi(P_{-\infty}) = \{-\infty\}.$$

Proof : Let $\phi : S \longrightarrow U$ be a biholomorphic map with $\phi(z + l') = \phi(z) + l$, where $\frac{\pi}{l'} = \operatorname{mod}(U/l \cdot \mathbb{Z})$. We shall prove that for all $c > 0$

$$I_\phi(\Upsilon(c \cdot x^2)) = \{\infty\} \qquad \text{in} \quad \widehat{\mathbb{C}}.$$

Then the Theorem follows because for any $0 < c_1 < c < c_2$ and any $k \in \mathbb{R}$

$$c_2 \cdot s^2 - (c \cdot s^2)_k, (c \cdot s^2)_k - c_1 \cdot s^2 \longrightarrow \infty \qquad \text{as} \qquad s \longrightarrow \infty.$$

We shall prove more precisely that there exists a constant $K_c > 0$ depending only on $c > 0$, such that

$$\phi(\varpi(c \cdot x^2) \cap \{z = x + iy | x \leq x_0\}) \subset \{z = x + iy | x \leq \frac{l}{2l'} x_0 + K_c\}. \quad (5.2)$$

Let $K_0 = \sup\{|\phi(x + i\pi/2) - xl/l'||0 \leq x \leq l'\}$ so that

$$\forall x \in \mathbb{R} : \qquad \Re(\phi(x + i\pi/2) - xl/l') \geq -K_0.$$

Let $0 < c$ be arbitrary and choose $r_c > \log 4/\pi$ such that

$$\text{Area}(\phi(\{z = x + iy | \, \mathrm{e}^{-r_c} < y < \pi - \mathrm{e}^{-r_c}\})/l \cdot \mathbb{Z}) \geq A - \frac{l^2}{4cl'^2}$$

and such that the hyperbolic geodesic orthogonal to $]0, i\pi[$ at $i\,\mathrm{e}^{-r_c}$ is contained in the strip $\{z = x + iy||x| < \frac{l'}{2}\}$. Let $x_0 = -\sqrt{r_c/c}$ so that $cx^2 \geq r_c$ for all $x \leq x_0$. Moreover let $L = 2\,\mathrm{d}_S(i\pi/2, i(\pi - 1)/2)$.

Let g_x denote the vertical hyperbolic geodesic $]x, x + i\pi[$, $x \in \mathbb{R}$. For $x \leq x_0$ and $s \geq r_c$ let $R_x(r_c, s)$ denote the quadrilateral, which is symmetric with respect to g_x, whose a-sides are the length L segments, of the geodesics orthogonal to g_x at $x + i\,\mathrm{e}^{-r_c}$ and $x + i\,\mathrm{e}^{-s}$ and whose b-sides are the segments of the (hyperbolic) distance $L/2$ parallels to g_x connecting the a-sides. Then

$$\frac{1}{\text{mod}(R_x(r_c, s))} \leq s - r_c + \log 4/\pi \leq s.$$

In fact under the unique automorphism ψ_x of S with $\psi_x(x + i\pi/2) = i\pi/2$ and $\psi'(x + i\pi/2) = i$ the quadrilateral $R_x(r_c, s)$ maps to the (Euclidean) width 1 rectangular quadrilateral symmetric with respect g, whose a-sides are the vertical geodesic segments with x-coordinates $x_1 = \mathrm{d}_S(i\pi/2, i\,\mathrm{e}^{-r_c})$ and $x_2 = \mathrm{d}_S(i\pi/2, i\,\mathrm{e}^{-s})$. Hence the modulus estimate follows from (5.1).

