# The Determinant of Elliptic Boundary Problems for Dirac Operators Reader on <br> The Scott-Wojciechowski Theorem <br> Version December 1999 <br> Uncorrected, Incomplete <br> Compiled, Edited, Revised, Augmented by <br> <br> B. Booss-Bavnbek 

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## Part 1

Introductory Material: Determinants of Dirac Operators, Spectral Asymmetry, and Grassmannians of Elliptic Boundary Projections

## CHAPTER 1

## The Idea of the Determinant

We introduce various notions and problems which will be fundamental for the book. In particular, we present the basic concepts of regularization, geometrization, and variation in a non-technical way. First, we recall how Gaussian integrals can be expressed by the determinant. Our model are partition functions of classical statistical mechanics involving a positive definit quadratic form on a finite-dimensional space. We discuss various regularization concepts which yield a welldefined finite determinant even when dropping the conditions of positiveness and of finite dimension, as required by functional methods in quantum field theory centred around the Dirac operator. We explain the role of the $\zeta$-function in the regularization process and show how the $\eta$-invariant naturally appears which measures the asymmetry of the spectrum and becomes the phase of the determinant.

Next we explain how the concept of the Fredholm determinant can be applied by a geometrization of the underlying operator space and without any regularization arbitrariness.

### 1.1. Functional Integrals and Spectral Asymmetry

Several important quantities in quantum mechanics and quantum field theory are expressed i terms of quadratic functionals and functional integrals. The concept of the determinant for Dirac operators arises naturally when one wants to evaluate the corresponding path integrals. As Itzykson and Zuber report in the chapter on Functional Methods of their monograph [55]: "The path integral formalism of Feynman and Kac provides a unified view of quantum mechanics, field theory, and statistical models. ... The original suggestion of an alternative presentation of quantum mechanical amplitudes in terms of path integrals stems from the work of Dirac (1933) and was brilliantly elaborated by Feynman in the 1940s. ... This work was first regarded with some suspicion due to the difficult mathematics required to give it a decent status. In the 1970s it has, however, proved to be the most flexible tool in suggesting new developments in field theory and therefore deserves a thorough presentation."

[^0]We shall restrict our discussion to the most easy variant of that complex matter focusing on the partition function of a quadratic functional given by the Euclidean action of a Dirac operator which is assumed to be elliptic with imaginary time due to Wick rotation and coupled to continuously varying vector potentials (sources, fields, connections), for the ease of presentation in vacuum. We refer to Bertlmann, $[\mathbf{1 2}]$ and Schwarz, $[\mathbf{9 0}]$ for an introduction into the quantum theoretic language for mathematicians and for a more extensive treatment of general aspects of quadratic functionals and functional integrals involving the relations to the Lagrangian and Hamiltonian formalism.

There are various alternative notions around, some are more sophisticated, some less. In the long run, physics perhaps will show which notions are the "correct", the most meaningful ones. At present, mathematics has already confirmed the path integral and the $\zeta$-regularized determinant for the (Euclidean) Dirac operator as notions which lead to reasonable expressions, permit precise calculations, and can be understood as canonical objects, independent of particular choices made for regularization. That is what we want to show in this book.

A special feature of Dirac operators is that their determinants involve a phase, the imaginary part of the determinant's logarithm. As we will see now, this is a consequence of the fact that, unlike the halfbounded Laplacian, Dirac operators as operators of first order have an infinite number of both positive and negative eigenvalues. Then the phase of the determinant reflects the spectral asymmetry of the corresponding Dirac operator.

The simplest path integral we meet in quantum field theory takes the form of the partition function and can be written formally as the integral

$$
\begin{equation*}
Z(\beta):=\int_{\Gamma} e^{-\beta S(\omega)} d \omega \tag{1.1.1}
\end{equation*}
$$

where $d \omega$ denotes functional integration over the space $\Gamma:=\Gamma(M ; E)$ of sections of a Euclidean vector bundle $E$ over a Riemannian manifold $M$.

In quantum theoretic language, $M$ is space or space-time; a $\omega \in \Gamma$ is a position function of a particle or a spinor field. The scaling parameter $\beta$ is a real or complex parameter, most often $\beta=1$. The functional $S$ is a quadratic real-valued functional on $\Gamma$ defined by $S(\omega):=\langle\omega, T \omega\rangle$ with a fixed linear symmetric operator $T: \Gamma \rightarrow \Gamma$. Typically $T$ is a Dirac operator and $S(\omega)$ is the action $\left.S(\omega)=\int_{M} l l a \omega, \mathcal{D} \omega\right\rangle$.

Mathematically speaking, the integral 1.1.1 is an oscillating integral like the Gaussian integral. It is ill-defined in general because
(I) as it stands, it is meaningless when $\operatorname{dim} \Gamma(M ; E)=+\infty$ (i.e., when $\operatorname{dim} M \geq 1$ ); and,
(II) even when $\operatorname{dim} \Gamma(M ; E)<\infty$ (i.e. when $\operatorname{dim} M=0$ and $M$ consists of a finite set of points), the integral $Z(\beta)$ diverges unless $\beta S(\omega)$ is positive and non-degenerate.

Newertheless, these expressions have been used and construed in quantum field theory. As a matter of fact, reconsidering the physicists use and interpretation of these mathematically ill-defined quantities, one can describe certain formal manipulations which lead to normalizing and evaluating $Z(\beta)$ in a mathematically rigorous way.

We begin with a few calculations in Case II, inspired by Adams and Sen, $[\mathbf{1}]$, to show how spectral asymmetry is naturally entering into the calculations even in the finite-dimensional case and how this suggests the non-standard definition of the determinant in the infinite-dimensional case for the Dirac operator.

Then, let $\operatorname{dim} \Gamma=d<\infty$ and set

$$
S(\omega)=\langle\omega, T \omega\rangle \text { for all } \omega \in \Gamma
$$

with a symmetric endomorphism $T$.

1. calculation. We assume $S$ positive and non-degenerate, i.e. $T$ strictly positive, $T>0$ with $\operatorname{spec} T=\left\{\lambda_{1}, \ldots \lambda_{d}\right\}$ with all $\lambda_{j}>0$. That is the classical case. We choose an orthonormal system of eigenvectors $\left(e_{1}, \ldots, e_{d}\right)$ of $T$ as basis for $\Gamma$. We have $S(\omega)=\sum \lambda_{j} x_{j}^{2}$ for $\omega=\sum x_{j} e_{j}$ and get for real $\beta>0$

$$
\begin{aligned}
Z(\beta) & =\int_{\Gamma} e^{-\beta S(\omega)} d \omega=\int_{\mathbf{R}^{d}} d x_{1} \ldots d x_{d} e^{-\beta \sum \lambda_{j} x_{j}^{2}} \\
& =\int_{-\infty}^{\infty} e^{-\beta \lambda_{1} x_{1}^{2}} d x_{1} \int_{-\infty}^{\infty} e^{-\beta \lambda_{2} x_{2}^{2}} d x_{2} \ldots \int_{-\infty}^{\infty} e^{-\beta \lambda_{d} x_{d}^{2}} d x_{d} \\
& =\sqrt{\frac{\pi}{\beta \lambda_{1}}} \sqrt{\frac{\pi}{\beta \lambda_{2}}} \cdots \sqrt{\frac{\pi}{\beta \lambda_{d}}}=\pi^{d / 2} \cdot \beta^{-d / 2} \cdot(\operatorname{det} T)^{-\frac{1}{2}}
\end{aligned}
$$

In that way the determinant appears when evaluating the simplest quadratic integral.
2. calculation. If the functional $S$ is positive and degenerate, $T \geq 0$, the partition function is given by

$$
Z(\beta)=\pi^{\zeta / 2} \cdot \beta^{-\zeta / 2} \cdot(\operatorname{det} \widetilde{T})^{-\frac{1}{2}} \cdot \operatorname{vol}(\operatorname{ker} T)
$$

where $\zeta:=\operatorname{dim} \Gamma-\operatorname{dim} \operatorname{ker} T$ and $\widetilde{T}:=\left.T\right|_{(\operatorname{ker} T)^{\perp}}$, but, of course $\operatorname{vol}(\operatorname{ker} T)=$ $\infty$. For approaches to renormalize this quantity in quantum chromodynamics, we refer to $[\mathbf{1}],[\mathbf{2 2}],[\mathbf{9 0}]$. Still we can take $\pi^{\zeta / 2} \beta^{-\zeta / 2}(\operatorname{det} \widetilde{T})^{-\frac{1}{2}}$ as our definition of the integral by discarding $\operatorname{vol}(\operatorname{ker} T)$.
3. calculation. Now we assume that the functional $S$ is non-degenerate, i.e. $T$ invertible, but $S$ is neither positive nor negative. We decompose $\Gamma=\Gamma_{+} \times \Gamma_{-}$and $T=T_{+} \oplus T_{-}$with $T_{+},-T_{-}$strictly positive in $\Gamma_{ \pm}$.

Formally, we obtain

$$
\begin{aligned}
Z(\beta) & =\left(\int_{\Gamma_{+}} d \omega_{+} e^{-\beta\left\langle\omega_{+}, T_{+} \omega\right\rangle}\right)\left(\int_{\Gamma_{-}} d \omega_{-} e^{-(-\beta)\left\langle\omega_{-},-T_{-} \omega\right\rangle}\right) \\
& =\pi^{d_{+} / 2} \beta^{-d_{+} / 2}\left(\operatorname{det} T_{+}\right)^{-\frac{1}{2}} \pi^{d_{-} / 2}(-\beta)^{-d_{-} / 2}\left(\operatorname{det}-T_{-}\right)^{-\frac{1}{2}} \\
& =\pi^{\zeta / 2} \beta^{-d_{+} / 2}(-\beta)^{-d_{-} / 2}(\operatorname{det}|T|)^{-\frac{1}{2}}
\end{aligned}
$$

where $d_{ \pm}:=\operatorname{dim} \Gamma_{ \pm}$, hence $\zeta=d_{+}+d_{-}$and $|T|:=\sqrt{\widetilde{T}^{2}}=T_{+} \oplus-T_{-}$.
4. calculation. In the preceding formula, the term $(\beta)^{-d_{+} / 2}(-\beta)^{-d_{-} / 2}$ is undefined for $\beta \in \mathbf{R}_{ \pm}$. We shall replace it by a more intelligible term for $\beta=1$ by first expanding $Z(\beta)$ into the upper complex halfplanes and then formally setting $\beta=1$. More precisely, let $\beta \in \mathbf{C}_{+}=\{z \in \mathbf{C} \mid \Im z>0\}$ and write $\beta=|\beta| e^{i \theta}$ with $\theta \in[0, \pi]$, hence $-\beta=|\beta| e^{i(\theta-\pi)}$ with $\theta-\pi \in[-\pi, 0]$. We set $\beta^{a}:=|\beta|^{a} e^{i \theta a}$ and get

$$
\begin{aligned}
\beta^{-d_{+} / 2}(-\beta)^{-d_{-} / 2} & =\left(|\beta| e^{i \theta}\right)^{-d_{+} / 2}\left(|\beta| e^{i(\theta-\pi)}\right)^{-d_{-} / 2} \\
& =|\beta|^{-\zeta / 2} e^{-i \frac{d_{+}}{2} \theta} e^{-i \frac{d_{-}}{2} \theta} e^{i \pi \frac{d_{-}}{2}}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
-\frac{d_{+}}{2} \theta-\frac{d_{-}}{2} \theta+\pi \frac{d_{-}}{2} & =-\frac{\theta}{2}\left(d_{+}+d_{-}\right)+\frac{\pi}{2}\left(\frac{d_{-}}{2}+\frac{d_{+}}{2}+\frac{d_{-}}{2}-\frac{d_{+}}{2}\right) \\
& =-\frac{\theta}{2} \zeta+\frac{\pi}{4}(\zeta-\eta)=-\frac{\pi}{4}\left(\frac{2 \theta \zeta}{\pi}+(\eta-\zeta)\right)
\end{aligned}
$$

where $\zeta:=d_{+}+d_{-}$is the finite-dimensional equivalent of the $\zeta$-invariant, counting the eigenvalues, and $\eta:=d_{+}-d_{-}$the finite-dimensional equivalent of the $\eta$-invariant, measuring the spectral asymmetry of $T$. We obtain

$$
Z(\beta)=\pi^{\zeta / 2}|\beta|^{-\zeta / 2} e^{-i \frac{\pi}{4}\left(\frac{2 \zeta \theta}{\pi}+(\zeta-\eta)\right)}(\operatorname{det}|T|)^{-\frac{1}{2}}
$$

and, formally, for $\beta=1$, i.e. $\theta=0$,

$$
\begin{equation*}
Z(1)=\pi^{\zeta / 2} \underbrace{e^{-i \frac{\pi}{4}(\zeta-\eta)}(\operatorname{det}|T|)^{-\frac{1}{2}}}_{=: \operatorname{det} T} \tag{1.1.2}
\end{equation*}
$$

Remark 1.1.1. (a) The methods and results of this section also apply to real-valued quadratic functionals on complex vector spaces. Since the integration in (1.1.1) in this case is over the real vector space underlying $\Gamma$, which has twice the dimension of $\Gamma$, the expressions for the partition functions in this case become the square of those above.
(b) In the preceding calculations we worked with ordinary commuting numbers and functions. The resulting Gaussian integrals are also called bosonic integrals. If we consider fermionic integrals, we work with Grassmannian variables and obtain the determinant not in the denominator, but in the nominator (see e.g. [11] or [12]). We shall exploit that aspect later in Chapter ???.
(c) Another problem appears even in finite dimensions, namely when a determinant shall not be defined for a endomorphism but for a homomorphism.........

Equation (1.1.2) suggests a non-standard definition of the determinant for the infinite-dimensional case.

### 1.2. The $\zeta$-Determinant for Operators of Infinite Rank

Once again, our point of departure is finite-dimensional linear algebra. Let $T: \mathbf{C}^{d} \rightarrow \mathbf{C}^{d}$ be an invertible, positive operator with eigenvalues $0<$ $\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{d}$. We have the equality

$$
\begin{aligned}
\operatorname{det} T & =\prod \lambda_{j}=\exp \left\{\left.\sum \ln \lambda_{j} e^{-s \ln \lambda_{j}}\right|_{s=0}\right\} \\
& =\exp \left(-\left.\frac{d}{d s}\left(\sum \lambda_{j}^{-s}\right)\right|_{s=0}\right)=e^{-\left.\frac{d}{d s} \zeta_{T}(s)\right|_{s=0}}
\end{aligned}
$$

where $\zeta_{T}(s):=\sum_{j=1}^{d} \lambda_{j}^{-s}$.
We show that the preceding formula generalizes naturally, when $T$ is replaced by a positive definite self-adjoint elliptic operator $L$ (for the ease of presentation, of second order, like the Laplacian) acting on sections of a Hermitian vector bundle over a closed manifold $M$ of dimension $m$. Then $L$ has a discrete spectrum spec $L=\left\{\lambda_{j}\right\}_{j \in \mathbf{N}}$ with $0<\lambda_{1} \leq \lambda_{2} \leq \ldots$, satisfying the asymptotic formula $\lambda_{n} \sim n^{m / 2}$ (see e.g. [45], Lemma 1.6.3). We extend $\zeta_{L}(s):=\sum_{j=1}^{\infty} \lambda_{j}^{-s}$ in the complex plane by

$$
\zeta_{L}(s):=\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t L} d t
$$

with the $\Gamma$-function $\Gamma(s):=\int_{0}^{\infty} t^{s-1} e^{-t} d t$. Note that $e^{-t L}$ is the heat operator transforming any initial section $f_{0}$ into a section $f_{t}$ satisfying the heat equation $\frac{\partial}{\partial t} f+L f=0$. Clearly $\operatorname{Tr} e^{-t L}=\sum e^{-t \lambda_{j}}$.

One shows that the original definition of $\zeta_{L}(s)$ yields a holomorphic function for $\Re(s)$ large and that its preceding extension is meromorphic in the entire complex plane with simple poles only. The point $s=0$ is a regular point and $\zeta_{L}(s)$ is a holomorphic function at $s=0$. From the asymptotic expansion of $\Gamma(s) \sim \frac{1}{s}+g+s h(s)$ close to $s=0$ with the Euler number $g$ and a suitable holomorphic function $h$ we obtain an explicit formula

$$
\zeta_{L}^{\prime}(0) \sim \int_{0}^{\infty} \frac{1}{t} \operatorname{Tr} e^{-t L} d t-g \zeta_{L}(0)
$$

This is proved in xxx / will be proved in xxx .
Therefore, Ray and Singer in [84] could introduce $\operatorname{det}_{\zeta}(L)$ by defining:

$$
\operatorname{det}_{\zeta} L:=e^{-\left.\frac{d}{d s} \zeta_{L}(s)\right|_{s=0}}=e^{-\zeta_{L}^{\prime}(0)} .
$$

The preceding definition does not apply immediately to the main hero here, the Dirac operator $\mathcal{D}$ which has infinitely many positive $\lambda_{j}$ and negative eigenvalues $-\mu_{j}$. Clearly by the preceding argument

$$
\operatorname{det}_{\zeta} \mathcal{D}^{2}=e^{-\zeta_{\mathcal{D}}{ }^{2}} \quad \text { and } \quad \operatorname{det}_{\zeta}|\mathcal{D}|=e^{-\zeta_{|\mathcal{D}|}^{\prime}}=e^{-\frac{1}{2} \zeta_{\mathcal{D}}{ }^{2}}
$$

For the Dirac operator we set

$$
\ln \operatorname{det} \mathcal{D}:=-\left.\frac{d}{d s} \zeta_{\mathcal{D}}(s)\right|_{s=0}
$$

with, choosing the branch $(-1)^{-s}=e^{i \pi s}$,

$$
\begin{aligned}
\zeta_{\mathcal{D}}(s)= & \sum \lambda_{j}^{-s}+\sum(-1)^{-s} \mu_{j}^{-s}=\sum \lambda_{j}^{-s}+e^{i \pi s} \sum \mu_{j}^{-s} \\
= & \frac{\sum \lambda_{j}^{-s}+\sum \mu_{j}^{-s}}{2}+\frac{\sum \lambda_{j}^{-s}-\sum \mu_{j}^{-s}}{2} \\
& \quad+e^{i \pi s}\left\{\frac{\sum \lambda_{j}^{-s}+\sum \mu_{j}^{-s}}{2}-\frac{\sum \lambda_{j}^{-s}-\sum \mu_{j}^{-s}}{2}\right\} \\
= & \frac{1}{2}\left\{\zeta_{\mathcal{D}^{2}}\left(\frac{s}{2}\right)+\eta_{\mathcal{D}}(s)\right\}+\frac{1}{2} e^{i \pi s}\left\{\zeta_{\mathcal{D}^{2}}\left(\frac{s}{2}\right)-\eta_{\mathcal{D}}(s)\right\},
\end{aligned}
$$

where $\eta_{\mathcal{D}}(s):=\sum \lambda_{j}^{-s}-\sum \mu_{j}^{-s}$. Later we will show that $\eta_{T}(s)$ is a holomorphic function of $s$ for $\Re(s)$ large with a meromorphic extension to the whole complex plane which is holomorphic in the neighborhood of $s=0$. We obtain

$$
\begin{aligned}
\zeta_{\mathcal{D}}^{\prime}(s)=\frac{1}{4} \zeta_{\mathcal{D}^{2}}^{\prime}\left(\frac{s}{2}\right)+\frac{1}{2} \eta_{\mathcal{D}}^{\prime}(s)+\frac{1}{2} i \pi e^{i \pi s}\left\{\zeta_{\mathcal{D}^{2}}\left(\frac{s}{2}\right)\right. & \left.-\eta_{\mathcal{D}^{2}}(s)\right\} \\
& +\frac{1}{2} e^{i \pi s}\left\{\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}\left(\frac{s}{2}\right)-\eta_{\mathcal{D}}^{\prime}(s)\right\}
\end{aligned}
$$

It follows:

$$
\zeta_{\mathcal{D}}^{\prime}(0)=\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)+\frac{i \pi}{2}\left\{\zeta_{\mathcal{D}^{2}}(0)-\eta_{\mathcal{D}}(0)\right\}
$$

and

$$
\begin{aligned}
\operatorname{det}_{\zeta} \mathcal{D} & =e^{-\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)} e^{-\frac{i \pi}{2}\left\{\zeta_{\mathcal{D}^{2}}(0)-\eta_{\mathcal{D}}(0)\right\}} \\
& =e^{-\frac{i \pi}{2}\left\{\zeta_{|\mathcal{D}|}(0)-\eta_{\mathcal{D}}(0)\right\}} e^{-\zeta_{|\mathcal{D}|}^{\prime}(0)} \\
& =e^{-\frac{i \pi}{2}\left\{\zeta_{|\mathcal{D}|}(0)-\eta_{\mathcal{D}}(0)\right\}} \operatorname{det}_{\zeta}|\mathcal{D}|,
\end{aligned}
$$

and for the Dirac operator's 'partition function' in the sense of (1.1.1):

$$
Z(1)=\pi^{\zeta|\mathcal{D}|}(0)\left(\operatorname{det}_{\zeta} \mathcal{D}\right)^{-\frac{1}{2}} .
$$

Remark 1.2.1. In the preceding formulas three spectral invariants enter of the Dirac operator $\mathcal{D}$ :
(1) $\zeta_{\mathcal{D}^{2}}(0)$, it is given by $\int_{M} \alpha(x) d x$, where $\alpha(x)$ denotes the index density which is a certain coefficient in the heat kernel expansion and is locally expressed by the coefficients of $\mathcal{D}$. In particular, $\zeta_{\mathcal{D}^{2}}(0)$ remains unchanged for small changes of the spectrum. Actually, $\zeta_{L}(0)$ vanishes when $L$ is the square of an self-adjoint elliptic operator on a closed manifold.
(2) $\eta_{\mathcal{D}}(0)$, it is not given by an integral, not by a local formula. It depends, however, only on finitely many terms of the symbol of the resolvent $(\mathcal{D}-\lambda)^{-1}$ and will not change when one changes or removes a finite number of eigenvalues.
(3) $\zeta_{\mathcal{D}^{2}}^{\prime}(0)$, it is the most delicate of the invariants inmvolved: Neither it is a local invariant, nor does it depend only on the total symbol of the Dirac operator. Below in xxx we will show that even small changes of the eigenvalues will change the $\zeta^{\prime}$-invariant and hence the determinant.
Although Felix Klein in [58] rated the determinant as the most simple example of an invariant, today we must give an inverse rating. For the present authors, not the transformation groups and invariants which reveal the widest symmetries or display the greatest stability are at the centre of focus, but, according to Dirac's approach to elementary particle physics, the finest invariants which can detect small anomalies and will be changed out of nearly nothing deserve the highest interest. Correspondingly, the determinant and its amplitude (3) are the most subtle and the most fascinating objects of our study. They are much more difficult to comprehend than (2); and (2) is much more difficult to comprehend than (1).

For this book, this suggests a scale of stages of investigations. First we have to show that all three invariants are well defined for Dirac operators on closed manifolds. Then we shall concentrate on investigating the properties of (2), the main ingredient into the determinant's phase. Then we have to show that all of them make sense for boundary problems belonging to a certain Grassmannian. Finally, we have to investigate the stability properties under variation of the coefficients.

### 1.3. The Determinant as a Canonical Element of a Complex Line

Before we do that we discuss another definition of the determinant. This one is more algebraic than analytic.

- Fredholm determinant
- Segal determinant line
- Quillen determinant line
- Families


## CHAPTER 2

## The $\zeta$-Determinant on the Circle

The goal of this chapter is to show how the things works out in the case of a simple example of the operator $\mathcal{D}_{f}=-i \frac{d}{d x}+f(x)$ on the circle $S^{1}$. We study the case of the operator $\mathcal{D}_{a}=$ $-i \frac{d}{d x}+a: C^{\infty}\left(S^{1}\right) \rightarrow C^{\infty}\left(S^{1}\right)$. We show that all ingredients in the formula (7.4.6) are well-defined and that in fact the $\zeta^{-}$ determinant is a true algebraic determinant.
First let us explain why it is enough to study operators $\mathcal{D}_{a}$, when it seems that we should investigate operators of the form

$$
-i \frac{d}{d x}+f(x)
$$

where $f(x)$ denotes a smooth real-valued function on $S^{1}$. Let us consider two such operators

$$
\mathcal{D}_{f_{i}}=-i \frac{d}{d x}+f_{i}(x) .
$$

and us introduce functions

$$
g_{i}(x)=\int_{0}^{x} f_{i}(s) d s .
$$

We have now the following result

Proposition 2.0.1. Assume that

$$
\begin{equation*}
g_{1}(2 \pi)=g_{2}(2 \pi), \tag{2.0.1}
\end{equation*}
$$

then operators $\mathcal{D}_{f_{1}}$ and $\mathcal{D}_{f_{2}}$ are unitary equivalent.

Proof. We define operator $U$ acting on functions on $S^{1}$ via formula

$$
(U s)(x)=e^{i\left(g_{1}(x)-g_{2}(x)\right)} s(x) .
$$

The operator $U$ is a unitary operator on $L^{2}\left(S^{1}\right)$, and the straightforward computations shows

$$
U \mathcal{D}_{f_{1}} U^{-1}=-i \frac{d}{d x}+f_{2}(x)
$$

hence $\mathcal{D}_{f_{1}}$ and $\mathcal{D}_{f_{2}}$ are unitary equivalent, which among the other things shows that they have the same spectrum.

Corollary 2.0.2. Operator $-i \frac{d}{d x}+f(x)$ is unitary equivalent to the operator $-i \frac{d}{d x}+a$, where

$$
\begin{equation*}
a=\frac{\int_{0}^{2 \pi} f(s) d s}{2 \pi} \tag{2.0.2}
\end{equation*}
$$

Corollary 2.0.3. The spectrum of the operator $-i \frac{d}{d x}+f(x)$ is equal to $\{k+a\}_{k \in \mathbf{Z}}$, where $a$ is given by the formula (2.0.2).

The last corollary follows from the fact that we know spectrum of the operator $\mathcal{D}_{a}=-i \frac{d}{d x}+a$. It has eigenvalues $k+a$ corresponding to the eigenfunctions $\phi_{k}(x)=\frac{1}{\sqrt{2 \pi}} e^{i k x}$.

## 2.1. $\zeta-$ and $\eta$-Function of a Dirac operator and the Heat Kernel

We study the $\zeta$-determinant of the operator $\mathcal{D}_{a}$ using Heat Operator determined by $\mathcal{D}_{a}^{2}$. We start with the discussion of the situation in the case of a general Dirac operator. Later we prove all results for the operator $\mathcal{D}_{a}$ on $S^{1}$. Let $\mathcal{D}: C^{\infty}(M ; S) \rightarrow C^{\infty}(M ; S)$ denote a Dirac operator acting on sections of bundle of Clifford modules over a closed manifold $M$. We want to solve the Heat Propagation problem for the operator $\mathcal{D}^{2}$, which means that having given $f_{0} \in C^{\infty}(M ; S)$, we want to solve the problem:

$$
\begin{equation*}
\left(\frac{d}{d t}+\mathcal{D}^{2}\right) f(t, x)=0 \text { for } t>0 \text { with } f(0, x)=f_{0}(x) \tag{2.1.1}
\end{equation*}
$$

The problem (2.1.1) has a unique solution for each smooth initial data $f_{0}(x)$ (see for instance [?, ?]). The usual way to get the solution is to construct a family of operators $e^{-t \mathcal{D}^{2}}=E(t): C^{\infty}(M ; S) \rightarrow C^{\infty}(M ; S)$ such that $E(0)=I d$ and for each $f_{0}$ and $t>0$ we have

$$
\left(E(t) f_{0}\right)(x)=f(t, x)
$$

It is not difficult to see that $E(t)$ staisfies semigroup property i.e. for each $s, t>0$ we have $E(t+s)=E(t) E(s)$. Moreover operator $E(t)$ has a smooth kernel, which means that there exists a smooth function $e(t ; x, y)$, where for each $x, y \in M e(t ; x, y)$ is a linear map from $S_{x}$ to $S_{y}$ such that

$$
\begin{equation*}
\left(e^{-t \mathcal{D}^{2}} f_{0}\right)(x)=f(t, x)=\int_{M} e(t ; x, y) f_{0}(y) d y \tag{2.1.2}
\end{equation*}
$$

Assuming that we know a spectral decomposition of the operator $\mathcal{D}$ we have a nice abstract formula which gives kernel of the Heat Operator. Let us denote by $\lambda_{k}$ an eigenvalue of $\mathcal{D}$, which corresponds to the eigensection $\phi_{k}$. Then we have

$$
\begin{equation*}
e(t ; x, y)=\sum_{-\infty}^{+\infty} e^{-t \lambda_{k}^{2}} \phi_{k}(x) \otimes \phi_{k}(y) . \tag{2.1.3}
\end{equation*}
$$

In other words we have equality

$$
(E(t) s)(x)=\sum_{-\infty}^{+\infty}\left(\int_{M}<s(y) ; \phi_{k}(y)>d y\right) \phi_{k}(x)
$$

where $<\cdot ; \cdot>$ denote an inner product on the fibre $S_{y}$. We refer to [45] and $[?, ?]$ for more details. In the following we concentrate on a very easy special case. Although formula (2.1.3) looks nice it has a relatively small value in the case we want explicit formula for the kernel $e(t ; x, y)$. We start with the differential operator, for which we know exact formula for the kernel and then study how the perturbation of the operator affects the Heat Kernel. This is what we do in the next Section in order to study $\zeta$-function of the operator $\mathcal{D}_{0}^{2}=-\frac{d^{2}}{d x^{2}}$ on the circle. Now we finally justify the introduction of the Heat Operator. We use this operator in order to study $\zeta$-function and $\eta$-function of the Dirac operators.

Proposition 2.1.1. The following equalities hold for a Dirac operator $\mathcal{D}$

$$
\begin{equation*}
\zeta_{\mathcal{D}^{2}}(s)=\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}^{2}} d t \text { for } \operatorname{Re}(s)>\frac{\operatorname{dim} M}{2} \tag{2.1.4}
\end{equation*}
$$

and

$$
\eta_{\mathcal{D}}(s)=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}^{2}} d t \quad \text { for } \quad \operatorname{Re}(s)>\frac{1+\operatorname{dim} M}{2}
$$

where in the discussion of $\zeta_{\mathcal{D}^{2}}(s)$ we assume that $\mathcal{D}$ is invertible. This assumption is not necessary in the case of $\eta_{\mathcal{D}}(s)$.

Proof. We prove second equality in (2.1.4). The proof of the first one is completely analogous. We have

$$
\begin{gathered}
\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}^{2}} d t=\sum_{-\infty}^{+\infty} \int_{0}^{\infty} t^{\frac{s-1}{2}} \lambda_{k} e^{-t \lambda_{k}^{2}} d t=\sum_{-\infty}^{+\infty} \lambda_{k}\left(\lambda_{k}\right)^{-\frac{s+1}{2}} \int_{0}^{\infty}\left(t \lambda_{k}^{2}\right)^{\frac{s-1}{2}} e^{-t \lambda_{k}^{2}} d\left(t \lambda_{k}^{2}\right)= \\
\sum_{-\infty}^{+\infty} \operatorname{sign} \lambda_{k} \cdot\left|\lambda_{k}\right|^{-s} \cdot \int_{0}^{\infty} r^{\frac{s-1}{2}} e^{-r} d r=\Gamma\left(\frac{s+1}{2}\right) \cdot \eta_{\mathcal{D}}(s)
\end{gathered}
$$

We discuss the suitable domain of $s$ for which (2.1.4) is valid only for the operator $\mathcal{D}_{f}$.

Remark 2.1.2. If we assume that $\mathcal{D}$ has eigenvalue 0 , i.e. there exists nontrivial solution of the equation $\mathcal{D} s=0$, then of course the first formula in (2.1.4) does not hold as the integral on the right side is divergent. To cure this problem we proceed as follows. The operator $\mathcal{D}$ is an elliptic operator hence ker $\mathcal{D}$ the space of all solutions of $\mathcal{D}$ is finite dimensional and consists of only smooth sections. This is obvious for the operator $\mathcal{D}_{f}$ and we refer to [45], or [?, ?] for the proof of this fact for general Dirac operator. We consider integral in (2.1.4) on the orthogonal complement of this space, but to stay consistent with the definition (7.1.7) we have to add the dimension of $\operatorname{ker} \mathcal{D}$. More precisely if we denote by $\Pi_{\mathcal{D}}$ orthogonal projection onto ker $\mathcal{D}$, then first formula in (2.1.4) is replaced by
$\zeta_{\mathcal{D}^{2}}(s)=\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr}\left(e^{-t \mathcal{D}^{2}}-\Pi_{\mathcal{D}}\right) d t+\operatorname{dim} \operatorname{ker} \mathcal{D} \quad$ for $\operatorname{Re}(s)>\frac{\operatorname{dim} M}{2}$.

### 2.2. Heat Kernel and $\zeta$-Function on $S^{1}$

We discuss $\zeta$-function of the operator $\mathcal{D}_{0}^{2}=\Delta=-\frac{d^{2}}{d x^{2}}$ on $S^{1}$. We use formula (2.1.4) hence we need information on the kernel of the operator $e^{-t \Delta}$. It is not difficult to check that $e_{\mathbf{R}^{1}}(t ; x, y)$ kernel of the corresponding operator on $\mathbf{R}^{1}$ is given by the formula

$$
e_{\mathbf{R}^{1}}(t ; x, y)=\frac{1}{\sqrt{4 \pi t}} e^{-\frac{(x-y)^{2}}{4 t}}
$$

Let me repeat again that this means that the function $f(t, x)$ given by the formula

$$
f(t, x)=\left(e^{-t\left(-\frac{d^{2}}{d x^{2}}\right)} f_{0}\right)(t, x)=\int_{\mathbf{R}^{1}} \frac{1}{\sqrt{4 \pi t}} e^{-\frac{(x-y)^{2}}{4 t}} f_{0}(y) d y
$$

solves the problem (2.1.1) with the initial data $f_{0}(x)$. Now we define kernel on $S^{1}$ using the formula

$$
\begin{equation*}
e_{S^{1}}(t ; x, y)=\sum_{n \in \mathbf{Z}} e_{\mathbf{R}^{1}}(t ; x, y+2 \pi n) \tag{2.2.1}
\end{equation*}
$$

In (2.2.1) we use the representation of $S^{1}$ as $\mathbf{R} / 2 \pi \mathbf{Z}$. Now as the exercise reader may check the following fact

Proposition 2.2.1. The kernel of the operator $e^{-t \Delta}$ on $S^{1}$ is equal to $e_{S^{1}}(t ; x, y)$.

The Proposition 11.3.2 has the following extremely important consequence

Theorem 2.2.2. Let us assume that $0<t<1$, then there exists positive constants $c_{1}, c_{2}$, such that the following equality holds

$$
\left|e_{S^{1}}(t: x, y)-e_{\mathbf{R}^{1}}(t ; x, y)\right|<c_{1} \cdot e^{-\frac{c_{2}}{t}}
$$

Remark 2.2.3. (1) The statement of the Theorem 2.2 .2 is usually written as

$$
\begin{equation*}
e_{s^{1}}(t ; x, y)=\frac{1}{\sqrt{4 \pi t}} e^{-\frac{(x-y)^{2}}{t}}+O\left(e^{-\frac{c}{t}}\right) \text { for small } t \tag{2.2.2}
\end{equation*}
$$

(2) In the following we really need (7.2.6) for $(x, y)$ close to the diagonal hence the distance between $x$ and $y$ is indeed given by $|x-y|$.

Proof. We have

$$
e_{s^{1}}(t ; x, y)=\frac{1}{\sqrt{4 \pi t}} e^{-\frac{(x-y)^{2}}{t}}+\frac{1}{\sqrt{4 \pi t}} \cdot \sum_{n \neq 0} e^{-\frac{(x-y-2 \pi n)^{2}}{t}}
$$

and we estimate sum as follows

$$
\begin{gathered}
\sum_{n \neq 0} e^{-\frac{(x-y-2 \pi n)^{2}}{t}}=\sum_{k=1}^{\infty}\left(e^{-\frac{(x-y-2 \pi n)^{2}}{t}}+e^{-\frac{(x-y+2 \pi n)^{2}}{t}}\right)<2 \cdot \sum_{k=1}^{\infty} e^{-\frac{(\pi n)^{2}}{4 t}}< \\
2 \cdot \sum_{k=1}^{\infty}\left(e^{-\frac{\pi^{2}}{4 t}}\right)^{n}=2 \cdot \frac{1}{e^{\frac{\pi^{2}}{4 t}}} \cdot \frac{1}{1-\frac{1}{e^{\frac{\pi^{2}}{4 t}}}}=\frac{2}{e^{\frac{\pi^{2}}{4 t}-1}}<\frac{2}{e^{\frac{\pi^{2}}{8 t}}}=2 \cdot e^{-\frac{\pi^{2}}{8 t}} .
\end{gathered}
$$

Now (7.2.6) follows from the elementary estimate

$$
\frac{1}{\sqrt{t}} \cdot e^{-\frac{c}{t}}<c_{1} \cdot e^{-\frac{c}{2 t}}
$$

Let $f, g:(0, \infty) \rightarrow \mathbf{R}$ are smooth functions, then we write $f \sim g$ if and only if for any natural number $m$ we have

$$
\lim _{t \rightarrow 0} \frac{f(t)-g(t)}{t^{m}}=0
$$

In particular we have just shown that for each $x, y \in S^{1}$

$$
e_{S^{1}}(t ; x, y) \sim \frac{1}{\sqrt{4 \pi t}} e^{-\frac{(x-y)^{2}}{t}}
$$

The next Corollary is the first result, which ties spectral geometry we study with the number theory. It is also at that point that we introduce trace of the heat operator.

Corollary 2.2.4.

$$
\begin{equation*}
\sum_{n \in \mathbf{N}} e^{-t n^{2}} \sim \frac{\sqrt{\pi}}{2} \cdot t^{-1 / 2}-\frac{1}{2} . \tag{2.2.3}
\end{equation*}
$$

Proof. We study the trace of the Heat Operator $e^{-t \Delta}$ on $S^{1}$. The operator $\Delta$ has a complete set of eigenvalues with corresponding eigensections given a standard basis of $L^{2}\left(S^{1}\right)$. Namely for any integer $k, k^{2}$ is an eigenvalue with corresponding eigenfunction $\phi_{k}=\frac{1}{\sqrt{2 \pi}} e^{i k x}$. We know that the trace of the operator can be represented by the eigensections and eigenfunctions as follows (see for instance one of our standard references like [45], and for more of related Functional Analysis [?], or [?, ?]). In our particular case we have

$$
\operatorname{Tr} e^{-t \Delta}=\sum_{k \in \mathbf{Z}}\left(e^{-t \Delta} \phi_{k} ; \phi_{k}\right)=\sum_{k \in \mathbf{Z}} e^{-t k^{2}}=1+2 \cdot \sum_{n \in \mathbf{N}} e^{-t n^{2}}
$$

where

$$
(f ; g)=\int_{S^{1}} f(x) \bar{g}(x) d x
$$

is standard $L^{2}$ product on $S^{1}$. On the other hand trace of the operator with a smooth kernel is also given by the integral of this kernel over the diagonal

$$
\operatorname{Tr} e^{-t \Delta}=\int_{S^{1}} e_{s^{1}}(t ; x, x) d x
$$

Therefore we have
$\sum_{n=1}^{\infty} e^{-t n^{2}}=\frac{1}{2}\left(\operatorname{Tr} e^{-t \Delta}-1\right)=\frac{1}{2}\left(\int_{S^{1}} e_{S^{1}}(t ; x, x) d x-1\right)=\frac{1}{2}\left(\frac{1}{\sqrt{4 \pi t}} \int_{S^{1}} d x-1\right)+O\left(e^{-\frac{c}{t}}\right)$,
which finally gives (2.2.3).

Now we investigate $\zeta_{\Delta}(s) \zeta$-function of the operator. Let us observe that this is immediately related to the number theory. Namely if we introduce Riemann $\zeta$-function

$$
\zeta_{\mathcal{R}}(s)=\sum_{n=1}^{\infty} n^{-s}
$$

then we have equality

$$
\begin{equation*}
\zeta_{\Delta}(s)=\sum_{k \in \mathbf{Z}}\left(k^{2}\right)^{-s}=2 \cdot \zeta_{\mathcal{R}}(2 s)+1 \tag{2.2.4}
\end{equation*}
$$

This is the reason that we can formulate the main result of this Section in terms of the $\zeta_{\mathcal{R}}(s)$. We have following Theorem

Theorem 2.2.5. Function $\zeta_{\mathcal{R}}(s)$ is a holomorphic function of $s$ for $R e(s)>$ 1. It has a meromorphic extension to complex plane. Point $s=1$ is the only pole of $\zeta_{\mathcal{R}}(s)$. It is a simple pole and we have

$$
\begin{equation*}
\operatorname{Res}_{s=1} \zeta_{\mathcal{R}}(s)=1 \quad, \quad \zeta_{\mathcal{R}}(0)=-\frac{1}{2} \text { and } \zeta_{\mathcal{R}}(-2 l)=0 \quad \text { for } l=1,2,3, \ldots \tag{2.2.5}
\end{equation*}
$$

Theorem 2.2.5 follows from the corresponding result for the $\zeta$-function of the operator $\mathcal{D}_{0}$ on $S^{1}$. We use representation (2.1.5). We also need two properties of the function $\Gamma(s)$. First let us recall that in the neighborhood of $s=0, \Gamma(s)$ has the following form

$$
\begin{equation*}
\Gamma(s)=\frac{1}{s}+\gamma+s f(s)=\frac{1+s \gamma+s^{2} f(s)}{s} \tag{2.2.6}
\end{equation*}
$$

where $\gamma$ denote Euler constant. We also use the identity

$$
\begin{equation*}
s \Gamma(s)=\Gamma(s+1), \tag{2.2.7}
\end{equation*}
$$

in order to extend $\Gamma(s)$, from the holomorphic function on $\operatorname{Re}(s)>1$ to a meromorphic function on the whole complex plane. Points $s_{k}=-k k=$ $0,1,2, \ldots$ are the only poles and

$$
\begin{equation*}
\operatorname{Res}_{s=-k} \Gamma(s)=\frac{(-1)^{k}}{k!}, \tag{2.2.8}
\end{equation*}
$$

as follows from (2.2.6) and (2.2.7). Let us also observe that now we can easily show that function $\frac{1}{\Gamma(s)}$ is a holomorphic function of $s$ on the whole complex plane and the only zeros of $\frac{1}{\Gamma(s)}$ are points $s_{k}=-k$. Now we are ready to analyze function $\zeta_{\Delta}(s)$.

Proposition 2.2.6. Let function $h(s)$ be given by the formula

$$
\begin{equation*}
h(s)=\frac{1}{\Gamma(s)} \cdot \int_{1}^{\infty} t^{s-1} \operatorname{Tr}\left(e^{-t \mathcal{D}^{2}}-\Pi_{\mathcal{D}}\right) d t \tag{2.2.9}
\end{equation*}
$$

Then $h(s)$ is a holomorphic function of $s$ on the whole complex plane.

Proof. We estimate $\operatorname{Tr}\left(e^{-t \mathcal{D}^{2}}-\Pi_{\mathcal{D}}\right)$ for $1<t$ as follows

$$
\begin{gathered}
\operatorname{Tr}\left(e^{-t \mathcal{D}^{2}}-\Pi_{\mathcal{D}}\right)=2 \sum_{k \in \mathbf{N}} e^{-t k^{2}}=2 \sum_{k \in \mathbf{N}} e^{-\frac{t}{2} k^{2}} \cdot e^{-\frac{t}{2} k^{2}}< \\
2 \cdot e^{-\frac{t}{2}} \cdot \sum_{k \in \mathbf{N}} e^{-\frac{t}{2} k^{2}}<2 \cdot e^{-\frac{t}{2}} \cdot \sum_{k \in \mathbf{N}} e^{-\frac{k^{2}}{2}}<c \cdot e^{-\frac{t}{2}}
\end{gathered}
$$

and now result follows from elementary complex ananlysis.

It is integral on the interval $0<t<1$, which determines singularities of $\zeta_{\Delta}(s)$. We have

$$
\begin{aligned}
& \frac{1}{\Gamma(s)} \cdot \int_{0}^{1} t^{s-1} \operatorname{Tr}\left(e^{-t \mathcal{D}^{2}}-\Pi(\mathcal{D})\right) d t=\frac{1}{\Gamma(s)} \cdot \int_{0}^{1} t^{s-1}\left(\int_{S^{1}}\left(\frac{1}{\sqrt{4 \pi t}}-1+O\left(e^{-\frac{c}{t}}\right) d x\right) d t=\right. \\
& \quad \frac{\sqrt{\pi}}{\Gamma(s)} \cdot \int_{0}^{1} t^{s-\frac{1}{2}} d t+g(s)=\frac{\sqrt{\pi}}{\Gamma(s)} \cdot \frac{1}{s-\frac{1}{2}}-\frac{1}{\Gamma(s)} \cdot \int_{0}^{1} t^{s-1} d t+\frac{1}{\Gamma(s)} \cdot g(s)
\end{aligned}
$$

where $g(s)$ is yet another function holomorphic on the whole complex plane and we use the fact that

$$
\operatorname{Tr} \Pi(\mathcal{D})=\operatorname{dim} \operatorname{ker} \mathcal{D}=1
$$

We have proved the following Theorem

Theorem 2.2.7. There exists a function $g(s)$ holomorphic on the whole complex plane such that $\zeta_{\Delta}(s)$ has the following representation

$$
\begin{equation*}
\zeta_{\Delta}(s)=\frac{\sqrt{\pi}}{\Gamma(s)} \cdot \frac{1}{s-\frac{1}{2}}-\frac{1}{\Gamma(s)} \cdot \frac{1}{s}+1+\frac{1}{\Gamma(s)} \cdot g(s) \tag{2.2.10}
\end{equation*}
$$

We combine this result with

$$
\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \text { and } \frac{1}{\Gamma(-k)}=0 \quad \text { for } k=0,1,2 . .
$$

in order to obtain next result

Corollary 2.2.8. The only pole of $\zeta_{\Delta}(s)$ is located at $s=\frac{1}{2}$. It is a simple pole and the reesiduum of $\Gamma(s)$ at $s=1$ is equal to 1 . Moreover $\zeta_{\Delta}(0)=0$ and $\zeta_{\Delta}(-k)=1$ for $k=1,2,3, \ldots$.

Theorem 2.2.5 follows immediately from Corollary 2.2.8 because now we can use (2.2.10) to represent $\zeta_{\mathcal{R}}(s)$ as

$$
\begin{equation*}
\zeta_{\mathcal{R}}(s)=\frac{1}{2}\left(\zeta_{\Delta}\left(\frac{s}{2}\right)-1\right)=\frac{\sqrt{\pi}}{\Gamma\left(\frac{s}{2}\right)} \cdot \frac{1}{s-1}-\frac{1}{\Gamma\left(\frac{s}{2}\right)} \cdot \frac{1}{s}+\frac{1}{2 \Gamma\left(\frac{s}{2}\right)} \cdot g\left(\frac{s}{2}\right) \tag{2.2.11}
\end{equation*}
$$

### 2.3. Duhamel's Principle and $\zeta$-Function of the Operators $\Delta_{f}$

In this Section we study the $\zeta$-function of the perturbations of the operator $\Delta=\Delta_{0}=\mathcal{D}_{0}^{2}$. We discuss in order with increasing technical difficulties operators $\Delta_{a}=\Delta+a, \Delta_{f}=\Delta+f(x)$, where $f(x)$ is a smooth function on $S^{1}$, and the operator $\mathcal{D}_{a}^{2}=-\frac{d^{2}}{d x^{2}}-2 i a \frac{d}{d x}+a^{2}$. The easiest example here is of course operator $\Delta_{a}$. It seems quite natural, that we expect the equality

$$
e^{-t \Delta_{a}}=e^{-t \Delta-t a}=e^{-t a} e^{-t \Delta}
$$

to hold, especially becuse the bounded operator $(B f)(x)=a f(x)$ commutes with the operator $\Delta$ and the equality

$$
e^{-t(A+B)}=e^{-t A} e^{-t B}
$$

holds for commuting matrices $A, B$ and for bounded commuting operators in a Hilbert space as well. The operator $\Delta$ however is an unbounded operator on $L^{2}\left(S^{1}\right)$, which creates technical problems. We discuss standard way of getting the Heat Operator for the perturbation of the given (Dirac) operator. We introduce now Duhamel's Principle. We offer a formal formulation without given detailed assumptions and later on we will show that everything can be make rigorous in the case of Dirac operators on $S^{1}$

Duhamel's Principle Let $\mathcal{A}$ and $\mathcal{A}+B$ are the operators acting on a separable Hilbert space such that the operators $e^{-t \mathcal{A}}$ and $e^{-t(\mathcal{A}+B)}$ exist. Then

$$
\begin{equation*}
e^{-t(\mathcal{A}+B)}=e^{-t \mathcal{A}}-\int_{0}^{t} e^{-s(\mathcal{A}+B)} B e^{-(t-s) \mathcal{A}} d s \tag{2.3.1}
\end{equation*}
$$

Proof. The Heat Operator is equal to identity at $t=0$, therefore

$$
e^{-t(\mathcal{A}+B)}-e^{-t \mathcal{A}}=\int_{0}^{t} \frac{d}{d s}\left(e^{-s(\mathcal{A}+B)} e^{-(t-s) \mathcal{A}}\right) d s=\int_{0}^{t} e^{-s(\mathcal{A}+B)}(-(\mathcal{A}+B)+\mathcal{A}) e^{-(t-s) \mathcal{A}} d s .
$$

Now we apply formula (2.3.1) to $e^{-s(\mathcal{A}+B)}$ in $\int_{0}^{t} e^{-s(\mathcal{A}+B)} B e^{-(t-s) \mathcal{A}} d s$ and we obtain
$e^{-t(\mathcal{A}+B)}=e^{-t \mathcal{A}}-\int_{0}^{t} e^{-s \mathcal{A}} B e^{-(t-s) \mathcal{A}} d s+\int_{0}^{t} d s \int_{0}^{s} d r e^{-r(\mathcal{A}+B)} B e^{-(s-r) \mathcal{A}} B e^{-(t-s) \mathcal{A}}$.

The discussion above shows the way of constructing the operator $e^{-t(\mathcal{A}+B)}$ having given operator $e^{-t / \mathcal{A}}$ as the infinite series in terms of the operators $e^{-t \mathcal{A}}$ and $B$. We introduce some notation before getting more specific. Let $\mathcal{B}(t)$ and $\mathcal{C}(t)$ denote two 1-parameter families of bounded operators in a Hilbert space. We introduce $(\mathcal{B} * \mathcal{C})(t)$ convolution of $\mathcal{B}(t)$ and $\mathcal{C}(t)$ as the operator

$$
(\mathcal{B} * \mathcal{C})(t)=\int_{0}^{t} \mathcal{B}(s) \mathcal{C}(t-s) d s
$$

We take

$$
\mathcal{B}(t)=e^{-t \mathcal{A}}, \quad \mathcal{C}(t)=B e^{-t \mathcal{A}} \text { and } \mathcal{C}_{n}(t)=(\mathcal{C} * \mathcal{C} * . . * \mathcal{C})(t)
$$

where we convoluted $\mathcal{C}(t) n$-times in the last formula. Then the following formal equation gives the Heat Operator of $\mathcal{A}+B$

$$
\begin{equation*}
e^{-t(\mathcal{A}+B)}=e^{-t \mathcal{A}}+\sum_{n=1}^{\infty}(-1)^{n}\left(\mathcal{B} * \mathcal{C}_{n}\right)(t) \tag{2.3.2}
\end{equation*}
$$

At this point it is a formal expression, which becomes rigorous identity whenever we are able to show that the series on the right side of (2.3.2) is absolutely convergent. Actually in our case we show even more. We use (2.3.2) in order to construct kernel of the operator $e^{-t \Delta_{f}}$. We take $\mathcal{A}=\Delta$ and $\mathcal{B}=f(x)$ then $e_{n, f}(t ; x, y)$ kernel of the operator $\left(\mathcal{B} * \mathcal{C}_{n}\right)(t)$ is equal to

$$
\begin{align*}
& e_{n, f}(t ; x, y)=\int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \int_{0}^{s_{1}} d s_{2} \int_{S^{1}} d u_{2} \ldots . \int_{0}^{s_{n-1}} d s_{n} \int_{S^{1}} d u_{n}  \tag{2.3.3}\\
& e_{\Delta}\left(s_{n} ; x, u_{n}\right) f\left(u_{n}\right) e_{\Delta}\left(s_{n-1} ; u_{n}, u_{n-1}\right) \ldots . . f\left(u_{1}\right) e_{\Delta}\left(t-s_{1} ; u_{1}, y\right)
\end{align*}
$$

We use this representation in order to prove the absolute and uniform convergence of the series, which represents kernel of the operator $e^{-t \Delta_{f}}$.

Theorem 2.3.1. There exists $t_{0}$ such that series

$$
\begin{equation*}
e_{\Delta}(t ; x, y)+\sum_{n=1}^{\infty}(-1)^{n} e_{n, f}(t ; x, y) \tag{2.3.4}
\end{equation*}
$$

converges uniformly on $S^{1}$ for $0<t \leq t_{0}$.

Remark 2.3.2. 1. Although we prove Theorem 2.3.1 for small $t_{0}$ the choice of $t_{0}$ is unsignificant. We can easily prove the following variant of the Theorem:

For any $t_{0}$ there exists a constant $M\left(t_{0}\right)$ such that for any $0<t \leq t_{0}$ and any $x, y \in S^{1}$

$$
\begin{equation*}
\left|e_{\Delta_{f}}(t ; x, y)-e_{\Delta}(t ; x, y)\right|=\left|\sum_{n=1}^{\infty}(-1)^{n} e_{n}(t ; x, y)\right|<M\left(t_{0}\right) \tag{2.3.5}
\end{equation*}
$$

We leave fun with the estimates to the reader.
2. We do not present the optimal result. In fact estimates in the proof were written down half an hour before the lecture. This is however not very important matter. We want to show the uninitiated reader the most crude way of proving that from the fact that $e^{-t \Delta}$ is well-defined follows the existence of the Heat Operator for the operator $\Delta_{f}$. We will use an extra information on the kernel $e_{S^{1}}(t ; x, y)$ later on.

Proof. There exist positive constants $c_{1}, c_{2}$ such that for any $x, y \in S^{1}$ and for any $0<t \leq 1$ we have

$$
\left|e_{\Delta}(t ; x, y)\right|<\frac{c_{1}}{\sqrt{t}},|f(x)|<c_{2}
$$

We start with the term $e_{1, f}(t ; x, y)$

$$
\begin{gathered}
\left|e_{1}, f(t ; x, y)\right|=\left|\int_{0}^{t} d s \int_{S^{1}} d u e_{\Delta}(s ; x, u) f(u) e_{\Delta}(t-s ; u, y)\right|< \\
\frac{c_{1}^{2} c_{2}}{4 \pi} \int_{0}^{t} d s \int_{S^{1}} d u \frac{1}{\sqrt{s(t-s)}}=\frac{c_{1}^{2} c_{2}}{2} \int_{0}^{t} \frac{d s}{\sqrt{s(t-s)}} \leq c_{1}^{2} c_{2} \int_{0}^{\frac{t}{2}} \frac{d s}{\sqrt{s \cdot \frac{t}{2}}}= \\
c_{1}^{2} c_{2} \cdot \sqrt{\frac{t}{2}} \cdot \int_{0}^{\frac{t}{2}} \frac{d s}{\sqrt{s}}=2 c_{1}^{2} c_{2} .
\end{gathered}
$$

The second inequality is even easier. Let $K>0$ denote a constant, then we have

$$
\left|\int_{0}^{t} d s \int_{S^{1}} d u K f(u) e_{\Delta}(t-s ; u, y)\right|<2 \pi c_{1} c_{2} K \cdot \int_{0}^{t} \frac{d s}{\sqrt{t-s}}=4 \pi c_{1} c_{2} K \sqrt{t}
$$

Now, we can deal with $e_{n, f}(t ; x, y)$

$$
\begin{gathered}
\left|e_{n, f}(t ; x, y)\right|=\mid \int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \int_{0}^{s_{1}} d s_{2} \int_{S^{1}} d u_{2} \ldots . \int_{0}^{s_{n-1}} d s_{n} \int_{S^{1}} d u_{n} \\
e_{\Delta}\left(s_{n} ; x, u_{n}\right) f\left(u_{n}\right) e_{\Delta}\left(s_{n-1}-s_{n} ; u_{n}, u_{n-1}\right) \ldots . . f\left(u_{1}\right) e_{\Delta}\left(t-s_{1} ; u_{1}, y\right) \mid< \\
2 c_{1}^{2} c_{2} \int_{0}^{t} d s_{1} \int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \int_{0}^{s_{1}} d s_{2} \int_{S^{1}} d u_{2} \ldots \int_{0}^{s_{n-2}} d s_{n-1} \int_{S^{1}} d u_{n-1} \\
\left|f\left(u_{n-1}\right) e_{\Delta}\left(s_{n-2}-s_{n-1} ; u_{n-1}, u_{n-2}\right) \ldots . . f\left(u_{1}\right) e_{\Delta}\left(t-s_{1} ; u_{1}, y\right)\right|< \\
\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2}\right) \int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \int_{0}^{s_{1}} d s_{2} \int_{S^{1}} d u_{2} \ldots \int_{0}^{s_{n-3}} d s_{n-2} \int_{S^{1}} d u_{n-2} \\
\sqrt{s_{n-2}} \cdot\left|f\left(u_{n-2}\right) e_{\Delta}\left(s_{n-3}-s_{n-2} ; u_{n-2}, u_{n-3}\right) \ldots . . f\left(u_{1}\right) e_{\Delta}\left(t-s_{1} ; u_{1}, y\right)\right|< \\
\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2}\right) \int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \int_{0}^{s_{1}} d s_{2} \int_{S^{1}} d u_{2} \ldots \sqrt{s_{n-3}} \int_{0}^{s_{n-3}} d s_{n-2} \int_{S^{1}} d u_{n-2} \\
\left|f\left(u_{n-2}\right) e_{\Delta}\left(s_{n-3}-s_{n-2} ; u_{n-2}, u_{n-3}\right) \ldots . . f\left(u_{1}\right) e_{\Delta}\left(t-s_{1} ; u_{1}, y\right)\right|< \\
\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2}\right)^{2} \int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \ldots . . \int_{0}^{s_{n-4}} d s_{n-3} \int_{S^{1}} d u_{n-3} s_{n-3} \\
\left|f\left(u_{n-3}\right) e_{\Delta}\left(s_{n-4}-s_{n-3} ; u_{n-3}, u_{n-4}\right) \ldots . . f\left(u_{1}\right) e_{\Delta}\left(t-s_{1} ; u_{1}, y\right)\right|< \\
<\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2}\right)^{n-1} \cdot t^{\frac{n-1}{2}}=\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2} \sqrt{t}\right)^{n-1} .
\end{gathered}
$$

Now we finally can estimate the whole series.

$$
\left|\sum_{n=1}^{\infty}(-1)^{n} e_{n}(t ; x, y)\right|<\sum_{n=1}^{\infty}\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2} \sqrt{t}\right)^{n-1}=\left(2 c_{1}^{2} c_{2}\right)\left(4 \pi c_{1} c_{2} \sqrt{t}\right) \cdot \frac{1}{1-4 \pi c_{1} c_{2} \sqrt{t}}
$$

We proved that the series (2.3.4) is uniformly convergent for $0<t<\frac{1}{\left(4 \pi c_{1} c_{2}\right)^{2}}$

We saw that correction terms entered in the succesive powers of $\sqrt{t}$, which leads us to the following fact

Corollary 2.3.3. There exists a family $\left\{u_{n}(x)\right\}_{n \in \mathbf{N}}$ of smooth functions on $S^{1}$ such that for any natural number $N$ we have

$$
\begin{equation*}
e_{\Delta_{f}}(t ; x, x)-\sum_{n=0}^{N} t^{\frac{n-1}{2}} u_{n}(x)=O\left(t^{N}\right) . \tag{2.3.6}
\end{equation*}
$$

The last result is not really the best we can get. We know (see for instance $[45, ?]$ ) that actually we have expansion in powers of $t$. This result is acxtually quite difficult to establish in the case of a general Dirac operator. In the case of $S^{1}$ we can use the fact that, for any $0<t$ and any $x, y \in S^{1}$ $e_{\Delta}(t ; x, y)>0$ (see (2.2.1). We also use the fact that identity $e^{-s \Delta} e^{-(t-s) \Delta}=$ $e^{-t \Delta}$ reads as follows on the level of the kernel of the operators

$$
\begin{equation*}
\int_{S^{1}} e_{\Delta}(s ; x, u) e_{\Delta}(t-s ; u, y) d u=e_{\Delta}(t ; x, y) \tag{2.3.7}
\end{equation*}
$$

Theorem 2.3.4. There exists a family $\left\{v_{n}(x)\right\}_{n \in \mathbf{N}}$ of smooth functions on $S^{1}$ such that for any natural number $N$ we have

$$
\begin{equation*}
e_{\Delta_{f}}(t ; x, x)-\sum_{n=0}^{N} t^{n-\frac{1}{2}} v_{n}(x)=O\left(t^{N}\right) . \tag{2.3.8}
\end{equation*}
$$

Proof. As I promised we show that indeed correction terms enter in powers of $t$. Let us explain that on the level of the first term

$$
\begin{gathered}
\left|\int_{0}^{t} d s \int_{S^{1}} d u e_{\Delta}(s ; x, u) f(u) e_{\Delta}(t-s ; u, y)\right|<c_{2} \cdot \int_{0}^{t} d s \int_{S^{1}} d u e_{\Delta}(s ; x, u) e_{\Delta}(t-s ; u, y)= \\
c_{2} \cdot \int_{0}^{t} e_{\Delta}(t ; x, y) d s=c_{2} t \cdot e_{\Delta}(t ; x, y)
\end{gathered}
$$

The estimate on the $n$ - th term of the expansion follows in the same way.

Corollary 2.3.5. For any $f(x)$ smooth function on $S^{1}$ we have

$$
\zeta_{\Delta_{f}}(0)=0 .
$$

Proof. The result follows from the corresponding result for $\Delta$ as we have just proved that there exists a positive constant $c$ such that the following estimate holds

$$
\left|e_{\Delta_{f}}(t ; x, y)-e_{\Delta}(t ; x, y)\right|<c \sqrt{t}
$$

for any $x, y \in S^{1}$ and any $0<t \leq 1$. Let $\Pi_{\Delta_{f}}$ denote orthogonal projection onto the kernel of the operator $\Delta_{f}$. Then we have

$$
\begin{aligned}
& \zeta_{\Delta_{f}}(s)=\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr}\left(e^{-t \Delta_{f}}-\Pi_{\Delta_{f}}\right) d t+\operatorname{dim} \operatorname{ker} \Delta_{f}= \\
& \frac{1}{\Gamma(s)} \int_{0}^{1} t^{s-1} \operatorname{Tr}\left(e^{-t \Delta_{f}}-\Pi_{\Delta_{f}}\right) d t+\operatorname{dim} \operatorname{ker} \Delta_{f}+\frac{1}{\Gamma(s)} \cdot h(s)
\end{aligned}
$$

where $h(s)$ is a function holomorphic on the whole complex plane. Trace of $\Pi_{\Delta_{f}}$ is equal to dim ker $\Delta_{f}$ and therefore we have

$$
\begin{aligned}
& \zeta_{\Delta_{f}}(s)=\frac{1}{\Gamma(s)} \int_{0}^{1} t^{s-1} \operatorname{Tr} e^{-t \Delta_{f}} d t+\left(\operatorname{dim} \operatorname{ker} \Delta_{f}\right) \cdot\left(1-\frac{1}{\Gamma(s)} \int_{0}^{1} t^{s-1} d s\right)+\frac{1}{\Gamma(s)} \cdot h(s)= \\
& \quad \frac{1}{\Gamma(s)} \int_{0}^{1} t^{s-1}\left(\frac{\sqrt{\pi}}{\sqrt{t}}+O(\sqrt{t})\right) d t+\left(\operatorname{dim} \operatorname{ker} \Delta_{f}\right) \cdot\left(1-\frac{s}{\Gamma(s)}\right)+\frac{1}{\Gamma(s)} \cdot h(s)= \\
& \frac{\sqrt{\pi}}{\Gamma(s)} \cdot \frac{1}{s-\frac{1}{2}}+\frac{1}{\Gamma(s)} \int_{0}^{1} t^{s-1} O(\sqrt{t}) d t+\left(\operatorname{dim} \operatorname{ker} \Delta_{f}\right) \cdot\left(1-\frac{s}{\Gamma(s)}\right)+\frac{1}{\Gamma(s)} \cdot h(s) .
\end{aligned}
$$

We take $\lim _{s \rightarrow 0}$ and obtain 0.

As the exercise reader may use Duhamel's Principle to check that

$$
\begin{equation*}
e_{\Delta+a}(t ; x, y)=e^{-t a} e_{\Delta}(t ; x, y) \tag{2.3.9}
\end{equation*}
$$

### 2.4. Heat Kernel of the Operator $\mathcal{D}_{a}^{2}$

In this Section we use Duhamel's Principle to construct kernel of the operator $e^{-t \mathcal{D}_{a}^{2}}$ on $S^{1}$. Our goal is to prove analogue of Theorem 2.3.4 in this new situation. We have

$$
\mathcal{D}_{a}^{2}=-\Delta-2 i a \frac{d}{d x}+a^{2}=-\Delta+2 a \mathcal{D}_{0}+a^{2}
$$

We know that we do not have any problem with $a^{2}$. We simply have:

$$
e_{-\Delta+2 a \mathcal{D}_{0}+a^{2}}(t ; x, y)=e^{-t a^{2}} e_{-\Delta+2 a \mathcal{D}_{0}}(t ; x, y)
$$

The problem here is that the perturbation is now of order 1 , hence it is not a bounded operator on $L^{2}\left(S^{1}\right)$. Stil we can study kernel of the operator $e^{-\Delta+2 a \mathcal{D}_{0}}$ the way we did it in a previous Section. We have to show absolute convergence of the series

$$
\begin{equation*}
e_{\Delta}(t ; x, y)+\sum_{n=1}^{\infty} e_{n, \mathcal{D}_{a}}(t ; x, y) \tag{2.4.1}
\end{equation*}
$$

where

$$
\begin{gathered}
e_{n, \mathcal{D}_{a}}(t ; x, y)=\int_{0}^{t} d s_{1} \int_{S^{1}} d u_{1} \int_{0}^{s_{1}} d s_{2} \int_{S^{1}} d u_{2} \ldots . \int_{0}^{s_{n-1}} d s_{n} \int_{S^{1}} d u_{n} \\
e_{\Delta}\left(s_{n} ; x, u_{n}\right) 2 a \mathcal{D}_{0} e_{\Delta}\left(s_{n-1} ; u_{n}, u_{n-1}\right) \ldots . .2 a \mathcal{D}_{0} e_{\Delta}\left(t-s_{1} ; u_{1}, y\right) .
\end{gathered}
$$

Let us first figure out the straightforward estimates on $e_{n, \mathcal{D}_{a}}(t ; x, y)$. We have

$$
\begin{gathered}
\left|e_{1, \mathcal{D}_{a}}(t ; x, y)\right|=2 a \cdot\left|\int_{0}^{t} d s \int_{S^{1}} e_{\Delta}(s ; x, u)\left(\frac{d}{d u}\right) e_{\Delta}(t-s ; u, y)\right| \leq \\
2 a \frac{1}{4 \pi} \cdot \int_{0}^{t} \frac{d s}{\sqrt{s(t-s)}} \int_{S^{1}} e^{-\frac{(x-u)^{2}}{4 t}} \cdot \frac{|u-y|}{2(t-s)} e^{-\frac{(u-y)^{2}}{4 t}} \leq \\
2 a \frac{1}{4 \pi} \cdot \int_{0}^{t} \frac{d s}{\sqrt{s(t-s)}} \int_{S^{1}} \frac{|u-y|}{2(t-s)} e^{-\frac{(u-y)^{2}}{4 t}}<\frac{2}{4 \pi} \cdot \int_{0}^{t} \frac{d s}{\sqrt{s(t-s)}} \int_{0}^{\infty} \frac{z}{2(t-s)} e^{-\frac{z^{2}}{4 t}}= \\
2 a \frac{1}{2 \pi} \int_{0}^{t} \frac{d s}{\sqrt{s(t-s}} \leq 2 a \frac{1}{2 \pi} \frac{2}{t} \cdot \int_{0}^{t} d s=2 a \frac{1}{\pi} .
\end{gathered}
$$

More general we have

$$
\begin{equation*}
\left|e_{n, \mathcal{D}_{a}}(t ; x, y)\right|<\frac{1}{\pi}(2 a)^{n} \frac{1}{\frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{n-1}{2}}\left(\frac{1}{\sqrt{\pi}}\right)^{n-1} t^{\frac{n-1}{2}} \tag{2.4.2}
\end{equation*}
$$

and as in Section 2.3 we proved the uniform and absolute convergence of the series which formally gives kernel of the operator $e^{-\Delta+2 a \mathcal{D}_{0}}$, hence we have just constructed this kernel. Actually again we expect that terms with odd $n$ should disappear. This follows from the general theory (see for instance [45] Chapter 1, or [?]). However in our case we can offer a simple argument which shows that $\operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}}$ expands in powers of $t$ rather than $\sqrt{t}$.

Theorem 2.4.1. There exists a sequence of real numbers $\left\{r_{k}\right\}_{k=0}^{\infty}$ such that for any natural number $N$ we have

$$
\begin{equation*}
\operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}}-\sum_{k=0}^{N} t^{k-\frac{1}{2}} r_{k}=O\left(t^{N}\right) \tag{2.4.3}
\end{equation*}
$$

Remark 2.4.2. If we work harder we would be able to prove that there exist a sequence of smooth functions $\left\{f_{k}(x)\right\}$ such that

$$
e_{\mathcal{D}_{a}^{2}}(t ; x, x)-\sum_{k=0}^{N} t^{k-\frac{1}{2}} f_{k}(x)=O\left(t^{N}\right) .
$$

This "Local Variant" of Theorem 2.4.1 for the general Dirac operator is proved in $[45, ?]$.

Proof. We use the fact that operators $\Delta$ and $\mathcal{D}_{0}=-i \frac{d}{d x}$ commute. Now we write a series which gives the operator $e^{-t\left(\Delta+2 a \mathcal{D}_{0}\right)}$

$$
e^{-t\left(\Delta+2 a \mathcal{D}_{0}\right)}=e^{-t \Delta}+(-1)^{n} E_{n}(t)
$$

where

$$
\begin{gathered}
E_{n}(t)=(2 a)^{n} \int_{0}^{t} d s_{1} \int_{0}^{s_{1}} d s_{2} \ldots \int_{0}^{s_{n-1}} d s_{n} e^{-s_{n} \Delta} \mathcal{D}_{0} e^{-\left(s_{n-1}-s_{n}\right) \Delta} \mathcal{D}_{0} \ldots \mathcal{D}_{0} e^{-\left(t-s_{1}\right) \Delta}= \\
\left.(2 a)^{n}\left(\mathcal{D}_{0}\right)^{n} e^{-t \Delta} \cdot \int_{0}^{t} d s_{1} \int_{0}^{s_{1}} d s_{2} \ldots \int_{0}^{s_{n-1}} d s_{n}=(2 a)^{n} \cdot \frac{t^{n}}{n!} \mathcal{D}_{0}\right)^{n} e^{-t \Delta}
\end{gathered}
$$

Now Theorem 2.4.1 follows from the fact that $\mathcal{D}_{0}=-i \frac{d}{d x}$ has symmetric spectrum which implies the equality

$$
\operatorname{Tr} \mathcal{D}_{0}^{2 n+1} e^{-t \mathcal{D}^{2}}=0
$$

for any natural number $n$.

