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STABILIZATION OF PARTIAL DIFFERENTIAL EQUATIONS BY FINITE DIMENSIONAL BOUNDARY FEEDBACK CONTROL:

A pseudo-differential approach.

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ABSTRACT.
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STABILIZATION OF PARTIAL DIFFERENTIAL EQUATIONS BY FINITE DIMENSIONAL BOUNDARY FEEDBACK CONTROL:
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ABSTRACT

We consider stabilization of Dirichlet evolution boundary problems by finite dimensional boundary feedback. A pseudo-differential operator method is used to transform the original operator domains into "classical" domains, where standard operator theory can be employed.

§ 1: INTRODUCTION.

We consider stabilization of parabolic and hyperbolic partial differential equations of the form

$$\begin{cases} \partial_t u + Au = 0 & \text{in } \Omega, \text{ for } t > 0 \\ \gamma u = 0 & \text{on } \Gamma, \text{ for } t > 0 \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0. \end{cases} \tag{1.1}$$

$$\begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega, \text{ for } t \in \mathbb{R} \\ \gamma u = 0 & \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1.2}$$

A is formally selfadjoint, uniformly strongly elliptic differential operator of order $2m$ with C^∞-coefficients on a bounded, open domain $\Omega \subset \mathbb{R}^n$, with smooth boundary $\partial\Omega = \Gamma$. Here $\gamma = \{\gamma_j\}_{0 \le j < m}$ is the <u>Dirichlet trace operator</u>.

$$\gamma_j u = \left(\frac{\partial}{\partial n}\right)^j u\Big|_\Gamma \tag{1.3}$$

We denote similarly $\nu = \{\gamma_j\}_{m \leq j < 2m}$ the <u>Neumann trace operator</u>.

The Dirichlet realization A_γ of A is the operator acting like A in $L^2(\Omega)$ and with domain

$$D(A_\gamma) = \{u \in H^{2m}(\Omega) \,|\, \gamma u = 0\} = H^{2m}(\Omega) \cap H_0^m(\Omega) \ , \tag{1.4}$$

where $H^s(\Omega)$ is the Sobolev space of L^2-functions with L^2-derivatives up to order s. It is well known that A_γ is an unbounded, selfadjoint operator in $L^2(\Omega)$, with a sequence of real eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \to \infty$. We see that (1.1) and (1.2) are the time dependent evolution problems associated with A_γ, generalizing the heat equation resp. the wave equation. When $\lambda_1 > 0$, all solutions $u(t,x)$ of (1.1) are exponentially decreasing for $t \to \infty$, and all solutions $u(t,x)$ of (1.2) are bounded, we call this the <u>stable</u> case. However, if some eigenvalues are negative, there are solutions, both of (1.1) and (1.2), that blow up exponentially for $t \to \infty$. It is therefore of interest to investigate how one can change the systems to obtain the stable case.

By a <u>perturbation of the first kind</u> of the system (1.1), we will understand a system of the form

$$\begin{cases} \partial_t u + Au + Gu = 0 & \text{in } \Omega \text{, for } t > 0 \\ \gamma u = 0 & \text{on } \Gamma \text{, for } t > 0 \\ u = u_0 & \text{in } \Omega \text{, at } t = 0 \end{cases} \tag{1.5}$$

where the interior operator A is replaced by $A + G$, where G has finite rank.

Stabilization of the a priori unstable system (1.1) by a perturbation of the first kind has been studied e.g. in Nambu [9] and Triggiani [11].

It is shown there that under suitable circumstances, it is possible to choose G of finite rank, such that (1.5) is stable.

By a <u>perturbation of the second kind</u> of the system (1.1), we will understand a system of the form

$$\begin{cases} \partial_t u + Au = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \gamma u = T'u & \text{on } \Gamma \text{ , for } t > 0 \\ u = u_0 & \text{in } \Omega \text{ , at } t = 0 \end{cases} \tag{1.6}$$

where the boundary operator γ is replaced by $\gamma - T'$, T' being of finite rank. More specifically, T' is an operator applied to functions u on Ω , wheras G should be an operator on the boundary values at Γ (the relevant boundary values are the Cauchy-data, $\zeta u = \{\gamma u, \nu u\}$, i.e. G is of the form $K\zeta$, where K maps from the boundary to the interior.
We define perturbations of the system (1.2) analogously.

We will refer to (1.5) and (1.6) as boundary feedback systems, when T' is of the special form

$$T'u = \sum_{j=1}^{N} (u|w_j)g_j \tag{1.7}$$

(here $(\cdot|\cdot)$ is the usual $L^2(\Omega)$-inner product, $w_j \in C^\infty(\bar{\Omega})$, $g_j \in C^\infty(\Gamma)^m$, $j = 1,2,\dots,N$) and T' is called a finite dimensional feedback operator.
Boundary feedback systems have been studied in a number of papers by I. Lasiecka and R. Triggiani (see e.g. Lasiecka & Triggiani [6] and [7]). The main result is, that it is possible to choose the functions w_j and g_j , appearing in (1.7), such that the system (1.6) is stable. Lasiecka & Triggiani take a semigroup approach to investigate the system (1.6), using developments on the semigroup approach presented in Washburn [12] and Balakrishnan [1]. (The basic idea of a semigroup model is presented in Fattorini [3], where ordinary differential equations are considered.)
In the following we present a pseudo-differential operator method to investigate the system (1.6), with T' given by (1.7). This gives, in a simple way, the results of Lasiecka & Triggiani, as well as a similar result for the associated hyperbolic problem

$$\begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \gamma u = T'u & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega \text{ , at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases} \tag{1.8}$$