The quadrilateral $R_x(r_c, s)$ is contained in a fundamental domain and is located outside $\{z = x + iy | \, \mathrm{e}^{-r_c} < y < \pi - \mathrm{e}^{-r_c}\}$. Thus

$$\text{Area}(R_x(r_c, s)) \leq A - (A - \frac{l^2}{4cl'^2}) = \frac{l^2}{4cl'^2}.$$

Let $\Delta_{x,s}$ denote the set of curves γ in $R_x(r_c, s)$ connecting the a-sides and let $l(\cdot)$ denote the Euclidean curve length. Then Rengels inequality implies that for $r_c < s \leq cx^2$

$$\inf_{\gamma\in\Delta_{x,s}} (l(\gamma))^2 \leq \frac{\text{Area}(R_x(r_c,s))}{\text{mod }(R_x(r_c,s))} \leq \frac{l^2}{4cl'^2}\cdot(s-r_c+\log 4/\pi)$$
$$\leq \frac{l^2}{4cl'^2}\cdot s \leq \frac{l^2}{4cl'^2}(c\cdot x^2) \leq \left(\frac{l}{2l'}x\right)^2$$

Hence there exists $\gamma \in \Delta_{x,s}$ with $l(\gamma) \leq \frac{l}{2l'}x$. The hyperbolic length of each half each of the a-sides of $R_x(r_c,s)$ is $\frac{L}{2}$ and the hyperbolic length of the arc $\phi([x+i\pi/2, x+i\,\mathrm{e}^{-r_c}])$ is bounded by $r_c + \log 2$. Moreover $\lambda_U(z) \geq 1/\lambda_0$, for all $z \in U$, (where λ_U denotes the coefficient of the infinitessimal hyperbolic metric and $\lambda_0 = \max\{l, A/l\}$) because U is simply connected and l-periodic with quotient area A. Thus

$$\phi([x+i\pi/2, x+i\,\mathrm{e}^{-cx^2}]) \subset \{u+iv|u \leq \frac{l}{2l'}x + K_0 + \lambda_0(L+r_c+\log 2)\}.$$

This proves (5.2) with $K_c = K_0 + \lambda_0(L + r_c + \log 2)$ from which the Theorem follows. **q.e.d.**

Theorem and Example 5.3 *Let $F \in C(\mathbb{R}_+)$ be a continuously differentiable with $F' \in C^+(\mathbb{R}_+)$ non zero and increasing. Suppose*

$$\int_0^\infty \frac{1}{F'(x)}dx < \infty.$$

Then given $A, l, l' > 0$ with $\frac{\pi}{l'} < \frac{A}{l}$, there exists a simply connected, l periodic domain $U \subset \mathbb{C}$, with $\text{Area}(U/l\cdot\mathbb{Z}) = A$ and $\text{mod}(U/l\cdot\mathbb{Z}) = \pi/l'$ for which

$$\overline{I_{P_{-\infty}}(F)} = \mathbb{R}.$$

Proof : We shall prove that there exists a biholomorphic map $\phi : S \longrightarrow U$ with $\phi(z + l') = \phi(z) + l$ and

$$\mathbb{R}_- \subsetneq I_\phi(\varpi(F)), \tag{5.3}$$

from which the Theorem follows. Define a decreasing function $f : [l, \infty] \longrightarrow \mathbb{R}_+$ by $f(x) = \dfrac{8}{l'\cdot F'((x-l)\cdot l'/l)}$ and extend f to all of $\mathbb{R}$ by defining $f(x) = f(l)$

for all $x \leq l$. Moreover define a decreasing homeomorphism $H : \mathbb{R} \longrightarrow \mathbb{R}_+$ by

$$H(x) = \int_x^\infty f(t)dt = \frac{8l}{l'^2} \int_{\frac{l'x}{l}-l'}^\infty \frac{1}{F'(t)} dt$$

so that $H'(x) = -f(x)$.