Corollary 2.4.3. For any real a

$$
\begin{equation*}
\zeta_{\mathcal{D}_{a}^{2}}(0)=0 . \tag{2.4.4}
\end{equation*}
$$

Proof. The result follows from the fact that we have just proved

$$
\operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}}-\operatorname{Tr} e^{-t \mathcal{D}_{0}^{2}}<\frac{c}{\sqrt{t}}
$$

for $0<t<1$.

## 2.5. $\eta$-Invariant - The Phase of the $\zeta$-Determinant

In this Section we study the $\eta$-invariant of the operator $\mathcal{D}_{a}$. The main result is

Theorem 2.5.1. The function $\eta_{\mathcal{D}_{a}}(s)$ is a holomorphic function of $s$ for $\operatorname{Re}(s)>-2$.

Once again we prove this result by studying Heat Kernels. We remember formula

$$
\eta_{\mathcal{D}_{a}}(s)=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \cdot \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t
$$

We now decompose $\operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}}$ and use the fact that $\operatorname{Tr} \mathcal{D}_{0} e^{-t \mathcal{D}_{0}^{2}}$ is equal to 0

$$
\operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}}=\operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}}-\operatorname{Tr} \mathcal{D}_{0} e^{-t \mathcal{D}_{0}^{2}}=
$$

$\operatorname{Tr}\left(\mathcal{D}_{a}-\mathcal{D}_{0}\right) e^{-t \mathcal{D}_{a}^{2}}-\operatorname{Tr} \mathcal{D}_{0}\left(e^{-t \mathcal{D}_{a}^{2}}-e^{-t \mathcal{D}_{0}^{2}}\right)=a \cdot \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}}-\operatorname{Tr} \mathcal{D}_{0}\left(e^{-t \mathcal{D}_{a}^{2}}-e^{-t \mathcal{D}_{0}^{2}}\right)$.
We know the expansion of the first summand on the right side

$$
\begin{equation*}
a \cdot \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}}-a \cdot \sqrt{\frac{\pi}{t}}+c_{1} \cdot \sqrt{t}=O\left(t^{\frac{3}{2}}\right) . \tag{2.5.1}
\end{equation*}
$$

Now we remember that

$$
e^{-t \mathcal{D}_{a}^{2}}=e^{-t a^{2}} e^{-t\left(\Delta-2 a \mathcal{D}_{0}\right)}=e^{-t a^{2}}\left(e^{-t\left(\Delta-2 a \mathcal{D}_{0}\right)}-e^{-t \Delta}\right)+e^{-t a^{2}} e^{-t \Delta},
$$

and we have to study now
$\operatorname{Tr} \mathcal{D}_{0}\left(e^{-t \mathcal{D}_{a}^{2}}-e^{-t \mathcal{D}_{0}^{2}}\right)=\operatorname{Tr} e^{-t a^{2}} \mathcal{D}_{0}\left(e^{-t\left(\Delta-2 a \mathcal{D}_{0}\right)}-e^{-t \Delta}\right)+\operatorname{Tr}\left(e^{-t a^{2}}-I d\right) \mathcal{D}_{0} e^{-t \Delta}$.

The second term on the right side is again equal to 0 and finally we only have to show that $\operatorname{Tr} e^{-t a^{2}} \mathcal{D}_{0}\left(e^{-t\left(\Delta-2 a \mathcal{D}_{0}\right)}-e^{-t \Delta}\right)$ has the correct asymptotic. It was already observed in Section 2.4 that

$$
\begin{equation*}
e^{-t\left(\Delta-2 a \mathcal{D}_{0}\right)}-e^{-t \Delta}=\sum_{n=1}^{\infty} \frac{(2 a t)^{n}}{n!} \mathcal{D}_{0}^{n} e^{-t \Delta} \tag{2.5.3}
\end{equation*}
$$

hence we do have

$$
\begin{gathered}
\operatorname{Tr} \mathcal{D}_{0}\left(e^{-t \mathcal{D}_{a}^{2}}-e^{-t \mathcal{D}_{0}^{2}}\right)=\operatorname{Tr} e^{-t a^{2}} \mathcal{D}_{0} e^{-t\left(\Delta-2 a \mathcal{D}_{0}\right)}= \\
\operatorname{Tr} e^{-t a^{2}} \sum_{n=1}^{\infty} \frac{(2 a t)^{n}}{n!} \mathcal{D}_{0}^{n+1} e^{-t \Delta}=\operatorname{Tr} e^{-t a^{2}} \sum_{k=1}^{\infty} \frac{(2 a t)^{2 k-1}}{(2 k-1)!} \mathcal{D}_{0}^{2 k} e^{-t \Delta} .
\end{gathered}
$$

Now let us use an extra symmetry we have in this formula

$$
\begin{equation*}
\operatorname{Tr} \mathcal{D}_{0}^{2 k} e^{-t \Delta}=(-1)^{k}\left(\frac{d}{d t}\right)^{k}\left(\operatorname{Tr} e^{-t \mathcal{D}^{2}}\right)=(-1)^{k}\left(\frac{d}{d t}\right)^{k}\left(\frac{\sqrt{\pi}}{\sqrt{t}}+O\left(e^{-\frac{c}{t}}\right)\right) . \tag{2.5.4}
\end{equation*}
$$

We have proved following result

Proposition 2.5.2. There exist a sequence of constants $\left\{b_{k}\right\}$ such that

$$
\begin{equation*}
\operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} \sim \frac{b_{0}}{\sqrt{t}}+b_{1} \sqrt{t}+\ldots \tag{2.5.5}
\end{equation*}
$$

However we are not out of trouble if the situation in the neighborhood of $s=0$ is concerned. This is due to the fact that we have to study

$$
\lim _{s \rightarrow 0} \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} D d_{a} e^{-t \mathcal{D}_{a}^{2}} d t
$$

and in this situation factor $\frac{1}{\Gamma(s)}$ is replaced by $\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}$, hence the singularity which comes from the heat kernel is not cancelled by singularity of $\Gamma(s)$. Now, when you study more precisely formulas (2.5.1) and (7.2.3) . We come to a conclusion that actually we have

$$
\begin{equation*}
b_{0}=0 \tag{2.5.6}
\end{equation*}
$$

in (2.5.5), but as in these notes a focus is on different methods related to Heat Kernel we offer yet another argument. This type of argument is used quite often in different contexts in Spectral Geometry. We define a function

$$
\mathcal{R}(a)=\operatorname{Res}_{s=0} \eta_{\mathcal{D}_{a}}(0)
$$

Theorem 2.5.3.

$$
\begin{equation*}
\frac{d \mathcal{R}}{d a}=0 \tag{2.5.7}
\end{equation*}
$$

We need the next Lemma in the proof of Theorem 2.5.3.

Lemma 2.5.4.

$$
\begin{equation*}
e^{-\dot{t} \mathcal{D}_{a}^{2}}=\frac{d}{d a}\left(e^{-t \mathcal{D}_{a}^{2}}\right)=-\int_{0}^{t} e^{-s \mathcal{D}_{a}^{2}}\left(\dot{\mathcal{D}}_{a} \mathcal{D}_{a}+\mathcal{D}_{a} \dot{\mathcal{D}}_{a}\right) e^{-(t-s) \mathcal{D}_{a}^{2}} d s \tag{2.5.8}
\end{equation*}
$$

REmARK 2.5.5. Of course (2.5.8) is the formula which holds for the smooth, 1-parameter family of Dirac operators over closed manifold. In the particular case of the operator $\mathcal{D}_{a}=-i \frac{d}{d x}+a$ on $S^{1}$, we have $\dot{\mathcal{D}}_{a}=\frac{d D_{a}}{d a}=1$ and (2.5.8) becomes

$$
e^{-t \mathcal{D}_{a}^{2}}=-2 \int_{0}^{t} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d s=-2 t \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}}
$$

Proof. In the Lemma 2.5.4 we study variation of the Heat Kernel under the smooth change of the Dirac operator

$$
\begin{aligned}
& \frac{d}{d a}\left\{e^{-t \mathcal{D}_{a}^{2}}\right\}=\lim _{r \rightarrow 0} \frac{1}{r} \cdot\left(e^{-t \mathcal{D}_{a+r}^{2}}-e^{-t \mathcal{D}_{a}^{2}}\right)=\lim _{r \rightarrow 0} \frac{1}{r} \cdot \int_{0}^{t} \frac{d}{d s}\left(e^{-s \mathcal{D}_{a+r}^{2}} e^{-(t-s) \mathcal{D}_{a}^{2}}\right) d s= \\
& \lim _{r \rightarrow 0} \frac{1}{r} \cdot \int_{0}^{t} e^{-s \mathcal{D}_{a+r}^{2}}\left(\mathcal{D}_{a}^{2}-\mathcal{D}_{a+r}^{2}\right) e^{-(t-s) \mathcal{D}_{a}^{2}} d s= \\
& \lim _{r \rightarrow 0} \int_{0}^{t} e^{-s \mathcal{D}_{a}^{2}} \frac{\mathcal{D}_{a}^{2}-\mathcal{D}_{a+r}^{2}}{r} e^{-(t-s) \mathcal{D}_{a}^{2}} d s+\lim _{r \rightarrow 0} \int_{0}^{t} \frac{1}{r}\left(e^{-s \mathcal{D}_{a+r}^{2}}-e^{-s \mathcal{D}_{a}^{2}}\right)\left(\mathcal{D}_{a}^{2}-\mathcal{D}_{a+r}^{2}\right) e^{-(t-s) \mathcal{D}_{a}^{2}} d s=
\end{aligned}
$$

$$
-\int_{0}^{t} e^{-s \mathcal{D}_{a}^{2}} \dot{\mathcal{D}}_{a}^{2} e^{-(t-s) \mathcal{D}_{a}^{2}} d s+0=-\int_{0}^{t} e^{-s \mathcal{D}_{a}^{2}}\left(\dot{\mathcal{D}}_{a} \mathcal{D}_{a}+\mathcal{D}_{a} \dot{\mathcal{D}}_{a}\right) e^{-(t-s) \mathcal{D}_{a}^{2}} d s
$$

Proof. Now we are ready to prove Theorem 2.5.3.
We have

$$
\begin{equation*}
\mathcal{R}(a)=\lim _{s \rightarrow 0} s \cdot \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t \tag{2.5.9}
\end{equation*}
$$

and we study the variation of the integral on the right side of (2.5.9).

$$
\begin{aligned}
& \frac{d}{d a}\left\{\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t\right\}=\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t+\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a}\left\{e^{-t \mathcal{D}_{a}^{2}}\right\} d t= \\
& \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t+\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a} e^{-\dot{t} \mathcal{D}_{a}^{2}} d t= \\
& \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t-2 \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} t \dot{\mathcal{D}}_{a} \mathcal{D}_{a}^{2} e^{-t \mathcal{D}_{a}^{2}}=\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t+ \\
& \left.2 \int_{0}^{\infty} t^{\frac{s+1}{2}} \frac{d}{d t}\left(\operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}}\right) d t=\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t+2 \lim _{\epsilon \rightarrow 0} t^{\frac{s+1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-\epsilon \mathcal{D}_{a}^{2}}\right)\left|\left.\right|_{0} ^{\frac{1}{\varepsilon}}-\right. \\
& 2 \int_{0}^{\infty} \frac{d}{d t}\left(t^{\frac{s+1}{2}}\right) \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t
\end{aligned}
$$

The limit

$$
\left.\lim _{\epsilon \rightarrow 0}\left(t^{\frac{s+1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-\epsilon \mathcal{D}_{a}^{2}}\right)\right|_{\varepsilon} ^{\frac{1}{\varepsilon}}
$$

is equal to 0 for the invertible operator $\mathcal{D}_{a}$ and $s>0$ and we obtain a crucial formula

$$
\begin{equation*}
\frac{d}{d a}\left\{\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t\right\}=-s \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t \tag{2.5.10}
\end{equation*}
$$

Once again formula simplifies in the case of $\mathcal{D}_{a}$ on $S^{1}$ as $\dot{\mathcal{D}}_{a}=1$. Now we finally have information about the variation of the residuum

$$
\frac{d \mathcal{R}}{d a}=\frac{d}{d a} \lim _{s \rightarrow 0} s \cdot \eta_{\mathcal{D}_{a}}(s)=-\lim _{s \rightarrow 0} \frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \frac{d}{d a}\left\{\int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t\right\}=
$$

$$
-\frac{1}{\sqrt{\pi}} \lim _{s \rightarrow 0} s^{2} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t=0
$$

Let us observe that in fact we also obtain the formula for the variation of the $\eta$-invariant i.e. the number $\eta_{\mathcal{D}_{a}}(0)$.

$$
\begin{gathered}
\dot{\eta}_{\mathcal{D}_{a}}(0)=\lim _{s \rightarrow 0} \frac{-s}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t=-\frac{1}{\sqrt{\pi}} \cdot \lim _{s \rightarrow 0} s \cdot \int_{0}^{1} t^{\frac{s-1}{2}} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t= \\
-\frac{1}{\sqrt{\pi}} \cdot \lim _{s \rightarrow 0} s \cdot \int_{0}^{1} t^{\frac{s-1}{2}}\left(\sqrt{\frac{\pi}{t}}+r_{1} \sqrt{t}+O\left(t^{\frac{3}{2}}\right)\right) d t=-\frac{1}{\sqrt{\pi}} \cdot \lim _{s \rightarrow 0} s \cdot \int_{0}^{1} t^{\frac{s-1}{2}} \sqrt{\frac{\pi}{t}} d t= \\
\quad-\lim _{s \rightarrow 0} s \cdot \int_{0}^{1} t^{\frac{s}{2}-1} d t=-\lim _{s \rightarrow 0} s \cdot \frac{1}{\frac{s}{2}}=-2 .
\end{gathered}
$$

Theorem 2.5.3 implies that $\eta_{\mathcal{D}_{a}}(s)$ is a holomorphic function of $s$ in the neighborhood of $s=0$, beacause we know that

$$
\mathcal{R}(0)=0
$$

and by Theorem 2.5.3, this extends to any real number $a$. In fact we have proved more, because equality (2.5.5) implies that there exists constant $c>0$ such that for any $0<t \leq 1$ we have

$$
\begin{equation*}
\left|\operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}}\right|<c \sqrt{t}, \tag{2.5.11}
\end{equation*}
$$

which gives us representation

$$
\operatorname{Tr} D d_{a} e^{-t \mathcal{D}_{a}^{2}}=b_{1} \sqrt{t}+b_{2} t^{\frac{3}{2}}+O\left(t^{\frac{5}{2}}\right) .
$$

Now we can discussed structure of the $\eta$-function of $\mathcal{D}_{a}$

$$
\begin{gathered}
\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}_{a}^{2}} d t=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}}\left(b_{1} \sqrt{t}+b_{2} t^{\frac{3}{2}}+O\left(t^{\frac{5}{2}}\right)\right) d t= \\
\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{1} t^{\frac{s-1}{2}}\left(b_{1} \sqrt{t}+b_{2} t^{\frac{3}{2}}+O\left(t^{\frac{5}{2}}\right)\right) d t+h(s)= \\
\frac{b_{1}}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{1} t^{\frac{s}{2}} d t+\frac{b_{2}}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{1} t^{\frac{s}{2}+1}+h_{1}(s)+h(s)
\end{gathered}
$$

where $h(s)$ is a holomorphic function on the whole complex plane and $h_{1}(s)$ is holomorphic for $\operatorname{Re}(s)>-6$. This shows that there exists $h_{2}(s)$ a function holomorphic for $\operatorname{Re}(s)>-6$, such that

$$
\begin{equation*}
\eta_{\mathcal{D}_{a}}(s)=\frac{b_{1}}{\Gamma\left(\frac{s+1}{2}\right)} \cdot \frac{2}{s+2}+\frac{b_{2}}{\Gamma\left(\frac{s+1}{2}\right)} \cdot \frac{2}{s+4}+h_{2}(s) . \tag{2.5.12}
\end{equation*}
$$

In particular Theorem 2.5.1 is proved.

Corollary 2.5.6.

$$
\begin{equation*}
\eta_{\mathcal{D}_{a}}(0)=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t \tag{2.5.13}
\end{equation*}
$$

Proof. Theorem 2.5.1 implies that we can apply second formula in (2.1.4) for any $s$ with $\operatorname{Re}(s)>-2$.

Corollary 2.5.7.

$$
\begin{equation*}
\dot{\eta}_{\mathcal{D}_{a}}(0)=-\frac{2}{\sqrt{\pi}} \cdot \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \cdot \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-\varepsilon \mathcal{D}_{a}^{2}} \tag{2.5.14}
\end{equation*}
$$

which in the case of operator $\mathcal{D}_{a}$ on $S^{1}$ gives $\dot{\eta}_{\mathcal{D}_{a}}(0)=-2$.

Proof. We differentiate equation (2.5.14)

$$
\begin{aligned}
& \frac{d}{d a} \eta_{\mathcal{D}_{a}}(0)=\frac{d}{d a}\left(\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t\right)=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t+ \\
& \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \sqrt{t} \cdot \frac{d}{d t}\left(\operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}}\right) d t=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t+ \\
& \left.\frac{2}{\sqrt{\pi}} \cdot \lim _{\varepsilon \rightarrow 0}\left(\sqrt{t} \cdot \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}}\right)\right|_{\varepsilon} ^{\frac{1}{\varepsilon}}-\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-t \mathcal{D}_{a}^{2}} d t=-\frac{2}{\sqrt{\pi}} \cdot \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \cdot \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-\varepsilon \mathcal{D}_{a}^{2}} .
\end{aligned}
$$

In the case of $\mathcal{D}_{a}$ on $S^{1}$ we have

$$
-\frac{2}{\sqrt{\pi}} \cdot \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \cdot \operatorname{Tr} \dot{\mathcal{D}}_{a} e^{-\varepsilon \mathcal{D}_{a}^{2}}=-\frac{2}{\sqrt{\pi}} \cdot \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \cdot\left(\sqrt{\frac{\pi}{\varepsilon}}+O(\sqrt{\varepsilon})\right)=-2 .
$$

Theorem 2.5.8.

$$
\begin{equation*}
\eta_{\mathcal{D}_{a}}(0)=1-2 a . \tag{2.5.15}
\end{equation*}
$$

Proof. We have just proved that

$$
\eta_{\mathcal{D}_{a}}(0)=-2 a \bmod \mathbf{Z}
$$

More precisely $\tilde{\eta}_{\mathcal{D}_{a}}(0)$ continous part of $\eta_{\mathcal{D}_{a}}(0)$ (as function of $a$ ) is equal to $-2 a$. Now we see that we can argue that formula

$$
\eta_{\mathcal{D}_{a}}(s)=\frac{1}{a^{s}}+\sum_{k \neq 0} \operatorname{sign}(k+a)|k+a|^{s}
$$

invites us to put $\lim _{s \rightarrow 0} \eta_{\mathcal{D}_{a}}(s)=1+\tilde{\eta}_{\mathcal{D}_{a}}(0)=1-2 a$. However, we can argue for some other choice as well. As we can see in the next section our choice of the integer comes from the determinant theory.

It follows now from Theorem 2.5.8 and Corollary 2.4.3 that we know the phase of the $\zeta$-determinant of the operator $\mathcal{D}_{a}$

Corollary 2.5.9. The pahse of the $\operatorname{det}_{\zeta} \mathcal{D}_{a}$ is eqqual to $e^{2 a-1}$.

In the next Section we deal with the modulus of the $\zeta$-determinant.

### 2.6. Modulus of $\operatorname{det}_{\zeta} \mathcal{D}_{a}$

There are some reason beyond the scope of this notes (see for instance $[\mathbf{9 4}, \mathbf{9 5}]$, which tell us that we should pick up the operator $\mathcal{D}_{\frac{1}{2}}=-i \frac{d}{d x}+\frac{1}{2}$ and assume that its $\zeta$-determinant is equal to 1 and then obtain the value of the $\zeta$-determinant for other $\mathcal{D}_{a}$ by studying variation of the determinant with respect to $a$. We will show later on that we also obtain equality

$$
\operatorname{det}_{\zeta} \mathcal{D}_{\frac{1}{2}}=1
$$

from the argument from Elementary Number Theory. We start however with study of the variation of the modulus with respect to the parameter $a$. Now, then phase is $-h a l f$ of $\zeta_{\mathcal{D}_{a}^{2}}^{\prime}(0)=\left.\frac{d}{d s} z_{\mathcal{D}_{a}^{2}}^{\prime}(s)\right|_{s=0}$. We know that $\zeta_{D_{a}^{2}}(s)$ is holomorphic in the neighborhood of $s=0$ and we can take the derivative with respect to $s$, which gives

$$
\begin{aligned}
& \lim _{s \rightarrow 0} \frac{d}{d s}\left(\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t\right)=\lim _{s \rightarrow 0}\left(-\frac{\Gamma^{\prime}(s)}{\Gamma(s)^{2}} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t\right)+ \\
& \lim _{s \rightarrow 0}\left(\frac{1}{\Gamma(s)} \frac{d}{d s}\left(\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t\right)\right)=\lim _{s \rightarrow 0}\left(-\frac{\Gamma^{\prime}(s)}{\Gamma(s)^{2}} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t\right)
\end{aligned}
$$

Remark 2.6.1. The fact that $\lim _{s \rightarrow 0}\left(\frac{1}{\Gamma(s)} \frac{d}{d s}\left(\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t\right)\right)$ dissapear is due to the absence of the singularity of the function $\mathcal{K}(s)=\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t$ at $s=0$ and it is characteristic for dimension 1 , or more general for odd dimensional manifold $M$. In the even-dimensional case this part may produce additonal contribution.

We know that $\Gamma^{\prime}(s)=-\frac{1}{s^{2}}+h(s)$, where $h(s)$ is a holomorphic function in the neighborhood of $s=0$, hence we arrived at the equation

$$
\begin{equation*}
\left.\frac{d}{d s}\left(\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t\right)\right|_{s=0}=\lim _{s \rightarrow 0} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t=\int_{0}^{\infty} \frac{1}{t} \cdot \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t \tag{2.6.1}
\end{equation*}
$$

We use Lemma 2.5.4 to study the variation of the right side of (2.6.1) and obtain

$$
\begin{equation*}
\frac{d}{d a} \int_{0}^{\infty} \frac{1}{t} \cdot \operatorname{Tr} e^{-t \mathcal{D}_{a}^{2}} d t=-2 \cdot \int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}}_{a} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t=-2 \cdot \int_{0}^{\infty} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t=-2 \cdot \eta_{\mathcal{D}_{a}}(1) \tag{2.6.2}
\end{equation*}
$$

We work on the expression $\int_{0}^{\infty} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t$ in order to obtain the variation. We may assume that $\mathcal{D}_{a}$ is invertible, hence

$$
\begin{gathered}
\int_{0}^{\infty} \operatorname{Tr} \mathcal{D}_{a} e^{-t \mathcal{D}_{a}^{2}} d t=\int_{0}^{\infty} \operatorname{Tr} \mathcal{D}_{a}^{-1} \mathcal{D}_{a}^{2} e^{-t \mathcal{D}_{a}^{2}} d t=-\int_{0}^{\infty} \frac{d}{d t}\left(\operatorname{Tr} \mathcal{D}_{a}^{-1} e^{-t \mathcal{D}_{a}^{2}}\right) d t= \\
\left.\lim _{\varepsilon \rightarrow 0}\left(\operatorname{Tr} \mathcal{D}_{a}^{-1} e^{-t \mathcal{D}_{a}^{2}}\right)\right|_{\varepsilon} ^{\frac{1}{\varepsilon}}=-\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \mathcal{D}_{a}^{-1} e^{-\varepsilon \mathcal{D}_{a}^{2}}
\end{gathered}
$$

We have just proved

Proposition 2.6.2.

$$
\begin{equation*}
\frac{d}{d a} \zeta_{\mathcal{D}_{a}^{2}}^{\prime}(0)=2 \cdot \lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \mathcal{D}_{a}^{-1} e^{-\varepsilon \mathcal{D}_{a}^{2}} \tag{2.6.3}
\end{equation*}
$$

We need a formula for the operator $\mathcal{D}_{a}^{-1}$ in order to get the right side of (2.6.3) . The point is that we have an explicit formula for $k_{a}(x, y)$ kernel of the operator $\mathcal{D}_{a}^{-1}$

$$
k_{a}(x, y):=\left\{\begin{array}{ll}
-\frac{i e^{-i(x-y)}}{1-e^{2 \pi i a}} & \text { for } x<y  \tag{2.6.4}\\
+\frac{i e^{-i(x-y)}}{1-e^{-2 \pi i a}} & \text { for } x>y
\end{array} .\right.
$$

Now we have

$$
\begin{gathered}
\frac{d}{d a} \zeta_{\mathcal{D}_{a}^{2}}^{\prime}(0)=2 \cdot \lim _{\varepsilon \rightarrow 0} \int_{S^{1}} d x \int_{S^{1}} d y k_{a}(x, y) e_{\mathcal{D}_{a}^{2}}(t ; y, x)= \\
2 \cdot \lim _{\varepsilon \rightarrow 0} \int_{S^{1}} d x \int_{|x-y|<\delta} d y k_{a}(x, y) e_{\mathcal{D}_{a}^{2}}(t ; y, x)
\end{gathered}
$$

The last equality follows from the fact that the Heat Kernel $e(t ; x, y)$ is exponentially dying when the distance between $x$ and $y$ is bounded away from 0 and time is going to 0 . In other words

$$
\lim _{\varepsilon \rightarrow 0} k_{a}(x, y) e_{\mathcal{D}_{a}^{2}}(t ; y, x)=0 \text { for }|x-y|>\delta
$$

Therefore we do have

$$
\begin{gathered}
\frac{d}{d a} \zeta_{\mathcal{D}_{a}^{2}}^{\prime}(0)=2 \cdot \lim _{\varepsilon \rightarrow 0} \int_{S^{1}} d x \int_{r<\delta} d r\left\{k_{a}(x ; x+r) e_{\mathcal{D}_{a}^{2}}(\varepsilon ; x+r, x)+\left\{k_{a}(x ; x-r) e_{\mathcal{D}_{a}^{2}}(\varepsilon ; x-r, x)\right\}=\right. \\
2 \cdot \lim _{\varepsilon \rightarrow 0} \int_{S^{1}} d x \int_{r<\delta} d r e^{-i r a}\left\{-\frac{i}{1-e^{2 \pi i a}}+\frac{i}{1-e^{-2 \pi i a}}\right\} \frac{1}{\sqrt{4 \pi \varepsilon}} e^{-\frac{r^{2}}{4 \varepsilon}}= \\
4 \pi \cdot \frac{\sin 2 \pi a}{1-\cos 2 \pi a} \cdot \lim _{\varepsilon \rightarrow 0} \int_{r<\delta} d r e^{-i r a} \frac{1}{\sqrt{4 \pi \varepsilon}} e^{-\frac{r^{2}}{4 \varepsilon}}=2 \pi \cdot \frac{\sin 2 \pi a}{1-\cos 2 \pi a} .
\end{gathered}
$$

and we have proved

Proposition 2.6.3.

$$
\begin{equation*}
\frac{d}{d a} \zeta_{\mathcal{D}_{a}^{2}}^{\prime}(0)=2 \pi \cdot \frac{\sin 2 \pi a}{1-\cos 2 \pi a} \tag{2.6.5}
\end{equation*}
$$

## CHAPTER 3

## The $\zeta$-Determinant on the Interval

We illustrate both constructions of the determinant, the analytical and the geometrical by the most simple conceivable example, the determinant of the Dirac operator $-i \frac{d}{d x}+r$ on the interval and determine the variation of the determinant under change of the parameter $r$ and the coupling condition.

### 3.1. Introduction

A long standing question in mathematics and mathematical physics is: How natural is the $\zeta$-renormalization procedure leading to the definition of the determinant of the Dirac operator?

We offer a detailed discussion of this question in the general case in the paper [94] (for work in progress, see also [93]). In this note we give a presentation of the 1-dimensional toy model for the general theory. The answer is positive, in the sense that we show using heat kernel methods that the $\zeta$-determinant is, up to a multiplicative constant, equal to a canonically defined algebraic determinant. Moreover, we are able to demonstrate all our analytical tools at work in this simple situation and also to explain several conceptual problems which arise in the theory. Therefore this note serves as an announcement and a pilot for a general analysis of the $\zeta$-determinant of an elliptic boundary value problem to be presented in $[\mathbf{9 4}]$ and to be elaborated in the sequel.

We avoid discussion of the general theory of the $\zeta$-determinant in dimension 1. Our analytical results, though they were not published before, can be obtained by using different methods. We refer to [?] for related results and an extensive bibliography of the subject.

We study the $\zeta$-determinant of the operator $-i \frac{d}{d x}$, or more generally $-i \frac{d}{d x}+B(x)$, where $B(x)$ is a self-adjoint $n \times n$ matrix, acting on $\mathbf{C}^{n}$-valued functions on the interval $[0,2 \pi]$. We have to pose a boundary condition in order to obtain a self-adjoint operator with a discrete spectrum. Such boundary conditions are parameterised by unitary operators $T: \mathbf{C}^{n} \rightarrow \mathbf{C}^{n}$, defining $\mathcal{D}_{T}$ as the closed self-adjoint extension of the operator $\mathcal{D}:=-i \frac{d}{d x}$ with the domain

$$
\begin{equation*}
\left\{s \in C^{\infty}\left([0,2 \pi] ; \mathbf{C}^{n}\right) \mid s(2 \pi)=T s(0)\right\} \tag{3.1.1}
\end{equation*}
$$

To make a connection with the Grassmannian description of the space of boundary conditions used in [93], let us observe that the space of boundary
data can be identified with $\mathbf{C}^{n} \oplus \mathbf{C}^{n}=\left\{(s(0), s(2 \pi)) \mid s \in C^{\infty}\left([0,2 \pi] ; \mathbf{C}^{n}\right)\right\}$ and there is an orthogonal projection onto the set of boundary data determined by condition $T$

$$
P_{T}=\frac{1}{2}\left(\begin{array}{cc}
\operatorname{Id} & T^{-1} \\
T & \mathrm{Id}
\end{array}\right) .
$$

We define the canonical determinant $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{T}$ of the operator $\mathcal{D}_{T}$, by the formula

$$
\begin{equation*}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{T}:=\operatorname{det} \frac{\mathrm{Id}_{\mathbf{C}^{n}}-T^{-1}}{2} \tag{3.1.2}
\end{equation*}
$$

Let us recall that the $\zeta$-determinant $\operatorname{det}_{\zeta} \mathcal{D}_{T}$ of the operator $\mathcal{D}_{T}$, introduced by Ray and Singer in [84] (see also [100], [?]), is given by the formula

$$
\begin{equation*}
\operatorname{det}_{\zeta} \mathcal{D}_{T}=e^{\frac{i \pi}{2}\left(\eta_{\mathcal{D}_{T}}(0)-\zeta_{\mathcal{D}_{T}^{2}}(0)\right)} \cdot e^{-1 / 2 \cdot\left(d /\left.d s\left(\zeta_{\mathcal{D}_{T}^{2}}\right)\right|_{s=0}\right)} \tag{3.1.3}
\end{equation*}
$$

Here is the main result of this note:

Theorem 3.1.1. There exists a constant $C$ such that for any unitary $T$ the following equality holds

$$
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{T}=C \cdot \operatorname{det}_{\zeta} \mathcal{D}_{T}
$$

Actually, for the choices of (3.1.2) and (4.3.1), we shall find (in Proposition 3.3.5 below) that the constant $C$ takes the value $2^{n}$.

The Theorem shows that the $\zeta$-determinant can be obtained by a "healthy" algebraic procedure.

Remark 3.1.2. (a) The canonical determinant, as defined above, appears naturally in the higher dimensional case (see [93] and [94]). The determinant line bundle over the infinite-dimensional Grassmannian of elliptic boundary conditions for a Dirac operator is a non-trivial complex line bundle with canonical determinant section, as defined by Quillen. This bundle restricted to the sub-Grassmannian of self-adjoint conditions becomes a trivial line bundle, and the canonical section becomes a function once we fix a trivialization. In [93] it was shown that there is a natural choice of trivialization; the canonical determinant of an elliptic boundary value problem for the Dirac operator is precisely the value of the canonical section in this trivialization.

Now, in the 1-dimensional case, the Grassmannian is finite-dimensional and there are no trivializations determined by the Calderón projection or the tangential operator. Nevertheless we observe that

- The operator $\mathcal{D}_{- \text {Id }}$ has spectrum $\left\{\frac{2 k+1}{2}\right\}_{k \in \mathbf{Z}}$, hence it corresponds to the operator $-i \frac{d}{d x}+\frac{1}{2}$ on $S^{1}$, which is the Dirac operator defined by the non-trivial Spin-structure on $S^{1}$. This makes it somewhat natural to assume that $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{\text {-Id }}$ is equal to 1 .
- The operator $\mathcal{D}_{\text {Id }}$ has spectrum equal to the set of integers (more precisely equal to the direct sum of $n$ copies of $\mathbf{Z}$ ), hence it is noninvertible and its determinant should be equal to 0 . More generally, $\mathcal{D}_{T}$ is non-invertible, whenever 1 is an eigenvalue of the matrix $T$. This narrows a reasonable choice to

$$
\operatorname{det} \frac{\mathrm{Id}_{\mathbf{C}^{n}}-T^{ \pm 1}}{2}
$$

- The choice of the -1 in the exponent of $T$ in the formula is also motivated by the higher dimensional case. It makes (3.1.2) consistent with the definition of the canonical determinant given in formula (2.4) of [93] if we assume the transformation $K$ in (2.4) of [93], which determines the Calderon projection, corresponds to -Id, and the transformation $S$ defining the boundary condition corresponds to $T$.
(b) The essential novelty of the results presented here is the method. The variational formula of [?] is with respect to a variation of an operator with a fixed boundary condition and is obtained via a contour integral. Here we deal with the harder problem of proving the variational equality with respect to the boundary condition and using the heat kernel representation of the spectral $\zeta$-function. Thus we prove the projective equality of the determinants as functions on the unitary group $U(n)$ considered as the parameter space of self-adjoint boundary conditions.

We study the variation of the $\zeta$-determinant to prove Theorem 3.1.1. We actually show that the variation of the phase of $\operatorname{det}_{\zeta}$ is equal to the variation of the phase of $\operatorname{det}_{\mathcal{C}}$ and that the variation of the modulus of $\operatorname{det}_{\zeta}$ is equal to the variation of the modulus of $\operatorname{det}_{\mathcal{C}}$.

Remark 3.1.3. The proof suggests that one can fix the value of $\operatorname{det}_{\zeta}$ of the operator $\mathcal{D}_{\text {-Id }}$ as 1 and use the integral from the variation in order to define $\operatorname{det}_{\zeta}$ on the whole of $U(n)$. This modified $\zeta$-determinant is equal to $\operatorname{det}_{\mathcal{C}}$.

In Section 1 we present formulas for the variation of $\operatorname{det}_{\mathcal{C}}$. In Section 2 we discuss the variation of $\operatorname{det}_{\zeta}$ and, in order to determine the constant $C$, compute $\operatorname{det}_{\zeta}\left(\mathcal{D}_{- \text {Id }}\right)$.

### 3.2. The Variation of the Canonical Determinant

In this Section we discuss the variation of $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{T}$ at a fixed boundary condition $T$. We replace $T$ by $T_{r}:=e^{i r \alpha} T$, where $\alpha=\alpha^{*}$ is a self-adjoint $n \times n$ matrix and compute

$$
d /\left.d r\left\{\operatorname{det}_{\mathcal{C}} \mathcal{D}_{T_{r}}\right\}\right|_{r=0} .
$$

The result is stated in the following Proposition.

Proposition 3.2.1. Let $\mathcal{R}_{T}(\alpha)$ denote $d /\left.d r\left\{\ln \operatorname{det}_{\mathcal{C}} \mathcal{D}_{T_{r}}\right\}\right|_{r=0}$, then the phase of the variation $d /\left.\operatorname{dr}\left\{\operatorname{det}_{\mathcal{C}} \mathcal{D}_{T_{r}}\right\}\right|_{r=0}$ is given by the formula

$$
\begin{equation*}
\operatorname{Im} \mathcal{R}_{T}(\alpha)=-\frac{\operatorname{tr} \alpha}{2} \tag{3.2.1}
\end{equation*}
$$

and the modulus is equal to

$$
\begin{equation*}
\operatorname{Re} \mathcal{R}_{T}(\alpha)=-\frac{i}{2} \cdot \operatorname{tr} \alpha(\operatorname{Id}+T)(\operatorname{Id}-T)^{-1} \tag{3.2.2}
\end{equation*}
$$

Proof. We use the formula

$$
\begin{equation*}
\frac{d}{d r}\left\{l n \operatorname{det} S_{r}\right\}=\operatorname{Tr}\left(\frac{d}{d r}\left\{S_{r}\right\} S_{r}^{-1}\right) \tag{3.2.3}
\end{equation*}
$$

which in the case $S_{r}:=\frac{\mathrm{Id}-T_{r}^{-1}}{2}$ gives

$$
\mathcal{R}_{T}(\alpha)=d /\left.d r\left\{l n \operatorname{det}_{\mathcal{C}} \mathcal{D}_{T_{r}}\right\}\right|_{r=0}=-i \cdot \operatorname{tr} \alpha(\operatorname{Id}-T)^{-1}
$$

It follows that the phase of the variation of the determinant is equal to

$$
\begin{aligned}
\operatorname{Im} \mathcal{R}_{T}(\alpha) & =\frac{1}{2 i} \cdot \operatorname{tr}\left\{-i \alpha(\operatorname{Id}-T)^{-1}-\left(-i \alpha(\operatorname{Id}-T)^{-1}\right)^{*}\right\} \\
& =-\frac{1}{2} \cdot \operatorname{tr}\left\{\alpha(\operatorname{Id}-T)^{-1}+\left(\operatorname{Id}-T^{-1}\right)^{-1} \alpha\right\} \\
& =-\frac{1}{2} \cdot \operatorname{tr}\left\{\alpha(\operatorname{Id}-T)^{-1}-\alpha(\operatorname{Id}-T)^{-1} T\right\}=-\frac{\operatorname{tr} \alpha}{2}
\end{aligned}
$$

We compute the modulus of $\mathcal{R}_{T}(\alpha)$ in the same way

$$
\begin{aligned}
& \operatorname{Re} \mathcal{R}_{T}(\alpha)=\frac{1}{2} \cdot \operatorname{tr}\left\{-i \alpha(\operatorname{Id}-T)^{-1}+\left(-i \alpha(\operatorname{Id}-T)^{-1}\right)^{*}\right\} \\
= & -\frac{i}{2} \cdot \operatorname{tr}\left\{\alpha(\operatorname{Id}-T)^{-1}-\alpha\left(\operatorname{Id}-T^{-1}\right)^{-1}\right\}=-\frac{i}{2} \cdot \operatorname{tr} \alpha(\operatorname{Id}-T)^{-1}(\operatorname{Id}+T) .
\end{aligned}
$$

### 3.3. The Variation of the $\zeta$-Determinant

Now we study the variation of the $\zeta$-determinant under the change of the boundary condition described at the beginning of Section 1. We use a unitary twist in order to keep the boundary condition fixed and vary the operator inside the interval away from the boundary. This method was used for the first time in this context in [40] and since then has been crucial in obtaining several interesting results in spectral geometry (see [62], [93], $[?])$. We introduce a smooth cut-off function $f:[0,2 \pi] \rightarrow[0,1]$ equal to 1 for $0 \leq x \leq \pi / 2$ and equal to 0 for $3 \pi / 2 \leq x \leq 2 \pi$ and define a unitary transformation of the (trivial) bundle $S^{1} \times \mathbf{C}^{n}$ as follows

$$
\begin{equation*}
U_{r}(x):=T^{-1} e^{i r f(x) \alpha} T \tag{3.3.1}
\end{equation*}
$$

The operator $\mathcal{D}_{T_{r}}$ is unitarily equivalent to the operator

$$
\mathcal{D}_{r}:=\left(U_{r}\left(-i \frac{d}{d x}\right) U_{r}^{-1}\right)_{T}
$$

hence we can compute the variation of the $\zeta$-determinant of $\mathcal{D}_{T_{r}}$ by computing the variation of the family of operators

$$
\mathcal{D}_{r}=-i \frac{d}{d x}-r f^{\prime}(x) T^{-1} \alpha T
$$

with fixed domain $\{s \mid s(2 \pi)=T s(0)\}$. Equivalently, we can study the operator

$$
-i \frac{d}{d x}-r f^{\prime}(x) T^{-1} \alpha T: C^{\infty}\left(S^{1} ; V_{T}\right) \rightarrow C^{\infty}\left(S^{1} ; V_{T}\right)
$$

where $V_{T}$ denotes the complex bundle over $S^{1}$ defined by $T$. Now, we can (as in [93]) directly compute the value of the invariants of the operator $\mathcal{D}_{r}$ which contribute to the phase. Alternatively, we can use the result from the case of closed manifolds, see [45]. In any case we have the following well-known result.

Lemma 3.3.1. For any $T \in U(n)$ and for any self-adjoint $n \times n$ matrix $\alpha$ the following equalities hold

$$
\begin{equation*}
\zeta_{\mathcal{D}_{T}^{2}}(0)=0 \quad \text { and }\left.\quad \frac{d}{d r}\left\{\eta_{\mathcal{D}_{r}}(0)\right\}\right|_{r=0}=-\frac{\operatorname{tr} \alpha}{\pi} \tag{3.3.2}
\end{equation*}
$$

Corollary 3.3.2. The variation of the phase of $\operatorname{det}_{\zeta}$ is equal to the variation of the phase of $\operatorname{det}_{\mathcal{C}}$.

We need to do more extensive work in order to compute the variation of the modulus of $\operatorname{det}_{\zeta}$. We have to study the variation of $\zeta_{\mathcal{D}_{T}^{2}}^{\prime}(0)$, which is given by the regularized integral

$$
\int_{0}^{\infty} \frac{1}{t} \cdot \operatorname{Tr} e^{-t \mathcal{D}_{T}^{2}} d t=\lim _{s \rightarrow 0}\left\{\int_{0}^{\infty} t^{s-1} \cdot \operatorname{Tr} e^{-t \mathcal{D}_{T}^{2}} d t-\frac{\zeta_{\mathcal{D}_{T}^{2}}(0)}{s}\right\}
$$

Actually, $\zeta_{\mathcal{D}_{T}^{2}}(0)$ vanishes by Lemma 4.3 .2 and the integral on the left side of the identity is well-defined. Now we differentiate with respect to the parameter and obtain

$$
\begin{aligned}
\left.\frac{d}{d r}\left\{\zeta_{\mathcal{D}_{r}^{2}}^{\prime}(0)\right\}\right|_{r=0} & =\int_{0}^{\infty} \frac{1}{t} \cdot \operatorname{Tr}\left(-2 t \dot{\mathcal{D}}_{0} \mathcal{D}_{0} e^{-t \mathcal{D}_{0}^{2}}\right) d t \\
& =-2 \int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{0}^{-1} \mathcal{D}_{0}^{2} e^{-t \mathcal{D}_{0}^{2}} d t \\
& =2 \int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{0}^{-1} \frac{d}{d t}\left(e^{-t \mathcal{D}_{0}^{2}}\right) d t \\
& =\left.2 \cdot \lim _{\varepsilon \rightarrow 0}\left(\operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{0}^{-1} e^{-t \mathcal{D}_{0}^{2}}\right)\right|_{t=\varepsilon} ^{t=1 / \varepsilon}=-2 \cdot \lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{0}^{-1} e^{-\varepsilon \mathcal{D}_{0}^{2}}
\end{aligned}
$$

This gives us the following formula for the variation of the modulus

$$
\begin{equation*}
\left.\frac{d}{d r}\left\{-\frac{1}{2} \zeta_{\mathcal{D}_{r}^{2}}^{\prime}(0)\right\}\right|_{r=0}=\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{0}^{-1} e^{-\varepsilon \mathcal{D}_{0}^{2}} \tag{3.3.3}
\end{equation*}
$$

We only need an explicit formula for the kernel of the operator $\mathcal{D}_{0}^{-1}=$ $\mathcal{D}_{T}^{-1}$, to evaluate this formula.

Lemma 3.3.3. Let $k_{T}(x, y)$ denote the kernel of the operator $\mathcal{D}_{T}^{-1}$. Then

$$
k_{T}(x, y)= \begin{cases}-i(\operatorname{Id}-T)^{-1} & \text { for } x<y \\ i\left(\operatorname{Id}-T^{-1}\right)^{-1} & \text { for } x>y\end{cases}
$$

Now we can evaluate the variation of the modulus of $\operatorname{det}_{\zeta}$. We have:

$$
\begin{aligned}
&\left.\frac{d}{d r}\left\{-\frac{1}{2} \zeta_{\mathcal{D}_{r}^{2}}^{\prime}(0)\right\}\right|_{r=0}=\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{0}^{-1} e^{-\varepsilon \mathcal{D}_{0}^{2}} \\
&=-\lim _{\varepsilon \rightarrow 0} \int_{0}^{2 \pi} d x \operatorname{tr}\left(f^{\prime}(x) T^{-1} \alpha T \int_{0}^{2 \pi} k_{T}(x, y) \mathcal{E}(\varepsilon ; y, x) d y\right)
\end{aligned}
$$

where $\mathcal{E}(\varepsilon ; y, x)$ denotes the kernel of the operator $e^{-\varepsilon \mathcal{D}_{0}^{2}}$. It is immediate from the properties of the heat kernel that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{0}^{2 \pi} d x \operatorname{tr}\left(f^{\prime}(x) T^{-1} \alpha T \int_{\{y ;|x-y|>\sigma\}} k_{T}(x, y) \mathcal{E}(\varepsilon ; y, x) d y\right)=0 \tag{3.3.4}
\end{equation*}
$$

for any $\sigma>0$. Moreover, $f^{\prime}(x)$ is equal to 0 for $x$ outside of the interval [ $\pi / 2,3 \pi / 2$ ] , hence we use Duhamel's Principle and replace the original heat kernel by the heat kernel of the operator $-d^{2} / d x^{2}$ on the real line. We have

$$
\begin{aligned}
& \frac{d}{d r}\{ \left.-\frac{1}{2} \zeta_{\mathcal{D}_{r}^{2}}^{\prime}(0)\right\}\left.\right|_{r=0} \\
&=-\lim _{\varepsilon \rightarrow 0} \int_{0}^{2 \pi} d x \operatorname{tr}\left(f^{\prime}(x) T^{-1} \alpha T \int_{\{y ;|x-y|<\sigma\}} k_{T}(x, y) \mathcal{E}(\varepsilon ; y, x) d y\right) \\
&=-\lim _{\varepsilon \rightarrow 0} \int_{0}^{2 \pi} d x \operatorname{tr}\left(f ^ { \prime } ( x ) T ^ { - 1 } \alpha T \left\{-i(\operatorname{Id}-T)^{-1} \int_{0}^{\sigma} \frac{1}{\sqrt{4 \pi \varepsilon}} e^{-r^{2} / 4 \varepsilon} d r\right.\right. \\
&\left.\left.\quad+i\left(\operatorname{Id}-T^{-1}\right)^{-1} \int_{0}^{\sigma} \frac{1}{\sqrt{4 \pi \varepsilon}} e^{-r^{2} / 4 \varepsilon} d r\right\}\right) \\
&=-\frac{i}{\sqrt{\pi}} \cdot \int_{0}^{2 \pi} \operatorname{tr}\left(f^{\prime}(x) T^{-1} \alpha T\left\{\left(I d-T^{-1}\right)^{-1}-(\operatorname{Id}-T)^{-1}\right\}\right) d x \\
& \quad \cdot \lim _{\varepsilon \rightarrow 0} \int_{0}^{\sigma} e^{-r^{2} / 4 \varepsilon} \frac{d r}{2 \sqrt{\varepsilon}} \\
&= \frac{i}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot \operatorname{tr}\left(T^{-1} \alpha T\left\{\left(\operatorname{Id}-T^{-1}\right)^{-1}-(\operatorname{Id}-T)^{-1}\right\}\right) \\
&= \frac{i}{2} \cdot \operatorname{tr} T^{-1} \alpha T(\operatorname{Id}-T)^{-1}(\operatorname{Id}+T) .
\end{aligned}
$$

This ends the proof of Theorem 0.1.

A particular corollary of our computations, which may be of independent interest, (at least in dimensions higher than 1) is that the only critical point of the modulus of the determinant is the 'normalizing' and 'diagonalizing' boundary condition given in dimension 1 by $T=-\mathrm{Id}$ :

Corollary 3.3.4. The variation of the modulus of the determinants $\operatorname{det}_{\zeta}$ and $\operatorname{det}_{\mathcal{C}}$ at $T=-\mathrm{Id}$ is equal to 0 .

We determine the proportionality constant between $\operatorname{det}_{\zeta}$ and $\operatorname{det}_{\mathcal{C}}$ by calculating the precise value of $\operatorname{det}_{\zeta}\left(\mathcal{D}_{T}\right)$ at $T=-\mathrm{Id}$. Recall that by definition $\operatorname{det}_{\mathcal{C}}\left(\mathcal{D}_{\text {-Id }}\right)=1$.

Proposition 3.3.5. For the operator $\mathcal{D}=-i \frac{d}{d x}$ acting on $\mathbf{C}^{n}$-valued functions on the interval $[0,2 \pi]$ we have

$$
\operatorname{det}_{\zeta}\left(\mathcal{D}_{- \text {Id }}\right)=2^{n} .
$$

Proof. We use well-known formulas for the Hurwitz $\zeta$ function. As noted in Remark 0.2, $\mathcal{D}_{\text {-Id }}$ has the same spectrum as $-i \frac{d}{d x}+\frac{1}{2}$ Id acting on $C^{\infty}\left(S^{1} ; \mathbf{C}^{n}\right)$. More generally, consider the operator $D_{A}=-i \frac{d}{d x}+A$ where $A$ is a $n \times n$ diagonal matrix with diagonal entries $0<a_{i} \leq 1$. Since obviously

$$
\begin{equation*}
\operatorname{det}_{\zeta} D_{A}=\prod_{i=1}^{n} \operatorname{det}_{\zeta} D_{a_{i}} \tag{3.3.5}
\end{equation*}
$$

where $D_{a_{i}}=-i \frac{d}{d x}+a_{i}$ is acting on $C^{\infty}\left(S^{1} ; \mathbf{C}\right)$, it will be enough for us to consider the rank $n=1$ case. So we are going to compute the $\zeta$ determinant

$$
\begin{equation*}
\operatorname{det}_{\zeta}\left(D_{a}\right)=e^{\frac{i \pi}{2} \eta_{D_{a}}(0)} \cdot\left(\operatorname{det}_{\zeta} D_{a}^{2}\right)^{1 / 2} \tag{3.3.6}
\end{equation*}
$$

of $D_{a}=-i \frac{d}{d x}+a$ with $0<a \leq 1$. To do so we use the Hurwitz zeta-function

$$
\begin{equation*}
\zeta(s, a)=\sum_{n=0}^{\infty} \frac{1}{(n+a)^{s}} \tag{3.3.7}
\end{equation*}
$$

So $\zeta(s, 1)=\zeta(s)$. This has the analytically continued values

$$
\begin{equation*}
\zeta(0, a)=\frac{1}{2}-a, \quad \zeta^{\prime}(0, a)=\log \Gamma(a)-\frac{1}{2} \log (2 \pi) \tag{3.3.8}
\end{equation*}
$$

We then have (choosing $\left.(-1)^{-s}=e^{i \pi s}\right)$ :

$$
\begin{equation*}
\sum_{n \in \mathbf{Z}} \frac{1}{(n+a)^{s}}=\zeta(s, a)+e^{i \pi s} \zeta(s, 1-a) \tag{3.3.9}
\end{equation*}
$$

We also have
(3.3.10) $\operatorname{spec}\left(D_{a}\right)=\{n+a \mid n \in \mathbf{Z}\} \quad$ and $\quad \operatorname{spec}\left(D_{a}^{2}\right)=\left\{(n+a)^{2} \mid n \in \mathbf{Z}\right\}$.

So that:

$$
\begin{equation*}
\zeta_{D_{a}}(s)=\zeta(s, a)+e^{i \pi s} \zeta(s, 1-a) \quad \text { and } \quad \zeta_{D_{a}^{2}}(s)=\zeta(2 s, a)+\zeta(2 s, 1-a) \tag{3.3.11}
\end{equation*}
$$

Hence using (11.2.1)

$$
\begin{equation*}
\zeta_{D_{a}}(0)=0 \quad \text { and } \quad \zeta_{D_{a}^{2}}(0)=0 \tag{3.3.12}
\end{equation*}
$$

Using $\Gamma(z) \Gamma(1-z)=\pi / \sin (\pi z)$ and (11.2.1), an elementary computation from (11.2.6) gives

$$
\begin{equation*}
\zeta_{D_{a}}^{\prime}(0)=-\log (2 \sin (\pi a))-\frac{i \pi}{2}(1-2 a) \tag{3.3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\zeta_{D_{a}^{2}}^{\prime}(0)=-\log \left(4 \sin ^{2}(\pi a)\right) \tag{3.3.14}
\end{equation*}
$$

Thus by (3.3.6)

$$
\begin{equation*}
\operatorname{det}_{\zeta} D_{a}=e^{\frac{i \pi}{2}(1-2 a)} \cdot 2 \sin (\pi a)=e^{\frac{i \pi}{2}(1-2 a)}\left(\operatorname{det}_{\zeta} D_{a}^{2}\right)^{1 / 2} \tag{3.3.15}
\end{equation*}
$$

One can easily check the exponent is $\eta(0)$. In particular, then for $a=1 / 2$

$$
\begin{equation*}
\left(\operatorname{det}_{\zeta} D_{a=1 / 2}^{2}\right)^{1 / 2}=2 \sin (\pi / 2)=2 \tag{3.3.16}
\end{equation*}
$$

and from (10.2.10) we arrive at $2^{n}$ for the modulus of the zeta determinant at $T=-\mathrm{Id}$.

Remark 3.3.6. Proposition 3.3.5 also follows from [91] where contour integral methods were used to prove the equality

$$
\begin{equation*}
\operatorname{det}_{\zeta}\left(\mathcal{D}_{T}\right)=\operatorname{det}\left(\operatorname{Id}+R^{-1}\right) \tag{3.3.17}
\end{equation*}
$$

[Notice the different sign of $R$ in (10.2.9) , since in this note we are considering the boundary condition (i.e. $f(2 \pi)=T f(0)$ ) orthogonal to the one considered in $[\mathbf{9 1}]$ (i.e. $f(2 \pi)=-R f(0)$ or, equivalently, $\left.\left.P_{R}(f(0), f(2 \pi))=0\right)\right]$.

### 3.4. The Operator $-i \frac{d}{d x}+B(x)$ - still erronneous

Let us observe that, in fact, the case of the operator $\mathcal{D}=-i \frac{d}{d x}+$ $B(x)$, where $\{B(x)\}_{x \in[0,2 \pi]}$ denotes a smooth family of self-adjoint $n \times n$ matrices does not introduce new features. We can explain this by using the unitary twist again. We define a holonomy operator (holonomy of a connection $\nabla=i \mathcal{D}$ ) by the formula

$$
\begin{equation*}
h(x)=e^{i \int_{0}^{x} B(u) d u} \tag{3.4.1}
\end{equation*}
$$

Proposition 3.4.1. The operator $\mathcal{D}_{T}$ is unitary equivalent to the operator $\left(-i \frac{d}{d x}\right)_{H T}$, where the matrix $H$ is equal to $h(2 \pi)$.

Proof. Let $\mathcal{D} s=\lambda s$ and $s(2 \pi)=T s(0)$, then we have

$$
\begin{aligned}
-i \frac{d}{d x}(h s) & =h\left(h^{-1}\left(-i \frac{d}{d x}\right) h\right) s=h\left(-i \frac{d}{d x}+h^{-1}(-i) i B(x) h\right) s \\
& =h\left(-i \frac{d}{d x}+B(x)\right) s=\lambda(h s)
\end{aligned}
$$

and

$$
(h s)(2 \pi)=H(s(2 \pi))=H T(s(0)) .
$$

## CHAPTER 4

## EBVP and Grassmannians

### 4.1. Dirac Operators - Clifford, Compatible Conn., Product <br> Structure, Index on Closed Mf., Green's Formula, Calderon Projector

We now give a more detailed presentation of the situation discussed in this paper. Let $\mathcal{D}: C^{\infty}(M ; S) \rightarrow C^{\infty}(M ; S)$ denote a compatible Dirac operator acting on the space of sections of a bundle of Clifford modules $S$ over compact manifold $M$ with boundary $Y$. It is not actually necessary to assume that $\mathcal{D}$ is a Compatible Dirac Operator. Further technical comments are made in the final Section of the paper.

In the present paper we always assume that $M$ is an odd-dimensional manifold; the even-dimensional case will be discussed separately. And we discuss only the Product Case. Namely we assume that the Riemannian metric on $M$ and the Hermitian structure on $S$ are products in a certain collar neighborhood of the boundary. Let us fix a parameterization $N=$ $[0,1] \times Y$ of the collar. Then in $N$ the operator $\mathcal{D}$ has the form

$$
\begin{equation*}
\mathcal{D}=G\left(\partial_{u}+B\right), \tag{4.1.1}
\end{equation*}
$$

where $G: S|Y \rightarrow S| Y$ is a unitary bundle isomorphism (Clifford multiplication by the unit normal vector) and $B: C^{\infty}(Y ; S \mid Y) \rightarrow C^{\infty}(Y ; S \mid Y)$ is the corresponding Dirac operator on $Y$, which is an elliptic self-adjoint operator of first order. Furthermore, $G$ and $B$ do not depend on the normal coordinate $u$ and they satisfy the identities

$$
\begin{equation*}
G^{2}=-I d \quad \text { and } \quad G B=-B G \tag{4.1.2}
\end{equation*}
$$

Since $Y$ has dimension $2 m$ the bundle $S \mid Y$ decomposes into its positive and negative chirality components $S \mid Y=S^{+} \bigoplus S^{-}$and we have a corresponding splitting of the operator $B$ into $B^{ \pm}: C^{\infty}\left(Y ; S^{ \pm}\right) \rightarrow C^{\infty}\left(Y ; S^{\mp}\right)$, where $\left(B^{+}\right)^{*}=B^{-}$. Equation (7.1.2) can be rewritten in the form

$$
\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right)\left(\partial_{u}+\left(\begin{array}{cc}
0 & B^{-} \\
B^{+} & 0
\end{array}\right)\right) .
$$

### 4.2. Global Elliptic Boundary Conditions and Grassmannian(s)

In order to obtain an unbounded Fredholm operator with good elliptic regularity properties we have to impose a boundary condition on the operator $\mathcal{D}$. Let $\Pi_{>}$denote the spectral projection of $B$ onto the subspace of $L^{2}(Y ; S \mid Y)$ spanned by the eigenvectors corresponding to the nonnegative eigenvalues of $B$. It is well known that $\Pi_{\geq}$is an elliptic boundary condition for the operator $\mathcal{D}$ (see [5], [27]). The meaning of the ellipticity is as follows. We introduce the unbounded operator $\mathcal{D}_{\Pi_{\geq}}$equal to the operator $\mathcal{D}$ with domain

$$
\operatorname{dom} \mathcal{D}_{\Pi_{\geq}}=\left\{s \in H^{1}(M ; S) ; \Pi_{\geq}(s \mid Y)=0\right\}
$$

where $H^{1}$ denotes the first Sobolev space. Then the operator

$$
\mathcal{D}_{\Pi_{\geq}}=\mathcal{D}: \operatorname{dom}\left(\mathcal{D}_{\Pi_{\geq}}\right) \rightarrow L^{2}(M ; S)
$$

is a Fredholm operator with kernel and cokernel consisting only of smooth sections.

The orthogonal projection $\Pi_{\geq}$is a pseudodifferential operator of order 0 (see [27]). In fact we can take any pseudodifferential operator $R$ of order 0 with principal symbol equal to the principal symbol of $\Pi_{\geq}$and obtain an operator $\mathcal{D}_{R}$ which satisfies the aforementioned properties.

In order to explain this phenomenon, we give a short exposition of the necessary facts from the theory of elliptic boundary problems. In contrast to the case of an elliptic operator on a closed manifold, the operator $\mathcal{D}$ has an infinite-dimensional space of solutions. More precisely, the space

$$
\left\{s \in C^{\infty}(M: S) ; \mathcal{D} s=0 \text { in } M \backslash Y\right\}
$$

is infinite-dimensional. We introduce Calderon projection, which is the projection onto $\mathcal{H}(\mathcal{D})$ the Cauchy Data space of the operator $\mathcal{D}$
$\mathcal{H}(\mathcal{D})=\left\{f \in C^{\infty}(Y ; S \mid Y) ; \exists_{s \in C^{\infty}(M ; S)} \mathcal{D}(s)=0\right.$ in $M \backslash Y$ and $\left.s \mid Y=f\right\}$.
The projection $P(\mathcal{D})$ is a pseudodifferential operator with principal symbol equal to the symbol of $\Pi_{\geq}$. It is also an orthogonal projection in the case of a Dirac operator on an odd-dimensional manifold (see [27]). The operator $\mathcal{D}$ has the Unique Continuation Property, and hence we have a one to one correspondence between solutions of the operator $\mathcal{D}$ and the traces of the solutions on the boundary $Y$. This explains roughly, why only the projection $\mathcal{P}_{R}$ onto the kernel of the boundary conditions $R$ matters. If the difference $\mathcal{P}_{R}-P(\mathcal{D})$ is an operator of order -1 , then it follows that we must choose the domain of the operator $\mathcal{D}_{R}$ in such a way that we throw away almost all solutions of the operator $\mathcal{D}$ on $M \backslash Y$, with the possible exception of a finite dimensional subspace. The above condition on $\mathcal{P}_{R}$ allows us also to construct a parametrix for the operator $\mathcal{D}_{R}$, hence
we obtain regularity of the solutions of the operator $\mathcal{D}_{R}$. We refer to the monograph [27] for more details.

We can therefore restrict ourselves to the study of the Grassmannian $\operatorname{Gr}(\mathcal{D})$ of all pseudodifferential projections which differ from $\Pi_{\geq}$by an operator of order -1 . The space $\operatorname{Gr}(\mathcal{D})$ has infinitely many connected components and two boundary conditions $P_{1}$ and $P_{2}$ belong to the same connected component if and only if

$$
\text { index } \mathcal{D}_{P_{1}}=\text { index } \mathcal{D}_{P_{2}}
$$

We are interested, however, in self-adjoint realizations of the operator $\mathcal{D}$. The involution $G: S|Y \rightarrow S| Y$ equips $L^{2}(Y ; S \mid Y)$ with a symplectic structure, and Green's formula (see [27])

$$
\begin{equation*}
\left(\mathcal{D} s_{1}, s_{2}\right)-\left(s_{1}, \mathcal{D} s_{2}\right)=-\int_{Y}<G\left(s_{1} \mid Y\right) ; s_{2} \mid Y>d y \tag{4.2.1}
\end{equation*}
$$

shows that the boundary condition $R$ provides a self-adjoint realization $\mathcal{D}_{R}$ of the operator $\mathcal{D}$ if and only if $\operatorname{ker} R$ is a Lagrangian subspace of $L^{2}(Y ; S \mid Y)$ (see [26], $[\mathbf{2 7}],[\mathbf{4 0}]$ ). We may therefore restrict our attention to those elements of $\operatorname{Gr}(\mathcal{D})$ which are Lagrangian subspaces of $L^{2}(Y ; S \mid Y)$. More precisely, we introduce $G r^{*}(\mathcal{D})$, the Grassmannian of orthogonal, pseudodifferential projections $P$ such that $P-\Pi_{\geq}$is an operator of order -1 and

$$
\begin{equation*}
-G P G=I d-P \tag{4.2.2}
\end{equation*}
$$

The space $G r^{*}(\mathcal{D})$ is contained in the connected component of $\operatorname{Gr}(\mathcal{D})$ parameterizing projections $P$ with index $\mathcal{D}_{P}=0$.

For analytical reasons associated with the existence of the $\zeta$-determinant, in this paper we discuss only the Smooth, Self-adjoint Grassmannian, a dense subset of the space $G r^{*}(\mathcal{D})$, defined by

$$
\begin{equation*}
G r_{\infty}^{*}(\mathcal{D})=\left\{P \in G r^{*}(\mathcal{D}) ; P-\Pi_{\geq} \text {has a smooth kernel }\right\} \tag{4.2.3}
\end{equation*}
$$

The spectral projection $\Pi_{\geq}$is an element of $G r_{\infty}^{*}(\mathcal{D})$ if and only if ker $B=$ $\{0\}$. However, it is well-known that $P(\mathcal{D})$ the (orthogonal) Calderon projection is an element of $G r^{*}(\mathcal{D})$ (see for instance [26]), and it was proved by the first author that $P(\mathcal{D})-\Pi_{\geq}$is a smoothing operator (see [91], Proposition 2.2.), and hence that $\bar{P}(\mathcal{D})$ is an element of $G r_{\infty}^{*}(\mathcal{D})$. The finite-dimensional perturbations of $\Pi_{\geq}$discussed below (see also [40], [62] and $[\mathbf{1 1 2}]$ ) provide further examples of boundary conditions from $G r_{\infty}^{*}(\mathcal{D})$. The latter were introduced by Jeff Cheeger, who called them Ideal Boundary Conditions (see [34], [35]).

For any $P \in G r^{*}(\mathcal{D})$ the operator $\mathcal{D}_{P}$ has a discrete spectrum nicely distributed along the real line (see [26], [40]). It was shown by the second author that for any $P \in G r_{\infty}^{*}(\mathcal{D}), \eta_{\mathcal{D}_{P}}(s)$ and $\zeta_{\mathcal{D}_{P}^{2}}(s)$ are well-defined functions, holomorphic for $\operatorname{Re}(s)$ large and with meromorphic extensions to the whole complex plane with only simple poles. In particular both functions are holomorphic in a neighborhood of $s=0$. Therefore $\operatorname{det}_{\zeta} \mathcal{D}_{P}$ defined by formula (??) is a well-defined, smooth function on $G r_{\infty}^{*}(\mathcal{D})$ (see [113]).

### 4.3. Boundary Problems defined by $G r_{\infty}^{*}(\mathcal{D})$ : Inverse Operator and Poisson Maps

For any $P \in \operatorname{Gr}(\mathcal{D})$ the operator $\mathcal{D}_{P}$ is a Fredholm operator, hence it has closed range. As a consequence we can define an inverse to the induced operator $\operatorname{dom} \mathcal{D}_{P} / \operatorname{ker} \mathcal{D}_{P} \rightarrow L^{2}(M ; S) /$ coker $\mathcal{D}_{P}$. If we assume that $P$ is an element of $G r^{*}(\mathcal{D})$, the operator $\mathcal{D}_{P}$ is self-adjoint and $\operatorname{ker} \mathcal{D}_{P}=$ coker $\mathcal{D}_{P}$. It follows that if we assume $\operatorname{ker} \mathcal{D}_{P}=\{0\}$, then there exists an inverse $\mathcal{D}_{P}^{-1}$ to the operator $\mathcal{D}_{P}$.

In this section we give an explicit formula for the operator $\mathcal{D}_{P}^{-1}$. This formula plays a key role in the proof of the main result of the paper. The operator $\mathcal{D}_{P}^{-1}$ is a sum of two operators. The first is the interior inverse of $\mathcal{D}^{-1}$. The second is a correction term which lives on the boundary.

We start with the "interior" part of the inverse. Let $\tilde{M}=M_{-} \cup_{Y} M$ denote the closed double of the manifold $M\left(M_{-}\right.$is a copy of $M$ with reversed orientation). The bundle of Clifford modules $S$ extends to a bundle $\tilde{S}$ of Clifford modules over $\tilde{M}$ and the operator $\mathcal{D}$ determines a compatible Dirac operator $\tilde{\mathcal{D}}$ over $\tilde{M}$ (equal to $\mathcal{D}$ on $M$ and $-\mathcal{D}$ on $M_{-}$). We refer to [109], [39] for the details of these constructions and applications to the analytic realization of $K$ homology. The operator $\tilde{\mathcal{D}}: C^{\infty}(\tilde{M} ; \tilde{S}) \rightarrow$ $C^{\infty}(\tilde{M} ; \tilde{S})$ is an invertible self-adjoint operator, hence its inverse $\tilde{\mathcal{D}}^{-1}$ is a well-defined elliptic operator of order -1 over the manifold $\tilde{M}$. We also have natural extension and restriction maps acting on sections of $S$ and $\tilde{S}$. The extension (by zero) operator $e_{+}: L^{2}(M ; S) \rightarrow L^{2}(\tilde{M} ; \tilde{S})$ is given by the formula:

$$
e_{+}(s):=\left\{\begin{array}{ll}
s & \text { on } \tilde{M} \backslash M_{-}  \tag{4.3.1}\\
0 & \text { on } M_{-}
\end{array} .\right.
$$

The restriction operator $r_{+}: H^{s}(\tilde{M} ; \tilde{S}) \rightarrow H^{s}(M ; S)$, where $H^{s}$ denotes the $s$ - th Sobolev space, is given by $\tilde{f} \rightarrow f=\tilde{f} \mid M$. To simplify the notation in the following we always write

$$
\begin{equation*}
\mathcal{D}^{-1}=r_{+} \tilde{\mathcal{D}}^{-1} e_{+} \tag{4.3.2}
\end{equation*}
$$

The operator $\mathcal{D}^{-1}$ is the interior part of the inverse. It is used in several crucial constructions in the theory of boundary problems. It maps $L^{2}(M$ : $S$ ) into $H^{1}(M ; S)$, however the range is not necessarily inside the domain of $\mathcal{D}_{P}$. For this reason we have to introduce an additional term to obtain an operator with the correct range. To do this, we need to study the situation in a neighborhood of the boundary $Y$. The restriction of smooth sections tom the boundary extends to a continuous map:

$$
\gamma_{0}: H^{s}(M ; S) \rightarrow H^{s-\frac{1}{2}}(Y ; S \mid Y)
$$

which is well-defined for $s>1 / 2$ (see $[\mathbf{2 7}]$ ). The corresponding adjoint operator $\gamma_{0}^{*}$ (in the distributional sense) provides us with a well-defined map

$$
\gamma_{0}^{*}: H^{s}(Y ; S \mid Y) \rightarrow H^{s-\frac{1}{2}}(M ; S)
$$

for $s<-1 / 2$. Now for any real $s$ the mapping

$$
\mathcal{K}=r_{+} \tilde{\mathcal{D}}^{-1} \gamma_{0}^{*} \Gamma: C^{\infty}(Y ; S \mid Y) \rightarrow C^{\infty}(M ; S)
$$

extends to a continuous map $\mathcal{K}: H^{s-1 / 2}(Y ; S \mid Y) \rightarrow H^{s}(M ; S)$, with range equal to the space

$$
\operatorname{ker}(\mathcal{D}, s)=\left\{f \in H^{s}(M ; S): \mathcal{D} f=0 \text { in } M \backslash Y\right\}
$$

In fact, the map

$$
\begin{equation*}
\mathcal{K}: \mathcal{H}(\mathcal{D}, s)=\operatorname{Ran}\left\{P(\mathcal{D}): H^{s-1 / 2}(Y ; S \mid Y) \rightarrow H^{s-1 / 2}(Y ; S \mid Y)\right\} \rightarrow \operatorname{ker}(\mathcal{D}, s) \tag{4.3.3}
\end{equation*}
$$

is an isomorphism (see [27]). We have the following equality:

$$
\begin{equation*}
\mathcal{D}^{-1} \mathcal{D}=I d-\mathcal{K} \gamma_{0} \tag{4.3.4}
\end{equation*}
$$

which holds on the space of smooth sections (see [27] Lemma 12.7).
The operator $\mathcal{K}$ is called the Poisson operator of $\mathcal{D}$. It defines the Calderon projection:

$$
\begin{equation*}
P(\mathcal{D})=\gamma_{0} \mathcal{K} \tag{4.3.5}
\end{equation*}
$$

(see [27] Theorem 12.4).