Here we would like to point out that there now exists a quite general theory that includes all the above perturbations when they have "smooth coefficients", namely the theory of pseudo-differential boundary problems. For these, the solvability of parabolic problems like (1.6) (and far more general cases) has been discussed in great detail in Grubb [4]. However, in the work that follows, we use only some basic results of the pseudo-differential methods, but the techniques are crucial for the simplicity of the proofs, and the theory is very helpful for the understanding of the underlying problems.

Our main result is that we can, in general, transform a perturbation of the second kind into a perturbation of the first kind, whenever the boundary condition is normal, in the sense described in Grubb [4]. This includes the "classical" normal boundary conditions, as well as the feedback boundary condition $\gamma u - T'u = 0$, with T' given by (1.7). In this case, however, a special transformation of the systems (1.6) and (1.8) into systems of the first kind proves to be very useful. It turns out that the transformation can be regarded as a generalized change of coordinates, and the resulting, transformed system operator $A + G$ is merely a finite-dimensional, A-bounded perturbation of A. In this case, stabilization theory for perturbations of the first kind is straightforward, as we can apply the well known "pole assignment theorem", due to Wonham (see Wonham [13]). The operator G in $A + G$ is a so-called singular Green operator, it is of a pseudo-differential nature, and since our developments also involve manipulations with other pseudo-differential operators, we will briefly introduce some facts and terminology concerning them, for the benefit of the reader unfamiliar with these operators. For details we refer to Grubb [4], (and Hörmander [5] for the interior pseudo-differential operators).

§ 2: SOME FACTS ABOUT PSEUDO-DIFFERENTIAL OPERATORS.

A pseudo-differential operator P of order $d \in \mathbb{R}$, defined on $\mathbb{R}^n$ (or on a neighborhood of $\bar{\Omega}$) is a special kind of (singular) integral operator, defined by the formula

$$Pu(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}p(x,\xi)\hat{u}(\xi)d\xi \; , \qquad (2.1)$$

where $\hat{u}$ is the Fourier transform of u, and where the symbol $p(x,\xi)$ is a C^∞-function, developed in a series of terms $p^j(x,\xi)$, $j \in \mathbb{N}$, homogeneous of degree $d-j$ in ξ, for $|\xi| \geq 1$, of which $p(x,\xi)$ is a kind of asymptotic sum.

Example 2.1.

The differential operator

$$Au(x) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha u(x) \qquad (2.2)$$

with $C^\infty(\bar{\Omega})$ *coefficients, can be written* (F^{-1} *is the inverse Fourier transform*)

$$Au(x) = \sum_{|\alpha|\leq 2m} a_\alpha(x)F^{-1}[\xi^\alpha \hat{u}(\xi)] \qquad (2.3)$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi} \sum_{|\alpha|\leq 2m} a_\alpha(x)\xi^\alpha \hat{u}(\xi)d\xi \ ,$$

so A *is a pseudo-differential operator of order* $2m$, *with symbol* $p(x,\xi) = \sum_{|\alpha|\leq 2m} a_\alpha(x)\xi^\alpha$.

■

[Note also that (2.1) can be written

$$Pu(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\xi}p(x,\xi)u(y)dyd\xi\Big] \ . \qquad (2.4)$$

The truncated operator P_Ω over Ω is defined by

$$P_\Omega = r_\Omega P e_\Omega \qquad (2.5)$$

where e_Ω denotes extension by zero on $\mathbb{R}^n\setminus\Omega$ and r_Ω denotes restriction from $\mathbb{R}^n$ to Ω . Pseudo-differential operators over the $(n-1)$-dimensional manifold Γ are defined from pseudo-differential operators on $\mathbb{R}^{n-1}$, using local coordinates.

Poisson operators K from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$ of order $d \in \mathbb{R}$ basically takes on the following form

$$Kv(x) = (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} e^{ix'\xi'}\tilde{k}(x',x_n,\xi')\ \hat{v}\ (\xi')d\xi' \ , \qquad (2.6)$$

where we apply the standard notation

$$x = (x_1, x_2, \dots, x_n) = (x', x_n) . \qquad (2.7)$$

Here $\tilde{k} \in C^\infty$ satisfies suitable estimates, and is also a series of (quasi) homogeneous terms. Poisson operators from Γ to Ω are defined similarly, by the help at local coordinates.

