Spectral Invariants of Operators of Dirac Type on Partitioned Manifolds

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Abstract. We review the concepts of the *index* of a Fredholm operator, the *spectral flow* of a curve of self-adjoint Fredholm operators, the *Maslov index* of a curve of Lagrangian subspaces in symplectic Hilbert space, and the *eta invariant* of operators of Dirac type on closed manifolds and manifolds with boundary. We emphasize various (occasionally overlooked) aspects of rigorous definitions and explain the quite different stability properties. Moreover, we utilize the heat equation approach in various settings and show how these topological and spectral invariants are mutually related in the study of additivity and nonadditivity properties on partitioned manifolds.

Contents

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Introduction

For many decades now, workers in differential geometry and mathematical physics have been increasingly concerned with differential operators (exterior differentiation, connections, Laplacians, Dirac operators, etc.) associated to underlying Riemannian or space–time manifolds. Of particular interest is the interplay between the spectral decomposition of such operators and the geometry/topology of the underlying manifold. This has become a large, diverse field involving index theory, the distribution of eigenvalues, zero sets of eigenfunctions, Green functions, heat and wave kernels, families of elliptic operators and their determinants, canonical sections, etc.. Moreover, Donaldson's analysis of moduli of solutions of the nonlinear Yang–Mills equations and Seiberg–Witten theory have led to profound insights into the classification of four–manifolds, which were not accessible by techniques that are effective in higher dimensions.

We shall touch upon many of these topics, but we focus on three invariants characterizing the asymmetry of the spectrum of operators of Dirac type: the *index* which gives the chiral asymmetry of the kernel or null space (i.e., the difference in the number of independent left and right zero modes) of a total Dirac operator; the *spectral flow* of a curve of Dirac operators that counts the net number of eigenvalues moving from the negative half line over to the positive; and the *eta invariant* which describes the overall asymmetry of the spectrum of a Dirac operator. The three invariants appear both for Dirac operators and curves of Dirac operators on a closed manifold or on a smooth compact manifold with boundary subject to suitable boundary conditions.

Index and spectral flow can be described in general functional analytic terms, namely for bounded and unbounded, closed Fredholm operators and *curves* of bounded and unbounded self–adjoint Fredholm operators. Correspondingly, stability properties of index and spectral flow and their topological and geometric meaning are relatively well understood. There is, however, not an easily identifiable operator class for eta invariant, and its behavior under perturbations is rather delicate.

Most significant differences between the behavior of these three invariants are met when we address splitting properties on partitioned manifolds (with product metrics assumed near the partitioning hypersurface). It is well known that the index can be described in local terms. This is one aspect of the Atiyah–Singer Index Theorem. So, splitting formulas for the index are relatively easily obtainable, once one specifies and understands boundary and transmission conditions for the gluing of the parts of the underlying manifold. In this process, precise additivity is obtained only for a few invariants, Euler characteristic and signature, which actually can be characterized by cutting and pasting invariance. In general, an error term appears which can be expressed either by spectral flow or, equivalently, by the index of a boundary value problem on a cylinder over the separating hypersurface or, alternatively, by the index of a suitable Fredholm pair.

Surprisingly, simple splitting formulas can be obtained also for the spectral flow and eta invariant. This is particularly surprising for the eta invariant where we only have a local formula for the first derivative. Now the integer error terms are expressed by the Maslov index for curves of Cauchy data spaces. These formulas relate the symmetric category of self–adjoint Dirac operators over closed partitioned manifolds (and self–adjoint boundary value problems over compact manifolds with boundary) to the symplectic analysis of Lagrangian subspaces (the Cauchy data spaces).

One message of this review is, that index, spectral flow, and eta invariant, in spite of their quite different appearance, share various features which become most visible on partitioned manifolds. Roughly speaking, one reason for that is that the spectral flow of a path of Dirac type operators (say, on a closed manifold) with unitarily equivalent ends $A_1 = gA_0g^{-1}$ equals the index of the induced suspension operator $\partial_t + A_t$ on the underlying mapping torus (see [**BoWo93**, Theorems 17.3, 17.17, and Proposition 25.1]). Another reason is that the integer part of the derivative of the eta invariant along a path of Dirac operators or boundary problems can be expressed by the spectral flow (see, e.g., [**DoWo91**], [**LeWo96**, p. 39], and [**KiLe00**, Section 3]).

We also summarize recent discussion on additivity and non–additivity of the zeta regularized determinant. It should be mentioned that the various splitting formulas all depend decisively on the well–established unique continuation property (UCP) for operators of Dirac type. We give a full proof of weak UCP below in Section 1.4. Moreover (modulo some technicalities for the computation of the index density form, to appear in our forthcoming book [**BlBo03**]), we provide proofs of the Atiyah–Singer and Atiyah–Patodi–Singer Index Theorems in important special cases using heat equation methods. That we explain concrete calculations only for the case of the Atiyah–Patodi–Singer (spectral – “APS”) boundary condition is no big loss of generality since each admissible boundary condition for an operator of Dirac type $\mathcal{D}$ with tangential part $\mathcal{B}$ can be written as the APS projection of a perturbed operator $\mathcal{B}'$ (see below Lemma 2.29, following a recent result, Gerd Grubb [**Gr02**]).

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1. Basic Notations and Results

1.1. Index of Fredholm Operators and Spectral Flow of Curves of Self–Adjoint Fredholm Operators.

1.1.1. *Notation.* Let H be a separable complex Hilbert space. First let us introduce some notation for various spaces of operators in H:

$$\begin{aligned}
\mathcal{C}(H) &:= && \text{closed, densely defined operators on } H,\\
\mathcal{B}(H) &:= && \text{bounded linear operators } H \to H,\\
\mathcal{U}(H) &:= && \text{unitary operators } H \to H,\\
\mathcal{K}(H) &:= && \text{compact linear operators } H \to H,\\
\mathcal{F}(H) &:= && \text{bounded Fredholm operators } H \to H,\\
\mathcal{CF}(H) &:= && \text{closed, densely defined Fredholm operators on } H.
\end{aligned}$$

If no confusion is possible we will omit "(H)" and write $\mathcal{C}$, $\mathcal{B}, \mathcal{K}$, etc.. By $\mathcal{C}^{\mathrm{sa}}, \mathcal{B}^{\mathrm{sa}}$ etc., we denote the set of self–adjoint elements in $\mathcal{C}$, $\mathcal{B}$, etc..

1.1.2. *Operators With Index – Fredholm Operators.* The topology of the operator spaces $\mathcal{U}(H)$, $\mathcal{F}(H)$, and $\mathcal{F}^{\mathrm{sa}}(H)$ is quite well understood. The key results are

1. the *Kuiper Theorem* which states that $\mathcal{U}(H)$ is contractible;
2. the *Atiyah–Jänich Theorem* which states that $\mathcal{F}(H)$ is a classifying space for the functor K. Explicitly, the construction of the index bundle of a continuous family of bounded Fredholm operators parametrized over a compact topological space X yields a homomorphism of semigroups, namely index : $[X, \mathcal{F}] \to K(X)$. In particular, the index is a homotopy invariant, and it provides a one–to–one correspondence of the connected components $[point, \mathcal{F}]$ of $\mathcal{F}$ with $K(point) = \mathbb{Z}$.
3. the corresponding observation that $\mathcal{F}^{\mathrm{sa}}(H)$ consists of three connected components, the contractible subsets $\mathcal{F}^{\mathrm{sa}}(H)_+$ and $\mathcal{F}^{\mathrm{sa}}(H)_-$ of essentially positive, respectively essentially negative, Fredholm operators, and the topologically nontrivial component $\mathcal{F}^{\mathrm{sa}}_*(H)$ which is a classifying space for the functor K^{-1}. In particular, the spectral flow gives an isomorphism of the fundamental group $\pi_1(\mathcal{F}^{\mathrm{sa}}_*(H))$ onto the integers.

Full proofs can be found in [**BlBo03**] for items 1 and 2, and in [**AtSi69**], [**BoWo93**], and [**Ph96**] for item 3.

So much for the bounded case. Of course, the Dirac operators of interest to us are not bounded in L^2. On closed manifolds, however, they can be considered as bounded operators from the first Sobolev space H^1 into L^2, and by identifying these two Hilbert spaces we can consider a Dirac operator as a bounded operator of a Hilbert space in itself. The same philosophy can be applied to Dirac operators on compact manifolds with boundary when we consider the domain not as dense subspace in L^2 but as Hilbert space (with the graph inner product) and then identify.

Strictly speaking, the concept of unbounded operator is dispensable here. However, for varying boundary conditions it is necessary to keep the distinction and to follow the variation of the domain as a variation of subspaces in L^2.

Following [**CorLab**], we generalize the concept of *Fredholm operators* to the unbounded case.

Definition 1.1. Let H be a complex separable Hilbert space. A linear (not necessarily bounded) operator F with domain $\operatorname{Dom}(F)$, null–space $\operatorname{Ker}(F)$, and range $\operatorname{Im}(F)$ is called *Fredholm* if the following conditions are satisfied.

(i) $\operatorname{Dom}(F)$ is dense in H.
(ii) F is closed.
(iii) The range $\operatorname{Im}(F)$ of F is a closed subspace of H.
(iv) Both $\dim \operatorname{Ker}(F)$ and $\operatorname{codim} \operatorname{Im}(F) = \dim \operatorname{Im}(F)^\perp$ are finite. The difference of the dimensions is called $\operatorname{index}(F)$.

So, a closed operator F is characterized as a Fredholm operator by the same properties as in the bounded case. Moreover, as in the bounded case, F is Fredholm if and only if F^* is Fredholm (proving the closedness of $\operatorname{Im}(F^*)$ is delicate: see [**CorLab**, Lemma 1.4]) and (clearly) we have

$$\operatorname{index} F = \dim \operatorname{Ker} F - \dim \operatorname{Ker} F^* = -\operatorname{index} F^*.$$

In particular, $\operatorname{index} F = 0$ in case F is self–adjoint.

The composition of (not necessarily bounded) Fredholm operators yields again a Fredholm operator. More precisely, we have the following composition rule. For the proof, which is considerably more involved than that in the bounded case, we refer to [**GoKr57**], [**CorLab**, Lemma 2.3 and Theorem 2.1].

Theorem 1.2. (Gohberg, Krein) *If F and G are (not necessarily bounded) Fredholm operators then their product GF is densely defined with*

$$\operatorname{Dom}(GF) = \operatorname{Dom}(F) \cap F^{-1} \operatorname{Dom}(G)$$

and is a Fredholm operator. Moreover,

$$\operatorname{index} GF = \operatorname{index} F + \operatorname{index} G\,.$$

1.1.3. *Metrics on the Space of Closed Operators.* For $S, T \in \mathcal{C}(H)$ the orthogonal projections $P_{\mathfrak{G}(S)}$, $P_{\mathfrak{G}(T)}$ onto the graphs of S, T in $H \oplus H$ are bounded operators and

$$\gamma(S,T) := \|P_{\mathfrak{G}(T)} - P_{\mathfrak{G}(S)}\|$$

defines a metric for $\mathcal{C}(H)$, the *projection metric*.

It is also called the *gap metric* and it is (uniformly) equivalent with the metric given by measuring the distance between the (closed) graphs, namely

$$d(\mathfrak{G}(S), \mathfrak{G}(T)) := \sup_{\{x \in \mathfrak{G}(S) : \|x\| = 1\}} d(x, \mathfrak{G}(T)) + \sup_{\{x \in \mathfrak{G}(T) : \|x\| = 1\}} d(x, \mathfrak{G}(S)).$$

For details and the proof of the following Lemma and Theorem, we refer to [**CorLab**, Section 3].

LEMMA 1.3. *For $T \in \mathcal{C}(H)$ the orthogonal projection onto the graph of T in $H \oplus H$ can be written (where $R_T := (\mathrm{I} + T^*T)^{-1}$) as*

$$P_{\mathfrak{G}(T)} = \begin{pmatrix} R_T & R_T T^* \\ T R_T & T R_T T^* \end{pmatrix} = \begin{pmatrix} R_T & T^* R_{T^*} \\ T R_T & T T^* R_{T^*} \end{pmatrix} = \begin{pmatrix} R_T & T^* R_{T^*} \\ T R_T & \mathrm{I} - R_{T^*} \end{pmatrix}.$$

THEOREM 1.4. (Cordes, Labrousse) *a) The space $\mathcal{B}(H)$ of bounded operators on H is dense in the space $\mathcal{C}(H)$ of all closed operators in H. The topology induced by the projection ($\cong$ gap) metric on $\mathcal{B}(H)$ is equivalent to that given by the operator norm.*
b) Let $\mathcal{CF}(H)$ denote the space of closed (not necessarily bounded) Fredholm operators. Then the index is constant on the connected components of $\mathcal{CF}(H)$ and yields a bijection between the integers and the connected components.