Remark 4.3.1. Formula (4.3.5) gives, a priori, only a projector, not an orthogonal projection, onto $\mathcal{H}(\mathcal{D})$. In the situation discussed in this paper, however, the resulting projector is orthogonal. We refer to [27] for the details of the construction, which is originally due to Calderon and Seeley.

To construct the correction term to the parametrix we require that the operator $\mathcal{S}(P)$ be invertible, but this requirement is equivalent to $\mathcal{D}_{P}$ being invertibility:

LEmma 4.3.2. The operator $\mathcal{D}_{P}$ is an invertible operator if and only if the operator $\mathcal{S}(P)=P P(\mathcal{D}): \mathcal{H}(\mathcal{D}) \rightarrow$ Ran $P$ is invertible.

Proof. Grassmannian $G r_{\infty}^{*}(\mathcal{D})$ is a subspace of the "big" Grassmannian $\operatorname{Gr}(\mathcal{D})$ (see [110], [26], [40] Appendix B). The space $\operatorname{Gr}(\mathcal{D})$ has countably many connected components distinguished by the index of the operator $\mathcal{S}(P)$, i.e. $P_{1}$ and $P_{2}$ belong to the same connected component of $\operatorname{Gr}(\mathcal{D})$ if and only if index $\mathcal{S}\left(P_{1}\right)=\operatorname{index} \mathcal{S}\left(P_{2}\right)$. The space $G r_{\infty}^{*}(\mathcal{D})$ is contained in the index zero component of $\operatorname{Gr}(\mathcal{D})$. Now we have

$$
\operatorname{ker} \mathcal{S}(P)=\{f: P(\mathcal{D}) f=f \quad \text { and } \quad P(f)=0\}
$$

and

$$
\text { coker } \mathcal{S}(P)=\{g: P g=g \text { and } P(\mathcal{D}) g=0\}
$$

If $P$ is an element of $G r^{*}(\mathcal{D})$, then index $\mathcal{S}(P)=$ index $\mathcal{D}_{P}=0$. We see that the operator $\mathcal{S}(P)$ is invertible if and only if it has trivial kernel. Similarly a self-adjoint Fredholm operator $\mathcal{D}_{P}$ is invertible only if it has trivial kernel. On the other hand, the operator $\mathcal{K}$ defines an isomorphism

$$
\mathcal{K}: \operatorname{ker} \mathcal{S}(P) \rightarrow \operatorname{ker} \mathcal{D}_{P}
$$

This ends the proof of Lemma.

Remark 4.3.3. Note that the lemma proves a somewhat stronger statement: via the Poisson operator $\mathcal{K}$, constructing solutions for the operator $\mathcal{S}(P)$ is equivalent to constructing solutions to the elliptic boundary value problem $\mathcal{D}_{P}$ (and the same for the adjoints). In particular this implies that the index of the two operators coincide. This is the underlying reason why
it is easier to compute determinants on manifolds with boundary than on closed manifolds.

From now on we assume that $\mathcal{D}_{P}$ is invertible. The operator $\mathcal{S}(P)^{-1}$ is not a pseudodifferential operator, as it acts from Ran $P$ into $\mathcal{H}(\mathcal{D})$. However, we can show that it is a restriction of an elliptic pseudodifferential operator of order 0 . More precisely, the operator $P P(\mathcal{D})+(I d-P)(I d-$ $P)(\mathcal{D})$ is an elliptic pseudodifferential operator, which can be written as

$$
\mathcal{S}(P) \oplus(I d-P)(I d-P(\mathcal{D})): \mathcal{H}(\mathcal{D}) \oplus \mathcal{H}(\mathcal{D})^{\perp} \rightarrow W \oplus W^{\perp}
$$

where $W=\operatorname{Ran} P$. It can be seen that:

$$
\operatorname{ker} \mathcal{S}(P)=\operatorname{coker}(I d-P)(I d-P(\mathcal{D}))
$$

and

$$
\text { coker } \mathcal{S}(P)=\operatorname{ker}(I d-P)(I d-P(\mathcal{D})) .
$$

Therefore if we assume that $\operatorname{ker} \mathcal{S}(P)=\{0\}$, then the operator $P P(\mathcal{D})+$ $(I d-P)(I d-P)(\mathcal{D}))$ is invertible. Its inverse is an elliptic operator (see for instance $[\mathbf{2 7}])$ and it follows that

$$
\begin{equation*}
\mathcal{S}(P)^{-1}=P(\mathcal{D})[P P(\mathcal{D})+(I d-P)(I d-P(\mathcal{D}))]^{-1} P \tag{4.3.6}
\end{equation*}
$$

We can now present the formula for the inverse of the operator $\mathcal{D}_{P}$.

Theorem 4.3.4. Assume that the operator $\mathcal{D}_{P}: \operatorname{dom} \mathcal{D}_{P} \rightarrow L^{2}(M ; S)$ is invertible, then its inverse is given by the formula:

$$
\begin{equation*}
\mathcal{D}_{P}^{-1}=\mathcal{D}^{-1}-\mathcal{K} \mathcal{S}(P)^{-1} P \gamma_{0} \mathcal{D}^{-1} \tag{4.3.7}
\end{equation*}
$$

Proof. The equality (4.3.3) shows that $\mathcal{D K}$ is equal to 0 in $M \backslash Y$, and hence that $\mathcal{D} \mathcal{D}_{P}^{-1}$ is equal to $I d$ on $L^{2}(M ; S)$. Now let $f \in L^{2}(M ; S)$, then:

$$
\begin{gathered}
P \gamma_{0}\left(\mathcal{D}_{P}^{-1} f\right)=P\left(\gamma_{0}\left(\mathcal{D}^{-1} f\right)-P \gamma_{0} \mathcal{K} \mathcal{S}(P)^{-1} P \gamma_{0} \mathcal{D}^{-1}(f)=\right. \\
P\left(\gamma_{0}\left(\mathcal{D}^{-1} f\right)-P P(\mathcal{D}) \mathcal{S}(P)^{-1} P \gamma_{0} \mathcal{D}^{-1}(f)=P\left(\gamma_{0}\left(\mathcal{D}^{-1} f\right)-P\left(\gamma_{0}\left(\mathcal{D}^{-1} f\right)=0\right.\right.\right.
\end{gathered}
$$

and hence $\mathcal{D}_{P}^{-1} f \in \operatorname{dom}\left(\mathcal{D}_{P}\right)$. We have shown that $\mathcal{D}_{P} \mathcal{D}_{P}^{-1}: L^{2}(M ; S) \rightarrow$ $L^{2}(M ; S)$ is equal to $I d_{L^{2}}$ and that $\mathcal{D}_{P}^{-1}: L^{2}(M ; S) \rightarrow \operatorname{dom} \mathcal{D}_{P}$, hence the operator $\mathcal{D}_{P}^{-1}$ is indeed a right inverse of $\mathcal{D}_{P}$, and obviously it is also a left inverse.

Corollary 4.3.5. Let $P_{1}, P_{2} \in \operatorname{Gr}_{\infty}^{*}(\mathcal{D})$ such that the operators $\mathcal{D}_{P_{1}}$ and $\mathcal{D}_{P_{2}}$ are invertible. Then the difference $\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}$ is an operator with smooth kernel.

Proof. It follows from Theorem 4.3.4 that

$$
\begin{equation*}
\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}=\mathcal{K}\left(\mathcal{S}\left(P_{2}\right)^{-1} P_{2}-\mathcal{S}\left(P_{1}\right)^{-1} P_{1}\right) \gamma_{0} \mathcal{D}^{-1} \tag{4.3.8}
\end{equation*}
$$

Now the fact that $P_{1}-P_{2}$ is an operator with a smooth kernel and equation (4.3.6) implies that the operator $\mathcal{S}\left(P_{2}\right)^{-1} P_{2}-\mathcal{S}\left(P_{1}\right)^{-1} P_{1}$ also has a smooth kernel.

For the rest of this Section we take a closer look at the operator $\mathcal{D}_{P}^{-1} \mathcal{D}_{P}$, as it allows us to introduce another important operator $\mathcal{K}(P)$-the Poisson operator of the operator $\mathcal{D}_{P}$. From formula (4.3.4) we have that

$$
\begin{gathered}
\mathcal{D}_{P}^{-1} \mathcal{D}_{P}=I d-\mathcal{K} \gamma_{0}-\mathcal{K} \mathcal{S}(P)^{-1} P \gamma_{0}\left(I d-\mathcal{K} \gamma_{0}\right)= \\
I d-\mathcal{K} \gamma_{0}+\mathcal{K} \mathcal{S}(P)^{-1} P P(\mathcal{D}) \gamma_{0}-\mathcal{K} \mathcal{S}(P)^{-1} P \gamma_{0}=I d-\mathcal{K} \mathcal{S}(P)^{-1} P \gamma_{0}
\end{gathered}
$$

Hence if $f \in \operatorname{dom} \mathcal{D}_{P}$, then $P \gamma_{0}(f)=0$ and

$$
\mathcal{D}_{P}^{-1} \mathcal{D}_{P} f=\left(I d-\mathcal{K} \mathcal{S}(P)^{-1} P \gamma_{0}\right) f=f
$$

We define the Poisson operator of $\mathcal{D}_{P}$ by

$$
\begin{equation*}
\mathcal{K}(P)=\mathcal{K} \mathcal{S}(P)^{-1} P \tag{4.3.9}
\end{equation*}
$$

which appeared in the second term of the operator $\mathcal{D}_{P}^{-1} \mathcal{D}_{P}$ and vanishes on $\operatorname{dom} \mathcal{D}_{P}$.

Let $g$ denote an element in the range of the projection $P$. More precisely, assume that $g \in H^{s}(Y ; S \mid Y)$ and $P g=g$. Then $\mathcal{K}(P) g$ is an element of $\operatorname{ker}(\mathcal{D}, s+1 / 2)$ and hence $\gamma_{0} \mathcal{K}(P) g$ is an element of the space $\mathcal{H}(\mathcal{D}, s)$, in
general not equal to $g$. However, the part of $\gamma_{0} \mathcal{K}(P) g$ along $P$ is in fact equal to the original element $g$ :

$$
P \gamma_{0} \mathcal{K}(P) g=P \gamma_{0} \mathcal{K} \mathcal{S}(P)^{-1} P g=P P(\mathcal{D}) \mathcal{S}(P)^{-1} P g=P g
$$

In Section 4 we also need the following result.

Proposition 4.3.6. Let $P_{1}, P_{2} \in G r_{\infty}^{*}(\mathcal{D})$ such that the operators $\mathcal{D}_{P_{1}}$ and $\mathcal{D}_{P_{2}}$ are invertible. Let $f_{1}, f_{2} \in \operatorname{Ran} P_{2}$ and assume that

$$
P_{1} \gamma_{0} \mathcal{K}\left(P_{2}\right) f_{1}=P_{1} \gamma_{0} \mathcal{K}\left(P_{2}\right) f_{2}
$$

Then, $f_{1}=f_{2}$ and $\mathcal{K}\left(P_{2}\right) f_{1}=\mathcal{K}\left(P_{2}\right) f_{2}$.

Proof. We have

$$
P_{1} \gamma_{0}\left(\mathcal{K}\left(P_{2}\right) f_{i}\right)=\mathcal{S}\left(P_{1}\right) \mathcal{S}\left(P_{2}\right)^{-1} f_{i}
$$

hence the first equality follows from the invertibility of the operators $\mathcal{S}\left(P_{1}\right)$ and $\mathcal{S}\left(P_{2}\right)$. The second is a consequence of the Unique Continuation Property for Dirac operators. We have

$$
\begin{gathered}
\gamma_{0}\left(\mathcal{K}\left(P_{2}\right) f_{1}=P(\mathcal{D}) \mathcal{S}\left(P_{2}\right)^{-1} f_{1}=\mathcal{S}\left(P_{1}\right)^{-1}\left(\mathcal{S}\left(P_{1}\right) \mathcal{S}\left(P_{2}\right)^{-1} f_{1}\right)=\right. \\
\mathcal{S}\left(P_{1}\right)^{-1}\left(\mathcal{S}\left(P_{1}\right) \mathcal{S}\left(P_{2}\right)^{-1} f_{2}\right)=\gamma_{0}\left(\mathcal{K}\left(P_{2}\right) f_{2}\right.
\end{gathered}
$$

and hence two solutions of $\mathcal{D}$ with the same Cauchy data, hence they are equal.

Remark 4.3.7. (1) Let us observe that the construction of the inverse presented in this Section gives, in fact, a parametrix for any elliptic boundary problem for the Dirac operator. First of all if $P$ is an element of $\operatorname{Gr}(\mathcal{D})$ we still can use formula (4.3.7) in order to construct the aforementioned parametrix. The operator $\mathcal{S}(P)^{-1}$ has to be replaced by the operator $\mathcal{R}(P)$ , which is of the form

$$
P(\mathcal{D}) R P: \operatorname{Ran} P \rightarrow \mathcal{H}(\mathcal{D})
$$

where $R$ denotes any parametrix of the operator $P P(\mathcal{D})+(I d-P)(I d-$ $P(\mathcal{D})$ ). The formula

$$
\mathcal{C}_{P}=\mathcal{D}^{-1}-\mathcal{K} \mathcal{R}(P) P \gamma_{0} \mathcal{D}^{1}
$$

now gives the operator, such that $\mathcal{D}_{P} \mathcal{C}_{P}-I d$ and $\mathcal{C}_{P} \mathcal{D}_{P}-I d$ have smooth kernels.
(2) More generally, this formula gives a parametrix for any elliptic boundary problem $\mathcal{D}_{T}$ as defined in [27] (where the authors were following Seeley's exposition $[\mathbf{9 7}])$. The reason is that $\mathcal{N}_{T}$, the kernel of the boundary condition $T$, and $\mathcal{H}(\mathcal{D})$ form a Fredholm pair of subspaces of $L^{2}(Y ; S \mid Y)$, which allows a parametrix $R$ to be constructed. This fact was well-known to Booss and Wojciechowski and is implicit in their work [?] and [25] (see also Proposition 1.4. in [22]. Last but not least, we are not really restricted in this construction of the parametrix only to Dirac operators. This construction holds for any first order elliptic differential operator on a compact manifold with boundary. The details will be presented elsewhere.

## CHAPTER 5

## Determinant Line Bundles and the Canonical Determinant

### 5.1. Determinant Bundle

Associated to the family of elliptic boundary value problems $\left\{\mathcal{D}_{P}: P \in\right.$ $\left.G r_{\infty}(\mathcal{D})\right\}$ one has a determinant line bundle $\operatorname{DET}(\mathcal{D})$ over $G r_{\infty}(\mathcal{D})$, as explained in Section 1, which is non-trivial over $G r_{\infty}(\mathcal{D})$. Further for each choice of basepoint $P_{0} \in G r_{\infty}(\mathcal{D})$ one has a smooth family of Fredholm operators

$$
\left\{\mathcal{S}_{P_{0}}(P):=P P_{0}: \text { Ran } P_{0} \rightarrow \text { Ran } P P \in G r_{\infty}(\mathcal{D})\right\}
$$

with associated (Segal) determinant line bundle $D E T_{P_{0}}$ equipped with its canonical determinant section $P \rightarrow \operatorname{det} \mathcal{S}_{P_{0}}(P) \in \operatorname{Det}_{P_{0}}(P)$, where $\operatorname{Det}_{P_{0}}(P)$ is the determinant line of the Fredholm operator $\mathcal{S}_{P_{0}}(P)$. Moreover, for $P_{0}, P_{1} \in G r_{\infty}(\mathcal{D})$ there is a canonical line bundle isomorphism

$$
\begin{equation*}
D E T_{P_{0}}=\operatorname{Det}_{P_{0}}\left(P_{1}\right) \otimes D E T_{P_{1}} \tag{5.1.1}
\end{equation*}
$$

defined where the operators are invertible by

$$
\begin{equation*}
\operatorname{det}\left[\mathcal{S}_{P_{1}}(P) \mathcal{S}_{P_{0}}\left(P_{1}\right)\right]=\operatorname{det} \mathcal{S}_{P_{0}}\left(P_{1}\right) \otimes \operatorname{det} \mathcal{S}_{P_{1}}(P) . \tag{5.1.2}
\end{equation*}
$$

The first factor on the right-side of (11.1.1) refers to the trivial bundle with fibre $\operatorname{Det}_{P_{0}}\left(P_{1}\right)$. The determinant line bundle of the family of elliptic boundary value problems is classified in this sense by

$$
\begin{equation*}
\operatorname{DET}(\mathcal{D})=D E T_{P(\mathcal{D})} \tag{5.1.3}
\end{equation*}
$$

where $P(\mathcal{D})$ is the Calderon projection, preserving the canonical determinant sections

$$
\begin{equation*}
\operatorname{det} \mathcal{D}_{P} \longleftrightarrow \operatorname{det} \mathcal{S}(P), \tag{5.1.4}
\end{equation*}
$$

where we have written $\mathcal{S}(P)$ for $\mathcal{S}_{P(\mathcal{D})}(P)$. We may therefore rewrite (11.1.1) fibrewise as

$$
\begin{equation*}
D E T \mathcal{D}_{P}=D E T \mathcal{D}_{P_{0}} \otimes \operatorname{Det}_{P_{0}}(P) . \tag{5.1.5}
\end{equation*}
$$

We refer to [91] for all these facts.
Let $\sigma\left(\mathcal{D}_{P_{0}}\right)$ denote the image of the canonical element $\operatorname{det} \mathcal{S}_{P_{0}}\left(P_{1}\right) \otimes$ $\operatorname{det} \mathcal{D}_{P_{0}} \in \operatorname{DET} \mathcal{D}_{P_{0}} \otimes \operatorname{Det}_{P_{0}}(P)$ under the isomorphism (11.1.5). Relative to the choice of the basepoint $P_{0}$, we therefore have two canonical elements in $D E T \mathcal{D}_{P}$, namely $\operatorname{det} \mathcal{D}_{P}$ and $\sigma\left(\mathcal{D}_{P_{0}}\right)$. Thus over the open subset where the operators are invertible, according to our earlier discussion we obtain a regularized determinant of $D_{P}$ by taking the quotient of these elements. The point however is to make a canonical choice of the basepoint $P_{0}$.

In the following, to make the presentation smoother we assume that ker $B=\{0\}$. This is in fact not a serious restriction and we can easily relax this condition. The point is that now the operators

$$
B^{ \pm}: F^{ \pm}=C^{\infty}\left(Y ; S^{ \pm}\right) \rightarrow F^{\mp}=C^{\infty}\left(Y ; S^{\mp}\right)
$$

are invertible. (We use also $F^{ \pm}$to denote the space of $L^{2}$ sections of the bundle of spinors of "positive' (resp. "negative") chirality.)

Coming back to the canonical choice of the basepoint, in our situation we are interested just in the real submanifold $G r_{\infty}^{*}(\mathcal{D})$ of self-adjoint boundary conditions and the 'correct' choice is indicated by the fact any elliptic boundary condition $P \in G r_{\infty}^{*}(\mathcal{D})$ is described precisely by the property that its range is the graph of an elliptic unitary isomorphism $T: F^{+} \rightarrow F^{-}$ such that $T-\left(B^{+} B^{-}\right)^{-1 / 2} B^{+}$has a smooth kernel.

There is a further subtlety that the corresponding orthogonal projection $P^{+}$onto $F^{+}$is not actually an element of the Grassmannian. But from (11.1.2) the isomorphism (11.1.5) is well-defined if we include the correction factor $\tau=\operatorname{det}\left(\mathcal{S}(P(\mathcal{D})) / \operatorname{det}\left[\mathcal{S}_{P(\mathcal{D})}\left(P^{+}\right) \mathcal{S}_{P^{+}}(P(\mathcal{D}))\right]\right.$, which introduces a factor of $1 / 2$ in the final formula (see (5.2.6)). The canonical determinant is then defined to be the quotient taken in $D E T \mathcal{D}_{P}$

$$
\begin{equation*}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}=\frac{\operatorname{det} \mathcal{D}_{P}}{\sigma\left(D_{P^{+}}\right)} . \tag{5.1.6}
\end{equation*}
$$

Roughly speaking this is the quotient $\operatorname{det} \mathcal{D}_{P} / \operatorname{det} \mathcal{D}_{P}^{+}$, the precise definition takes account of the fact that the domains of the operators $\mathcal{D}_{P}$ and $\mathcal{D}_{P}^{+}$ are different and hence that their canonical determinant elements live in different complex lines. In Section 1 we carry out a precise computation and we see that $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}$ is actually the Fredholm determinant of an operator living on the boundary $Y$ constructed from projections $P$ and $P(\mathcal{D})$.The main result of the paper is the following Theorem:

Theorem 5.1.1. The following equality holds over $\operatorname{Gr}_{\infty}^{*}(\mathcal{D})$

$$
\begin{equation*}
\operatorname{det}_{\zeta} \mathcal{D}_{P}=\operatorname{det}_{\zeta} \mathcal{D}_{P(\mathcal{D})} \cdot \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P} \tag{5.1.7}
\end{equation*}
$$

Remark 5.1.2. (1) Theorem 11.1.1 shows that, at least on $G r_{\infty}^{*}(\mathcal{D})$, the $\zeta$-determinant is an object which is a natural extension of the well-defined algebraic concept of the determinant.
(2) Our results show that the $\zeta$-determinant of the boundary problem $\mathcal{D}_{P}$ is actually equal to the Fredholm determinant of the operator $\mathcal{S}(P)$ living on the boundary. This extends the corresponding result for the index , which is well-known (see Theorem 20.8 [27]).
(3) With Theorem 11.1.1 at our disposal we can now try a new approach to the pasting formula for the $\zeta$-determinant with respect to a partitioning of a closed manifold. The pasting formula for $\operatorname{det}_{\mathcal{C}}$ was introduced in [91]. It is hoped that a new insight into the pasting mechanism of the $\zeta$-determinant will be obtained by combining results of [91] and formula (11.1.7).

### 5.2. Canonical Determinant on the Grassmannian $G r_{\infty}^{*}(\mathcal{D})$

In this Section we give a brief review of the construction of the determinant line bundle and give an explicit construction of the canonical determinant.

The determinant line bundle over the space of Fredholm operators was first introduced in a seminal paper of Quillen [83]. An equivalent construction which is better suited to our purposes here was subsequently given by Segal (see [98]), and so we follow his approach. Let $\operatorname{Fred}(\mathcal{H})$ denote the space of Fredholm operators on a separable Hilbert space $\mathcal{H}$. We work first in the connected component $\operatorname{Fred}_{0}(\mathcal{H})$ of this space parameterizing operators of index zero. For $A \in \operatorname{Fred}_{0}(\mathcal{H})$ define

$$
\operatorname{Fred}_{A}=\{S \in \operatorname{Fred}(\mathcal{H}) ; S-A \text { is trace }- \text { class }\}
$$

Fix a trace-class operator $\mathcal{A}$ such that $S=A+\mathcal{A}$ is an invertible operator. Then the determinant line of the operator $A$ is defined as

$$
\begin{equation*}
\text { Det } A=\text { Fred }_{A} \times \mathbf{C} / \cong, \tag{5.2.1}
\end{equation*}
$$

where the equivalence relation is defined as follows

$$
(R, z)=\left(S\left(S^{-1} R\right), z\right) \simeq\left(S, z \cdot \operatorname{det}_{F r}\left(S^{-1} R\right)\right) .
$$

The Fredholm determinant of the operator $S^{-1} R$ is well-defined, as it is of the form $I d_{\mathcal{H}}$ plus a trace class operator. Denoting the equivalence class of a pair $(R, z)$ by $[R, z]$, complex multiplication is defined on $\operatorname{Det} A$ by

$$
\begin{equation*}
\lambda \cdot[R, z]=[R, \lambda z] . \tag{5.2.2}
\end{equation*}
$$

The canonical determinant element is defined by

$$
\begin{equation*}
\operatorname{det} A:=[A, 1] \tag{5.2.3}
\end{equation*}
$$

and is non-zero if and only if $A$ is invertible.
The complex lines fit together over $\operatorname{Fred}_{0}(\mathcal{H})$ to define a complex line bundle $\mathcal{L}$, the determinant line bundle. To see this, observe first that over the open set $U_{\mathcal{A}}$ in $\operatorname{Fred}_{0}(\mathcal{H})$ defined by

$$
U_{\mathcal{A}}=\left\{F \in \operatorname{Fred}_{0}(\mathcal{H}) ; F+\mathcal{A} \text { is invertible }\right\}
$$

the assignment $F \rightarrow \operatorname{det} F$ defines a trivializing (non-vanishing) section of $\mathcal{L}_{\mid U_{\mathcal{A}}}$. The transition map between the canonical determinant elements over $U_{\mathcal{A}} \cap U_{\mathcal{B}}$ is the smooth (holomorphic) function

$$
g_{\mathcal{A B}}(F)=\operatorname{det}_{F r}\left((F+\mathcal{A})(F+\mathcal{B})^{-1}\right) .
$$

This defines $\mathcal{L}$ globally as a complex line bundle over $\operatorname{Fred}_{0}(\mathcal{H})$, endowed with the canonical section $A \rightarrow \operatorname{det} A$. If ind $A=d$ we define $\operatorname{Det} A$ to be the determinant line of $A \oplus 0$ as an operator $\mathcal{H} \longrightarrow \mathcal{H} \oplus \mathbf{C}^{d}$ if $d>0$ , or $\mathcal{H} \oplus \mathbf{C}^{-d} \longrightarrow \mathcal{H}$ if $d<0$ and the construction extends in the obvious way to the other components of $\operatorname{Fred}(\mathcal{H})$. Note that the canonical section is zero outside of $\operatorname{Fred}_{0}(\mathcal{H})$.

We use this construction in order to define the determinant line bundle over $G r_{\infty}(\mathcal{D})$. For each projection $P \in G r_{\infty}(\mathcal{D})$ we have the (Segal) determinant line $\operatorname{Det}(P(\mathcal{D}), P)$ of the operator $P P(\mathcal{D}): \mathcal{H}(\mathcal{D}) \rightarrow$ Ran $P$ and the determinant line $\operatorname{Det} \mathcal{D}_{P}$ of the boundary value problem $\mathcal{D}_{P}$ : $\operatorname{dom}\left(\mathcal{D}_{P}\right) \longrightarrow L^{2}(M ; S)$. These lines fit together in the manner explained above to define determinant line bundles $D E T_{P(\mathcal{D})}$ and $D E T \mathcal{D}$, respectively, over the Grassmannian (some care has to be taken as the operator acts between two different Hilbert spaces, but with the obvious notational modifications we once again obtain well-defined determinant line bundles). The canonical isomorphism (11.1.3) identifies the two line bundles and preserves the determinant elements (equation (11.1.4)). The bundle $D E T_{P(\mathcal{D})}$
is a non-trivial line bundle over $G r_{\infty}(\mathcal{D})$, but when restricted to the Grassmannian $G r_{\infty}^{*}(\mathcal{D})$ it is canonically trivial.

We use the specific trivialization introduced in [91]. Recall that we work here with orthogonal projections onto the Lagrangian subspaces of $L^{2}(Y ; S \mid Y)$, which are a compact perturbation of the Cauchy data space $\mathcal{H}(\mathcal{D})$. We have assumed that ker $B=\{0\}$, and hence $\Pi_{>}(B)$ is an element of $G r_{\infty}^{*}(\mathcal{D})$. The range of $\Pi_{>}(B)$ is actually the graph of the unitary operator $V_{>}: F^{+} \rightarrow F^{-}$given by the formula:

$$
V_{>}=\left(B^{+} B^{-}\right)^{-1 / 2} B^{+}
$$

This identification extends to the whole Grassmannian $G r_{\infty}^{*}(\mathcal{D})$ : elements are in $1-$ to -1 correspondence with unitary maps $V: F^{+} \rightarrow F^{-}$, such that the difference $V-V_{>}$is an operator with a smooth kernel. The corresponding orthogonal projection $P$ is given by the formula

$$
P=\frac{1}{2}\left(\begin{array}{cc}
I d_{F^{+}} & V^{-1} \\
V & I d_{F^{-}}
\end{array}\right)
$$

By choosing a basepoint, the correspondence defined above allows us to establish an isomorphism between $G r_{\infty}^{*}(\mathcal{D})$ and the group $U^{\infty}\left(F^{-}\right)$of unitaries acting on $F^{-}=L^{2}\left(Y ; S^{-}\right)$which differ from $I d_{S^{-}}$by an operator with a smooth kernel. It is convenient for us to work with the Calderon projection as basepoint, hence let $K: C^{\infty}\left(Y ; S^{+}\right) \rightarrow C^{\infty}\left(Y ; S^{-}\right)$denote the unitary such that $\mathcal{H}(\mathcal{D})$ is equal to the $\operatorname{graph}(K)$. Now for any projection $P \in G r_{\infty}^{*}(\mathcal{D})$ there exists $T=T(P): F^{+} \rightarrow F^{-}$such that $P=\operatorname{graph}(T)$, and so we have a natural isomorphism $G r_{\infty}^{*}(\mathcal{D}) \cong U^{\infty}\left(F^{-}\right)$defined by the map $P \rightarrow T K^{-1}$. This is expressed in terms of the homogeneous structure of the Grassmannian by

$$
P=\left(\begin{array}{cc}
I d_{F^{+}} & 0  \tag{5.2.4}\\
0 & T K^{-1}
\end{array}\right) P(\mathcal{D})\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & K T^{-1}
\end{array}\right)
$$

Now we can define a non-vanishing section $l$ of the determinant line bundle $D E T_{P(\mathcal{D})}$ over $G r_{\infty}^{*}(\mathcal{D})$. The value of $l$ at the projection $P$ is the class in $\operatorname{Det}_{P(\mathcal{D})}(P)$ of the couple

$$
\left(U(P)=\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & T K^{-1}
\end{array}\right) ; 1\right)
$$

where the operator $U(P)$ acts from $\mathcal{H}(\mathcal{D})$ to $\operatorname{Ran}(P)$. That is, $l(P)=$ det $U(P)$. The fact that $l(P)$ is an element of $\operatorname{Det}_{P(\mathcal{D})}(P) \cong \operatorname{Det} \mathcal{D}_{P}$ follows from the following elementary result.

Lemma 5.2.1. The difference between $U(P)$ and the operator $S(P)=P P(\mathcal{D})$ : $\mathcal{H}(\mathcal{D}) \rightarrow$ RanP is an operator with a smooth kernel, hence det $U(P)=$ $[U(P), 1]$ is an element of Det $P$.

Proof. The operator $U(P)$ acts from $\operatorname{graph}(\mathcal{K})=\mathcal{H}(\mathcal{D})$ to $\operatorname{graph}(T)=$ $\operatorname{Ran}(P)$ and acts by

$$
\binom{x}{K x} \rightarrow\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & T K^{-1}
\end{array}\right)\binom{x}{K x}=\binom{x}{T x} .
$$

The operator $P P(\mathcal{D})$ is given by the following formula

$$
P P(\mathcal{D})=\frac{1}{4}\left(\begin{array}{cc}
I d_{F^{+}}+T^{-1} K & T^{-1}+K-1 \\
T+K & I d_{F^{-}}+T K^{-1}
\end{array}\right)
$$

leading to the following expression for the operator $S(P)=P P(\mathcal{D})$ : $\mathcal{H}(\mathcal{D}) \rightarrow \operatorname{Ran}(P)$

$$
S(P)\binom{x}{K x}=\binom{\frac{I d_{F^{+}}+T^{-1} K}{2} x}{\frac{I d_{F^{-}}+T K^{-1}}{2} K x}=\left(\begin{array}{cc}
\frac{I d_{F^{+}}+T^{-1} K}{2} & 0  \tag{5.2.5}\\
0 & \frac{I d_{F^{-}}+T K^{-1}}{2}
\end{array}\right)\binom{x}{K x} .
$$

Because $T^{-1} K$ (resp. $T K^{-1}$ ) differs from $I d_{F^{+}}$(resp. $I d_{F^{+}}$) by a smoothing operator, it is now obvious that the difference $U(P)-S(P)$ is an operator with a smooth kernel.

The discussion above allows us to now define the Canonical Determinant over $G r_{\infty}^{*}(\mathcal{D})$. Let $A: \mathcal{H}(\mathcal{D}) \rightarrow$ Ran $P$ denote an invertible Fredholm operator such that $A-S(P)$ is an operator of trace class. We have:

$$
\begin{aligned}
\operatorname{det} A & :=[(A, 1)] \\
& =\left[\left(U(P)\left(U(P)^{-1} A\right), 1\right)\right] \\
& =\left[\left(U(P) ; \operatorname{det}_{F r}\left(U(P)^{-1} A\right)\right]\right. \\
& =\operatorname{det}_{F r}\left(U(P)^{-1} A\right)[(U(P) ; 1] \\
& :=\operatorname{det}_{F r}\left(U(P)^{-1} A\right) \cdot \operatorname{det} U(P) .
\end{aligned}
$$

where we use equations (5.2.2) and (5.2.2). The above identity means we can define the determinant of the operator $A$ as the ratio in $\operatorname{Det} A$ of the non-vanishing canonical elements $\operatorname{det} A$ and $\operatorname{det} U(P)$, or equivalently as the Fredholm determinant of the operator $U(P)^{-1} A$. This leads to the following definition of the Canonical Determinant of the operator $\mathcal{D}_{P}$.

Definition 5.2.2. We define the Canonical determinant of the elliptic boundary value problem $\mathcal{D}_{P}$ by:

$$
\begin{equation*}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}:=\operatorname{det}_{\mathcal{C}} S(P):=\operatorname{det}_{F r}\left(U(P)^{-1} S(P)\right) \tag{5.2.6}
\end{equation*}
$$

where $\mathcal{S}(P):=\mathcal{S}(P, P(\mathcal{D}))$.

The essential point is that the determinants of the Fredholm operators $\mathcal{D}_{P}$ and $\mathcal{S}(P)$ regarded as elements of the complex lines $\operatorname{Det} \mathcal{D}_{P}$ and $\operatorname{Det}_{P(\mathcal{D})}(P)$ are canonically identified by the isomorphism (11.1.3). It is straightforward to check that the Fredholm determinant on the right-side of (5.2.6) is precisely the quotient (11.1.6). In fact, from the proof of Lemma 5.2.1 we see that the determinant on the right side of the equality (5.2.6) is the Fredholm determinant of the operator

$$
\left(\begin{array}{cc}
\frac{I d_{F+}+T^{-1} K}{2} & 0 \\
0 & \frac{I d_{F-}+T K^{-1}}{2}
\end{array}\right)
$$

acting on the graph of the operator $K$. Hence we obtain:

Lemma 5.2.3.

$$
\begin{equation*}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}=\operatorname{det}_{F r}\left(\frac{I d+K T^{-1}}{2}\right) \tag{5.2.7}
\end{equation*}
$$

where the Fredholm determinant on the right-side is taken on $F^{-}$.

We may therefore reformulate Theorem 11.1.1 as:

Theorem 5.2.4. The following equality holds over $G r_{\infty}^{*}(\mathcal{D})$

$$
\begin{equation*}
\operatorname{det}_{\zeta} \mathcal{D}_{P}=\operatorname{det}_{\zeta} \mathcal{D}_{P(\mathcal{D})} \cdot \operatorname{det}_{F r} \frac{I d+K T^{-1}}{2} \tag{5.2.8}
\end{equation*}
$$

Equivalently, $\left(\right.$ since $\left.\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P(\mathcal{D})}=1\right)$

$$
\begin{equation*}
\frac{\operatorname{det}_{\zeta} \mathcal{D}_{P}}{\operatorname{det}_{\zeta} \mathcal{D}_{P(\mathcal{D})}}=\frac{\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}}{\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P(\mathcal{D})}} \tag{5.2.9}
\end{equation*}
$$

Remark 5.2.5. In Section 4 we also use the determinants of operators of the form

$$
S\left(P_{1}\right) S\left(P_{2}\right)^{-1}: \operatorname{Ran} P_{2} \rightarrow \operatorname{Ran} P_{1},
$$

under the assumption that the operator $S\left(P_{2}\right)$ is invertible. It follows from the discussion presented above that for any Fredholm operator $A$ : Ran $P_{2} \rightarrow$ Ran $P_{1}$, such that the difference between the operator $A$ and the operator $P_{1} P_{2}: \operatorname{Ran} P_{2} \rightarrow \operatorname{Ran} P_{1}$ is of trace class we can define the determinant of $A$ using the formula

$$
\operatorname{det} A=\operatorname{det}_{F r}\left(\begin{array}{cc}
I d_{F^{+}} & 0  \tag{5.2.10}\\
0 & T_{2} T_{1}^{-1}
\end{array}\right) A
$$

where $\operatorname{Ran} P_{i}$ is equal to graph $T_{i}$.

### 5.3. Canonical Determinant and Metric

## Part 2

## Spectral Invariants - The Heat Kernel Approach

## CHAPTER 6

## The Heat Kernel on $\mathcal{G} r_{\infty}$

In this chapter we first summarize the basic properties of operators with heat kernels, recall the definition of $\Gamma$-regularized spectral functions and invariants, and derive variation formulas for them. Then we explain Duhamel's principle and discuss various simple, but characteristic examples for its application. Also by Duhamel's principle we establish the $\zeta$-function, the $\eta$-function, and the $\zeta$-determinant first for ideal boundary conditions and subsequently on the smooth self-adjoint Grassmannian $\mathcal{G r}_{\infty}^{*}$ (and on $U_{\text {graph }}$ ).

### 6.1. Introduction

The spectral invariants discussed in this book are constructed from the kernels of the heat operators determined by a Dirac operator $\mathcal{D}$. In this chapter we discuss the construction of the kernels of the operators $e^{-t \mathcal{D}_{P}{ }^{2}}$ and $\mathcal{D} e^{-t \mathcal{D}_{P}^{2}}$ where $\mathcal{D}_{P}$ is a $L_{2}-$ realization of $\mathcal{D}$ subject to a boundary condition $P$.

Our approach in this book is based on Duhamel's principle in the spirit of the classical paper of McKean and Singer [67]. The heat kernel is either constructed from the heat kernel of one or several slightly modified, related operators or it is patched together from heat kernels coming from a decomposition of the manifold into parts. For manifolds with boundary and with product metric close to the boundary it suffices to glue two heat kernels together, one coming from the interior of the manifold (or from the closed double) and one coming from a half infinite cylinder over the boundary with suitable boundary conditions imposed. The difficulty we face here is due to the fact that the heat kernels are not local objects and we have the error term. The good news is that the differences between the precise and the glued (approximate) heat kernels can be estimated in a sufficiently nice and operational way. The aforementioned estimates allows us to prove the existence of the spectral invariants, we use in the construction of the $\zeta$-determinant. Those invariants can be also constructed in a different ways using parametrix of $\mathcal{D}$. Both methods are essentially equivalent. However, we prefer Duhamel's principle because it leads us 'naturally' to estimates of the differences between true and approximate kernels; to estimates about

[^1]the true kernels; to formulas and proofs for additivity theorems for spectral invariants over partitioned manifolds; and to variational formulas.

### 6.2. Duhamel's Principle and Heat Kernel Estimates

Let $\mathcal{D}: C^{\infty}(M ; S) \rightarrow C^{\infty}(M ; S)$ denote a compatible Dirac operator acting on sections of a bundle of Clifford modules $\$$ over a compact manifold $M$ (possibly with boundary $Y$ ). Let $\mathcal{D}_{0}$ denote the operator $D d$ in the boundaryless case and $\mathcal{D}_{P}$ the operator where $P$ is an ideal boundary condition in the case of a nontrivial boundary $Y$.

Whenever we can split a manifold into parts we can approximate the heat kernel by the heat kernel of the parts. This follows from Duhamel's principle. More precisely we have

Theorem 6.2.1 (Duhamel's Formula). Let $M$ be a compact manifold with or without boundary and $\mathcal{D}: \operatorname{dom}(\mathcal{D}) \rightarrow L_{2}(M ; \$)$ be a $L_{2}$-extension of an elliptic differential operator with heat kernel $\mathrm{e}\left(t ; x, x^{\prime}\right)$. Let $\left\{U_{1}, U_{2}\right\}$ be a covering of $M$ by two open submanifolds. For $j=1,2$ we assume that $U_{j}$ is isometric to an open submanifold of a (not necessarily compact) manifold $M_{j}$ and that there is given on $M_{j}$ a $L_{2}$-extension $\mathcal{D}_{j}: \operatorname{dom}\left(\mathcal{D}_{j}\right) \rightarrow L_{2}\left(M_{j} ; \mathbb{K}_{j}\right)$ of a differential operator with heat kernel $\mathrm{e}_{j}\left(t ; x, x^{\prime}\right)$ such that $\mathcal{D}_{j}$ and $\mathcal{D}$ coincide over $U_{j} .{ }^{2}$ Then there exist positive real constants $c_{1}, c_{2}$ such that for any $0<t<1$

$$
\begin{equation*}
\left|\mathrm{e}\left(t ; x, x^{\prime}\right)-\mathrm{e}_{j}\left(t ; x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}} \quad \text { for } x, x^{\prime} \in U_{j} \tag{6.2.1}
\end{equation*}
$$

and, more generally,

$$
\begin{equation*}
\left|\mathrm{e}\left(t ; x, x^{\prime}\right)-Q\left(t ; x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}} \quad \text { for arbitrary } x, x^{\prime} \in M \tag{6.2.2}
\end{equation*}
$$

where $Q\left(t ; x, x^{\prime}\right)$ is an integral kernel over $M$ obtained by suitably gluing the heat kernels $\mathrm{e}_{1}$ and $\mathrm{e}_{2}$.

Proof. First we define on $C^{\infty}(M ; \$)$ for any parameter $t>0$ an operator $Q(t)$ with a smooth kernel, given by

$$
Q\left(t ; x, x^{\prime}\right):=\sum_{j=1}^{2} \psi_{j}(x) \mathrm{e}_{j}\left(t ; x, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right),
$$

where $\left\{\varphi_{1}, \varphi_{2}\right\}$ is a smooth partition of unity on $M$ suitable for the covering $\left\{U_{1}, U_{2}\right\}$ and $\psi_{1}, \psi_{2}$ are non-negative smooth functions such that

$$
\begin{aligned}
\psi_{j} & \equiv 1 & & \text { on }\left\{x \in M \mid d\left(x, \operatorname{supp} \varphi_{j}\right)<\delta\right\} \\
\text { and } \psi_{j} & \equiv 0 & & \text { on }\left\{x \in M \mid d\left(x, \operatorname{supp} \varphi_{j}\right) \geq 2 \delta\right\}
\end{aligned}
$$

[^2]for a suitable $\delta>0$. We notice
\[

$$
\begin{equation*}
\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime}, \operatorname{supp} \varphi_{j}\right)=\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime \prime}, \operatorname{supp} \varphi_{j}\right) \geq \delta \tag{6.2.3}
\end{equation*}
$$

\]

Then, for $x^{\prime} \in U_{j}$ with $\varphi_{j}\left(x^{\prime}\right)=1$, we have by definition:

$$
Q\left(t ; x, x^{\prime}\right)= \begin{cases}e_{j}\left(t ; x, x^{\prime}\right) & \text { if } d\left(x, \operatorname{supp} \varphi_{j}\right)<\delta, \text { and }  \tag{6.2.4}\\ 0 & \text { if } d\left(x, \operatorname{supp} \varphi_{j}\right) \geq 2 \delta\end{cases}
$$

For fixed $t>0$, we determine the difference between the precise heat kernel $\mathrm{e}\left(t ; x, x^{\prime}\right)$ and the approximate one $Q\left(t ; x, x^{\prime}\right)$. By Duhamel's Formula (Proposition 6.3.1), we have

$$
\mathrm{e}\left(t ; x, x^{\prime}\right)-Q\left(t ; x, x^{\prime}\right)=-\int_{0}^{t} d s \int_{M} d z \mathrm{e}(s ; x, z) C\left(t-s ; z, x^{\prime}\right)
$$

with

$$
\begin{aligned}
& C\left(t-s ; z, x^{\prime}\right)=\left(\mathcal{D}_{0(z)}^{2}+\frac{d}{d(t-s)}\right) Q\left(t-s ; z, x^{\prime}\right) \\
&=\left(\mathcal{D}_{0(z)}^{2}-\frac{d}{d s}\right) Q\left(t-s ; z, x^{\prime}\right) \\
&=\sum_{j=1}^{2}\left\{\psi_{j}^{\prime \prime}(z) \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)+2 \psi_{j}^{\prime}(z) \frac{d}{d z}\left(\mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right)\right) \varphi_{j}\left(x^{\prime}\right)\right. \\
&\quad+\psi_{j}(z) \underbrace{\left(\mathcal{D}_{0(z)}^{2}-\frac{d}{d s}\right) \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right)}_{=0} \varphi_{j}\left(x^{\prime}\right)\} .
\end{aligned}
$$

As stated in (8.2.5), the supports of $\varphi_{j}$ and $\psi_{j}^{\prime}$ (and, equally, $\psi_{j}^{\prime \prime}$ ) are disjoint and separated from each other by a distance $\geq \delta$. Then, in fine correspondence to (6.3.10), the error term $C\left(t-s ; z, x^{\prime}\right)$ vanishes both for $d\left(z, x^{\prime}\right)<\delta$ and, as well, for sufficiently large geodesic distance, say if $x^{\prime} \in M \backslash \operatorname{supp} \varphi_{2}$ and $z \in M \backslash \operatorname{supp} \varphi_{1}$. Moreover, since we have $t \geq s \geq 0$, we obtain

$$
(t-s)^{-m / 2} e^{-c^{\prime} \frac{d^{2}\left(z, x^{\prime}\right)}{(t-s)}} \leq c t^{-m / 2} e^{-c^{\prime} \frac{d^{2}\left(z, x^{\prime}\right)}{t}} \leq \tilde{c} e^{-c^{\prime} \frac{d^{2}\left(z, x^{\prime}\right)}{2 t}}
$$

Thus we can estimate the first sum

$$
\left|\psi_{j}^{\prime \prime}(z) \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)\right| \leq c_{1} e^{-c^{\prime} \delta^{2} / 2 t}=c_{1} e^{-\frac{c_{2}}{t}}
$$

Similarly we estimate the second sum. We have

$$
2\left|\psi_{j}^{\prime}(z) \frac{d}{d z} \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)\right| \leq \frac{c t^{-m / 2}}{t} e^{-\frac{c^{\prime}}{t}} \leq c_{1} e^{-\frac{c_{2}}{t}}
$$

Renaming the constants, we have obtained the crucial estimate

$$
\left|C\left(t, x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}} \quad \text { for } d\left(x, x^{\prime}\right)>\delta
$$

We recall from Lemma 6.2.2 the other crucial estimate

$$
\left|\mathrm{e}\left(t, x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}}
$$

This gives

$$
\begin{aligned}
\mid \mathrm{e}\left(t ; x, x^{\prime}\right) & -Q\left(t ; x, x^{\prime}\right)\left|\leq \int_{0}^{t} d s \int_{M} d z\right| \mathrm{e}(s ; x, z) C\left(t-s ; z, x^{\prime}\right) \mid \\
& \leq \int_{0}^{t} d s \int_{M} d z c_{1} e^{-\frac{c_{2}}{t}} c_{1} e^{-\frac{c_{2}}{t}}=\operatorname{vol}(M) c_{1}^{2} \int_{0}^{t} e^{-\frac{c_{2} t}{s(t-s)}} d s \\
& s(t-s)=(t-s) s \\
= & \operatorname{vol}(M) c_{1}^{2} \int_{0}^{\frac{t}{2}} e^{-\frac{c_{2} t}{s(t-s)}} d s \\
& \stackrel{t-s \leq t / 2}{\leq} 2 \operatorname{vol}(M) c_{1}^{2} \int_{0}^{\frac{t}{2}} e^{-\frac{c_{2} t}{s t / 2}} d s \\
& =2 \operatorname{vol}(M) c_{1}^{2} \int_{0}^{\frac{t}{2}} e^{-\frac{2 c_{2}}{s}} d s \leq 2 \operatorname{vol}(M) c_{1}^{2} e^{-\frac{4 c_{2}}{t}} \int_{0}^{\frac{t}{2}} d s \\
& =\operatorname{vol}(M) c_{1}^{2} t e^{-\frac{4 c_{2}}{t}}<c_{3} e^{-\frac{c_{4}}{t}} .
\end{aligned}
$$

We make a review of the estimates for the kernel $e_{0}(t ; x, y)$ of the operator $e_{0}(t)=e^{-t \mathcal{D}_{0}{ }^{2}}$ and $\mathcal{E}_{0}(t ; x, y)$ the kernel of the operator $\mathcal{D} e^{-t \mathcal{D}_{0}{ }^{2}}$. First of all, there exists positive constants $c_{1}$ and $c_{2}$ such that for any $0<t<8$ we have

$$
\begin{equation*}
\left\|e_{0}(t ; x, y)\right\|<c_{1} t^{-m / 2} e^{-c_{2} d^{2}(x, y) / t} a n d\left\|\mathcal{E}_{0}(t ; x, y)\right\|<c_{1} t^{-(m+1) / 2} e^{-c_{2} d^{2}(x, y) / t} \tag{6.2.5}
\end{equation*}
$$

The estimate for the heat kernels of the operator $\mathcal{D}_{P}$ was proved in [BBKW93] (see [BBKW93], Theorem 22.14) using the variant of the Duhamel's principle stated above. In the following we also need estimates, which implies vanishing of the heat kernels as time goes to $\infty$.

Lemma 6.2.2. There exist positive constants $c_{1}$ and $c_{2}$ such that for all $x, y \in M$ and any $t>0$

$$
\left|\mathcal{E}_{0}\left(t ; x, x^{\prime}\right)\right| \leq c_{1} t^{-\frac{1+m}{2}} \cdot e^{-\frac{t}{4} \lambda_{0}^{2}} \cdot e^{-c_{2} \frac{d^{2}\left(x, x^{\prime}\right)}{t}} \leq c_{1} t^{-\frac{1+m}{2}} \cdot e^{-c_{2} \frac{d^{2}\left(x, x^{\prime}\right)}{t}}
$$

where $d\left(x, x^{\prime}\right)$ denotes the geodesic distance. This results holds also for the $e_{0}(t, x, y)$ in the case of the invertible operator $\mathcal{D}_{0}$ (otherwise limit of the operator $e_{0}(t)$ as $t \rightarrow \infty$ is equal to the projection onto kernel of $\mathcal{D}_{0}$ )

$$
\left|\mathrm{e}_{0}\left(t ; x, x^{\prime}\right)\right| \leq c_{1} t^{-\frac{m}{2}} \cdot e^{-\frac{t}{4} \lambda_{0}^{2}} \cdot e^{-c_{2} \frac{d^{2}\left(x, x^{\prime}\right)}{t}} \leq c_{1} t^{-\frac{m}{2}} \cdot e^{-c_{2} \frac{d^{2}\left(x, x^{\prime}\right)}{t}}
$$

Proof. Let $\left\{f_{k} ; \lambda_{k}\right\}_{k \in \mathbf{Z}}$ denote a discrete spectral resolution of $\mathcal{D}_{0}$ (see also Remark ??c). We have

$$
\left|\mathcal{E}_{0}\left(t ; x, x^{\prime}\right)\right| \leq \sum_{k: \lambda_{k}=0}\left|\lambda_{k}\right| e^{-t \lambda_{k}^{2}} \cdot\left|f_{k}(x)\right| \cdot\left|f_{k}\left(x^{\prime}\right)\right| .
$$

According to the Sobolev Lemma (see for instance [45], Lemma 1.1.4), the imbedding $\mathcal{H}^{m}(M ; \$) \hookrightarrow C^{0}(M ; \$)$ is continuous. Thus we get a uniform pointwise estimate for the eigensections:

$$
\begin{equation*}
\left\|f_{k}(x)\right\| \leq b \cdot\left(1+\lambda_{k}^{2 m}\right) \tag{6.2.6}
\end{equation*}
$$

where the constant $b>0$ does not depend on $k$. Clearly

$$
\left(\frac{1+\lambda_{k}^{2 m}}{m!}\right)^{2} \leq e^{2 \lambda_{k}^{2}}
$$

It follows

$$
\begin{aligned}
\left|\mathcal{E}_{0}\left(t ; x, x^{\prime}\right)\right| & \leq b^{2} \sum_{k}\left|\lambda_{k}\right| e^{-t \lambda_{k}^{2}} \cdot\left(1+\lambda_{k}^{2 m}\right)^{2} \leq b^{2}(m!)^{2} \sum_{k}\left|\lambda_{k}\right| e^{-(t-2) \lambda_{k}^{2}} \\
& \leq b_{1} e^{-(t-4) \lambda_{0}^{2}} \cdot \sum_{k} e^{-\lambda_{k}^{2}} \leq b_{2} e^{-(t / 2) \lambda_{0}^{2}}
\end{aligned}
$$

for $t \geq 8$. Let us observe that there exist positive constants $b_{3}, b_{4}, b_{5}$ such that

$$
\begin{aligned}
t^{m / 2} \cdot e^{-(t / 4) \cdot \lambda_{0}^{2}} \leq b_{3} & \text { for any } 0<t<+\infty \\
b_{4} \leq e^{-c_{2} \frac{d^{2}\left(x, x^{\prime}\right)}{t}} \leq b_{5} & \text { for } t \geq 6
\end{aligned}
$$

which proves for $t \geq 8$ the assertion for the kernel $\mathcal{E}_{0}$ of $e^{-t \mathcal{D}_{0}^{2}}$. In the same way we prove the assertion for the kernel $\varepsilon_{0}$.

### 6.3. Duhamel's Principle

In the functional analytical setting, Duhamel's Principle can be considered as expressing the difference between a given heat kernel and any arbitrary symmetric integral kernel in the following simple form.

Proposition 6.3.1 (Duhamel's Formula). Let $M$ be a Riemannian manifold with or without boundary and $\$$ a Hermitian vector bundle over $M$. Let $\mathcal{D}_{0}: \operatorname{dom} \mathcal{D}_{0} \rightarrow L_{2}(M ; \$)$ be an operator with heat kernel(s). Let $Q\left(t ; x, x^{\prime}\right)$ be a smooth symmetric kernel for a smooth family of operators of trace class $Q(t)$ such that $Q\left(t ; \cdot, x^{\prime}\right) \in \operatorname{dom} \mathcal{D}_{0}$ for all $t>0$ and $\lim _{t \rightarrow 0} Q(t)=\mathrm{Id}$. Then we have for all $t>0, x, x^{\prime} \in M$

$$
\begin{align*}
& \mathrm{e}_{0}\left(t ; x, x^{\prime}\right)=Q\left(t ; x, x^{\prime}\right)  \tag{6.3.1}\\
& \begin{aligned}
&\left.-\int_{0}^{t} d s \int_{M} d z \mathrm{e}_{0}(s ; x, z)\left(\begin{array}{l}
\mathcal{D}_{0(z)}^{2}
\end{array}\right) \frac{d}{d(t-s)}\right) Q\left(t-s ; z, x^{\prime}\right) \\
&=Q\left(t ; x, x^{\prime}\right)-\left(\mathrm{e}_{0} * C_{1}\right)\left(t ; x, x^{\prime}\right)
\end{aligned}
\end{align*}
$$

and

$$
\begin{equation*}
\mathrm{e}_{0}\left(t ; x, x^{\prime}\right)=Q\left(t ; x, x^{\prime}\right)+\sum_{k=1}^{\infty}(-1)^{k}\left(Q * C_{k}\right)\left(t ; x, x^{\prime}\right) \tag{6.3.2}
\end{equation*}
$$

Here

$$
(\alpha * \beta)\left(t ; x, x^{\prime}\right):=\int_{0}^{t} d s \int_{M} d z \alpha(s ; x, z) \beta\left(t-s ; z, x^{\prime}\right)
$$

denotes the convolution; and

$$
C_{1}(t):=\left(\mathcal{D}_{0}^{2}+\frac{d}{d t}\right) Q(t) \quad \text { with } \quad C_{1}\left(t ; x, x^{\prime}\right):=\left(\mathcal{D}_{0(x)}^{2}+\frac{d}{d t}\right) Q\left(t ; x, x^{\prime}\right)
$$

denotes the error term with the recursive formula

$$
C_{k}(t):=\left(C_{1} * C_{k-1}\right)(t) \quad \text { for } k>1
$$

Note . In (6.3.2), the infinite series of kernels converges uniformly on compact subsets of $\mathbf{R}_{+} \times M \times M$ and defines an absolute convergent series of operators under suitable assumptions about the kernel $Q$, e.g. when $Q$ itself is a true heat kernel.

Proof. We fix $t>0$ and define for $s \in(0, t]$ the operator $\Theta(s):=$ $\mathrm{e}_{0}(s) Q(t-s)$. Since $\mathrm{e}_{0}(0)=Q(0)=\mathrm{Id}$, we have

$$
\begin{aligned}
\mathrm{e}_{0}(t)-Q(t) & =\Theta(t)-\Theta(0)=\int_{0}^{t} \frac{d}{d s} \Theta(s) d s \\
& =\int_{0}^{t}\left\{\left(\frac{d}{d s} \mathrm{e}_{0}(s)\right) Q(t-s)+\mathrm{e}_{0}(s) \frac{d}{d s} Q(t-s)\right\} d s \\
& =\int_{0}^{t} \mathrm{e}_{0}(s)\left(-\mathcal{D}_{0}^{2}+\frac{d}{d s}\right) Q(t-s) d s,
\end{aligned}
$$

which proves (6.3.1). To prove (6.3.2) we substitute recursively $Q(t)-$ $\left(\mathrm{e}_{0} * C\right)(t)$ for $\mathrm{e}_{0}(t)$ on the right side of (6.3.1).

Remark 6.3.2. (a) The preceding proposition remains valid when we replace the kernel $\mathrm{e}_{0}$ of $e^{-t \mathcal{D}_{0}^{2}}$ by the kernel $\mathcal{E}_{0}$ of the operator $\mathcal{D}_{0} e^{-t \mathcal{D}_{0}^{2}}$. Recall that $\mathcal{E}_{0}\left(t ; x, x^{\prime}\right)=\mathcal{D}_{0} \mathrm{e}_{0}\left(t ; x, x^{\prime}\right)$. Then equation (6.3.1) has to be replaced by

$$
\begin{equation*}
\mathcal{E}_{0}\left(t ; x, x^{\prime}\right)=\mathcal{D}_{0} Q\left(t ; x, x^{\prime}\right)-\left(\mathrm{e}_{0} * \mathcal{D}_{0} C_{1}\right)\left(t ; x, x^{\prime}\right) \tag{6.3.3}
\end{equation*}
$$

and equation (6.3.2) by

$$
\begin{equation*}
\mathcal{E}_{0}\left(t ; x, x^{\prime}\right)=\mathcal{D}_{0} Q\left(t ; x, x^{\prime}\right)+\sum_{k=1}^{\infty}(-1)^{k}\left(Q * \mathcal{C}_{k}\right)\left(t ; x, x^{\prime}\right) \tag{6.3.4}
\end{equation*}
$$

with

$$
\mathcal{C}_{1}\left(t ; x, x^{\prime}\right):=\mathcal{D}_{0} C_{1}\left(t ; x, x^{\prime}\right) \quad \text { and } \quad \mathcal{C}_{k}(t):=\left(C_{1} * \mathcal{C}_{k-1}\right)(t) \quad \text { for } k>1 .
$$

(b) To apply Duhamel's principle a typical situation is met when we want to estimate the difference between the heat kernels $\mathrm{e}_{0}$ and $\mathrm{e}_{a}$ of two operators,
say a given $\mathcal{D}_{0}$ and an alternative $\mathcal{D}_{a}$. Clearly $\left(\mathcal{D}_{a}+\frac{d}{d s}\right) \mathrm{e}_{a}(s)=0$. Thus we can replace (6.3.1) and (6.3.2) by

$$
\begin{equation*}
\mathrm{e}_{0}=\mathrm{e}_{a}-\mathrm{e}_{0} *\left(\mathcal{D}_{0}^{2}-\mathcal{D}_{a}^{2}\right) \mathrm{e}_{a} \tag{6.3.5}
\end{equation*}
$$

and

$$
\begin{align*}
\mathrm{e}_{0}=\mathrm{e}_{a}-\mathrm{e}_{a} * & \left(\mathcal{D}_{0}^{2}-\mathcal{D}_{a}^{2}\right) \mathrm{e}_{a}+\mathrm{e}_{a} *\left(\mathcal{D}_{0}^{2}-\mathcal{D}_{a}^{2}\right) \mathrm{e}_{a} *\left(\mathcal{D}_{0}^{2}-\mathcal{D}_{a}^{2}\right) \mathrm{e}_{a}-\ldots  \tag{6.3.6}\\
& +(-1)^{k} \mathrm{e}_{a} *\left(\mathcal{D}_{0}^{2}-\mathcal{D}_{a}^{2}\right) \mathrm{e}_{a} * \cdots *\left(\mathcal{D}_{0}^{2}-\mathcal{D}_{a}^{2}\right) \mathrm{e}_{a} \pm \ldots
\end{align*}
$$

Whenever we can split a manifold into parts we can approximate the heat kernel by the heat kernel of the parts. This follows from Duhamel's principle. More precisely we have

Theorem 6.3.3 (Duhamel's Splitting Formula). Let $M$ be a compact manifold with or without boundary and $\mathcal{D}: \operatorname{dom}(\mathcal{D}) \rightarrow L_{2}(M ; \$)$ be a $L_{2}-$ extension of an elliptic differential operator with heat kernel $\mathrm{e}\left(t ; x, x^{\prime}\right)$. Let $\left\{U_{1}, U_{2}\right\}$ be a covering of $M$ by two open submanifolds. For $j=1,2$ we assume that $U_{j}$ is isometric to an open submanifold of a (not necessarily compact) manifold $M_{j}$ and that there is given on $M_{j}$ a $L_{2}$-extension $\mathcal{D}_{j}: \operatorname{dom}\left(\mathcal{D}_{j}\right) \rightarrow L_{2}\left(M_{j} ; \mathbb{X}_{j}\right)$ of a differential operator with heat kernel $\mathrm{e}_{j}\left(t ; x, x^{\prime}\right)$ such that $\mathcal{D}_{j}$ and $\mathcal{D}$ coincide over $U_{j} .{ }^{3}$ Then there exist positive real constants $c_{1}, c_{2}$ such that for any $0<t<1$

$$
\begin{equation*}
\left|\mathrm{e}\left(t ; x, x^{\prime}\right)-\mathrm{e}_{j}\left(t ; x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}} \quad \text { for } x, x^{\prime} \in U_{j} \tag{6.3.7}
\end{equation*}
$$

and, more generally,

$$
\begin{equation*}
\left|\mathrm{e}\left(t ; x, x^{\prime}\right)-Q\left(t ; x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}} \quad \text { for arbitrary } x, x^{\prime} \in M \tag{6.3.8}
\end{equation*}
$$

where $Q\left(t ; x, x^{\prime}\right)$ is an integral kernel over $M$ obtained by suitably gluing the heat kernels $\mathrm{e}_{1}$ and $\mathrm{e}_{2}$.