<u>A trace operator T</u> of order $d \in \mathbb{R}$ from Ω to Γ, is an operator of the form

$$Tu = \sum_{0 \le j \le \ell-1} S_j \gamma_j u + T'u \qquad (2.8)$$

where the γ_j are the usual trace operators

$$\gamma_j = \left(\frac{\partial}{\partial n}\right)^j \qquad (2.9)$$

and the S_j are pseudo-differential operators on Γ of order $d-j$. T' is a special kind of trace operator that in local coordinates (where Ω and Γ are replaced by $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$) has the form

$$T'u(x') = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} \int_0^\infty \tilde{t}'(x', x_n, \xi') \acute{u}(\xi', x_n) dx_n d\xi' . \qquad (2.10)$$

Here $\acute{u}(\xi', x_n)$ denotes the partial Fourier transform $F_{x' \to \xi'} u(x', x_n)$, and $\tilde{t}'$ is a function of the same type as the $\tilde{k}$ above. The number ℓ in (2.8) is called <u>the class</u> of T .

Finally, a <u>singular Green operator G</u> of order $d \in \mathbb{R}$ and class $\ell \in \mathbb{N}$ on Ω is an operator

$$Gu = \sum_{0 \le j \le \ell-1} K_j \gamma_j u + G'u \qquad (2.11)$$

where the K_j are Poisson operators of order $d-j$.

G' has the form (in local coordinates)

$$G'u(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} \int_0^\infty \tilde{g}'(x', x_n, y_n, \xi') \acute{u}(\xi', y_n) dy_n d\xi' . \qquad (2.12)$$

where $\tilde{g}'$ is a C^∞-function, satisfying appropriate estimates. Singular Green operators arize typically when Poisson and trace operators are composed.

One of the advantages of using the pseudo-differential calculus is that composition and inversion of operators is worked out in a systematic way, once and for all; and that these operations are closely linked with the "symbolic calculus", i.e. the calculus for the associated constant coefficient cases by the help of the Fourier transform.

§ 3: TRANSFORMATION.

We will now introduce a transformation that will help us in solving the stabilization problem (see Grubb [4], lemma 1.6.8). Define the trace operator T by

$$T = \gamma - T' \tag{3.1}$$

where T' is given by (1.7). The realization A_T of A is the operator acting like A, with domain

$$D(A_T) = \{u \in H^{2m}(\Omega) \,|\, Tu = 0\} . \tag{3.2}$$

Now according to Grubb, there exists an operator Λ, which is a homeomorphism in $H^s(\Omega)$ for any $s \geq 0$, such that Λ defines a bijection

$$\Lambda\colon D(A_T) \xrightarrow{\sim} D(A_\gamma) . \tag{3.3}$$

where $D(A_\gamma)$ is the domain of the Dirichlet realization introduced in § 1. Then, applying the techniques from Grubb [4] we find that A_T is (obviosly) densely defined, and closed.

Now consider, for $\ell = 1,2$, the parabolic, resp. hyperbolic perturbation of the second kind

$$\partial_t^\ell u + A_T u = 0 \quad , \quad u \in D(A_T). \tag{3.4}$$

Using (3.3), this can be transformed into

$$\partial_t^\ell \Lambda^{-1} v + A_T \Lambda^{-1} v = 0 \quad , \quad v \in D(A_\gamma) \tag{3.5}$$

where $v = \Lambda u$

Acting with Λ from the left in (3.5) we find

$$\partial_t^\ell v + \Lambda A_T \Lambda^{-1} v = 0 \quad , \quad v \in D(A_\gamma) \ . \tag{3.6}$$

It can be shown that $\Lambda A_T \Lambda^{-1}$ has the form

$$\Lambda A_T \Lambda^{-1} = A + G \tag{3.7}$$

where G is a singular Green operator of finite rank, hence (3.6) is a perturbation of the first kind.

In this case, where T' is of the form (1.7), we have another, very useful, factorization

$$A_T = A_\gamma(1-K_\gamma T') \ . \tag{3.8}$$

Here K_γ is the Poisson operator that solves the Dirichlet problem for A, i.e. $u = K_\gamma \varphi$ is the solution of

$$\begin{cases} Au = 0 & \text{in } \Omega \\ \gamma u = \varphi & \text{on } \Gamma \end{cases} \tag{3.9}$$

(We assume, for the moment, that 0 is not an eigenvalue of A_γ).

With our application in mind it is now important to notice that we can <u>choose</u> T', such that $1-K_\gamma T'$, like Λ, defines a homeomorphism

$$1-K_\gamma T' : D(A_T) \tilde{\rightarrow} D(A_\gamma) \ . \tag{3.10}$$

The factorization (3.8) is now evident, since for $v = (1-K_\gamma T')u$, $u \in D(A_T)$, we have

$$Av = Au \tag{3.11}$$

because $A_\gamma K_\gamma = 0$.

Proceeding as above, we see that the problem

$$\partial_t^\ell u + A_T u = 0 \quad , \quad u \in D(A_T) \tag{3.12}$$

transforms into

$$\partial_t^\ell v + (1-K_\gamma T')A_\gamma v = 0 \quad , \quad v \in D(A_\gamma) \tag{3.13}$$

where $v = (1-K_\gamma T')u$.

Thus we have transformed the perturbation of the second kind (3.12) into a perturbation of the first kind (3.13), and we are able to calculate the system operator.