EXAMPLE 1.5. Let H be a Hilbert space with $e_1, e_2, \dots$ a complete orthonormal system (i.e., an orthonormal basis for H). Consider the multiplication operator M_{id}, given by the domain

$$D := \mathrm{Dom}(M_{\mathrm{id}}) := \left\{ \sum_{j=1}^{\infty} c_j e_j \mid \sum_{j=1}^{\infty} j^2 |c_j|^2 < +\infty \right\}$$

and the operation

$$D \ni u = \sum c_j e_j \quad \mapsto \quad M_{\mathrm{id}}(u) := \sum j c_j e_j .$$

It is a densely defined closed operator which is injective and surjective. Let P_n denote the orthogonal projection of H onto the linear span of the n-th orthonormal basis element e_n. Clearly the sequence (P_n) does not converge in $\mathcal{B}(H)$ in the operator norm. However, the sequence $M_{\mathrm{id}} - 2nP_n$ of self–adjoint Fredholm operators converges in $\mathcal{C}(H)$ with the projection metric to M_{id}.

This can be seen by the following argument: On the subset of self–adjoint (not necessarily bounded) operators in the space $\mathcal{C}^{\mathrm{sa}}(H)$ the projection metric is uniformly equivalent to the metric γ given by

$$\gamma(A_1, A_2) := \|(A_1 + i)^{-1} - (A_2 + i)^{-1}\|,$$

(see below Theorem 1.10). Then for $T_n := M_{\mathrm{id}} - 2nP_n$,

$$\left\|(T_n + i)^{-1} - (M_{\mathrm{id}} + i)^{-1}\right\| = \left\|(-n+i)^{-1}\, e_n - (n+i)^{-1}\, e_n\right\| = \tfrac{2n}{n^2+1} \to 0.$$

Remark 1.6. The results by Heinz Cordes and Jean–Philippe Labrousse may appear to be rather counter–intuitive. For (a), it is worth mentioning that the operator–norm distance and the projection metric on the set of bounded operators are equivalent but not uniformly equivalent since the operator norm is complete while the projection metric is not complete on the set of bounded operators. Actually, this is the point of the first part of (a); see also the preceding example.

Assertion (b) says two things: (i) that the index is a homotopy invariant, i.e., two Fredholm operators have the same index if they can be connected by a continuous curve in $\mathcal{CF}(H)$; (ii) that two Fredholm operators having the same index always can be connected by a continuous curve in $\mathcal{CF}(H)$. Note that the topological results are not as far reaching as for bounded Fredholm operators.

1.1.4. *Self–Adjoint Fredholm Operators and Spectral Flow.* We investigate the topology of the subspace of self–adjoint (not necessarily bounded) Fredholm operators. Many users of the notion of spectral flow feel that the definition and basic properties are too trivial to bother with. However, there are some difficulties both with extending the definition of spectral flow from loops to paths and from curves of bounded self–adjoint Fredholm operators to curves of not necessarily bounded self–adjoint Fredholm operators.

To overcome the second difficulty, the usual way is to apply the *Riesz transformation* which yields a bijection

$$\begin{aligned} &\mathcal{R} : \mathcal{C}^{\mathrm{sa}} \longrightarrow \{S \in \mathcal{B}^{\mathrm{sa}} \mid \|S\| \le 1 \text{ and } S \pm I \text{ both injective}\}, \text{ where} \\ &\mathcal{R}\,(T) := T(I+T^2)^{-1/2} \end{aligned} \tag{1.1}$$

In [**BoFu98**] the following theorem was proved:

Theorem 1.7. *Let S be a self–adjoint operator with compact resolvent in a real separable Hilbert space $\mathcal{H}$ and let C be a bounded self–adjoint operator. Then the sum $S + C$ also has compact resolvent and is a closed Fredholm operator. We have*

$$\|\mathcal{R}(S+C) - \mathcal{R}(S)\| \le c\|C\|\,,$$

where the constant c does not depend on S or on C.

The preceding theorem is applied in the following form:

Corollary 1.8. *Curves of self–adjoint (unbounded) Fredholm operators in a separable real Hilbert space of the form $\{S + C_t\}_{t\in I}$ are mapped into continuous curves in $\mathcal{F}^{\mathrm{sa}}$ by the transformation $\mathcal{R}$ when S is a self–adjoint operator with compact resolvent and $\{C_t\}_{t\in I}$ is a continuous curve of bounded self–adjoint operators.*

Remark 1.9. Define

$$\mathcal{T}_S : \mathcal{B}^{\mathrm{sa}} \longrightarrow \mathcal{CF}^{\mathrm{sa}} \text{ by } \ \mathcal{T}_S\,(C) := S + C$$

This is translation by S, mapping bounded self–adjoint operators on $\mathcal{H}$ into self–adjoint Fredholm operators in $\mathcal{H}$. On $\mathcal{CF}^{\mathrm{sa}}$, the gap topology is defined by the metric

$$g(A_1, A_2) := \sqrt{\|R_{A_1} - R_{A_2}\|^2 + \|A_1 R_{A_1} - A_2 R_{A_2}\|^2}\,,$$

where $R_{A_{\cdot}} := (\mathrm{I} + A_{\cdot}^2)^{-1}$ as before (see Cordes and Labrousse, [**CorLab**] and also Kato, [**Kat**]). Theorem 1.7 says that the composition $\mathcal{R} \circ \mathcal{T}_S$ is continuous.

$$\begin{tikzcd} \mathcal{B}^{\mathrm{sa}} \arrow[r, "\mathcal{T}_S"] \arrow[dr, "\mathcal{R}\circ\mathcal{T}_S"'] & \mathcal{CF}^{\mathrm{sa}} \arrow[d, "\mathcal{R}"] \\ & \mathcal{F}^{\mathrm{sa}} \end{tikzcd}$$

Further, we can prove that the translation operator $\mathcal{T}_S$ is a continuous operator from $\mathcal{B}^{\mathrm{sa}}$ onto the subspace $\mathcal{B}^{\mathrm{sa}} + S \subset \mathcal{CF}^{\mathrm{sa}}$.

The preceding arguments permit to treat continuous curves of Dirac operators in the same way as continuous curves of self–adjoint bounded Fredholm operators under the precondition that the domain is fixed and the perturbation is only by bounded self–adjoint operators. That precondition is satisfied when we have a curve of Dirac operators on a closed manifold which differ only by the underlying connection. It is also satisfied for curves of Dirac operators on a manifold with boundary as long the perturbation is bounded. In particular, this demands that the domain remains fixed.

A closer look at Example 1.5 shows that the preceding argument cannot be generalized: The sequence of the Riesz transforms $\mathcal{R}(T_n)$ of $T_n := M_{\mathrm{id}} - 2nP_n$ does not converge, since $\mathcal{R}(T_n)$ is not a Cauchy sequence:

$$\begin{aligned} &\|\mathcal{R}(T_n) - \mathcal{R}\,(T_{n+1})\| \geq \|\mathcal{R}(T_n)e_n - \mathcal{R}\,(M_{\mathrm{id}})\,e_n\| \\ &= \left\| \tfrac{n-2n}{\sqrt{1+(n-2n)^2}} e_n - \tfrac{n}{\sqrt{1+n^2}} e_n \right\| = \tfrac{2n}{\sqrt{1+n^2}} \to 2 \ \text{ as } \ n \to \infty. \end{aligned}$$

In particular, the Riesz transformation is not continuous on the whole space $\mathcal{C}^{\mathrm{sa}}$, nor on $\mathcal{CF}^{\mathrm{sa}}$, neither on the whole space of self–adjoint operators with compact resolvent. Other methods are needed for working with varying domains.

Of course, we can define a different metric in $\mathcal{C}^{\mathrm{sa}}$; e.g. the metric which makes the Riesz transformation a homeomorphism. That approach was chosen by L. Nicolaescu in [**Ni00**]. He shows that quite a large class of naturally arising curves of Dirac operators with varying domain are continuous under this 'Riesz metric', as opposed to the aforementioned example.

Here, we choose a different approach and adopt the gap metric. Note that continuity in the gap metric is much easier to establish than continuity in the

Riesz metric. We follow [**BoLePh01**] where the proofs can be found. A feature of this approach is the use of the Cayley Transform:

THEOREM 1.10. *(a) On $\mathcal{C}^{\mathrm{sa}}$ the gap metric is (uniformly) equivalent to the metric γ given by*

$$\gamma(T_1, T_2) = \|(T_1 + i)^{-1} - (T_2 + i)^{-1}\|.$$

(b) Let $\kappa : \mathbb{R} \to S^1 \setminus \{1\}$, $x \mapsto \frac{x-i}{x+i}$ denote the Cayley transform. Then κ induces a homeomorphism

$$\begin{aligned} \kappa : \mathcal{C}^{\mathrm{sa}}(H) &\longrightarrow \{U \in \mathcal{U}(H) \mid U - I \text{ is injective } \} =: \mathcal{U}_{\mathrm{inj}} \\ T &\mapsto \kappa(T) = (T - i)(T + i)^{-1}. \end{aligned} \tag{1.2}$$

More precisely, the gap metric is (uniformly) equivalent to the metric $\tilde{\delta}$ defined by $\tilde{\delta}(T_1, T_2) = \|\kappa(T_1) - \kappa(T_2)\|$.

We note some immediate consequences of the Cayley picture:

COROLLARY 1.11. *(a) With respect to the gap metric the set $\mathcal{B}^{\mathrm{sa}}(H)$ is dense in $\mathcal{C}^{\mathrm{sa}}(H)$.*
(b) For $\lambda \in \mathbb{R}$ the sets

$$\{T \in \mathcal{C}^{\mathrm{sa}}(H) \mid \lambda \notin \operatorname{spec} T\} \qquad \text{and} \qquad \{T \in \mathcal{C}^{\mathrm{sa}}(H) \mid \lambda \notin \operatorname{spec}_{\mathrm{ess}} T\}$$

are open in the gap topology.
(c) The set $\mathcal{CF}^{\mathrm{sa}} = \{T \in \mathcal{C}^{\mathrm{sa}} \mid 0 \notin \operatorname{spec}_{\mathrm{ess}} T\} = \kappa^{-1}({}_{\mathcal{F}}\mathcal{U})$, where ${}_{\mathcal{F}}\mathcal{U} := \{U \in \mathcal{U} \mid -1 \notin \operatorname{spec}_{\mathrm{ess}} U\} = \{U \in \mathcal{U} \mid U + I \text{ Fredholm operator}\}$, of (not necessarily bounded) self–adjoint Fredholm operators is open in $\mathcal{C}^{\mathrm{sa}}$.

The preceding Corollary implies that the set $\mathcal{F}^{\mathrm{sa}}$ is dense in $\mathcal{CF}^{\mathrm{sa}}$ with respect to the gap metric.
Contrary to the bounded case, we have the following somewhat surprising result in the unbounded case: In particular, it shows that not every gap continuous path in $\mathcal{CF}^{\mathrm{sa}}$ with endpoints in $\mathcal{F}^{\mathrm{sa}}$ can be continuously deformed into an operator norm continuous path in $\mathcal{F}^{\mathrm{sa}}$, in spite of the density of $\mathcal{F}^{\mathrm{sa}}$ in $\mathcal{CF}^{\mathrm{sa}}$.

THEOREM 1.12. *(a) $\mathcal{CF}^{\mathrm{sa}}$ is path connected with respect to the gap metric.*
(b) Moreover, its Cayley image

$${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}} := \{U \in \mathcal{U} \mid U + I \text{ Fredholm and } U - I \text{ injective}\} = \kappa(\mathcal{CF}^{\mathrm{sa}})$$

is dense in ${}_{\mathcal{F}}\mathcal{U}$.

PROOF. (a) Once again we look at the Cayley transform picture. Note that so far we have introduced three different subsets of unitary operators

$$\mathcal{U}_{\mathrm{inj}} := \{U \in \mathcal{U} \mid U - I \ \text{ injective}\} = \kappa(\mathcal{C}^{\mathrm{sa}}) \tag{1.3}$$

$${}_{\mathcal{F}}\mathcal{U} := \{U \in \mathcal{U} \mid U + I \ \text{ Fredholm}\}, \text{ and} \tag{1.4}$$

$${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}} := {}_{\mathcal{F}}\mathcal{U} \cap \mathcal{U}_{\mathrm{inj}} = \kappa(\mathcal{CF}^{\mathrm{sa}}). \tag{1.5}$$

Let $U \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$. Then H is the direct sum of the spectral subspaces $H_\pm$ of U corresponding to $[0,\pi)$ and $[\pi,2\pi]$ respectively and we may decompose $U = U_+ \oplus U_-$. More precisely, we have

$$\mathrm{spec}(U_+) \subset \{e^{it} \mid t \in [0,\pi)\} \text{ and } \mathrm{spec}(U_-) \subset \{e^{it} \mid t \in [\pi, 2\pi]\}\,.$$

Note that there is no intersection between the spectral spaces in the endpoints: if -1 belongs to $\mathrm{spec}(U)$, it is an isolated eigenvalue by our assumption and hence belongs only to $\mathrm{spec}(U_-)$; if 1 belongs to $\mathrm{spec}(U)$, it can belong both to $\mathrm{spec}(U_+)$ and $\mathrm{spec}(U_-)$, but in any case, it does not contribute to the decomposition of U since, by our assumption, 1 is not an eigenvalue at all.

By spectral deformation ("squeezing the spectrum down to $+i$ and $-i$") we contract U_+ to iI_+ and U_- to $-iI_-$, where $I_\pm$ denotes the identity on $H_\pm$. We do this on the upper half arc and the lower half arc, respectively, in such a way that 1 does not become an eigenvalue under the course of the deformation: actually it will no longer belong to the spectrum; neither will -1 belong to the spectrum. That is, we have connected U and $iI_+ \oplus -iI_-$ within $\kappa\,(\mathcal{CF}^{\mathrm{sa}})$.