Proof. First we define on $C^{\infty}(M ; \$)$ for any parameter $t>0$ an operator $Q(t)$ with a smooth kernel, given by

$$
Q\left(t ; x, x^{\prime}\right):=\sum_{j=1}^{2} \psi_{j}(x) \mathrm{e}_{j}\left(t ; x, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)
$$

where $\left\{\varphi_{1}, \varphi_{2}\right\}$ is a smooth partition of unity on $M$ suitable for the covering $\left\{U_{1}, U_{2}\right\}$ and $\psi_{1}, \psi_{2}$ are non-negative smooth functions such that

$$
\begin{aligned}
& \psi_{j} \equiv 1 \\
& \text { on }\left\{x \in M \mid d\left(x, \operatorname{supp} \varphi_{j}\right)<\delta\right\} \\
& \text { and } \psi_{j} \equiv 0 \\
& \text { on }\left\{x \in M \mid d\left(x, \operatorname{supp} \varphi_{j}\right) \geq 2 \delta\right\}
\end{aligned}
$$

[^3]for a suitable $\delta>0$. We notice
\[

$$
\begin{equation*}
\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime}, \operatorname{supp} \varphi_{j}\right)=\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime \prime}, \operatorname{supp} \varphi_{j}\right) \geq \delta \tag{6.3.9}
\end{equation*}
$$

\]

Then, for $x^{\prime} \in U_{j}$ with $\varphi_{j}\left(x^{\prime}\right)=1$, we have by definition:

$$
Q\left(t ; x, x^{\prime}\right)= \begin{cases}e_{j}\left(t ; x, x^{\prime}\right) & \text { if } d\left(x, \operatorname{supp} \varphi_{j}\right)<\delta, \text { and }  \tag{6.3.10}\\ 0 & \text { if } d\left(x, \operatorname{supp} \varphi_{j}\right) \geq 2 \delta\end{cases}
$$

For fixed $t>0$, we determine the difference between the precise heat kernel $\mathrm{e}\left(t ; x, x^{\prime}\right)$ and the approximate one $Q\left(t ; x, x^{\prime}\right)$. By Duhamel's Formula (Proposition 6.3.1), we have

$$
\mathrm{e}\left(t ; x, x^{\prime}\right)-Q\left(t ; x, x^{\prime}\right)=-\int_{0}^{t} d s \int_{M} d z \mathrm{e}(s ; x, z) C\left(t-s ; z, x^{\prime}\right)
$$

with

$$
\begin{aligned}
& C\left(t-s ; z, x^{\prime}\right)=\left(\mathcal{D}_{0(z)}^{2}+\frac{d}{d(t-s)}\right) Q\left(t-s ; z, x^{\prime}\right) \\
&=\left(\mathcal{D}_{0(z)}^{2}-\frac{d}{d s}\right) Q\left(t-s ; z, x^{\prime}\right) \\
&=\sum_{j=1}^{2}\left\{\psi_{j}^{\prime \prime}(z) \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)+2 \psi_{j}^{\prime}(z) \frac{d}{d z}\left(\mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right)\right) \varphi_{j}\left(x^{\prime}\right)\right. \\
&\quad+\psi_{j}(z) \underbrace{\left(\mathcal{D}_{0(z)}^{2}-\frac{d}{d s}\right) \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right)}_{=0} \varphi_{j}\left(x^{\prime}\right)\} .
\end{aligned}
$$

As stated in (8.2.5), the supports of $\varphi_{j}$ and $\psi_{j}^{\prime}$ (and, equally, $\psi_{j}^{\prime \prime}$ ) are disjoint and separated from each other by a distance $\geq \delta$. Then, in fine correspondence to (6.3.10), the error term $C\left(t-s ; z, x^{\prime}\right)$ vanishes both for $d\left(z, x^{\prime}\right)<\delta$ and, as well, for sufficiently large geodesic distance, say if $x^{\prime} \in M \backslash \operatorname{supp} \varphi_{2}$ and $z \in M \backslash \operatorname{supp} \varphi_{1}$. Moreover, since we have $t \geq s \geq 0$, we obtain

$$
(t-s)^{-m / 2} e^{-c^{\prime^{2}\left(z, x^{\prime}\right)}} \frac{(t-s)}{(t-s)} \leq c t^{-m / 2} e^{-c^{\prime} \frac{d^{2}\left(z, x^{\prime}\right)}{t}} \leq \tilde{c} e^{-c^{\prime^{2}} \frac{d^{2}\left(z, x^{\prime}\right)}{2 t}}
$$

Thus we can estimate the first sum

$$
\left|\psi_{j}^{\prime \prime}(z) \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)\right| \leq c_{1} e^{-c^{\prime} \delta^{2} / 2 t}=c_{1} e^{-\frac{c_{2}}{t}}
$$

Similarly we estimate the second sum. We have

$$
2\left|\psi_{j}^{\prime}(z) \frac{d}{d z} \mathrm{e}_{j}\left(t-s ; z, x^{\prime}\right) \varphi_{j}\left(x^{\prime}\right)\right| \leq \frac{c t^{-m / 2}}{t} e^{-\frac{c^{\prime}}{t}} \leq c_{1} e^{-\frac{c_{2}}{t}}
$$

Renaming the constants, we have obtained the crucial estimate

$$
\left|C\left(t, x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}} \quad \text { for } d\left(x, x^{\prime}\right)>\delta
$$

We recall from Lemma 6.2.2 the other crucial estimate

$$
\left|\mathrm{e}\left(t, x, x^{\prime}\right)\right|<c_{1} e^{-\frac{c_{2}}{t}}
$$

This gives

$$
\begin{aligned}
\mid \mathrm{e}\left(t ; x, x^{\prime}\right) & -Q\left(t ; x, x^{\prime}\right)\left|\leq \int_{0}^{t} d s \int_{M} d z\right| \mathrm{e}(s ; x, z) C\left(t-s ; z, x^{\prime}\right) \mid \\
& \leq \int_{0}^{t} d s \int_{M} d z c_{1} e^{-\frac{c_{2}}{t}} c_{1} e^{-\frac{c_{2}}{t}}=\operatorname{vol}(M) c_{1}^{2} \int_{0}^{t} e^{-\frac{c_{2} t}{s(t-s)}} d s \\
& s(t-s)=(t-s) s \\
= & \operatorname{vol}(M) c_{1}^{2} \int_{0}^{\frac{t}{2}} e^{-\frac{c_{2} t}{s(t-s)}} d s \\
& \stackrel{t-s \leq t / 2}{\leq} 2 \operatorname{vol}(M) c_{1}^{2} \int_{0}^{\frac{t}{2}} e^{-\frac{c_{2} t}{s t / 2}} d s \\
& =2 \operatorname{vol}(M) c_{1}^{2} \int_{0}^{\frac{t}{2}} e^{-\frac{2 c_{2}}{s}} d s \leq 2 \operatorname{vol}(M) c_{1}^{2} e^{-\frac{4 c_{2}}{t}} \int_{0}^{\frac{t}{2}} d s \\
& =\operatorname{vol}(M) c_{1}^{2} t e^{-\frac{4 c_{2}}{t}}<c_{3} e^{-\frac{c_{4}}{t}} .
\end{aligned}
$$

Remark 6.3.4. For the calculation of the estimation of the poles and the limiting behaviour of $\operatorname{tr}\left(\mathrm{e}_{0}(t ; x, x)\right)$ and $\operatorname{tr}\left(\mathcal{E}_{0}(t ; x, x)\right)$ up to an exponentially small error, we shall replace $M$ by $\mathbf{R}^{m}$ and suppose that

$$
\mathcal{D}_{0}=\Delta=-\frac{\partial^{2}}{\partial x_{1}^{2}}-\cdots-\frac{\partial^{2}}{\partial x_{m}^{2}}
$$

far out. This is permissible, because of the preceding Splitting Theorem. Then we consider a coordinate patch $U$ of $M$ and consider it as $\mathbf{R}^{m}$. We define the alternative operator $\mathcal{D}_{a}$ to be $\mathcal{D}_{0}$ with its coefficients frozen at $x^{\prime} \in$ $U$ and denote the corresponding heat kernel by $\mathrm{e}_{a}$. We denote the difference $\left.\mathcal{D}_{0}^{2}\right|_{U}-\mathcal{D}_{a}^{2}$ by $\mathcal{R}$. Then we have one more reformulation of Duhamel's principle:

$$
\begin{equation*}
\mathrm{e}_{0}\left(t ; x, x^{\prime}\right)=\mathrm{e}_{a}\left(t ; x, x^{\prime}\right)-\int_{0}^{t} d s \int_{\mathbf{R}^{m}} \mathrm{e}_{0}(s ; x, z) \mathcal{R}_{(z)} \mathrm{e}_{a}\left(t-s ; z, x^{\prime}\right) d z \tag{6.3.11}
\end{equation*}
$$

and

$$
\begin{align*}
& \mathrm{e}_{0}\left(t ; x, x^{\prime}\right)=\mathrm{e}_{a}\left(t ; x, x^{\prime}\right)+\sum_{k=1}^{\infty}(-1)^{k} \int_{0}^{t} d s \int_{\mathbf{R}^{m}} d z \int_{0}^{s} d s_{1} \int_{\mathbf{R}^{m}} d x_{1} \ldots  \tag{6.3.12}\\
& \ldots \int_{0}^{s_{k-2}} d s_{k-1} \int_{\mathbf{R}^{m}} d x_{k-1} \mathrm{e}_{a}\left(s_{k-1} ; x, x_{k-1}\right) \mathcal{R}_{\left(x_{k-1}\right)} \mathrm{e}_{a}\left(s_{k-2}-s_{k-1} ; x_{k-1}, x_{k-2}\right) \\
& \ldots \mathcal{R}_{\left(x_{1}\right)} \mathrm{e}_{a}\left(s-s_{1} ; x_{1}, z\right) \mathcal{R}_{(z)} \mathrm{e}_{a}\left(t-s ; z, x^{\prime}\right)
\end{align*}
$$

Example 6.3.5. To give a simple example, we consider the Laplacian $\Delta_{0}$ with domain $L_{2}\left(\mathbf{R}^{m}\right)$ and its perturbation $\Delta_{1}:=\Delta_{0}+V(x)$ by a potential
$V \in C_{0}^{\infty}\left(\mathbf{R}^{m}\right)$. Then

$$
\mathrm{e}_{\Delta_{0}}\left(t ; x, x^{\prime}\right)=\mathrm{e}_{0}\left(t ; x, x^{\prime}\right)=(4 \pi t)^{-m / 2} e^{-\frac{\left\|x-x^{\prime}\right\|^{2}}{4 t}} .
$$

Let us assume that $e^{-t \Delta_{1}}$ exists. (This is not hard to prove). Then its integral kernel is unique and satisfies the conditions formulated in the Proposition 6.3.1. We apply equation (6.3.12) of the preceding Remark and obtain for $\mathcal{R}=V$ :

$$
\begin{aligned}
& e^{-t \Delta_{1}}=e^{-t \Delta_{0}}+ \\
& \sum_{k=1}^{\infty}(-1)^{k} \int_{0}^{t} d s \int_{0}^{s} d s_{1} \ldots \int_{0}^{s_{k-2}} d s_{k-1} e^{-s_{k-1} \Delta_{0}} V e^{-\left(s_{k-2}-s_{k-1}\right) \Delta_{0}} \ldots \\
& \ldots V e^{-\left(s-s_{1}\right) \Delta_{0}} V e^{-(t-s) \Delta_{0}} .
\end{aligned}
$$

We consider the first term

$$
q_{0}\left(t ; x, x^{\prime}\right)=\int_{0}^{t} e^{-s \Delta_{0}} V e^{-(t-s) \Delta_{0}} d s
$$

Setting

$$
V_{\max }:=\sup _{\mathbf{R}^{m}}|V(x)|
$$

we obtain on the diagonal

$$
\begin{aligned}
\left|q_{0}(t ; x, x)\right| & =\left|\int_{0}^{t} d s \int_{\mathbf{R}^{m}} d z \mathrm{e}_{0}(s ; x, z) V(z) \mathrm{e}_{0}(t-s ; z, x)\right| \\
& \leq \int_{0}^{t} d s \int_{\mathbf{R}^{m}} d z\left|\mathrm{e}_{0}(s ; x, z) V(z) \mathrm{e}_{0}(t-s ; z, x)\right| \\
& \leq V_{\max } \int_{0}^{t} d s \int_{\mathbf{R}^{m}} d z(4 \pi s)^{-m / 2}(4 \pi(t-s))^{-m / 2} e^{-\frac{\|x-z\|^{2}}{4 s}} e^{-\frac{\|x-z\|^{2}}{4(t-s)}} \\
& \leq V_{\max }(4 \pi)^{-m} \int_{0}^{t}(s(t-s))^{-m / 2} d s \int_{\mathbf{R}^{m}} e^{-\frac{t\|x-z\|^{2}}{4 s(t-s)}} d z
\end{aligned}
$$

Since $\int_{\mathbf{R}} e^{-x^{2} / \alpha} d x=\sqrt{\pi \alpha}$ we have

$$
\int_{\mathbf{R}^{m}} e^{-x^{2} / \alpha} d x=(\pi \alpha)^{m / 2} .
$$

Thus we have

$$
\begin{aligned}
\left|q_{0}(t ; x, x)\right| & \leq V_{\max }(4 \pi)^{-m} \pi^{m / 2} \cdot \int_{0}^{t}(s(t-s))^{-m / 2}\left(\frac{4 s(t-s)}{t}\right)^{m / 2} d s \\
& =V_{\max } 4^{-m / 2} \pi^{-m / 2} \cdot t^{-m / 2} \int_{0}^{t} d s=\frac{V_{\max }}{(4 \pi)^{m / 2}} t^{\frac{2-m}{2}}
\end{aligned}
$$

The same type of computations shows that the kernel $q_{k-1}(t ; x, x)$ of the operator

$$
\int_{0}^{t} d s \int_{0}^{s} d s_{1} \ldots \int_{0}^{s_{k-2}} d s_{k-1} e^{-s_{k-1} \Delta_{0}} V \ldots V e^{-(t-s) \Delta_{0}}
$$

is bounded as follows:

$$
\left|q_{k-1}(t ; x, x)\right| \leq\left(\frac{V_{\max }}{(4 \pi)^{m / 2}}\right)^{k} \frac{1}{k!} t^{\frac{2 k-m}{2}}
$$

This proves

Lemma 6.3.6. Let $\Delta_{1}:=\Delta_{0}+V(x)$ be a Laplacian with potential $V \in$ $C_{0}^{\infty}\left(\mathbf{R}^{m}\right)$. Then there exist positive real constants $c_{1}, c_{2}$ such that for all $t>0$ and $x \in \mathbf{R}^{m}$ the kernel $\mathrm{e}_{1}$ of the heat operator $e^{-t \Delta_{1}}$ can be estimated by

$$
\left|\mathrm{e}_{1}(t ; x, x)\right| \leq \frac{c_{1}}{(4 \pi t)^{m / 2}} \cdot e^{c_{2} t}
$$

## CHAPTER 7

## The $\zeta$-Determinant on the Smooth, Self-adjoint Grassmannian

In this chapter we discuss the existence of the $\zeta$-determinant of a Dirac operator $\mathcal{D}$ on a compact manifold with boundary. We show that the determinant is well defined in the case of the operator $\mathcal{D}$ with a domain determined by a boundary condition from the smooth, self-adjoint Grassmannian $\mathcal{G r}_{\infty}^{*}(\mathcal{D})$.

### 7.1. Introduction

Recent studies in Quantum Field Theory and Topology have stressed the importance of the correct definition of the renormalized determinant of the Dirac operator over a manifold with boundary. The renormalization successufully used in the case of a closed manifold is the $\zeta$-renormalization introduced by Ray and Singer in [84] (see also [100]). The $\zeta$-determinant of the Dirac operator $\mathcal{D}$ on a closed manifold is given by the formula:

$$
\begin{equation*}
\operatorname{det}_{\zeta} \mathcal{D}=e^{\frac{i \pi}{2}\left(\eta_{\mathcal{D}}(0)-\zeta_{\mathcal{D}^{2}}(0)\right)} \cdot e^{-1 / 2 \cdot\left(d /\left.d s\left(\zeta_{\mathcal{D}^{2}}(s)\right)\right|_{s=0}\right)} \tag{7.1.1}
\end{equation*}
$$

where $\zeta_{\mathcal{D}^{2}}(s)$ and $\eta_{\mathcal{D}}(s)$ are functions constructed from the eigenvalues of the operator $\mathcal{D}$.

Now let us assume that $\mathcal{D}: C^{\infty}(M ; S) \rightarrow C^{\infty}(M ; S)$ is a compatible Dirac operator acting on sections of a bundle of Clifford modules $S$ over a compact Riemannian manifold $M$ with boundary $Y$. We concentrate on the case of an odd-dimensional manifold $M$, and from now on we assume that $n=\operatorname{dim} M=2 m+1$.

Let us point out, however, that our results are true for Dirac operators on an even-dimensional manifold as well. The necessary modifications due to the different algebraic structure of the spinors in the odd and even case can be found in [?], where we discuss the applications of our results in the even-dimensional situation (see also [22] for an introductory discussion of applications of the $\zeta$-determinant of elliptic boundary problems in Quantum Chromodynamics).

We discuss only the Product Case. Namely we assume that the Riemannian metric on $M$ and the Hermitian structure on $S$ are products in a certain collar neighborhood of the boundary. Let us fix $N=[0,1] \times Y$ the collar. Then in $N$ the operator $\mathcal{D}$ has the form

$$
\begin{equation*}
\mathcal{D}=G\left(\partial_{u}+B\right) \tag{7.1.2}
\end{equation*}
$$

where $G: S|Y \rightarrow S| Y$ is a unitary bundle isomorphism (Clifford multiplication by the unit normal vector) and $B: C^{\infty}(Y ; S \mid Y) \rightarrow C^{\infty}(Y ; S \mid Y)$ is the corresponding Dirac operator on $Y$, which is an elliptic self-adjoint operator of first order. Furthermore, $G$ and $B$ do not depend on the normal coordinate $u$ and they satisfy the identities

$$
\begin{equation*}
G^{2}=-I d \quad \text { and } \quad G B=-B G \tag{7.1.3}
\end{equation*}
$$

Since $Y$ has dimension $2 m$ the bundle $S \mid Y$ decomposes into its positive and negative chirality components $S \mid Y=S^{+} \bigoplus S^{-}$and we have a corresponding splitting of the operator $B$ into $B^{ \pm}: C^{\infty}\left(Y ; S^{ \pm}\right) \rightarrow C^{\infty}\left(Y ; S^{\mp}\right)$, where $\left(B^{+}\right)^{*}=B^{-}$. The equation (7.1.2) can be rewritten in the following form

$$
\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right)\left(\partial_{u}+\left(\begin{array}{cc}
0 & B^{-} \\
B^{+} & 0
\end{array}\right)\right) .
$$

In order to obtain a nice unbounded Fredholm operator we have to impose a boundary condition on the operator $\mathcal{D}$. Let $\Pi_{\geq}$denote the spectral projection of $B$ onto the subspace of $L^{2}(Y ; S \mid Y)$ spanned by the eigenvectors corresponding to the nonnegative eigenvalues of $B$. It is well known that $\Pi_{\geq}$is an elliptic boundary condition for the operator $\mathcal{D}$ (see [5], [27]). The meaning of the ellipticity is as follows. We introduce the unbounded operator $\mathcal{D}_{\Pi_{\geq}}$equal to the operator $\mathcal{D}$ with domain

$$
\operatorname{dom} \mathcal{D}_{\Pi_{\geq}}=\left\{s \in H^{1}(M ; S) ; \Pi_{\geq}(s \mid Y)=0\right\}
$$

where $H^{1}$ denotes the first Sobolev space. Then the operator $\mathcal{D}_{\Pi_{\geq}}=\mathcal{D}$ : $\operatorname{dom}\left(\mathcal{D}_{\Pi_{\geq}}\right) \rightarrow L^{2}(M ; S)$ is a Fredholm operator with kernel and cokernel consisting only of smooth sections.

The orthogonal projection $\Pi_{\geq}$is a pseudodifferential operator of order 0 (see [27]). In fact we can take any pseudodifferential operator $R$ of order 0 with principal symbol equal to the principal symbol of $\Pi_{\geq}$and obtain an operator $\mathcal{D}_{R}$ which satisfies the aforementioned properties. Let us point out, however, that only the projection onto the kernel of the operator $R$ is used in the construction of the operator $\mathcal{D}_{R}$. Therefore we can restrict ourselves to the study of the Grassmannian $\operatorname{Gr}(\mathcal{D})$ of all pseudodifferential projections which differ from $\Pi_{\geq}$by an operator of order -1 . The space $\operatorname{Gr}(\mathcal{D})$ has infinitely many connected components and two boundary conditions $P_{1}$ and $P_{2}$ belong to the same connected component if and only if

$$
\text { index } \mathcal{D}_{P_{1}}=\text { index } \mathcal{D}_{P_{2}}
$$

We are interested however in self-adjoint realizations of the operator $\mathcal{D}$ . The involution $G: S|Y \rightarrow S| Y$ equips $L^{2}(Y ; S \mid Y)$ with a symplectic structure, and Green's formula (see [27])

$$
\begin{equation*}
\left(\mathcal{D} s_{1}, s_{2}\right)-\left(s_{1}, \mathcal{D} s_{2}\right)=-\int_{Y}<G\left(s_{1} \mid Y\right) ; s_{2} \mid Y>d y \tag{7.1.4}
\end{equation*}
$$

shows that the boundary condition $R$ provides a self-adjoint realization $\mathcal{D}_{R}$ of the operator $\mathcal{D}$ if and only if ker $R$ is a Lagrangian subspace of $L^{2}(Y ; S \mid Y)$ (see [26], [27], [40]). It is therefore reasonable to restrict ourselves to those elements of $\operatorname{Gr}(\mathcal{D})$ which are Lagrangian subspaces of $L^{2}(Y ; S \mid Y)$. More precisely we introduce $G r^{*}(\mathcal{D})$, the Grassmannian of orthogonal, pseudodifferential projections $P$ such that $P-\Pi_{\geq}$is an operator of order -1 and

$$
\begin{equation*}
-G P G=I d-P . \tag{7.1.5}
\end{equation*}
$$

The space $G r^{*}(\mathcal{D})$ is contained in the connected component of $\operatorname{Gr}(\mathcal{D})$ parameterizing projections $P$ with index $\mathcal{D}_{P}=0$.

We discuss only the Smooth, Self-adjoint Grassmannian, a dense subset of the space $G r^{*}(\mathcal{D})$, defined by

$$
\begin{equation*}
G r_{\infty}^{*}(\mathcal{D})=\left\{P \in G r^{*}(\mathcal{D}) ; P-\Pi_{\geq} \text {has a smooth kernel }\right\} . \tag{7.1.6}
\end{equation*}
$$

Remark 7.1.1. The spectral projection $\Pi_{\geq}$is an element of $G r_{\infty}^{*}(\mathcal{D})$ if and only if ker $B=\{0\}$. However, it is well-known that $P(\mathcal{D})$ the (orthogonal) Calderon projection is an element of $G r^{*}(\mathcal{D})$ (see for instance [26]), and it has been recently observed by Simon Scott (see [91], Proposition 2.2.) that $P(\mathcal{D})-\Pi_{\geq}$is a smoothing operator, and hence that $P(\mathcal{D})$ is an element of $G r_{\infty}^{*}(\mathcal{D})$. The finite-dimensional perturbations of $\Pi_{\geq}$discussed below (see also [40], [62] and [112]) provide further examples of boundary conditions from $G r_{\infty}^{*}(\mathcal{D})$. The latter were introduced by Jeff Cheeger, who called them Ideal Boundary Conditions (see [34], [35]).

For any $P \in G r^{*}(\mathcal{D})$ the operator $\mathcal{D}_{P}$ has a discrete spectrum nicely distributed along the real line (see [26], [40]). Therefore one might expect that $\operatorname{det}_{\zeta}\left(\mathcal{D}_{P}\right)$ is well defined. To see that, we have to study the asymptotic expansion of the heat kernels involved in the construction of the determinant, or equivalently the expansion of the operator $\left(\mathcal{D}_{P}-\lambda\right)^{-1}$. The existence of a nice asymptotic expansion of the trace of the heat kernels used in the constructions of $\eta_{\mathcal{D}_{P}}(s)$ and $\zeta_{\mathcal{D}_{P}^{2}}(s)$ was established in a recent work of Gerd Grubb [?] . She used the machinery developed in her earlier work and her joint work with Bob Seeley (see [?], [?], [?]). However, at the moment, the problem of explicit computation of the coefficents in the expansion is open. From this point of view the existence of the invariants used to define $\operatorname{det}_{\zeta}$ depends on the vanishing of particular coefficients in the corresponding expansions.

We choose a different route. It follows from our earlier work on Grassmannians (see [26], [27], and [40] Appendix B) that $G r_{\infty}^{*}(\mathcal{D})$ is a path
connected space. As a consequence we can perform a Unitary Twist and replace the operator $\mathcal{D}_{P}$ by a unitarily equivalent operator $(\mathcal{D}+\mathcal{R})_{\Pi_{\sigma}}$, where $\Pi_{\sigma} \in G r_{\infty}^{*}(\mathcal{D})$ denotes an appropriate finite-dimensional modification of $\Pi_{\geq}$defined below in Section 1. The operator $\mathcal{D}_{\Pi_{\sigma}}$ has a well-defined $\zeta$ determinant and the correction term $\mathcal{R}$ lives in the collar $N$. The operator $\mathcal{R}$ is no longer a differential operator, but for any $0 \leq u \leq 1, \mathcal{R}_{u}=\left.\mathcal{R}\right|_{\{u\} \times Y}$ is a pseudodifferential operator. If we assume that $P-\Pi_{\geq}$has a smooth kernel then the operator $\mathcal{R}_{u}$ has a smooth kernel for each $0 \leq u \leq 1$ and this is all that one needs in order to study the correction terms appearing in the corresponding heat kernels. This is the reason why we restrict attention to the space $G r_{\infty}^{*}(\mathcal{D})$. The arguments and the results hold also in the case of $P \in G r^{*}(\mathcal{D})$ and $P-\Pi_{\geq}$of trace class.

The main result of this chapter is the following Theorem.

Theorem 7.1.2. For any projection $P \in G r_{\infty}^{*}(\mathcal{D}), \eta_{\mathcal{D}_{P}}(s)$ and $\zeta_{\mathcal{D}_{P}^{2}}(s)$ are holomorphic functions of $s$ in a neighborhood of $s=0$.

Corollary 7.1.3. The $\zeta$-determinant is a well-defined smooth function on $G r_{\infty}^{*}(\mathcal{D})$.

Remark 7.1.4. (1) The result stated above implies the existence of the Quillen $\zeta$-function metric for families of elliptic boundary value problems. This metric was studied before by Piazza [81] in the context of $b$-calculus developed by Melrose and his collaborators.
(2) In fact, we are able to obtain complete asymptotic expansions of the heat kernels for the operator $\mathcal{D}_{P}$. The reason is that Duhamel's Principle allows one to study the interior contribution and the boundary contribution separately and identify the singularities caused by the boundary contribution. This procedure was used before in [40] (Section 4 and Appendix A) and $[\mathbf{6 0}]$ (Section 1). As the asymptotic expansion has been already studied (see [?], [?], [?]) we leave the details to the reader and concentrate instead on the analysis of the $\zeta$-determinant.
(3) A more difficult problem than the existence of the asymptotic expansion is to show that the invariants used in the construction of the determinant are well defined. For instance Grubb and Seeley showed the regularity of the $\eta$-function only for finite-dimensional perturbations of the

Atiyah-Patodi-Singer boundary condition (see [?]). A similar result was also obtained by Dai and Freed (see [37]). The $\eta$-invariant of a more general class of boundary problems was also studied recently by Brüning and Lesch (see [?]). There, however, the authors had to deal with the residuum of the $\eta$-function at $s=0$, which is not present in our situation.
(4) The results of this chapter were announced in a talk K.P. Wojciechowski gave at the Annual Meeting of the AMS in San Francisco in January 1995.

We also want to single out one particular result, which is related to the discussion of the dependence of spectral invariants on the symbol of the operator given in [?].

Proposition 7.1.5. The value of the $\zeta$-function at $s=0$ is constant on $G r_{\infty}^{*}(\mathcal{D})$, i.e.

$$
\begin{equation*}
\zeta_{\mathcal{D}_{P_{1}}^{2}}(0)=\zeta_{\mathcal{D}_{P_{2}}^{2}}(0) . \tag{7.1.7}
\end{equation*}
$$

for any $P_{1}, P_{2} \in G r_{\infty}^{*}(\mathcal{D})$.

The results of this chapter allow us to study the $\zeta$-determinant as a function on $G r_{\infty}^{*}(\mathcal{D})$. In particular, we are interested in the relation of the $\zeta$-determinant and the Quillen determinant defined as a canonical section of the determinant line bundle over the Grassmannian. It was observed by Scott [91] that when restricted to the self-adjoint Grassmannian the determinant line bundle over $\operatorname{Gr}(\mathcal{D})$ becomes trivial. Moreover, it has a natural trivialization over $G r_{\infty}^{*}(\mathcal{D})$. The Quillen determinant expressed in this trivialization becomes a function. We refer to the determinant obtained in this way as the canonical determinant and we denote it by $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}$ (see [91] for details). In recent work of the author and Simon Scott the relation between $\operatorname{det}_{\zeta} \mathcal{D}_{P}$ and $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}$ is studied. In fact, it has been shown that, up to a natural multiplicative constant, the two determinants are equal. Proposition 7.1.5 and Proposition ?? are used in an essential way in the proof of this result. We refer the reader to [?], [23], [93], [94] for details.

In this chapter we discuss another application of Theorem 0.2, the extension of the additivity formula for the $\eta$-invariant to boundary conditions from $G r_{\infty}^{*}(\mathcal{D})$. This formula has been previously known only for finite-dimensional perturbations of the Atiyah-Patodi-Singer condition (see [112]). Let us point out that the additivity formula for the $\eta$-invariant stated in Theorem ?? and Proposition 7.1.5 implies the additivity of the
phase of the $\zeta$-determinant under the pasting of two manifolds with the same boundary. This extends the result of Dai and Freed (see [37]).

In the following two sections we study the $\eta$-function of $\mathcal{D}_{P}$. We obtain the following result as a conclusion of our computations.

Theorem 7.1.6. For any $P \in G r_{\infty}^{*}(\mathcal{D})$ the function $\eta_{\mathcal{D}_{P}}(s)$ is a holomorphic function of $s$ in the half-plane $\operatorname{Re}(s)>-1$.

Section 4 contains a discussion of $\zeta_{\mathcal{D}_{P}^{2}}(s)$ and $d /\left.d s\left(\zeta_{\mathcal{D}_{P}^{2}}(s)\right)\right|_{s=0}$.
In Section 5 we present proofs of two technical results used in Section 3 and Section 4.

### 7.2. Boundary Contribution to the $\eta$-Function. I. Unitary Twist and Duhamel's Principle

Let us assume for a moment that the manifold $M$ does not have a boundary. The Dirac operator $\mathcal{D}$ is then a self-adjoint elliptic operator with a discrete spectrum $\left\{\lambda_{k}\right\}_{k \in \mathbf{Z}}$. We define the $\eta$-function of $\mathcal{D}$ as follows:

$$
\begin{equation*}
\eta_{\mathcal{D}}(s)=\sum_{\lambda_{k} \neq 0} \operatorname{sign}\left(\lambda_{k}\right)\left|\lambda_{k}\right|^{-s} . \tag{7.2.1}
\end{equation*}
$$

The function $\eta_{\mathcal{D}}(s)$ is a holomorphic function of $s$ for $\operatorname{Re}(s)>\operatorname{dim}(M)$ and it has a meromorphic extension to $\mathbf{C}$ with isolated simple poles on the real axis. The point $s=0$ is not a pole and $\eta_{\mathcal{D}}=\eta_{\mathcal{D}}(0)$ the $\eta$-invariant of the operator $\mathcal{D}$ is an important invariant, which has found numerous applications in geometry, topology and physics. In the case of a compatible Dirac operator $\mathcal{D}$ the $\eta$-function is actually a holomorphic function of $s$ for $\operatorname{Re}(s)>-2$. This was shown by Bismut and Freed $[\mathbf{1 7}]$, who used the heat kernel representation of the $\eta$-function

$$
\begin{equation*}
\eta_{\mathcal{D}}(s)=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \cdot \operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}^{2}} d t \tag{7.2.2}
\end{equation*}
$$

which in particular allows us to express the $\eta$-invariant as

$$
\begin{equation*}
\eta_{\mathcal{D}}(0)=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \cdot \operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}^{2}} d t \tag{7.2.3}
\end{equation*}
$$

It follows from (7.2.2) that the estimate

$$
\left|\operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}^{2}}\right|<c \sqrt{t}
$$

implies the regularity of the $\eta$-function. In fact, Bismut and Freed proved a sharper result: Let $\mathcal{E}(t ; x, y)$ denote the kernel of the operator $\mathcal{D} e^{-t \mathcal{D}^{2}}$, then there exists a positive constant $c$ such that for any $x \in M$ and for any $0<t<1$

$$
\begin{equation*}
|\operatorname{tr} \mathcal{E}(t ; x, x)|<c \sqrt{t} \tag{7.2.4}
\end{equation*}
$$

We argue along the same lines and prove the following Proposition, which implies Theorem 7.1.6.

Proposition 7.2.1. For any $P \in G r_{\infty}^{*}(\mathcal{D})$ there exists a positive constant $c>0$ such that for any $0<t<1$ the following estimate holds

$$
\begin{equation*}
\left|\operatorname{Tr} \mathcal{D}_{P} e^{-t \mathcal{D}_{P}^{2}}\right|<c \tag{7.2.5}
\end{equation*}
$$

The proof of Proposition 7.2.1 occupies Section 2 and Section 3 of the chapter. Proposition 7.2 .1 is a statement on the small time asymptotics, which by Duhamel's Principle allows us to replace the kernel of the operator by a suitable parametrix built from the heat kernel of the operator on $\tilde{M}$, the closed double of the manifold $M$, and the heat kernel of the operator $G\left(\partial_{u}+B\right)$ subject to the boundary condition $P$ on a cylinder $[0, \infty) \times Y$.

However, we need to start with a concrete representation of the heat kernel on the cylinder. Such a representation is well-known for the original Atiyah-Patodi-Singer condition $\Pi_{\geq}$(see [5], or [27] Section 22). In general the projection $\Pi_{\geq}$is not an element of the Grassmannian $\operatorname{Gr}_{\infty}^{*}(\mathcal{D})$. Nevertheless, one can find easily a finite-dimensional modification of $\Pi_{\geq}$which belongs to this Grassmannian and then use the explicit formulas for the heat kernel on the cylinder.

We obtain our modification of the Atiyah-Patodi-Singer condition in the following way. The involution $G$ (see (7.1.3)) restricted to $\operatorname{ker}(B)$ defines a symplectic structure on this subspace of $L^{2}(Y ; S \mid Y)$ and the Cobordism Theorem for Dirac Operators (see for instance [27], Corollary 21.16) implies

$$
\operatorname{dim} \operatorname{ker}\left(B^{+}\right)=\operatorname{dim} \operatorname{ker}\left(B^{-}\right)
$$

The last equality shows the existence of Lagrangian subspaces of $\operatorname{ker}(B)$. We choose such a subspace $W$ and let $\sigma: L^{2}(Y ; S \mid Y) \rightarrow L^{2}(Y ; S \mid Y)$ denote the orthogonal projection of $L^{2}(Y ; S \mid Y)$ onto $W$. Let $\Pi_{>}$denote the orthogonal projection of $L^{2}(Y ; S \mid Y)$ onto the subspace spanned by eigenvectors of $B$ corresponding to the positive eigenvalues. Then

$$
\begin{equation*}
\Pi_{\sigma}=\Pi_{>}+\sigma \in G r_{\infty}^{*}(\mathcal{D}) \tag{7.2.6}
\end{equation*}
$$

gives an element of $G r_{\infty}^{*}(\mathcal{D})$, which is a finite-dimensional perturbation of the Atiyah-Patodi-Singer condition. The operator $\mathcal{D}_{\sigma}=\mathcal{D}_{\Pi_{\sigma}}$ is a selfadjoint operator and the properties of its $\eta$-function were studied in [40] (see Section 4 and Appendix A). It follows that $\eta_{\mathcal{D}_{\sigma}}(s)$ is a holomorphic function for $\operatorname{Re}(s)>-2$. To make a connection with the operator $\mathcal{D}_{P}$ we need the following result, which is an easy consequence of the topological structure of the Grassmannians studied in [26], [27], [40] (Appendix B).

Lemma 7.2.2. For any $P \in G r_{\infty}^{*}(\mathcal{D})$ there exists a smooth path $\left\{g_{u}\right\}_{0 \leq u \leq 1}$ of unitary operators on $L^{2}(Y ; S \mid Y)$ which satisfies

$$
G g_{u}=g_{u} G \text { and } g_{u}-I d \text { has a smooth kernel },
$$

such that $g_{1}=I d$ and the path $\left\{P_{u}=g_{u} \Pi_{\sigma} g_{u}^{-1}\right\} \subset G r_{\infty}^{*}(\mathcal{D})$ connects $P_{0}=P$ with $P_{1}=\Pi_{\sigma}$.

We can always assume that the path $\left\{g_{u}\right\}$ is constant on subintervals $[0,1 / 4]$ and $[3 / 4,1]$. We introduce $U$ a unitary operator on $L^{2}(M ; S)$ using the formula

$$
U:=\left\{\begin{array}{ll}
\operatorname{Id} & \text { on } M \backslash N  \tag{7.2.7}\\
g_{u} & \text { on } N
\end{array} .\right.
$$

The following Lemma introduces the Unitary Twist, which allows us to replace the operator $\mathcal{D}_{P}$ by a modified operator $\mathcal{D}+\mathcal{R}$ subject to the boundary condition $\Pi_{\sigma}$. This makes possible an explicit construction of the heat kernels on a cylinder.

Lemma 7.2.3. The operators $\mathcal{D}_{P}$ and $\mathcal{D}_{U, \sigma}=\left(U^{-1} \mathcal{D} U\right)_{\Pi_{\sigma}}$ are unitarily equivalent operators.

Proof. Let $\left\{f_{k} ; \mu_{k}\right\}_{k \in \mathbf{Z}}$ denote a spectral resolution of the operator $\mathcal{D}_{P}$. This means that for each $k$ we have

$$
\mathcal{D} f_{k}=\mu_{k} f_{k} \quad \text { and } \quad P\left(f_{k} \mid Y\right)=0
$$

This implies

$$
U^{-1} \mathcal{D} U\left(U^{-1} f_{k}\right)=\mu_{k}\left(U^{-1} f_{k}\right) \quad \text { and } \Pi_{\sigma}\left(\left(U^{-1} f_{k}\right) \mid Y\right)=g_{0}^{-1} P\left(f_{k} \mid Y\right)=0
$$

hence $\left\{U^{-1} f_{k} ; \mu_{k}\right\}$ is a spectral resolution of $\left(U^{-1} \mathcal{D} U\right)_{\Pi_{\sigma}}$.

In the collar $N$, we have formulas

$$
U^{-1} \mathcal{D} U=\mathcal{D}+G U^{-1} \frac{\partial U}{\partial u}+G U^{-1}[B, U]
$$

and

$$
U^{-1} \mathcal{D}^{2} U=\mathcal{D}^{2}-2 U^{-1} \frac{\partial U}{\partial u} \partial_{u}-U^{-1} \frac{\partial^{2} U}{\partial u^{2}}+U^{-1}\left[B^{2}, U\right]
$$

which restricted to the collar $[0,1 / 4] \times Y$ give

$$
\begin{equation*}
U^{-1} \mathcal{D} U=\mathcal{D}+G U^{-1}[B, U] \text { and } U^{-1} \mathcal{D}^{2} U=\mathcal{D}^{2}+U^{-1}\left[B^{2}, U\right] \tag{7.2.8}
\end{equation*}
$$

It follows from Lemma 7.2.3 that we can study the operator $\mathcal{D}_{U, \sigma}$ instead of the operator $\mathcal{D}_{P}$. We use the representation (7.2.8) in the construction of the parametrix of the kernel of the operator $\mathcal{D}_{U, \sigma} e^{-t \mathcal{D}_{U, \sigma}^{2}}$. This parametrix is built from the heat kernel on the double manifold $\tilde{M}$ and the heat kernel on the cylinder. The bundle $S$ and the operator $\mathcal{D}$ extend to the corresponding objects $\tilde{S}$ and $\tilde{\mathcal{D}}$ on $\tilde{M}$ (see [40]; see [27] for a detailed discussion of the glueing constructions). There is also the obvious double $\tilde{U}$ of the unitary transformation $U$. We introduce the operator

$$
\tilde{U}^{-1} \tilde{\mathcal{D}} \tilde{U}: C^{\infty}(\tilde{M} ; \tilde{S}) \rightarrow C^{\infty}(\tilde{M} ; \tilde{S})
$$

which is unitarily equivalent to $\tilde{\mathcal{D}}$. Therefore the estimate (7.2.4) holds for the kernel $\tilde{\mathcal{E}}_{U}(t ; x, y)$ of the operator

$$
\tilde{U}^{-1} \tilde{\mathcal{D}} \tilde{U} e^{-t(\tilde{U}-1 \tilde{\mathcal{D}} \tilde{U})^{2}}=\tilde{U}^{-1} \tilde{\mathcal{D}} e^{-t \tilde{\mathcal{D}}^{2}} \tilde{U} .
$$

It follows from Duhamel's Principle that on $M \backslash N$ up to exponentially small error (in $t$ ), the kernel of $\tilde{U}^{-1} \tilde{\mathcal{D}} e^{-t \tilde{\mathcal{D}}^{2}} \tilde{U}$ is equal to the kernel of $\mathcal{D}_{U, \sigma} e^{-t \mathcal{D}_{U, \sigma}^{2}}$ for $0<t<1$ (see [40], [60]; see [27] for a detailed discussion of the variant of Duhamel's Principle we need in this chapter). More precisely, we have the following Lemma, which takes care of the situation in the interior of $M$

Lemma 7.2.4. Let $\mathcal{E}_{U, \sigma}(t ; x, y)$ denote the kernel of the operator

$$
\mathcal{D}_{U, \sigma} e^{-t \mathcal{D}_{U, \sigma}^{2}},
$$

then there exist positive constants $c_{1}, c_{2}$ such that for any $x \in M_{1 / 8}=M \backslash$ $[0,1 / 8] \times Y$ and any $0<t<1$ the following estimate holds

$$
\begin{equation*}
\left\|\mathcal{E}_{U, \sigma}(t ; x, x)-\tilde{\mathcal{E}}_{U}(t ; x, x)\right\| \leq c_{1} e^{-\frac{c_{2}}{t}} \tag{7.2.9}
\end{equation*}
$$

Hence the estimate (7.2.4) holds for the kernel of the operator $\mathcal{D}_{U, \sigma} e^{-t \mathcal{D}_{U, \sigma}^{2}}$ in $M_{1 / 8}$.

Now, we study the heat kernel in the collar neighborhood of $Y$. Once again we apply Duhamel's Principle to replace the kernel $\mathcal{E}_{U, \sigma}(t ; x, y)$ of the operator $\mathcal{D}_{U, \sigma} e^{-t \mathcal{D}_{U, \sigma}^{2}}$ by the corresponding kernel on $[0,+\infty) \times Y$. It follows from equation (7.2.8) that up to an exponentially small error we can use the kernel of the operator

$$
\begin{equation*}
\left(G\left(\partial_{u}+B\right)+\mathcal{K}_{1}\right) e^{-t\left(-\partial_{u}^{2}+B^{2}+\mathcal{K}_{2}\right)_{\sigma}} \tag{7.2.10}
\end{equation*}
$$

where

$$
\mathcal{K}_{1}=G U^{-1}[B, U] \quad \text { and } \mathcal{K}_{2}=U^{-1}\left[B^{2}, U\right]
$$

Let us observe that $\mathcal{K}_{1}$ anticommutes and $\mathcal{K}_{2}$ commutes with the involution $G$. The symbol $\exp \left(-t\left(-\partial_{u}^{2}+B^{2}+\mathcal{K}_{2}\right)_{\sigma}\right)$ in (7.2.10) denotes the following operator. We consider the operator $G\left(\partial_{u}+B\right)_{\Pi_{\sigma}}$ on the infinite cylinder $[0,+\infty) \times Y$ and its square, which we denote by $\left(-\partial_{u}^{2}+B^{2}\right)_{\sigma}$. The operator $\left(-\partial_{u}^{2}+B^{2}\right)_{\sigma}$ is an unbounded self-adjoint operator in $L^{2}([0,+\infty) \times Y ; S)$ and the kernel of the operator $\exp \left(-t\left(-\partial_{u}^{2}+B^{2}\right)_{\sigma}\right)$ is given by explicit formulas (see [5], [27]). We add the bounded operator $\mathcal{K}_{2}$ and obtain the operator $\left(-\partial_{u}^{2}+B^{2}+\mathcal{K}_{2}\right)_{\sigma}$. It follows from standard theory (see for instance [86]) that the semigroup $\exp \left(-t\left(-\partial_{u}^{2}+B^{2}+\mathcal{K}_{2}\right)_{\sigma}\right)$ is well defined. We study the trace of the kernel of $\left(G\left(\partial_{u}+B\right)+\mathcal{K}_{1}\right) e^{-t\left(-\partial_{u}^{2}+B^{2}+\mathcal{K}_{2}\right)_{\sigma}}$ in the next Section.

### 7.3. Boundary Contribution to the $\eta$-Function. II. Heat Kernel on the Cylinder

In this Section we continue the proof of the Proposition 7.2.1. We have to show that the boundary contribution to $\operatorname{Tr} \mathcal{D}_{P} e^{-t \mathcal{D}_{P}^{2}}$ is bounded for $t$ sufficiently small. Let $e(t)$ denote the operator $\exp \left(-t\left(-\partial_{u}^{2}+B^{2}+\mathcal{K}_{2}\right)_{\sigma}\right)$ and $e_{1}(t)$ denote the operator $\exp \left(-t\left(-\partial_{u}^{2}+B^{2}\right)_{\sigma}\right)$. We have the formula

$$
\begin{equation*}
e(t)=e_{1}(t)+\sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}(t) \tag{7.3.1}
\end{equation*}
$$

where the term $\mathcal{K}_{2} e_{1}$ appears n-times in the curly bracket under the summation sign and $*$ denotes convolution (see for instance [27]; Section 22C).

It follows from the explicit formulas giving the kernel of the operator $e_{1}(t)$ (see (7.3.5) and Appendix formula (7.5.1)) that

$$
\int_{Y} \operatorname{tr} G\left(\partial_{u}+B\right) e_{1}(t ;(u, y),(v, z))_{\substack{y=z \\ u=v}} d y=0 \quad, y, z \in Y
$$

We want to show that there exists a positive constant $C$ such that for any $0 \leq u \leq 1 / 8$
(7.3.2) $\left|\int_{Y} \operatorname{tr}\left(G\left(\partial_{u}+B\right)+\mathcal{K}_{1}\right) e(t ;(u, y),(v, z))_{\left\lvert\, \begin{array}{l}y=z \\ u=v \\ v\end{array}\right.} d y\right|<C \quad, y, z \in Y$.

It follows from Formula (7.3.1) that we have to study the trace

$$
\begin{aligned}
\int_{Y} \operatorname{tr}( & \left.G\left(\partial_{u}+B\right)+\mathcal{K}_{1}\right) \\
& \left\{e_{1}+\sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t ;(u, y),(v, z))_{\substack{y=z \\
u=v}} d y
\end{aligned}
$$

The involution $G$ commutes with the operators $e_{1}$ and $\mathcal{K}_{2}$ and anticommutes with $B$ and $\mathcal{K}_{1}$, which gives us

$$
\begin{aligned}
& \int_{Y} \operatorname{tr} G B\left\{e_{1}(t)+\right. \\
& \left.\qquad \sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t ;(u, y),(v, z))_{\mid \substack{\mid=z \\
u=v}} d y=0
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{Y} \operatorname{tr} \mathcal{K}_{1}\left\{e_{1}(t)+\right. \\
& \left.\qquad \sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t ;(u, y),(v, z))_{\left\lvert\, \begin{array}{l}
y=z \\
u=v \\
\hline
\end{array}\right.} d y=0 .
\end{aligned}
$$

Therefore we have the equality

$$
\begin{equation*}
\left.\int_{Y} \operatorname{tr} G\left(\partial_{u}+B\right)+\mathcal{K}_{1}\right) e(t ;(u, y),(v, z))_{\substack{y=z \\ u=v}} d y= \tag{7.3.3}
\end{equation*}
$$

$$
\int_{Y} \operatorname{tr} G \partial_{u}\left\{e_{1}(t)+\right.
$$

$$
\left.\sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t ;(u, y),(v, z))_{\substack{y=z \\ u=v}} d y=
$$

$$
\int_{Y} \operatorname{tr} G \partial_{u}\left\{\sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t ;(u, y),(v, z))_{\substack{y=z \\ u=v}} d y
$$

The last equality in (7.3.3) follows from the fact that

$$
\int_{Y} \operatorname{tr} G\left(\partial_{u} e_{1}\right)(t ;(u, y),(v, z))_{\left\lvert\, \begin{array}{l}
\mid=z \\
u=v \\
\end{array}\right.} d y=0
$$

(see formula (7.5.1)). We have to study the right side of (7.3.3). The crucial point here is to estimate the first term

$$
\int_{Y} \operatorname{tr} G\left(\partial_{u} e_{1}\right) * \mathcal{K}_{2} e_{1}(t ;(u, y),(v, z))_{\substack{y=z \\ u=v}} d y .
$$

We estimate the trace in the $Y$ - direction of the operator

$$
G\left(\partial_{u} e_{1}\right) * \mathcal{K}_{2} e_{1}(t)=\int_{0}^{t} G\left(\partial_{u} e_{1}(s)\right) \mathcal{K}_{2} e_{1}(t-s) d s
$$

Our result essentially follows from the fact that $\mathcal{E}_{1}(t-s ;(u, y),(v, z))$, the kernel of the operator $e_{1}(t-s)$, and $\mathcal{F}(s ;(u, y),(v, z))$, the kernel of the operator $\partial_{u} e_{1}(s)$, have nice "diagonal" representations on the cylinder. We can choose a spectral resolution $\left\{\varphi_{n} ; \mu_{n}\right\}_{n \in \mathbf{Z} \backslash\{0\}}$ of the tangential operator $B$, such that $\varphi_{n}$ corresponds to a positive eigenvalue or is an element of Ran $\sigma$ and $G \varphi_{n}=\varphi_{-n}$. This means that one has

$$
\begin{equation*}
B \varphi_{n}=\mu_{n} \varphi_{n} \text { and } \Pi_{\sigma} \varphi_{n}=0 \tag{7.3.4}
\end{equation*}
$$

for $\mu_{n} \geq 0$, and

$$
B\left(G \varphi_{n}\right)=-\mu_{n}\left(G \varphi_{n}\right) \text { and } \Pi_{\sigma}\left(G \varphi_{n}\right)=G \varphi_{n} .
$$

Now we can represent our kernels in the following way.

$$
\begin{equation*}
\mathcal{E}_{1}(t ;(u, y),(v, z))=\sum_{n \in \mathbf{Z} \backslash\{0\}} g_{n}(t ; u, v) \varphi_{n}(y) \otimes \varphi_{n}^{*}(z), \tag{7.3.5}
\end{equation*}
$$

and

$$
\mathcal{F}(t ;(u, y),(v, z))=\sum_{n \in \mathbf{Z} \backslash\{0\}} h_{n}(t ; u, v) \varphi_{n}(y) \otimes \varphi_{n}^{*}(z),
$$

where $g_{n}(t ; u, v)$ and $h_{n}(t ; u, v)$ are given by explicit formulas (see (7.5.1)). We have

$$
\operatorname{Tr}_{Y} G\left(\partial_{u} e_{1}(s)\right) \mathcal{K}_{2} e_{1}(t-s)_{\mid u=u_{0}}=
$$

and

$$
\begin{aligned}
& \left(\partial_{u} e_{1}(s)\right) \mathcal{K}_{2} e_{1}(t-s)\left(\varphi_{n}\right)(y)_{\mid u=u_{0}}= \\
& \quad \sum_{m \in \mathbf{Z} \backslash\{0\}} \int_{0}^{\infty} d v \cdot g_{m}\left(s ; u_{0}, v\right) h_{n}\left(t-s ; v, u_{0}\right)\left(\varphi_{m} ; \mathcal{K}_{2} \varphi_{n}\right) \varphi_{m}(y) .
\end{aligned}
$$

This gives us the following expressions
(7.3.6) $\operatorname{Tr}_{Y} G\left(\partial_{u} e_{1}(s)\right) \mathcal{K}_{2} e_{1}(t-s)_{\mid u=u_{0}}=$

$$
\sum_{m \in \mathbf{Z} \backslash\{0\}} \int_{0}^{\infty} h_{-n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v \cdot\left(\varphi_{n} ; \mathcal{K}_{2} \varphi_{n}\right),
$$

and

$$
\begin{aligned}
\operatorname{Tr}_{Y} e_{1}(s) \mathcal{K}_{2} e_{1}(t-s)_{\mid u=u_{0}} & = \\
& \sum_{m \in \mathbf{Z} \backslash\{0\}} \int_{0}^{\infty} g_{n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v \cdot\left(\varphi_{n} ; \mathcal{K}_{2} \varphi_{n}\right)
\end{aligned}
$$

The existence of the $\eta$-invariant for the operator $\mathcal{D}_{P}$ is now a consequence of the first part of the following Theorem. The second part of the Theorem is used below in Section 3, where we deal with the $\zeta$-function and its derivative.

Theorem 7.3.1. There exists a positive constant $c>0$ such that for any $n \neq 0$ and for any $0<t<1$ we have the following estimates

$$
\begin{equation*}
\left|\int_{0}^{\infty} h_{-n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v\right|<\frac{c}{\sqrt{s(t-s)}}, \tag{7.3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\int_{0}^{\infty} g_{n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v\right|<\frac{c}{\sqrt{t}} . \tag{7.3.8}
\end{equation*}
$$

The proof of the Theorem 7.3.1 is completely elementary and consists of long and tedious computations. We present the proof in the closing section of this chapter. Theorem 7.3.1 has the following immediate Corollary.

Corollary 7.3.2. Let $\gamma(u)$ denote a non-increasing smooth function equal to 1 for $u \leq 1 / 8$, and equal to 0 for $u \geq 1 / 4$. Then there exists a positive constant $c$ such that
(7.3.9) $\left|\operatorname{Tr} \gamma(u)\left\{G\left(\partial_{u} e_{1}\right) * \mathcal{K}_{2} e_{1}\right\}(t)\right| \leq c \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|$ and

$$
\left|\operatorname{Tr} \gamma(u)\left\{e_{1} * \mathcal{K}_{2} e_{1}\right\}(t)\right| \leq c \sqrt{t} \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|
$$

Proof. We prove the first estimate in (7.3.9), the proof of the second is completely analogous.

$$
\begin{aligned}
& \left|\operatorname{Tr} \gamma(u)\left\{G\left(\partial_{u} e_{1}\right) * \mathcal{K}_{2} e_{1}\right\}(t)\right| \leq \\
& \mid \sum_{n \in \mathbf{Z} \backslash\{0\}} \int_{0}^{t} d s \int_{0}^{\infty} \gamma(u) d u \\
& \int_{0}^{\infty} h_{-n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v \cdot\left(\varphi_{n} ; \mathcal{K}_{2} \varphi_{n}\right) \mid \leq \\
& c_{1} \cdot \sum_{n \in \mathbf{Z} \backslash\{0\}} \int_{0}^{t} d s \mid \int_{0}^{\infty} \gamma(u) d u \\
& \int_{0}^{\infty} h_{-n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v|\cdot|\left(\varphi_{n} ; \mathcal{K}_{2} \varphi_{n}\right) \mid< \\
& c_{2} \cdot\left(\sum_{n \in \mathbf{Z} \backslash\{0\}}\left|\left(\varphi_{n} ; \mathcal{K}_{2} \varphi_{n}\right)\right|\right) \cdot \int_{0}^{t} \frac{d s}{\sqrt{s(t-s)}} \leq c_{3} \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right| \cdot \int_{0}^{t / 2} \frac{d s}{\sqrt{s(t-s)}}< \\
& c_{3} \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right| \cdot \frac{1}{\sqrt{t / 2}} \cdot \int_{0}^{t / 2} \frac{d s}{\sqrt{s}}<c_{4} \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right| .
\end{aligned}
$$

## Proof of Proposition 7.2.1

We have

$$
\begin{aligned}
& \left.\mid \operatorname{Tr} \gamma(u)\left\{G\left(\partial_{u}+B\right)+\mathcal{K}_{1}\right) e\right\}(t) \mid \\
& =\mid \operatorname{Tr} \gamma(u) G \partial_{u}\left\{\sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t) \\
& \leq\left|\operatorname{Tr} \gamma(u)\left\{G\left(\partial_{u} e_{1}\right) * \mathcal{K}_{2} e_{1}\right\}(t)\right| \\
& +\left|\operatorname{Tr} \sum_{n=2}^{\infty} \gamma(u)\left\{G\left(\partial_{u} e_{1}\right) * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}(t)\right| \\
& \leq c \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|+\sum_{n=2}^{\infty} \mid \operatorname{Tr} \int_{0}^{t} d s_{1} \int_{0}^{s_{1}} d s_{2} \ldots \int_{0}^{s_{n-1}} d s_{n} . \\
& \gamma(u)\left(G \partial_{u} e_{1}\right)\left(s_{n}\right) \circ\left(\mathcal{K}_{2} e_{1}\right)\left(s_{n-1}-s_{n}\right) \circ \ldots \circ\left(\mathcal{K}_{2} e_{1}\right)\left(t-s_{1}\right) \mid \\
& \leq c \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|+\sum_{n=2}^{\infty} \int_{0}^{t} d s_{1} \int_{0}^{s_{1}} d s_{2} . . \int_{0}^{s_{n-1}} d s_{n} \\
& \left\{\operatorname{Tr}\left|\gamma(u)\left(G \partial_{u} e_{1}\right)\left(s_{n}\right) \circ\left(\mathcal{K}_{2} e_{1}\right)\left(s_{n-1}-s_{n}\right)\right| \cdot\right. \\
& \|\left(\mathcal{K}_{2} e_{1}\right)\left(s_{n-2}-s_{n-1}\|. .\|\left(\mathcal{K}_{2} e_{1}\right)\left(t-s_{1}\right) \|\right\} \\
& \leq c \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|+\sum_{n=2}^{\infty} \int_{0}^{t} d s_{1} \int_{0}^{s_{1}} d s_{2} \ldots \int_{0}^{s_{n-1}} d s_{n} . \\
& \operatorname{Tr}\left|\gamma(u)\left(G \partial_{u} e_{1}\right)\left(s_{n}\right) \circ\left(\mathcal{K}_{2} e_{1}\right)\left(s_{n-1}-s_{n}\right)\right| \cdot\left\|\mathcal{K}_{2}\right\|^{n-1} \\
& \leq c \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|\left\{1+c \cdot \sum_{n=2}^{\infty}\left\|\mathcal{K}_{2}\right\|^{n-1} \int_{0}^{t} d s_{1} \int_{0}^{s_{1}} d s_{2} \ldots \int_{0}^{s_{n-2}} d s_{n-1}\right\} \\
& =c \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right|\left\{1+c \cdot\left\|\mathcal{K}_{2}\right\| \cdot \sum_{n=2}^{\infty} \frac{\left(\left\|\mathcal{K}_{2}\right\| t\right)^{n-2}}{(n-2)!}\right\} \leq c_{1} \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right| \cdot e^{c_{2} t\left\|\mathcal{K}_{2}\right\|}
\end{aligned}
$$

for some positive constants $c_{1}$ and $c_{2}$. This ends the proof of Proposition 7.2.1.

### 7.4. The Modulus of the $\zeta$-Determinant on the Grassmannian

In this Section we study the spectral invariants of the operator $\mathcal{D}_{P}^{2}$ used in the construction of the $\zeta$-determinant, namely $\zeta_{\mathcal{D}_{P}^{2}}(0)$ and $d /\left.d s\left(\zeta_{\mathcal{D}_{P}^{2}}(s)\right)\right|_{s=0}$ (see (7.1.1)). Let us review briefly the situation in the case of a closed manifold $M$. We follow here the presentation in $[\mathbf{1 0 0}]$ and the necessary technicalities can be found in [45]. We assume that $\mathcal{D}$ is an invertible operator. Otherwise $\operatorname{det}_{\zeta} \mathcal{D}=0$. We have

$$
\begin{equation*}
\zeta_{\mathcal{D}^{2}}(s)=\operatorname{Tr}\left(\mathcal{D}^{2}\right)^{-s}=\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}^{2}} d t \tag{7.4.1}
\end{equation*}
$$

which is a well defined holomorphic function of $s$ for $\operatorname{Re}(s)>\frac{n}{2}$, where $n=\operatorname{dim} M$, and has a meromorphic extension to the whole complex plane with only simple poles. The poles and residues are determined by the small time asymptotics of the heat kernel. Let $\mathcal{E}(t ; x, y)$ denote the kernel of the operator $\exp \left(-t \mathcal{D}^{2}\right)$. The pointwise trace $\operatorname{tr} \mathcal{E}(t ; x, x)$ has an asymptotic expansion as $t \rightarrow 0$

$$
\begin{equation*}
\operatorname{tr} \mathcal{E}(t ; x, x)=t^{-n / 2} \sum_{k=0}^{N} t^{k / 2} a_{k}\left(\mathcal{D}^{2} ; x\right)+o\left(t^{\frac{N-n}{2}}\right) \tag{7.4.2}
\end{equation*}
$$

where $a_{k}\left(\mathcal{D}^{2} ; x\right)$ are computed from the coefficients of the operator $\mathcal{D}^{2}$ at the point $x$ (see [45]). It follows that the meromorphic extension of $\zeta_{\mathcal{D}^{2}}(s)$ has poles at the points $s_{k}=\frac{n-k}{2}$ with residues given by

$$
\begin{equation*}
\operatorname{Res}_{s=s_{k}} \zeta_{\mathcal{D}^{2}}(s)=\frac{1}{\Gamma\left(\frac{n-k}{2}\right)} a_{k}\left(\mathcal{D}^{2}\right) \tag{7.4.3}
\end{equation*}
$$

where $a_{k}\left(\mathcal{D}^{2}\right)$ denotes the integral

$$
a_{k}\left(\mathcal{D}^{2}\right)=\int_{M} a_{k}\left(\mathcal{D}^{2} ; x\right) d x
$$

In particular, there are no poles at non-positive integers and $\zeta_{\mathcal{D}^{2}}(0)$ is given by

$$
\begin{equation*}
\zeta_{\mathcal{D}^{2}}(0)=a_{n}\left(\mathcal{D}^{2}\right) \tag{7.4.4}
\end{equation*}
$$

The functions $\Gamma(s)$ and $\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}^{2}} d t$ have the following asymptotic expansion in a neighborhood of $s=0$.

$$
\begin{equation*}
\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}^{2}} d t=\frac{a_{n}\left(\mathcal{D}^{2}\right)}{s}+b+s f(s) \text { and } \Gamma(s)=\frac{1}{s}+\gamma+\operatorname{sh}(s) \tag{7.4.5}
\end{equation*}
$$

where $f(s)$ and $h(s)$ are holomorphic functions of $s$ and $\gamma$ denotes Euler's constant. The number $b$ denotes the regularized value of the integral $\int_{0}^{\infty} t^{-1} \operatorname{Tr} e^{-t \mathcal{D}^{2}} d t$. Now we differentiate

$$
\begin{align*}
&-\ln \operatorname{det}_{\zeta}\left(\mathcal{D}^{2}\right)= d /\left.d s\left(\zeta_{\mathcal{D}^{2}}\right)\right|_{s=0}  \tag{7.4.6}\\
&=\frac{d}{d s}\left\{\frac{a_{n}\left(\mathcal{D}^{2}\right)}{s}+b+s f(s)\right. \\
& \frac{1}{s}+\gamma+\operatorname{sh}(s) \\
&\left.\right|_{s=0}=b-\gamma \cdot a_{n}\left(\mathcal{D}^{2}\right),
\end{align*}
$$

and obtain the formula for the derivative of $\zeta_{\mathcal{D}^{2}}(s)$ at $s=0$.
We want to study the corresponding invariants on a manifold with boundary for the operator $\mathcal{D}_{P}$, where $P \in \operatorname{Gr}_{\infty}^{*}(\mathcal{D})$. We show that despite the additonal poles, caused by the boundary contribution, at least in a neighborhood of $s=0$ the situation is not different from the case of a
closed manifold. First we have the following result which holds in the case of the operator $\mathcal{D}_{\sigma}$.

Proposition 7.4.1. The function $\Gamma(s) \zeta_{\mathcal{D}_{\sigma}}(s)=\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{\sigma}^{2}} d t$ has a simple pole at $s=0$. Hence $\zeta_{\mathcal{D}_{\sigma}^{2}}(0)$ and, according to formula (7.4.6),

$$
\ln \operatorname{det}_{\zeta}\left(\mathcal{D}_{\sigma}\right)^{2}=-d /\left.d s\left(\zeta_{\mathcal{D}_{\sigma}^{2}}(s)\right)\right|_{s=0}
$$

are well defined.

The proof of Proposition 7.4 .1 consists of a straightforward computation of the boundary contribution and is included in the Appendix.

Now the fact that $\zeta_{\mathcal{D}_{P}^{2}}(0)$ and $d /\left.d s\left(\zeta_{\mathcal{D}_{P}^{2}}(s)\right)\right|_{s=0}$ are well defined is an immediate consequence of the next Theorem.

Theorem 7.4.2. For any $P \in G r_{\infty}^{*}(\mathcal{D})$ there exists a constant $c>0$ such that the following estimate holds for any $0<t<1$ :

$$
\begin{equation*}
\left|\operatorname{Tr} e^{-t \mathcal{D}_{P}^{2}}-\operatorname{Tr} e^{-t \mathcal{D}_{\sigma}^{2}}\right|<c \sqrt{t} \cdot \operatorname{Tr}\left|\mathcal{K}_{2}\right| e^{t\left\|\mathcal{K}_{2}\right\|} \tag{7.4.7}
\end{equation*}
$$

Proof. We essentially repeat the proof of Proposition 7.2.1. We replace the operator $\mathcal{D}_{P}$ by the operator $\mathcal{D}_{U, \sigma}$ and use Duhamel's Principle to obtain

$$
\begin{align*}
& \left|\operatorname{Tr} e^{-t \mathcal{D}_{P}^{2}}-\operatorname{Tr} e^{-t \mathcal{D}_{\Pi_{s}}^{2}}\right|=  \tag{7.4.8}\\
& \left.\quad \mid \operatorname{Tr} \sum_{n=1}^{\infty}\left\{e_{1} * \mathcal{K}_{2} e_{1} * \mathcal{K}_{2} e_{1} * \ldots * \mathcal{K}_{2} e_{1}\right\}\right\}(t) \mid+O\left(e^{-c / t}\right) .
\end{align*}
$$

Now we use the second part of Theorem 7.3.1 in order to estimate the sum on the right side of (7.4.8) in exactly the same way as in the proof of Proposition 7.2.1.

Theorem 7.4.2 shows that the difference $\zeta_{\mathcal{D}_{P}^{2}}(s)-\zeta_{\mathcal{D}_{\sigma}^{2}}(s)$ is a holomorphic function of $s$ for $\operatorname{Re}(s)>-\frac{1}{2}$. Therefore $\zeta_{\mathcal{D}_{P}^{2}}(s)$ is a holomorphic
function of $s$ in a neighborhood of $s=0$. The proof of Theorem 7.1.2 is now complete.

## Proof of Proposition 7.1.5

The Proposition is an easy Corollary of Theorem 7.4.2. It follows from (7.4.1) and (7.4.5) that we have the equality

$$
\begin{gathered}
\zeta_{\mathcal{D}_{P}^{2}}(0)-\zeta_{\mathcal{D}_{\sigma}^{2}}(0)=\lim _{s \rightarrow 0} \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \operatorname{Tr}\left(e^{-t \mathcal{D}_{P}^{2}}-e^{-t \mathcal{D}_{\sigma}^{2}}\right) d t= \\
\lim _{s \rightarrow 0} s \int_{0}^{1} t^{s-1} \operatorname{Tr}\left(e^{-t \mathcal{D}_{P}^{2}}-e^{-t \mathcal{D}_{\sigma}^{2}}\right) d t
\end{gathered}
$$

Now we apply Theorem 7.4.2 and obtain

$$
\begin{aligned}
\left|\zeta_{\mathcal{D}_{P}^{2}}(0)-\zeta_{\mathcal{D}_{\sigma}^{2}}(0)\right|<\lim _{s \rightarrow 0} s \int_{0}^{1} t^{s-1} \mid \operatorname{Tr} e^{-t \mathcal{D}_{P}^{2}} & -\operatorname{Tr} e^{-t \mathcal{D}_{\sigma}^{2}} \mid d t \\
& <c \cdot \lim _{s \rightarrow 0} s \int_{0}^{1} t^{s-1 / 2} d t=0
\end{aligned}
$$

This ends the Proof of Proposition 7.1.5.

### 7.5. Proof of Theorem 7.3.1 and Proposition 7.4.1

We start with a discussion of Theorem 7.3.1. Recall the formulas for the functions $g_{n}(t ; u, v)$ (see for instance [27], (22.33) and (22.35))

$$
\begin{equation*}
g_{n}(t ; u, v)=\frac{e^{-\mu_{n}^{2} t}}{2 \sqrt{\pi t}} \cdot\left\{e^{-\frac{(u-v)^{2}}{4 t}}-e^{-\frac{(u+v)^{2}}{4 t}}\right\} \text { for } n>0 \tag{7.5.1}
\end{equation*}
$$

and

$$
\begin{gathered}
g_{n}(t ; u, v)=\frac{e^{-\left(-\mu_{n}\right)^{2} t}}{2 \sqrt{\pi t}} \cdot\left\{e^{-\frac{(u-v)^{2}}{4 t}}+e^{-\frac{(u+v)^{2}}{4 t}}\right\}+ \\
\left(-\mu_{n}\right) e^{-\left(-\mu_{n}\right)(u+v)} \cdot \operatorname{erfc}\left(\frac{u+v}{2 \sqrt{t}}-\left(-\mu_{n}\right) \sqrt{t}\right) \text { for } n<0
\end{gathered}
$$

where

$$
\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-r^{2}} d r<\frac{2}{\sqrt{\pi}} e^{-x^{2}} .
$$

We begin with the estimate of the integral $\int_{0}^{\infty} g_{n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v$ . The most singular term is

$$
\begin{gathered}
\int_{0}^{\infty} \frac{e^{-\mu_{n}^{2} s}}{2 \sqrt{\pi s}} \cdot e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} \cdot \frac{e^{-\mu_{n}^{2}(t-s)}}{2 \sqrt{\pi(t-s)}} \cdot e^{-\frac{\left(u_{0}+v\right)^{2}}{4(t-s)}} d v= \\
\frac{1}{4 \pi} \frac{e^{-\mu_{n}^{2} t}}{\sqrt{s(t-s)}} \int_{0}^{\infty} e^{-\frac{t\left(u_{0}-v\right)^{2}}{4 s(t-s)}} d v< \\
\frac{1}{4 \pi} \frac{e^{-\mu_{n}^{2} t}}{\sqrt{s(t-s)}} 2 \sqrt{\frac{s(t-s}{t}} \int_{-\infty}^{+\infty} e^{-r^{2}} d r=\frac{1}{2 \sqrt{\pi t}} e^{-\mu_{n}^{2} t}<\frac{1}{2 \sqrt{\pi t}} .
\end{gathered}
$$

We also have the inequality

$$
e^{-\frac{(u+v)^{2}}{4 t}} \leq e^{-\frac{(u-v)^{2}}{4 t}},
$$

which holds for $u, v \geq 0$ and implies the estimate

$$
\int_{0}^{\infty} \frac{e^{-\mu_{n}^{2} s}}{2 \sqrt{\pi s}} \cdot e^{-\frac{\left(u_{0} \mp v\right)^{2}}{4 s}} \cdot \frac{e^{-\mu_{n}^{2}(t-s)}}{2 \sqrt{\pi(t-s)}} \cdot e^{-\frac{\left(u_{0} \pm v\right)^{2}}{4(t-s)}} d v \leq \frac{1}{2 \sqrt{\pi t}}
$$

This gives

$$
\int_{0}^{\infty} g_{n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v<\frac{1}{2 \sqrt{\pi t}}
$$

for positive $n$. If $n<0$ we also have to discuss the terms of the form

$$
\int_{0}^{\infty} \frac{e^{-\mu_{n}^{2} s}}{2 \sqrt{\pi s}} \cdot e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} \cdot \mu_{n} e^{\mu_{n}\left(u_{0}+v\right)} \cdot \operatorname{erfc}\left(\frac{u_{0}+v}{2 \sqrt{t-s}}+\mu_{n} \sqrt{t-s}\right) d v
$$

We have

$$
\begin{gathered}
\int_{0}^{\infty} \frac{e^{-\mu_{n}^{2} s}}{2 \sqrt{\pi s}} \cdot e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} \cdot \mu_{n} e^{\mu_{n}(u+v)} \cdot \operatorname{erfc}\left(\frac{u+v}{2 \sqrt{t-s}}+\mu_{n} \sqrt{t-s}\right)< \\
\frac{\mu_{n} e^{-\mu_{n}^{2} t}}{\pi \sqrt{s}} \int_{0}^{\infty} e^{-\frac{t\left(u_{0}-v\right)^{2}}{4 s(t-s)}} d v<\frac{\mu_{n} e^{-\mu_{n}^{2} t}}{\sqrt{\pi}} \sqrt{\frac{t-s}{t}}<c e^{-\mu_{n}^{2} t}
\end{gathered}
$$

Finally, we have to estimate the term in which the $\operatorname{erfc}$ function appears twice.

$$
\begin{aligned}
& \int_{0}^{\infty} \mu_{n} e^{\mu_{n}\left(u_{0}+v\right)} \operatorname{erfc}\left(\frac{u_{0}+v}{2 \sqrt{s}}+\mu_{n} \sqrt{s}\right) \cdot \\
& \quad \mu_{n} e^{\mu_{n}\left(u_{0}+v\right)} \operatorname{erfc}\left(\frac{u_{0}+v}{2 \sqrt{t-s}}+\mu_{n} \sqrt{t-s}\right) d v \\
& \quad<\frac{4}{\pi} \int_{0}^{\infty} \mu_{n}^{2} e^{2 \mu_{n}\left(u_{0}+v\right)} e^{-\left(\frac{u_{0}+v}{2 \sqrt{s}}+\mu_{n} \sqrt{s}\right)^{2}} \cdot e^{-\left(\frac{u_{0}+v}{2 \sqrt{t-s}}+\mu_{n} \sqrt{t-s}\right)^{2}} d v \\
& =\frac{4}{\pi} \mu_{n}^{2} e^{-\mu_{n}^{2} t} \int_{0}^{\infty} e^{-\frac{t\left(u_{0}+v\right)^{2}}{4 s(t-s)}} d v=\frac{4}{\pi} \mu_{n}^{2} e^{-\mu_{n}^{2} t} 2 \sqrt{\frac{s(t-s)}{t}} \int_{-\infty}^{+\infty} e^{-r^{2}} d r \\
& \quad \leq \frac{8}{\sqrt{\pi}} \mu_{n}^{2} e^{-\mu_{n}^{2} t} 2 \sqrt{t} \sqrt{\pi}<c \sqrt{t} e^{-\mu_{n}^{2} t} .
\end{aligned}
$$

The computations given above finish the proof of (7.3.8). We work the same way in order to obtain the estimate (7.3.7). The only difference is the appearence of $h_{n}\left(t ; u_{0}, v\right)=\frac{\partial g_{n}}{\partial u}\left(t ; u_{0}, v\right)$. The first term we have to consider has the form

$$
\begin{gathered}
\int_{0}^{\infty} \frac{e^{-\mu_{n}^{2} s}}{2 \sqrt{\pi s}} \frac{\left|u_{0}-v\right|}{2 s} e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} \cdot \frac{e^{-\mu_{n}^{2}(t-s)}}{2 \sqrt{\pi(t-s)}} e^{-\frac{\left(u_{0}-v\right)^{2}}{4(t-s)}} d v< \\
\frac{1}{4 \pi} \frac{e^{-\mu_{n}^{2} t}}{2 \sqrt{s(t-s)}} \int_{0}^{\infty} \frac{\left|u_{0}-v\right|}{2 s} e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} d v= \\
\frac{1}{4 \pi} \frac{e^{-\mu_{n}^{2} t}}{\sqrt{s(t-s)}}\left\{\int_{0}^{u_{0}} \frac{u_{0}-v}{2 s} e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} d v+\int_{u_{0}}^{\infty} \frac{v-u_{0}}{2 s} e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}} d v\right\}= \\
\frac{1}{4 \pi} \frac{e^{-\mu_{n}^{2} t}}{\sqrt{s(t-s)}}\left\{\int_{0}^{u_{0}} d\left(e^{-\frac{\left(u_{0}-v\right)^{2}}{4 s}}\right)-\int_{u_{0}}^{\infty} d\left(e^{-\frac{t\left(u_{0}-v\right)^{2}}{4 s}}\right\}<\frac{e^{-\mu_{n}^{2} t}}{2 \pi} \cdot \frac{1}{\sqrt{s(t-s)}} .\right.
\end{gathered}
$$

We work on the other terms which appear in $\int_{0}^{\infty} h_{-n}\left(s ; u_{0}, v\right) g_{n}\left(t-s ; v, u_{0}\right) d v$ in the same way. The details are left to the reader.