We have

<u>Theorem 3.1.</u>

The boundary feedback systems

$$\begin{cases} \partial_t u + Au = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \gamma u = T'u & \text{on } \gamma \text{ , for } t > 0 \\ u = u_0 & \text{in } \Omega \text{ , for } t = 0 \end{cases} \tag{3.14}$$

and

$$\begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \gamma u = T'u & \text{on } \gamma \text{ , for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega \text{ , at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases} \tag{3.15}$$

with T' given by by (1.7), transforms into the systems

$$\begin{cases} \partial_t v + Av - K_\gamma T'Av = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \gamma v = 0 & \text{on } \Gamma \text{ , for } t > 0 \\ v = v_0 & \text{in } \Omega \text{ , at } t = 0 \end{cases} \tag{3.14'}$$

and

$$\begin{cases} \partial_t^2 v + Av - K_\gamma T'Av = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \gamma v = 0 & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\ v = v_0 & \text{in } \Omega \text{ , at } t = 0 \\ \partial_t v = v_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases} \tag{3.15'}$$

■

Since

$$K_\gamma T'Av = \sum_{j=1}^{N} (Av|w_j)K_\gamma g_j \tag{3.16}$$

for $v \in H^{2m}(\Omega)$, $K_\gamma T'A$ has finite rank, and we see that $\tilde{A} = A - K_\gamma T'A$ can be regarded as a finite dimensional perturbation of A .
We obviosly have

$$\|K_\gamma T'Av\|_{L^2} \leq c\|Av\|_{L^2} \leq c\|Av\|_{L^2} + \|v\|_{L^2} \tag{3.17}$$

for $v \in D(A_\gamma)$, so $K_\gamma T'A$ is A-bounded.
Since A_γ is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$ so is $\tilde{A}_\gamma$, using the perturbation result in Zabczyk [14], proposition 1.
We have

<u>Theorem 3.2.</u>
The realization $\tilde{A}_\gamma$ of the operator

$$\tilde{A} = A - K_\gamma T'A \tag{3.18}$$

with domain

$$D(\tilde{A}_\gamma) = H^{2m}(\Omega) \cap H_0^m(\Omega) \quad (= D(A_\gamma)) \tag{3.19}$$

is the infinitesimal generator of an analytic semigroup $e^{-\tilde{A}_\gamma t}$, $t \geq 0$, on $L^2(\Omega)$, giving the solution to (3.14') as

$$v(t,x) = e^{-\tilde{A}_\gamma t} v_0(x) \quad , \quad x \in \Omega \, , \; t \geq 0 \, , \tag{3.20}$$

when $v_0 \in L^2(\Omega)$. The solution to the original system (3.14) is then

$$u(t,x) = (1-K_\gamma T')^{-1} e^{-\tilde{A}_\gamma t} (1-K_\gamma T')u_0(x) \quad , \quad x \in \Omega \, , \; t \geq 0 \tag{3.21}$$

when $u_0 \in L^2(\Omega)$.

■

§ 4. STABILIZATION.

An application of the pseudo-differential transformation.

We will now show how the transformation from § 3 can be used as a shortcut to the results of Lasiecka and Triggiani ([6], [7], [8]), which have been of great inspiration to us.

The assumed instability of the systems (1.1) and (1.2), is caused by the negative eigenvalues in the pure point spectrum of A_γ, and we will show that we can choose a <u>finite dimensional feedback</u> boundary condition

$$\gamma u = T'u \tag{4.1}$$

where T' is defined by

$$T'u = \sum_{j=1}^{N} (u|w_j)g_j \tag{4.2}$$

(see (1.7)), such that the systems

$$\begin{cases} \partial_t u + Au = 0 & \text{in } \Omega, \quad \text{for } t > 0 \\ \gamma u = T'u & \text{on } \Gamma, \quad \text{for } t > 0 \\ u = u_0 & \text{in } \Omega, \quad \text{at } t = 0 \end{cases} \tag{4.3}$$

and

$$\begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega, \quad \text{for } t \in \mathbb{R} \\ \gamma u = T'u & \text{on } \Gamma, \quad \text{for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega, \quad \text{at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega, \quad \text{at } t = 0 \end{cases} \tag{4.4}$$

are <u>stable</u> systems (in the sense described in § 1). We will apply the pseudo-differential transformation to the perturbations of the second kind (4.3) and (4.4) and then apply Wonhams "pole assignment theorem" on the resulting systems of the first kind. This, combined with a classical resolvent analysis, gives us the desired results.