We distinguish two cases: If H_- is *finite-dimensional*, we now rotate $-iI_-$ up through -1 into iI_-: More precisely, we consider $\{iI_+ \oplus e^{i(\pi/2+(1-t)\pi)} I_-\}_{t\in[0,1]}$. This proves that we can connect U with $iI_+ \oplus iI_- = iI$ within $\kappa\,(\mathcal{CF}^{\mathrm{sa}})$ in this first case.

If H_- is *infinite-dimensional*, we "dilate" $-iI_-$ in such a way that no eigenvalues remain. To do this, we identify H_- with $L^2([0,1])$. Now multiplication by $-i$ on $L^2([0,1])$ can be connected to multiplication by a function whose values are a short arc centered on $-i$ and so that the resulting operator V_- on H_- has no eigenvalues. This will at no time introduce spectrum near $+1$ or -1. We then rotate this arc up through $+1$ (which keeps us in the right space) until it is centered on $+i$. Then we contract the spectrum on H_- to be $+i$. That is, also in this case we have connected our original operator U to $+iI$. To sum up this second case (see also Figure 1):

$$\begin{aligned} U \sim iI_+ \oplus -iI_- \sim iI_+ \oplus V_- \sim iI_+ \oplus e^{it\pi} V_- \text{ for } t \in [0,1] \\ \sim iI_+ \oplus -V_- \sim iI_+ \oplus -(-iI_-) \sim iI\,. \end{aligned}$$

To prove (b), we just decompose any $V \in {}_{\mathcal{F}}\mathcal{U}$ into $V = U \oplus I_1$, where $U \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}(H_0)$ and I_1 denotes the identity on the 1-eigenspace $H_1 = \mathrm{Ker}(V-I)$ of V with $H = H_0 \oplus H_1$ an orthogonal decomposition. Then for $\varepsilon > 0$, $U \oplus e^{i\varepsilon} I_1 \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$ approaches U for $\varepsilon \to 0$. $\square$

Remark 1.13. The preceding proof shows also that the two subsets of $\mathcal{CF}^{\mathrm{sa}}$

$$\mathcal{CF}^{\mathrm{sa}}_\pm = \{T \in \mathcal{CF}^{\mathrm{sa}} \mid \mathrm{spec}_{\mathrm{ess}}(T) \subset \mathbb{C}_\mp\},$$

the spaces of all essentially positive, resp. all essentially negative, self-adjoint Fredholm operators, are no longer open. The third of the three complementary subsets

$$\mathcal{CF}^{\mathrm{sa}}_* = \mathcal{CF}^{\mathrm{sa}} \setminus \left(\mathcal{CF}^{\mathrm{sa}}_+ \cup \mathcal{CF}^{\mathrm{sa}}_-\right)$$

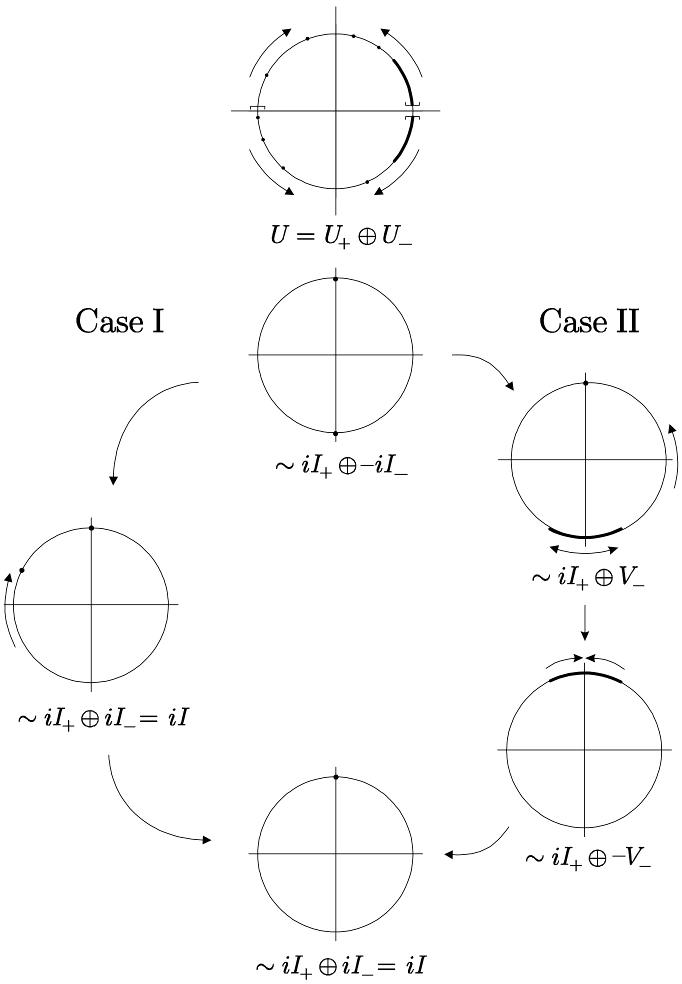

FIGURE 1. Connecting a fixed U in ${}_{\mathcal{F}}\mathcal{U}_{\text{inj}}$ to iI. Case I (finite rank U_-) and Case II (infinite rank U_-)

is also not open. We do not know whether the two "trivial" components are contractible as in the bounded case nor whether the whole space is a classifying space for K^1 as the nontrivial component in the bounded case. Independently of Example 1.5, the connectedness of $\mathcal{CF}^{\text{sa}}$ and the disconnectedness of $\mathcal{F}^{\text{sa}}$ show that the Riesz map is not continuous on $\mathcal{CF}^{\text{sa}}$ in the gap topology.

In analogy to [**Ph96**], we can give an explicit description of the winding number (spectral flow across -1) $\text{wind}(f)$ of a curve f in ${}_{\mathcal{F}}\mathcal{U}$. Alternatively, it can be used as a definition of wind:

Proposition 1.14. *Let* $f:[0,1]\to {}_{\mathcal{F}}\mathcal{U}$ *be a continuous path.*
(a) There is a partition $\{0=t_0<t_1<\cdots<t_n=1\}$ *of the interval and positive real numbers* $0<\varepsilon_j<\pi$, $j=1,\dots,n$, *such that* $\operatorname{Ker}(f(t)-e^{i(\pi\pm\varepsilon_j)})=\{0\}$ *for* $t_{j-1}\le t\le t_j$.
(b) Then

$$\operatorname{wind}(f)=\sum_{j=1}^{n} k(t_j,\varepsilon_j)-k(t_{j-1},\varepsilon_j),$$

where

$$k(t,\varepsilon_j):=\sum_{0\le\theta<\varepsilon_j}\dim\operatorname{Ker}(f(t)-e^{i(\pi+\theta)}).$$

(c) In particular, this calculation of $\operatorname{wind}(f)$ *is independent of the choice of the partition of the interval and of the choice of the barriers.*

Proof. In (a) we use the continuity of f and the fact that $f(t)\in {}_{\mathcal{F}}\mathcal{U}$. (b) follows from the path additivity of wind. (c) is immediate from (b). $\square$

This idea of a *spectral flow across* -1 was introduced first in [**BoFu98**, Sec. 1.3], where it was used to give a definition of the Maslov index in an infinite dimensional context (see also Definition 1.24 further below).

After these explanations the definition of spectral flow for paths in $\mathcal{CF}^{\mathrm{sa}}$ is straightforward:

Definition 1.15. Let $f:[0,1]\to\mathcal{CF}^{\mathrm{sa}}(H)$ be a continuous path. Then the *spectral flow* $\operatorname{SF}(f)$ is defined by

$$\operatorname{SF}(f):=\operatorname{wind}(\kappa\circ f).$$

From the properties of κ and of the winding number we infer immediately:

Proposition 1.16. SF *is path additive and homotopy invariant in the following sense: let* $f_1,f_2:[0,1]\to\mathcal{CF}^{\mathrm{sa}}$ *be continuous paths and let* $h:[0,1]\times[0,1]\to\mathcal{CF}^{\mathrm{sa}}$ *be a homotopy such that* $h(0,t)=f_1(t), h(1,t)=f_2(t)$ *and such that* $\dim\operatorname{Ker}h(s,0),\dim\operatorname{Ker}h(s,1)$ *are independent of* s. *Then* $\operatorname{SF}(f_1)=\operatorname{SF}(f_2)$. *In particular,* SF *is invariant under homotopies leaving the endpoints fixed.*

From Proposition 1.14 we get

Proposition 1.17. *For a continuous path* $f:[0,1]\to\mathcal{F}^{\mathrm{sa}}$ *our definition of spectral flow coincides with the definition in* [**Ph96**].

Note that also the conventions coincide for $0\in\operatorname{spec}f(0)$ or $0\in\operatorname{spec}f(1)$.

COROLLARY 1.18. *For any $S \in \mathcal{CF}^{\mathrm{sa}}$ with compact resolvent and any continuous path $C : [0,1] \to \mathcal{B}^{\mathrm{sa}}$ we have* $\mathrm{SF}(S + C) = \mathrm{SF}(\mathcal{R}\,(S + C))$ *where $\mathcal{R}$ denotes the Riesz transformation of (1.1).*

Note that the curve $S+C$ is in $\mathcal{CF}^{\mathrm{sa}}$, so that $\mathrm{SF}(S+C)$ is defined via Cayley transformation, whereas the curve $\mathcal{R}\,(S+C)$ of the Riesz transforms is in $\mathcal{F}^{\mathrm{sa}}$.

REMARK 1.19. The spectral flow induces a surjection of $\pi_1\,(\mathcal{CF}^{\mathrm{sa}})$ onto $\mathbb{Z}$. Because $\mathbb{Z}$ is free, there is a right inverse of SF and a normal subgroup G of $\pi_1\,(\mathcal{CF}^{\mathrm{sa}})$ such that we have a split short exact sequence

$$0 \to G \to \pi_1\,(\mathcal{CF}^{\mathrm{sa}}) \to \mathbb{Z} \to 0.$$

For now, an open question is whether G is trivial: does the spectral flow distinguish the homotopy classes? That is, the question is whether each loop with spectral flow 0 can be contracted to a constant point, or equivalently, whether two continuous paths in $\mathcal{CF}^{\mathrm{sa}}$ with same endpoints and with same spectral flow can be deformed into each other? Or is $\pi_1\,(\mathcal{CF}^{\mathrm{sa}}) \cong \mathbb{Z} \times\!| \; G$ the semi-direct product of a nontrivial factor G with $\mathbb{Z}$? In that case, homotopy invariants of a curve in $\mathcal{CF}^{\mathrm{sa}}$ are not solely determined by the spectral flow (contrary to the folklore behind parts of the topology and physics literature). For now, we can only speculate about the existence of an additional invariant and its possible definition. For example, one can try to define a *spectral flow at infinity.* Then, continuity of the Riesz transformation $\mathcal{R}$ on a subclass $\mathcal{S} \subset \mathcal{CF}^{\mathrm{sa}}$ would imply vanishing spectral flow at infinity. Non-vanishing spectral flow at infinity will typically appear with families of the type discussed in Example 1.5 (and after Remark 1.9). One may also expect it with curves of differential operators of second order. However, the results of [**Ni00**], though only obtained under quite restrictive conditions, may indicate that perhaps spectral flow at infinity will not be exhibited for continuous curves of Dirac operators. If this is true, it will also explain why the mentioned unfounded folklore has not yet led to clear contradictions.

1.2. Symmetric Operators and Symplectic Analysis. In an interview with Victor M. Buchstaber in the Newsletter of the European Mathematical Society [**No01**, p. 20], Sergej P. Novikov recalls his idea of the late 1960's and the early 1970's, which were radically different of the main stream in topology at that time, "that the explanation of higher signatures and of other deep properties of multiply connected manifolds had a symplectic origin... In 1971 I. Gelfand went into my algebraic ideas: they impressed him greatly. In particular, he told me of his observation that the so-called von Neumann theory of self-adjoint extensions of symmetric operators is simply the choice of a Lagrangian subspace in a Hilbert space with symplectic structure." Closely following [**BoFu98**], [**BoFu99**], and [**BoFuOt01**] (see also the Krein–Vishik–Birman theory summarized in [**AlSi80**] and [**LaSnTu75**], for first pointing to symplectic aspects), we will elaborate on that thought.

1.2.1. *Symplectic Hilbert Space, Fredholm Lagrangian Grassmannian, and Maslov Index.* We fix the following notation. Let $(\mathcal{H}, \langle .,. \rangle)$ be a separable real Hilbert space with a fixed symplectic form ω, i.e., a skew-symmetric bounded bilinear form on $\mathcal{H}\times\mathcal{H}$ which is nondegenerate. We assume that ω is compatible with $\langle .,. \rangle$ in the sense that there is a corresponding almost complex structure $J : \mathcal{H} \to \mathcal{H}$ defined by

$$\omega(x,y) = \langle Jx, y\rangle \tag{1.6}$$

with $J^2 = -\mathrm{I}$, ${}^tJ = -J$, and $\langle Jx, Jy\rangle = \langle x, y\rangle$. Here ${}^tJ$ denotes the transpose of J with regard to the (real) inner product $\langle x, y\rangle$. Let $\mathcal{L} = \mathcal{L}(\mathcal{H})$ denote the set of all Lagrangian subspaces λ of $\mathcal{H}$ (i.e., $\lambda = (J\lambda)^\perp$, or equivalently, let λ coincide with its annihilator λ^0 with respect to ω). The topology of $\mathcal{L}$ is defined by the operator norm of the orthogonal projections onto the Lagrangian subspaces.