Now, we show that the function $\Gamma(s) \zeta_{\mathcal{D}_{\sigma}}(s)$ has a simple pole at $s=0$. We follow the method applied in Section 4 of [40] to study the $\eta$-invariant of $\mathcal{D}_{\sigma}$. Let us point out that the situation is simpler in the case of the $\eta$-invariant due to the absence of the boundary contribution. Nevertheless, the result corresponding to Lemma 4.2 of [40] holds also in the present case. Namely modulo a function holomorphic on the whole complex plane, $\Gamma(s) \zeta_{\mathcal{D}_{\sigma}}(s)$ splits into an interior contribution and a cylinder contribution. This again follows from Duhamel's Principle. First of all,

$$
\int_{1}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{\sigma}} d t
$$

is a holomorphic function on the whole complex plane. For $0<t<1$, we replace $\exp \left(-t \mathcal{D}_{\sigma}^{2}\right)$ by the operator $\exp \left(-t \tilde{\mathcal{D}}^{2}\right)$ inside of $M$ and by the
operator $\exp \left(-t\left(-\partial_{u}^{2}+B^{2}\right)_{\sigma}\right)$ on $N$. The interior contribution produces simple poles at the points $s_{k}=\frac{k-n}{2}$ with residues given by the formula

$$
\int_{M} a_{k}\left(\tilde{\mathcal{D}}^{2} ; x\right) d x
$$

where $\tilde{\mathcal{D}}$ denotes the double of the Dirac operator $\mathcal{D}$ on $\tilde{M}$, the closed double of $M$ (see formulas (7.4.2) and (7.4.3)). In particular the contribution to the residuum at $s=0$ is equal to

$$
\int_{M} a_{n}\left(\tilde{\mathcal{D}}^{2} ; x\right) d x=0
$$

This is due to the point-wise vanishing of $a_{n}\left(\tilde{\mathcal{D}}^{2} ; x\right)$, which follows from the fact that $n=\operatorname{dim} M$ is odd (see for instance [45]). The cylinder contribution has the form

$$
\begin{aligned}
& \int_{0}^{\infty} t^{s-1} \operatorname{Tr} \gamma(u) e^{-t\left(-\partial_{u}^{2}+B^{2}\right)_{\Pi_{\sigma}}} d t \\
&=\sum_{n \in \mathbf{Z} \backslash\{0\}} \int_{0}^{\infty} t^{s-1} d t \int_{0}^{\infty} \gamma(u) g_{n}(t ; u, u) d u
\end{aligned}
$$

where $\gamma(u)$ denotes the cut-off function. The integral

$$
\int_{0}^{\infty} t^{s-1} d t \int_{0}^{\infty} \gamma(u) g_{n}(t ; u, u) d u
$$

consists of two terms. The first term produces the contribution

$$
\int_{0}^{\infty} t^{s-1} d t\left\{\sum_{n \in \mathbf{Z} \backslash\{0\}} \int_{0}^{\infty} \gamma(u) \frac{e^{-\mu_{n}^{2} t}}{2 \sqrt{\pi t}} \cdot\left(1-\operatorname{sign}(n) e^{-\frac{u^{2}}{t}}\right) d u\right\}
$$

This is convergent for $\operatorname{Re}(s)>n / 2$ and in fact it is equal to

$$
\begin{gathered}
\int_{0}^{\infty} t^{s-1} d t\left\{\sum_{n \in \mathbf{Z} \backslash\{0\}} \int_{0}^{\infty} \gamma(u) \frac{e^{-\mu_{n}^{2} t}}{2 \sqrt{\pi t}} \cdot\left(1-\operatorname{sign}(n) e^{-\frac{u^{2}}{t}}\right) d u\right\}= \\
\frac{\int_{0}^{\infty} \gamma(u) d u}{2 \sqrt{\pi}} \cdot \int_{0}^{\infty} t^{s-3 / 2} \operatorname{Tr} e^{-t B^{2}} d t
\end{gathered}
$$

It follows now from (7.4.2) and (7.4.3) that the expression on the right side has a meromorphic extension to the whole complex plane with simple poles. Moreover, it is regular at $s=0$. The reason is that the residuum at $s=0$ is given by the formula

$$
\frac{\int_{0}^{\infty} \gamma(u) d u}{2 \sqrt{\pi}} a_{\operatorname{dim}(Y)+1}\left(B^{2}\right)
$$

and $a_{\operatorname{dim}(Y)+1}\left(B^{2}\right)$ is equal to 0 due to the fact that $\operatorname{dim}(Y)+1$ is an odd number (see for instance [45]). We are left with

$$
\int_{0}^{\infty} t^{s-1} d t \int_{0}^{\infty} \gamma(u)\left\{\sum_{n>0} \mu_{n} e^{2 u \mu_{n}} \operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\mu_{n} \sqrt{t}\right)\right\} d u
$$

We only have to show that this term produces at most a simple pole at $s=0$. We can neglect the presence of the cut-off function $\gamma(u)$ and then we obtain for large $\operatorname{Re}(s)$

$$
\begin{gathered}
\int_{0}^{\infty} t^{s-1} d t \int_{0}^{\infty}\left\{\sum_{n>0} \mu_{n} e^{2 u \mu_{n}} \operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\mu_{n} \sqrt{t}\right)\right\} d u= \\
\frac{1}{2} \int_{0}^{\infty} t^{s-1} d t \int_{0}^{\infty}\left\{\sum_{n>0} \frac{d}{d u}\left(e^{2 u \mu_{n}}\right) \operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\mu_{n} \sqrt{t}\right)\right\} d u= \\
\frac{1}{2} \int_{0}^{\infty} t^{s-1} d t\left\{\left.\sum_{n>0}\left\{e^{2 u \mu_{n}} \operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\mu_{n} \sqrt{t}\right)\right\}\right|_{0} ^{\infty}-\right. \\
\left.\int_{0}^{\infty} e^{2 u \mu_{n}}\left(\frac{d}{d u}\left\{\operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\mu_{n} \sqrt{t}\right)\right\}\right) d u\right\}= \\
\frac{1}{2} \int_{0}^{\infty} t^{s-1}\left\{\sum_{n>0} e^{-\mu_{n}^{2} t} \frac{2}{\sqrt{\pi}}\left(\int_{0}^{\infty} e^{-u^{2} / t} \frac{d u}{\sqrt{t}}\right)-\sum_{n>0} \operatorname{erfc}\left(\mu_{n} \sqrt{t}\right)\right\} d t= \\
\frac{1}{2} \int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t B^{2}} d t-\frac{1}{2} \int_{0}^{\infty} t^{s-1}\left\{\sum_{n>0} \operatorname{erfc}\left(\mu_{n} \sqrt{t}\right)\right\} d t .
\end{gathered}
$$

The first sum on the right side is equal to $\frac{1}{2} \Gamma(s) \zeta_{B^{2}}(s)$, hence it produces the correct asymptotic expansion with a simple pole at $s=0$. We need the next identity in order to deal with the second sum .

$$
\begin{gathered}
\int_{0}^{\infty} t^{s-1} \operatorname{erfc}\left(\mu_{n} \sqrt{t}\right) d t=\frac{1}{s} \int_{0}^{\infty} \frac{d}{d t}\left(t^{s}\right) \operatorname{erfc}\left(\mu_{n} \sqrt{t}\right) d t= \\
\left.\left\{\frac{1}{s} t^{s} \operatorname{erfc}\left(\mu_{n} \sqrt{t}\right)\right\}\right|_{0} ^{\infty}-\frac{1}{s} \int_{0}^{\infty} t^{s} e^{-\mu_{n}^{2} t} \frac{\mu_{n}}{2 \sqrt{t}} d t= \\
-\frac{1}{2 s} \mu_{n} \int_{0}^{\infty} t^{s-1 / 2} e^{-\mu_{n}^{2} t} d t=-\frac{\Gamma(s+1 / 2)}{2 s}\left(\mu_{n}^{2}\right)^{-s}
\end{gathered}
$$

Therefore we obtain

$$
\begin{aligned}
&-\frac{1}{2} \int_{0}^{\infty} t^{s-1}\left\{\sum_{n>0} \operatorname{erfc}\left(\mu_{n} \sqrt{t}\right)\right\} d t \\
&=\frac{\Gamma(s+1 / 2)}{4 s} \frac{1}{2} \zeta_{B^{2}}(s)=\frac{\Gamma(s+1 / 2)}{8 s} \zeta_{B^{2}}(s)
\end{aligned}
$$

and the expression on the right side has a meromorphic extension to the whole complex plane, with a simple pole at $s=0$ with residuum

$$
\operatorname{Res}_{s=0} \frac{\Gamma(s+1 / 2)}{8 s} \zeta_{B^{2}}(s)=\frac{\sqrt{\pi}}{8} a_{\operatorname{dim}(Y)}\left(B^{2}\right) .
$$

We have shown that the cylinder contribution to the trace integral $\int_{0}^{\infty} t^{s-1} \mathrm{Tr} e^{-t \mathcal{D}_{\Pi_{\sigma}}^{2}} d t$ has a meromorphic extension to the whole complex plane with an isolated simple pole at $s=0$, which ends the proof of Proposition 7.4.1.

Part 3
Pasting of $\eta$-Invariants

## CHAPTER 8

## The Adiabatic Duhamel Principle

1
Let $\mathcal{D}=\mathcal{D}_{1} \cup \mathcal{D}_{2}$ be a compatible Dirac operator on a closed manifold $M$ which is partitioned $M=M_{1} \cup_{Y} M_{2}$ into two compact manifolds with common boundary $Y$. Consider its cylindrical prolongation $\mathcal{D}^{R}=\mathcal{D}_{1}^{R} \cup \mathcal{D}_{2}^{R}$ on the stretched manifold

$$
M^{R}=M_{1} \cup([-R, R] \times Y) \cup M_{2}=M_{1}^{R} \cup M_{2}^{R}
$$

Imposing the Atiyah-Patodi-Singer boundary condition $\Pi_{>}$on the right part $M_{2}$, we prove a different version of the Duhamel Principle and derive an adiabatic formula for the $\eta$-invariant

$$
\lim _{R \rightarrow \infty}\left\{\eta_{\mathcal{D}_{2, \Pi_{>}}^{R}}(0)-\int_{M_{2}^{R}} \eta_{\mathcal{D}^{R}}(0 ; x) d x\right\}=0
$$

Since neither the $\eta$-invariant nor the $\eta$-density depend, modulo $\mathbf{Z}$, on the length $R$, we obtain the splitting formula

$$
\eta_{\mathcal{D}}(0)=\eta_{\mathcal{D}_{1, \Pi_{<}}}(0)+\eta_{\mathcal{D}_{2, \Pi}}(0)
$$

### 8.1. Introduction

In this and the following chapter, the third part of our book, we derive pasting formulas for the $\eta$-invariant. Below, in the fourth part, we then derive pasting formulas for the determinant line bundle, the $\zeta$-determinant , and for the curvature of the $\zeta$-determinant : how do these invariants split when the manifold is split? One reason for this question is, of course, that we wish to understand the invariants of a complicated manifold in terms of more simple building blocks.

Various pasting formulas for topological invariants are well-established. The master formula is the Novikov additivity for the signatures of partitioned $4 k$-dimensional manifolds. It can be generalized to additivity theorems for the index and to Bojarski type formulas. That are formulas which give the index on the closed partitioned manifold in terms of the Cauchy data spaces along the separating hypersurface in even dimension and, correspondingly, the spectral flow of a continuous family of Dirac operators in terms of the Maslov index of the Cauchy data spaces in odd dimension.

[^4]From a topological viewpoint these splitting formulas are not very surprising although the precise determination of the correction terms can demand considerable effort. The Novikov additivity, for instance, where all correction terms are cancelled, is an easy consequence of Poincaré duality and the Mayer-Vietoris sequence of singular homology.

However, analytical explanations for the various additivity formulas for the index can also be obtained. They help to understand the nature of the correction terms. Here the master problem is the Riemann-Hilbert problem of complex analysis, expressing the index of a coupling problem along a curve solely by the coupling data.

Unfortunately, we need a little more topology, differential geometry, functional analysis, and elliptic partial differential equation analysis when we base our pasting laws on the decomposition of a given closed manifold into two manifolds with boundary (that is the analytical approach) instead of basing the calculations on triangulations of the manifold (as it is the singular homology's approach). The benefit of the analytical approach is that we can describe and follow the cutting and pasting process explicitly on the level of the operators and their eigenvalues and not only on the level of the homology groups and their invariants.

By now, it seems that splitting formulas for the index are rather well understood. The reason is that the index is given by a local formula and the splitting is rather obvious as soon as the process of cutting and pasting of operators is understood. To understand the underlying idea it suffices to read the concluding Chapter 26 of our monograph [27]. One major topic of that book has been the cutting and pasting of Dirac operators over partitioned manifolds and the various corresponding index additivity theorems.

For a different approach we refer to the work by N. Teleman (see, in particular, [103], [104]) and others on a true discretisation of the Dirac operator over a triangulated manifold. These authors carry out the analysis in intimate analogy to the topology. The discrete methods have been proved successful in some instances (see e.g. Müller [72] and Lück [63]), but a discrete description of spectral and other global elliptic boundary conditions and the corresponding calculation of topological and spectral invariants is missing (1997).

Splitting formulas for spectral quantities like the $\eta$-invariant and the determinant which are not topological invariants are a veritable challenge. Surprisingly, here the process of understanding was running in reverse order. First the analysis was understood, i.e. the additivity of the $\eta$-invariant, later the underlying procedure of algebraic topology, i.e. the pasting formula for the canonical determinant. All that will be explained in the following.

In this chapter we shall use the following notations and make the following assumptions.

Figure 8.1.1. a) Partitioned manifold $M=M_{1} \cup M_{2}$ with hypersurface $Y$, neck $\mathcal{N}$, and normal coordinate $u$.
b) Stretched partitioned manifold $M^{R}=M_{1}^{R} \cup M_{2}^{R}$

Assumptions 8.1.1. (a) Let $M$ be an odd-dimensional closed partitioned Riemannian manifold $M=M_{1} \cup_{Y} M_{2}$ with $M_{1}, M_{2}$ compact manifolds with common boundary $Y$. Let $\mathcal{S}$ be a bundle of Clifford modules over $M$.
(b) To begin with we assume that $\mathcal{D}$ is a compatible ( $=$ true) Dirac operator over $M$. Thus, in particular, $\mathcal{D}$ is symmetric and has a unique self-adjoint extension in $L_{2}(M ; \mathcal{S})$.
(c) We assume that there exists a bi-collar cylindrical neighbourhood ( $a$ neck) $\mathcal{N} \simeq(-1,1) \times Y$ of the separating hypersurface $Y$ such that the Riemannian structure on $M$ and the Hermitian structure on $\mathcal{S}$ are product in $\mathcal{N}$, i.e. they do not depend on the normal coordinate $u$, when restricted to $Y_{u}=\{u\} \times Y$. Our convention for the orientation of the coordinate $u$ is that it runs from $M_{1}$ to $M_{2}$, i.e. $M_{1} \cap \mathcal{N}=(-1,0] \times Y$ and $\mathcal{N} \cap M_{2}=[0,1) \times Y$. Then the operator $\mathcal{D}$ takes the following form on $\mathcal{N}$ :

$$
\begin{equation*}
\left.\mathcal{D}\right|_{\mathcal{N}}=G\left(\partial_{u}+B\right), \tag{8.1.1}
\end{equation*}
$$

where the principal symbol in $u$-direction $G:\left.\left.\mathcal{S}\right|_{Y} \rightarrow \mathcal{S}\right|_{Y}$ is a unitary bundle isomorphism (Clifford multiplication by the normal vector $d u$ ) and the tangential operator $B: C^{\infty}\left(Y ;\left.\mathcal{S}\right|_{Y}\right) \rightarrow C^{\infty}\left(Y ;\left.\mathcal{S}\right|_{Y}\right)$ is the corresponding Dirac operator on $Y$. Note that $G$ and $B$ do not depend on the normal coordinate $u$ in $\mathcal{N}$ and they satisfy the following identities

$$
\begin{equation*}
G^{2}=-\operatorname{Id}, G^{*}=-G, G \cdot B=-B \cdot G, B^{*}=B \tag{8.1.2}
\end{equation*}
$$

Hence, $G$ is a skew-adjoint involution and $\mathcal{S}$, the bundle of spinors, decomposes in $\mathcal{N}$ into $\pm i$-eigenspaces of $G,\left.\mathcal{S}\right|_{\mathcal{N}}=\mathcal{S}^{+} \oplus \mathcal{S}^{-}$. It follows that (8.1.1) leads to the following representation of the operator $\mathcal{D}$ in $\mathcal{N}$

$$
\left.\mathcal{D}\right|_{\mathcal{N}}=\left(\begin{array}{cc}
i & 0  \tag{8.1.3}\\
0 & -i
\end{array}\right) \cdot\left(\partial_{u}+\left(\begin{array}{cc}
0 & B_{-}=B_{+}^{*} \\
B_{+} & 0
\end{array}\right)\right)
$$

where $B_{+}: C^{\infty}\left(Y ; \mathcal{S}^{+}\right) \rightarrow C^{\infty}\left(Y ; \mathcal{S}^{-}\right)$maps the spinors of positive chirality into the spinors of negative chirality.
(d) To begin with we consider only the case of $\operatorname{ker} B=\{0\}$. That implies that $B$ is an invertible operator. More precisely, there exists a pseudodifferential elliptic operator $L$ of order -1 such that $B L=\operatorname{Id}_{\mathcal{S}}=L B$ (see, for instance, [27], Proposition 9.5).
(e) For real $R>0$ we study the closed stretched manifold $M^{R}$ which we obtain from $M$ by inserting a cylinder of length $2 R$, i.e. replacing the collar $\mathcal{N}$ by the cylinder $(-2 R-1,+1) \times Y$

$$
M^{R}=M_{1} \cup([-2 R, 0] \times Y) \cup M_{2}
$$

We extend the bundle $\mathcal{S}$ to the stretched manifold $M^{R}$ in a natural way. The extended bundle will be also denoted by $\mathcal{S}$. The Riemannian structure on $M$ and the Hermitian structure on $\mathcal{S}$ are product in $\mathcal{N}$, hence we can extend them to smooth metrics on $M^{R}$ in a natural way and, at the end, we can extend the operator $\mathcal{D}$ to an operator $\mathcal{D}^{R}$ on $M^{R}$ by using formula (8.1.1). Then $M^{R}$ splits into two manifolds with boundary: $M^{R}=M_{1}^{R} \cup M_{2}^{R}$ with $M_{1}^{R}=M_{1} \cup((-2 R, R] \times Y), M_{2}^{R}=([-R, 0) \times Y) \cup M_{2}$, and $\partial M_{1}=\partial M_{2}^{R}=$ $\{-R\} \times Y$. Consequently, the operator $\mathcal{D}^{R}$ splits into $\mathcal{D}^{R}=\mathcal{D}_{1}^{R} \cup \mathcal{D}_{2}^{R}$. We shall impose spectral boundary conditions to obtain self-adjoint operators $\mathcal{D}_{1, \Pi_{<}}, \mathcal{D}_{1, \Pi_{<}}^{R}, \mathcal{D}_{2, \Pi_{>}}$, and $\mathcal{D}_{2, \Pi_{>}}^{R}$ in the corresponding $L_{2}$ spaces on the parts (see (8.1.4)).
(f) We also introduce the complete, non-compact Riemannian manifold with cylindrical end

$$
M_{2}^{\infty}:=((-\infty, 0] \times Y) \cup M_{2}
$$

by gluing the half-cylinder $(-\infty, 0] \times Y$ to the boundary $Y$ of $M_{2}$. Clearly, the Dirac operator $\mathcal{D}$ extends also to $C^{\infty}\left(M_{2}^{\infty}, \mathcal{S}\right)$.

To fix our notation we recall from part II of this book. Let $\Pi_{>}$(respectively $\Pi_{<}$) denote the spectral projection of $B$ onto the subspace of $L_{2}\left(Y ;\left.\mathcal{S}\right|_{Y}\right)$ spanned by the eigensections corresponding to the positive (respectively negative) eigenvalues. Then $\Pi_{>}$is a self-adjoint elliptic boundary condition for the operator $\mathcal{D}_{2}=\left.\mathcal{D}\right|_{M_{2}}$ (see [27], Proposition 20.3). This means that the operator $\mathcal{D}_{2, \Pi_{>}}$defined by

$$
\begin{cases}\mathcal{D}_{2, \Pi_{>}} & =\left.\mathcal{D}\right|_{M_{2}}  \tag{8.1.4}\\ \operatorname{dom}\left(\mathcal{D}_{2, \Pi_{>}}\right) & =\left\{s \in H^{1}\left(M_{2} ;\left.\mathcal{S}\right|_{M_{2}}\right) \mid \Pi_{>}\left(\left.s\right|_{Y}\right)=0\right\}\end{cases}
$$

| manifolds | operators | integral kernels |
| :---: | :---: | :---: |
| $M=M_{1} \cup_{Y} M_{2}$ | $\mathcal{D}, e^{-t \mathcal{D}^{2}}, \mathcal{D} e^{-t \mathcal{D}^{2}}$ | $\mathrm{e}\left(t ; x, x^{\prime}\right), \mathcal{E}\left(t ; x, x^{\prime}\right)$ |
| $M^{R}=M_{1}^{R} \cup_{Y} M_{2}^{R}$ | $\mathcal{D}^{R}, e^{-t\left(\mathcal{D}^{R}\right)^{2}}, \mathcal{D}^{R} e^{-t\left(\mathcal{D}^{R}\right)^{2}}$ | $\mathrm{e}^{R}\left(t ; x, x^{\prime}\right), \mathcal{E}^{R}\left(t ; x, x^{\prime}\right)$ |
| $M_{2}$ | $\mathcal{D}_{2}, e^{-t \mathcal{D}_{2, \Pi}^{2}{ }^{2}{ }^{2}}, \mathcal{D}_{2} e^{-t \mathcal{D}_{2}{ }^{2} \Pi_{>}{ }^{2}}$ | $\mathrm{e}_{2}\left(t ; x, x^{\prime}\right), \mathcal{E}_{2}\left(t ; x, x^{\prime}\right)$ |
| $M_{2}^{R}=([-R, 0] \times Y) \cup M_{2}$ | $\mathcal{D}_{2}^{R}, e^{-t\left(\mathcal{D}_{2, \Pi}^{R}\right)^{2}}, \mathcal{D}_{2}^{R} e^{\left.-t\left(\mathcal{D}_{2, \Pi}^{R}\right)^{\prime}\right)^{2}}$ | $\mathrm{e}_{2}^{R}\left(t ; x, x^{\prime}\right), \mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$ |
|  | $Q_{2}^{R}(t), C^{R}(t)=\left(\left(\mathcal{D}_{2, \Pi_{>}}^{R}\right)^{2}+\frac{d}{d t}\right) Q_{2}^{R}(t)$ | $Q_{2}^{R}\left(t ; x, x^{\prime}\right), C^{R}\left(t ; x, x^{\prime}\right)$ |
| $M_{2}^{\infty}=((-\infty, 0] \times Y) \cup M_{2}$ | $\mathcal{D}_{2}^{\infty}, e^{-t\left(\mathcal{D}_{2}^{\infty}\right)^{2}}, \mathcal{D}_{2}^{\infty} e^{-t\left(\mathcal{D}_{2}^{\infty}\right)^{2}}$ | $\mathrm{e}_{2}^{\infty}\left(t ; x, x^{\prime}\right), \mathcal{E}_{2}^{\infty}\left(t ; x, x^{\prime}\right)$ |
| $Y_{\text {cyl }}^{\infty}=(-\infty,+\infty) \times Y$ | $D_{\text {cyl }}, e^{-t D_{\text {cyl }}^{2}}, D_{\text {cyl }} e^{-t D_{\text {cyl }}^{2}}$ | $\mathrm{e}_{\text {cyl }}\left(t ; x, x^{\prime}\right), \mathcal{E}_{\text {cyl }}\left(t ; x, x^{\prime}\right)$ |
| $Y_{\mathrm{cyl} / 2}^{\infty}=[0,+\infty) \times Y$ | $D_{\text {aps }}, e^{-t D_{\text {aps }}^{2}}, D_{\text {aps }} e^{-t D_{\text {aps }}^{2}}$ | $\mathrm{e}_{\text {aps }}\left(t ; x, x^{\prime}\right), \mathcal{E}_{\text {aps }}\left(t ; x, x^{\prime}\right)$ |

Table 1. Table of manifolds, operators, and integral kernels
is an unbounded self-adjoint operator in $L_{2}\left(M_{2} ;\left.\mathcal{S}\right|_{M_{2}}\right)$ with compact resolvent. In particular,

$$
\mathcal{D}_{2, \Pi_{>}}: \operatorname{dom}\left(\mathcal{D}_{2, \Pi_{>}}\right) \longrightarrow L_{2}\left(M_{2} ;\left.\mathcal{S}\right|_{M_{2}}\right)
$$

is a Fredholm operator with discrete real spectrum and the kernel of $\mathcal{D}_{2, \Pi\rangle}$ consists of smooth sections of $\left.\mathcal{S}\right|_{M_{2}}$. As shown before (see Chapter 6), the $\eta$-function of $\mathcal{D}_{2, \Pi>}$ is well defined and enjoys all the properties of the $\eta$-function of the Dirac operator defined on a closed manifold. In particular, $\eta_{\mathcal{D}_{2, \Pi\rangle}}(0)$, the $\eta$-invariant of $\mathcal{D}_{2, \Pi>}$, is well defined. Similarly, $\Pi_{<}$is a self-adjoint boundary condition for the operator $\left.\mathcal{D}\right|_{M_{1}}$, and we define the operator $\mathcal{D}_{1, \Pi_{<}}$using a formula corresponding to (8.1.4). To keep track of the various manifolds, operators, and integral kernels we refer to Table 1 where we have collected the major notations.

The main results of this chapter are the following theorem on the adiabatic limits of the $\eta$-invariants and its additivity corollary:

Theorem 8.1.2. Attaching a cylinder of length $R>0$ at the boundary of the manifold $M_{2}$, we can approximate the $\eta$-invariant of the spectral boundary condition on the prolonged manifold $M_{2}^{R}$ by the corresponding integral of the 'local' $\eta$-function of the closed stretched manifold $M^{R}$ :

$$
\lim _{R \rightarrow \infty}\left\{\eta_{\mathcal{D}_{2, \Pi}^{R},}(0)-\int_{M_{2}^{R}} \eta_{\mathcal{D}^{R}}(0 ; x) d x\right\} \equiv 0 \quad \bmod \mathbf{Z}
$$

and

Corollary 8.1.3. $\eta_{\mathcal{D}}(0) \equiv \eta_{\mathcal{D}_{1, \Pi_{<}}}(0)+\eta_{\mathcal{D}_{2, \Pi},}(0) \bmod \mathbf{Z}$.

Remark 8.1.4. With hindsight, it is not surprising that modulo the integers the preceding additivity formula for the $\eta$-invariant on a partitioned manifold is precise. An intuitive argument runs along the following lines: 'almost' all eigensections and eigenvalues of the operator $\mathcal{D}$ on the closed partitioned manifold $M=M_{1} \cup M_{2}$ can be traced back either to eigensections $\psi_{1, k}$ and eigenvalues $\mu_{1, k}$ of the spectral boundary problem $\mathcal{D}_{1, \Pi_{<}}$ on the part $M_{1}$ or to eigensections $\psi_{2, \ell}$ and eigenvalues $\mu_{2, \ell}$ of the spectral boundary problem $\mathcal{D}_{2, \Pi>}$ on the part $M_{2}$. Unfortunately, we have no explicit method for doing an exact correspondence. But we have an approximate method: due to the product form of the Dirac operator in a neighbourhood of the separating hypersurface, eigensections on one part $M_{1}$ or $M_{2}$ of the manifold $M$ can be extended to smooth sections on the whole of $M$. Not as true eigensections of $\mathcal{D}$, but with a relative error which is rapidly decreasing with $R \rightarrow \infty$ when we attach cylinders of length $R$ to the part manifolds or, equivalently, insert a cylinder of length $2 R$ in $M$. So much about the great majority of eigensections and eigenvalues.

There is also a residual set $\left\{\mu_{0, j}\right\}$ of eigenvalues of $\mathcal{D}$ which can neither be traced back to eigenvalues of $\mathcal{D}_{1, \Pi_{<}}$nor of $\mathcal{D}_{2, \Pi_{>}}$. These eigenvalues can, however, be traced back to the kernel of the Dirac operators $\mathcal{D}_{1}^{\infty}$ and $\mathcal{D}_{2}^{\infty}$ on the part manifolds with cylindrical ends $M_{1}^{\infty}$ and $M_{2}^{\infty}$. Because of Fredholm properties the residual set is finite and, hence (by Appendix 2) can be discarded for calculating the $\eta$-invariant.

Therefore no $R$, no prolongation of the bicollar neighbourhood $\mathcal{N}$ enters the formula. Nevertheless, our arguments rely on an adiabatic argument to strain the spectrum of $\mathcal{D}$ into its three parts

$$
\begin{equation*}
\operatorname{spec} \mathcal{D} \sim\left\{\mu_{0, j}\right\} \cup\left\{\mu_{1, k}\right\} \cup\left\{\mu_{2, \ell}\right\} . \tag{8.1.5}
\end{equation*}
$$

For the most part of this chapter, however, we shall not make all arguments explicit on the level of the single eigenvalue. It suffices to keep to the level of the $\eta$-invariant. Roughly speaking, the reason is the following. Contrary to the index, the $\eta$-invariant can not be described by a local formula. Nevertheless, it can be described by an integral over the manifold. The integrand, however, is not defined in local terms solely. In particular, when writing the $\eta$-function in integral form and decomposing the $\eta$-integral

$$
\eta_{\mathcal{D}}(s)=\int_{M} \eta_{\mathcal{D}}(s ; x, x) d x=\int_{M_{1}} \eta_{\mathcal{D}}\left(s ; x_{1}, x_{1}\right) d x_{1}+\int_{M_{2}} \eta_{\mathcal{D}}\left(s ; x_{2}, x_{2}\right) d x_{2}
$$

there is no geometrical interpretation of the integrals on the right over the two parts of the manifold. This is very unfortunate. But for sufficiently large $R$, the integrals become intelligible and can be read as the $\eta$-invariants of $\mathcal{D}_{1, \Pi_{<}}^{R}$ and $\mathcal{D}_{2, \Pi_{>}}^{R}$. That is the meaning of the adiabatic limit.

### 8.2. Heat Kernels on the Manifold $M_{2}^{R}$

Let $\mathcal{E}_{2}^{R}(t)$ denote the integral kernel of the operator $\mathcal{D}_{2}^{R} e^{\left.-t\left(\mathcal{D}_{2, \Pi}^{R}\right\rangle\right)^{2}}$ defined on the manifold $M_{2}^{R}=([-R, 0] \times Y) \cup M_{2}$. According to Theorem ?? the $\eta$-invariant of the self-adjoint operator $\mathcal{D}_{2, \Pi>}^{R}$ is well defined and we have

$$
\begin{align*}
\eta_{\mathcal{D}_{2, \Pi}^{R}}(0)= & \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x \\
= & \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x  \tag{8.2.1}\\
& +\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x \tag{8.2.2}
\end{align*}
$$

We first deal with the integral of (8.2.1) and show that it splits into an interior contribution and a cylinder contribution as $R \rightarrow \infty$. Then we will show that the integral of (8.2.2) disappears as $R \rightarrow \infty$.

The most simple construction of a parametrix for $\mathcal{E}_{2}^{R}(t)$, i.e. of an approximate heat kernel is the one which we introduced in Theorem ?? (Duhamel's Splitting Formula of Chapter 6 and which we applied later to establish the $\eta$-invariant: we glue the kernel $\mathcal{E}$ of the operator $\mathcal{D} e^{-t \mathcal{D}^{2}}$, given on the whole, closed manifold $M$ and the kernel $\mathcal{E}_{\text {aps }}^{\infty}$ of the $L_{2}$-extension of the operator $G\left(\partial_{u}+B\right) e^{-t\left(G\left(\partial_{u}+B\right)\right)^{2}}$, given on the half-infinite cylinder $[-R, \infty) \times Y$ and subject to the Atiyah-Patodi-Singer boundary condition at the end $u=-R$. In that construction the gluing happens on the neck $\mathcal{N}=[0,1) \times Y$ with suitable cut-off functions (see Figure 8.2.1a).

Locally, the heat kernel is always of the form $(4 \pi t)^{-m / 2} e^{c_{1} t} e^{-\left|x-x^{\prime}\right|^{2} / 4 t}$ (see Chapter 5.??). By Duhamel's Principle we get after gluing a similar global result for the kernel $\mathrm{e}_{2}^{R}\left(t ; x, x^{\prime}\right)$ of the operator $e^{-t\left(\mathcal{D}_{2, \Pi\rangle}^{R}\right)^{2}}$ and, putting a factor $t^{-1 / 2}$ in front, for the kernel of the combined operator $\mathcal{D} e^{-t\left(\mathcal{D}_{2, \Pi>}^{R}\right)^{2}}$ (see, e.g. Gilkey [45], Lemma 1.9.1). That proves two crucial estimates:

Lemma 8.2.1. There exist positive reals $c_{1}, c_{2}$, and $c_{3}$ which do not depend on $R$ such that for all $x, x^{\prime} \in M_{2}^{R}$ and any $t>0$ and $R>0$

$$
\begin{align*}
& \left|\mathrm{e}_{2}^{R}\left(t ; x, x^{\prime}\right)\right| \leq c_{1} \cdot t^{-\frac{m}{2}} \cdot e^{c_{2} t} \cdot e^{-c_{3} \frac{d^{2}\left(x, x^{\prime}\right)}{t}}  \tag{8.2.3}\\
& \left|\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)\right| \leq c_{1} t^{-\frac{1+m}{2}} \cdot e^{c_{2} t} \cdot e^{-c_{3} \frac{d^{2}\left(x, x^{\prime}\right)}{t}} \tag{8.2.4}
\end{align*}
$$

Here $d\left(x, x^{\prime}\right)$ denotes the geodesic distance.

Note . Notice that exactly the same type of estimate is also valid for the kernel $\mathcal{E}^{R}\left(t ; x, x^{\prime}\right)$ on the stretched closed manifold $M^{R}$ and for the kernel $\mathcal{E}_{\text {aps }}^{\infty}\left(t ; x, x^{\prime}\right)$ on the infinite cylinder. For details see also [27], Theorem

Figure 8.2.1. Two constructions of a parametrix for $\mathcal{E}_{2}^{R}$ over $\left.M_{2}^{R} . a\right)$ Gluing $\mathcal{E}_{\text {aps }}^{\infty}$ and $\mathcal{E}$ over $\mathcal{N}$. b) Gluing $\mathcal{E}_{\text {aps }}^{\infty}$ and $\mathcal{E}^{R}$ over $\mathcal{N}^{R}$

Figure 8.2.2. The choice of the cut-off functions
22.14. There, however, the term $e^{c_{2} t}$ was suppressed in the final formula because the emphasis was on small time asymptotic.

As mentioned before, for $R \rightarrow \infty$ we want to separate the contribution to the kernel $\mathcal{E}_{2}^{R}$ which comes from the cylinder and the contribution from the interior by a gluing process. Unfortunately, the inequality 8.2.4 does not suffice to show that the contribution to the $\eta$-invariant, more precisely to the integral (8.2.1), which comes from the 'error' term vanishes with $R \rightarrow \infty$. Therefore, we introduce a different parametrix for the kernel $\mathcal{E}_{2}^{R}$.

Instead of gluing over the fixed neck $\mathcal{N}=[0,1) 2 \times Y$ we glue over a segment $\mathcal{N}^{R}$ of growing length of the attached cylinder, say $\mathcal{N}^{R}:=$ $\left(-\frac{4}{7} R,-\frac{3}{7} R\right) \times Y$ (see Figure 8.2.1b). Thus, we choose a smooth partition of unity $\left\{\chi_{\text {aps }}, \chi_{\text {int }}\right\}$ on $M_{2}^{R}$ suitable for the covering $\left\{U_{\text {aps }}, U_{\text {int }}\right\}$ with $U_{\text {aps }}:=\left[-R,-\frac{3}{7} R\right) \times Y$ and $U_{\text {int }}:=\left(\left(-\frac{4}{7} R, 0\right] \times Y\right) \cup M_{2}$, hence $U_{\text {aps }} \cap U_{\text {int }}=\mathcal{N}^{R}$. Moreover, we choose non-negative smooth cut-off functions $\left\{\psi_{\text {aps }}, \psi_{\text {int }}\right\}$ such that

$$
\begin{aligned}
\psi_{j} \equiv 1 & \text { on }\left\{x \in M_{2}^{R} \left\lvert\, \operatorname{dist}\left(x, \operatorname{supp} \chi_{j}\right)<\frac{1}{7} R\right.\right\} \\
\text { and } \psi_{j} \equiv 0 & \text { on }\left\{x \in M_{2}^{R} \left\lvert\, \operatorname{dist}\left(x, \operatorname{supp} \chi_{j}\right) \geq \frac{2}{7} R\right.\right\}
\end{aligned}
$$

for $j \in\{$ aps, int $\}$ (see Figure 8.2.2). We notice

$$
\begin{equation*}
\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime}, \operatorname{supp} \chi_{j}\right)=\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime \prime}, \operatorname{supp} \chi_{j}\right) \geq \frac{1}{7} R \tag{8.2.5}
\end{equation*}
$$

Moreover, we may assume that

$$
\begin{equation*}
\left|\frac{\partial^{k} \psi_{j}}{\partial u^{k}}\right| \leq c_{0} / R \tag{8.2.6}
\end{equation*}
$$

for all $k$, where $c_{0}$ is a certain positive constant.
For any parameter $t>0$ we define an operator $Q_{2}^{R}(t)$ on $C^{\infty}\left(M_{2}^{R} ; \mathcal{S}\right)$ with a smooth kernel, given by

$$
\begin{equation*}
Q_{2}^{R^{\prime}}\left(t ; x, x^{\prime}\right):=\psi_{\mathrm{aps}}(x) \mathcal{E}_{\mathrm{aps}}^{\infty}\left(t ; x, x^{\prime}\right) \chi_{\mathrm{aps}}\left(x^{\prime}\right)+\psi_{\mathrm{int}}(x) \mathcal{E}^{R}\left(t ; x, x^{\prime}\right) \chi_{\mathrm{int}}\left(x^{\prime}\right) . \tag{8.2.7}
\end{equation*}
$$

Recall that $\mathcal{E}^{R}$ denotes the kernel of the operator $\mathcal{D}^{R} e^{-t\left(\mathcal{D}^{R}\right)^{2}}$, given on the stretched closed manifold $M^{R}$. Notice that, by construction, $Q_{2}^{R}(t)$ maps $L_{2}\left(M_{2}^{R} ; \mathcal{S}\right)$ into the domain of the operator $\mathcal{D}_{2, \Pi_{>}}^{R}$.

Then, for $x^{\prime} \in U_{\text {aps }}$ with $\chi_{\text {aps }}\left(x^{\prime}\right)=1$, we have by definition:

$$
Q_{2}^{R}\left(t ; x, x^{\prime}\right)= \begin{cases}\mathcal{E}_{\text {aps }}^{\infty}\left(t ; x, x^{\prime}\right) & \text { if } d\left(x, \operatorname{supp} \chi_{\text {aps }}\right)<\frac{1}{7} R, \text { and }  \tag{8.2.8}\\ 0 & \text { if } d\left(x, \operatorname{supp} \chi_{\text {aps }}\right) \geq \frac{2}{7} R .\end{cases}
$$

Correspondingly, we have for $x^{\prime} \in U_{\text {int }}$ with $\chi_{\text {int }}\left(x^{\prime}\right)=1$

$$
Q_{2}^{R}\left(t ; x, x^{\prime}\right)= \begin{cases}\mathcal{E}^{R}\left(t ; x, x^{\prime}\right) & \text { if } d\left(x, \operatorname{supp} \chi_{\mathrm{inn}}\right)<\frac{1}{7} R, \text { and }  \tag{8.2.9}\\ 0 & \text { if } d\left(x, \operatorname{supp} \chi_{\mathrm{int}}\right) \geq \frac{2}{7} R\end{cases}
$$

For fixed $t>0$, we determine the difference between the precise kernel $\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$ and the approximate one $Q_{2}^{R}\left(t ; x, x^{\prime}\right)$. Let $C^{R}(t)$ denote the operator $\left(\left(\mathcal{D}_{2, \Pi}^{R}\right)^{2}+\frac{d}{d t}\right) \circ Q_{2}^{R}(t)$ and $C^{R}\left(t ; x, x^{\prime}\right)$ its kernel. By definition, we
have $\left(\left(\mathcal{D}_{2, \Pi\rangle}^{R}\right)^{2}+\frac{d}{d t}\right) \circ \mathcal{E}_{2}^{R}(t)=0$. Thus, $C^{R}(t)$ 'measures' the error we make when replacing the precise kernel $\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$ by the glued, approximate one.

More precisely, we have by Duhamel's Formula (Proposition 6.3.1)

$$
\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)-Q_{2}^{R}\left(t ; x, x^{\prime}\right)=-\int_{0}^{t} d s \int_{M_{2}^{R}} d z \mathcal{E}_{2}^{R}(s ; x, z) C^{R}\left(t-s ; z, x^{\prime}\right)
$$

with

$$
\begin{aligned}
& C^{R}\left(t-s ; z, x^{\prime}\right)=\left(\left(\mathcal{D}_{2,(z)}^{R}\right)^{2}+\frac{d}{d(t-s)}\right) Q_{2}^{R}\left(t-s ; z, x^{\prime}\right) \\
&=\left(\left(\mathcal{D}_{2(z)}^{R}\right)^{2}-\frac{d}{d s}\right) Q_{2}^{R}\left(t-s ; z, x^{\prime}\right) \\
&= \psi_{\mathrm{aps}}^{\prime \prime}(z) \mathcal{E}_{\mathrm{aps}}^{R}\left(t-s ; z, x^{\prime}\right) \chi_{\mathrm{aps}}\left(x^{\prime}\right)+2 \psi_{\mathrm{aps}}^{\prime}(z) \frac{\partial}{\partial u}\left(\mathcal{E}_{\mathrm{aps}}^{R}\left(t-s ; z, x^{\prime}\right)\right) \chi_{\mathrm{aps}}\left(x^{\prime}\right) \\
&+\psi_{\mathrm{aps}}(z) \underbrace{\left(\mathcal{D}_{(z)}^{2}-\frac{d}{d s}\right) \mathcal{E}_{\mathrm{aps}}^{R}\left(t-s ; z, x^{\prime}\right)}_{=0} \chi_{\mathrm{aps}}\left(x^{\prime}\right) \\
&+\psi_{\mathrm{int}}^{\prime \prime}(z) \mathcal{E}^{R}\left(t-s ; z, x^{\prime}\right) \chi_{\mathrm{int}}\left(x^{\prime}\right)+2 \psi_{\mathrm{int}}^{\prime}(z) \frac{\partial}{\partial u}\left(\mathcal{E}^{R}\left(t-s ; z, x^{\prime}\right)\right) \chi_{\mathrm{int}}\left(x^{\prime}\right) \\
&+\psi_{\mathrm{int}}(z) \underbrace{\left(\left(\mathcal{D}_{(z)}^{R}\right)^{2}-\frac{d}{d s}\right) \mathcal{E}^{R}\left(t-s ; z, x^{\prime}\right)}_{=0} \chi_{\mathrm{int}}\left(x^{\prime}\right)
\end{aligned}
$$

Here, $\mathcal{D}_{(z)}$ denotes the operator $\mathcal{D}$ acting on the $z$ variable; and in the partial derivative $\frac{\partial}{\partial u}$ the letter $u$ denotes the normal coordinate of the variable $z$.

As stated in (8.2.5), the supports of $\chi_{j}$ and $\psi_{j}^{\prime}$ (and, equally, $\psi_{j}^{\prime \prime}$ ) are disjoint and separated from each other by a distance $R / 7$ in the normal variable for $j \in\{$ aps, int $\}$. Then the error term $C^{R}\left(t-s ; z, x^{\prime}\right)$ vanishes both for the distance in the normal variable $d\left(z, x^{\prime}\right)<R / 7$ and, actually, whenever $z$ or $x^{\prime}$ are outside the segment $\left[-\frac{6}{7} R, \frac{1}{7} R\right] \times Y$.

Let $z$ and $x^{\prime}$ be on the cylinder and $|u-v|>R / 7$ where $u$ and $v$ denote their normal coordinates. We investigate the error term $C^{R}\left(t-s ; z, x^{\prime}\right)$ which consists of six summands. Two of them vanish as we have pointed out above. The remaining four summands involve the kernels $\mathcal{E}_{\text {aps }}^{\infty}\left(t-s ; z, x^{\prime}\right)$ on the infinite cylinder $[-R, \infty) \times Y$ and $\mathcal{E}^{R}\left(t-s ; z, x^{\prime}\right)$ on the stretched closed manifold $M^{R}$. We shall use that both kernels can be estimated according to inequality (8.2.4).

We estimate the first summand

$$
\begin{aligned}
\left|\psi_{\text {aps }}^{\prime \prime}(z) \mathcal{E}_{\text {aps }}^{\infty}\left(t-s ; z, x^{\prime}\right) \chi_{\text {aps }}\left(x^{\prime}\right)\right| & \leq \frac{c_{0}}{R} c_{1}(t-s)^{-\frac{1+m}{2}} e^{c_{2} t} e^{-c_{3} \frac{d^{2}\left(z, x^{\prime}\right)}{t-s}} \\
& \leq c_{1}^{\prime} e^{c_{2}^{\prime} t} e^{-c_{3}^{\prime} R^{2} / t} .
\end{aligned}
$$

Here we have exploited that $t \geq s \geq 0$ and

$$
(t-s)^{-(1+m) / 2} e^{-c_{2} \frac{d^{2}\left(z, x^{\prime}\right)}{(t-s)}} \leq c t^{-(1+m) / 2} e^{-c_{2} \frac{d^{2}\left(z, x^{\prime}\right)}{t}} \leq \tilde{c} e^{-c_{2} \frac{d^{2}\left(z, x^{\prime}\right)}{2 t}}
$$

Similarly we estimate the second summand

$$
\begin{aligned}
2 \left\lvert\, \psi_{\mathrm{aps}}^{\prime}(z) \frac{\partial}{\partial u} \mathcal{E}_{\mathrm{aps}}^{\infty}(t-s ;\right. & \left.z, x^{\prime}\right) \chi_{\mathrm{aps}}\left(x^{\prime}\right) \mid \\
& \leq \frac{c_{0}}{R} c_{1} \frac{(t-s)^{-\frac{1+m}{2}}}{\sqrt{t}} e^{c_{2} t} e^{-c_{3} \frac{d^{2}\left(z, x^{\prime}\right)}{t-s}} \leq c_{1}^{\prime} e^{c_{2}^{\prime} t} e^{-c_{3}^{\prime} R^{2} / t}
\end{aligned}
$$

where the factor $1 / \sqrt{t}$ comes from the differentiation of the kernel as explained before.

The third and forth summands, involving the kernel $\mathcal{E}^{R}$ of the closed stretched manifold $M^{R}$, are treated in exactly the same way.

Altogether we have proved

Lemma 8.2.2. The error kernel $C^{R}(t ; u, v)$ vanishes for $u \notin\left[-\frac{6}{7} R,-\frac{1}{7} R\right]$. Moreover, $C^{R}(t ; u, v)$ vanishes whenever $|u-v| \leq R / 7$. For arbitrary $x, x^{\prime} \in$ $M_{2}^{R}$ we have the estimate

$$
\left|C^{R}\left(t ; x, x^{\prime}\right)\right| \leq c_{1} e^{c_{2} t} e^{-c_{3} R^{2} / t}
$$

with constants $c_{1}, c_{2}, c_{3}$ independent of $x, x^{\prime}, t, R$.

We consider the pointwise error

$$
\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)=\int_{0}^{t} d s \int_{M_{2}^{R}} d z \mathcal{E}_{2}^{R}(s ; x, z) C^{R}(t-s ; z, x) .
$$

We obtain the following proposition as a consequence of the preceding lemma.

Proposition 8.2.3. For all $x \in M_{2}^{R}$ and all $t>0$ we have

$$
\operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x)-\operatorname{tr} Q_{2}^{R}(t ; x, x)=\operatorname{tr}\left(\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)\right) .
$$

Moreover, there exist positive constants $c_{1}, c_{2}, c_{3}$, independent of $R$, such that the 'error' term satisfies the inequality

$$
\left|\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)\right| \leq c_{1} \cdot e^{c_{2} t} \cdot e^{-c_{3}\left(R^{2} / t\right)} .
$$

Proof. We estimate the error term

$$
\begin{aligned}
\mid \mathcal{E}_{2}^{R}(t ; x, x) & -Q_{2}^{R}(t ; x, x) \mid \\
& \leq \int_{0}^{t} d s \int_{M_{2}^{R}} d z\left|\mathcal{E}_{2}^{R}(s ; x, z) C^{R}(t-s ; z, x)\right| \\
& \leq c_{1}^{2} e^{c_{2} t} \cdot \int_{0}^{t} d s \int_{M_{2}^{R}} d z\left\{s^{-\frac{d+1}{2}} \cdot e^{-c_{3} \frac{d^{2}(x, z)}{s}}\right\} \cdot e^{-c_{3} \frac{d^{2}(x, z)}{t-s}} \\
& \leq c_{1}^{2} e^{c_{2} t} \cdot \int_{0}^{t} d s \int_{\operatorname{supp}_{z} C^{R(t-s ; z, x)}} d z e^{-c_{4} \frac{d^{2}(x, z)}{s}} \cdot e^{-c_{3} \frac{d^{2}(x, z)}{t-s}} \\
& \leq c_{1}^{2} e^{c_{2} t} \cdot \int_{0}^{t} d s \int_{\operatorname{supp}_{z} C^{R}(t-s ; z, x)} d z e^{-c_{5} \frac{t \cdot R^{2}}{s(t-s)}} \\
& \leq c_{1}^{2} e^{c_{2} t} \cdot c R \cdot \int_{0}^{t} d s e^{-c_{5} \frac{t \cdot R^{2}}{s(t-s)}} \leq c_{1}^{2} e^{c_{2} t} \cdot 2 c R \cdot \int_{0}^{t / 2} d s e^{-c_{5} \frac{t \cdot R^{2}}{s(t / 2)}} \\
& =c_{1}^{2} e^{c_{2} t} \cdot 2 c R \cdot \int_{0}^{t / 2} d s e^{-2 c_{5} \frac{R^{2}}{s}} .
\end{aligned}
$$

Here we have used that $\operatorname{vol}\left(\operatorname{supp}_{z} C^{R}(t-s ; z, x)\right) \sim \operatorname{vol}(Y) \cdot R$ according to Lemma 8.2.2.

We investigate the last integral.

$$
\begin{aligned}
\int_{0}^{t} e^{-\frac{c}{s}} d s=-\int_{0}^{t} & \frac{s^{2}}{c} \cdot e^{-\frac{c}{s}} \cdot\left(-\frac{c}{s^{2}}\right) d s \\
& <-\int_{0}^{t} \frac{t^{2}}{c} \cdot e^{-\frac{c}{s}} \cdot\left(-\frac{c}{s^{2}}\right) d s=-\frac{t^{2}}{c} \int_{\infty}^{\frac{c}{t}} e^{-r} d r=\frac{t^{2}}{c} e^{-\frac{c}{t}}
\end{aligned}
$$

Thus we have

$$
\left|\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)\right| \leq c_{1}^{2} e^{c_{2} t} \cdot 2 c R \cdot \frac{t^{2}}{c_{6} R^{2}} e^{-\frac{c_{6} R^{2}}{t}} \leq c_{7} e^{c_{2} t} \cdot e^{-c_{8}\left(R^{2} / t\right)}
$$

The preceding proposition shows that, for $t$ smaller than $\sqrt{R}$, the trace $\operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x)$ of the kernel of the operator $\mathcal{D}_{2}^{R} e^{\left.-t\left(\mathcal{D}_{2, \Pi}^{R}\right\rangle\right)^{2}}$ approaches the trace $\operatorname{tr} Q_{2}^{R}(t ; x, x)$ of the approximative kernel pointwise as $R \rightarrow \infty$. In particular, we have:

Corollary 8.2.4. The following equality holds

$$
\begin{aligned}
& \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x \\
&=\frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} Q_{2}^{R}(t ; x, x) d x+\mathcal{O}\left(e^{-c R}\right)
\end{aligned}
$$

as $R \rightarrow \infty$.

Proof. We have

$$
\begin{aligned}
& \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} Q_{2}^{R}(t ; x, x) d x \\
&+\frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr}\left(\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)\right) d x
\end{aligned}
$$

and we have to show that the second summand on the right side is $\mathcal{O}\left(e^{-c R}\right)$ as $R \rightarrow \infty$. We estimate

$$
\begin{aligned}
& \left|\frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr}\left(\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)\right) d x\right| \\
& \quad \leq \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}}\left|\mathcal{E}_{2}^{R}(t ; x, x)-Q_{2}^{R}(t ; x, x)\right| d x \\
& \quad \leq \frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} c_{1} \cdot e^{c_{2} t} \cdot e^{-c_{3}\left(R^{2} / t\right)} d x \\
& \quad \leq \frac{c_{1} \operatorname{vol}\left(M_{2}^{R}\right)}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{e^{c_{2} t} \cdot e^{-c_{3}\left(R^{2} / t\right)}}{\sqrt{t}} d t \\
& \quad \leq c_{4} R \int_{0}^{\sqrt{R}} e^{c_{2} \sqrt{R}} \cdot e^{-c_{5} R^{3 / 2}} d t \leq c_{4} R^{3 / 2} \cdot e^{-c_{6} R} \leq c_{7} \cdot e^{-c_{8} R}
\end{aligned}
$$

Corollary 8.2 .4 shows that the essential part of the local $\eta$-function of the spectral boundary condition on the half manifold with attached cylinder of length $R$, namely the 'small-time' integral from 0 to $\sqrt{R}$ can be replaced, as $R \rightarrow \infty$, by the corresponding integral over the $\operatorname{trace} \operatorname{tr} Q_{2}^{R}(t ; x, x)$ of the approximate kernel, constructed in (8.2.7). Now we show that $\operatorname{tr} Q_{2}^{R}(t ; x, x)$ can be replaced pointwise (for $x \in M_{2}^{R}$ ) by the trace $\operatorname{tr} \mathcal{E}^{R}(t ; x, x)$ of the kernel of the operator $\mathcal{D}^{R} e^{-t\left(\mathcal{D}^{R}\right)^{2}}$ which is defined on the stretched closed manifold $M^{R}$.

Consider the Dirac operator

$$
G\left(\partial_{u}+B\right): C^{\infty}([0, \infty) \times Y ; \mathcal{S}) \longrightarrow C^{\infty}([0, \infty) \times Y ; \mathcal{S})
$$

on the half-infinite cylinder with the domain

$$
\left\{s \in C_{0}^{\infty}([0, \infty) \times Y ; \mathcal{S}) \mid \Pi_{>}\left(\left.s\right|_{\{0\} \times Y}=0 .\right.\right.
$$

It has a unique self-adjoint extension which we denote by $D_{\text {aps }}$. Recall that the integral kernel $\mathcal{E}_{\text {aps }}^{\infty}$ of the operator $D_{\text {aps }} e^{-t\left(D_{\text {aps }}\right)^{2}}$ enters in the definition
of the approximative kernel $Q_{2}^{R}$ as given in (8.2.7). We show that $\mathcal{E}_{\text {aps }}^{\infty}(t ; x, x)$ is traceless for all $x \in[0, \infty) \times Y$. Then

$$
\begin{equation*}
\operatorname{tr} Q_{2}^{R}(t ; x, x)=\operatorname{tr} \mathcal{E}^{R}(t ; x, x) \text { for all } x \in M_{2}^{R} \tag{8.2.10}
\end{equation*}
$$

follows.
To prove that a product $T V$ is traceless, the following quick argument can be applied occasionally.

Lemma 8.2.5. Let $G$ be unitary with $G^{2}=-\mathrm{Id}$. We consider an operator $V$ of trace class which is 'even', i.e. it commutes with $G$. Moreover, $T$ is odd, i.e. it anticommutes with $G$. Then

$$
\operatorname{Tr}(T V)=0
$$

Proof. We have, by unitary equivalence,

$$
\operatorname{Tr}(T V)=\operatorname{Tr}(-G(T V) G)=\operatorname{Tr}(-G T G V)=\operatorname{Tr}\left(G^{2} T V\right)=\operatorname{Tr}(-T V)
$$

Lemma 8.2.6. Let $\chi:[0, \infty) \rightarrow \mathbf{R}$ be a smooth function with compact support and $t>0$. Then the trace of the operator $\chi \cdot D_{\mathrm{aps}} e^{-t\left(D_{\mathrm{aps}}\right)^{2}}$ vanishes. In particular,

$$
\int_{Y} \operatorname{tr} \mathcal{E}_{\mathrm{aps}}^{\infty}(t ; u, y ; u, y) d y=0
$$

for all $u \in[0, \infty)$.

Proof. Clearly, $D_{\text {aps }}^{2}=\left(G\left(\partial_{u}+B\right)\right)^{2}=-\partial_{u}^{2}+B^{2}$ is even, hence also the power series $e^{-t\left(D_{\text {aps }}\right)^{2}}$ is even. On the other side, with $B$ also $G B$ is odd. Thus, by Lemmma 9.2.6:

$$
\operatorname{Tr}\left(\chi \cdot G B e^{-t\left(D_{\mathrm{aps}}\right)^{2}}\right)=0
$$

To show that

$$
\operatorname{Tr}\left(\chi \cdot G \partial_{u} e^{-t\left(D_{\mathrm{aps}}\right)^{2}}\right)=0
$$

we need a slightly more specific argument: Let $\mathrm{e}_{\text {aps }}^{\infty}$ denote the heat kernel of the operator $D_{\text {aps }}$. For $u, v \in[0, \infty)$ and $y, z \in Y$ it has the following form (see e.g. [27], Formulae 22.33 and 22.35):

$$
\mathrm{e}_{\mathrm{aps}}(t ; u, y ; v, z)=\sum_{k \in \mathbf{Z}} e_{k}(t ; u, v) \varphi_{k}(y) \otimes \varphi_{k}^{*}(z)
$$

for an orthonormal system $\left\{\varphi_{k}\right\}$ of eigensections of $B$. Hence,

$$
G \partial_{u} \mathrm{e}_{\mathrm{aps}}(t ; u, y ; v, z)=\sum_{k \in \mathbf{Z}} e_{k}^{\prime}(t ; u, v) G \varphi_{k}(y) \otimes \varphi_{k}^{*}(z)
$$

$\operatorname{But}\left\langle G \varphi_{k} ; \varphi_{k}\right\rangle=0$ on $Y$ since $G \varphi_{k}$ is orthogonal to $\varphi_{k}$.

So far we found

$$
\begin{aligned}
& \eta_{\mathcal{D}_{2, \Pi\rangle}^{R}}(0)=\frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}^{R}(t ; x, x) d x+\mathcal{O}\left(e^{-c R}\right) \\
& +\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x
\end{aligned}
$$

as $R \rightarrow \infty$. To prove Theorem 8.1.2, we still have to show

$$
\begin{align*}
& \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x=\mathcal{O}\left(e^{-c R}\right)  \tag{8.2.11}\\
& \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}^{R}(t ; x, x) d x=\mathcal{O}\left(e^{-c R}\right) \tag{8.2.12}
\end{align*}
$$

as $R \rightarrow \infty$. Recall that $\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$ denotes the kernel of the operator $\mathcal{D}_{2}^{R} e^{-t\left(\mathcal{D}_{2, \Pi}^{R},\right)^{2}}$ on the compact manifold $M_{2}^{R}$ with boundary $\{-R\} \times Y$, and $\mathcal{E}^{R}\left(t ; x, x^{\prime}\right)$ the kernel of the operator $\mathcal{D}^{R} e^{-t\left(\mathcal{D}^{R}\right)^{2}}$ on the closed stretched manifold $M^{R}$.

In the following we show (8.2.11), i.e. that we can neglect the contribution to the $\eta$-invariant of $\mathcal{D}_{2, \Pi \text {, }}^{R}$ which comes from the large time asymptotic of $\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$. The key to that is that the eigenvalue of $\mathcal{D}_{2, \Pi\rangle}^{R}$ with the smallest absolute value is uniformly bounded away from zero.

Theorem 8.2.7. Let $\mu_{0}(R)$ denote the smallest (in absolute value) nonvanishing eigenvalue of the operator $\mathcal{D}_{2, \Pi}^{R}$, on the manifold $M_{2}^{R}$. Let us assume, as always in this chapter, that $\operatorname{ker} B=\{0\}$. Then there exists a positive constant $c_{0}$, which does not depend on $R$ such that

$$
\mu_{0}(R)>c_{0}
$$

for $R$ sufficiently large.

Remark 8.2.8. The preceding result differs from the behaviour of the small eigenvalues on the stretched, closed manifold $M^{R}$. On the manifold with boundary $M_{2}^{R}$ with the attached cylinder of length $R$, the eigenvalues are bounded away from 0 when $R \rightarrow \infty$ due to the spectral boundary condition. That is the statement of the preceding theorem which we are going to prove in the next two sections. On $M^{R}$, on the contrary, the set of eigenvalues splits into one set of eigenvalues becoming exponentially small and another one of eigenvalues being uniformly bounded away from 0 as $R \rightarrow \infty$. This we are going to show further below. Clearly, the reason for the different behaviour is that on $M_{2}^{R}$ the eigensections must satisfy the spectral boundary condition. Therefore they are exponentially decreasing on the cylinder. But
on $M^{R}$ we have to do with eigensections on a closed manifold which need not decrease, but require part of the eigenvalues to decrease exponentially (for details see Theorem 8.5.1 below).

### 8.3. Dirac Operators on Manifolds with Cylindrical Ends

To prove Theorem 8.2.7 we first recall a few properties of the cylindrical Dirac operator $D_{\text {cyl }}:=G\left(\partial_{u}+B\right)$ on the infinite cylinder $Y_{\text {cyl }}^{\infty}:=$ $(-\infty,+\infty) \times Y$. A special feature of the cylindrical manifold $Y_{\mathrm{cyl}}^{\infty}$ is that we may apply the theory of Sobolev spaces exactly as in the case of $\mathbf{R}^{m}$. The point is that we can choose a covering of the open manifold $Y_{\text {cyl }}^{\infty}$ by a finite number of coordinate charts. We can also choose a finite trivialization of the bundle $\left.\mathcal{S}\right|_{Y_{\mathrm{cy} 1}^{\infty}}$. Let $\left\{U_{\iota}, \kappa_{\iota}\right\}_{\iota=1}^{K}$ be such a trivialization, where $\kappa_{\iota}:\left.\mathcal{S}\right|_{U_{\iota}} \rightarrow V_{\iota} \times \mathbf{C}^{N}$ is a bundle isomorphism and $V_{\iota}$ an open (possibly noncompact) subset of $\mathbf{R}^{m}$. Let $\left\{f_{\iota}\right\}$ be a corresponding partition of unity. We assume that for any $\iota$ the derivatives of the function $f_{\iota}$ are bounded.

Definition 8.3.1. We say that a section (or distribution) $s$ of the bundle $\mathcal{S}$ over $Y_{\text {cyl }}^{\infty}$ belongs to the $p$-th Sobolev space $\mathcal{H}^{p}\left(Y_{\text {cyl }}^{\infty} ; \mathcal{S}\right), p \in \mathbf{R}$, if and only if $f_{\iota} \cdot s$ belongs to the Sobolev space $\mathcal{H}^{p}\left(\mathbf{R}^{m} ; \mathbf{C}^{N}\right)$ for any $\iota$. We define the p-th Sobolev norm

$$
\|s\|_{p}:=\sum_{\iota=1}^{K}\left\|\left(\operatorname{Id}+\Delta_{\iota}\right)^{p / 2}\left(f_{\iota} \cdot s\right)\right\|_{L_{2}\left(\mathbf{R}^{m}\right)}
$$

where $\Delta_{\iota}$ denotes the Laplacian on the trivial bundle $V_{\iota} \times \mathbf{C}^{N} \subset \mathbf{R}^{m} \times \mathbf{C}^{N}$.

Lemma 8.3.2. (a) For the unique self-adjoint $L_{2}$-extension of $D_{\mathrm{cyl}}$ (denoted by the same symbol) we have

$$
\operatorname{dom}\left(D_{\mathrm{cyl}}\right)=\mathcal{H}^{1}\left(Y_{\mathrm{cy1}}^{\infty} ; \mathcal{S}\right)
$$

(b) Let $\lambda_{1}$ denote the smallest positive eigenvalue of the operator $B$ on the manifold $Y$. Then we have

$$
\begin{equation*}
\left\langle\left(D_{\mathrm{cyl}}\right)^{2} s ; s\right\rangle \geq \lambda_{1}^{2}\|s\|^{2} \tag{8.3.1}
\end{equation*}
$$

for all $s \in \operatorname{dom}\left(D_{\text {cyl }}\right)$, and for any $\mu \in\left(-\lambda_{1},+\lambda_{1}\right)$ the operator

$$
D_{\mathrm{cyl}}-\mu: \mathcal{H}^{1}\left(Y_{\mathrm{cy} 1}^{\infty} ; \mathcal{S}\right) \longrightarrow L_{2}\left(Y_{\mathrm{cy} 1}^{\infty} ; \mathcal{S}\right)
$$

is an isomorphism of Hilbert spaces.
(c) Let $\mathcal{R}_{\mathrm{cyl}}(\mu)$ denote the inverse of the operator $D_{\mathrm{cyl}}-\mu$. Then the family $\left\{\mathcal{R}_{\text {cyl }}(\mu)\right\}_{\mu \in\left(-\lambda_{1}, \lambda_{1}\right)}$ is a smooth family of elliptic pseudo-differential operators of order -1 .

Proof. (a) follows immediately from the corresponding result on the model manifold $\mathbf{R}^{m}$.
To prove (b) we consider a spectral resolution $\left\{\varphi_{k}, \lambda_{k}\right\}_{k \in \mathbf{Z} \backslash 0}$ of $L_{2}(Y ; \mathcal{S})$ generated by the tangential operator $B$. Because of (8.1.2) we have $\lambda_{-k}=$ $-\lambda_{k}$. We consider a section $s$ belonging to the dense subspace $C_{0}^{\infty}\left(Y_{\mathrm{cy} 1}^{\infty} ; \mathcal{S}\right)$ of $\operatorname{dom}\left(D_{\text {cyl }}\right)$, and expand it in terms of the preceding spectral resolution

$$
s(u, y)=\sum_{k \in \mathbf{Z} \backslash\{0\}} f_{k}(u) \varphi_{k}(y) .
$$

Since $\left(D_{\text {cyl }}\right)^{2}=-\partial_{u}^{2}+B^{2}$, we obtain

$$
\left(D_{\mathrm{cyl}}\right)^{2} s=\sum\left(\lambda_{k}^{2} f_{k}-f_{k}^{\prime \prime}\right) \varphi_{k},
$$

hence

$$
\begin{aligned}
\left\langle\left(D_{\mathrm{cyl}}\right)^{2} s ; s\right\rangle & =\sum \int_{-\infty}^{\infty}\left(\lambda_{k}^{2} f_{k}(u)-f_{k}^{\prime \prime}(u)\right) \bar{f}_{k}(u) d u \\
& \geq \lambda_{1}^{2}\|s\|^{2}-\sum \int_{-\infty}^{\infty} f_{k}^{\prime \prime}(u) \bar{f}_{k}(u) d u \\
& =\lambda_{1}^{2}\|s\|^{2}+\sum \int_{-\infty}^{\infty} f_{k}^{\prime}(u) \bar{f}_{k}^{\prime}(u) d u \geq \lambda_{1}^{2}\|s\|^{2}
\end{aligned}
$$

It follows that $\left(D_{\text {cyl }}\right)^{2}$ (and therefore $\left.D_{\text {cyl }}\right)$ has bounded inverse in $L_{2}\left(Y_{\text {cyl }}^{\infty} ; \mathcal{S}\right)$ and, more generally, that $\left(D_{\text {cyl }}\right)^{2}-\mu$ is invertible for $\mu \in\left(-\lambda_{1}, \lambda_{1}\right)$.
To prove (c) we apply the symbolic calculus and construct a parametrix $S$ for the operator $D_{\text {cyl }}$, i.e. $S$ is an elliptic pseudo-differential operator of order -1 such that

$$
S D_{\mathrm{cyl}}=\mathrm{Id}+T
$$

where $T$ is a smoothing operator. Thus

$$
D_{\mathrm{cyl}}^{-1}=S-T D_{\mathrm{cyl}}^{-1} .
$$

The operator $T D_{\text {cyl }}^{-1}$ is a smoothing operator, hence $D_{\text {cyl }}^{-1}$ is an elliptic pseudodifferential operator of order -1 . The same argument can be applied to the resolvent $\mathcal{R}_{\text {cyl }}(\mu)=\left(D_{\text {cyl }}-\mu\right)^{-1}$ for arbitrary $\mu \in\left(-\lambda_{1}, \lambda_{1}\right)$. The smoothness of the family follows by standard calculation.

To prove Theorem 8.2.7 we need to refine the preceding results on the infinite cylinder $Y_{\text {cyl }}^{\infty}$ to the Dirac operator, naturally extended to the manifold $M_{2}^{\infty}=((-\infty, 0] \times Y) \cup M_{2}$ with cylindrical end. Let $C_{0}^{\infty}\left(M_{2}^{\infty}, \mathcal{S}\right)$ denote the space of compactly supported smooth sections of $\mathcal{S}$ over $M_{2}$. Then

$$
\begin{equation*}
\left.\mathcal{D}_{2}^{\infty}\right|_{C_{0}^{\infty}\left(M_{2}^{\infty}, \mathcal{S}\right)}: C_{0}^{\infty}\left(M_{2}^{\infty}, \mathcal{S}\right) \rightarrow L_{2}\left(M_{2}^{\infty}, \mathcal{S}\right) \tag{8.3.2}
\end{equation*}
$$

is symmetric. Moreover, we have

Lemma 8.3.3. Let $s \in C^{\infty}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ be an eigensection of $\mathcal{D}_{2}^{\infty}$. Then there exist $C, c>0$ such that, on $(-\infty, 0] \times Y$, we have $|s(u, y)| \leq C e^{c u}$.