Let the eigenvalues of A_γ be arranged in a non-decreasing sequence

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{K-1} \leq 0 < \lambda_K \leq \dots\dots, \tag{4.5}$$

each eigenvalue repeated according to multiplicity, and let $\{\varphi_j\}_{j\geq 1}$ be a corresponding set of orthonormalized eigenfunctions of A_γ. Now define P_u and P_s as the orthogonal projections of $L^2(\Omega)$ on the orthogonal subspaces X_u, resp. X_s, defined by

$$\begin{cases} X_u = \text{span } \{\varphi_j\}_{1\leq j<K} \\ X_s = \overline{\text{span}} \ \{\varphi_j\}_{j\geq K} \end{cases}. \tag{4.6}$$

<u>*Remark 4.1.*</u>

The results in Lasiecka & Triggiani [6], [7] and [8] are formulated as if nonselfadjoint realizations are treated as well, but on the other hand the treatment is based heavily on the orthogonal projections on the eigenspaces X_u and X_s of A_γ. Orthogonality of eigenspaces in general requires at least that A_γ be normal, i.e. $A_\gamma A_\gamma^* = A_\gamma^* A_\gamma$, but we know of no Dirichlet realization A_γ satisfying this without being selfadjoint. ■

Since X_u and $X_s \cap D(A_\gamma)$ are invariant subspaces for A_γ, we can define the restrictions

$$\begin{cases} A_u = A_\gamma|X_u \\ A_s = A_\gamma|X_s \cap D(A_\gamma) \end{cases}. \tag{4.7}$$

Then A_u is a bounded operator on X_u, and A_s is an unbounded operator on $D(A_s) = X_s \cap D(A_\gamma)$. Notice that P_u and P_s commute with A_γ on $D(A_\gamma)$.

Now writing $f_u = P_u f$, $f_s = P_s f$ for $f \in L^2(\Omega)$, we have that when $u \in D(A_T)$ (see (3.2) and (3.8)), then $v = (1-K_\gamma T')u \in D(A_\gamma)$ satisfies

$$\begin{cases} Av = Au \\ v_u \in X_u \\ v_s \in D(A_s) \ . \end{cases} \tag{4.8}$$

We now use the factorization

$$A_T = A_\gamma(1-K_\gamma T') \tag{4.9}$$

in the discussion of the resolvent equation

$$(A_T-\lambda)u = f \quad , \quad f \in L^2(\Omega) \ . \tag{4.10}$$

First we consider the case, where we are allowed to <u>decouple</u> the feedback by assuming that

$$P_s w_j = 0 \quad , \quad j = 1,2,\ldots,N \ . \tag{4.11}$$

(i.e. the w_j are in X_u ; the "unstable" eigenspace).
Then the equation (4.10) reduces to the system

$$A_u u_u - A_u P_u K_\gamma T' u_u - \lambda u_u = f_u \tag{4.12}$$

$$-AP_s K_\gamma T' u_u + Au_s - \lambda u_s = f_s \tag{4.13}$$

where we observe that (4.12) is a finite dimensional resolvent equation for the matrix operator

$$\bar{A}_u = A_u - A_u P_u K_\gamma T' \ . \tag{4.14}$$

To this we can apply Wonhams theorem to stabilize the unstable part of the system. We then have one of Lasiecka and Triggiani's results (in the case m = 1):

<u>Theorem 4.2.</u>

Assume that the Neumann traces $\{\nu\varphi_j\}_{1\leq j<K}$ are linearly independent, so that

$$\dim (\nu X_u) = \dim(X_u) \ (= K-1) \tag{4.15}$$

and let $\{c_j\}_{1\leq j<K}$ be an arbitrary given set of K-1 distinct, real numbers.

Then there exists a number N and a set $\{w_j, g_j\}_{1\leq j\leq N}$ where $w_j \in X_u$ and $g_j \in C^\infty(\Gamma)^m$, such that with

$$T'u = \sum_{j=1}^{N} (u|w_j)g_j \ . \tag{4.16}$$

the eigenvalues of the matrix operator

$$\bar{A}_u = A_u - A_u P_u K_\gamma T' \tag{4.17}$$

on X_u, are $\{c_j\}_{1\leq j<K}$.

The number N of feedback terms can be taken as the largest multiplicity of the eigenvalues $\{\lambda_j\}_{1\leq j<K}$ of A_u. In particular, N = 1 when the eigenvalues are simple.

■

<u>*Remark 4.3.*</u>

For the application of the Wonham theorem here, it is important that the range of $P_u K_\gamma$ fills out all of X_u; this can be reformulated as the question of whether the Neumann traces of the Dirichlet eigenfunctions in X_u are linearly independent. In that case the results are easy to formulate and allow N to be very low, otherwise the results become increasingly complicated and require higher N, the more linear dependence there is.

■

If we chose the poles $\{c_j\}_{1\leq j<K}$ occurring in Theorem 4.2 such that $c_j \geq \lambda_K (> 0)$ and choose T' according to the theorem, we obtain after a classical resolvent analysis:

Lemma 4.4.

The resolvent $R(\lambda, A_T)$ of A_T satisfies the inequality

$$\|R(\lambda, A_T)\|_{L^2, L^2} \leq \frac{c}{\operatorname{dist}(\lambda, \operatorname{co}(\operatorname{sp}(A_T)))} \tag{4.18}$$

as an operator in $L^2(\Omega)$. Here $c > 0$ is a constant independent of λ, and $\operatorname{co}(\operatorname{sp}(A_T))$ is the convex hull of the spectrum of A_T.

■

We then have (for $m = 1$) some of Lasiecka and Triggianis main results:

Theorem 4.5.