Let $\lambda_0 \in \mathcal{L}$ be fixed. Then any $\mu \in \mathcal{L}$ can be obtained as the image of $\lambda_0^\perp$ under a suitable unitary transformation

$$\mu = U(\lambda_0^\perp)$$

(see also Figure 2a). Here we consider the real symplectic Hilbert space $\mathcal{H}$ as a complex Hilbert space via J. The group $\mathcal{U}(\mathcal{H})$ of unitary operators of $\mathcal{H}$ acts transitively on $\mathcal{L}$; i.e., the mapping

$$\begin{array}{rccc} \rho: & \mathcal{U}(\mathcal{H}) & \longrightarrow & \mathcal{L} \\ & U & \mapsto & U({\lambda_0}^\perp) \end{array} \tag{1.7}$$

is surjective and defines a principal fibre bundle with the group of orthogonal operators $\mathcal{O}(\lambda_0)$ as structure group.

Example 1.20. (a) In finite dimensions one considers the space $\mathcal{H} := \mathbb{R}^n \oplus \mathbb{R}^n$ with the symplectic form

$$\omega\left((x,\xi),(y,\eta)\right) := -\langle x,\eta\rangle + \langle \xi, y\rangle \qquad \text{for } (x,\xi),\ (y,\eta) \in \mathcal{H}.$$

To emphasize the finiteness of the dimension we write $\mathrm{Lag}(\mathbb{R}^{2n}) := \mathcal{L}(\mathcal{H})$. For linear subspaces of $\mathbb{R}^{2n}$ one has

$$l \in \mathrm{Lag}(\mathbb{R}^{2n}) \iff \dim l = n \text{ and } l \subset l^0 := \omega\text{-annihilator of } l;$$

i.e., Lagrangian subspaces are true half-spaces which are maximally isotropic (‘isotropic’ means $l \subset l^0$).

Note that $\mathbb{R}^{2n} = \mathbb{R}^n \otimes \mathbb{C}$ with the Hermitian product

$$\left((x,\xi),(y,\eta)\right)_{\mathbb{C}} = (x+i\xi)(y-i\eta) := \langle x,y\rangle + \langle \xi,\eta\rangle + i\,\langle \xi, y\rangle - i\,\langle x,\eta\rangle\,.$$

Then every $U \in \mathrm{U}(n)$ can be written in the form $U = A + iB = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ with ${}^tAB = {}^tBA$, $A^tB = B^tA$, and $A^tA + B^tB = I$, and

$$\mathrm{O}(n) \ni A \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = A + 0i \in \mathrm{U}\,(n)$$

gives the embedding of $\mathrm{O}(n)$ in $\mathrm{U}(n)$. One finds $\mathrm{Lag}(\mathbb{R}^{2n}) \cong \mathrm{U}(n)/\mathrm{O}(n)$ with the fundamental group

$$\pi_1(\mathrm{Lag}(\mathbb{R}^{2n})), \lambda_0) \cong \mathbb{Z}.$$

The mapping is given by the 'Maslov index' of loops of Lagrangian subspaces which can be described as an intersection index with the 'Maslov cycle'. There is a rich literature on the subject, see e.g. the seminal paper [**Ar67**], the systematic review [**CaLeMi94**], or the cohomological presentation [**Go97**].

(b) Let $\{\varphi_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ be a complete orthonormal system for $\mathcal{H}$. We define an almost complex structure, and so by (1.6) a symplectic form, by

$$J\varphi_k := \mathrm{sign}(k)\,\varphi_{-k}\,.$$

Then the spaces $\mathcal{H}_- := \mathrm{span}\{\varphi_k\}_{k<0}$ and $\mathcal{H}_+ := \mathrm{span}\{\varphi_k\}_{k>0}$ are complementary Lagrangian subspaces of $\mathcal{H}$.

In infinite dimensions, the space $\mathcal{L}$ is contractible due to Kuiper's Theorem (see [**BlBo03**], Part I) and therefore topologically not interesting. Also, we need some restrictions to avoid infinite-dimensional intersection spaces when counting intersection indices. Therefore we replace $\mathcal{L}$ by a smaller space. This problem can be solved, as first suggested in Swanson [**Sw78**], by relating symplectic functional analysis with the space $\mathrm{Fred}(\mathcal{H})$ of Fredholm operators, we obtain finite dimensions for suitable intersection spaces and at the same time topologically highly nontrivial objects.

Definition 1.21. (a) The space of *Fredholm pairs* of closed infinite-dimensional subspaces of $\mathcal{H}$ is defined by

$$\mathrm{Fred}^2(\mathcal{H}) := \{(\lambda,\mu) \mid \dim(\lambda\cap\mu) < +\infty \text{ and } \lambda{+}\mu \subset \mathcal{H} \text{ closed} \\ \text{and } \dim \mathcal{H}/(\lambda{+}\mu) < +\infty\}.$$

Then

$$\mathrm{index}(\lambda,\mu) := \dim(\lambda\cap\mu) - \dim \mathcal{H}/(\lambda{+}\mu)$$

is called the *Fredholm intersection index of* (λ,μ).

(b) The *Fredholm-Lagrangian Grassmannian* of a real symplectic Hilbert space $\mathcal{H}$ at a fixed Lagrangian subspace λ_0 is defined as

$$\mathcal{FL}_{\lambda_0} := \{\mu \in \mathcal{L} \mid (\mu,\lambda_0) \in \mathrm{Fred}^2(\mathcal{H})\}.$$

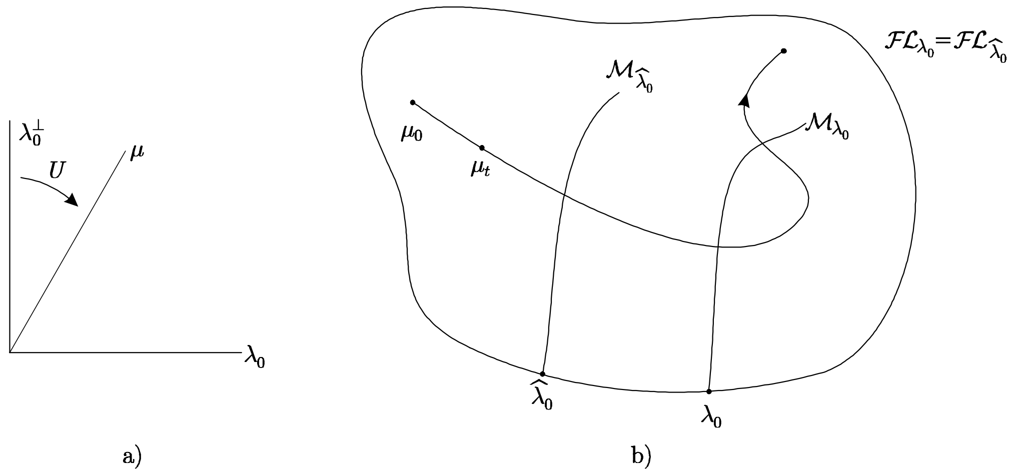

FIGURE 2. a) The generation of $\mathcal{L} = \left\{\mu = U(\lambda_0^\perp) | U \in \mathcal{U}(\mathcal{H})\right\}$
b) One curve and two Maslov cycles in $\mathcal{FL}\lambda_0 = \mathcal{FL}\widehat{\lambda}_0$

(c) The *Maslov cycle* of λ_0 in $\mathcal{H}$ is defined as

$$\mathcal{M}_{\lambda_0} := \mathcal{FL}_{\lambda_0} \setminus \mathcal{FL}^{(0)}_{\lambda_0},$$

where $\mathcal{FL}^{(0)}_{\lambda_0}$ denotes the subset of Lagrangians intersecting λ_0 transversally; i.e., $\mu \cap \lambda_0 = \{0\}$.

Recall the following algebraic and topological characterization of general and Lagrangian Fredholm pairs (see [**BoFu98**] and [**BoFu99**], inspired by [**BoWo85**], Part 2, Lemma 2.6).

PROPOSITION 1.22. *(*a*) Let $\lambda, \mu \in \mathcal{L}$ and let π_λ, π_μ denote the orthogonal projections of $\mathcal{H}$ onto λ respectively μ. Then*

$$(\lambda, \mu) \in \mathrm{Fred}^2(\mathcal{H}) \Longleftrightarrow \pi_\lambda + \pi_\mu \in \mathrm{Fred}(\mathcal{H}) \Longleftrightarrow \pi_\lambda - \pi_\mu \in \mathrm{Fred}(\mathcal{H}).$$

*(*b*) The fundamental group of $\mathcal{FL}_{\lambda_0}$ is $\mathbb{Z}$, and the mapping of the loops in $\mathcal{FL}_{\lambda_0}(\mathcal{H})$ onto $\mathbb{Z}$ is given by the Maslov index*

$$\mathbf{mas} : \pi_1(\mathcal{FL}_{\lambda_0}(\mathcal{H}))) \xrightarrow{\cong} \mathbb{Z}.$$

To define the Maslov index, one needs a systematic way of counting, adding and subtracting the dimensions of the intersections $\mu_s \cap \lambda_0$ of the curve $\{\mu_s\}$ with the Maslov cycle $\mathcal{M}_{\lambda_0}$. We recall from [**BoFu98**], inspired by [**Ph96**], a functional analytical definition for continuous curves without additional assumptions.

First we recall from (1.7) that any $\mu \in \mathcal{FL}_{\lambda_0}(\mathcal{H})$ can be obtained as the image of $\lambda_0^\perp$ under a suitable unitary transformation $\mu = U(\lambda_0^\perp)$. As noted before, the operator U is not uniquely determined by μ. Actually, from $\beta \cong \lambda_0 \otimes \mathbb{C}$ we obtain a complex conjugation so that we can define the transpose by the following formula

$$^{T}U := \overline{U^*}$$

and obtain a unitary operator $W(\mu) := U\,{}^{T}U$ which can be defined invariantly as the *complex generator* of the Lagrangian space μ relative to λ_0.

PROPOSITION 1.23. *The composed mapping W*

$$\mathcal{FL}_{\lambda_0}(\mathcal{H}) \ni \mu \longmapsto U \longmapsto W(\mu) := U\,{}^{T}U \in {}_{\mathcal{F}}\mathcal{U}\left(\mathcal{H}\right)$$

is well defined. In particular, we have

$$\operatorname{Ker}\left(I+W\right) = \left(\mu \cap \lambda_0\right) \otimes \mathbb{C} = \left(\mu \cap \lambda_0\right) + J\left(\mu \cap \lambda_0\right)$$

PROOF. The operator $I+W$ is a Fredholm operator because (μ, λ_0) is a Fredholm pair. $\square$

DEFINITION 1.24. Let $\mathcal{H}$ be a symplectic Hilbert space and $\lambda_0 \in \mathcal{L}(\mathcal{H})$. Let

$$[0,1] \ni s \longmapsto \mu_s \in \mathcal{FL}_{\lambda_0}(\mathcal{H})$$

be a continuous curve. Then $W \circ \mu$ is a continuous curve in ${}_{\mathcal{F}}\mathcal{U}\left(\mathcal{H}\right)$, and the Maslov index can be defined by

$$\mathbf{mas}\left(\mu, \lambda_0\right) := \operatorname{wind}\left(W \circ \mu\right),$$

where wind is defined as in Proposition 1.14.

REMARK 1.25. To define the Maslov (intersection) index $\mathbf{mas}\left(\{\mu_s\}, \lambda_0\right)$, we count the change of the eigenvalues of W_s near -1 little by little. For example, between $s=0$ and $s=s'$ we plot the spectrum of the complex generator W_s close to $e^{i\pi}$. In general, there will be no parametrization available of the spectrum near -1. For sufficiently small s', however, we can find barriers $e^{i(\pi+\theta)}$ and $e^{i(\pi-\theta)}$ such that no eigenvalues are lost through the barriers on the interval $[0,s']$. Then we count the number of eigenvalues (with multiplicity) of W_s between $e^{i\pi}$ and $e^{i(\pi+\theta)}$ at the right and left end of the interval $[0,s']$ and subtract. Repeating this procedure over the length of the whole s-interval $[0,1]$ gives the Maslov intersection index $\mathbf{mas}\left(\{\mu_s\}, \lambda_0\right)$ without any assumptions about smoothness of the curve, 'normal crossings', or non-invertible endpoints.
It is worth mentioning that the construction can be simplified for a *complex* symplectic Hilbert space $\mathcal{H}$. Then each Lagrangian subspace of $\mathcal{H}$ is the graph of a uniquely determined unitary operator from $\operatorname{Ker}\left(J-iI\right)$ to $\operatorname{Ker}\left(J+iI\right)$. Moreover, a pair (λ, μ) of Lagrangian subspaces is a Fredholm pair if and only if $U^{-1}V$ is Fredholm, where $\lambda = \mathfrak{G}\left(U\right)$ and $\mu = \mathfrak{G}\left(V\right)$. We have $\mathbf{mas}\left(\{\mu_t\}, \lambda_0\right) = \operatorname{SF}\left(U_t^{-1}V, 1\right)$ where the 1 indicates that the spectral flow is taken at the eigenvalue 1.