For $x \in \mathbb{R}$ define a "hair" $\Delta_x = \{z = u + iv | u \geq x, v = H(u)\}$. Let $h = A/l$ and define

$$U_x = \{z = u + iv | 0 < v < h\} \setminus \left(\bigcup_{n \in \mathbb{Z}} (n \cdot l + \Delta_x) \right)$$

so that U_x is l-periodic and $\mathrm{Area}(U_x/l \cdot \mathbb{Z}) = A$. Let $x_0 = H^{-1}(h)$ then U_x is a simply connected domain for $x > x_0$. Moreover the modulus $\mod (U_x/l \cdot \mathbb{Z})$ varies continuously and monotonicly from 0 to A/l as x varies from x_0 to ∞. Let $x_1 > x_0$ denote the value for which the modulus equals π/l'. Define $U = U_{x_1}$ and let $\phi : S \longrightarrow U$ be a univalent map with $\phi(z + l') = \phi(z) + l$.

Let $g_U = \phi(g)$ denote the unique $z \mapsto z + l$ invariant hyperbolic geodesic in U. Let δ denote the unique hyperbolic geodesic in U orthogonal to g_U and connecting $x_1 + iF(x_1)$ to the upper boundary of U. Let z_0 denote the unique intersection point of g_U and δ. Precomposing ϕ by a real translation if necessary, we can suppose $\phi(i\pi/2) = z_0$. We shall see that the pair U, ϕ satisfies (5.3).

Define a differentiable curve $\gamma : [x_1, \infty[\longrightarrow U$ by

$$\gamma(t) = t + \frac{i}{2}(H(t) + H(t + l)).$$

We shall prove that there exists $c_0 \in \mathbb{R}$ such that

$$\forall\, t \geq x_1 : \qquad d_U(\gamma(l \cdot t), g_U) \leq \frac{7}{8} F(l' \cdot t) + c_0. \tag{5.4}$$

The inclusion (5.3) easily follows from (5.4) as follows. Let n_0 be bigger than or equal to x_1/l. For $n \geq n_0$ define $w_n = \gamma(n \cdot l) - n \cdot l \in U \cap i\mathbb{R}_+$. Let κ_n denote the unique hyperbolic geodesic segment realizing $d_U(w_n, g_U)$ and let z_n denote the endpoint of κ_n on g_U. Then $\Re(\phi^{-1}(z_n)) < -n \cdot l'$, by the definition of w_n and the chosen normalization of ϕ. Increasing n_0, if necessary we can suppose $F(n_0 \cdot l')/8 > c_0$. Then for all $n \geq n_0$ $\kappa_n \subset \phi(\varpi(r))$. Evidently this implies $\mathbb{R}_- \subset \overline{\cup_{n \geq 1} \kappa_n}$, so that (5.3) and hence the Theorem follows.

Let us proceed to prove (5.4). Choose $t_0 \geq \max\{x_1, 0\}$ such that $f(t_0) \leq \frac{1}{2}$. Then $0 < f(t + l) \leq f(t_0) < \frac{1}{2}$ for all $t \geq t_0$.

Claim 4 *For all $t \geq t_0$ the set U contains a Euclidean disc with center $\gamma(t)$ and radius $\frac{l}{3} \cdot f(t+l)$.*

Proof of Claim: Let

$$s(t) = \frac{1}{2}(H(t) - H(t+l)) = \frac{1}{2}\int_t^{t+l} f(x)dx \geq \frac{l}{2}f(t+l).$$

Then U contains the set

$$\begin{aligned}\Delta(t) = \{z = x + iy | t - s(t) < x < t + s(t), \\ H(t+l) - \frac{1}{2}(x-t) < y < H(t) \quad \text{for} \quad x \leq t, \\ H(t+l) < y < H(t) - \frac{1}{2}(x-t) \quad \text{for} \quad x \geq t\}.\end{aligned}$$

Evidently the set $\Delta(t)$ contains a circle of radius

$$\frac{1}{\sqrt{2}}s(t) \geq \frac{l}{2\sqrt{2}}f(t+l) \geq \frac{l \cdot f(t+l)}{3}$$

around $\gamma(t)$. This proves the Claim **q.e.d. (of Claim)**

Secondly we compute

$$\begin{aligned}|\gamma'(t)| &= \sqrt{1 + \frac{1}{4}(f(t) + f(t+l))^2} \\ &\leq \sqrt{1 + f(t)^2} \leq 1 + \frac{1}{2}f(t)^2 \\ &\leq 1 + \frac{1}{8} = \frac{9}{8} \quad \text{for} \quad t \geq t_0.\end{aligned}$$