There exists a finite dimensional boundary condition

$$\gamma u = T'u \quad \text{on} \quad \Gamma \tag{4.19}$$

where

$$T'u = \sum_{j=1}^{N} (u|w_j)g_j \ , \tag{4.20}$$

$w_j \in X_u$, $g_j \in C^\infty(\Gamma)^m$, $j = 1,2,\ldots,N$, such that the realization A_T of A , with domain

$$D(A_T) = \{u \in H^{2m}(\Omega) \,|\, Tu = \gamma u - T'u = 0\} \tag{4.21}$$

is the infinitesimal generator of an analytic semigroup $e^{-A_T t}$, $t \geq 0$ on $L^2(\Omega)$, giving the solution the Dirichlet boundary feedback parabolic system (4.3) as

$$u(t,x) = e^{-A_T t} u_o(x) \ , \ x \in \Omega \ , \ t \geq 0 \ , \tag{4.22}$$

when $u_0 \in L^2(\Omega)$. The solution (4.22) satisfies

$$\|u(t,\cdot)\|_{L^2} \leq M e^{-\lambda_K t} \|u_0\|_{L^2} \ , \ t \geq o \ , \ M > 0 \ , \tag{4.23}$$

where λ_K is the first positive Dirichlet eigenvalue of A. Moreover, the operators

$$\cos(A_T^{1/2}t) \quad \text{and} \quad \sin(A_T^{1/2}t) \tag{4.24}$$

are well defined, and we can write the solution to the hyperbolic system (4.4) as

$$u(t,x) = \cos(A_T^{1/2}t)u_0(x) + A_T^{-1/2}\sin(A_T^{1/2}t)u_1(x) \tag{4.25}$$

$x \in \Omega$., $t \in \mathbb{R}$, when $u_0, u_1 \in L_2(\Omega)$.

■

Remark .4.6.

One of the slightly mysterious facts about the perturbation of the second kind, is that the operator A_T *can never be semibounded (i.e. satisfy an inequality* $\operatorname{Re} e^{i\theta}(A_T u|u) \geq c\|u\|^2_{L^2}$ *for some* c *and* θ *), when* $T' \neq 0$ *.. This is a consequence of Proposition 1.7.11 in Grubb [4]. In particular,* A_T *can never be selfadjoint. Since* A_T *is never semibounded, the semigroup* $e^{-A_T t}$, $t \geq 0$ *is never a contraction semigroup, hence the constant* M *in (4.23) is always greater than 1. This is also noticed by Lasiecka and Triggiani, who consider the translated Laplacian.*
This is however not true for the Neumann problem.

■

Remark 4.7.

Comparison of Eqs. (4.22) and (3.21) shows that for the semigroups we have

$$e^{-A_T t} = (1-K_\gamma T')^{-1} e^{-\tilde{A}_\gamma t}(1-K_\gamma T') \ . \tag{4.26}$$

This justifies the term "generalized change of coordinates" from § 1.

■

Now, it is straightforward to extend the theory to include more general cases, with only "weak" decoupling. If $w_j \in C^{\infty}(\bar{\Omega})$, with $P_s w_j \neq 0$, we can also obtain stabilization for small $\|P_s w_j\|_{L^2}$; the pseudo-differential transformation can again be applied to prove the existence of a feedback semigroup $e^{-A_T t}$, $t \geq 0$ on $L^2(\Omega)$, giving the solution to the parabolic system (4.3) exactly as in (4.22). In the more general case, the solution satisfies

$$\|u(t,\cdot)\|_{L^2} \leq M' e^{-(\lambda_K - \varepsilon)t} \|u_0\|_{L^2}, \quad t \geq 0, \tag{4.27}$$

$M' > 1$, $\varepsilon > 0$.

What is probably more surprising, is that if moreover the w_j's are chosen so that $P_s w_j$ are in $D(A_\gamma)$ for $j = 1,2,\ldots,N$, the solution of the hyperbolic system (4.4) can be represented as in (4.25). This is basically be-cause we can apply <u>bounded</u> perturbation theory (Sova [10], Fattorini [2]) in this case.

One of the forces of the pseudo-differential transformation, is that the problems are reduced to classical resolvent discussions in $L^2(\Omega)$, and it requires no use of the fractional powers spaces as e.g. $D(A_\gamma^{-\alpha})$, $0 \leq \alpha < ½$, that Lasiecka and Triggiani use. Therefore, our resolvent estimate (4.18), in the decoupled case, is sharper than the corresponding estimate in Lasiecka and Triggiani [7], hence our version of the exponential damping result (4.23) is a bet sharper than the original.

<u>*Remark 4.8.*</u>

If we allow simultaneous interference in the operator equation and the boundary condition, we obtain a <u>perturbation of the third kind:</u>

$$\begin{cases} \partial_t u + Au + Gu = 0 & \text{in } \Omega, \text{ for } t > 0 \\ \gamma u = T'u & \text{on } \Gamma, \text{ for } t > 0 \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0. \end{cases} \tag{4.28}$$

where the operators G and T' are of the types considered above. In this way we can stabilize the system and we can construct the feedback so that the operator realization

$$(A + G)_{\gamma\text{-}T}. \qquad (4.29)$$

considered is selfadjoint. ■

Final remark.

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100/85 "TALSYSTEMETS OPBYGNING".
Af: Mogens Niss.