REMARK 1.26. (a) By identifying $\mathcal{H} \cong \lambda_0 \otimes \mathbb{C} \cong \lambda_0 \oplus \sqrt{-1}\,\lambda_0$, we split in [**BoFuOt01**] any $U \in \mathcal{U}(\mathcal{H})$ into a real and imaginary part

$$U = X + \sqrt{-1}\,Y$$

with $X, Y : \lambda_0 \to \lambda_0$. Let $\mathcal{U}(\mathcal{H})^{\mathrm{Fred}}$ denote the subspace of unitary operators which have a Fredholm operator as real part. This is the total space of a principal fibre bundle over the Fredholm Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0}$ as base space and with the orthogonal group $\mathcal{O}(\lambda_0)$ as structure group. The projection is given by the restriction of the trivial bundle $\rho : \mathcal{U}(\mathcal{H}) \to \mathcal{L}$ of (1.7) . This bundle

$$\mathcal{U}(\mathcal{H})^{\mathrm{Fred}} \xrightarrow{\rho} \mathcal{FL}_{\lambda_0}$$

may be considered as the infinite-dimensional generalization of the familiar bundle $\mathrm{U}(n) \to \mathrm{Lag}(\mathbb{R}^{2n})$ for finite n and provides an alternative proof of the homotopy type of $\mathcal{FL}_{\lambda_0}$.

(b) The Maslov index for curves depends on the specified Maslov cycle $\mathcal{M}_{\lambda_0}$. It is worth emphasizing that two *equivalent* Lagrangian subspaces λ_0 and $\widehat{\lambda}_0$ (i.e., $\dim \lambda_0/(\lambda_0 \cap \widehat{\lambda}_0) < +\infty$) always define the same Fredholm Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0} = \mathcal{FL}_{\widehat{\lambda}_0}$ but may define different Maslov cycles $\mathcal{M}_{\lambda_0} \neq \mathcal{M}_{\widehat{\lambda}_0}$. The induced Maslov indices may also become different

$$\mathbf{mas}(\{\mu_s\}_{s\in[0,1]}, \lambda_0) - \mathbf{mas}(\{\mu_s\}_{s\in[0,1]}, \widehat{\lambda}_0) \overset{\text{in general}}{\neq} 0 \tag{1.8}$$

(see [**BoFu99**], Proposition 3.1 and Section 5). However, if the curve is a loop, then the Maslov index does not depend on the choice of the Maslov cycle. From this property it follows that the difference in (1.8), beyond the dependence on λ_0 and $\widehat{\lambda}_0$, depends only on the initial and end points of the path $\{\mu_s\}$ and may be considered as the infinite-dimensional generalization $\sigma_{\mathbf{H\ddot{o}r}}(\mu_0, \mu_1; \lambda_0, \widehat{\lambda}_0)$ of the Hörmander index. It plays a part as the transition function of the universal covering of the Fredholm Lagrangian Grassmannian (see also Figure 2b).

1.2.2. *Symmetric Operators and Symplectic Analysis.* Let H be a real separable Hilbert space and A an (unbounded) closed symmetric operator defined on the domain $D_{\min}$ which is supposed to be dense in H. Let A^* denote its adjoint operator with domain $D_{\max}$. We have that $A^*|_{D_{\min}} = A$ and that A^* is the maximal closed extension of A in H. Note that $D_{\max}$ is a Hilbert space with the graph scalar product

$$(x, y)_{\mathcal{G}} := (x, y) + (A^*x, A^*y)\,,$$

and $D_{\min}$ is a closed subspace of this $D_{\max}$ since A is closed (each sequence in $D_{\min}$ which is Cauchy relative to the graph norm defines a sequence in the graph $\mathfrak{G}\,(A)$ which is Cauchy relative to the simple norm in $H \times H$).

We form the space β of *natural boundary values* with the *natural trace map* γ in the following way:

$$\begin{aligned} D_{\max} &\xrightarrow{\gamma} D_{\max}/D_{\min} =: \beta \\ x \ &\longmapsto \gamma\,(x) = [x] := x + D_{\min}. \end{aligned}$$

The space β becomes a symplectic Hilbert space with the induced scalar product and the symplectic form given by Green's form

$$\omega\left([x],[y]\right) := (A^*x, y) - (x, A^*y) \quad \text{for } [x],[y] \in \beta.$$

We define the *natural Cauchy data space* $\Lambda := \gamma\left(\operatorname{Ker} A^*\right)$. It is a Lagrangian subspace of β under the assumption that A admits at least one self-adjoint Fredholm extension A_D. Actually, we shall assume that A has a self-adjoint extension A_D with compact resolvent. Then $(\Lambda, \gamma\left(D\right))$ is a Fredholm pair of subspaces of β; i.e., $\Lambda \in \mathcal{FL}_{\gamma(D)}\left(\beta\right)$.

We consider a continuous curve $\{C_s\}_{s\in[0,1]}$ in the space of bounded self-adjoint operators on H. We assume that the operators A^*+C_s-r have no 'inner solutions'; i.e., they satisfy the *weak inner unique continuation property* (UCP)

$$\operatorname{Ker}\left(A^* + C_s - r\right) \cap D_{\min} = \{0\} \tag{1.9}$$

for $s \in [0,1]$ and $|r| < \varepsilon_0$ with $\varepsilon_0 > 0$. For a discussion of UCP see Section 1.4 below.

Clearly, the domains $D_{\max}$ and $D_{\min}$ are unchanged by the perturbation C_s for any s. So, β does not depend on the parameter s. Moreover, the symplectic form ω is invariantly defined on β and so also independent of s. It follows (see [**BoFu98**, Theorem 3.9]) that the curve $\{\Lambda_s := \gamma\left(\operatorname{Ker}\left(A^* + C_s\right)\right)\}$ is continuous in $\mathcal{FL}_{\gamma(D)}\left(\beta\right)$.

We summarize the basic findings:

Proposition 1.27. (a) *Assume that there exists a self–adjoint Fredholm extension A_D of A with domain D. Then the Cauchy data space $\Lambda(A)$ is a closed Lagrangian subspace of β and belongs to the Fredholm–Lagrangian Grassmannian $\mathcal{FL}_{\gamma(D)}\left(\beta\right)$.*
(b) *For arbitrary domains D with $D_{\min} \subset D \subset D_{\max}$ and $\gamma(D)$ Lagrangian, the extension $A_D := A_{\max}|_D$ is self–adjoint. It becomes a Fredholm operator, if and only if the pair $(\gamma(D), \Lambda(A))$ of Lagrangian subspaces of β becomes a Fredholm pair.*
(c) *Let $\{C_t\}_{t\in I}$ be a continuous family (with respect to the operator norm) of bounded self–adjoint operators. Here the parameter t runs within the interval $I = [0,1]$. Assume the weak inner UCP for all operators $A^* + C_t$. Then the spaces $\gamma(\operatorname{Ker}(A^* + C_t, 0))$ of Cauchy data vary continuously in β.*

Given this, the family $\{A_D + C_s\}$ can be considered at the same time in the spectral theory of self–adjoint Fredholm operators, defining a spectral flow, and in the symplectic category, defining a Maslov index. Under the preceding assumptions, the main result obtainable at that level is the following general spectral flow formula (proved in [**BoFu98**, Theorem 5.1] and inviting to generalizations for varying domains D_s instead of fixed domain D):

Theorem 1.28. *Let A_D be a self-adjoint extension of A with compact resolvent and let $\{A_D + C_s\}_{s\in[0,1]}$ be a family satisfying the weak inner UCP assumption. Let Λ_s denote the Cauchy data space $\gamma\left(\operatorname{Ker}\left(A^* + C_s\right)\right)$ of $A^* + C_s$. Then*

$$\operatorname{SF}\{A_D + C_s\} = \mathbf{mas}\left(\{\Lambda_s\}, \gamma(D)\right).$$

1.3. Operators of Dirac Type and their Ellipticity. There are different notions of operators of Dirac type. We shall not discuss the original hyperbolic Dirac operator (in the Minkowski metric), but restrict ourselves to the elliptic case related to Riemannian metrics.

Recall that, if (M,g) is a compact smooth Riemannian manifold (with or without boundary) with $\dim M = m$, we denote by $\mathfrak{Cl}(M) = \{\mathfrak{Cl}(TM_x, g_x)\}_{x\in M}$ the bundle of Clifford algebras of the tangent spaces. For $S \to M$ a smooth complex vector bundle of Clifford modules, the *Clifford multiplication* is a bundle map $\mathbf{c} : \mathfrak{Cl}(M) \to \operatorname{Hom}(S,S)$ which yields a representation $\mathbf{c} : \mathfrak{Cl}(TM_x, g_x) \to \operatorname{Hom}_{\mathbb{C}}(S_x, S_x)$ in each fiber. We may assume that the bundle S is equipped with a Hermitian metric which makes the Clifford multiplication skew-symmetric

$$\langle \mathbf{c}(v)s, s'\rangle = -\langle s, \mathbf{c}(v)s'\rangle \qquad \text{for } v \in TM_x \text{ and } s \in S_x.$$

We note that it is not necessary to assume that (M,g) admits a spin structure in order that $\mathfrak{Cl}(M)$ and S exist. Indeed, one special case is obtained by taking $S = \Lambda^*(TM)$ and letting $\mathbf{c}$ be the extension of $\mathbf{c}(v)(\alpha) = v \wedge \alpha - v \llcorner \alpha$ for $v \in TM_x \subset \mathfrak{Cl}(TM_x, g_x)$ and $\alpha \in \Lambda^*(TM_x)$, where "$\llcorner$" denotes interior product (i.e., the dual of the exterior product $\wedge$). The extension is guaranteed by the fact that $\mathbf{c}(v)^2 = -g_x(v,v)\,\mathrm{I}$. However, we do need a spin structure in the case where S is a bundle of spinors.

Any choice of a smooth connection

$$\nabla : \mathrm{C}^\infty(M;S) \to \mathrm{C}^\infty(M; T^*M \otimes S)$$

defines an *operator of Dirac type* $\mathcal{D} := \mathbf{c} \circ \nabla$ under the Riemannian identification of the bundles TM and T^*M. In local coordinates we have $\mathcal{D} := \sum_{j=1}^m \mathbf{c}(e_j)\nabla_{e_j}$ for any orthonormal base $\{e_1, \dots, e_m\}$ of TM_x. Actually, we may choose a local frame in such a way that

$$\nabla_{e_j} = \frac{\partial}{\partial x_j} + \text{ zero order terms}$$

for all $1 \le j \le m$. So, locally, we have

$$\mathcal{D} := \sum\nolimits_{j=1}^m \mathbf{c}(e_j)\frac{\partial}{\partial x_j} + \text{ zero order terms}\,. \tag{1.10}$$

It follows at once that the principal symbol $\sigma_1(\mathcal{D})(x,\xi)$ is given by Clifford multiplication with $i\xi$, so that any operator of Dirac type is elliptic with symmetric principal symbol. If the connection ∇ is *compatible* with Clifford multiplication (i.e. $\nabla\mathbf{c} = 0$), then the operator $\mathcal{D}$ itself becomes symmetric. We shall, however, admit incompatible metrics. Moreover, the *Dirac Laplacian* $\mathcal{D}^2$ has principal symbol

$\sigma_2(\mathcal{D}^2)(x,\xi)$ given by scalar multiplication by $\|\xi\|^2$ using the Riemannian metric. So, the principal symbol of $\mathcal{D}^2$ is a real multiple of the identity, and $\mathcal{D}^2$ is elliptic. In the special case above where $S = \Lambda^*(TM)$ and we identify $\Lambda^*(TM)$ with $\Lambda^*(T^*M)$ by means of the metric g, $\mathcal{D}$ becomes $d+\delta : \Omega^*(M,\mathbb{C}) \to \Omega^*(M,\mathbb{C})$ and $\mathcal{D}^2 = d\delta + \delta d$ is the Hodge Laplacian on the space $\Omega^*(M,\mathbb{C})$ of complex-valued forms on M.

On a closed manifold M, a key result for any operator $\mathcal{D}$ of Dirac type (actually, for any linear elliptic operator of first order) is the *a priori* estimate (*Gårding's inequality*)

$$\|\psi\|_1 \leq C\left(\|\mathcal{D}\psi\|_0 + \|\psi\|_0\right) \text{ for all } \psi \in H^1(M;S)\,. \tag{1.11}$$

Here $\|\cdot\|_0$ denotes the L^2 norm and $H^1(M;S)$ denotes the first Sobolev space with the norm $\|\cdot\|_1$. Note that the same symbol $\mathcal{D}$ is used for the original operator (defined on smooth sections) and its closed L^2 extension with domain $H^1(M;S)$.

Combined with the simple continuity relation $\|\mathcal{D}\psi\|_0 \leq C'\|\psi\|_1$, inequality (1.11) shows that the first Sobolev norm $\|\cdot\|_1$ and the graph norm coincide on $H^1(M;S)$.