Proof. Let $\left\{\varphi_{k}, \lambda_{k}\right\}_{k \in \mathbf{Z} \backslash 0}$ be a spectral resolution of the tangential operator $B$. Because of (8.1.2) we have $\lambda_{-k}=-\lambda_{k}$ and we can assume that $\varphi_{-k}=G \varphi_{k}$ for $k \in \mathbf{N}$. Then

$$
\begin{equation*}
\left\{\varphi_{k}^{ \pm}=\frac{1}{\sqrt{2}}\left(\varphi_{k} \pm G \varphi_{k}\right), \pm \lambda_{k}\right\}_{k \in \mathbf{N}} \tag{8.3.3}
\end{equation*}
$$

is a spectral resolution of the composed operator $G B$ on $Y$. Notice that we have

$$
\begin{equation*}
G \varphi_{k}^{+}=-\varphi_{k}^{-} \quad \text { and } \quad G \varphi_{k}^{-}=\varphi_{k}^{+} \tag{8.3.4}
\end{equation*}
$$

Let $s \in C^{\infty}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ and

$$
\begin{equation*}
\mathcal{D}_{L_{2}}^{\infty} \psi=\mu \psi \tag{8.3.5}
\end{equation*}
$$

with $\mu \in \mathbf{R}$. We expand $\left.s\right|_{(-\infty, 0] \times Y}$ in terms of the spectral resolution of GB just constructed:

$$
s(u, y)=\sum_{k=1}^{\infty} f_{k}(u) \varphi_{k}^{+}(y)+g_{k}(u) \varphi_{k}^{-} .
$$

Because of (8.3.3), (8.3.4), and (8.3.5) the coefficients $f_{k}, g_{k}$ must satisfy the system of ordinary differential equations

$$
\left(\begin{array}{cc}
\lambda_{k} & \partial / \partial u \\
-\partial / \partial u & -\lambda_{k}
\end{array}\right)\binom{f_{k}}{g_{k}}=\mu\binom{f_{k}}{g_{k}}
$$

or, equivalently,

$$
\binom{f_{k}^{\prime}}{g_{k}^{\prime}}=\mathbf{A}\binom{f_{k}}{g_{k}} \quad \text { with } \mathbf{A}:=\left(\begin{array}{cc}
0 & -\left(\mu+\lambda_{k}\right) \\
\mu-\lambda_{k} & 0
\end{array}\right) .
$$

Since $s \in L_{2}$, of the eigenvalues $\pm \sqrt{\lambda_{k}^{2}-\mu^{2}}$ of $\mathbf{A}$ only those which are on the positive real line enter in the construction of $s$ by solving the preceding differential equation. In particular, all coefficients $f_{k}, g_{k}$ must vanish identically for $\lambda_{k} \leq \mu$. Thus

$$
\begin{align*}
& s(u, y)=\sum_{\lambda_{k}>\mu} a_{k}\left(\exp \left(\sqrt{\lambda_{k}^{2}-\mu^{2}} u\right) \varphi_{k}^{+}\right.  \tag{8.3.6}\\
&\left.\quad-\frac{\lambda_{k}-\mu}{\sqrt{\lambda_{k}^{2}-\mu^{2}}} \exp \left(\sqrt{\lambda_{k}^{2}-\mu^{2}} u\right) \varphi_{k}^{-}\right)
\end{align*}
$$

and, in particular,

$$
|s(u, y)| \leq C \exp \left(\sqrt{\lambda_{k_{0}}^{2}-\mu^{2}} \frac{u}{2}\right), \quad u<0
$$

for some constant $C$ when $\lambda_{k_{0}}$ denotes the smallest positive eigenvalue of $B$ such that $\lambda_{k_{0}}>|\mu|$.

In spectral theory we are looking for self-adjoint $L_{2}$-extensions of a symmetric operator. We recall: on a closed manifold, the Dirac operator is essentially self-adjoint; i.e. its minimal closed extension is self-adjoint (and therefore there do not exist other self-adjoint extensions) and it is a Fredholm operator. On a compact manifold with boundary, the situation is much more complicated. There is a huge variety of dense domains to which the Dirac operator can be extended such that it becomes self-adjoint; and there is a smaller, but still large variety where the extension of the Dirac operator becomes self-adjoint and Fredholm (see e.g. Booß-Bavnbek and Furutani $[20]$ ); a special type of self-adjoint and Fredholm domains are the domains specified by the boundary conditions belonging to the Grassmannian of all self-adjoint generalized Atiyah-Patodi-Singer projections. These boundary conditions are treated in this book.

Now we shall show that the situation on manifolds with (infinite) cylindrical ends resembles the situation on closed manifolds.

We recall the following simple lemma (see also Reed and Simon [85], Theorem VIII.3, Corollary, p. 257).

Lemma 8.3.4. Let $A$ be a densely defined symmetric operator in a separable complex Hilbert space $\mathcal{H}$. We assume that range $(A+i)$ is dense in $\mathcal{H}$. Then $A$ is essentially self-adjoint.

Proof. Since $A$ is symmetric, the operator $A+i$ is injective and the operator $(A+i)^{-1}$ is well defined and bounded on the dense subspace range $(A+i)$ of $\mathcal{H}$. Then the closure $R_{i}$ of $(A+i)^{-1}$ has the whole space $\mathcal{H}$ as domain and $R_{i}$ is bounded and injective. Now a standard argument of functional analysis (see e.g. Pedersen [80], Proposition 5.1.7) says that the inverse $R_{i}^{-1}$ of a densely defined, closed, and injective operator $R_{i}$ has the same properties. Thus our $R_{i}^{-1}$ is closed; and by construction it is the minimal closed extension of $A+i$. Therefore, $R_{i}^{-1}-i$ is symmetric and the minimal closed extension of $A$, hence self-adjoint and equal $A^{*}$.

We apply the lemma for $\mathcal{H}=L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ and take for $A$ the operator of (8.3.2). To prove that the range $\left(\mathcal{D}_{2}^{\infty}+i\right)\left(C_{0}^{\infty}\left(M_{2}^{\infty} ; \mathcal{S}\right)\right)$ is dense in $L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ we consider a section $s \in L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ which is orthogonal to $\left(\mathcal{D}_{2}^{\infty}+i\right)\left(C_{0}^{\infty}\left(M_{2}^{\infty} ; \mathcal{S}\right)\right)$; i.e. the distribution $\left(\mathcal{D}_{2}^{\infty}-i\right) s$ vanishes when applied to any test function, hence

$$
\begin{equation*}
\left(\mathcal{D}_{2}^{\infty}-i\right) s=0 \tag{8.3.7}
\end{equation*}
$$

Since $\mathcal{D}_{2}^{\infty}-i$ is elliptic, by elliptic regularity $s$ is smooth in all inner points, that is for our complete manifold in all points. On the cylinder $(-\infty, 0] \times Y$ we expand $s$ in terms of the eigensections of the composed operator $G B$ on $Y$. It follows that $s$ satisfies an estimate of the form

$$
\begin{equation*}
|s(u, y)| \leq C e^{c u}, \quad(u, y) \in(-\infty, 0] \times Y \tag{8.3.8}
\end{equation*}
$$

for some constants $C, c>0$ (according to Lemma 8.3.3). On the manifold $M_{2}^{R}$ with cylindrical end of finite length $R$ we apply Green's formula and get

$$
\begin{equation*}
\left\langle\mathcal{D}_{2}^{R} s^{R} ; s^{R}\right\rangle-\left\langle s^{R} ; \mathcal{D}_{2}^{R} s^{R}\right\rangle=-\int_{\{-R\} \times Y}\left(\left.G s\right|_{\{-R\} \times Y} d y,\left.s\right|_{\{-R\} \times Y}\right) \tag{8.3.9}
\end{equation*}
$$

where $s^{R}$ denotes the restriction of $s$ to the manifold $M_{2}^{R}$ with boundary $\{-R\} \times Y$. For $R \rightarrow \infty$, the right side of (8.3.9) vanishes; and the left side becomes $2 i\|s\|^{2}$ by (8.3.7). Hence $s=0$.

Thus we have proved

Lemma 8.3.5. The operator (8.3.2) is essentially self-adjoint.

We denote the (unique) self-adjoint $L_{2}$-extension by the same symbol $\mathcal{D}_{2}^{\infty}$.

We define the Sobolev spaces on the manifold $M_{2}^{\infty}$ like in Definition 8.3.1. Once again, the point is that manifolds with cylindrical ends, even they are not compact but only complete, are like the infinite cylinder sufficient simple to be covered by a finite system of local charts. Clearly

$$
\operatorname{dom}\left(\mathcal{D}_{2}^{\infty}\right)=\mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right)
$$

and

$$
\mathcal{D}_{2}^{\infty}: \mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right) \rightarrow L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)
$$

is bounded. There are, however, substantial differences between the properties of the simple Dirac operator $D_{\text {cyl }}$ on the infinite cylinder and the Dirac operator $\mathcal{D}_{2}^{\infty}$ on the manifold with cylindrical end. For instance, from $B$ the discreteness of the spectrum and the regularity at 0 (i.e., 0 is not an eigenvalue) are passed on to $D_{\text {cyl }}$, but not to $\mathcal{D}_{2}^{\infty}$. Yet we can prove the following result:

## Proposition 8.3.6. The operator

$$
\mathcal{D}_{2}^{\infty}: \operatorname{dom}\left(\mathcal{D}_{2}^{\infty}\right)=\mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right) \longrightarrow L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)
$$

is a Fredholm operator and its spectrum in the interval $\left(-\lambda_{1}, \lambda_{1}\right)$ consists of finitely many eigenvalues of finite multiplicity. Here $\lambda_{1}$ denotes the smallest positive eigenvalue of $B$.

Note . The first part of the proposition is false, if we drop the assumption of invertible tangential operator $B$, see Example ?? below. The second part of the proposition remains true also for singular $B$. Actually, using more advanced methods one can show that the essential spectrum of $\mathcal{D}_{2}^{\infty}$ is equal to $\left(-\infty,-\lambda_{1}\right] \cup\left[\lambda_{1}, \infty\right)$ (see for instance Müller [73], Section 4).

Before proving the proposition we shall collect various criteria for the compactness of a bounded operator between Sobolev spaces on an open manifold. Let $X$ be a complete (not necessarily compact) Riemannian manifold with a fixed Hermitian bundle. Recall the three cornerstones of the Sobolev analysis of Dirac operators for $X$ closed.

Rellich Lemma: The inclusion $\mathcal{H}^{1}(X) \hookrightarrow L_{2}(X)$ is compact.
Compact Resolvent: To each Dirac operator $\mathcal{D}$ we have a parametrix $\mathcal{R}$ which is an elliptic pseudo-differential operator of order -1 with principal symbol equal to the inverse of the principal symbol of $\mathcal{D}$. So it is a bounded operator from $L_{2}(X)$ to $\mathcal{H}^{1}(X)$, hence compact in $L_{2}(X)$. In particular, for $\mu$ in the resolvent set the resolvent $(\mathcal{D}-\mu)^{-1}$ is compact as operator in $L_{2}(X)$.
Smoothing Operator: Any integral operator over $X$ with smooth kernel is a smoothing operator, i.e. it maps distributional sections of arbitrary low order into smooth sections. Moreover, it is of trace class and thus compact.

In the general case, i.e. for not necessarily compact $X$, the Rellich Lemma remains valid for sections with compact support. A compact resolvent is not attainable, hence the essential spectrum appears. Operators with smooth kernel remain smoothing operators, but in general they are no longer of trace class nor compact. We recall:

Lemma 8.3.7. Let $X$ be a complete (not necessarily compact) Riemannian manifold with fixed Hermitian bundle. Let $K$ be a compact subset of $X$.
(a) The injection $\mathcal{H}^{1}(X) \hookrightarrow L_{2}(X)$ defines a compact operator when we restrict it to all sections with support in $K$. In particular, for any cutoff function $\chi$ with support in $K$ and any bounded operator $\mathcal{R}: L_{2}(X) \rightarrow$ $\mathcal{H}^{1}(X)$ the operator $\chi \mathcal{R}$ is compact in $L_{2}(X)$.
(b) Let $T: L_{2}(X) \rightarrow L_{2}(X)$ be an integral operator with a kernel $k(x, y) \in$ $L_{2}\left(X^{2}\right)$. Then the operator $T$ is a bounded, compact operator (in fact it is of Hilbert-Schmidt class).
(c) Let $T: L_{2}(X) \rightarrow L_{2}(X)$ be a bounded compact operator and $\mathcal{H}^{\prime}$ a closed subspace of $L_{2}(X)$, e.g. $\mathcal{H}^{\prime}:=L_{2}\left(X^{\prime}\right)$ where $X^{\prime}$ is a submanifold of $X$ of codimension 0. Assume that $T\left(\mathcal{H}^{\prime}\right) \subset \mathcal{H}^{\prime}$. Then $\left.T\right|_{\mathcal{H}^{\prime}}$ is compact as operator from $\mathcal{H}^{\prime}$ to $\mathcal{H}^{\prime}$.

Proof. (a) follows immediately from the local Rellich Lemma. (b) is the famous Hilbert-Schmidt Lemma. Also (c) is well known, see e.g. Hörmander [54], Proposition 19.1.13 where (c) is proved within the category of trace class operators.

In general an integral operator $T$ with smooth kernel is not compact even if either $\operatorname{supp}_{x} k\left(x, x^{\prime}\right)$ or $\operatorname{supp}_{x^{\prime}} k\left(x, x^{\prime}\right)$ are contained in a compact subset $K \subset X$. Consider for instance on $Y_{\text {cyl }}^{\infty}=(-\infty,+\infty) \times Y$ an integral
operator $T$ with a smooth kernel of the form

$$
k\left(x, x^{\prime}\right)=\chi(x) d\left(x, x^{\prime}\right)
$$

where $d\left(x, x^{\prime}\right)$ denotes the distance and $\chi$ is a function with support in a ball of radius 1 (and equal 1 in a smaller ball). Then $T$ is not a compact operator on $L_{2}\left(Y_{\mathrm{cyl}}^{\infty}\right)$ : choose a sequence $\left\{s_{n}\right\}$ of $L_{2}$ functions of norm 1 and with $\operatorname{supp} s_{n}$ contained in a ball of radius 1 such that $d\left(\operatorname{supp} \chi, \operatorname{supp} s_{n}\right)=n$. Then for any $n$ we have $\left\|T s_{n}\right\|>C n$. Thus $T$ is not compact, in fact not even bounded.

For the bounded resolvent (see Lemma 8.3.2)

$$
\mathcal{R}_{\mathrm{cyl}}: L_{2}\left(Y_{\mathrm{cyl}}^{\infty} ; \mathcal{S}\right) \rightarrow \mathcal{H}^{1}\left(Y_{\mathrm{cy1}}^{\infty} ; \mathcal{S}\right)
$$

we have, however, the following corollary to the preceding lemma. It provides an example of a compact integral operator on an open manifold with a smooth kernel which is compactly supported only in one variable.

Corollary 8.3.8. Let $\chi$ and $\psi$ be smooth cut-off functions on $Y_{\text {cyl }}^{\infty}$ with support contained in the half-cylinder $(-\infty, 0) \times Y$. Let supp $\chi$ be compact. Then the operators $\chi \mathcal{R}_{\mathrm{cy} 1} \psi$ and $\psi \mathcal{R}_{\mathrm{cyl}} \chi$ are compact in $L_{2}\left(Y_{\mathrm{cyl}}^{\infty} ; \mathcal{S}\right)$.

Proof. The operator $\chi \mathcal{R}_{\text {cyl }} \psi$ is compact according to the preceding lemma, claim (a). Its adjoint operator is $\psi \mathcal{R}_{\text {cyl }} \chi$, since $\mathcal{R}_{\text {cyl }}$ is self-adjoint. Thus it is also compact (even if its range is not compactly supported).

Proof of the Proposition. Let $\mu \in\left(-\lambda_{1}, \lambda_{1}\right)$. We show that the operator $\mathcal{D}_{2}^{\infty}-\mu$ is an (unbounded) Fredholm operator in $L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)$. To do that we construct a parametrix $\mathcal{R}_{2}^{\infty}(\mu)$ for $\mathcal{D}_{2}^{\infty}-\mu$. Let $\mathcal{R}_{\text {cyl }}(\mu)$ denote the inverse operator of $D_{\mathrm{cyl}}-\mu$ on the infinite cylinder $Y_{\mathrm{cyl}}^{\infty}$ introduced in Lemma 8.3.2. Let $\mathcal{R}(\mu)$ be a parametrix for the operator $\mathcal{D}-\mu$ on the closed partitioned manifold $M$. We may assume that $\mathcal{R}(\mu)$ is an elliptic pseudo-differential operators of order -1 such that

$$
\mathcal{R}(\mu)(\mathcal{D}-\mu)=\operatorname{Id}-T(\mu) \text { and }(\mathcal{D}-\mu) \mathcal{R}(\mu)=\operatorname{Id}-T^{\prime}(\mu)
$$

where $T(\mu), T^{\prime}(\mu)$ are operators with smooth kernels, hence compact and even of trace class because they are on a closed manifold. Moreover, we may assume that the operator family $\{\mathcal{R}(\mu)\}$ is a smooth family.

We glue the two parametrices $\mathcal{R}_{\text {cyl }}(\mu)$ and $\mathcal{R}(\mu)$ over a narrow segment of the neck, say $\mathcal{N}^{1}=\left(-\frac{4}{7},-\frac{3}{7}\right) \times Y$ (see Figure 8.3.1) in a similar way as we have done before for the heat kernels. Thus, we choose a smooth partition of unity $\left\{\chi_{\text {cyl }}, \chi_{\text {int }}\right\}$ on $M_{2}^{\infty}$ suitable for the covering $\left\{U_{\text {cyl }}, U_{\text {int }}\right\}$ with $U_{\mathrm{cyl}}:=\left(-\infty,-\frac{3}{7}\right) \times Y$ and $U_{\mathrm{int}}:=\left(-\frac{4}{7}, 0\right] \times Y \cup M_{2}$, hence $U_{\mathrm{cyl}} \cap U_{\mathrm{int}}=$ $\mathcal{N}^{1}$. Moreover, we choose non-negative smooth cut-off functions $\left\{\psi_{\text {cyl }}, \psi_{\text {int }}\right\}$

Figure 8.3.1. The construction of a parametrix for $\mathcal{D}_{2}^{\infty}-\mu$ over $M_{2}^{\infty}$
such that

$$
\begin{aligned}
\psi_{j} \equiv 1 & \text { on }\left\{x \in M_{2}^{\infty} \left\lvert\, \operatorname{dist}\left(x, \operatorname{supp} \chi_{j}\right)<\frac{1}{7}\right.\right\} \\
\text { and } \psi_{j} \equiv 0 & \text { on }\left\{x \in M_{2}^{\infty} \left\lvert\, \operatorname{dist}\left(x, \operatorname{supp} \chi_{j}\right) \geq \frac{2}{7}\right.\right\}
\end{aligned}
$$

for $j \in\{\mathrm{cyl}, \mathrm{int}\}$ (see Figure 8.2.2 with $R=1$ ). Like before, we have

$$
\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime}, \operatorname{supp} \chi_{j}\right)=\operatorname{dist}\left(\operatorname{supp} \psi_{j}^{\prime \prime}, \operatorname{supp} \chi_{j}\right) \geq \frac{1}{7} .
$$

We define the parametrix $\mathcal{R}_{2}^{\infty}$ by the formula

$$
\begin{equation*}
\mathcal{R}_{2}^{\infty}(\mu):=\psi_{\mathrm{cyl}} \mathcal{R}_{\mathrm{cyl}}(\mu) \chi_{\mathrm{cyl}}+\psi_{\mathrm{int}} \mathcal{R}(\mu) \chi_{\mathrm{int}} . \tag{8.3.10}
\end{equation*}
$$

Clearly, on the cylinder $(-\infty,+1) \times Y$ we have

$$
\begin{aligned}
\chi_{j} \mathcal{D} s=\chi_{j} G\left(\partial_{u}\right. & +B) s \\
& =G\left(\partial_{u}+B\right)\left(\chi_{j} s\right)-G \cdot\left(\frac{\partial \chi_{j}}{\partial u}\right) s=\mathcal{D}\left(\chi_{j} s\right)-G \cdot\left(\frac{\partial \chi_{j}}{\partial u}\right) s
\end{aligned}
$$

for $j \in\{\mathrm{cyl}$, int $\}$. Thus

$$
\begin{aligned}
\mathcal{R}_{2}^{\infty}(\mu) & \left(\mathcal{D}_{2}^{\infty}-\mu\right) s \\
= & \psi_{\text {cyl }} \mathcal{R}_{\text {cyl }}(\mu) \chi_{\text {cyl }}\left(\mathcal{D}_{2}^{\infty}-\mu\right) s+\psi_{\text {int }} \mathcal{R}(\mu) \chi_{\text {int }}\left(\mathcal{D}_{2}^{\infty}-\mu\right) s \\
= & \left\{\begin{array}{c}
\mathcal{R}_{\text {cyl }}(\mu)\left(D_{\text {cyl }}-\mu\right) s=s \\
\text { on the cylinder for } u<-\frac{4}{7} \\
s-\psi_{\text {cyl }} \mathcal{R}_{\text {cyl }}(\mu) G \cdot\left(\frac{\partial \chi_{\text {cyl }}}{\partial u}\right) s-\psi_{\text {int }} \mathcal{R}(\mu) G \cdot\left(\frac{\partial \chi_{\text {int }}}{\partial u}\right) s \\
\text { on the neck } \mathcal{N}^{1} \\
\mathcal{R} \mathcal{D} s=s-T \chi_{\text {int }} s \\
\text { in the interior, for } u>-\frac{3}{7}
\end{array}\right.
\end{aligned} .
$$

Now we show that the operator

$$
\begin{align*}
& \mathrm{Id}-\mathcal{R}_{2}^{\infty}(\mu)\left(\mathcal{D}_{2}^{\infty}-\mu\right)  \tag{8.3.11}\\
& \quad=\psi_{\mathrm{cyl}} \mathcal{R}_{\mathrm{cyl}}(\mu) G \cdot\left(\frac{\partial \chi_{\mathrm{cyl}}}{\partial u}\right)+\psi_{\mathrm{int}} \mathcal{R}(\mu) G \cdot\left(\frac{\partial \chi_{\mathrm{int}}}{\partial u}\right)+T \chi_{\mathrm{int}}
\end{align*}
$$

is compact. Since $M$ is closed, the operators $T \chi_{\text {int }}$ and $\psi_{\text {int }} \mathcal{R}(\mu) G \cdot\left(\frac{\partial \chi_{\text {int }}}{\partial u}\right)$ are compact in $L_{2}(M ; \mathcal{S})$. Thus, they are compact in $L_{2}(M ; \mathcal{S}) \cap L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ and hence in $L_{2}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ (see also Lemma 8.3.7.c). The first summand on the right side of (8.3.11) is also compact by Corollary 8.3.8. This proves that the operator of (8.3.11) is compact. Restricted to the eigenspace $\operatorname{ker}\left(\mathcal{D}_{2}^{\infty}-\mu\right)$ it is the identity by definition, hence the eigenspace is finite-dimensional. This proves the main part of the proposition.

To see that there are only finitely many eigenvalues in the interval we recall that $\mathcal{D}_{2}^{\infty}-\mu$ is self-adjoint and, by the preceding argument, Fredholm. In particular, its range is closed. So we can apply the standard argument (see e.g. [27], Proposition 16.1) showing that the spectrum is discrete.

### 8.4. The Estimate of the Lowest Non-Trivial Eigenvalue

In this section we prove Theorem 8.2.7. Recall that the tangential operator $B$ is assumed to be non-singular and that $\lambda_{1}$ denotes the smallest positive eigenvalue of $B$. So far, we have established that

I: the operator $D_{\mathrm{cyl}}$ on the infinite cylinder $Y_{\mathrm{cyl}}^{\infty}$ has no eigenvalues in the interval $\left(-\lambda_{1},+\lambda_{1}\right)$, and
II: the operator $\mathcal{D}_{2}^{\infty}$ on the manifold $M_{2}^{\infty}$ with infinite cylindrical end has only finitely many eigenvalues in the interval $\left(-\lambda_{1},+\lambda_{1}\right)$, each of finite multiplicity.
We have to show that
III: the non-vanishing eigenvalues of $\left(\mathcal{D}_{2}^{R}\right)_{\Pi_{>}}$are bounded away from 0 by a bound independent of $R$.

Proof of Theorem 8.2.7. The idea of the proof is the following. We define a positive constant $\mu_{1}$ independent of $R$. Then let $R$ be a positive real (more precisely $R>R_{0}$ for a suitable positive $R_{0}$ ), and $s \in L_{2}\left(M_{2}^{R} ; \mathcal{S}\right)$ any eigensection with eigenvalue $\mu \in\left(-\lambda_{1} / \sqrt{2},+\lambda_{1} / \sqrt{2}\right)$, i.e.

$$
s \in \operatorname{dom}\left(\mathcal{D}_{2}^{R}\right)_{\Pi_{>}} \text {i.e. } \Pi_{>}\left(\left.s\right|_{\{-R\} \times Y}\right)=0 \text { and } \mathcal{D}_{2}^{R} s=\mu s
$$

Then we show that

$$
\mu^{2}>\mu_{1} / 2
$$

for a certain real $\mu_{1}>0$ which is independent of $R$ and $s$. A natural choice of $\mu_{1}$ is

$$
\begin{equation*}
\mu_{1}=\min \left\{\left.\frac{\left\|\mathcal{D}_{2}^{\infty} \Psi\right\|^{2}}{\|\Psi\|^{2}} \right\rvert\, \Psi \in \mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right) \text { and } \Psi \perp \operatorname{ker} \mathcal{D}_{2}^{\infty}\right\} \tag{8.4.1}
\end{equation*}
$$

Figure 8.4.1. a) Continuous extension of a given eigensection by a nullsection. b) With growing $R$ the enlargement $\alpha$ and the cosine $\beta$ decrease exponentially and the norm of the projection $\widetilde{s}$ goes to 1

Note that by II above (Proposition 8.3.6) the nullspace $\operatorname{ker} \mathcal{D}_{2}^{\infty}$ is of finite dimension. We shall define a certain extension $s^{\infty} \in \mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ of $s$.

Quite a comfortable reasoning would be achieved, if we could extend $s$ to an eigensection of $\mathcal{D}_{2}^{\infty}$ on all of $M_{2}^{\infty}$. Then it follows at once that the discrete part of the spectrum of $\mathcal{D}_{2}^{\infty}$ is not empty, $\mu$ belongs to it, $\sqrt{\mu_{1}}$ is the smallest eigenvalue $>0$, and hence we have $\mu^{2}>\mu_{1} / 2$ as wanted.

In general, such a convenient extension of the given eigensection $s$ can not be achieved. But due to the spectral boundary condition satisfied by $s$ in the hypersurface $\{-R\} \times Y$, the eigensection $s$ over $M_{2}^{R}$ can be continuously extended by a section over $(-\infty,-R] \times Y$ on which the Dirac operator vanishes (see Figure 8.4.1a). By construction, both the enlargement $\alpha$ of the $L_{2}$ norm of $s$ by the chosen extension and the cosine $\beta$ of the angle between $s^{\infty}$ and $\operatorname{ker} \mathcal{D}_{2}^{\infty}$ can be estimated independently of the specific choice of $s$ and $\mu$. It turns out that they both decrease exponentially with growing $R$ (see Figure 8.4.1b).

Let $\left\{s_{1}, \ldots, s_{q}\right\}$ be an orthonormal basis of $\operatorname{ker} \mathcal{D}_{2}^{\infty}$ and set

$$
\widetilde{s}:=s^{\infty}-\sum_{j=1}^{q}\left\langle s^{\infty} ; s_{j}\right\rangle s_{j} .
$$

Clearly, the section $\widetilde{s}$ belongs to $\mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right)$ and is orthogonal to $\operatorname{ker} \mathcal{D}_{2}^{\infty}$. Hence, on one side,

$$
\begin{equation*}
\frac{\left\|\mathcal{D}_{2}^{\infty} \widetilde{s}\right\|^{2}}{\|\widetilde{s}\|^{2}} \geq \mu_{1} \tag{8.4.2}
\end{equation*}
$$

On the other side, we have by construction

$$
\left\|\mathcal{D}_{2}^{\infty} \widetilde{s}\right\|^{2}=\left\|\mathcal{D}_{2}^{\infty} s^{\infty}\right\|^{2}=\left\|\mathcal{D}_{2}^{R} s\right\|_{M_{2}^{R}}^{2}=\mu^{2}
$$

Finally, we shall prove that

$$
\begin{equation*}
\|\widetilde{s}\| \rightarrow 1 \text { as } R \rightarrow \infty \tag{8.4.3}
\end{equation*}
$$

Then the estimate

$$
\begin{equation*}
\mu^{2}>\frac{\mu_{1}}{2} \tag{8.4.4}
\end{equation*}
$$

follows for sufficiently large $R$. Since we have assumed that $\mu^{2}<\frac{\lambda_{1}^{2}}{2}$, we have also $\mu_{1}<\lambda_{1}^{2}$, hence $\mu_{1}$ belongs to the discrete part of the spectrum of $\left(\mathcal{D}_{2}^{\infty}\right)^{2}$ and, by the Min-Max Principle (see, for instance, [87]), must be its smallest eigenvalue $>0$.

Thus, to prove the theorem we are left with the task of first constructing a suitable extension $\widetilde{s}$ of $s$ and then proving (8.4.3).

We expand $\left.s\right|_{[-R, 0] \times Y}$ in terms of a spectral resolution

$$
\left\{\varphi_{k}, \lambda_{k} ; G \varphi_{k},-\lambda_{k}\right\}_{k \in \mathbf{N}}
$$

of $L_{2}(Y ; \mathcal{S})$ generated by $B$ :

$$
s(u, y)=\sum_{k=1}^{\infty} f_{k}(u) \varphi_{k}(y)+g_{k}(u) G \varphi_{k} .
$$

Since $\left.G\left(\partial_{u}+B\right) s\right|_{[-R, 0] \times Y}=0$, the coefficients must satisfy the system of ordinary differential equations

$$
\binom{f_{k}^{\prime}}{g_{k}^{\prime}}=\mathbf{A}_{k}\binom{f_{k}}{g_{k}} \quad \text { with } \mathbf{A}_{k}:=\left(\begin{array}{cc}
-\lambda_{k} & \mu  \tag{8.4.5}\\
-\mu & \lambda_{k}
\end{array}\right)
$$

Moreover, since $\left.\Pi_{>} s\right|_{\{-R\} \times Y}=0$ we have

$$
\begin{equation*}
f_{k}(-R)=0 \quad \text { for any } k \geq 1 \tag{8.4.6}
\end{equation*}
$$

Thus, for each $k$ the pair $\left(f_{k}, g_{k}\right)$ is uniquely determined up to a constant $a_{k}$. More explicitly, since the eigenvalues of $\mathbf{A}_{k}$ are $\pm\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}$, a suitable
choice of the eigenvectors of $\mathbf{A}_{k}$ gives

$$
\begin{gathered}
f_{k}(u)=a_{k} \frac{\mu}{\sqrt{\lambda_{k}^{2}-\mu^{2}}} \sinh \sqrt{\lambda_{k}^{2}-\mu^{2}}(R+u) \quad \text { and } \\
g_{k}(u)=a_{k}\left(\cosh \sqrt{\lambda_{k}^{2}-\mu^{2}}(R+u)+\frac{\lambda_{k}}{\sqrt{\lambda_{k}^{2}-\mu^{2}}} \sinh \left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}(R+u)\right)
\end{gathered}
$$

We assume $\|s\|_{L_{2}}=1$. Then we have, with $v:=\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}(R+u)$ :

$$
\begin{aligned}
1 \geq & \int_{[-R, 0] \times Y}|s(u, y)|^{2} d u d y=\sum_{k=1}^{\infty} \int_{-R}^{0}\left(\left|f_{k}(u)\right|^{2}+\left|g_{k}\right|^{2}\right) d u \\
= & \sum_{k=1}^{\infty}\left|a_{k}\right|^{2} \frac{1}{\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}} \int_{0}^{\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R}\left(\frac{\mu^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot \sinh ^{2} v\right. \\
& \left.\quad+\cosh ^{2} v+2 \frac{\lambda_{k}}{\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}} \cdot \cosh v \cdot \sinh v+\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot \sinh ^{2} v\right) d v \\
= & \sum_{k=1}^{\infty}\left|a_{k}\right|^{2}\left\{-\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot R+(1 / 4) \cdot \frac{\mu^{2}}{\left(\lambda_{k}^{2}-\mu\right)^{3 / 2}} \cdot \sinh \left(2\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R\right)\right. \\
& \quad+(1 / 4) \cdot\left(1+\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}}\right) \cdot\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} \cdot \sinh \left(2\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R\right) \\
& \left.\quad+\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot \cosh ^{2}\left(\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R\right)\right\} .
\end{aligned}
$$

Since $\lambda_{k}^{2} \geq \lambda_{1}^{2}>2 \mu^{2}$ we have $2\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}>\sqrt{2} \lambda_{k}>\lambda_{k}$. Moreover, we have for all $k \geq 1$

$$
\begin{aligned}
-\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot R & +\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot \frac{\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2}}{4} \cdot \sinh \left(2\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R\right) \\
& >-R+\frac{\left(\lambda_{1}^{2}-\mu^{2}\right)^{1 / 2}}{4} \cdot \sinh \left(2\left(\lambda_{1}^{2}-\mu^{2}\right)^{1 / 2} R\right) \\
& >-R+\frac{\sqrt{2}}{8} \lambda_{1} \cdot \sinh \left(\sqrt{2} \lambda_{1} R\right)>0,
\end{aligned}
$$

if $R \geq R_{0}$ for some positive $R_{0}$ which depends only on $\lambda_{1}$ and not on $\mu$ nor on $s$ nor on $k$.

Thus, for any $k$ the sum in the braces can be estimated in the following way:

$$
\{\ldots\}>\frac{\lambda_{k}^{2}}{\lambda_{k}^{2}-\mu^{2}} \cdot \cosh ^{2}\left(\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R\right)>\frac{1}{4} e^{2\left(\lambda_{k}^{2}-\mu^{2}\right)^{1 / 2} R}>\frac{1}{4} e^{\lambda_{k} R}
$$

Hence, we have

$$
\begin{equation*}
\sum_{k=1}^{\infty}\left|a_{k}\right|^{2} \cdot e^{\lambda_{k} R} \leq 4 \tag{8.4.7}
\end{equation*}
$$

Note that the preceding estimate does not depend on $R$ (provided that $R>R_{0}$ ) nor on $k$ nor on the specific choice of $s$, and that $R_{0}$ only depends on $\lambda_{1}$.

According to (8.4.7) the absolute value of the coefficients $a_{k}$ is rapidly decreasing in such a way that, in particular, we can extend the eigensection $s$ of $\mathcal{D}_{2, \Pi_{>}}^{R}$, given on $M_{2}^{R}$ to a continuous section on $M_{2}^{\infty}$ by the formula

$$
s^{\infty}(x):= \begin{cases}s(x) & \text { for } x \in M_{2}^{R} \\ \sum_{k=1}^{\infty} a_{k} e^{\lambda_{k}(R+u)} G \varphi_{k}(y) & \text { for } x=(u, y) \in(-\infty,-r] \times Y\end{cases}
$$

By construction, $s^{\infty}$ is smooth on $M_{2}^{\infty} \backslash(\{-R\} \times Y)$ and belongs to the Sobolev space $\mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right)$. It follows from (8.4.7) that

$$
\begin{align*}
\left\|s^{\infty}\right\|_{L_{2}}^{2} & =\|s\|_{L_{2}}^{2}+\sum_{k=1}^{\infty}\left|a_{k}\right|^{2} \int_{-\infty}^{-R} e^{2 \lambda_{k}(R+u)} d u=1+\sum_{k=1}^{\infty}\left|a_{k}\right|^{2} \frac{1}{2 \lambda_{k}} \\
& \leq 1+\frac{1}{2 \lambda_{1}} \cdot \sum_{k=1}^{\infty}\left|a_{k}\right|^{2} \leq 1+\frac{1}{2 \lambda_{1}} \cdot\left(\sum_{k=1}^{\infty}\left|a_{k}\right|^{2} \cdot e^{\lambda_{k} R}\right) \cdot e^{-\lambda_{1} R} \\
& \leq 1+\frac{2}{\lambda_{1}} \cdot e^{-\lambda_{1} R} . \tag{8.4.8}
\end{align*}
$$

Next, let $\Psi \in \operatorname{ker} \mathcal{D}_{2}^{\infty}$ and assume that $\|\Psi\|=1$. By (8.3.6), the section $\Psi$ has on $(-\infty, 0] \times Y$ the form

$$
\begin{equation*}
\Psi\left((u, y)=\sum_{k=1}^{\infty} b_{k} e^{\lambda_{k} u} G(y) \varphi_{k}(y)\right. \tag{8.4.9}
\end{equation*}
$$

with

$$
\sum \int_{-\infty}^{0}\left|b_{k}\right|^{2} e^{2 \lambda_{k} u} d u=\sum \frac{1}{2 \lambda_{k}} \cdot\left|b_{k}\right|^{2}<+\infty
$$

Set $l:=\left.\Psi\right|_{M_{2}^{R}}$. Then $l$ satisfies the equations

$$
\mathcal{D}_{2}^{R} l=0 \text { and } \Pi_{>}\left(\left.l\right|_{\{-R\} \times Y}\right)=0
$$

Hence, $l$ belongs to $\operatorname{ker} \mathcal{D}_{2, \Pi\rangle}^{R}$. This implies the following equality:

$$
\begin{aligned}
& \int_{M_{2}^{R}}\left\langle s^{\infty}(x) ; \Psi(x)\right\rangle d x=\frac{1}{\mu} \cdot \int_{M_{2}^{R}}\left\langle\mathcal{D}_{2}^{R} s^{\infty}(x) ; l(x)\right\rangle d x \\
&=\frac{1}{\mu} \cdot \int_{M_{2}^{R}}\left\langle s^{\infty}(x) ; \mathcal{D}_{2}^{R} l(x)\right\rangle d x-\frac{1}{\mu} \cdot \int_{Y}\left\langle G s^{\infty}(-R, y) ; l(-R, y)\right\rangle d x .
\end{aligned}
$$

On the other hand,

$$
\int_{(-\infty,-r] \times Y}\left\langle s^{\infty}(x) ; \Psi(x)\right\rangle d x=\sum_{k>0} \frac{a_{k} \overline{{b_{k}}_{k}}}{2 \mu} \cdot e^{-\lambda_{k} R} \leq C_{1} e-\lambda_{1} R .
$$

Therefore

$$
\begin{equation*}
\left|\left\langle s^{\infty} ; \Psi\right\rangle\right| \leq C_{1} e^{-\lambda_{1} R} . \tag{8.4.10}
\end{equation*}
$$

Hence, we have proved

Lemma 8.4.1. Any eigensection $s \in \mathcal{H}^{1}\left(M_{2}^{R} ; \mathcal{S}\right)$ of $\mathcal{D}_{2, \Pi}^{R}$, with eigenvalue $\mu \in\left(-\lambda_{1} / \sqrt{2}, \lambda_{1} / \sqrt{2}\right)$ can be extended to a continuous section $s^{\infty}$ on $M_{2}^{\infty}$ which is smooth on $M_{2}^{\infty} \backslash(\{-R\} \times Y)$ and belongs to the first Sobolev space $\mathcal{H}^{1}\left(M_{2}^{\infty} ; \mathcal{S}\right)$. Moreover, the enlargement of the norm of $s$ by the extension and the cosine of the angle between $s^{\infty}$ and $\operatorname{ker} \mathcal{D}_{2}^{\infty}$ are exponentially decreasing by formulae (8.4.8) and (8.4.10).

The final step in proving the theorem follows at once from the preceding lemma. We recall: by definition of $\mu_{1}$ we have $\left\|\mathcal{D}_{2}^{\infty} \widetilde{s}\right\|^{2} /\|\widetilde{s}\|^{2} \geq \mu_{1}$ and by construction of $\widetilde{s}$ we have $\left\|\mathcal{D}_{2}^{\infty} \widetilde{s}\right\|^{2}=\mu^{2}$. Thus, we have $\mu^{2} \geq\|\widetilde{s}\|^{2} \cdot \mu_{1}$. To establish the wanted bound $\mu^{2}>\mu_{1} / 2$, it remains to show that $\|\widetilde{s}\|^{2}>1 / 2$ for sufficiently large $R$.

Since $\widetilde{s}$ is the orthogonal projection of $s^{\infty}$ onto $\left(\operatorname{ker} \mathcal{D}_{2}^{\infty}\right)^{\perp}$ and by the orthonormality of the basis $\left\{s_{1}, \ldots, s_{q}\right\}$ of ker $\mathcal{D}_{2}^{\infty}$, we have

$$
\|\widetilde{s}\|^{2}=\left\|s^{\infty}\right\|^{2}-\sum_{j=1}^{q}\left|\left\langle s^{\infty} ; s_{j}\right\rangle\right|^{2} \leq 1+\frac{2}{\lambda_{1}} e^{-\lambda_{1} R}-\sum_{j=1}^{q}\left|\left\langle s^{\infty} ; s_{j}\right\rangle\right|^{2}
$$

Thus,

$$
\left|\|\widetilde{s}\|^{2}-1\right| \leq \frac{2}{\lambda_{1}} e^{-\lambda_{1} R}+q C_{1} e^{-2 \lambda_{1} R} \leq C_{2} e^{-C_{3} R}
$$

which proves the theorem.

We finish this section by proving the asymptotic estimate (8.2.11). Recall that $\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$ denotes the kernel of the operator $\mathcal{D}_{2, \Pi}^{R} e^{-t\left(\mathcal{D}_{2, \Pi}^{R}\right)^{2}}$ where $\mathcal{D}_{2, \Pi\rangle}^{R}$, denotes the Dirac operator over the manifold $M_{2}^{R}$ with the spectral boundary condition at the boundary $\{-R\} \times Y$. Then we have:

Lemma 8.4.2.

$$
\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}_{2}^{R}(t ; x, x) d x=\mathcal{O}\left(e^{-c R}\right)
$$

Proof. For any eigenvalue $\mu \neq 0$ of $\mathcal{D}_{2, \Pi\rangle}^{R}$ and $R>0$ sufficiently large ( $R \cdot c_{0}^{2} \geq 1$, where $c_{0}$ denotes the lower uniform bound for $\mu^{2}$ of Theorem 8.2.7) we have

$$
\begin{align*}
\left|\int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \mu e^{-t \mu^{2}} d t\right| & \leq \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}}|\mu| e^{-t \mu^{2}} d t=\int_{|\mu| R^{1 / 4}}^{\infty} e^{-\tau^{2}} d \tau  \tag{8.4.11}\\
& \leq \int_{|\mu| R^{1 / 4}}^{\infty} \tau e^{-\tau^{2}} d \tau=\left[-\frac{1}{2} e^{-\tau^{2}}\right]_{|\mu| R^{1 / 4}}^{\infty} \frac{1}{2} e^{-\mu^{2} \sqrt{R}}
\end{align*}
$$

which gives

$$
\begin{aligned}
& \left\lvert\, \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}}\right. \left.\int_{M_{2}^{R}} \operatorname{Tr}\left(\mathcal{D}_{2, \Pi\rangle}^{R} e^{-t\left(\mathcal{D}_{2, \Pi}^{R}\right)^{2}}\right)\left|\leq \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \sum_{\mu \neq 0}\right| \mu \right\rvert\, e^{-t \mu^{2}} d t \\
& \quad \leq \frac{1}{2} \cdot \sum_{\mu \neq 0} e^{-\mu^{2} \sqrt{R}}=\frac{1}{2} \cdot \sum_{\mu \neq 0} e^{-(\sqrt{R}-1) \mu^{2}} \cdot e^{-\mu^{2}} \\
& \quad \leq C_{1} \cdot e^{-\sqrt{R} \mu_{0}^{2}} \operatorname{Tr}\left(e^{-\left(\mathcal{D}_{2, \Pi>}^{R}\right)^{2}}\right) \leq C_{2} \cdot e^{-\sqrt{R} \mu_{0}^{2}} \operatorname{vol}\left(M_{2}^{R}\right) \\
& \leq C_{3} \cdot e^{-\sqrt{R} \mu_{0}^{2}} \leq C_{3} \cdot e^{-C_{4} \sqrt{R}}
\end{aligned}
$$

Here we have exploited that the heat kernel $\mathrm{e}_{2}^{R}\left(t ; x, x^{\prime}\right)$ of the operator $\mathcal{D}_{2, \Pi\rangle}^{R}$ can be estimated by

$$
\left|\mathrm{e}_{2}^{R}\left(t ; x, x^{\prime}\right)\right| \leq c_{1} \cdot t^{-\frac{m}{2}} \cdot e^{c_{2} t} \cdot e^{-c_{3} \frac{d^{2}\left(x, x^{\prime}\right)}{t}}
$$

according to (8.2.3). Thus,

$$
\begin{equation*}
\left|\operatorname{Tr}\left(e^{-\left(\mathcal{D}_{2, \Pi\rangle}^{R}\right)^{2}}\right)\right| \leq \int_{M_{2}^{R}}\left|\operatorname{tr} \mathrm{e}_{2}^{R}(1 ; x, x)\right| d x \leq c_{1} \cdot e^{c_{2}} \cdot \int_{M_{2}^{R}} d x . \tag{8.4.12}
\end{equation*}
$$

### 8.5. The Spectrum on the Closed Stretched Manifold

So far we proved the asymptotic equation

$$
\frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}^{R}(t ; x, x) d x+\mathcal{O}\left(e^{-c R}\right)=\eta_{\mathcal{D}_{2, \Pi>}^{R}}(0)
$$

as $R \rightarrow \infty$. It follows that

$$
\lim _{R \rightarrow \infty} \eta_{R}=\lim _{R \rightarrow \infty}\left(\eta_{\mathcal{D}_{1, \Pi_{<}}^{R}}(0)+\eta_{\mathcal{D}_{2, \Pi_{>}}^{R}}(0)\right),
$$

where

$$
\eta_{R}:=\frac{1}{\sqrt{\pi}} \int_{0}^{\sqrt{R}} \frac{d t}{\sqrt{t}} \int_{M^{R}} \operatorname{tr} \mathcal{E}^{R}(t ; x, x) d x
$$

To prove Theorem 8.1.2, we still have to show (8.2.12), i.e., that we can extend the integration from $\sqrt{R}$ to infinity:

$$
\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{d t}{\sqrt{t}} \int_{M_{2}^{R}} \operatorname{tr} \mathcal{E}^{R}(t ; x, x) d x=\mathcal{O}\left(e^{-c R}\right) \text { as } R \rightarrow \infty .
$$

Recall that $\mathcal{E}^{R}\left(t ; x, x^{\prime}\right)$ denotes the kernel of the operator $\mathcal{D}^{R} e^{-t\left(\mathcal{D}^{R}\right)^{2}}$ on the closed stretched manifold $M^{R}$.

Formally, our task of proving the preceding estimate reminds of our previous task of proving the corresponding estimate for the kernel $\mathcal{E}_{2}^{R}\left(t ; x, x^{\prime}\right)$ of the operator $\mathcal{D}_{2}^{R} e^{-t\left(\mathcal{D}_{2, \Pi}^{R}\right)^{2}}$ (see Lemma 8.4.2). Both integrals are over the same prolonged compact manifold $M_{2}^{R}$ with boundary $\{-R\} \times Y$. The methods we can apply are, however, different: In the previous case, we had a
uniform positive bound for the absolute value of the smallest non-vanishing eigenvalue of the boundary value problem $\mathcal{D}_{2, \Pi\rangle}^{R}$ for sufficiently large $R$.

As mentioned above in Remark 8.2.8, such a bound does not exist for the Dirac operator $\mathcal{D}^{R}$ on the closed stretched manifold $M^{R}$. Moreover, for the spectral boundary condition we shall show

$$
\operatorname{dim} \operatorname{ker} \mathcal{D}_{2, \Pi_{>}}^{R}=\operatorname{dim} \operatorname{ker} \mathcal{D}_{2, \Pi_{>}} \text {and } \eta_{\mathcal{D}_{2, \Pi\rangle}^{R}}(0)=\eta_{\mathcal{D}_{2, \Pi\rangle}}(0)
$$

for any $R$ (see Proposition 8.6.2 below). For $\mathcal{D}^{R}$, on the contrary, the dimension of the kernel can change and, thus, $\eta_{\mathcal{D}^{R}}$ can admit an integer jump in value as $R \rightarrow \infty$. This is due to the presence of 'small' eigenvalues created by $L_{2}$-solutions of the operators $\mathcal{D}_{1}^{\infty}$ and $\mathcal{D}_{2}^{\infty}$ on the half-manifolds with cylindrical ends. We use a straightforward analysis of small eigenvalues inspired by the proof of Theorem 8.2.7 to prove the following result

Theorem 8.5.1. There exists $R_{0}>0$ and positive constants $a_{1}, a_{2}$, and $a_{3}$, such that for any $R>R_{0}$, the eigenvalue $\mu$ of the operator $\mathcal{D}^{R}$ is either bounded away from 0 with $a_{1}<|\mu|$, or is exponentially small $|\mu|<a_{2} e^{-a_{3} R}$. Let $\mathcal{W}^{R}$ denote the subspace of $L_{2}\left(M^{R} ; \mathcal{S}\right)$ spanned by the eigensections of $\mathcal{D}^{R}$ corresponding to the exponentially small eigenvalues. Then $\operatorname{dim} \mathcal{W}^{R}=$ $q$, where $q=\operatorname{dim}\left(\operatorname{ker} \mathcal{D}_{1}^{\infty}\right)+\operatorname{dim}\left(\operatorname{ker} \mathcal{D}_{2}^{\infty}\right)$.

Recall from Proposition 8.3.6 that the operator $\mathcal{D}_{j}^{\infty}$, acting on the first Sobolev space $\mathcal{H}^{1}\left(M_{j}^{\infty} ; \mathcal{S}\right)$, is an (unbounded) self-adjoint Fredholm operator in $L_{2}\left(M_{j}^{\infty} ; \mathcal{S}\right)$ which has a discrete spectrum in the interval $\left(-\lambda_{1},+\lambda_{1}\right)$ where $\lambda_{1}$ denotes the smallest positive eigenvalue of the tangential operator $B$. Thus, the space $\operatorname{ker} \mathcal{D}_{j}^{\infty}$ of $L_{2}$-solutions is of finite dimension.

To prove the theorem we first investigate the small eigenvalues of the operator $\mathcal{D}^{R}$ and the pasting of $L_{2}$-solutions. Let $R>0$. We re-parametrize the normal coordinate $u$ such that $M_{1}^{R}=M_{1} \cup((-R, 0] \times Y)$ and $M_{2}^{R}=$ $([0, R) \times Y) \cup M_{2}$. We introduce the subspace $\mathcal{V}^{R} \subset L_{2}\left(M^{R} ; \mathcal{S}\right)$ spanned by $L_{2}$-solutions of the operators $\mathcal{D}_{j}^{\infty}$. We choose an auxiliary smooth real function $f^{R}=f_{1}^{R} \cup f_{2}^{R}$ on $M^{R}$. We assume that $f^{R}$ is equal 1 outside the cylinder $[-R, R] \times Y$, and $f^{R}$ is a function of the normal variable $u$ on the cylinder. Moreover, $f^{R}(-u)=f^{R}(u)$, or, in other words, $f_{1}^{R}(-u)=f_{2}^{R}(u)$. We assume that $f_{2}^{R}$ is an increasing function of $u$ such that

$$
f_{2}^{R}(u)=\left\{\begin{array}{ll}
0 & \text { for } 0 \leq u \leq \frac{R}{4} \\
1 & \text { for } \frac{R}{2} \leq u \leq R
\end{array} .\right.
$$

We also assume that there exists a constant $\gamma>0$ such that $\left|\frac{\partial^{p} f_{2}^{R}}{\partial u^{p}}(u)\right|<$ $\gamma R^{-p}$. If $s_{j} \in C^{\infty}\left(M_{j}^{\infty} ; \mathcal{S}\right)$, we define $s_{1} \cup_{f^{R}} s_{2}$ by the formula

$$
\left(s_{1} \cup_{f^{R}} s_{2}\right)(x):= \begin{cases}f_{1}^{R}(x) s_{1}(x) & \text { for } x \in M_{1}^{R} \\ f_{2}^{R}(x) s_{2}(x) & \text { for } x \in M_{2}^{R}\end{cases}
$$

Clearly, we have

$$
\begin{align*}
s_{1} \cup_{f^{R}} s_{2} & =s_{1} \cup_{f^{R}} 0+0 \cup_{f^{R}} s_{2}  \tag{8.5.1}\\
\mathcal{D}^{R}\left(s_{1} \cup_{f R} s_{2}\right) & =\left(\mathcal{D}_{1}^{\infty} s_{1}\right) \cup_{f^{R}}\left(\mathcal{D}_{2}^{\infty} s_{2}\right)+s_{1} \cup_{g^{R}} s_{2} \text { and }  \tag{8.5.2}\\
\left\|s_{1} \cup_{f^{R}} s_{2}\right\|^{2} & =\left\|s_{1} \cup_{f^{R}} 0\right\|^{2}+\left\|0 \cup_{f^{R}} s_{2}\right\|^{2}, \tag{8.5.3}
\end{align*}
$$

where $g^{R}:=g_{1}^{R} \cup g_{2}^{R}$ with $g_{j}^{R}(u, y)=G(y) \frac{\partial f_{j}^{R}}{\partial u}(u, y)$ and $\|\cdot\|$ denotes the $L_{2}$-norm on the manifold $M^{R}$.

Definition 8.5.2. We denote by $\mathcal{V}^{R}$ the subspace of $C^{\infty}\left(M^{R} ; \mathcal{S}\right)$ defined by

$$
\mathcal{V}^{R}:=\operatorname{span}\left\{s_{1} \cup_{f R} s_{2} \mid s_{j} \in \operatorname{ker} \mathcal{D}_{j}^{\infty}\right\}
$$

Let $\left\{s_{1,1}, \ldots, s_{1, q_{1}}\right\}$ be a basis of $\operatorname{ker} \mathcal{D}_{1}^{\infty}$ and $\left\{s_{2,1}, \ldots, s_{2, q_{2}}\right\}$ a basis of $\operatorname{ker} \mathcal{D}_{2}^{\infty}$. Then the $q=q_{1}+q_{2}$ sections $\left\{s_{1, \nu_{1}} \cup_{f R} 0\right\} \cup\left\{0 \cup_{f R} s_{2, \nu_{2}}\right\}$ form a basis of $\mathcal{V}^{R}$. We want to show that $\mathcal{V}^{R}$ approximates the space $\mathcal{W}^{R}$ of eigensections of $\mathcal{D}^{R}$ corresponding to the 'small' eigenvalues, for $R$ sufficiently large. We begin with an elementary result:

Lemma 8.5.3. There exists $R_{0}$ such that for any $R>R_{0}$ and any $s \in \mathcal{V}^{R}$, the following estimate holds

$$
\left\|\mathcal{D}^{R} s\right\| \leq e^{-\lambda_{1} R}\|s\|
$$

Proof. It suffices to prove the estimate for basis sections of $\mathcal{V}^{R}$. Thus, let $s=s_{1} \cup_{f^{R}} 0$ with $s_{1} \in \operatorname{ker} \mathcal{D}_{1}^{\infty}$. By (8.5.2) we have

$$
\mathcal{D}^{R} s(x)= \begin{cases}0 & \text { for } x \in M_{1} \cup M_{2}  \tag{8.5.4}\\ G(y) \frac{\partial f_{1}^{R}}{\partial u}(u, y) \cdot s_{1}(u, y) & \text { for } x=(u, y) \in[-R, R] \times Y .\end{cases}
$$

Here $f_{1}^{R}$ is continued in a trivial way on the whole cylinder $[-R, R] \times Y$. Now, $s_{1}$ is a $L_{2}$-solution of $\mathcal{D}_{1}^{\infty}$, hence $s_{1}(u, y)=\sum_{k} c_{k} e^{-(R+u) \lambda_{k}} \varphi_{k}(y)$ on this cylinder where $\left\{\varphi_{k}, \lambda_{k} ; G \varphi_{k},-\lambda_{k}\right\}_{k \in \mathbf{N}}$ is, as above, a spectral resolution
of $L_{2}(Y ; \mathcal{S})$ generated by $B$. We estimate the norm of $\mathcal{D}^{R} S$ :

$$
\begin{aligned}
\left\|\mathcal{D}^{R} S\right\|^{2} & =\left\|G \frac{\partial f_{1}^{R}}{\partial u} \cdot s_{1}\right\|^{2} \\
& =\sum_{k} \int_{-\frac{R}{2}}^{-\frac{R}{4}} \int_{Y}\left(\frac{\partial f_{1}^{R}}{\partial u}\right)^{2} \cdot\left|c_{k}\right|^{2} \cdot e^{-2(R+u) \lambda_{k}}\left(\varphi_{k}(y) ; \varphi_{k}(y)\right) d y d u \\
& =\int_{-\frac{R}{2}}^{-\frac{R}{4}}\left(\frac{\partial f_{1}^{R}}{\partial u}\right)^{2} \cdot \sum_{k}\left|c_{k}\right|^{2} \cdot e^{-2(R+u) \lambda_{k}} \cdot 1 \cdot d u \\
& \leq \frac{\gamma^{2}}{R^{2}} \cdot \sum_{k}\left(\left|c_{k}\right|^{2} \cdot \int_{-\frac{R}{2}}^{-\frac{R}{4}} e^{-2(R+u) \lambda_{k}} d u\right) \\
& =\frac{\gamma^{2}}{R^{2}} \cdot \sum_{k}\left(\left|c_{k}\right|^{2} \cdot \int_{R \lambda_{k}}^{\frac{3}{2} R \lambda_{k}} e^{-v} \frac{d v}{2 \lambda_{k}}\right) \\
& \leq \frac{\gamma^{2}}{R^{2}} \cdot \sum_{k}\left|c_{k}\right|^{2} \cdot \frac{e^{-R \lambda_{k}}-e^{-\frac{3}{2} R \lambda_{k}}}{2 \lambda_{k}} \\
& \leq \frac{\gamma^{2}}{R^{2}} \cdot \sum_{k} \frac{e^{-R \lambda_{k}}}{2 \lambda_{k}}\left|c_{k}\right|^{2} \leq \frac{\gamma^{2}}{R^{2}} e^{-R \lambda_{1}} \cdot \sum_{k} \frac{\left|c_{k}\right|^{2}}{2 \lambda_{k}} .
\end{aligned}
$$

On the other hand we have the elementary inequality

$$
\begin{aligned}
&\|s\|^{2}=\left\|s_{1} \cup_{f^{R}} 0\right\|^{2} \geq \int_{-R}^{-R+1} \int_{Y}\left|s_{1}(u, y)\right|^{2} d y d u \\
&=\sum\left|c_{k}\right|^{2} \cdot \frac{1-e^{-2 \lambda_{k}}}{2 \lambda_{k}} \geq d \cdot \sum \frac{\left|c_{k}\right|^{2}}{2 \lambda_{k}}
\end{aligned}
$$

with $0<d \leq 1-e^{-2 \lambda_{1}}$. Thus, we have the following estimate for any $s \in \mathcal{V}^{R}$ of the form $s_{1} \cup_{f^{R}} 0$ and for sufficiently large $R$

$$
\left\|\mathcal{D}^{R} s\right\|^{2} \leq \frac{\gamma^{2}}{R^{2} d} e^{-R \lambda_{1}} \cdot d \cdot \sum_{k} \frac{\left|c_{k}\right|^{2}}{2 \lambda_{k}} \leq \frac{\gamma^{2}}{R^{2} d} e^{-R \lambda_{1}} \cdot\|s\|^{2} \leq e^{-R \lambda_{1}} \cdot\|s\|^{2}
$$

For $s=0 \cup_{f^{R}} s_{2}$, we estimate the norm of $\mathcal{D}_{R} s$ in the same way, taking regard that $s_{2}$ has the form $s_{2}(u, y)=\sum_{k} d_{k} e^{(u+R) \lambda_{k}} G(y) \varphi_{k}(y)$ on the cylinder.

Let $\left\{\rho_{k} ; \psi_{k}\right\}$ denote a spectral decomposition of the space $L_{2}\left(M^{R} ; \mathcal{S}\right)$ generated by the operator $\mathcal{D}^{R}$. For $a>0$, let $P_{a}$ denote the orthogonal projection onto the space $\mathcal{H}_{a}:=\operatorname{span}\left\{\psi_{k}| | \rho_{k} \mid>a\right\}$.

Lemma 8.5.4. For sufficiently large $R$, the following estimate holds for any $s \in \mathcal{V}^{R}$

$$
\left\|\left(\operatorname{Id}-P_{e^{-\frac{R \lambda_{1}}{4}}}\right)\right\| \leq e^{-\frac{R \lambda_{1}}{2}} \cdot\|s\| .
$$

Proof. We represent $s$ as the series $s=\sum_{k} a_{k} \psi_{k}$. We have

$$
\begin{aligned}
\left\|\operatorname{Id}-P_{e^{-\frac{R \lambda_{1}}{4}}}\right\| & =\sum_{\rho_{k}^{2}>e^{-R \lambda_{1} / 2}} a_{k}^{2} \leq \sum_{\rho_{k}^{2}>e^{-R \lambda_{1} / 2}} e^{\frac{R \lambda_{1}}{2}} \cdot \rho_{k}^{2} a_{k}^{2} \\
& \leq \sum^{\frac{R \lambda_{1}}{2}} \cdot \rho_{k}^{2} a_{k}^{2}=e^{\frac{R \lambda_{1}}{2}}\left\|\mathcal{D}^{R} s\right\|^{2} \\
& \leq e^{\frac{R \lambda_{1}}{2}} e^{-R \lambda_{1}}\|s\|^{2}=e^{-\frac{R \lambda_{1}}{2}}\|s\|^{2}
\end{aligned}
$$

Proposition 8.5.5. The spectral projection $P_{e^{-\frac{R \lambda_{1}}{4}}}$ restricted to the subspace $\mathcal{V}^{R}$ is an injection. In particular, $\mathcal{D}^{R}$ has at least $q$ eigenvalues $\rho$ such that $|\rho| \leq e^{-\frac{R \lambda_{1}}{4}}$, where $q$ is the sum of the dimensions of the spaces $\operatorname{ker} \mathcal{D}_{j}^{\infty}$ of $L_{2}$-solutions of the operators $\mathcal{D}_{1}^{\infty}$ and $\mathcal{D}_{2}^{\infty}$.

Proof. Let $s \in \mathcal{V}^{R}$, and assume that $P_{e^{-R \lambda_{1} / 4}}(s)=0$. We have

$$
\|s\|=\left\|\left(\operatorname{Id}-P_{e^{-\frac{R \lambda_{1}}{4}}}\right) s\right\| \leq e^{-\frac{R \lambda_{1}}{2}} \cdot\|s\| \leq \frac{1}{2}\|s\|
$$

for $R$ sufficiently large.

The proposition shows that the operator $\mathcal{D}^{R}$ has at least $q$ exponentially small eigensections, which we can approximate by pasting together $L_{2}$-solutions. Now we will show that this makes the list of eigenvalues approaching 0 as $R \rightarrow+\infty$ complete.

Let $\psi$ be an eigensection of $\mathcal{D}^{R}$ corresponding to an eigenvalue $\mu$, where $|\mu|<\lambda_{1}$. Like in the proof of Theorem 8.2.7 we expand $\left.\psi\right|_{[-R, R] \times Y}$ in terms of a spectral resolution

$$
\left\{\varphi_{k}, \lambda_{k} ; G \varphi_{k},-\lambda_{k}\right\}_{k \in \mathbf{N}}
$$

of $L_{2}(Y ; \mathcal{S})$ generated by $B$ :

$$
\psi(u, y)=\sum_{k=1}^{\infty} f_{k}(u) \varphi_{k}(y)+g_{k}(u) G \varphi_{k}
$$

where the coefficients must satisfy the system of ordinary differential equations of (8.4.5)

$$
\binom{f_{k}^{\prime}}{g_{k}^{\prime}}=\mathbf{A}_{k}\binom{f_{k}}{g_{k}} \quad \text { with } \mathbf{A}_{k}:=\left(\begin{array}{cc}
-\lambda_{k} & \mu \\
-\mu & \lambda_{k}
\end{array}\right)
$$

For the eigenvalues $\pm \sqrt{\lambda_{k}^{2}-\mu^{2}}$ of $\mathbf{A}_{k}$ and the eigenvectors

$$
\binom{\lambda_{k}+\sqrt{\lambda_{k}^{2}-\mu^{2}}}{\mu} \text { and }\binom{\mu}{\lambda_{k}+\sqrt{\lambda_{k}^{2}-\mu^{2}}}
$$

we get a natural splitting of $\psi(u, y)$ in the form $\psi(u, y)=\psi_{+}(u, y)+\psi_{-}(u, y)$ with

$$
\begin{aligned}
& \psi_{+}(u, y)=\sum_{k} a_{k} e^{-\sqrt{\lambda_{k}^{2}-\mu^{2}} u}\left\{\left(\lambda_{k}+\sqrt{\lambda_{k}^{2}-\mu^{2}}\right) \varphi_{k}(y)+\mu G(y) \varphi_{k}(y)\right\}, \text { and } \\
& \psi_{-}(u, y)=\sum_{k} b_{k} e^{\sqrt{\lambda_{k}^{2}-\mu^{2}} u}\left\{\mu \varphi_{k}(y)+\left(\lambda_{k}+\sqrt{\lambda_{k}^{2}-\mu^{2}}\right) G(y) \varphi_{k}(y)\right\} .
\end{aligned}
$$

Then we have the following estimate of the $L_{2}$ - norm of $\psi$ in the $y-$ direction on the cylinder:

Lemma 8.5.6. Assume that $\|\psi\|=1$. There exist positive constants $c_{1}, c_{2}$ such that $\left\|\left.\psi\right|_{\{u\} \times Y}\right\| \leq c_{1} e^{-c_{2} R}$ for $-\frac{3}{4} R \leq u \leq \frac{3}{4} R$.

Proof. We have

$$
\begin{aligned}
& \left\|\left.\psi_{+}\right|_{\{-R+r\} \times Y}\right\|^{2} \\
& \quad \leq e^{-2 r \sqrt{\lambda_{k}^{2}-\mu^{2}}} \cdot\left\|\sum_{k} a_{k} e^{-R \sqrt{\lambda_{k}^{2}-\mu^{2}}}\left\{\left(\lambda_{k}+\sqrt{\lambda_{k}^{2}-\mu^{2}}\right) \varphi_{k}+\mu G \varphi_{k}\right\}\right\|^{2} \\
& \\
& \quad=e^{-2 r \sqrt{\lambda_{k}^{2}-\mu^{2}}} \cdot\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|^{2} .
\end{aligned}
$$

In the same way we get

$$
\left\|\left.\psi_{-}\right|_{\{R-r\} \times Y}\right\|^{2} \leq e^{-2 r \sqrt{\lambda_{k}^{2}-\mu^{2}}} \cdot\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\|^{2} .
$$

Let us observe that, in fact, the used argument proves that

$$
\begin{aligned}
& \left\|\left.\psi_{+}\right|_{\{r\} \times Y}\right\| \leq e^{-(r-s)} \sqrt{\lambda_{k}^{2}-\mu^{2}} \cdot\left\|\left.\psi_{+}\right|_{\{s\} \times Y}\right\|, \text { and } \\
& \left\|\left.\psi_{-}\right|_{\{s\} \times Y}\right\| \leq e^{-(r-s) \sqrt{\lambda_{k}^{2}-\mu^{2}}} \cdot\left\|\left.\psi_{-}\right|_{\{r\} \times Y}\right\|,
\end{aligned}
$$

for any $-R<s<r<R$. We also have another elementary inequality

$$
\left\|\left.\psi\right|_{\{r\} \times Y}\right\|^{2} \geq\left\|\left.\psi_{+}\right|_{\{r\} \times Y}\right\|^{2}-2 \cdot\left\|\left.\psi_{+}\right|_{\{r\} \times Y}\right\| \cdot\left\|\left.\psi_{-}\right|_{\{r\} \times Y}\right\| \text {. }
$$

This helps estimate the $L_{2}$-norm of $\psi_{ \pm}$in the $y$-direction. We have

$$
\begin{aligned}
\|\psi\|^{2} \geq & \int_{-R}^{-R+1}\left\|\left.\psi\right|_{\{u\} \times Y}\right\|^{2} d u \\
\geq & \int_{-R}^{-R+1}\left(\left\|\left.\psi_{+}\right|_{\{u\} \times Y}\right\|^{2}-2\left\|\left.\psi_{+}\right|_{\{u\} \times Y}\right\|\left\|\left.\psi_{-}\right|_{\{u\} \times Y}\right\|\right) d u \\
\geq & \left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|^{2} \\
& -2 \int_{-R}^{-R+1}\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\| e^{-2 R \sqrt{\lambda_{k}^{2}-\mu^{2}}}\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\| d u \\
\geq & \left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|^{2}-2 e^{-2 R \sqrt{\lambda_{k}^{2}-\mu^{2}}}\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\| .
\end{aligned}
$$

In the same way we have

$$
\|\psi\|^{2} \geq\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\|^{2}-2 e^{-2 R \sqrt{\lambda_{k}^{2}-\mu^{2}}}\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\| .
$$

We add the last two inequalities and use

$$
2\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\| \leq\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|^{2}+\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\|^{2}
$$

to obtain

$$
2\|\psi\|^{2} \geq\left(1-e^{-2 R \sqrt{\lambda_{k}^{2}-\mu^{2}}}\right)\left(\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|^{2}+\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\|^{2}\right) .
$$

This gives us the inequality we need, namely

$$
\left\|\left.\psi_{ \pm}\right|_{\{\mp R\} \times Y}\right\|^{2} \leq 4\|\psi\|^{2} .
$$

Now we finish the proof of the lemma.

$$
\begin{aligned}
& \left\|\left.\psi\right|_{\{u\} \times Y}\right\|=\left\|\left.\psi_{+}\right|_{\{u\} \times Y}+\left.\psi_{-}\right|_{\{u\} \times Y}\right\| \\
& \quad \leq e^{-(u+R)} \sqrt{\lambda_{k}^{2}-\mu^{2}}\left\|\left.\psi_{+}\right|_{\{-R\} \times Y}\right\|+e^{-(R-u)} \sqrt{\lambda_{k}^{2}-\mu^{2}}\left\|\left.\psi_{-}\right|_{\{R\} \times Y}\right\| \\
& \quad \leq 2\left(e^{-(u+R) \sqrt{\lambda_{k}^{2}-\mu^{2}}}+e^{-(R-u) \sqrt{\lambda_{k}^{2}-\mu^{2}}}\right)\|\psi\| \leq c_{1} e^{-c_{2} R},
\end{aligned}
$$

for certain positive constants $c_{1}, c_{2}$ when $-\frac{3}{4} R \leq u \leq \frac{3}{4} R$.
We are ready to state the technical main result of this section.

Theorem 8.5.7. Let $\psi$ denote an eigensection of the operator $\mathcal{D}^{R}$ corresponding to an eigenvalue $\mu$, where $|\mu|<\lambda_{1}$. Assume that $\psi$ is orthogonal to the subspace $P_{e^{-R \lambda_{1} / 4}} \mathcal{V}^{R} \subset L_{2}\left(M^{R}, \mathcal{S}\right)$. Then there exists a positive constant $c$ such that $|\lambda|>c$.

To prove the theorem we may assume that $\|\psi\|=1$. We begin with an elementary consequence of Lemma 8.5.4.