101/85 "EXTENDED MOMENTUM THEORY FOR WINDMILLS IN PERTURBATIVE FORM".
Af: Ganesh Sengupta.

102/85 OPSTILLING OG ANALYSE AF MATEMATISKE MODELLER, BELYST VED MODELLER OVER KØERS FODEROPTAGELSE OG - OMSÆTNING".
Projektrapport af: Lis Eilertzen, Kirsten Habekost, Lill Røn og Susanne Stender.
Vejleder: Klaus Grünbaum.

103/85 "ØDSLE KOLDKRIGERE OG VIDENSKABENS LYSE IDEER".
Projektrapport af: Niels Ole Dam og Kurt Jensen.
Vejleder: Bent Sørensen.

104/85 "ANALOGREGNEMASKINEN OG LORENZLIGNINGER".
Af: Jens Jæger.

105/85 "THE FREQUENCY DEPENDENCE OF THE SPECIFIC HEAT AF THE GLASS REANSITION".
Af: Tage Christensen.

"A SIMPLE MODEL AF AC HOPPING CONDUCTIVITY".
Af: Jeppe C. Dyre.
Contributions to the Third International Conference on the Structure of Non - Crystalline Materials held in Grenoble July 1985.

106/85 "QUANTUM THEORY OF EXTENDED PARTICLES".
Af: Bent Sørensen.

107/85 "EN MYG GØR INGEN EPIDEMI".
- flodblindhed som eksempel på matematisk modellering af et epidemiologisk problem.
Projektrapport af: Per Hedegård Andersen, Lars Boye, CarstenHolst Jensen, Else Marie Pedersen og Erling Møller Pedersen.
Vejleder: Jesper Larsen.

108/85 "APPLICATIONS AND MODELLING IN THE MATEMATICS CUR - RICULUM" - state and trends -
Af: Mogens Niss.

109/85 "COX I STUDIETIDEN" - Cox's regressionsmodel anvendt på studenteroplysninger fra RUC.
Projektrapport af: Mikael Wennerberg Johansen, Poul Katler og Torben J. Andreasen.
Vejleder: Jørgen Larsen.

110/85 "PLANNING FOR SECURITY".
Af: Bent Sørensen

111/85 JORDEN RUNDT PÅ FLADE KORT".
Projektrapport af: Birgit Andresen, Beatriz Quinones og Jimmy Staal.
Vejleder: Mogens Niss.

112/85 "VIDENSKABELIGGØRELSE AF DANSK TEKNOLOGISK INNOVATION FREM TIL 1950 - BELYST VED EKSEMPLER".
Projektrapport af: Erik Odgaard Gade, Hans Hedal, Frank C. Ludvigsen, Annette Post Nielsen og Finn Physant.
Vejleder: Claus Bryld og Bent C. Jørgensen.

113/85 "DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS 11".
Af: Bernhelm Booss og Krzysztof Wojciechowski.

114/85 "ANVENDELSE AF GRAFISKE METODER TIL ANALYSE AF KONTIGENSTABELLER".
Projektrapport af: Lone Biilmann, Ole R. Jensen og Anne-Lise von Moos.
Vejleder: Jørgen Larsen.

115/85 "MATEMATIKKENS UDVIKLING OP TIL RENÆSSANCEN".
Af: Mogens Niss.

116/85 "A PHENOMENOLOGICAL MODEL FOR THE MEYER-NELDEL RULE".
Af: Jeppe C. Dyre.

117/85 "KRAFT & FJERNVARMEOPTIMERING"
Af: Jacob Mørch Pedersen.
Vejleder: Bent Sørensen

118/85 "TILFÆLDIGHEDEN OG NØDVENDIGHEDEN IFØLGE PEIRCE OG FYSIKKEN".
Af: Peder Voetmann Christiansen

119/86 "DET ER GANSKE VIST - - EUKLIDS FEMTE POSTULAT KUNNE NOK SKABE RØRE I ANDEDAMMEN".
Af: Iben Maj Christiansen
Vejleder: Mogens Niss.

120/86 "ET ANTAL STATISTISKE STANDARDMODELLER".
Af: Jørgen Larsen

121/86 "SIMULATION I KONTINUERT TID".
Af: Peder Voetmann Christiansen.

122/86 "ON THE MECHANISM OF GLASS IONIC CONDUCTIVITY".
Af: Jeppe C. Dyre.

123/86 "GYMNASIEFYSIKKEN OG DEN STORE VERDEN".
Fysiklærerforeningen, IMFUFA, RUC.

124/86 "OPGAVESAMLING I MATEMATIK".
Samtlige opgaver stillet i tiden 1974-jan. 1986.

125/86 "UVBY,β - systemet - en effektiv fotometrisk spektralklassifikation af B-,A- og F-stjerner".
Projektrapport af: Birger Lundgren.

126/86 "OM UDVIKLINGEN AF DEN SPECIELLE RELATIVITETSTEORI".
Projektrapport af: Lise Odgaard & Linda Szkotak Jensen
Vejledere: Karin Beyer & Stig Andur Pedersen.

127/86 "GALOIS' BIDRAG TIL UDVIKLINGEN AF DEN ABSTRAKTE ALGEBRA".
Projektrapport af: Pernille Sand, Heine Larsen & Lars Frandsen.
Vejleder: Mogens Niss.