1.4. Weak Unique Continuation Property. A linear or non-linear operator $\mathfrak{D}$, acting on functions or sections of a bundle over a compact or non-compact manifold M has the *weak Unique Continuation Property (UCP)* if any solution ψ of the equation $\mathfrak{D}\psi = 0$ has the following property: if ψ vanishes on a nonempty open subset Ω of M, then it vanishes on the whole connected component of M containing Ω. Note that weak inner UCP, as defined in (1.9), follows from weak UCP, but not vice versa.

There is also a notion of *strong UCP*, where, instead of assuming that a solution ψ vanishes on an open subset, one assumes only that ψ vanishes 'of high order' at a point. The concepts of weak and strong UCP extend a fundamental property of analytic functions to *some* elliptic equations other than the Cauchy-Riemann equation.

Up to now, (almost) all work on UCP goes back to two seminal papers [**Ca33**], [**Ca39**] by Torsten Carleman, establishing an inequality of Carleman type (see our inequality 1.16 below). In this approach, the difference between weak and strong UCP and the possible presence of more delicate nonlinear perturbations are related to different choices of the weight function in the inequality, and to whether L^2 estimates suffice or L^p and L^q estimates are required.

The weak UCP is one of the basic properties of an operator of Dirac type $\mathcal{D}$. Contrary to common belief, UCP is *not* a general fact of life for elliptic operators. See [**Pl61**] where counter-examples are given with smooth coefficients. Lack of UCP invalidates the continuity of the Cauchy data spaces and of the Calderón projection (Proposition 1.27c and Theorem 2.28b) and of the main continuity lemma (Lemma 2.27). It corrupts the invertible double construction (Section 2.3.4) and threatens Bojarski type theorems (like Proposition 2.33 and Theorem 2.37). For partitioned manifolds $M = M_1 \cup_\Sigma M_2$ (see Subsection 2.3), it guarantees that there are no *ghost* solutions of $\mathcal{D}\psi = 0$; that is, there are no solutions which vanish on M_1 and have nontrivial support in the interior of M_2. This property is also called UCP *from open subsets* or *across any hypersurface*. For Euclidean (classical) Dirac operators (i.e., Dirac operators on $\mathbb{R}^m$ with constant coefficients and without perturbation), the property follows by squaring directly from the well-established UCP for the classical (constant coefficients and no potential) Laplacian.

From [**BoWo93**, Chapter 8] we recall a very simple proof of the weak UCP for operators of Dirac type, inspired by [**Ni73**, Sections 6-7, in particular the proof of inequality (7.11)] and [**Tr80**, Section II.3]. We refer to [**Boo00**] for a further slight simplification and a broader perspective, and to [**BoMaWa02**] for perturbed equations.

The proof does not use advanced arguments of the Aronszajn/Cordes type (see [**Ar57**] and [**Co56**]) regarding the diagonal and real form of the principal symbol of the Dirac Laplacian nor any other reduction to operators of second order (like [**We82**]), but only the following product property of Dirac type operators (besides Gårding's inequality).

LEMMA 1.29. *Let Σ be a closed hypersurface of M with orientable normal bundle. Let u denote a normal variable with fixed orientation such that a bicollar neighborhood N of Σ is parameterized by $[-\varepsilon, +\varepsilon] \times \Sigma$. Then any operator of Dirac type can be rewritten in the form*

$$\mathcal{D}|_N = \mathbf{c}(du)\left(\frac{\partial}{\partial u} + B_u + C_u\right), \tag{1.12}$$

where B_u is a self-adjoint elliptic operator on the parallel hypersurface Σ_u, and $C_u : S|_{\Sigma_u} \to S|_{\Sigma_u}$ a skew-symmetric operator of 0-th order, actually a skew-symmetric bundle homomorphism.

PROOF. Let (u, y) denote the coordinates in a tubular neighborhood of Σ. Locally, we have $y = (y_1, \dots, y_{m-1})$. Let $\mathbf{c}_u, \mathbf{c}_1, \dots, \mathbf{c}_{m-1}$ denote Clifford multiplication by the unit tangent vectors in normal, resp. tangential, directions. By (1.10), we have

$$\mathcal{D} = \mathbf{c}_u \frac{\partial}{\partial u} + \sum\nolimits_{k=1}^{m-1} \mathbf{c}_k \frac{\partial}{\partial y_k} + \text{ zero order terms} = \mathbf{c}_u \left(\frac{\partial}{\partial u} + \mathcal{B}_u\right), \text{ where}$$
$$\mathcal{B}_u := \sum\nolimits_{k=1}^{m-1} -\mathbf{c}_u \mathbf{c}_k \frac{\partial}{\partial y_k} + \text{ zero order terms.}$$

We shall call $\mathcal{B}_u$ the *tangential* operator component of the operator A. Clearly it is an elliptic differential operator of first order over Σ_u. From the skew-hermicity of $\mathbf{c}_u$ and $\mathbf{c}_k$, we have

$$\begin{aligned}\left(\mathbf{c}_u\mathbf{c}_k\frac{\partial}{\partial y_k}\right)^* &= \left(-\frac{\partial}{\partial y_k}\right)(-\mathbf{c}_k)(-\mathbf{c}_u) = -\mathbf{c}_k\,\mathbf{c}_u\frac{\partial}{\partial y_k} + \text{ zero order terms}\\ &= \mathbf{c}_u\mathbf{c}_k\,\frac{\partial}{\partial y_k} + \text{ zero order terms.}\end{aligned}$$

So,

$$\mathcal{B}_u^* = \mathcal{B}_u + \text{ zero order terms.}$$

Hence, the principal symbol of $\mathcal{B}_u$ is self-adjoint. Then the assertion of the lemma is proved by setting

$$B_u := \frac{1}{2}\left(\mathcal{B}_u + \mathcal{B}_u^*\right) \qquad \text{and} \qquad C_u := \frac{1}{2}\left(\mathcal{B}_u - \mathcal{B}_u^*\right) \tag{1.13}$$

□

REMARK 1.30. (a) It is worth mentioning that the product form (1.12) is invariant under perturbation by a bundle homomorphism. More precisely: Let $\mathfrak{D}$ be an operator on M which can be written in the form (1.12) close to any closed hypersurface Σ, with B_u and C_u as explained in the preceding Lemma. Let R be a bundle homomorphism. Then

$$(\mathfrak{D} + R)\,|_N = \mathbf{c}(du)\left(\frac{\partial}{\partial u} + B_u + C_u\right) + \mathbf{c}(du)S|_N$$

with $T|_N := \mathbf{c}(du)^* R|_N$. Splitting $T = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*)$ into a symmetric and a skew-symmetric part and adding these parts to B_u and C_u, respectively, yields the desired form of $(\mathfrak{D} + R)\,|_N$.
(b) For operators of Dirac type, it is well known that a perturbation by a bundle homomorphism is equivalent to modifying the underlying connection of the operator. This gives an alternative argument for the invariance of the form (1.12) for operators of Dirac type under perturbation by a bundle homomorphism.
(c) By the preceding arguments (a), respectively (b), establishing weak UCP for sections belonging to the kernel of a Dirac type operator, respectively an operator which can be written in the form (1.12), implies weak UCP for all eigensections. Warning: for general linear elliptic differential operators, weak UCP for "zero-modes" does not imply weak UCP for all eigensections.

To prove the weak UCP, in combination with the preceding lemma, the standard lines of the UCP literature can be radically simplified, namely with regard to the weight functions and the integration order of estimates. These simplifications make it also very easy to generalize the weak UCP to the perturbed case.

We replace the equation $\mathcal{D}\psi = 0$ by

$$\widetilde{\mathcal{D}}\psi := \mathcal{D}\psi + \mathfrak{P}_A(\psi) = 0\,, \tag{1.14}$$

where $\mathfrak{P}_A$ is an *admissible* perturbation, in the following sense.

DEFINITION 1.31. A perturbation is *admissible* if it satisfies the following estimate:

$$|\mathfrak{P}_A(\psi)|_x| \leq P(\psi, x)|\psi(x)| \qquad \text{for } x \in M \tag{1.15}$$

for a real-valued function $P(\psi, \cdot)$ which is locally bounded on M for each fixed ψ.

EXAMPLE 1.32. Some typical examples of perturbations satisfying the admissibility condition of Definition 1.31 are:
1. Consider a nonlinear perturbation

$$\mathfrak{P}_A(\psi)|_x := \omega(\psi(x)) \cdot \psi(x)\,,$$

where $\omega(\psi(x))|_{x\in M}$ is a (bounded) function which depends continuously on $\psi(x)$, for instance, for a fixed (bounded) spinor section $a(x)$ we can take

$$\omega(\psi(x)) := \langle \psi(x), a(x)\rangle$$

with $\langle\cdot,\cdot\rangle$ denoting the Hermitian product in the fiber of the spinor bundle over the base point $x \in M$. This satisfies (1.15) .
2. Another interesting example is provided by (linear) nonlocal perturbations with

$$\omega(\psi, x) = \left| \int k(x, z)\psi(z)dz \right|$$

with suitable integration domain and integrability of the kernel k. These also satisfy (1.15) .
3. Clearly, an unbounded perturbation may be both nonlinear and global at the same time. This will, in fact, be the case in our main application. In all these cases the only requirement is the estimate (1.15) with bounded $\omega(\psi(\cdot))$.

We now show that (admissible) perturbed Dirac operators always satisfy the weak Unique Continuation Property. In particular, we show that this is true for unperturbed Dirac operators.

THEOREM 1.33. *Let $\mathcal{D}$ be an operator of Dirac type and $\mathfrak{P}_A$ an admissible perturbation. Then any solution ψ of the perturbed equation* (1.14) *vanishes identically on any connected component of the underlying manifold M if it vanishes on a nonempty open subset of the connected component.*

PROOF. Without loss of generality, we assume that M is connected. Let ψ be a solution of the perturbed (or, in particular) unperturbed equation which vanishes on an open, nonempty set Ω. First we localize and convexify the situation and we introduce spherical coordinates (see Figure 3). Without loss of generality we may assume that Ω is maximal, namely the union of all open subsets on which ψ vanishes; i.e., $\Omega = M \setminus \operatorname{supp} \psi$.

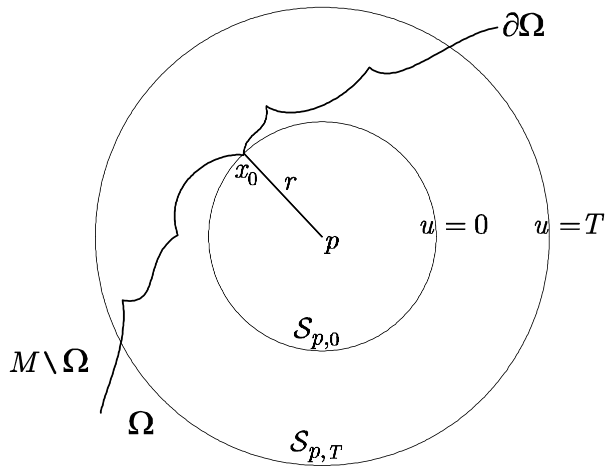

FIGURE 3. Local specification for the Carleman estimate

Since M is connected, to prove that $\Omega = M$ it suffices to show that Ω is closed (i.e., $\overline{\Omega} = \Omega$). If $\overline{\Omega} \neq \Omega$, then let $y_0 \in \partial\Omega := \overline{\Omega} \setminus \Omega$, and let B be an open, normal coordinate ball about y_0. Let $p \in \Omega \cap B$ and let $x_0 \in \operatorname{supp}\psi$ be a point of the non-empty, compact set $\overline{B} \setminus \Omega = \overline{B} \cap \operatorname{supp}\psi$ which is closest to p. Let

$$r := d\,(p, x_0) = \min_{x \in \overline{B}\setminus\Omega} d\,(p, x)\,.$$

For $z \in B$, let $u\,(z) := d\,(p, z) - r$ be a "radial coordinate". Note that $u = 0$ defines a sphere, say $\mathcal{S}_{p,0}$, of radius r about x_0. We have larger hyperspheres $\mathcal{S}_{p,u} \subset M$ for $0 \le u \le T$ with $T > 0$ sufficiently *small*. In such a way we have parameterized an annular region $N_T := \{\mathcal{S}_{p,u}\}_{u\in[0,T]}$ around p of width T and inner radius r, ranging from the hypersphere $\mathcal{S}_{p,0}$ which is contained in $\overline{\Omega}$, to the hypersphere $\mathcal{S}_{p,T}$. Note that N_T contains some points where $\psi \neq 0$, for otherwise $x_0 \in \Omega$. Let y denote a variable point in $\mathcal{S}_{p,0}$ and note that points in N_T may be identified with $(u, y) \in [0, T] \times \mathcal{S}_{p,0}$. Next, we replace the solution $\psi|N_T$ by a cutoff

$$v(u, y) := \varphi(u)\psi(u, y)$$

with a smooth bump function φ with $\varphi(u) = 1$ for $u \le 0.8\,T$ and $\varphi(u) = 0$ for $u \ge 0.9\,T$. Then $\operatorname{supp} v$ is contained in N_T. More precisely, it is contained in the annular region $N_{0.9\,T}$. Now our proof goes in two steps: first we establish a Carleman inequality for any spinor section v in the domain of $\mathcal{D}$ which satisfies $\operatorname{supp}(v) \subset N_T$. More precisely, we are going to show that for T sufficiently small there exists a constant C, such that

(1.16)
$$R\int_{u=0}^{T}\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\,|v(u,y)|^2\,dy\,du \le C\int_{u=0}^{T}\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\,|\mathcal{D}v(u,y)|^2\,dy\,du$$

holds for any real R sufficiently large. In the second step we apply (1.16) to our cutoff section v and conclude that then ψ is equal 0 on $N_{T/2}$.