Lemma 8.5.8. For any $s \in \mathcal{V}^{R}$ we have

$$
|\langle\psi ; s\rangle| \leq e^{-\frac{R \lambda_{1}}{2}}\|s\| .
$$

Proof. We have

$$
\begin{aligned}
|\langle\psi ; s\rangle| & =\left|\left\langle\psi ; P_{e^{-\frac{R \lambda_{1}}{4}}}(s)+\left(P_{e^{-R \lambda_{1} / 4}}\right)(s)\right\rangle\right|=\left|\left\langle\psi ;\left(P_{e^{-\frac{R \lambda_{1}}{4}}}\right)(s)\right\rangle\right| \\
& \leq\|\psi\|\left\|\left(P_{e^{-\frac{R \lambda_{1}}{4}}}\right)(s)\right\| \leq e^{-\frac{R \lambda_{1}}{2}}\|s\| .
\end{aligned}
$$

We want to compare $\psi$ with the eigensections on the corresponding manifolds with cylindrical ends. We use $\psi$ to construct a suitable section on $M_{2}^{\infty}=((-\infty, R] \times Y) \cup M_{2}$ (Note the re-parametrization compared with the convention chosen in the beginning of this chapter). Let $h: M_{2}^{\infty} \rightarrow \mathbf{R}$ be a smooth increasing function such that $h$ is equal to 1 on $M_{2}$ and $h$ is a function of the normal variable on the cylinder, equal to 0 for $u \leq \frac{1}{2} R$, and equal to 1 for $\frac{3}{4} R \leq u$. We also assume, as usual, that $\left|\frac{\partial^{p} h}{\partial u^{p}}\right| \leq \gamma R^{-p}$ for a certain constant $\gamma>0$. We define

$$
\psi_{2}^{\infty}(x):= \begin{cases}h(x) \psi(x) & \text { for } x \in M_{2}^{R} \\ 0 & \text { for } x \in(-\infty, 0] \times Y\end{cases}
$$

Proposition 8.5.9. There exist positive constants $c_{1}, c_{2}$ such that

$$
\left|\left\langle\psi_{2}^{\infty} ; s\right\rangle\right| \leq c_{1} e^{-c_{2} R}\|s\|
$$

for any $s \in \operatorname{ker} \mathcal{D}_{2}^{\infty}$.

Proof. For a suitable cut-off function $f_{2}^{R}$ we have

$$
\begin{aligned}
& \left|\left\langle\psi_{2}^{\infty} ; s\right\rangle\right|=\left|\int_{M_{2}^{\infty}}\left(\psi_{2}^{\infty}(x) ; s(x)\right) d x\right|=\left|\int_{M_{2}^{R}}\left(h(x) \psi(x) ; f_{2}^{R}(x) s(x)\right) d x\right| \\
& \quad \leq\left|\int_{M_{2}^{R}}\left(\psi(x) ; f_{2}^{R}(x) s(x)\right) d x\right|+\left|\int_{M_{2}^{R}}\left((1-h(x)) \psi(x) ; f_{2}^{R}(x) s(x)\right) d x\right| .
\end{aligned}
$$

We use Lemma 8.5.8 to estimate the first summand:

$$
\begin{aligned}
\left|\int_{M_{2}^{R}}\left(\psi(x) ; f_{2}^{R}(x) s(x)\right) d x\right| & =\left|\int_{M_{2}^{R}}\left(\psi(x) ;\left(0 \cup_{f R} s\right)(x)\right) d x\right| \\
& =\left|\left\langle\psi ; 0 \cup_{f^{R}} s\right\rangle\right| \leq e^{-\frac{R \lambda_{1}}{2}}\|s\| .
\end{aligned}
$$

We use Lemma 8.5.6 to estimate the second summand:

$$
\begin{aligned}
\mid \int_{M_{2}^{R}}((1-h(x)) & \left.\psi(x) ; f_{2}^{R}(x) s(x)\right) d x \mid \\
& \leq \int_{M_{2}^{R}}\left|\left((1-h(x)) \psi(x) ; f_{2}^{R}(x) s(x)\right)\right| d x \\
& \leq \int_{M_{2}^{R}}\|(1-h(x)) \psi(x)\|\left\|f_{2}^{R}(x) s(x)\right\| d x \\
& \leq\left(\int_{M_{2}^{R}}\|(1-h(x)) \psi(x)\|^{2} d x\right)^{\frac{1}{2}}\|s\| \\
& \leq\left(\int_{0}^{\frac{3}{4} R}\left\|\left.\psi\right|_{\{u\} \times Y}\right\|^{2}\right)^{\frac{1}{2}}\|s\| d u \\
& \leq\left(c_{1}^{2} e^{-2 c_{2} R} \frac{3}{4} R\right)^{\frac{1}{2}}\|s\| \leq c_{3} e^{-c_{4} R}\|s\|
\end{aligned}
$$

Proof of Theorem 8.5.7. Now we estimate $\mu^{2}$ from below by following the proof of Theorem 8.2.7. We choose

$$
\left\{s_{k}\right\}_{k=1}^{q_{2}}
$$

an orthonormal basis of the kernel of the operator $\mathcal{D}_{2}^{\infty}$. Let us define

$$
\widetilde{\psi}:=\psi_{2}^{\infty}-\sum_{k=1}^{q_{2}}\left\langle\psi_{2}^{\infty} ; s_{k}\right\rangle s_{k}
$$

Then $\tilde{\psi}$ is orthogonal to $\operatorname{ker} \mathcal{D}_{2}^{\infty}$ and it follows from Proposition 8.5.9 that $\|\widetilde{\psi}\| \geq \frac{1}{3}\left\|\psi_{2}^{\infty}\right\|>\kappa>0$ for $R$ large enough where $\kappa$ is independent of $R$, of the specific choice of the eigensection $\psi$, and of the cut-off function $h$. Let $\mu_{1}{ }^{2}$ denote the smallest non-zero eigenvalue of the operator $\left(\mathcal{D}_{2}^{\infty}\right)^{2}$. Once again, it follows from the Min-Max Principle, that $\left\langle\left(\mathcal{D}_{2}^{\infty}\right)^{2} \widetilde{\psi} ; \widetilde{\psi}\right\rangle \geq \mu^{2} \kappa^{2}$. We have

$$
\begin{aligned}
\mu^{2} & =\left\langle\left(\mathcal{D}^{R}\right)^{2} \psi ; \psi\right\rangle=\int_{M^{R}}\left\|\left(\mathcal{D}^{R} \psi\right)(x)\right\|^{2} d x \geq \int_{M_{2}^{R}}\left\|\left(\mathcal{D}^{R} \psi\right)(x)\right\|^{2} d x \\
& =\int_{M_{2}^{R}}\left\|\mathcal{D}^{R}(h(x) \psi(x)+(1-h(x)) \psi(x))\right\|^{2} d x \\
& \geq \int_{M_{2}^{\infty}}\left\|\left(\mathcal{D}_{2}^{\infty} \psi_{2}^{\infty}\right)(x)\right\|^{2} d x-\int_{M_{2}^{R}}\left\|\mathcal{D}^{R}((1-h) \psi)(x)\right\|^{2} d x
\end{aligned}
$$

It is not difficult to estimate the first term from below. We have

$$
\int_{M_{2}^{\infty}}\left\|\left(\mathcal{D}_{2}^{\infty} \psi_{2}^{\infty}\right)(x)\right\|^{2} d x=\left\langle\left(\mathcal{D}_{2}^{\infty}\right)^{2} \psi_{2}^{\infty} ; \psi_{2}^{\infty}\right\rangle=\left\langle\left(\mathcal{D}_{2}^{\infty}\right)^{2} \widetilde{\psi} ; \widetilde{\psi}\right\rangle \geq \mu_{1}^{2} \kappa^{2}
$$

We estimate the second term as follows:

$$
\begin{aligned}
& \int_{M_{2}^{R}}\left\|\mathcal{D}^{R}((1-h) \psi)(x)\right\|^{2} d x \\
& \quad=\int_{M_{2}^{R}}\left\|(1-h(x))\left(\mathcal{D}^{R} \psi\right)(x)-G(x) \frac{\partial h}{\partial u}(x) \psi(x)\right\|^{2} d x \\
& \leq
\end{aligned}
$$

Now we use Lemma 8.5.6 successively to estimate each summand on the right side by $c_{1} e^{-c_{2} R}$. This gives us

$$
\int_{M_{2}^{R}}\left\|\mathcal{D}^{R}((1-h) \psi)(x)\right\|^{2} d x \leq c_{3} e^{-c_{4} R}
$$

and finally we have $\mu^{2} \geq \mu_{1}^{2} \kappa^{2}-c_{3} e^{-c_{4} R} \geq \frac{\mu_{1}^{2} \kappa^{2}}{2}$, for $R$ large enough.
Theorem 8.5.1 is an easy consequence of Theorem 8.5.7.

### 8.6. The Additivity for Spectral Boundary Conditions

We finish the proof of Theorem 8.1.2. We still have to show equation (8.2.12), i.e.

Lemma 8.6.1. We have $\eta^{R}=\mathcal{O}\left(e^{-c R}\right)$ where

$$
\eta^{R}:=\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\mathcal{D}^{R} e^{-t \mathcal{D}_{R}^{2}}\right) d t
$$

Proof. It follows from Theorem 8.5.1 that we have

- 'exponentially small' eigenvalues corresponding to the eigensections from the subspace $\mathcal{W}^{R}$
- and the eigenvalues $\mu$ bounded away from 0 , with $|\mu| \geq a_{1}$, corresponding to the eigensections from the orthogonal complement of $\mathcal{W}^{R}$.

First we show that we can neglect the contribution due to the eigenvalues that are bounded away from 0 . We are precisely in the same situation as with the large time asymptotic of the corresponding integral for the Atiyah-Patodi-Singer boundary problem on the half manifold with the cylinder attached. Literally, we can repeat the proof of Lemma 8.4.2 by replacing $\mathcal{D}_{2, \Pi}^{R}$, by $\mathcal{D}^{R}$ and the uniform bound for the smallest positive eigenvalue of $\mathcal{D}_{2, \Pi\rangle}^{R}$ by our present bound $a_{1}$. Thus, we have

$$
\begin{aligned}
& \left|\int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\left.\mathcal{D}^{R} e^{-t \mathcal{D}_{R}^{2}}\right|_{\left(\mathcal{W}^{R}\right)^{\perp}}\right) d t\right| \leq \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}}\left\{\sum_{|\mu| \geq a_{1}}|\mu| e^{-t \mu^{2}}\right\} d t \\
& \leq \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}}\left\{\sum_{|\mu| \geq a_{1}} e^{-(t-1) \mu^{2}}\right\} d t \leq \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}}\left\{\sum_{|\mu| \geq a_{1}} e^{-\mu^{2}}\right\} e^{-(t-2) a_{1}^{2}} d t \\
& \leq e^{2 a_{1}^{2}} \operatorname{Tr}\left(e^{-t \mathcal{D}_{R}^{2}}\right) \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} e^{-t a_{1}^{2}} d t=e^{2 a_{1}^{2}} \operatorname{Tr}\left(e^{-t \mathcal{D}_{R}^{2}}\right) \frac{1}{a_{1}} \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} e^{-t a_{1}^{2}} a_{1} d t \\
& \leq \frac{e^{2 a_{1}^{2}}}{2 a_{1}} \operatorname{Tr}\left(e^{-t \mathcal{D}_{R}^{2}}\right) e^{-a_{1}^{2} \sqrt{R}} .
\end{aligned}
$$

For the last inequality see (8.4.11). A standard estimate on the heat kernel of the operator $\mathcal{D}^{R}$ gives (like in (8.4.12)) the inequality $\operatorname{Tr}\left(e^{-t \mathcal{D}_{R}^{2}}\right) \leq b_{3}$. $\operatorname{vol}\left(M^{R}\right) \leq b_{4} R$, which implies that

$$
\begin{equation*}
\left|\int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\left.\mathcal{D}^{R} e^{-t \mathcal{D}_{R}^{2}}\right|_{\left(\mathcal{W}^{R}\right)^{\perp}}\right) d t\right| \leq b_{5} e^{-b_{6} \sqrt{R}} \tag{8.6.1}
\end{equation*}
$$

That proves that the contribution from the large eigenvalues disappears as $R \rightarrow \infty$. The essential part of $\eta^{R}$ comes from the subspace $\mathcal{W}^{R}$ :

$$
\begin{align*}
& \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\left.\mathcal{D}^{R} e^{-t \mathcal{D}_{R}^{2}}\right|_{\mathcal{W}^{R}}\right) d t  \tag{8.6.2}\\
& =\sum_{|\mu|<a_{1}} \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \mu e^{-t \mu^{2}} d t=\sum_{|\mu|<a_{1}} \operatorname{sign}(\mu) \frac{2}{\sqrt{\pi}} \int_{|\mu| R^{1 / 4}}^{\infty} e^{-v^{2}} d v
\end{align*}
$$

It follows from Theorem 8.5.1 that $\lim _{R \rightarrow \infty}|\mu| R^{1 / 4}=0$. Thus, the right side of (8.6.2) is equal to

$$
\operatorname{sign}_{R}(\mathcal{D}):=\sum_{|\mu|<a_{1}} \operatorname{sign}(\mu)
$$

plus the smooth error term which is rapidly decreasing as $R \rightarrow \infty$.
Thus we have proved Theorem 8.1.2. In particular, we have proved

$$
\lim _{R \rightarrow \infty} \eta_{\mathcal{D}^{R}}(0) \equiv \lim _{R \rightarrow \infty}\left\{\eta_{\mathcal{D}_{1, \Pi_{<}}^{R}}(0)+\eta_{\mathcal{D}_{2, \Pi_{>}}^{R}}(0)\right\} \quad \bmod \mathbf{Z}
$$

To establish the true additivity assertion of Corollary 8.1.3, we show that the preceding $\eta$-invariants do not depend on $R$ modulo integers.

Proposition 8.6.2. (W. Müller.) The $\eta$-invariant $\eta_{\mathcal{D}_{2, \Pi}^{R}}(0) \in \mathbf{R} / \mathbf{Z}$ is independent of the cylinder length $R$.

Proof. Near to the boundary of $M_{2}^{R}$ we parametrize the normal coordinate $u \in[-R, 1)$ with the boundary at $u=-R$. First we show that $\operatorname{dim} \operatorname{ker} \mathcal{D}_{2, \Pi\rangle}^{R} \quad$ is independent of $R$. Let $s \in \operatorname{ker} \mathcal{D}_{2, \Pi_{>}}^{R}$. This is equivalent to

$$
\begin{equation*}
s \in C^{\infty}\left(M_{2}^{R} ; \mathcal{S}\right), \quad \mathcal{D}_{2}^{R} s=0, \text { and } \Pi_{>}\left(\left.s\right|_{\{-R\} \times Y}\right)=0 \tag{8.6.3}
\end{equation*}
$$

As in equation (8.3.6) (and in equation (8.4.9) of the proof of Theorem 8.2.7) we may expand $\left.s\right|_{[-R, 0] \times Y}$ in terms of the eigensections of the tangential operator $B$ :

$$
s(u, y)=\sum_{k=1}^{\infty} e^{\lambda_{k} u} G(y) \varphi_{k}(y)
$$

Let $R^{\prime}>R$. Then $s$ can be extended in the obvious way to $\widetilde{s} \in \operatorname{ker} \mathcal{D}_{2, \Pi_{>}}^{R^{\prime}}$, and the map $s \mapsto \widetilde{s}$ defines an isomorphism of $\operatorname{ker} \mathcal{D}_{2, \Pi\rangle}^{R}$ onto $\operatorname{ker} \mathcal{D}_{2, \Pi\rangle}^{R_{>}^{\prime}}{ }^{2}$.

Next, observe that there exists a smooth family of diffeomorphism $f_{R}$ : $[0,1) \rightarrow[-R, 1)$ which have the following cut-off properties

$$
f_{R}(u)= \begin{cases}u & \text { for } \frac{2}{3}<u<1 \\ u+R & \text { for } 0 \leq u<\frac{1}{3}\end{cases}
$$

Let $\psi_{R}:[0,1) \times Y \rightarrow[-R, 1) \times Y$ be defined by $\psi_{R}(u, y):=\left(f_{R}(u), y\right)$, and extend $\psi_{R}$ to a diffeomorphism $\psi_{R}: M_{2} \rightarrow M_{2}^{R}$ in the canonical
way, i.e., $\psi_{R}$ becomes the identity on $M_{2} \backslash((0,1) \times Y)$. There is also a bundle isomorphism which covers $\psi_{R}$. This induces an isomorphism $\psi_{R}^{*}: C^{\infty}\left(M_{2}^{R} ; \mathcal{S}\right) \rightarrow C^{\infty}\left(M_{2} ; \mathcal{S}\right)$. Let $\widetilde{\mathcal{D}}_{2}^{R}:=\psi_{R}^{*} \circ \mathcal{D}_{2}^{R} \circ\left(\psi_{R}^{*}\right)^{-1}$. Then $\left\{\widetilde{\mathcal{D}}_{2}^{R}\right\}_{R}$ is a family of Dirac operators on $M_{2}$, and $\widetilde{\mathcal{D}}_{2}^{R}=G\left(\partial_{u}+B\right)$ near $Y$. We pick the self-adjoint $L_{2}$-extension defined by dom $\widetilde{\mathcal{D}}_{2, \Pi_{>}}^{R}:=\psi_{R}^{*}\left(\operatorname{dom} \mathcal{D}_{2, \Pi_{>}}^{R}\right)$. Hence,

$$
\eta_{\widetilde{\mathcal{D}}_{2, \Pi>}^{R}}(s)=\eta_{\mathcal{D}_{2, \Pi\rangle}^{R}}(s) \text { and } \operatorname{ker} \widetilde{\mathcal{D}}_{2, \Pi_{>}}^{R}=\psi_{R}^{*} \operatorname{ker} \mathcal{D}_{2, \Pi_{>}}^{R}
$$

In particular, $\operatorname{dim} \widetilde{\mathcal{D}}_{2, \Pi\rangle}^{R}$ is constant, and we apply our variation formula Proposition ??? (identical with the formula 8.1.5 of Remark 8.1.3 - though established only for closed manifolds and families of invertible Dirac operators in Chapter 8) to get

$$
\frac{d}{d R}\left(\eta_{\mathcal{D}_{2, \Pi}^{R}}(0)\right)=-\frac{2}{\sqrt{\pi}} c_{m}(R)
$$

where $c_{m}(R)$ is determined by the asymptotic expansion ???.
Now let $S_{R}^{1}$ denote the circle of radius $2 R$. We lift the Clifford bundle from $Y$ to the torus $\mathbf{T}_{R}:=S_{R}^{1} \times Y$. We define the action of $\widehat{D}_{R}$ : $C^{\infty}\left(\mathbf{T}_{R}, \mathcal{S}\right) \rightarrow C^{\infty}\left(\mathbf{T}_{R}, \mathcal{S}\right)$ by $\widehat{D}^{R}=G\left(\partial_{u}+B\right)$. Since $c_{m}(R)$ is locally computable, it follows in the same way as above that

$$
\frac{d}{d R}\left(\eta_{\widehat{D}^{R}}(0)\right)=-\frac{2}{\sqrt{\pi}} c_{m}(R) .
$$

But a direct computation shows that the spectrum of $\widehat{D}^{R}$ is symmetric. Hence $\eta_{\widehat{D}^{R}}(s)=0$ and, therefore, $c_{m}(R)=0$.

In the same way we show that $\eta_{\mathcal{D}^{R}}$ is independent of $R$. This proves the additivity assertion of Corollary 8.1.3. We have proved, in fact, a little bit more:

ThEOREM 8.6.3. The following formula holds for $R$ large enough

$$
\eta_{\mathcal{D}}(0)=\eta_{\mathcal{D}_{1, \Pi_{<}}}(0)+\eta_{\mathcal{D}_{2, \Pi_{>}}}(0)+\operatorname{sign}_{R}(\mathcal{D}) .
$$

Theorem 8.6.3 has an immediate corollary which describes the case in which our additivity formula holds in $\mathbf{R}$, not just in $\mathbf{R} / \mathbf{Z}$.

Corollary 8.6.4. If $\operatorname{ker} \mathcal{D}_{1}^{\infty}=\{0\}=\operatorname{ker} \mathcal{D}_{2}^{\infty}$, then

$$
\eta_{\mathcal{D}}(0)=\eta_{\mathcal{D}_{1, \Pi_{<}}}(0)+\eta_{\mathcal{D}_{2, \Pi}}(0) .
$$

## CHAPTER 9

## $\eta$-Invariant and Variation of the Boundary Condition

1
In this section we study the variation of the $\eta$-invariant. First we recall the variation formula for a smooth 1-parameter family of Dirac operators over a closed manifold. Then we fix one Dirac operator and study a path in the connected Grassmannian of self-adjoint boundary conditions of generalized Atiyah-Patodi-Singer type, starting at the Atiyah-Patodi-Singer boundary condition. We apply the variation formula for the $\eta$-invariant and determine the difference between the two $\eta$ invariants. It turns out that, modulo integers, this real number is precisely the $\eta$-invariant of the finite cylinder subject to the given boundary condition on one end and the Atiyah-PatodiSinger boundary condition on the other end. Together with the adiabatic Duhamel's formula of the preceding Chapter this yields the general additivity formula

$$
\eta_{\mathcal{D}}=\eta_{\mathcal{D}_{1}, P_{1}}+\eta_{\mathcal{D}_{2},-G\left(\mathrm{Id}-P_{2}\right) G}+\eta_{P_{1}, P_{2}}^{\mathcal{N}}
$$

for the $\eta$-invariant on a closed partitioned manifold $M=$ $M_{1} \cup_{Y} M_{2}$ with $M_{1}, M_{2}$ compact manifolds with common boundary $Y$. Generalizations for singular tangential operators and non-compatible Dirac operators are given.

### 9.1. Variation of $\eta$ on a Closed Manifold

In this section we study the variation of the $\eta$-invariant on a closed manifold. Let $\left\{\mathcal{D}_{r}\right\}_{r \in[0,1]}$ be a smooth 1 -parameter family of Dirac operators. To begin with we assume that all of them are compatible. Afterwards we show what to do for non-compatible operators.

We recall the formula for large $\Re(s)$

$$
\begin{equation*}
\eta_{\mathcal{D}}(s)=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr}\left(\mathcal{D} e^{-t \mathcal{D}^{2}}\right) d t \tag{9.1.1}
\end{equation*}
$$

For compatible Dirac operators we have derived (see Theorem A.0.5) the estimate

$$
\begin{equation*}
\left|\operatorname{Tr} \mathcal{D} e^{-t \mathcal{D}^{2}}\right|<C \sqrt{t} \tag{9.1.2}
\end{equation*}
$$

[^5]for small $t>0$. Hence the $\eta$-function is a holomorphic function of $s$ for Re $s>-2$, and we get
\[

$$
\begin{equation*}
\eta_{\mathcal{D}}(0)=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\mathcal{D} e^{-t \mathcal{D}^{2}}\right) d t . \tag{9.1.3}
\end{equation*}
$$

\]

In the following computations we assume that the operators $\mathcal{D}_{r}$ are invertible. It was explained earlier that we can modify the operator $\mathcal{D}_{r}$ to an invertible operator which modulo integers does not change the value of the $\eta$-function at $s=0$. Therefore, in general, the formula of Lemma 9.1.1 holds only mod $Z$.

Now we differentiate equation (9.1.3) with respect to the parameter $r$ (we write shorthand $\mathcal{D}$ for $\mathcal{D}_{r}$ and dot for differentiation with regard to $r$ ):

$$
\begin{aligned}
\dot{\eta}_{\mathcal{D}_{r}}(0)= & \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}-2 t \dot{\mathcal{D}} \mathcal{D}^{2} e^{-t \mathcal{D}^{2}}\right) d t \\
= & \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t+\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \sqrt{t} \frac{d}{d t}\left(\operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}\right) d t \\
= & \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t+\frac{2}{\sqrt{\pi}}\left[\sqrt{t} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}\right]_{0}^{\infty} \\
& \quad-\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{2 \sqrt{t}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t \\
= & \frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0}\left[\sqrt{t} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}\right]_{\varepsilon}^{1 / \varepsilon}=-\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr} \dot{\mathcal{D}} e^{-\varepsilon \mathcal{D}^{2}} .
\end{aligned}
$$

Note that the vanishing of $\lim _{t \rightarrow \infty} \sqrt{t} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}$ follows from the assumption of invertibility of $\mathcal{D}$. This proves

Lemma 9.1.1. Let $M$ be a closed Riemannian manifold. Set $I:=[0,1]$ and let $\left\{\mathcal{D}_{r}\right\}_{r \in I}$ be a smooth family of invertible compatible Dirac operators on $M$. Then

$$
\frac{d}{d r} \eta_{\mathcal{D}_{r}}(0)=-\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr} \dot{\mathcal{D}}_{r} e^{-\varepsilon \mathcal{D}_{r}^{2}}
$$

For not necessarily compatible Dirac operators the estimate (9.1.2) is not available, and thus nor is the formula (9.1.3). Therefore, to determine the variation of $\eta(0)$ we go back to $\eta(s)$ for $\Re(s)$ sufficiently large, differentiate with regard to the parameter $r$, exploit the asymptotic expansion of the corresponding heat kernel traces, and then study the analytic extension to the point $s=0$. For the technical details see Gilkey [45], Section 1.13.

That way, we get

$$
\begin{aligned}
& \dot{\eta}_{\mathcal{D}_{r}}(s)= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr}\left(\dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}-2 t \dot{\mathcal{D}} \mathcal{D}^{2} e^{-t \mathcal{D}^{2}}\right) d t \\
&= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t \\
& \quad+\frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s+1}{2}} \frac{d}{d t}\left(\operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}\right) d t \\
&= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t+\frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \underbrace{\left[t^{\frac{s+1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}\right]_{0}^{\infty}}_{=0} \\
& \quad-\frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} \frac{s+1}{2} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t \\
&=-\frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t .
\end{aligned}
$$

By splitting the integral into a meromorphic part and a part holomorphic in 0 we obtain

$$
\begin{aligned}
\lim _{s \rightarrow 0} \dot{\eta}_{\mathcal{D}_{r}}(s) & =\lim _{s \rightarrow 0}-\frac{s}{\Gamma\left(\frac{s+1}{2}\right)}\{\int_{0}^{1} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t+\underbrace{\int_{1}^{\infty} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t}_{\text {bounded }}\} \\
& =\lim _{s \rightarrow 0}-\frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{1} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t
\end{aligned}
$$

Since the $\eta$-invariant $\eta_{\mathcal{D}_{r}}(0)$ depends on $r$ in a differentiable way, the preceding limit must be finite and equal to the variation $\dot{\eta}_{\mathcal{D}_{r}}(0)$.

We can determine the limit in terms of the heat kernel expansion. Assume that for any $r$ the operator $\mathcal{D}_{r}-\mathcal{D}_{0}$ is of order 0 , i.e. $\mathcal{D}_{r}=\mathcal{D}_{0}+T_{r}$, where $T_{r}: \mathbb{S} \rightarrow \$$ is a smooth family of endomorphisms of the spinor bundle. Then $\dot{\mathcal{D}}=\dot{T}$ and we have the following asymptotic expansion for small $t>0$

$$
\begin{equation*}
\operatorname{Tr} \dot{\mathcal{D}}_{r} e^{-t \mathcal{D}_{r}{ }^{2}} \sim t^{-\frac{m}{2}} \sum_{\iota=0}^{\infty} t^{\iota} b_{\iota}\left(\mathcal{D}_{r}\right), \tag{9.1.4}
\end{equation*}
$$

where $m:=\operatorname{dim} M$. We consider a single term $t^{\frac{-m+2 \iota}{2}} b_{\iota}$.

$$
\begin{aligned}
\int_{0}^{1} t^{\frac{s-1}{2}} \cdot t^{\frac{-m+2 \iota}{2}} b_{\iota} & =\int_{0}^{1} t^{\frac{s-1-m+2 \iota}{2}} b_{\iota}=\left[\frac{t^{s+1-m+2 \iota}}{\frac{s+1-m+2 \iota}{2}} b_{\iota}\right]_{0}^{1} \\
& =\frac{2 b_{\iota}}{s+1-m+2 \iota}
\end{aligned}
$$

Thus we have

$$
\lim _{s \rightarrow 0} \dot{\eta}_{\mathcal{D}_{r}}(s)=-\lim _{s \rightarrow 0} \frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \sum_{\iota=0}^{\infty} \frac{2 b_{\iota}}{s+1-m+2 \iota}=-\frac{2 b_{\frac{m-1}{2}}}{\sqrt{\pi}}
$$

because in the limit $s \rightarrow 0$ the denominator $1-m+2 \iota$ vanishes, if and only if $\iota=\frac{m-1}{2}$. This proves

Lemma 9.1.2. Let $M$ be a closed Riemannian manifold of dimension $m$. Let $\left\{\mathcal{D}_{r}\right\}_{r \in I}$ be a differentiable family of invertible Dirac operators on $M$. Then we have

$$
\begin{aligned}
\dot{\eta}_{\mathcal{D}_{r}}(0) & =-\lim _{s \rightarrow 0} \frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{1} t^{\frac{s-1}{2}} \operatorname{Tr} \dot{\mathcal{D}} e^{-t \mathcal{D}^{2}} d t \\
& =\left\{\begin{array}{cl}
-2 b_{\frac{m-1}{2}}\left(\mathcal{D}_{r}\right) / \sqrt{\pi} & \text { if } m \text { is odd }, \\
0 & \text { if } m \text { is even. }
\end{array}\right.
\end{aligned}
$$

Remark 9.1.3. Assume that $\dot{\mathcal{D}}$ is of the first order. In this case the corresponding expansion is

$$
\begin{equation*}
\operatorname{Tr} \dot{\mathcal{D}}_{r} e^{-t \mathcal{D}_{r}{ }^{2}} \sim t^{-\frac{m+1}{2}} \sum_{\iota=0}^{\infty} t^{(2 \iota+1) / 2} c_{2 \iota+1}\left(\mathcal{D}_{r}\right), \tag{9.1.5}
\end{equation*}
$$

and the result is

$$
\dot{\eta}_{\mathcal{D}_{r}}(0)=\left\{\begin{array}{cl}
-2 c_{m}\left(\mathcal{D}_{r}\right) / \sqrt{\pi} & \text { if } m \text { is odd, } \\
0 & \text { if } m \text { is even }
\end{array}\right.
$$

### 9.2. Variation of the Boundary Condition

In this section $\mathcal{D}$ denotes a fixed compatible Dirac operator on a smooth compact Riemannian manifold $X$ of odd dimension $m$ with boundary $Y$ and with product structure in a fixed collar neighbourhood $\mathcal{N}$ of $Y$.

Our calculation of the change of the $\eta$-invariant under a smooth variation of the boundary condition will be based on the following formula which has been proved in chapter ??? of this book.

Lemma 9.2.1. Let $\left\{P_{r}\right\}_{0 \leq r \leq 1}$ be a smooth path of projections in the smooth self-adjoint Grassmannian $\mathcal{G r}_{\infty}^{*}(\mathcal{D})$. We assume that all operators $\mathcal{D}_{r}:=$ $\mathcal{D}_{P_{r}}$ are invertible. Then we have

$$
\frac{d}{d r}\left(\eta_{\mathcal{D}_{r}}(0)\right)=-\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr} \dot{\mathcal{D}}_{r} e^{-\varepsilon \mathcal{D}_{r}^{2}}
$$

Let $P$ be a fixed projection in $\mathcal{G r}_{\infty}^{*}$. Without loss of generality (see Simon's observation on the domains in Chapter ???) we can assume that a smooth curve

$$
\left\{P_{r}=g_{r}^{\#} \Pi_{>}\left(g_{r}^{\#}\right)^{-1}\right\}_{r \in I}
$$

connects the spectral projection $\Pi_{>}$with $P$. Here $g_{r}: L_{2}\left(Y ;\left(\$_{Y}\right)^{-}\right) \rightarrow$ $L_{2}\left(Y ;\left(\left.\mathbb{S}\right|_{Y}\right)^{-}\right)$is a unitary operator of the form $\operatorname{Id}+K_{r}$ where $K_{r}$ is an operator with a smooth kernel and

$$
g_{r}^{\#}:=\left(\begin{array}{cc}
\operatorname{Id} & 0 \\
0 & g_{r}
\end{array}\right)
$$

and $g_{r}=\mathrm{Id}$, constant, for small $r(r$ close to 0$)$, and $g_{r}=g_{1}$ for large $r(r$ close to 1).

Example 9.2.2. Note that the operators with smooth kernel form the Lie algebra of the group

$$
\mathrm{GL}_{\infty}:=\{g \text { invertible } \mid g=\mathrm{Id}+\text { operator with smooth kernel }\} .
$$

Therefore, for a small real parameter $r \geq 0$, it would suffice to consider curves generated by a family

$$
\begin{equation*}
\left\{g_{r}:=e^{r \theta}\right\} \tag{9.2.1}
\end{equation*}
$$

where

$$
\theta: C^{\infty}\left(Y,\left.\$\right|_{Y}\right) \longrightarrow C^{\infty}\left(Y,\left.\$\right|_{Y}\right)
$$

is a fixed operator with smooth kernel specifying the 'direction' of the derivative, and then applying a cut-off function.

A technical problem arises, namely that the domain of the corresponding $L_{2}$-extensions $\mathcal{D}_{P_{r}}$ changes. To avoid dealing with this problem we apply a unitary twist. We replace the operator $\mathcal{D}_{r}:=\mathcal{D}_{P_{r}}$ by a unitarily equivalent operator $\widetilde{\mathcal{D}}_{\Pi_{>}}^{(r)}$. In that way all the original spectra are retained, and the domain of the operators $\widetilde{\mathcal{D}}_{\Pi_{>}}^{(r)}$ is now fixed.

This method was introduced in [40], Appendix A and used in a number of related papers (see [60], [62], [?]). It can be formulated as follows:

Lemma 9.2.3. For all $r \in I$ the operator $\mathcal{D}_{r}$ is unitarily equivalent with the operator $\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi_{>}}$, where

$$
U_{r}:= \begin{cases}\operatorname{Id} & \text { on } M \backslash \mathcal{N} \\ g_{r \cdot f(u)}^{\#} & \text { on } \mathcal{N}\end{cases}
$$

and $f$ is a smooth monoton-decreasing function equal to 0 close to $u=1$ and 1 near $u=0$.

Proof. By definition $U_{r}$ provides a bijection of $\operatorname{dom}\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi_{>}}$onto $\operatorname{dom} \mathcal{D}_{r}$; and for each $s \in \operatorname{dom}\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi_{>}}$the action of first applying $U_{r}$, then $\mathcal{D}$, and then $U_{r}^{-1}$ coincides precisely with the action of $U_{r}^{-1} \mathcal{D} U_{r}$.

Instead of analyzing the operators $\mathcal{D}_{r}, \dot{\mathcal{D}_{r}}$, and $\dot{\mathcal{D}}_{r} e^{-\varepsilon \mathcal{D}_{r}{ }^{2}}$, we analyze the corresponding unitarily equivalent operators obtained by the unitary twist with the chosen cut-off function $f(u)$. We shall denote the mapping $(u, r, y) \mapsto g_{r \cdot f(u)}(y)$ by the same letter $g$ and the partial derivative $\partial_{r}\left(g_{r \cdot f(u)}\right)$ by $\dot{g}_{r}$. Then we obtain

$$
\dot{g}_{r \cdot f(u)}^{-1}=-g_{r \cdot f(u)}^{-1} \dot{g}_{r \cdot f(u)} g_{r \cdot f(u)}^{-1}
$$

as derivative of the inverse family, and

$$
\left(g^{-1} \frac{\partial g}{\partial u}\right)=-g^{-1} \dot{g} g^{-1} \frac{\partial g}{\partial u}+g^{-1}\left(\frac{\dot{\partial} g}{\partial u}\right) .
$$

Proposition 9.2.4. (a) On the set $M \backslash \mathcal{N}$ the operator $U_{r}^{-1} \mathcal{D} U_{r}$ coincides with $\mathcal{D}$, while on the collar $\mathcal{N}$ we get

$$
\begin{aligned}
& U_{r}^{-1} \mathcal{D} U_{r}-\mathcal{D} \\
= & \left(\begin{array}{cc}
0 & 0 \\
0 & g_{r \cdot f(u)}^{-1} \partial_{u}\left(g_{r \cdot f(u)}\right)
\end{array}\right)+\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right)\left(\begin{array}{cc}
0 & B_{-}\left(g_{r \cdot f(u)}-\mathrm{Id}\right) \\
\left(g_{r \cdot f(u)}^{-1}-\mathrm{Id}\right) B_{+} & 0
\end{array}\right)
\end{aligned}
$$

(b) The first derivate of the twisted family vanishes on $M \backslash \mathcal{N}$, while we get on the collar $\mathcal{N}$

$$
\left(U_{r}^{-i} \mathcal{D} U_{r}\right)=\left(\begin{array}{cc}
0 & 0 \\
0 & -i\left(g^{-i} \frac{\partial g}{\partial u}\right)
\end{array}\right)+\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right)\left(\begin{array}{cc}
0 & B^{-} \dot{g} \\
-g^{-1} \dot{g} g^{-1} B^{+} & 0
\end{array}\right)
$$

Remark 9.2.5. The difference between the twisted operator and the original operator is a smoothing operator in 'tangential' direction (i.e. in the direction parallel with the boundary) and is an operator of order zero in normal direction; hence it is not a true pseudodifferential operator.

Proof. We prove (a) by straight forward calculation.

$$
\begin{aligned}
& U_{r}^{-1} \mathcal{D} U_{r} \\
&=\left(\begin{array}{cc}
1 & 0 \\
0 & g^{-1}
\end{array}\right)\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right)\left\{\partial_{u}+\left(\begin{array}{cc}
0 & B^{-} \\
B^{+} & 0
\end{array}\right)\right\}\left(\begin{array}{ll}
1 & 0 \\
0 & g
\end{array}\right) \\
&=\left(\begin{array}{cc}
i & 0 \\
0 & -i g^{-1}
\end{array}\right) \partial_{u}\left(\begin{array}{ll}
1 & 0 \\
0 & g
\end{array}\right) \\
& \quad+\left(\begin{array}{cc}
1 & 0 \\
0 & g^{-1}
\end{array}\right)\left(\begin{array}{cc}
i & 0 \\
0 & -i g^{-1}
\end{array}\right)\left(\begin{array}{cc}
0 & B^{-} \\
B^{+} & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & g
\end{array}\right) .
\end{aligned}
$$

The first summand is equal to

$$
\begin{aligned}
&\left(\begin{array}{cc}
i & 0 \\
0 & -i g^{-1}
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & g
\end{array}\right) \partial_{u}+\left(\begin{array}{cc}
i & 0 \\
0 & -i g^{-1}
\end{array}\right)\left(\begin{array}{cc}
0 & 0 \\
0 & -i g^{-1} \frac{\partial g}{\partial u}
\end{array}\right) \\
&=\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right) \partial_{u}+\left(\begin{array}{cc}
0 & 0 \\
0 & -i g^{-1} \frac{\partial g}{\partial u}
\end{array}\right) .
\end{aligned}
$$

The second summand yields

$$
\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right)\left(\begin{array}{cc}
0 & B^{-} g \\
g^{-1} B^{+} & 0
\end{array}\right)
$$

which proves assertion (a).
The proof of (b) follows at once.

For the chosen curve $\left\{P_{r}\right\}_{r \in I}$, connecting $P_{0}:=\Pi_{>}$and a given projection $P_{1}=P$, we write

$$
\dot{\eta}_{r_{0}}:=\left.\frac{d}{d r} \eta_{\mathcal{D}_{P_{r}}}(0)\right|_{r=r_{0}}
$$

for the variation of the $\eta$-invariant at a point $r_{0} \in I$. Choosing a cut-off function $f$ we obtain from the preceding proposition

$$
\begin{align*}
\dot{\eta}_{r_{0}} & =\dot{\eta}_{\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi_{>}}}(0)=-\left.\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(U_{r}^{-1} \mathcal{D} U_{r}\right) e^{-\varepsilon\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi_{>}}^{2}}\right|_{r=r_{0}} \\
2) & =\left.\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(\left\{\left(\begin{array}{cc}
0 & 0 \\
0 & R_{r_{0}}
\end{array}\right)+G S\right\} e^{-\varepsilon\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi_{>}}^{2}}\right)\right|_{r=r_{0}}, \tag{9.2.2}
\end{align*}
$$

where the operators in the brackets $\{\ldots\}$ are given by
$R_{r_{0}}:=\left.i \overbrace{\left(g^{-1} \frac{\partial g}{\partial u}\right)}\right|_{r=r_{0}}, \quad G:=\left(\begin{array}{cc}i & 0 \\ 0 & -i\end{array}\right), \quad S:=\left(\begin{array}{cc}0 & B^{-} \dot{g} \\ -g^{-1} \dot{g} g^{-1} B^{+} & 0\end{array}\right)$.

Fortunately, the preceding expression for $\dot{\eta}_{r_{0}}$ can be simplified substantially. Notice that in formula (9.2.2) the operators in the brackets $\{\ldots\}$ vanish outside of the neck $\mathcal{N}=[0,1] \times Y$ of the boundary. Actually, the first term, involving $R_{r_{0}}$ has support completely inside the neck since $\frac{\partial g}{\partial u}=0$ at $u=0$ and $u=1$. The second term $G S$ disappears only at $u=1$ where $\dot{g}$ vanishes.

First we shall consider the case $r=0$, i.e. the $\eta$-variation precisely at the spectral projection. By Duhamel's Principle we may replace the kernel of the heat operator $e^{-\varepsilon\left(U_{0}^{-1} \mathcal{D} U_{0}\right)_{\Pi}^{2}>}=e^{-\varepsilon \mathcal{D}_{\Pi}^{2}>}$ by the heat kernel $e^{-\varepsilon \mathcal{D}_{c y l}^{2}}$ on the infinite cylinder for analyzing the first term and by the heat kernel $e^{-\varepsilon \mathcal{D}_{\text {aps }}^{2}}$ on the half-infinite cylinder with the spectral boundary condition at $u=0$ for analyzing the second term. More precisely (see Chapter 6) we
have for cut-off functions $\varphi$ with $\varphi(u)$ equal 0 near 0 and 1 and $\psi$ with $\psi(u)$ equal 1 near 0 and equal 0 near $1 \psi(1)=0$

$$
\begin{align*}
& \varphi e^{-\varepsilon \mathcal{D}_{\Pi}^{2}>}=\varphi e^{-\varepsilon \mathcal{D}_{c y l}^{2}}+\mathcal{O}\left(e^{-\frac{c}{\varepsilon}}\right)  \tag{9.2.3}\\
& \psi e^{-\varepsilon \mathcal{D}_{\Pi}^{2}>}=\psi e^{-\varepsilon \mathcal{D}_{\text {aps }}^{2}}+\mathcal{O}\left(e^{-\frac{c}{\varepsilon}}\right) \tag{9.2.4}
\end{align*}
$$

We consider the second term of formula (9.2.2). Up to the error term which disappears for $\varepsilon \rightarrow 0$, it is of the form $\operatorname{Tr}(\psi T V)$ where the original heat operator $e^{-\varepsilon \mathcal{D}_{\Pi}^{2}}$ is according to (9.2.4) replaced by $V:=e^{-\varepsilon \mathcal{D}_{\text {aps }}^{2}} . V$ is of trace class and 'even', i.e. it commutes with $G$ (that is not the case for the original heat operator). Moreover, $T=G S$ is odd, i.e. it anticommutes with $G$.

Lemma 9.2.6.

$$
\operatorname{Tr}(\psi T V)=0
$$

Proof. We have

$$
\begin{aligned}
\operatorname{Tr}(\psi T V)=\operatorname{Tr}\left(-\psi G^{2} T V\right) & =\operatorname{Tr}(-\psi G(T V) G) \\
= & \operatorname{Tr}(-\psi G T G V)=\operatorname{Tr}\left(\psi G^{2} T V\right)=\operatorname{Tr}(-\psi T V)
\end{aligned}
$$

We consider the first term of the formula (9.2.2) for $\dot{\eta}_{r_{0}}$, still at $r_{0}=0$. According to (9.2.3), we can replace the true heat operator $e^{-\varepsilon \mathcal{D}_{\Pi>}^{2}}$ by the heat operator on the bi-infinite cylinder $e^{-\varepsilon \mathcal{D}_{\text {cyl }}^{2}}$ which can be written in the form

$$
\frac{1}{\sqrt{4 \pi \varepsilon}} e^{\frac{-(u-v)^{2}}{4 \varepsilon}} e^{-\varepsilon B^{2}}
$$

We obtain

$$
\begin{aligned}
& \frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{0}
\end{array}\right) e^{-\varepsilon \mathcal{D}_{\mathrm{H}}^{2}}\right) \\
& \quad=\frac{2}{\sqrt{\pi}}\left(\lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(\begin{array}{cc}
0 & 0 \\
0 & R_{0}
\end{array}\right) e^{-\varepsilon \mathcal{D}_{\text {cyl }}^{2}}+\mathcal{O}\left(e^{\frac{-c}{\varepsilon}}\right)\right) \\
& \quad=\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{0}
\end{array}\right) e^{-\varepsilon \mathcal{D}_{\text {cyl }}^{2}}\right) \\
& \quad=\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \int_{0}^{1} d u \operatorname{Tr}_{Y}\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{0}
\end{array}\right) e^{-\varepsilon B^{2}}\right) \frac{1}{\sqrt{4 \pi \varepsilon}} \\
& \quad=\frac{1}{\pi} \lim _{\varepsilon \rightarrow 0} \int_{0}^{1} d u \operatorname{Tr}_{Y}\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{0}
\end{array}\right) e^{-\varepsilon B^{2}}\right)=\frac{1}{\pi} \int_{0}^{1} d u \operatorname{Tr}_{Y}\left(\begin{array}{cc}
0 & 0 \\
0 & R_{0}
\end{array}\right) .
\end{aligned}
$$

Thus we have proved

$$
\left.\dot{\eta}\right|_{r=0}=\frac{1}{\pi} \int_{0}^{1} d u \operatorname{Tr}_{Y}\left(\begin{array}{ll}
0 & 0  \tag{9.2.5}\\
0 & \overbrace{\left(g^{-1} \frac{\partial g}{\partial u}\right)} \\
\left.\right|_{r=0}
\end{array}\right) .
$$

We show that the preceding formula remains valid in the general situation $r=r_{0}$. As in the case $r_{0}=0$ we begin with the second term of formula (9.2.2)

$$
\begin{align*}
& \left.\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(G S e^{-\varepsilon\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{\Pi}^{2}>}\right)\right|_{r=r_{0}}  \tag{9.2.6}\\
& \quad=\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr}\left(G S e^{-\varepsilon\left(U_{r_{0}}^{-1} \mathcal{D} U_{r_{0}}\right)_{\text {aps }}^{2}}+\mathcal{O}\left(e^{-\frac{c}{\varepsilon}}\right)\right)
\end{align*}
$$

according to (9.2.4). Recall that $G S$ is odd. Thus, we need only to show that the operator $e^{-\varepsilon\left(U_{r_{0}}^{-1} \mathcal{D} U_{r_{0}}\right)_{\text {aps }}^{2}}$ is even. Then we can apply Lemma 9.2.6 and the vanishing of (9.2.6) follows.

On the cylinder, we have

$$
\begin{aligned}
U_{r_{0}}^{-1} \mathcal{D}^{2} U_{r_{0}} & =U_{r_{0}}^{-1}\left(-\partial_{u}^{2}+B^{2}\right) U_{r_{0}}=U_{r_{0}}^{-1}\left(-\partial_{u}^{2} U_{r_{0}}+B^{2} U_{r_{0}}\right) \\
& =-\partial_{u}^{2}-2 U_{r_{0}}^{-1} \frac{\partial U_{r_{0}}}{\partial u} \partial_{u}-U_{r_{0}}^{-1} \frac{\partial^{2} U_{r_{0}}}{\partial u^{2}}+U_{r_{0}}^{-1} B^{2} U_{r_{0}} \\
& =-\partial_{u}^{2}+B^{2} \underbrace{-2 U_{r_{0}}^{-1} \frac{\partial U_{r_{0}}}{\partial u} \partial_{u}-U_{r_{0}}^{-1} \frac{\partial^{2} U_{r_{0}}}{\partial u^{2}}+U_{r_{0}}^{-1}\left[B^{2}, U_{r_{0}}\right]}_{=: W} .
\end{aligned}
$$

Clearly, the perturbation term $W$ is even. By Duhamel's recursive formula (Proposition 6.3.1, see also formula (6.3.2)) we have

$$
\begin{aligned}
& e^{-\varepsilon\left(U_{r_{0}}^{-1} \mathcal{D}^{2} U_{r_{0}}\right)_{\text {aps }}}=e^{-\varepsilon \mathcal{D}_{\text {aps }}^{2}}-\int_{0}^{\varepsilon} d s e^{-s \mathcal{D}_{\text {aps }}^{2}} W e^{-(\varepsilon-s) \mathcal{D}_{\text {aps }}^{2}} \\
& +\sum_{k=2}^{\infty}(-1)^{k} \int_{0}^{\varepsilon} d s \int_{0}^{s} d s_{1} \ldots \int_{0}^{s_{k-2}} d s_{k-1} e^{-s_{k-1} \mathcal{D}_{\text {aps }}^{2}} W e^{-\left(s_{k-2}-s_{k-1}\right) \mathcal{D}_{\text {aps }}^{2}} \\
& \ldots W e^{-\left(s-s_{1}\right) \mathcal{D}_{\text {aps }}^{2}} W e^{-(\varepsilon-s) \mathcal{D}_{\text {aps }}^{2}} .
\end{aligned}
$$

Thus, with $W$ also $e^{-\varepsilon\left(U_{r_{0}}^{-1} \mathcal{D}^{2} U_{r_{0}}\right)_{\text {aps }}}$ is even, and by Lemma 9.2.6

$$
\operatorname{Tr}\left(G S e^{-\varepsilon\left(U_{r_{0}}^{-1} \mathcal{D}^{2} U_{r_{0}}\right)_{\text {aps }}}\right)=0
$$

and the second term in the formula for $\left.\dot{\eta}\right|_{r=r_{0}}$ vanishes.

We are left with the first term in formula (9.2.2). For $\varepsilon$ sufficiently small we have $\operatorname{supp} R_{r_{0}} \subset(\varepsilon, 1-\varepsilon)$, hence

$$
\begin{aligned}
& \operatorname{Tr}\left.\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{r_{0}}
\end{array}\right) e^{-\varepsilon\left(U_{r}^{-1} \mathcal{D}^{2} U_{r}\right)_{\Pi_{>}}}\right)\right|_{r=r_{0}} \\
& \quad=\operatorname{Tr}(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{r_{0}}
\end{array}\right) e^{-\varepsilon\left(U_{r_{0}}^{-1}\right.} \overbrace{\left(-\partial_{u}^{2}+B^{2}\right)}^{\text {on }(-\infty,+\infty) \times Y} U_{r_{0}}) \\
&\left.\quad=\operatorname{O}\left(e^{\frac{-c}{\varepsilon}}\right)\right) \\
&=\operatorname{Tr}\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{r_{0}}
\end{array}\right) U_{r_{0}}^{-1} e^{-\varepsilon\left(-\partial_{u}^{2}+B^{2}\right)} U_{r_{0}}+\mathcal{O}\left(e^{\frac{-c}{\varepsilon}}\right)\right) \\
& \stackrel{(*)}{=} \operatorname{Tr}\left(U_{r_{0}}^{-1}\left(\begin{array}{cc}
0 & 0 \\
0 & R_{r_{0}}
\end{array}\right) e^{-\varepsilon\left(-\partial_{u}^{2}+B^{2}\right)} U_{r_{0}}+\mathcal{O}\left(e^{\frac{-c}{\varepsilon}}\right)\right) \\
&=\operatorname{Tr}\left(\left(\begin{array}{cc}
0 & 0 \\
0 & R_{r_{0}}
\end{array}\right) e^{-\varepsilon\left(-\partial_{u}^{2}+B^{2}\right)}+\mathcal{O}\left(e^{\frac{-c}{\varepsilon}}\right)\right) .
\end{aligned}
$$

In $\left(^{*}\right)$ we used that $R_{r_{0}}$ is made up by derivatives of $U_{r_{0}}$, hence $U_{r_{0}}^{-1}$ commutes with $\left(\begin{array}{cc}0 & 0 \\ 0 & R_{r_{0}}\end{array}\right)$.

Now we can argue as above for the case $r_{0}=0$ and obtain:

## Proposition 9.2.7.

$$
\left.\dot{\eta}\right|_{r=r_{0}}=\frac{1}{\pi} \int_{0}^{1} d u \operatorname{Tr}_{Y}\left(\begin{array}{l}
0 \\
0 \\
\left.i \overbrace{\left(g^{-1} \frac{\partial g}{\partial u}\right)}^{0}\right|_{r=r_{0}}
\end{array}\right) .
$$

The main result of this section is an immediate consequence of the preceding proposition:

Theorem 9.2.8. For any path in the smooth self-adjoint Grassmannian, beginning at the Atiyah-Patodi-Singer boundary condition we have

$$
\eta_{\mathcal{D}_{P}}(0)-\left.\eta_{\mathcal{D}_{\Pi_{>}}}(0) \equiv \frac{i}{\pi} \int_{0}^{1} d r \operatorname{Tr} \overbrace{\left(g^{-1} \frac{\partial g}{\partial u}\right)}\right|_{r} \bmod \mathbf{Z} .
$$

Remark 9.2.9. For the constant direction path of Example 9.2.2, we have $g(r, u)=e^{i r f(u) \theta}$, hence

$$
\overbrace{\left(g^{-1} \frac{\partial g}{\partial u}\right)}=\overbrace{\left(i r f^{\prime}(u) \theta\right)}^{i}=i f^{\prime}(u) \theta
$$

and therefore

$$
\eta_{\mathcal{D}_{P}}-\eta_{\mathcal{D}_{\mathrm{I}_{>}}} \equiv-\frac{1}{\pi} \operatorname{Tr} \theta \quad \bmod \mathbf{Z}
$$

## 9.3. $\eta$ on the Neck and Additivity Formula

Thus, given the $\eta$-invariant $\eta_{\mathcal{D}_{\Pi}>}$ of the Atiyah-Patodi-Singer boundary condition, we can obtain the $\eta$-invariant $\eta_{\mathcal{D}_{P}}$ of the new boundary condition $P \in \mathcal{G r}_{\infty}^{*}(\mathcal{D})$, roughly speaking, by attaching a second copy $[-1,0] \times Y$ of the neck $\mathcal{N}$ with boundary condition $\Pi_{<}$in 0 (i.e. on the end flanking the manifold $X$ ), and with the new boundary condition $P$ at -1 (i.e. at the new end).

To make the argument rigorous we consider the Grassmannian of selfadjoint boundary conditions of Atiyah-Patodi-Singer type for the Dirac operator $\mathcal{D}^{\mathcal{N}}=G\left(\partial_{u}+B\right)$ on the collar manifold $\mathcal{N}=[-1,0] \times Y$. Then all self-adjoint boundary conditions for $\mathcal{D}^{\mathcal{N}}$ at the left -1 -end are perturbations of $\Pi_{>}$(by symmetric smoothing operators); i.e. belong to the smooth self-adjoint Grassmannian $\mathcal{G r}_{\infty}^{*}(\mathcal{D})$ of the original $\mathcal{D}$ on the manifold $X$.

On the right 0 -end we rewrite the operator $\mathcal{D}^{\mathcal{N}}=-G\left(-\partial_{u}+B\right)$ so that we have an inward orientated normal derivative $-\partial_{u}$. Then all selfadjoint boundary conditions for $\mathcal{D}^{\mathcal{N}}$ at the 0 -end are perturbations of $\Pi_{<}=$ $-G \Pi_{>} G=\mathrm{Id}-\Pi$ (by symmetric smoothing operators); i.e can be written as Id $-P$ with $P$ as above.

More generally, let us consider the $L_{2}-$ realizations of a given Dirac operator on the neck $\mathcal{N}=[-1,0] \times Y$ of the form $G\left(\partial_{u}+B\right)$, defined by the smooth self-adjoint Grassmannian. These operators, which we denote by $\mathcal{D}_{P, \mathrm{Id}-P}^{\mathcal{N}}$, are specified by their domain

$$
\begin{aligned}
& \operatorname{dom}\left(\mathcal{D}_{P, \mathrm{Id}-P^{\prime}}^{\mathcal{N}}\right) \\
& \quad:=\left\{\psi \in \mathcal{H}^{1}(\mathcal{N} ; \$) \mid P\left(\left.\psi\right|_{\{-1\} \times Y}\right)=0 \text { and }\left(\operatorname{Id}-P^{\prime}\right)\left(\left.\psi\right|_{\{0\} \times Y}\right)=0\right\}
\end{aligned}
$$

where $P$ and $P^{\prime}$ belong to the smooth self-adjoint Grassmannian, defined by $B$.

Lemma 9.3.1. On the neck, the $\eta$-function of $\mathcal{D}_{P, \mathrm{Id}-P}^{\mathcal{N}}$ ) vanishes and in particular

$$
\eta_{\mathcal{D}_{P, I \mathrm{~d}-P}^{\mathcal{N}}}(0)=0
$$

Proof. We show that the spectrum is symmetric. Let a section $\psi$ and a real $\lambda$ be given such that

$$
\mathcal{D} \psi=\lambda \psi, \quad P \psi(-1, y)=0 \text { and }(\operatorname{Id}-P) \psi(0, y)=0
$$

for any $y \in Y$. We introduce the mirror operator

$$
(T \psi)(u, y):=G(y) \psi(-1-u, y)
$$

and obtain by the tangential identities of 8.1.2

$$
\begin{aligned}
\mathcal{D}(T \psi)(u, y) & =G\left(\partial_{u}+B\right) G(y) \psi(-1-u, y) \\
& =G \partial_{u} G \psi(-1-u, y)+G B G \psi(-1-u, y) \\
& =-\partial_{u} \psi(-1-u, y)+B \psi(-1-u, y) \\
& =-G^{2}\left(-\partial_{u} \psi(-1-u, y)+B \psi(-1-u, y)\right) \\
& =-G G\left(-\partial_{u} \psi(-1-u, y)+B \psi(-1-u, y)\right) \\
& =-G \lambda \psi(-1-u, y)=-\lambda(G \psi(-1-u, y))=-\lambda(T \psi)(u, y) .
\end{aligned}
$$

It remains to show that the section $T \psi$ belongs to the domain of the operator $\left.\mathcal{D}_{P, \mathrm{Id}-P}^{\mathcal{N}}\right)$. We have $-G P G=\mathrm{Id}-P$ since $P$ belongs to the self-adjoint Grassmannian of $B$. Thus at the left end

$$
P(\left.(T \psi)(u, y)\right|_{u=-1}=P G \psi(-1-1, y)=-G \underbrace{(\operatorname{Id}-P) \psi(0, y)}_{=0}
$$

and, similarly, at the right end

$$
(\operatorname{Id}-P)(\left.(T \psi)(u, y)\right|_{u=0}=(\operatorname{Id}-P) G \psi(-1, y)=-G \underbrace{P \psi(-1, y)}_{=0}
$$

All this together gives us

Theorem 9.3.2. Let $\mathcal{D}$ be a Dirac operator on a smooth compact manifold $X$ with boundary $Y$ and let $P \in \mathcal{G} r_{\infty}^{*}$. Then the difference between the $\eta-$ invariants on $X$, defined by the given $P$ and by the spectral projection $\Pi_{>}$, equals the corresponding $\eta$-invariant on the neck

$$
\eta_{\mathcal{D}_{P}}(0)-\eta_{\Pi_{>}} \equiv \eta_{\mathcal{D}_{P, \Pi_{<}}^{\mathcal{N}}}(0) .
$$

Proof. Let $\left\{g_{r}\right\}$ denote a family connecting $\Pi_{>}$and $P$ in $\mathcal{G r}_{\infty}^{*}(\mathcal{D})$ as in the beginning of this section. Then $\left(\begin{array}{cc}\operatorname{Id} & 0 \\ 0 & g_{r}\end{array}\right)$ connects the pairs $\Pi(B \oplus-B)=\left(\Pi_{>}, \Pi_{<}\right)$and $\left(P, \Pi_{<}\right)$in $\mathcal{G r}_{\infty}^{*}\left(\mathcal{D}^{\mathcal{N}}\right)$. By Theorem 9.2.8 the differences of the corresponding $\eta$-invariants are both expressed by the same analytical expression. Thus

$$
\eta_{\mathcal{D}_{P}}(0)-\eta_{\Pi_{>}} \equiv \eta_{\mathcal{D}_{P, \Pi_{<}}^{\mathcal{N}}}(0)-\eta_{\mathcal{D}_{\Pi_{>},\left(\mathrm{Id}-\Pi_{>}\right)}^{\mathcal{N}}}(0) .
$$

But the last term vanishes according to Lemma 9.3.1.
Together with the adiabatic Duhamel's formula of the preceding Chapter this yields the general additivity formula

Corollary 9.3.3. For the $\eta$-invariant on a closed partitioned manifold $M=M_{1} \cup_{Y} M_{2}$ with $M_{1}, M_{2}$ compact manifolds with common boundary $Y$ we have

$$
\eta_{\mathcal{D}}=\eta_{\mathcal{D}_{1}, P_{1}}+\eta_{\mathcal{D}_{2},-G\left(\mathrm{Id}-P_{2}\right) G}+\eta_{P_{1}, P_{2}}^{\mathcal{N}} .
$$

Part 4

## $\zeta$-Determinant and Fredholm Determinant

## CHAPTER 10

## The Variation of the Modulus

## 1

We derive formulas for the change of the modulus of the $\zeta_{-}^{-}$ determinant of a fixed Dirac operator on a compact smooth manifold with boundary under variation of the boundary condition.

### 10.1. Introduction

Before deriving our variational formulas of the modulus we shall emphasize the different levels of subtlety of the various spectral invariants involved into the definition of the determinant. To begin with we restrict ourselves to the case of a closed manifold. Recall that the $\zeta$-determinant is given by the following formula

$$
\operatorname{det} \mathcal{D}:=e^{\frac{i \pi}{2}\left(\eta_{\mathcal{D}}(0)-\zeta_{\mathcal{D}^{2}}(0)\right)} \cdot e^{-\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)}
$$

Let $\left\{\mathcal{D}_{r}\right\}_{r \in I}$ be a smooth family of compatible Dirac operators parametrized over the interval $I=[0,1]$. Then, as noticed before, we have the following formula for the variation $\dot{\eta}$ of the $\eta$-invariant $\eta_{\mathcal{D}_{r}}(0)$

$$
\begin{equation*}
\dot{\eta}=-\frac{2}{\sqrt{\pi}} \lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr} \dot{\mathcal{D}} e^{-\varepsilon \mathcal{D}^{2}} \tag{10.1.1}
\end{equation*}
$$

Assuming that $\mathcal{D}_{0}$ is invertible we find the variation of the modulus $-\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)$ of the determinant:

$$
\begin{align*}
\frac{d}{d r}\left(-\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)\right) & =\left(-\frac{1}{2} \int_{0}^{\infty} \frac{1}{t} e^{-t \mathcal{D}^{2}} d t\right)=\int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}} \mathcal{D} e^{-t \mathcal{D}^{2}} d t \\
& =\int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1} \mathcal{D}^{2} e^{-t \mathcal{D}^{2}} d t=\int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1} \frac{d}{d t}\left(e^{-t \mathcal{D}^{2}}\right) d t \\
& =\left.\lim _{\varepsilon \rightarrow 0}\left(\operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1} e^{-t \mathcal{D}^{2}} d t\right)\right|_{t=\varepsilon} ^{t=\frac{1}{\varepsilon}}=-\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}} . \tag{10.1.2}
\end{align*}
$$

Note . Formula (10.1.1) for the variation of the $\eta$-invariant requires that the family consists of compatible Dirac operators. In the general case, it can be replaced by another formula involving the heat kernel asymptotics of the operator $\dot{\mathcal{D}} e^{-t \mathcal{D}^{2}}$. Formula (10.1.2) for the variation of the modulus of the $\zeta$-determinant remains valid in the general non-compatible case.

[^6]Notice the very different sensitivity of the various invariants for changes of the underlying data:

- Recall that the index of an elliptic operator depends solely on the principal symbol and can be expressed by a local formula. Moreover, it is a homotopy invariant of the principal symbol.
- The variation $\dot{\eta}$ behaves, roughly speaking, like the index because it is local, i.e. it can be expressed by an integral, and it depends only on finitely many terms in the asymptotic expansion of the total symbol.
- Also the $\eta$-invariant depends only on finitely many terms of the asymptotic expansion of the total symbol of the operator $\mathcal{D}$ (or, equivalently, of the inverse $\mathcal{D}^{-1}$ ), but it is non-local. In particular, it is invariant under perturbation by pseudo-differential operators of order less than $-m-1$ and other operators of trace class. Here, $m$ denotes the dimension of the manifold.
- The same is true for the variation $\frac{d}{d r}\left(-\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)\right)$ : it depends only on finitely many terms of the asymptotic expansion of the total symbol, thus being a polynomial or, better, an algebraic function of the total symbol.
- Finally, the modulus $\zeta_{\mathcal{D}^{2}}^{\prime}(0)$ of the determinant depends on all (infinitely-many) terms of the asymptotic expansion of the symbol. As shown before, it can change by perturbation with a smoothing operator. This is a peculiar aspect of the widely studied nonmultiplicativity property of the $\zeta$-determinant .

Assuming that the derivative $\dot{\mathcal{D}}$ is of trace class we get

$$
\lim _{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \operatorname{Tr} \dot{\mathcal{D}} e^{-\varepsilon \mathcal{D}^{2}}=0
$$

and

$$
\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}}=\operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1}\left(\lim _{\varepsilon \rightarrow 0} e^{-\varepsilon \mathcal{D}^{2}}\right)=\operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1}
$$

By (10.1.1) and (10.1.2) it follows

Lemma 10.1.1. Let $\left\{\mathcal{D}_{r}\right\}$ be a smooth family of Dirac operators on a fixed closed manifold operating on sections in a fixed bundle $\mathcal{S}$ of Clifford modules and let $\left.\dot{\mathcal{D}}\right|_{r=0}$ be of trace class. Then we have

$$
\dot{\eta}=0 \quad \text { and } \quad \frac{d}{d r} \ln \operatorname{det}|\mathcal{D}|=\frac{d}{d r}\left(-\frac{1}{2} \zeta_{\mathcal{D}^{2}}^{\prime}(0)\right)=-\operatorname{Tr} \dot{\mathcal{D}} \mathcal{D}^{-1}
$$

all taken at $r=0$.

The vanishing of $\dot{\eta}$ is not surprising since

$$
\begin{equation*}
\eta_{\mathcal{D}}(0)=\eta_{\mathcal{D}+T}(0) \tag{10.1.3}
\end{equation*}
$$

for any $T$ of trace class, as mentioned before. For the modulus of the determinant, however, the preceding Lemma shows how delicate the modulus of the determinant really is and that it changes even under perturbation by a trace class operator - contrary to the $\eta$-invariant which remains according to (10.1.3) unchanged under such perturbation.

Having $\dot{\mathcal{D}}$ of trace class is, however, rather untypical. Variation of the connection would e.g. lead to a perturbation by a bundle endomorphism. Neither a variation of the boundary condition within the Grassmannian would lead to a $\dot{\mathcal{D}}$ of trace class. Actually, this is a very fortunate instance because the more substantial perturbations we are going to consider have the nice property that they permit the replacement of the precise inverse $\mathcal{D}^{-1}$ by any parametrix for $\mathcal{D}$ in the formula (10.1.2).

We proceed with our analysis on a closed manifold $M$ for a short while. We consider a fixed Dirac operator

$$
\mathcal{D}: C^{\infty}(M ; \mathcal{S}) \longrightarrow C^{\infty}(M ; \mathcal{S})
$$

and a not everywhere vanishing self-adjoint endomorphism $V: \mathcal{S} \rightarrow \mathcal{S}$ of the fixed bundle $\mathcal{S}$ of Clifford modules. Then the family

$$
\left\{\mathcal{D}_{r}:=\mathcal{D}+r V\right\}, \quad r \in[0,1]
$$

is a smooth family of self-adjoint operators of Dirac type. For $r \neq 0$, notice that $\mathcal{D}_{r}$ is not necessarily a true (compatible) Dirac operator because we admit any endomorphism $V$ which needs not to be compatible with the Clifford multiplication. For the ease of presentation we assume that $m:=\operatorname{dim} M$ is odd, hence $\zeta_{\mathcal{D}_{r}^{2}}(0)=0$.

First we study the variation $\dot{\eta}$. Since $V$ is not of trace class, $\operatorname{Tr} V e^{-\varepsilon \mathcal{D}^{2}}$ explodes for $\varepsilon \rightarrow 0$. But the factor $\sqrt{\varepsilon}$ keeps the trace bounded and gives finally the result (which replaces (10.1.1) for non-compatible Dirac operators, see our Chapter on the variation of eta)

$$
\dot{\eta}=\frac{a_{2 m-1}}{\sqrt{\pi}} .
$$

Here $m:=\operatorname{dim} M$ and $a_{2 m-1}$ denotes the $2 m-1$ coefficient in the asymptotic expansion of the kernel of the operator $V e^{-t \mathcal{D}^{2}}$.

For the variation of the modulus we meet a different situation due to the absence of the regularizing factor $\sqrt{\varepsilon}$. According to (10.1.2) we have to determine $\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} V \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}}$. We shall prove the following remarkable fact:

Proposition 10.1.2. For a family $\left\{\mathcal{D}_{r}:=\mathcal{D}+r V\right\}$ with self-adjoint bundle endomorphism $V$ the variation of the modulus of the determinant

$$
\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} V \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}}
$$

depends only on finitely many terms of the asymptotic expansion of the symbol of $\mathcal{D}^{-1}$.

Proof. First we express the integral kernel of the operator $\mathcal{D}^{-1}$ in local coordinates in terms of the symbol $\sigma(\mathcal{D})=d_{1}+d_{0}$ where $d_{1}$ denotes the principal symbol and $d_{0}$ the order -0 term. Then we have (see e.g. [45], Lemma 1.7.2.c)

$$
\sigma\left(\mathcal{D}^{-1}\right)=\sum_{j=1}^{\infty} q_{-j}
$$

with

$$
q_{-1}=d_{1}^{-1} \quad \text { and for } n>1
$$

$q_{-n}=-q_{-1}\left\{\sum_{|\alpha|+j=n, j<n} D_{\xi}^{\alpha} d_{0} \cdot D_{x}^{\alpha} q_{-j} / \alpha!+\sum_{|\alpha|+j=n+1, j<n} D_{\xi}^{\alpha} d_{1} \cdot D_{x}^{\alpha} q_{-j} / \alpha!\right\}$.
By construction, the term $q_{-n}$ is homogeneous of order $-n$ in the covariant variable $\xi$.