128/86 "SMÅKRYB" - om ikke-standard analyse.
Projektrapport af: Niels Jørgensen & Mikael Klintorp.
Vejleder: Jeppe Dyre.

129/86 "PHYSICS IN SOCIETY"
Lecture Notes 1983 (1986)
Af: Bent Sørensen

130/86 "Studies in Wind Power"
Af: Bent Sørensen

131/86 "FYSIK OG SAMFUND" - Et integreret fysik/historieprojekt om naturanskuelsens historiske udvikling og dens samfundsmæssige betingethed.
Projektrapport af: Jakob Heckscher, Søren Brønd, Andy Wierød.
Vejledere: Jens Høyrup, Jørgen Vogelius, Jens Højgaard Jensen.

132/86 "FYSIK OG DANNELSE"
Projektrapport af: Søren Brønd, Andy Wierød.
Vejledere: Karin Beyer, Jørgen Vogelius.

133/86 "CHERNOBYL ACCIDENT: ASSESSING THE DATA.
ENERGY SERIES NO. 15.
AF: Bent Sørensen.

134/87 "THE D.C. AND THE A.C. ELECTRICAL TRANSPORT IN AsSeTe SYSTEM"
Authors: M.B.El-Den, N.B.Olsen, Ib Høst Pedersen, Petr Viščor

135/87 "INTUITIONISTISK MATEMATIKS METODER OG ERKENDELSES-TEORETISKE FORUDSÆTNINGER"
MASTEMATIKSPECIALE: Claus Larsen
Vejledere: Anton Jensen og Stig Andur Pedersen

136/87 "Mystisk og naturlig filosofi: En skitse af kristendommens første og andet møde med græsk filosofi"
Projektrapport af Frank Colding Ludvigsen
Vejledere: Historie: Ib Thiersen
Fysik: Jens Højgaard Jensen

137/87 "HOPMODELLER FOR ELEKTRISK LEDNING I UORDNEDE FASTE STOFFER" - Resume af licentiatafhandling
Af: Jeppe Dyre
Vejledere: Niels Boye Olsen og Peder Voetmann Christiansen.

138/87 "JOSEPHSON EFFECT AND CIRCLE MAP."

Paper presented at The International Workshop on Teaching Nonlinear Phenomena at Universities and Schools, "Chaos in Education". Balaton, Hungary, 26 April-2 May 1987.

By: Peder Voetmann Christiansen

139/87 "Machbarkeit nichtbeherrschbarer Technik durch Fortschritte in der Erkennbarkeit der Natur"

Af: Bernhelm Booss-Bavnbek
Martin Bohle-Carbonell

140/87 "ON THE TOPOLOGY OF SPACES OF HOLOMORPHIC MAPS"

By: Jens Gravesen

141/87 "RADIOMETERS UDVIKLING AF BLODGASAPPARATUR - ET TEKNOLOGIHISTORISK PROJEKT"

Projektrapport af Finn C. Physant

Vejleder: Ib Thiersen

142/87 "The Calderón Projektor for Operators With Splitting Elliptic Symbols"

by: Bernhelm Booss-Bavnbek og
Krzysztof P. Wojciechowski

143/87 "Kursusmateriale til Matematik på NAT-BAS"

af: Mogens Brun Heefelt

144/87 "Context and Non-Locality - A Peircan Approach

Paper presented at the Symposium on the Foundations of Modern Physics The Copenhagen Interpretation 60 Years after the Como Lecture. Joensuu, Finland, 6 - 8 august 1987.

By: Peder Voetmann Christiansen

145/87 "AIMS AND SCOPE OF APPLICATIONS AND MODELLING IN MATHEMATICS CURRICULA"

Manuscript of a plenary lecture delivered at ICMTA 3, Kassel, FRG 8.-11.9.1987

By: Mogens Niss

146/87 "BESTEMMELSE AF BULKRESISTIVITETEN I SILICIUM"

- en ny frekvensbaseret målemetode.

Fysikspeciale af Jan Vedde

Vejledere: Niels Boye Olsen & Petr Viščor

147/87 "Rapport om BIS på NAT-BAS"

redigeret af: Mogens Brun Heefelt

148/87 "Naturvidenskabsundervisning med Samfundsperspektiv"

af: Peter Colding-Jørgensen DLH
Albert Chr. Paulsen

149/87 "In-Situ Measurements of the density of amorphous germanium prepared in ultra high vacuum"

by: Petr Viščor

150/87 "Structure and the Existence of the first sharp diffraction peak in amorphous germanium prepared in UHV and measured in-situ"

by: Petr Viščor

151/87 "DYNAMISK PROGRAMMERING"

Matematikprojekt af:
Birgit Andresen, Keld Nielsen og Jimmy Staal

Vejleder: Mogens Niss

152/87 "PSEUDO-DIFFERENTIAL PROJECTIONS AND THE TOPOLOGY OF CERTAIN SPACES OF ELLIPTIC BOUNDARY VALUE PROBLEMS"

by: Bernhelm Booss-Bavnbek
Krzysztof P. Wojciechowski