Step 1. First consider a few technical points. The Dirac operator $\mathcal{D}$ has the form $G(u)(\partial_u + \mathcal{B}_u)$ on the annular region $[0,T]\times\mathcal{S}_{p,0}$, and it is obvious that we may consider the operator $(\partial_u+\mathcal{B}_u)$ instead of $\mathcal{D}$. Moreover, we have by Lemma 1.29 that $\mathcal{B}_u = B_u + C_u$ with a self-adjoint elliptic differential operator B_u and an anti-symmetric operator C_u of order zero, both on $\mathcal{S}_{p,u}$. Note that the metric structures depend on the normal variable u.

Now make the substitution

$$v =: e^{-R(T-u)^2/2} v_0$$

which replaces (1.16) by

(1.17)
$$R\int_0^T\int_{\mathcal{S}_{p,u}} |v_0(u,y)|^2\;dy\,du \le C\int_0^T\int_{\mathcal{S}_{p,u}} \left|\frac{\partial v_0}{\partial u} + \mathcal{B}_u v_0 + R(T-u)v_0\right|^2\;dy\,du.$$

We denote the integral on the left side by J_0 and the integral on the right side by J_1. Now we prove (1.17). Decompose $\dfrac{\partial}{\partial u} + \mathcal{B}_u + R(T-u)$ into its symmetric part $B_u + R(T-u)$ and anti-symmetric part $\partial_u + C_u$. This gives

$$\begin{aligned}
J_1 &= \int_0^T\int_{\mathcal{S}_{p,u}} \left|\frac{\partial v_0}{\partial u} + \mathcal{B}_u v_0 + R(T-u)v_0\right|^2\;dy\,du\\
&= \int_0^T\int_{\mathcal{S}_{p,u}} \left|\frac{\partial v_0}{\partial u} + C_u v_0\right|^2\;dy\,du + \int_0^T\int_{\mathcal{S}_{p,u}} \left|(B_u + R(T-u))\,v_0\right|^2\;dy\,du\\
&\quad + 2\Re\int_0^T\int_{\mathcal{S}_{p,u}} \left\langle \frac{\partial v_0}{\partial u} + C_u v_0, B_u v_0 + R(T-u)v_0\right\rangle\,dy\,du\,.
\end{aligned}$$

Integrate by parts and use the identity for the real part

$$\Re\,\langle f, Pf\rangle = \frac{1}{2}\,\langle f, (P+P^*)f\rangle$$

in order to investigate the last and critical term which will be denoted by J_2. This yields (where we drop domains of integration)

$$\begin{aligned}
J_2 &= 2\Re \int\int \left\langle \frac{\partial v_0}{\partial u} + C_u v_0, B_u v_0 + R(T-u)v_0 \right\rangle dy\, du \\
&= 2\Re \int\int \left\langle \frac{\partial v_0}{\partial u}, B_u v_0 + R(T-u)v_0 \right\rangle dy\, du + 2\Re \int\int \langle C_u v_0, B_u v_0\rangle\, dy\, du \\
&= -2\Re \int\int \left\langle v_0, \left\{\frac{\partial}{\partial u}\left(B_u + R(T-u)\right)\right\} v_0 \right\rangle dy\, du - 2\Re \int\int \langle v_0, C_u B_u v_0\rangle\, dy\, du \\
&= 2 \int\int \left\langle v_0, -\frac{\partial B_u}{\partial u} v_0 + R v_0 \right\rangle dy\, du + \int\int \left(v_0; [B_u, C_u]\, v_0\right)\, dy\, du \\
&= 2R \int_0^T \|v_0\|_0^2\, du + \int\int \left\langle v_0, -2\frac{\partial B_u}{\partial u} v_0 + [B_u, C_u]\, v_0 \right\rangle dy\, du \\
&= 2R J_0 + J_3,
\end{aligned}$$

where $\|\cdot\|_m$ denotes the m-th Sobolev norm on $E|_{\mathcal{S}_{p,u}}$ and J_3 requires a careful analysis. It follows from the preceding decompositions of J_1 and J_2 that the proof of (1.17) will be completed with $C = \frac{1}{2}$ when $J_3 \geq 0$. If $J_3 < 0$ and $C = \frac{1}{2}$, it suffices to show that

$$|J_3| \leq \frac{1}{2}\left(R\int_0^T \|v_0\|_0^2\, du + \int_0^T \|(B_u + R(T-u))v_0\|^2\, du\right). \tag{1.18}$$

Since the operators B_u are elliptic of order 1, Gårding's inequality (1.11) yields

$$\|f\|_1 \leq c\left(\|f\|_0 + \|B_u f\|_0\right)$$

for any section f of E on $\mathcal{S}_{p,u}$ (and $0 \leq u \leq T$). Then, also using the fact that $-2\frac{\partial B_u}{\partial u} + [B_u, C_u]$ is a *first*-order differential operator on $E|\mathcal{S}_{p,u}$, we obtain

$$\begin{aligned}
|J_3| &\leq \int_0^T \|v_0\|_0 \left\|-2\tfrac{\partial B_u}{\partial u} v_0 + [B_u, C_u] v_0\right\|_0 du \leq c_1 \int_0^T \|v_0\|_0 \|v_0\|_1\, du \\
&\leq c_1 c \int_0^T \|v_0\|_0 \left(\|B_u v_0\|_0 + \|v_0\|_0\right) du \\
&\leq c_1 c \int_0^T \|v_0\|_0 \left\{\|(B_u + R(T-u))v_0\|_0 + (R(T-u)+1)\|v_0\|_0\right\} du \\
&\leq c_1 c (RT+1) \int_0^T \|v_0\|_0^2\, du + c_1 c \int_0^T \|(B_u + R(T-u))v_0\|_0\, \|v_0\|_0\, du\,.
\end{aligned}$$

The integrand of the second summand is equal to

$$\begin{aligned}
&\frac{\|(B_u + R(T-u))v_0\|_0}{\sqrt{c_1 c}}\left(\sqrt{c_1 c}\|v_0\|_0\right) \\
&\leq \tfrac{1}{2}\left\{\tfrac{1}{c_1 c}\|(B_u + R(T-u))v_0\|_0^2 + c_1 c\|v_0\|_0^2\right\}
\end{aligned} \tag{1.19}$$

with the inequality due to the estimate $ab \le \frac{1}{2}(a^2+b^2)$. By inserting (1.19) in the preceding inequality for $|J_3|$ we obtain

$$
\begin{aligned}
|J_3| &\le c_1 c(RT+1)\int_0^T \|v_0\|_0^2\,du \\
&\quad + c_1 c\int_0^T \left(\tfrac{1}{2}\left\{\tfrac{1}{c_1 c}\,\|(B_u+R(T-u))v_0\|_0^2 \,+\, c_1 c\|v_0\|_0^2\right\}\right)\,du \\
&= c_1 c(RT+1)\int_0^T \|v_0\|_0^2\,du + c_1 c\int_0^T \tfrac{1}{2}\,c_1 c\|v_0\|_0^2\,du \\
&\quad + \int_0^T \left(\tfrac{1}{2}\,\|(B_u+R(T-u))v_0\|_0^2\right)\,du \\
&= \int_0^T \tfrac{1}{2}\,\|(B_u+R(T-u))v_0\|_0^2\ du + c_1 c\left((RT+1)+\tfrac{1}{2}\,c_1 c\right)\int_0^T \|v_0\|_0^2\,du \\
&= \tfrac{1}{2}\int_0^T \|(B_u+R(T-u))v_0\|_0^2\ du + Rc_1 c\left(T+\frac{c_1 c+2}{2R}\right)\int_0^T \|v_0\|_0^2\,du
\end{aligned}
$$

So (1.18) holds for T and $\frac{1}{R}$ sufficiently small, and we then have the Carleman inequality (1.16) for $C=\frac{1}{2}$.

Step 2. To begin with, we have

(1.20)
$$
e^{RT^2/4}\int_0^{\frac{T}{2}}\int_{\mathcal{S}_{p,u}} |\psi(u,y)|^2\ dy\,du \le \int_0^T\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\,|\left(\varphi\psi\right)(u,y)|^2\,dy\,du =: I
$$

We apply our Carleman type inequality (1.16):

(1.21)
$$
I=\int_0^T\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\,|\left(\varphi\psi\right)(u,y)|^2\,dy\,du \le \frac{C}{R}\int_{u=0}^T\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\,|\mathcal{D}(\varphi\psi)(u,y)|^2\,dy\,du.
$$

Assuming that ψ is a solution of the perturbed equation $\mathcal{D}\psi+\mathfrak{P}_A(\psi)=0$, we get

$$
\mathcal{D}(\varphi\psi)=\varphi\mathcal{D}\psi+\mathbf{c}(du)\varphi'\psi=-\varphi\mathfrak{P}_A(\psi)+\mathbf{c}(du)\varphi'\psi.
$$

Using this in (1.21) and noting that $(a+b)^2\le 2\left(a^2+b^2\right)$, yields

$$
I\le\frac{2C}{R}\int_0^T\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\left(|\varphi(u)\mathfrak{P}_A(\psi)(u,y)|^2+|\mathbf{c}(du)\varphi'(u)\psi(u,y)|^2\right)\,dy\,du.
$$

Now we exploit our assumption

$$
|\mathfrak{P}_A(\psi)(x)|\le P(\psi,x)|\psi(x)|\quad\text{for } x\in M \tag{1.22}
$$

about the perturbation with locally bounded $P(\psi,\cdot)$, say

$$
|P(\psi,(u,y))|\le C_0:=\max_{x\in K}|P(\psi,x)|\quad\text{for all } y\in\mathcal{S}_{p,u},\ u\in[0,T]
$$

where K is a suitable compact set. We obtain at once

$$\begin{aligned}\left(1-\frac{2CC_0}{R}\right) I &\le \frac{2C}{R}\int_0^T\int_{\mathcal{S}_{p,u}} e^{R(T-u)^2}\,|\mathbf{c}(du)\varphi'(u)\psi(u,y)|^2\,dy\,du\\ &\le \frac{2C}{R}e^{RT^2/25}\int_0^T\int_{\mathcal{S}_{p,u}}|\mathbf{c}(du)\varphi'(u)\psi(u,y)|^2\,dy\,du.\end{aligned}$$

Here we use that $\varphi'(u)=0$ for $0\le u\le 0.8T$ so that we can estimate the exponential and pull it in front of the integral. Using (1.20),

$$\begin{aligned}&\int_0^{\frac{T}{2}}\int_{\mathcal{S}_{p,u}}|\psi(u,y)|^2\;dy\,du \le e^{-RT^2/4} I\\ &\le \frac{\frac{2C}{R}}{1-\frac{2CC_0}{R}}e^{RT^2\left(\frac{1}{25}-\frac{1}{4}\right)}\int_0^T\int_{\mathcal{S}_{p,u}}|\mathbf{c}(du)\varphi'(u)\psi(u,y)|^2\,dy\,du.\end{aligned}$$

As $R\to\infty$, we get $\int_0^{\frac{T}{2}}\int_{\mathcal{S}_{p,u}}|\psi(u,y)|^2\;dy\,du=0$ which contradicts $x_0\in\operatorname{supp}\psi$. □

2. The Index of Elliptic Operators on Partitioned Manifolds

2.1. Examples and the Hellwig–Vekua Index Theorem. We begin with some elementary examples.

Example 2.1. Consider the (trivially elliptic) ordinary differential operator on the unit interval $I=[0,1]$ defined by

$$P: C^\infty(I)\times C^\infty(I)\to C^\infty(I)\times C^\infty(I), \text{where }\ (f,g)\mapsto (f',-g')\,,$$

with the boundary conditions $C^\infty(I)\times C^\infty(I)\to C^\infty(\partial I)\cong\mathbb{C}\times\mathbb{C}$ (where $\partial I:=\{0,1\}$)

$$\begin{aligned}(i)\ & R_1:(f,g)\mapsto (f-g)\,|_{\partial I}\\ (ii)\ & R_2:(f,g)\mapsto f|_{\partial I}\\ (iii)\ & R_3:(f,g)\mapsto (f+g')|_{\partial I}\end{aligned}$$

We determine the index of the operators (for $i=1,2,3$)

$$(P,R_i): C^\infty(I)\times C^\infty(I)\to C^\infty(I)\times C^\infty(I)\times C^\infty(\partial I).$$

Clearly, $\dim\,\mathrm{Ker}(P,R_i)=1$. To determine the cokernel, one writes $P(f,g)=(F,G)$ and $R_i(f,g)=h$, with $F,G\in C^\infty(I)$ and $h=(h_0,h_1)\in\mathbb{C}\times\mathbb{C}$ obtaining,

$$f(t)=\int_0^t F(\tau)d\tau+c_1,\ \ g(t)=-\int_0^t G(\tau)d\tau+c_2$$

and two more equations for the boundary condition. The dimension of $\mathrm{Coker}(P,R_i)$ is then the number of linearly independent conditions on F, G, and h which must be imposed in order to eliminate the constants of integration. For each $i\in\{1,2,3\}$, there is only one condition, namely $h_0 = h_1 - \int_0^1 F(\tau)d\tau - \int_0^1 G(\tau)d\tau$; $h_0 = h_1 - \int_0^1 F(\tau)d\tau$; resp., $h_0 = h_1 - \int_0^1 F(\tau)d\tau - G(0) + G(1)$. So, the index vanishes in all three cases.