Thus for $lT \geq t_0$ we have

$$\begin{aligned}l_U(\gamma([t_0, lT])) &\leq \int_{t_0}^{lT} |\gamma'(t)| \cdot \lambda_U(\gamma(t))dt \\ &\leq \int_{t_0}^{lT} \frac{9}{8} \cdot \frac{2}{\frac{l}{3} \cdot f(t+l)}dt \\ &\leq \frac{54 \cdot l'}{8 \cdot 8l}\int_{t_0}^{lT} F'(t \cdot l'/l)dt \leq \frac{7}{8}(F(l' \cdot T) - F(t_0 \cdot l'/l)).\end{aligned}$$

To complete the proof of (5.4) we let $c_0 = \mathrm{d}_U(\gamma(t_0), g_U) - \frac{7}{8}F(t_0 \cdot l'/l)$. Then (5.4) holds for all $t \geq t_0$. **q.e.d.**

Conjecture 3 *Under the assumption $F \in C(\mathbb{R}_+)$ continuously differentiable with $F' \in C^+(\mathbb{R}_+)$ non zero and increasing the integral condition*

$$\int_0^\infty \frac{1}{F'(t)} dt = \infty \tag{5.5}$$

is optimal. That is $\overline{I_{P_{-\infty}}(F)} = \Pi(P_{-\infty})$ for any simply connected, $l > 0$ periodic and hyperbolic domain U, if and only if the condition (5.5) holds.

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"Lineære strukturer fra algebra og analyse"

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Eksamensopgaver fra 1976-93

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To Sigurdur Helgason on his sixtyfifth birthday
by: Jacques Faraut, Joachim Hilgert and Gestur Olafsson

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Fysikspeciale af: Anja Boisen, Peter Bøggild, Karen Birkelund
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af: Anders Marcussen, Anne C. Nilsson, Lone Michelsen, Per M. Hansen
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af: Tine Guldager Christiansen, Ken Andersen, Nikolaj Hermann, Jannik Rasmussen
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Opdaget eller opfundet
NAT-BAS-projekt
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af: Kristian Hoppe og Jeppe Guldager
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Verifikation af model
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(en kaotisk talgenerator)

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En specialerapport af Peter Hauge Jensen og Linda Kyndlev

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Et 3.modul matematik projekt

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291/95 1st Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

an example of using methods developed for the OECD/IEA and the US/EU fuel cycle externality study

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af: Bent Sørensen

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af: Lotte Ludvigsen & Jens Frandsen

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Af: Jesper Duelund og Birthe Friis

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Et matematisk modelprojekt

af: Claus Dræby, Michael Hansen, Tomas Højgård Jensen

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by: Bent Sørensen

301/95 TOK Modellering af træers vækst under påvirkning af ozon

af: Glenn Møller-Holst, Marina Johannessen, Birthe Nielsen og Bettina Sørensen

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302/95 KOMPRESSORER - Analyse af en matematisk model for aksialkompressorer

Projektrapport af: Stine Bøggild, Jakob Hilmer, Pernille Postgaard

Vejleder: Viggo Andreasen

303/95 Masterlignings-modeller af Glasovergangen
Termisk-Mekanisk Relaksation
Specialerapport udarbejdet af:

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304b/95 STATISTIKNOTER Simple normalfordelingsmodeller
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304c/95 STATISTIKNOTER Simple Poissonfordelingsmodeller
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Preprint of a chapter for the forth-
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Presented at the semiotic symposium