Locally, the $j$ th term in the expansion of the kernel

$$
K=\sum_{j=1}^{\infty} K_{-j}
$$

of the operator $\mathcal{D}^{-1}$ is completely determined by the $j$ th term $q_{-j}$ in the expansion of the symbol of $\mathcal{D}^{-1}$ :

$$
K_{-j}(x, y):=\int e^{i\langle x-y, \xi\rangle} q_{-j}(x, \xi) d \xi
$$

We shall write $K_{-j}(x, x-y)$ instead of $K_{-j}(x, y)$ because the $y$ does not appear as an isolated variable. We show that $K_{-j}(x, x-y)$ is a homogeneous function of the argument $x-y$ of order $j-m$. Recall yhat $m$ denotes the dimension of the manifold $M$. We have with the variable transformation $\xi=\left(\xi_{1}, \ldots \xi_{m}\right) \mapsto \eta:=r \xi$

$$
\begin{aligned}
K_{-j}(x, r(x-y)) & =\int e^{i\langle r(x-y), \xi\rangle} q_{-j}(x, \xi) d \xi \\
& =r^{-m} \int e^{i\langle x-y, \eta\rangle} q_{-j}\left(x, r^{-1} \eta\right) d \eta \\
& =r^{-m} r^{-j(-1)} \int e^{i\langle x-y, \eta\rangle} q_{-j}(x, \eta) d \eta \\
& =r^{j-m} \int e^{i\langle x-y, \eta\rangle} q_{-j}(x, \eta) d \eta
\end{aligned}
$$

In particular we have

$$
K_{-j}(x, x-y)=|x-y|^{j-m} K\left(x, \frac{x-y}{|x-y|}\right) .
$$

Next we consider the heat operator $e^{-t \mathcal{D}^{2}}$. It can be expressed in the form

$$
e^{-t \mathcal{D}^{2}}=\frac{1}{2 \pi i} \int_{\Gamma} e^{-t \lambda}\left(\mathcal{D}^{2}-\lambda\right)^{-1} d \lambda
$$

Expanding the symbol of $\left(\mathcal{D}^{2}-\lambda\right)^{-1}$ in homogeneous rational functions shows that the kernel of the heat operator has the following form for small $t>0$ (see e.g. [102], Proposition 13.3)

$$
E(t ; x, y) \sim \sum_{k=0}^{\infty} t^{(k-m) / 2} p_{k}\left(x, \frac{x-y}{\sqrt{t}}\right) e^{-\mathcal{Q}_{x}(x-y) / 4 t} .
$$

Here $p_{k}(x, z)$ are homogeneous polynomials of order $k$ in $z$ and $\mathcal{Q}_{x}$ denotes the positive definite quadratic form

$$
\mathcal{Q}_{x}(\xi)=\mathcal{Q}_{x}\left(\xi_{1}, \ldots, \xi_{m}\right)=\sigma_{2}\left(\mathcal{D}^{2}\right)(x, \xi)=\sum a_{i j}(x) \xi_{i} \xi_{j}
$$

given by the principal symbol of the operator $\mathcal{D}^{2}$ at the point $x \in M$. We shall write $E(t ; x, x-y)$ instead of $E(t ; x, y)$ because the $y$ does not appear as an isolated variable. We obtain

$$
E\left(t ; x, \frac{x-y}{\sqrt{t}}\right) \sim t^{-m / 2} \sum_{k=0}^{\infty}|x-y|^{k} p_{k}\left(x, \frac{x-y}{|x-y|}\right) e^{-\mathcal{Q}_{x}(x-y) / 4 t} .
$$

We return to our variational formula and find

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0} \operatorname{Tr} V \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}}=\lim _{\varepsilon \rightarrow 0} \int d x \operatorname{tr}\left(V(x) \operatorname{bigl}\left(\mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}}\right)(\varepsilon ; x, x)\right) \\
&= \lim _{\varepsilon \rightarrow 0} \int d x \int \operatorname{tr}(V(x) K(x, x-y) E(\varepsilon ; x, x-y)) d y \\
&= \lim _{\varepsilon \rightarrow 0} \int d x \operatorname{tr} V(x) \cdot\left\{\sum_{j \geq 1, k \geq 0} \varepsilon^{-m / 2}\right. \\
&\left.\quad \int|x-y|^{j-m} K_{-j}\left(x, \frac{x-y}{|x-y|}\right)|x-y|^{k} p_{k}\left(x, \frac{x-y}{|x-y|}\right) e^{-\mathcal{Q}_{x}(x-y) / 4 \varepsilon} d y\right\} .
\end{aligned}
$$

We investigate the sum in the big brackets

$$
\sum_{j \geq 1, k \geq 0} \int d y|x-y|^{j+k-m} K_{-j}\left(x, \frac{x-y}{|x-y|}\right) p_{k}\left(x, \frac{x-y}{|x-y|}\right) \varepsilon^{-m / 2} e^{-\mathcal{Q}_{x}(x-y) / 4 \varepsilon} .
$$

Clearly the factor $\varepsilon^{-m / 2}$ will blow up for $\varepsilon \rightarrow 0$. Each term in the preceding sum, however, is kept bounded by the Gaussian integral type expression

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \int_{0}^{\delta} e^{-a \frac{r^{2}}{4 \varepsilon}} \frac{1}{2 \sqrt{\varepsilon}} d r=\lim _{\varepsilon \rightarrow 0} \lim \int_{0}^{\delta / 2 \sqrt{\varepsilon}} & e^{-a \zeta^{2}} d \zeta  \tag{10.1.4}\\
& =\lim \int_{0}^{\infty} e^{-a \zeta^{2}} d \zeta=\frac{1}{2} \sqrt{\frac{\pi}{a}}
\end{align*}
$$

If $j+k<m$, the term $|x-y|^{j+k-m}$ will blow up for small $|x-y|$. But the variation exists and is finite. So, the whole integral must vanish if $j+k<m$.

For a pair $(j, k)$ with $j+k>m$, the summand

$$
\int d y|x-y|^{j+k-m} K_{-j}\left(x, \frac{x-y}{|x-y|}\right) p_{k}\left(x, \frac{x-y}{|x-y|}\right) \varepsilon^{-m / 2} e^{-\mathcal{Q}_{x}(x-y) / 4 \varepsilon}
$$

will, actually, vanish. To prove that we choose a $\delta>0$ and split the integration domain into $|x-y|>\delta$ which is bounded by the manifold's diameter and $|x-y|<\delta$. For each $x$, the homogeneous functions $K_{-j}\left(x, \frac{x-y}{|x-y|}\right)$ and $p_{k}\left(x, \frac{x-y}{|x-y|}\right)$ are actually functions on the unit sphere and hence baunded. We obtain

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \mid \int_{|x-y|>\delta} d y & \left.|x-y|^{\mid+k-m} K_{-j}\left(x, \frac{x-y}{|x-y|}\right) p_{k}\left(x, \frac{x-y}{|x-y|}\right) \varepsilon^{-m / 2} e^{-\mathcal{Q}_{x}(x-y) / 4 \varepsilon} \right\rvert\, \\
& \leq \lim _{\varepsilon \rightarrow 0} C \int_{|x-y|>\delta} \varepsilon^{-m / 2} e^{-\mathcal{Q}_{x}(x-y) / 4 \varepsilon} d y \quad \text { and with } z=\frac{x-y}{\sqrt{\varepsilon}} \\
& \leq \lim _{\varepsilon \rightarrow 0} C \int_{|z|>\frac{\delta}{\sqrt{\varepsilon}}} e^{-\mathcal{Q}_{x}(z) / 4} d z=0 .
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0}\left|\int_{|x-y|<\delta} \ldots d y\right| \leq \lim _{\varepsilon \rightarrow 0} d^{j+k-m} C & \int_{|z|<\frac{\delta}{\sqrt{\varepsilon}}} e^{-\mathcal{Q}_{x}(z) / 4} d z \\
& \leq \lim _{\varepsilon \rightarrow 0} d^{j+k-m} C \int e^{-\mathcal{Q}_{x}(z) / 4} d z=C^{\prime} d^{j+k-m}
\end{aligned}
$$

because of the finiteness of the Gaussian integral. But $\delta>0$ could be chosen arbitrary small and $j+k-m>0$, so also that integral must vanish.

So, only the finitely many terms with $j \geq 1, k \geq 0$, and $j+k=m$ contribute to the variation

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0} \operatorname{Tr} V \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}} \\
= & \lim _{\varepsilon \rightarrow 0} \sum_{j+k=m} \int d x \operatorname{tr} V(x) \int K_{-j}\left(x, \frac{x-y}{|x-y|}\right) p_{k}\left(x, \frac{x-y}{|x-y|}\right) \varepsilon^{-m / 2} e^{-\mathcal{Q}_{x}(x-y) / 4 \varepsilon} d y .
\end{aligned}
$$

Our proposition has two corollaries:

Corollary 10.1.3. (a) For the calculation of the variation of the modulus of the determiant we can replace $\mathcal{D}^{-1}$ by any parametrix $Q$ for $\mathcal{D}$, i.e. an operator with $\mathcal{D}^{-1}-Q$ is a smoothing operator.
(b) The computation of $\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} V \mathcal{D}^{-1} e^{-\varepsilon \mathcal{D}^{2}}$ localizes.

Proof. (a) is an obvious consequence of the proposition. To prove (b), we consider a smooth partition of unity $\left\{U_{\alpha}, \psi_{\alpha}\right\}$ and define $\mathcal{D}_{\alpha}:=\left.\mathcal{D}\right|_{U_{\alpha}}$ and $Q_{\alpha}:=\left(\mathcal{D}_{\alpha}\right)^{-1}$. Moreover, we set

$$
Q:=\sum_{\alpha} \varphi_{\alpha} Q_{\alpha} \psi_{\alpha}
$$

for suitable bumpfunctions $\varphi_{\alpha}$ with $\varphi_{\alpha} \equiv 1$ on $U_{\alpha}$ ). Then we have

$$
Q=\mathcal{D}^{-1} \mathcal{D} Q=\mathcal{D}^{-1}\left(\operatorname{Id}+\sum_{\alpha}\left[\mathcal{D}, \varphi_{\alpha}\right] Q_{\alpha} \psi_{\alpha}\right)
$$

hence

$$
\mathcal{D}^{-1}=Q-\sum_{\alpha} \mathcal{D}^{-1}\left[\mathcal{D}, \varphi_{\alpha}\right] Q_{\alpha} \psi_{\alpha}
$$

The commutator $\left[\mathcal{D}, \varphi_{\alpha}\right]$ has compact support in the complement of $U_{\alpha}$, hence disjoint of the support of $Q_{\alpha} \psi_{\alpha}$. Therefore, the combined operator $\left[\mathcal{D}, \varphi_{\alpha}\right] Q_{\alpha} \psi_{\alpha}$ is a smoothing operator, and so is the whole difference $\mathcal{D}^{-1}-$ $Q$.

### 10.2. Variation of the $\zeta$-Determinant on $G r_{\infty}^{*}(\mathcal{D})$

In this Section we study the variation of the $\zeta$-determinant of the operator $\mathcal{D}_{P}$, where $P \in G r_{\infty}^{*}(\mathcal{D})$, under a change of boundary condition. From Section 1 we know that the Grassmannian $\operatorname{Gr}_{\infty}^{*}(\mathcal{D})$ can be identified with the group $U^{\infty}\left(F^{-}\right)$. If we fix a base projection, for instance $P(\mathcal{D})$, then any other projection is of the form:

$$
P=\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g
\end{array}\right) P(\mathcal{D})\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g^{-1}
\end{array}\right)
$$

where $g: F^{-} \rightarrow F^{-}$is a unitary operator such that $g-I d$ has smooth kernel.

We introduce a smooth one-parameter family $\left\{g_{r}\right\}_{-\varepsilon<r<\varepsilon}$ of operators from $U^{\infty}\left(F^{-}\right)$with $g_{0}=I d_{F^{-}}$. Let $\left\{P_{r}\right\}$ denote the corresponding family of projections:

$$
P_{r}=\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}
\end{array}\right) P\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}^{-1}
\end{array}\right) .
$$

We want to compute the variation

$$
\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{r}}=\frac{d}{d r}\left\{\ln _{\left.\operatorname{det}_{\zeta} \mathcal{D}_{P_{r}}\right\}\left.\right|_{r=0} . . . . . . .}\right.
$$

For the purposes of this paper, it is enough to solve an easier problem. Let us fix two elements of the Grassmannian $P_{1}$ and $P_{2}$ such that $\mathcal{D}_{P_{1}}$ and $\mathcal{D}_{P_{2}}$ are invertible operators. The family $\left\{g_{r}\right\}$ determines two 1-parameter families of projections

$$
P_{i, r}=\left(\begin{array}{cc}
I d_{F^{+}} & 0  \tag{10.2.1}\\
0 & g_{r}
\end{array}\right) P_{i}\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}^{-1}
\end{array}\right)
$$

with respect to which we may study the relative variation

$$
\begin{equation*}
\left.\frac{d}{d r}\left\{\ln _{\operatorname{det}_{\zeta}} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0} \tag{10.2.2}
\end{equation*}
$$

The first obstacle here is that the domains of the unbounded operators $\mathcal{D}_{P_{i, r}}$ are varying with the parameter $r$. It was explained a long time ago how to solve this problem. We apply a "Unitary Twist" (see [40], [62]). The point is that we may extend the family of unitary isomorphisms $\left\{g_{r}\right\}$ on the boundary sections to a family of unitary transformations $\left\{U_{r}\right\}$ on $L^{2}(M ; S)$. To do that, fix a smooth non-decreasing function $\kappa(u)$ such that

$$
\kappa(u)=1 \text { for } u<1 / 4 \text { and } \kappa(u)=0 \text { for } u>3 / 4
$$

and for each $r$ introduce the 2-parameter family

$$
\begin{equation*}
g_{r, u}=g_{r \kappa(u)} \quad \text { for } 0 \leq u \leq 1 . \tag{10.2.3}
\end{equation*}
$$

Now we define a transformation $U_{r}$ as follows:

$$
U_{r}:=\left\{\begin{array}{ll}
\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r, u}
\end{array}\right) & \text { on }\{u\} \times Y \subset N=[0,1] \times Y  \tag{10.2.4}\\
I d & \text { on } M \backslash N
\end{array} .\right.
$$

We then have the following elementary result

LEMMA 10.2.1. The operators $\mathcal{D}_{P_{i, r}}$ and $\left(U_{r}^{-1} \mathcal{D} U_{r}\right)_{P_{i}}$ are unitary equivalent operators.

Clearly, the operators $U_{r}$ depend upon the choice of the extension function $\kappa$, however by Lemma 10.2.1 the $\zeta$-determinant does not, which is all that we need. In the following we use the notation

$$
\begin{equation*}
\mathcal{D}_{r}=U_{r}^{-1} \mathcal{D} U_{r} \quad \text { and } \quad \dot{\mathcal{D}}_{0}=\left.\frac{d}{d r} \mathcal{D}_{r}\right|_{r=0} . \tag{10.2.5}
\end{equation*}
$$

Note that the variation $\dot{\mathcal{D}}_{0}$ is localized in the collar neighbourhood $N$ of the boundary, using the representation (7.1.2) one has

$$
\begin{equation*}
\dot{\mathcal{D}}_{0}=G U_{r}^{-1}\left\{\frac{\partial}{\partial u} \widehat{U}_{r}+\left[B, \widehat{U}_{r}\right]\right\} U_{r} \tag{10.2.6}
\end{equation*}
$$

where $\widehat{U}_{r}=\frac{d U_{r}}{d r} U_{r}^{-1}$.

The Canonical determinant is also independent of the choice of the family $\left\{U_{r}\right\}$. This follows from the fact that $P_{i, r} P(\mathcal{D})$ and the operator $P_{i} P\left(\mathcal{D}_{r}\right)$ are unitarily equivalent

$$
\begin{equation*}
P_{i, r} P(\mathcal{D})=g_{r} P_{i} g_{r}^{-1} P(\mathcal{D})=g_{r}\left(P_{i} g_{r}^{-1} P(\mathcal{D}) g_{r}\right) g_{r}^{-1}=g_{r} P_{i} P\left(\mathcal{D}_{r}\right) g_{r}^{-1} \tag{10.2.7}
\end{equation*}
$$

The main result of this section is the following Theorem:

Theorem 10.2.2. The following equality holds for any pair of projections $P_{1}, P_{2} \in G r_{\infty}^{*}(\mathcal{D})$ such that $\mathcal{D}_{P_{1}}$ and $\mathcal{D}_{P_{2}}$ are invertible operators and for any smooth 1 - parameter family $\left\{g_{r}\right\}$ of operators from $U^{\infty}\left(F^{-}\right)$with $g_{o}=I d$ :

$$
\begin{equation*}
\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}=\operatorname{Tr} \dot{\mathcal{D}}_{0}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right) \tag{10.2.8}
\end{equation*}
$$

From equations (4.3.8) and (10.2.6) we can write as

$$
\begin{gathered}
\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}= \\
\operatorname{Tr}\left(G U_{r}^{-1}\left(\frac{\partial}{\partial u} \widehat{U}_{r}+\left[B, \widehat{U}_{r}\right]\right) U_{r} \mathcal{K}\left(\mathcal{S}\left(P_{2}\right)^{-1} P_{2}-\mathcal{S}\left(P_{1}\right)^{-1} P_{1}\right) \gamma_{0} \mathcal{D}^{-1}\right) .
\end{gathered}
$$

Notice that although the variation $\dot{\mathcal{D}}_{0}$ is localized in $N$, the variation of the $\zeta$-determinant is not, it depends on global data because of the term $\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}$. This is what makes the $\zeta$-determinant a more difficult spectral invariant than the $\eta$-invariant which corresponds to the phase of the determinant. Indeed, in the formula (10.2.8) a mathematician working in spectral geometry will recognize only a variation of the difference of logarithms of the modulus of the $\zeta$-determinant. The reason is that the variation of the phase of the determinant of $\mathcal{D}_{P_{1, r}}$ is equal to the variation of the phase of the determinant of $\mathcal{D}_{P_{2, r}}$.

Theorem 10.2.3. The variation of the phase of the $\zeta$-determinant of the operator $\mathcal{D}_{P_{r}}$ depends only on the family of the unitaries $\left\{g_{r}\right\}$ on $F^{-}$such that

$$
P_{r}=\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}
\end{array}\right) P_{0}\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}^{-1}
\end{array}\right)
$$

not on the choice of the base-point projection $P_{0}$. More specifically $\zeta_{\mathcal{D}_{P}^{2}}(0)$ is a constant function of the projection $P$ and the variation of the $\eta$-invariant depends only on the family $\left\{g_{r}\right\}$.

Proof. The Theorem follows from two technical results proved in the work of the second author $[\mathbf{1 1 3}]$. The phase of the determinant is equal to

$$
\exp \left\{\frac{i \pi}{2}\left(\zeta_{\mathcal{D}_{P_{r}}^{2}}(0)-\eta_{\mathcal{D}_{P_{r}}}(0)\right)\right\}
$$

It was shown in $[\mathbf{1 1 3}]$ (Proposition 0.5) that $\zeta_{\mathcal{D}_{P}^{2}}(0)$ is constant on $G r_{\infty}^{*}(\mathcal{D})$ , hence the variation of the logarithm of the phase is equal to the variation of the $\eta$-invariant times $-\left(\frac{i \pi}{2}\right)$. The formula for the variation of the $\eta$ invariant was derived in the proof of Theorem 4.3. in [113]. We have

$$
\begin{equation*}
\left.\frac{d}{d r} \eta_{\mathcal{D}_{P_{i, r}}}(0)\right|_{r=0}=\frac{i}{\pi} \int_{0}^{1} d u \operatorname{Tr}\left(\left.\frac{d}{d r}\left(g_{r, u}^{-1} \frac{\partial g_{r, u}}{\partial u}\right)\right|_{r=0}\right) . \tag{10.2.9}
\end{equation*}
$$

In particular the right side of (10.2.9) does not depend on $P_{i}$.

Remark 10.2.4. A special case of the formula (10.2.9) was discussed in the paper [93].

Next we study the logarithm of the modulus of the determinant

$$
\ln \left|\operatorname{det}_{\zeta} \mathcal{D}_{P}\right|=-\frac{1}{2} \zeta_{\mathcal{D}_{P}^{2}}^{\prime}(0)
$$

It is well-known (see Section 3 of [113]) that

$$
\begin{equation*}
\zeta_{\mathcal{D}_{P}^{2}}^{\prime}(0)=\lim _{s \rightarrow 0}\left\{\int_{0}^{\infty} t^{s-1} \operatorname{Tr} e^{-t \mathcal{D}_{P}^{2}} d t-\frac{\zeta_{\mathcal{D}_{P}^{2}}(0)}{s}\right\}-\gamma \cdot \zeta_{\mathcal{D}_{P}^{2}}(0), \tag{10.2.10}
\end{equation*}
$$

where $\gamma$ denotes the Euler constant. The fact that $\zeta_{\mathcal{D}_{P}^{2}}(0)$ does not depend on $P$ allows us to study just the variation of the integral in formula (10.2.10) and with the help of Duhamel's Principle we obtain

$$
\left.\frac{d}{d r}\left(\zeta_{\mathcal{D}_{P_{i, r}}^{2}}^{\prime}(0)\right)\right|_{r=0}=\int_{0}^{\infty} \frac{1}{t} \cdot \operatorname{Tr}\left(-2 t \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{i}} e^{-t \mathcal{D}_{P_{i}}^{2}}\right) d t=
$$

$$
\begin{gathered}
-2 \int_{0}^{\infty} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{i}}^{-1} \mathcal{D}_{P_{i}}^{2} e^{-t \mathcal{D}_{P_{i}}^{2}} d t=2 \int_{0}^{\infty} \frac{d}{d t}\left(\operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{i}}^{-1} e^{-t \mathcal{D}_{P_{i}}^{2}}\right) d t= \\
\left.2 \cdot \lim _{\varepsilon \rightarrow 0}\left(\operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{i}}^{-1} e^{-t \mathcal{D}_{P_{i}}^{2}}\right)\right|_{\varepsilon} ^{\frac{1}{\varepsilon}}=-2 \cdot \lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{i}}^{-1} e^{-\varepsilon \mathcal{D}_{P_{i}}^{2}} .
\end{gathered}
$$

We then have the following result:

Lemma 10.2.5.

$$
\begin{equation*}
\left.\frac{d}{d r}\left(-\frac{1}{2} \zeta_{\mathcal{D}_{P_{r}}^{2}}^{\prime}(0)\right)\right|_{r=0}=\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}_{0}} \mathcal{D}_{P_{i}}^{-1} e^{-\varepsilon \mathcal{D}_{P_{i}}^{2}} \tag{10.2.11}
\end{equation*}
$$

In general the limit on the right hand of the equation (10.2.11) is just the constant term in the asymptotic expansion of the heat kernel. However, since we discuss the difference (10.2.2), in this situation we actually obtain the true operator trace:

$$
\begin{gathered}
\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}=\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{1}}^{-1} e^{-\varepsilon \mathcal{D}_{P_{1}}^{2}}-\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}_{0}} \mathcal{D}_{P_{2}}^{-1} e^{-\varepsilon \mathcal{D}_{P_{2}}^{2}}= \\
\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right) e^{-\varepsilon \mathcal{D}_{P_{1}}^{2}}=\operatorname{Tr} \dot{\mathcal{D}}_{0}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right)
\end{gathered}
$$

where for the final step we use Corollary 7.3.2. This completes the proof of Theorem 10.2.2.

## CHAPTER 11

## Projective equality of the $\zeta$-determinant and Quillen determinant


#### Abstract

In this chapter we show that the $\zeta$-regularized determinant $\operatorname{det}_{\zeta} \mathcal{D}_{P}$ is equal to $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}$ modulo a natural multiplicative constant.


### 11.1. Introduction

The purpose of this paper is to explain a direct and precise identity between the $\zeta$-determinant of a self-adjoint elliptic boundary value problem for the Dirac operator over an odd-dimensional manifold with boundary and a regularization of the determinant as the Fredholm determinant of a canonically associated operator over the boundary. We consider an infinitedimensional Grassmannian of elliptic boundary conditions commensurable with Atiyah-Patodi-Singer condition. The latter regularization is defined, in the sense explained below, as the ratio of the determinant of the Dirac operator with given elliptic boundary condition to the determinant of the Dirac operator with the basepoint chiral spinor boundary condition. It is a regularization canonically constructed from the topology of the associated determinant line bundle and hence called the canonical determinant. The canonical determinant is a robust algebraic operator-theoretic object, while the $\zeta$-determinant is a highly delicate analytic object, and so it is surprising that they coincide. (Though, the equality of the torsions mentioned above at least suggests that the $\zeta$-determinant may be somehow related to Fredholm determinants.) Note however that the fundamental multiplicative property of the Fredholm determinant (??) does not hold for the $\zeta$-determinant; if $\mathrm{E}_{1}$ and $\mathrm{L}_{2}$ denote two positive elliptic operator of positive order on a Hilbert space $H$ then in general

$$
\operatorname{det}_{\zeta} \mathrm{L}_{1} \mathrm{~L}_{2} \neq \operatorname{det}_{\zeta} \mathrm{L}_{1} \cdot \operatorname{det}_{\zeta} \mathrm{L}_{2} .
$$

We refer to [61] for a detailed study of the so-called Multiplicative Anomaly. But this is not contradictory, the canonical determinant is also not multiplicative, due to the process of taking ratios. To formalise the construction of taking the ratios of determinants used to define the canonical determinant we need the machinery of the determinant line bundle. This was introduced in a fundamental paper of Quillen [83] for a family of Cauchy-Riemann operators acting on a Hermitian bundle over a Riemann surface, as the pull-back
of the corresponding 'universal' determinant bundle over the space of Fredholm operators on a separable Hilbert space. The determinant line bundle $D E T$ comes equipped with a canonical determinant section $A \mapsto \operatorname{det} A$, non-zero if and only if $A$ is invertible, where $\operatorname{det} A$ lives in the fibre over $A$, isomorphic to the complex line $D E T A:=\wedge^{\max } \operatorname{Ker} A \otimes \wedge^{\max }$ Coker $A$ . Quillen showed that in this context one can use the $\zeta$-function in order to construct a natural metric on this bundle and that the curvature of this metric provides a natural representative of the $1-s t$ Chern class. This was extended by Bismut and Freed to the context of the families of Dirac operators on closed manifolds (see [17]) and the curvature identified with the 2 -form component of the local families index density. Inspired by the Witten's work [107], they also studied the holonomy of the corresponding connection and provided the first proof of the Witten Holonomy Theorem. This result was also proved independently by Jeff Cheeger (see [35]).

The Bismut and Freed construction showed that the $\zeta$-regularization provides a natural metric on the determinant bundle, but did not provide a straightforward correspondence between the $\zeta$-determinant and the canonical determinant section. The problem is this: Given a non-zero section $\sigma$ of $D E T$ one can assign a complex function $\operatorname{det}_{\sigma}$ to the determinant section by taking the ratio of the two sections: $\operatorname{det} A=\operatorname{det}_{\sigma}(A) \cdot \sigma(A)$; given that $A$ is an operator with a $\zeta$-determinant, find $\sigma$ such that $\operatorname{det}_{\sigma}(A)=\operatorname{det}_{\zeta}(A)$. Clearly the global existence of such a section $\sigma$ is equivalent to the triviality of the determinant line bundle. The function det $_{\sigma}$ may regarded as a regularized determinant defined relative to the 'basepoint' $\sigma$. In order to link this up with Fredholm determinants we use an equivalent construction of the determinant line bundle due to Segal. This formalises the idea of defining a regularized determinant by taking the quotient of two comparable operators.

Associated to the family of elliptic boundary value problems $\left\{\mathcal{D}_{P}: P \in\right.$ $\left.G r_{\infty}(\mathcal{D})\right\}$ one has a determinant line bundle $\operatorname{DET}(\mathcal{D})$ over $G r_{\infty}(\mathcal{D})$, as explained in Section 1, which is non-trivial over $G r_{\infty}(\mathcal{D})$. Further for each choice of basepoint $P_{0} \in G r_{\infty}(\mathcal{D})$ one has a smooth family of Fredholm operators

$$
\left\{\mathcal{S}_{P_{0}}(P):=P P_{0}: \text { Ran } P_{0} \rightarrow \text { Ran } P P \in G r_{\infty}(\mathcal{D})\right\}
$$

with associated (Segal) determinant line bundle $D E T_{P_{0}}$ equipped with its canonical determinant section $P \rightarrow \operatorname{det} \mathcal{S}_{P_{0}}(P) \in \operatorname{Det}_{P_{0}}(P)$, where $\operatorname{Det}_{P_{0}}(P)$ is the determinant line of the Fredholm operator $\mathcal{S}_{P_{0}}(P)$. Moreover, for $P_{0}, P_{1} \in G r_{\infty}(\mathcal{D})$ there is a canonical line bundle isomorphism

$$
\begin{equation*}
D E T_{P_{0}}=\operatorname{Det}_{P_{0}}\left(P_{1}\right) \otimes D E T_{P_{1}}, \tag{11.1.1}
\end{equation*}
$$

defined where the operators are invertible by

$$
\begin{equation*}
\operatorname{det}\left[\mathcal{S}_{P_{1}}(P) \mathcal{S}_{P_{0}}\left(P_{1}\right)\right]=\operatorname{det} \mathcal{S}_{P_{0}}\left(P_{1}\right) \otimes \operatorname{det} \mathcal{S}_{P_{1}}(P) \tag{11.1.2}
\end{equation*}
$$

The first factor on the right-side of (11.1.1) refers to the trivial bundle with fibre $\operatorname{Det}_{P_{0}}\left(P_{1}\right)$. The determinant line bundle of the family of elliptic boundary value problems is classified in this sense by

$$
\begin{equation*}
\operatorname{DET}(\mathcal{D})=D E T_{P(\mathcal{D})} \tag{11.1.3}
\end{equation*}
$$

where $P(\mathcal{D})$ is the Calderon projection, preserving the canonical determinant sections

$$
\begin{equation*}
\operatorname{det} \mathcal{D}_{P} \longleftrightarrow \operatorname{det} \mathcal{S}(P), \tag{11.1.4}
\end{equation*}
$$

where we have written $\mathcal{S}(P)$ for $\mathcal{S}_{P(\mathcal{D})}(P)$. We may therefore rewrite (11.1.1) fibrewise as

$$
\begin{equation*}
D E T \mathcal{D}_{P}=D E T \mathcal{D}_{P_{0}} \otimes \operatorname{Det}_{P_{0}}(P) \tag{11.1.5}
\end{equation*}
$$

We refer to $[\mathbf{9 1}]$ for all these facts.
Let $\sigma\left(\mathcal{D}_{P_{0}}\right)$ denote the image of the canonical element $\operatorname{det} \mathcal{S}_{P_{0}}\left(P_{1}\right) \otimes$ $\operatorname{det} \mathcal{D}_{P_{0}} \in \operatorname{DET} \mathcal{D}_{P_{0}} \otimes \operatorname{Det}_{P_{0}}(P)$ under the isomorphism (11.1.5). Relative to the choice of the basepoint $P_{0}$, we therefore have two canonical elements in $D E T \mathcal{D}_{P}$, namely $\operatorname{det} \mathcal{D}_{P}$ and $\sigma\left(\mathcal{D}_{P_{0}}\right)$. Thus over the open subset where the operators are invertible, according to our earlier discussion we obtain a regularized determinant of $D_{P}$ by taking the quotient of these elements. The point however is to make a canonical choice of the basepoint $P_{0}$.

In the following, to make the presentation smoother we assume that ker $B=\{0\}$. This is in fact not a serious restriction and we can easily relax this condition. The point is that now the operators

$$
B^{ \pm}: F^{ \pm}=C^{\infty}\left(Y ; S^{ \pm}\right) \rightarrow F^{\mp}=C^{\infty}\left(Y ; S^{\mp}\right)
$$

are invertible. (We use also $F^{ \pm}$to denote the space of $L^{2}$ sections of the bundle of spinors of "positive' (resp. "negative") chirality.)

Coming back to the canonical choice of the basepoint, in our situation we are interested just in the real submanifold $G r_{\infty}^{*}(\mathcal{D})$ of self-adjoint boundary conditions and the 'correct' choice is indicated by the fact any elliptic boundary condition $P \in G r_{\infty}^{*}(\mathcal{D})$ is described precisely by the property that its range is the graph of an elliptic unitary isomorphism $T: F^{+} \rightarrow F^{-}$ such that $T-\left(B^{+} B^{-}\right)^{-1 / 2} B^{+}$has a smooth kernel.

There is a further subtlety that the corresponding orthogonal projection $P^{+}$onto $F^{+}$is not actually an element of the Grassmannian. But from
(11.1.2) the isomorphism (11.1.5) is well-defined if we include the correction factor $\tau=\operatorname{det}\left(\mathcal{S}(P(\mathcal{D})) / \operatorname{det}\left[\mathcal{S}_{P(\mathcal{D})}\left(P^{+}\right) \mathcal{S}_{P^{+}}(P(\mathcal{D}))\right]\right.$, which introduces a factor of $1 / 2$ in the final formula (see (5.2.6)). The canonical determinant is then defined to be the quotient taken in $D E T \mathcal{D}_{P}$

$$
\begin{equation*}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}=\frac{\operatorname{det} \mathcal{D}_{P}}{\sigma\left(D_{P^{+}}\right)} \tag{11.1.6}
\end{equation*}
$$

Roughly speaking this is the quotient $\operatorname{det} \mathcal{D}_{P} / \operatorname{det} \mathcal{D}_{P}^{+}$, the precise definition takes account of the fact that the domains of the operators $\mathcal{D}_{P}$ and $\mathcal{D}_{P}^{+}$ are different and hence that their canonical determinant elements live in different complex lines. In Section 1 we carry out a precise computation and we see that $\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}$ is actually the Fredholm determinant of an operator living on the boundary $Y$ constructed from projections $P$ and $P(\mathcal{D})$.The main result of the paper is the following Theorem:

Theorem 11.1.1. The following equality holds over $\operatorname{Gr}_{\infty}^{*}(\mathcal{D})$

$$
\begin{equation*}
\operatorname{det}_{\zeta} \mathcal{D}_{P}=\operatorname{det}_{\zeta} \mathcal{D}_{P(\mathcal{D})} \cdot \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P} \tag{11.1.7}
\end{equation*}
$$

Remark 11.1.2. (1) Theorem 11.1.1 shows that, at least on $G r_{\infty}^{*}(\mathcal{D})$, the $\zeta$-determinant is an object which is a natural extension of the well-defined algebraic concept of the determinant.
(2) Our results show that the $\zeta$-determinant of the boundary problem $\mathcal{D}_{P}$ is actually equal to the Fredholm determinant of the operator $\mathcal{S}(P)$ living on the boundary. This extends the corresponding result for the index , which is well-known (see Theorem 20.8 [27]).
(3) With Theorem 11.1.1 at our disposal we can now try a new approach to the pasting formula for the $\zeta$-determinant with respect to a partitioning of a closed manifold. The pasting formula for $\operatorname{det}_{\mathcal{C}}$ was introduced in [91]. It is hoped that a new insight into the pasting mechanism of the $\zeta$-determinant will be obtained by combining results of $[\mathbf{9 1}]$ and formula (11.1.7).

We study the variation of the determinants in order to prove Theorem 11.1.1. More precisely, we fix two projections $P_{1}, P_{2} \in G r_{\infty}^{*}(\mathcal{D})$ such that the operators $\mathcal{D}_{P_{i}}$ are invertible. Next we choose a family of unitary operators of the form

$$
\left\{\left(\begin{array}{cc}
I d_{S^{+}} & 0 \\
0 & g_{r}
\end{array}\right)\right\}_{0 \leq r \leq 1}
$$

where $g_{r}: F^{-} \rightarrow F^{-}$is a unitary operator, and such that $g_{r}-I d_{F^{-}}$is an operator with a smooth kernel for any $r$, and $g_{0}=I d_{F^{-}}$. We define two families of boundary conditions:

$$
\left.P_{i, r}=\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}
\end{array}\right)\right\} P_{i}\left(\begin{array}{cc}
I d_{F^{+}} & 0 \\
0 & g_{r}^{-1}
\end{array}\right)
$$

and study the relative variation:

$$
\left.\frac{d}{d r}\left\{\ln \operatorname{det} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}
$$

for both the Canonical determinant and the $\zeta$-determinant. Of course we face the technical problem of dealing with a family of unbounded operators with varying domain. To circumvent this and make sense of the variation with respect to the boundary condition we follow Douglas and Wojciechowski [40] and apply their "Unitary Trick". This defines an extension of our family of unitary operators on the boundary sections to a family $\left\{U_{r}\right\}$ of unitary operators acting on $L^{2}(M ; S)$ (see formula (10.2.4)). The operator $\mathcal{D}_{P_{i, r}}$ is unitarily equivalent to the operator $\left(\mathcal{D}_{r}\right)_{P_{i}}$, where

$$
\mathcal{D}_{r}=U_{r}^{-1} \mathcal{D} U_{r}
$$

Both the $\zeta$-determinant and the canonical determinant are invariant under this unitary twist which allows us to compute that both determinants have variation given by the expression

$$
\begin{equation*}
\left.\frac{d}{d r}\left\{\ln \operatorname{det} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}=\operatorname{Tr} \dot{D}_{0}\left(\mathcal{D}_{P_{1}}-\mathcal{D}_{P_{2}}\right) \tag{11.1.8}
\end{equation*}
$$

where $\dot{\mathcal{D}}_{0}$ denotes the operator $\left.\frac{d}{d r} \mathcal{D}_{r}\right|_{r=0}$. Now we use the fact that the set of projections $P \in G r_{\infty}^{*}(\mathcal{D})$, such that the operator $\mathcal{D}_{P}$ is invertible is actually path connected ( see [78]) and integrate the equality

$$
\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}=\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}
$$

in order to obtain formula (11.1.7) of Theorem 11.1.1.

Remark 11.1.3. The reader might think that formula (11.1.8) is incorrect as it does not contain the variation of the phase of the $\zeta$-determinant. The variation of logarithm of the modulus of $\zeta$-determinant (at $\mathcal{D}_{P_{i}}$ ) is known to be

$$
\lim _{\varepsilon \rightarrow 0} \operatorname{Tr} \dot{\mathcal{D}}_{0} \mathcal{D}_{P_{i}}^{-1} e^{-\varepsilon \mathcal{D}_{P_{i}}^{2}}
$$

and this leads to the right side of (11.1.8), understood as

$$
\frac{1}{2} \frac{d}{d r}\left(\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{1}}^{2}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2}}^{2}\right)
$$

However, it follows from a result of the work [113], that in the situation studied in this paper the variation of the phase of $\zeta$-determinant depends only on the family of unitaries $\left\{U_{r}\right\}$ and the operator $\mathcal{D}$. Therefore the phase contributions cancel each other. For more details we refer to Section 3.

In Section 1 we explain construction of the Canonical Determinant. We follow here the exposition of [91].

Assume that for given $P \in G r_{\infty}^{*}(\mathcal{D})$ the operator $\mathcal{D}_{P}$ is invertible. In Section 2 we present our construction of an inverse $\mathcal{D}_{P}^{-1}$. To do that we have to discuss certain aspects of the theory of elliptic boundary problems. We also introduce $\mathcal{K}$ the Poisson map of the operator $\mathcal{D}$ and $\mathcal{K}(P)$ the Poisson map of the operator $\mathcal{D}_{P}$. The first is used in the construction of the Calderon projection. The operator $\mathcal{K}(P)$ appears in several crucial places in our computation of the variation of the Canonical Determinant.

In Section 3 we discuss the variation of the $\zeta$-determinant and in Section 4 we study the variation of the Canonical Determinant. It has already been mentioned that the work [113] was crucial for the study here of the $\zeta$-determinant, while in the calculation of the variation of the Canonical Determinant we were influenced by the work of Robin Forman [43].

With (11.1.8) at hand, Section 5 contains the final steps of the proof of Theorem 0.1.

### 11.2. Variation of the Canonical Determinant

In this section we prove the corresponding result to Theorem 10.2.2 for the canonical determinant and show that it coincides with the relative variation of the $\zeta$-determinant (10.2.8). We begin with the following result:

Proposition 11.2.1. The following formula holds for any $P_{1}, P_{2} \in G r_{\infty}^{*}(\mathcal{D})$ such that $\mathcal{D}_{P_{1}}$ and $\mathcal{D}_{P_{2}}$ are invertible operators.

$$
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1, r}}\left(\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2, r}}\right)^{-1}=\operatorname{det}_{F r}\left(\begin{array}{cc}
I d & 0  \tag{11.2.1}\\
0 & T_{2} T_{1}^{-1}
\end{array}\right) \mathcal{S}_{r}\left(P_{1}\right) \mathcal{S}_{r}\left(P_{2}\right)^{-1}
$$

where $\mathcal{S}_{r}\left(P_{i}\right)$ denotes the operator $P_{i} P\left(\mathcal{D}_{r}\right): \mathcal{H}\left(\mathcal{D}_{r}\right) \rightarrow$ Ran $P_{i}$.

Proof. Let

$$
U_{T_{1}, T_{2}}=\left(\begin{array}{cc}
I d & 0 \\
0 & T_{2} T_{1}^{-1}
\end{array}\right): \operatorname{Ran} P_{1} \rightarrow \operatorname{RanP}_{2}
$$

and observe that

$$
\begin{equation*}
U_{T_{1}, T_{2}} U_{T_{3}, T_{1}}=U_{T_{3}, T_{2}}, \quad U_{T_{1}, T_{2}}^{-1}=U_{T_{2}, T_{1}} \tag{11.2.2}
\end{equation*}
$$

and that if $A: \operatorname{Ran} P_{1} \rightarrow \operatorname{Ran} P_{1}$ is of the form $I d$ plus trace-class then

$$
\begin{equation*}
\operatorname{det}_{F r} A=\operatorname{det}_{F r} U_{T_{1}, T_{2}}^{-1} A U_{T_{1}, T_{2}}, \tag{11.2.3}
\end{equation*}
$$

where the determinant on the left-side is taken on $\operatorname{Ran} P_{1}$ and the determinant on the right-side is taken on $\operatorname{Ran} P_{2}$. Then since $U(P)=U_{K, T}$, we have using the invariance (10.2.7) of the canonical determinant under a unitary twist and the multiplicativity (??) of the Fredholm determinant

$$
\begin{aligned}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1, r}}\left(\operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2, r}}\right)^{-1} & =\operatorname{det}_{F r}\left(U_{K_{r}, T_{1}}^{-1} \mathcal{S}_{r}\left(P_{1}\right)\right) \operatorname{det}_{F r}\left(\left(U_{K_{r}, T_{2}}^{-1} \mathcal{S}_{r}\left(P_{2}\right)\right)^{-1}\right) \\
& =\operatorname{det}_{F r}\left(U_{K_{r}, T_{1}}^{-1} \mathcal{S}_{r}\left(P_{1}\right) \mathcal{S}\left(P_{2}\right)^{-1} U_{K_{r}, T_{2}}\right) \\
& =\operatorname{det}_{F r}\left(U_{K_{r}, T_{2}}^{-1} U_{T_{1}, T_{2}} \mathcal{S}_{r}\left(P_{1}\right) \mathcal{S}\left(P_{2}\right)^{-1} U_{K_{r}, T_{2}}\right) \\
& =\operatorname{det}_{F r}\left(U_{T_{1}, T_{2}} \mathcal{S}_{r}\left(P_{1}\right) \mathcal{S}_{r}\left(P_{2}\right)^{-1}\right),
\end{aligned}
$$

where the last two lines use (11.2.2) and (11.2.3), respectively.

Hence setting

$$
\mathcal{S}_{r}=\left(\begin{array}{cc}
I d & 0 \\
0 & T_{2} T_{1}^{-1}
\end{array}\right) P_{1} P\left(\mathcal{D}_{r}\right)\left(P_{2} P\left(\mathcal{D}_{r}\right)\right)^{-1} P_{2}: \text { Ran } P_{2} \rightarrow \text { Ran } P_{2}
$$

we have proved

Corollary 11.2.2.

$$
\begin{equation*}
\left.\frac{d}{d r}\left\{l n \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}=\left.\operatorname{Tr}\left(\left(\frac{d}{d r} \mathcal{S}_{r}\right) \mathcal{S}_{r}^{-1}\right)\right|_{r=0}=\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1} \tag{11.2.4}
\end{equation*}
$$

Lemma 11.2.3.

$$
\begin{equation*}
\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1}=\operatorname{Tr} P_{1} \gamma_{0}\left(\left.\frac{d}{d r} \mathcal{K}_{r}\left(P_{2}\right)\right|_{r=0} P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right)\right. \tag{11.2.5}
\end{equation*}
$$

Proof. We compute

$$
\begin{gathered}
\left.\frac{d}{d r} \ln \operatorname{det} \mathcal{S}_{r}\right|_{r=0}=\left.\operatorname{Tr}\left(\left(\frac{d}{d r} \mathcal{S}_{r}\right) \mathcal{S}_{r}^{-1}\right)\right|_{r=0}=\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1} \\
=\operatorname{Tr}\left(\begin{array}{cc}
\operatorname{Id} & 0 \\
0 & T_{2} T_{1}^{-1}
\end{array}\right)\left\{\left.\frac{d}{d r}\left(P_{1} P\left(\mathcal{D}_{r}\right)\left(P_{2} P\left(\mathcal{D}_{r}\right)\right)^{-1} P_{2}\right)\right|_{r=0}\right\}\left(P_{2} P(\mathcal{D})\left(P_{1} P(\mathcal{D})\right)^{-1}\left(\begin{array}{cc}
I d & 0 \\
0 & T_{1} T_{2}^{-1}
\end{array}\right)\right. \\
=\left.\operatorname{Tr} \frac{d}{d r}\left(P_{1} \gamma_{0} \mathcal{S}_{r}\left(P_{2}\right)^{-1} P_{2}\right)\right|_{r=0} P_{2} \gamma_{0} \mathcal{K} \mathcal{S}\left(P_{1}\right)^{-1} P_{1} \\
=\left.\operatorname{Tr} \frac{d}{d r}\left(P_{1} \gamma_{0} \mathcal{K}_{r}\left(P_{2}\right)\right)\right|_{r=0} P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right) \\
= \\
\operatorname{Tr} P_{1} \gamma_{0}\left(\left.\frac{d}{d r} \mathcal{K}_{r}\left(P_{2}\right)\right|_{r=0} P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right) .\right.
\end{gathered}
$$

The lemma is proved.

The next Lemma takes care of the variation of the operator $\mathcal{K}_{r}\left(P_{2}\right)$

Lemma 11.2.4. The following formula holds at $r=0$

$$
\begin{equation*}
\dot{\mathcal{K}}_{0}\left(P_{2}\right):=\left.\frac{d}{d r} \mathcal{K}_{r}\left(P_{2}\right)\right|_{r=0}=-\mathcal{D}_{P_{2}} \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) . \tag{11.2.6}
\end{equation*}
$$

Proof. Let us fix $f \in \operatorname{Ran} P_{2}$ and let $s_{r}=\mathcal{K}_{r}\left(P_{2}\right) f$. We have

$$
\mathcal{D}_{r} s_{r}=0 \quad \text { and } P_{2} \gamma_{0} s_{r}=f,
$$

hence differentiation with respect to $r$ gives

$$
\left(\frac{d}{d r} \mathcal{D}_{r}\right) s_{r}=-\mathcal{D}_{r}\left(\frac{d}{d r} s_{r}\right) \text { and } \frac{d}{d r}\left(P_{2}\left(\gamma_{0} s_{r}\right)\right)=P_{2}\left(\gamma_{0} \frac{d}{d r} s_{r}\right)=0
$$

hence $\frac{d}{d r} s_{r} \in \operatorname{dom} \mathcal{D}_{P_{2}}$. We obtain

$$
\frac{d}{d r} \mathcal{K}_{r}\left(P_{2}\right) f=\frac{d}{d r} s_{r}=-\mathcal{D}_{r, P_{2}}^{-1}\left(\frac{d}{d r} \mathcal{D}_{r}\right) s_{r}=-\mathcal{D}_{r, P_{2}}^{-1}\left(\frac{d}{d r} \mathcal{D}_{r}\right) \mathcal{K}_{r}\left(P_{2}\right) f
$$

This gives at $r=0$

$$
\dot{\mathcal{K}}_{0}\left(P_{2}\right)=-\mathcal{D}_{P_{2}}^{-1} \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right)
$$

The trace of $\dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1}$ is therefore given by the following formula

$$
\begin{equation*}
\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1}=\operatorname{Tr} P_{1} \gamma_{0}\left(-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right) \tag{11.2.7}
\end{equation*}
$$

The next important step is to change the order of the operator under the trace:

$$
\begin{gathered}
\operatorname{Tr} P_{1} \gamma_{0}\left(-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right)=\operatorname{Tr}\left(P_{1} \gamma_{0}\left(-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) P_{2}\right)\left(P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right)\right) \\
=\operatorname{Tr}\left(P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right)\right)\left(P_{1} \gamma_{0}\left(-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) P_{2}\right)
\end{gathered}
$$

The exchange is justified by the fact that

$$
P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right)=P_{2} P P(\mathcal{D}) \mathcal{S}\left(P_{1}\right)^{-1} P_{1}
$$

is a pseudodifferential operator of order 0 (with the symbol equal to the symbol of $P(\mathcal{D})$ ), and hence that it is a bounded operator on $L^{2}(Y ; S \mid Y)$. Thus we have

$$
\begin{equation*}
\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1}=\operatorname{Tr}\left(P_{2} \gamma_{0} \mathcal{K}\left(P_{1}\right)\right)\left(P_{1} \gamma_{0}\left(-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) P_{2}\right) \tag{11.2.8}
\end{equation*}
$$

Now the formula for the variation of the Canonical determinant follows from the next result:

Lemma 11.2.5.

$$
\begin{equation*}
\mathcal{K}\left(P_{1}\right) P_{1} \gamma_{0} \mathcal{D}_{P_{2}}^{-1}=\mathcal{D}_{P_{2}}^{-1}-\mathcal{D}_{P_{1}}^{-1} . \tag{11.2.9}
\end{equation*}
$$

Proof. We fix $f \in L^{2}(M ; S)$. Let

$$
h=\mathcal{K}\left(P_{1}\right) P_{1} \gamma_{0}\left(\mathcal{D}_{P_{2}}^{-1} f\right) .
$$

Observe that the section $h$ is the unique solution of $\mathcal{D}$ with boundary data along $P_{1}$ equal to $P_{1} \gamma_{0}\left(\mathcal{D}_{P_{2}}^{-1} f\right)$. Indeed
$P_{1}\left(\gamma_{0} h\right)=P_{1}\left(g_{0} \mathcal{K}\left(P_{1}\right) P_{1} \gamma_{0}\left(\mathcal{D}_{P_{2}}^{-1} f\right)\right)=P_{1} P(\mathcal{D}) \mathcal{S}\left(P_{1}\right)^{-1} P_{1} \gamma_{0}\left(\mathcal{D}_{P_{2}}^{-1} f\right)=P_{1} \gamma_{0}\left(\mathcal{D}_{P_{2}}^{-1} f\right)$,
and uniqueness is a consequence of Proposition 4.3.6. Now, the section $g=\left(\mathcal{D}_{P_{2}}^{-1}-\mathcal{D}_{P_{1}}^{-1}\right) f$ is also a solution of $\mathcal{D}$ and the restriction of $g$ to the boundary has $P_{1}$-component equal to

$$
\begin{aligned}
P_{1}\left(\gamma_{0} g\right) & =P_{1}\left(\gamma_{0} \mathcal{K}\left(\mathcal{S}\left(P_{1}\right)^{-1} P_{1}-\mathcal{S}\left(P_{2}\right)^{-1} P_{2}\right) \gamma_{0} \mathcal{D}^{-1} f\right) \\
& =P_{1} P(\mathcal{D})\left(\mathcal{S}\left(P_{1}\right)^{-1} P_{1}-\mathcal{S}\left(P_{2}\right)^{-1} P_{2}\right) \gamma_{0} \mathcal{D}^{-1} f \\
& =P_{1} \gamma_{0} \mathcal{D}^{-1} f-P_{1} \gamma_{0}\left(\mathcal{K} \mathcal{S}\left(P_{2}\right)^{-1} P_{2} \gamma_{0} \mathcal{D}^{-1} f\right. \\
& =P_{1} \gamma_{0}\left(\mathcal{D}_{P_{2}}^{-1}\right) f
\end{aligned}
$$

and therefore $h$ and $g$ are the same section.

Hence we obtain from equation (11.2.8) and Lemma 11.2.5 that

$$
\begin{equation*}
\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1}=\operatorname{Tr} P_{2} \gamma_{0}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} \mathcal{K}\left(P_{2}\right) P_{2} \tag{11.2.10}
\end{equation*}
$$

The operator on the right side of (11.2.10) has a smooth kernel (see Corollary 7.3.2) and so we can again switch the order of operators:

$$
\begin{gathered}
\operatorname{Tr}\left(P_{2} \gamma_{0}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0}\right)\left(\mathcal{K}\left(P_{2}\right) P_{2}\right)=\operatorname{Tr}\left(\mathcal{K}\left(P_{2}\right) P_{2}\right)\left(P_{2} \gamma_{0}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2} 1}\right) \dot{\mathcal{D}}_{0}\right) \\
=\operatorname{Tr} \mathcal{K}\left(\mathcal{S}\left(P_{2}\right)^{-1} P_{2} \gamma_{0} \mathcal{K}\right)\left(\mathcal{S}\left(P_{2}\right)^{-1} P_{2}-\mathcal{S}\left(P_{1}\right)^{-1} P_{1}\right) \gamma_{0} \mathcal{D}^{-1} \dot{\mathcal{D}}_{0} \\
=\operatorname{Tr} \mathcal{K}\left(\mathcal{S}\left(P_{2}\right)^{-1} P_{2}-\mathcal{S}\left(P_{1}\right)^{-1} P_{1}\right) \gamma_{0} \mathcal{D}^{-1} \dot{\mathcal{D}}_{0}=\operatorname{Tr}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0}
\end{gathered}
$$

where we have used (4.3.5) and (4.3.8).
Thus we have

$$
\operatorname{Tr} \dot{\mathcal{S}}_{0} \mathcal{S}_{0}^{-1}=\operatorname{Tr}\left(\mathcal{D}_{P_{1}}^{-1}-\mathcal{D}_{P_{2}}^{-1}\right) \dot{\mathcal{D}}_{0} .
$$

This completes the proof of the following Theorem.

Theorem 11.2.6. With the assumptions of Theorem 10.2.2 one has
$\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\zeta} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}=\left.\frac{d}{d r}\left\{\ln \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1, r}}-\ln \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2, r}}\right\}\right|_{r=0}$.

### 11.3. Equality of the Determinants

In this Section we prove the main result of the paper. The key point here is the following elementary result, which allows us to integrate the equality (11.2.11). We refer to [78] for a more detailed discussion of the topology of $\tilde{G} r_{\infty}^{*}(\mathcal{D})$.

Proposition 11.3.1. The space $\tilde{G} r_{\infty}^{*}(\mathcal{D})$, which consists of projections $P \in G r_{\infty}^{*}(\mathcal{D})$ such that the operator $\mathcal{D}_{P}$ is invertible, is path connected.

Proof. We show that for any $P \in \tilde{G} r_{\infty}^{*}(\mathcal{D})$ there exists a path $\left\{P_{r}\right\}_{0 \leq r \leq 1} \in$ $\tilde{G} r_{\infty}^{*}(\mathcal{D})$ such that

$$
P_{0}=P \quad \text { and } \quad P_{1}=P(\mathcal{D}) .
$$

Let $H$ denote the range of the projection $P$. Lemma 4.3.2 tells us that if $\mathcal{D}_{P}$ is invertible then

$$
\begin{equation*}
H^{\perp} \oplus \mathcal{H}(\mathcal{D})=L^{2}(Y ; S \mid Y) \text { and } H^{\perp} \cap \mathcal{H}(\mathcal{D})=\{0\} \tag{11.3.1}
\end{equation*}
$$

Equivalently we can write the first equality in (11.3.1) as

$$
H \oplus \mathcal{H}(\mathcal{D})^{\perp}=L^{2}(Y ; S \mid Y)
$$

The equality above implies the existence of a linear operator $T: \mathcal{H}(\mathcal{D}) \rightarrow$ $\mathcal{H}(\mathcal{D})^{\perp}$, such that $H=\operatorname{graph}(T)$. The subspace $H$ is closed, and as a consequence of the Closed Graph Theorem $T$ is a continuous map. The fact that $H$ is Lagrangian gives the equality

$$
\begin{equation*}
T^{*} G+G T=0 \tag{11.3.2}
\end{equation*}
$$

where bundle anti-involution $G$ (see (7.1.2), (7.1.3)) determines symplectic structure on $L^{2}(Y ; S \mid Y)$. If we now write the projection $P$ with respect to the decomposition $L^{2}(Y ; S \mid Y)=\mathcal{H}(\mathcal{D}) \oplus \mathcal{H}(\mathcal{D})^{\perp}$, we obtain

$$
P=\left(\begin{array}{cc}
\left(I d+T^{*} T\right)^{-1} & \left(I d+T T^{*}\right)^{-1} T^{*}  \tag{11.3.3}\\
T\left(I d+T^{*} T\right)^{-1} & T\left(I d+T T^{*}\right)^{-1} T^{*}
\end{array}\right)
$$

Since $P \in G r_{\infty}^{*}(\mathcal{D})$ then $P-P(\mathcal{D})$ is a smoothing operator and so the operator $T$ has a smooth kernel. For each value of the parameter $r$ we define the operator $T_{r}=r T$ and the corresponding projection

$$
P_{r}=\left(\begin{array}{cc}
\left(I d+T_{r}^{*} T_{r}\right)^{-1} & \left(I d+T_{r} T_{r}^{*}\right)^{-1} T_{r}^{*} \\
T_{r}\left(I d+T_{r}^{*} T_{r}\right)^{-1} & T_{r}\left(I d+T_{r} T_{r}^{*}\right)^{-1} T_{r}^{*}
\end{array}\right)
$$

It is obvious that

$$
\operatorname{ker} P(\mathcal{D}) P_{r} \cong \operatorname{coker} S\left(P_{r}\right) \cong \operatorname{Graph}\left(T_{r}\right) \cap H(\mathcal{D})^{\perp}=\{0\}
$$

We know that index $S\left(P_{r}\right)$ is equal to 0 and hence that $S\left(P_{r}\right)$ also has a trivial kernel. The operators $T_{r}$ satisfy condition (11.3.2), hence $H_{r}=$ Ran $P_{r}$ is a Lagrangian subspace satisfying condition (11.3.1). It follows that the operators $\mathcal{D}_{P_{r}}$ are invertible. Moreover $P_{0}=P(\mathcal{D})$, which ends the proof.

The next result is a consequence of Theorem 11.2.6 and Proposition 11.3.1.

Proposition 11.3.2. Assume that we have $P_{1}, P_{2} \in G r_{\infty}^{*}(\mathcal{D})$ and $g \in$ $U^{\infty}\left(F^{-}\right)$such that all four operators $\mathcal{D}_{P_{1}}, \mathcal{D}_{U P_{1} U^{-1}}, \mathcal{D}_{P_{2}}, \mathcal{D}_{U P_{2} U^{-1}}$ are invertible, then

$$
\begin{equation*}
\frac{\operatorname{det}_{\zeta} \mathcal{D}_{U P_{1} U^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U P_{1} U^{-1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P_{1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1}}}=\frac{\operatorname{det}_{\zeta} \mathcal{D}_{U P_{2} U^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U P_{2} U^{-1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P_{2}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2}}} \tag{11.3.4}
\end{equation*}
$$

In particular, the ratio of the determinants does not depend on the choice of the base projection.

Proof. From Proposition 11.3.1, given any two projections from $G r_{\infty}^{*}(\mathcal{D})$ such that $\mathcal{D}_{P_{1}}$ and $\mathcal{D}_{P_{2}}$ are invertible operators, we can find a path $\left\{P_{r}\right\}$ in the subspace $\tilde{G} r_{\infty}^{*}(\mathcal{D})$ which connects $P_{1}$ and $P_{2}$. Hence we can use Theorem 11.2.6 and integrate equation (11.2.11), which gives the identity

$$
\begin{equation*}
\frac{\operatorname{det}_{\zeta} \mathcal{D}_{P_{1,1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1,1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P_{1,0}, 0} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{1,0}}}=\frac{\operatorname{tet}_{\zeta} \mathcal{D}_{P_{2,1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2,1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P_{2,0}, 0} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P_{2,0}}}, \tag{11.3.5}
\end{equation*}
$$

where by construction

$$
P_{i, 1}=g P_{i, 0} g^{-1}=g P_{i} g^{-1} \quad P_{i, 0}=P_{i} .
$$

We introduce an invariant $\mathcal{A}(g)$ using (11.3.4):

$$
\begin{equation*}
\mathcal{A}(g)=\frac{\operatorname{det}_{\zeta} \mathcal{D}_{U P U^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U P U^{-1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}} \tag{11.3.6}
\end{equation*}
$$

The next result follows from Proposition 11.3.2 and gives the first formula directly relating $\operatorname{det}_{\zeta}$ to the $\operatorname{det}_{\mathcal{C}}$.

Theorem 11.3.3. There is the following relation between $\operatorname{det}_{\zeta}$ and $\operatorname{det}_{\mathcal{C}}$ on $G r_{\infty}^{*}(\mathcal{D})$ :

$$
\begin{equation*}
\operatorname{det}_{\zeta} \mathcal{D}_{P}=\operatorname{det}_{\zeta} \mathcal{D}_{P(\mathcal{D})} \cdot \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P} \cdot \mathcal{A}(g) \tag{11.3.7}
\end{equation*}
$$

where, as before, $P=\left(\begin{array}{cc}I d & 0 \\ 0 & g\end{array}\right) P(\mathcal{D})\left(\begin{array}{cc}I d & 0 \\ 0 & g^{-1}\end{array}\right)$.

Proof. The result is immediate from the identity (11.3.4) with $P_{1}=$ $P(\mathcal{D})$ and $P_{2}=P=\left(\begin{array}{cc}I d & 0 \\ 0 & g\end{array}\right) P(\mathcal{D})\left(\begin{array}{cc}\text { Id } & 0 \\ 0 & g^{-1}\end{array}\right)$.

The main result of this Section is the following Theorem.

Theorem 11.3.4. The function $\mathcal{A}(g)$ is the trivial character on the group $U^{\infty}\left(F^{-}\right)$, i.e. for any $g \in U^{\infty}\left(F^{-}\right)$

$$
\mathcal{A}(g)=1
$$

Proof. Let $g$ and $h$ be elements of $G r_{\infty}^{*}(\mathcal{D})$ such that $\mathcal{D}_{U_{g} P(\mathcal{D}) U_{g}^{-1}}$, $\mathcal{D}_{U_{h} P(\mathcal{D}) U_{h}^{-1}}$ and $\mathcal{D}_{U_{h} U_{g} P(\mathcal{D}) U_{g}^{-1} U_{h}^{-1}}$ are invertible. We have

$$
\begin{gathered}
\mathcal{A}(h g)=\frac{\operatorname{det}_{\zeta} \mathcal{D}_{U_{h g} P U_{h g}^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U_{h g} P U_{h g}^{-1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}}= \\
\frac{\operatorname{det}_{\zeta} \mathcal{D}_{U_{h g} P P_{h g}^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U_{h g} P U_{h g}^{-1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{U_{g} P U_{g}^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U_{g} P U_{g}^{-1}}} \cdot \frac{\operatorname{det}_{\zeta} \mathcal{D}_{U_{g} P U_{g}^{-1}} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{U_{g} P U_{g}^{-1}}}{\operatorname{det}_{\zeta} \mathcal{D}_{P} / \operatorname{det}_{\mathcal{C}} \mathcal{D}_{P}}=\mathcal{A}(h) \mathcal{A}(g),
\end{gathered}
$$

hence $\mathcal{A}(g)$ is a character. It is well-known that there are only two nontrivial characters on the group $U^{\infty}\left(F^{-}\right)$

$$
\begin{equation*}
\mathcal{A}^{+}(g)=\operatorname{det}_{F r} g \quad \text { and } \mathcal{A}^{-}(g)=\left(\operatorname{det}_{F r} g\right)^{-1} . \tag{11.3.8}
\end{equation*}
$$

We study the variation of $\operatorname{det}_{\zeta}$ at the Calderon projection $P(\mathcal{D})$ to show that $\mathcal{A}(g)$ is actually the trivial character. Let $\alpha: F^{-} \rightarrow F^{-}$denote a selfadjoint operator with a smooth kernel. We define the 1-parameter smooth family of operators $\left\{g_{r}=e^{i r \alpha}\right\}$ in $U^{\infty}\left(F^{-}\right)$and the corresponding family of operators on $M$

$$
U_{r}=\left\{\begin{array}{ll}
I d & \text { on } M \backslash N \\
\left(\begin{array}{cc}
I d & 0 \\
0 & e^{i r \kappa(u) \alpha}
\end{array}\right) \text { on } N
\end{array} .\right.
$$

with $\kappa$ as in equation (10.2.3). The variation of the phase of the $\zeta$-determinant is equal to the variation of the $\eta$-invariant times the factor $-\left(\frac{i \pi}{2}\right)$. It follows now from formula (10.2.9) that

$$
\begin{gathered}
\left.\frac{d}{d r} \eta_{\mathcal{D}_{P_{i, r}}}(0)\right|_{r=0}=\frac{i}{\pi} \int_{0}^{1} d u \operatorname{Tr}\left(\left.\frac{d}{d r}\left(g_{r, u}^{-1} \frac{\partial g_{r, u}}{\partial u}\right)\right|_{r=0}\right)= \\
\frac{i}{\pi} \int_{0}^{1} d u \operatorname{Tr} \frac{d}{d r}\left(i r \kappa^{\prime}(u) \alpha\right)=-\frac{\operatorname{Tr} \alpha}{\pi} \int_{0}^{1} \kappa^{\prime}(u) d u=\frac{\operatorname{Tr} \alpha}{\pi},
\end{gathered}
$$

and so we see that variation of the $\zeta$-determinant in this case is equal to $-i \frac{\operatorname{Tr} \alpha}{2}$. On the other hand, the canonical determinant of $\mathcal{D}_{g_{r} P(\mathcal{D}) g_{r}^{-1}}$ is equal to

$$
\begin{gathered}
\operatorname{det}_{\mathcal{C}} \mathcal{D}_{g_{r} P(\mathcal{D}) g_{r}^{-1}}=\operatorname{det}_{F r} \frac{I d+K T_{r}^{-1}}{2}=\operatorname{det}_{F r} \frac{I d+e^{-i r \alpha}}{2} \\
=\operatorname{det}_{F r}\left(e^{-r \frac{i \alpha}{2}} \frac{e^{r \frac{i \alpha}{2}}+e^{-r \frac{i \alpha}{2}}}{2}\right)=\operatorname{det}_{F r}\left(e^{-r \frac{i \alpha}{2}} \cos r \frac{\alpha}{2}\right)=e^{-\frac{i r}{2} T r \alpha} \operatorname{det}_{F r}\left(\cos r \frac{\alpha}{2}\right) .
\end{gathered}
$$

Therefore the variation of the phase of the canonical determinant is equal to the variation of the phase of the $\zeta$-determinant. From equation (??), the variation of the only two non-trivial characters (11.3.8) of the group $U\left(F^{-}\right)$ are in our case equal to

$$
\left.\frac{d}{d r}\left(\mathcal{A}^{ \pm}\left(g_{r}\right)\right)\right|_{r=0}= \pm i \cdot \operatorname{Tr} \alpha
$$

and hence $\mathcal{A}(g)$ is the trivial character of the group $U^{\infty}\left(F^{-}\right)$.

This completes the proof of the main Theorem.

CHAPTER 12
Pasting of Determinants

## APPENDIX A

## The Regularity of the Local $\eta$-Function at $s=0$

We prove the regularity of the local $\eta$-function over a closed manifold at $s=0$, proved first by Bismut and Freed [17] using non-trivial results from stochastic analysis. The details of our proof are inspired by calculations presented in Bismut and Cheeger [14], Section 3.

Theorem A.0.5. Let $\mathcal{D}: C^{\infty}(M ; \$) \rightarrow C^{\infty}(M ; \mathbb{S})$ denote a compatible Dirac operator over a closed manifold $M$ of odd dimension m. Let $\mathrm{e}\left(t ; x, x^{\prime}\right)$ denote the integral kernel of the heat operator $e^{-t \mathcal{D}^{2}}$. Then there exists a positive constant $C$ such that

$$
\left|\operatorname{tr} \mathcal{D}_{x} e\left(t ; x, x^{\prime}\right)\right|_{x=x^{\prime}} \mid<C \sqrt{t}
$$

for all $x \in M$ and $0<t<1$.

We recall the definition of the 'local' $\eta$-function .

Definition A.0.6. Let $\left\{f_{k} ; \lambda_{k}\right\}_{k \in \mathbf{Z}}$ be a discrete spectral resolution of $\mathcal{D}$. Then we define

$$
\begin{aligned}
\eta_{\mathcal{D}}(s ; x) & :=\sum_{\lambda_{k} \neq 0} \operatorname{sign}\left(\lambda_{k}\right)\left|\lambda_{k}\right|^{-s}\left\langle f_{k}(x), f_{k}(x)\right\rangle \\
& =\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}}\left(\sum_{\lambda_{k} \neq 0} \lambda_{k} e^{-t \lambda_{k}^{2}}\left\langle f_{k}(x), f_{k}(x)\right\rangle\right) d t \\
& =\frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_{0}^{\infty} t^{\frac{s-1}{2}} \operatorname{tr} \mathcal{D} \mathrm{e}(t ; x, x) d t .
\end{aligned}
$$

Corollary A.0.7. Under the assumptions of the preceding theorem the 'local' $\eta$-function $\eta_{\mathcal{D}}(s ; x)$ is holomorphic in the halfplane $\Re(s) \geq-2$ for any $x \in M$.

Proof of the Corollary. ...

Proof of the Theorem. ${ }^{* * * * * * * * * * * * * * * * ~}$
We follow KPW, On the Bismut-Cheeger proof...

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[^0]:    ${ }^{1}$ Date: November 15, 2001. File name: BOOK1B.TEX, uses BOOKC.STY and BOOKREFE.TEX.

[^1]:    ${ }^{1}$ Date: November 15, 2001. File name: BOOKHEAT.TEX, uses BOOKC.STY and BOOKREFE.TEX.

[^2]:    ${ }^{2}$ The equality $\left.\mathcal{D}_{j}\right|_{U_{j}}=\left.\mathcal{D}\right|_{U_{j}}$ is to be understood with regard to the domains.

[^3]:    ${ }^{3}$ The equality $\left.\mathcal{D}_{j}\right|_{U_{j}}=\left.\mathcal{D}\right|_{U_{j}}$ is to be understood with regard to the domains.

[^4]:    $1_{\text {Date: }}$ November 15, 2001. File name: BOOK7C.TEX, uses BOOKC.STY and BOOKREFE.TEX.

[^5]:    ${ }^{1}$ Date: November 15, 2001. File name: BOOK8B.TEX, uses BOOKC.STY, BOOKAPP.TEX, and BOOKREFE.TEX.

[^6]:    $1_{\text {Date: November 15, 2001. File name: BOOKMOD1.TEX, uses BOOKC.STY and }}$ BOOKREFE.TEX.