The Emergence of Codes and Intensions as
a Basis of Sign Processes

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af: Marianne Wilcken Bjerregaard,
Frederik Voetmann Christiansen,
Jørn Skov Hansen, Klaus Dahl Jensen
Ole Schmidt

Vejledere: Peder Voetmann Christiansen og
Petr Viscor

309/95 Tiden og Tanken
En undersøgelse af begrebsverdenen Matematik
udført ved hjælp af en analogi med tid

af: Anita Stark og Randi Petersen

Vejleder: Bernhelm Booss-Bavnbek

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algebra og analyse" (E1)

af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH
ENERGY SYSTEM

by: Hélène Connor-Lajambe, Bernd Kuemmel,
Stefan Krüger Nielsen, Bent Sørensen

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by: B. Booss-Bavnbek, K.P.Wojciechowski

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THE OPENNESS OF THE FUTURE
The Logical Function of Idealism in Peirce's
Philosophy of Nature

By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian
Cardiovascular System

By: Johnny T. Ottesen

315/96 a + b "Rejsen til tidens indre" - Udarbejdelse af
et manuskript til en fjernsynsudsendelse
+ manuskript

af: Gunhild Hune og Karina Goyle

Vejledere: Peder Voetmann Christiansen og
Bruno Ingemann

316/96 Plasmaoscillation i natriumklynger
Specialerapport af: Peter Meibom, Mikko Østergård
Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer
af: Ulla Rasmussen
Vejleder: Anders Madsen

318/96 Modelling the Respiratory System
by: Tine Guldager Christiansen, Claus Dræby
Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming
Impacts

by: Bent Sørensen

320/96 Grassmannian and Boundary Contribution to the
-Determinant
by: K.P.Wojciechowski et al.

321/96 Modelkompetencer - udvikling og afprøvning
af et begrebsapparat

Specialerapport af: Nina Skov Hansen,
Christine Iversen, Kristin Troels-Smith

Vejleder: Morten Blomhøj

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Bredde-Kursus i Fysik 1976 - 1996

323/96 Structure and Dynamics of Symmetric Diblock
Copolymers
PhD Thesis
by: Christine Maria Papadakis

324/96 Non-linearity of Baroreceptor Nerves
by: Johnny T. Ottesen

325/96 Retorik eller realitet ?
Anvendelser af matematik i det danske
Gymnasiums matematikundervisning i
perioden 1903 - 88

Specialerapport af Helle Pilemann
Vejleder: Mogens Niss

326/96 Bevisteori
Eksemplificeret ved Gentzens bevis for
konsistensen af teorien om de naturlige tal

af: Gitte Andersen, Lise Mariane Jeppesen,
Klaus Frovin Jørgensen, Ivar Peter Zeck

Vejledere: Bernhelm Booss-Bavnbek og
Stig Andur Pedersen

327/96 NON-LINEAR MODELLING OF INTEGRATED ENERGY
SUPPLY AND DEMAND MATCHING SYSTEMS

by: Bent Sørensen

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329/96 The dynamics of cocirculating influenza strains conferring partial cross-immunity

and

A model of influenza A drift evolution

by: Viggo Andreasen, Juan Lin and Simon Levin

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by: Bent Sørensen

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Specialerapport af:

Vibeke Orlien og Christina Specht

Vejledere: Jacob M. Jacobsen og Jesper Larsen

332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG

Specialerapport af:

Stine Sofia Korremann

Vejleder: Dorthe Posselt

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an extension of methods found in the literature on monetisation of biodiversity

by: Bernd Kuemmel

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by: Bernd Kuemmel and Bent Sørensen

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by: Kathrine Legge

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- et projekt om matematisk modellering

af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen

Vejleder: Jørgen Larsen

338/97 Kvantisering af nanolederes elektriske ledningsevne

Første modul fysikprojekt

af: Søren Dam, Esben Danielsen, Martin Niss, Esben Friis Pedersen, Frederik Resen Steenstrup

Vejleder: Tage Christensen

339/97 Defining Discipline

by: Wolfgang Coy