For a more comprehensive treatment of the existence and uniqueness of boundary-value problems for ordinary differential equations (including systems), we refer to [**CodLev**] and [**Har64**, p. 322-403].

We now consider the Laplace operator $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$, as a linear elliptic differential operator from $C^\infty(X)$ to $C^\infty(X)$, where X is the unit disk $\{z = x + iy \mid |z| \leq 1\} \subset \mathbb{C}$.

EXAMPLE 2.2. For the Dirichlet boundary condition

$$R : C^\infty(X) \to C^\infty(\partial X), \text{ with } R(u) = u|_{\partial X} \; (\partial X := \{z \in \mathbb{C} \mid |z| = 1\}),$$

we show that

$$\text{a) } \operatorname{Ker}(\Delta, R) = \{0\} \quad \text{and} \quad \text{b) } \operatorname{Im}(\Delta, R)^\perp = \{0\},$$

where $\perp$ is orthogonal complement in $L^2(X) \times L^2(\partial X)$.[1] In particular, it follows that $\operatorname{index}(\Delta, R) = 0$.

For (a): $\operatorname{Ker}(\Delta, R)$ consists of functions of the form $u + iv$, where u and v are real-valued. Since the coefficients of the operators Δ and R are real, we may assume $v = 0$ without loss of generality. Thus, consider a real solution u with $\Delta u = 0$ in X and $u = 0$ on ∂X. Then (where $\nabla u := (u_x, u_y) := (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$)

$$0 = -\int_X u \Delta u \, dxdy = \int_X |\nabla u|^2 \, dxdy, \tag{2.1}$$

whence $\nabla u = 0$. Thus, u is constant, and indeed zero since $u = 0$ on ∂X. The trick lies in the equality (2.1), which follows from Stokes' formula, namely $\int_X d\omega = \int_{\partial X} \omega$, where ω is a 1-form. Indeed, setting $\omega := u \wedge *du$, where $*$ is the Hodge star operator ($*du = *(u_x dx + u_y dy) := u_x dy - u_y dx$), we obtain

$$d\omega = du \wedge *du + u \wedge d*du = |\nabla u|^2 \, dx \wedge dy + (u\Delta u) \, dx \wedge dy.$$

Using Stokes' formula and $u|_{\partial X} = 0$, we have

$$\int_X |\nabla u|^2 \, dxdy + \int_X (u\Delta u) \, dxdy = \int_X d\omega = \int_{\partial X} \omega = \int_{\partial X} u \wedge *du = 0.$$

[1] Here, consider that the intersection of the orthogonal complement of the range $\operatorname{Im}(\Delta, R)$ relative to the usual inner product in $L^2(X) \times L^2(\partial X)$ with the space $C^\infty(X) \times C^\infty(\partial X)$ is isomorphic to $\operatorname{Coker}(\Delta, R)$. This is true, since it turns out that the image of the natural Sobolev extension of (Δ, R) is closed in the L^2-norm, and its L^2-orthogonal complement is contained in $C^\infty(X) \times C^\infty(\partial X)$.

From this and $\Delta u = 0$, we conclude that $\nabla u = 0$ and u is constant.
For (b): Choose $L \in C^\infty(X)$ and $l \in C^\infty(\partial X)$ with (L, l) orthogonal to $\mathrm{Im}(\Delta, R)$, whence (relative to the usual measures on X and ∂X)

$$\int_X (\Delta u)\, L + \int_{\partial X} ul = 0 \text{ for all } u \in C^\infty(X). \tag{2.2}$$

Using a 2-fold integration by parts (in the exterior calculus), we obtain

(2.3)
$$\int_X u\Delta L - \int_X (\Delta u)\, L = \int_X (u(d * dL) - d\,(*du)\, L) = \int_{\partial X} (u * dL - L * du)\,.$$

First we consider u with support $\mathrm{supp}\,(u) :=$ the closure of $\{z \in X \mid u(z) \neq 0\}$ contained in the interior of X. Then

$$\int_X u\Delta L = \int_X (\Delta u)\, L = -\int_{\partial X} ul = 0,$$

and so $\Delta L = 0$. Now for $u \in C^\infty(X)$ we apply (2.2) and (2.3) to deduce that

$$\begin{aligned}\int_{\partial X} ul = -\int_X (\Delta u)\, L &= \int_{\partial X} (u * dL - L * du) \\ &= \int_{\partial X} (u\,(xL_x + yL_y) - L\,(xu_x + yu_y))\,.\end{aligned}$$

Thus, $l = xL_x + yL_y$ and $L|_{\partial X} = 0$, and we finally apply (a). Details are in [**Ho63**, p. 264].

The preceding result $\mathrm{index}(\Delta, R) = 0$ (for $Ru = u|_{\partial X}$) can also be obtained by proving the symmetry of Δ and that the L^2 extension on the domain defined by $Ru = 0$ is a self-adjoint Fredholm extension.

We now consider a C^∞ vector field $\nu : \partial X \to \mathbb{C}$ on the boundary $\partial X = \{z \in \mathbb{C} \mid |z| = 1\}$. For $u \in C^\infty(X)$, $z \in \partial X$, and $\nu(z) = \alpha(z) + i\beta(z)$, the "directional derivative" of the function u relative to ν at the point z is

$$\tfrac{\partial u}{\partial \nu}\,(z) := \alpha(z)\tfrac{\partial u}{\partial x}\,(z) + \beta(z)\tfrac{\partial u}{\partial y}\,(z)\,.$$

From the standpoint of differential geometry it is better, either to denote the vector field by $\frac{\partial}{\partial \nu}$ or to write the directional derivative as simply as $\nu\,[u]\,(z)$, since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ can be regarded as vector fields. The pair $(\Delta, \frac{\partial}{\partial \nu})$ defines a linear operator

$$(\Delta, \tfrac{\partial}{\partial \nu}) : C^\infty(X) \to C^\infty(X) \oplus C^\infty(\partial X) \text{ given by } u \mapsto (\Delta u, \tfrac{\partial u}{\partial \nu}).$$

THEOREM 2.3. *(I. N. Vekua 1952). For $p \in \mathbb{Z}$ and $\nu(z) := z^p$, we have that $(\Delta, \frac{\partial}{\partial \nu})$ is an operator with finite-dimensional kernel and cokernel, and*

$$\mathrm{index}(\Delta, \tfrac{\partial}{\partial \nu}) = 2\,(1 - p)\,.$$

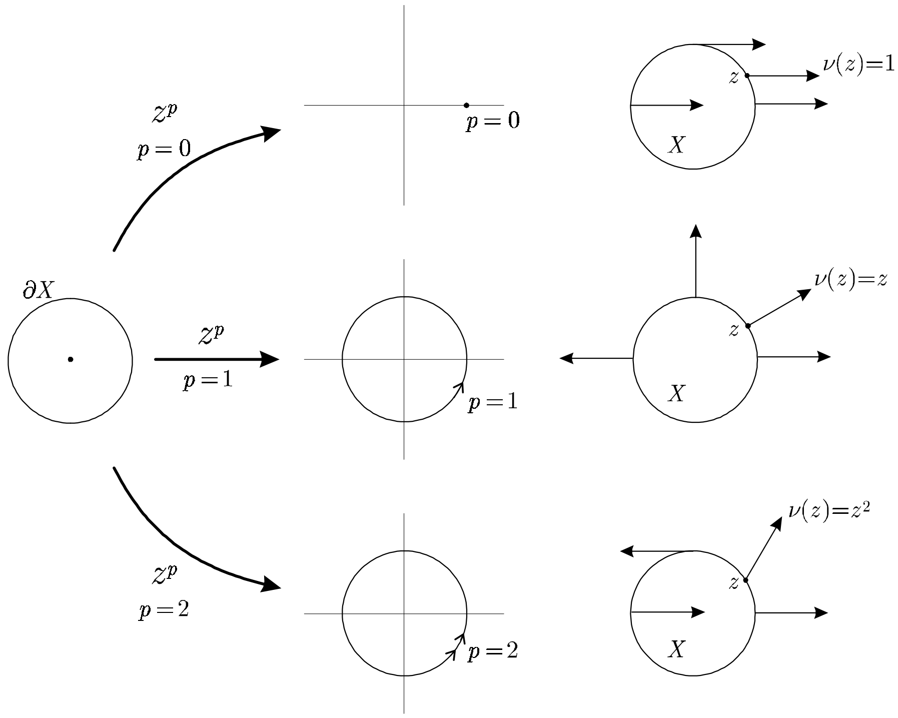

FIGURE 4. The vector field $\nu : \partial X \to \mathbb{C}$ with winding number $p = 0, 1, 2$

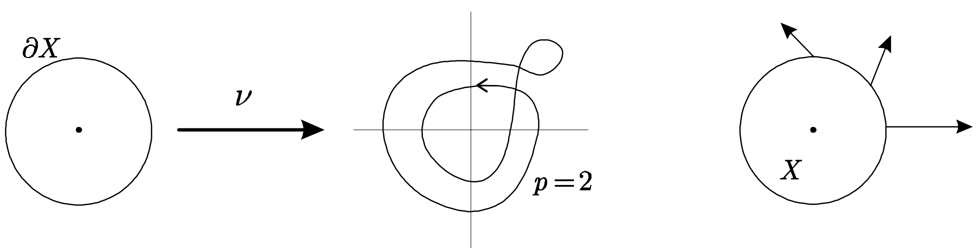

FIGURE 5. Another vector field ν with winding number 2

REMARK 2.4. The theorem of Vekua remains true, if we replace z^p by any nonvanishing "vector field" $\nu : \partial X \to \mathbb{C} \setminus \{0\}$ with "winding number" p:

Moreover, in place of the disk, we can take X to be any simply-connected domain in $\mathbb{C}$ with a "smooth" boundary ∂X. The reason is the homotopy invariance of the index.

REMARK 2.5. In the theory of Riemann surfaces (e.g., in Riemann–Roch Theorem), one also encounters the number $2(1 - p)$ as the Euler characteristic of a closed surface of genus p. This is no accident, but rather it is connected with the relation between elliptic boundary-value problems and elliptic operators on closed manifolds. Specifically, there is a relation between the index of $(\Delta, \frac{\partial}{\partial\nu})$ and the index of Cauchy-Riemann operator for complex line bundles over $S^2 = \mathbb{P}^1(\mathbb{C})$ with Chern number $1-p$ (e.g., see Example 2.19 for a start).

REMARK 2.6. Motivated by the method of replacing a differential equation by difference equations, David Hilbert and Richard Courant expected "linear problems of mathematical physics which are correctly posed to behave like a system of N linear algebraic equations in N unknowns... If for a correctly posed problem in linear differential equations the corresponding homogeneous problem possesses only the trivial solution zero, then a uniquely determined solution of the general inhomogeneous system exists. However, if the homogeneous problem has a nontrivial solution, the solvability of the inhomogeneous system requires the fulfillment of certain additional conditions." This is the "heuristic principle" which Hilbert and Courant saw in the Fredholm Alternative [**CoHi**]. Günter Hellwig [**He52**] (nicely explained in [**Ha52**]) in the real setting and Ilya Nestorovich Vekua [**Ve56**] in complex setting disproved it with their independently found example where the principle fails for $p \neq 1$.
We remark that in addition to these "oblique-angle" boundary-value problems, "coupled" oscillation equations, as well as restrictions of boundary-value problems, even with vanishing index, to suitable half-spaces, furnish further more or less elementary examples for index $\neq 0$. The simplest example of a system of first order differential operators on the disc is provided in Example 2.7 below. A world of more advanced, and for differential geometry much more meaningful examples, is approached by the Atiyah–Patodi–Singer Index Theorem, see Section 2.4 below.

PROOF OF THEOREM 2.3. (After [**Ho63**, p. 266 f.]) Since the coefficients of the differential operators $(\Delta, \frac{\partial}{\partial\nu})$ are real, we may restrict ourselves to real functions. Thus, $u \in C^\infty(X)$ denotes a single real-valued function, rather than a complex-valued function.

$\operatorname{Ker}(\Delta, \frac{\partial}{\partial\nu})$: It is well-known that $\operatorname{Ker}(\Delta)$ consists of real (or imaginary) parts of holomorphic functions on X (e.g., see [**Ah53**, p. 175 ff.]). Hence, $u \in \operatorname{Ker}(\Delta)$, exactly when $u = \mathfrak{Re}(f)$ where $f = u+iv$ is holomorphic; i.e., the Cauchy-Riemann equation $\frac{\partial f}{\partial \bar z} = 0$ holds, where $\frac{\partial}{\partial \bar z} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Explicitly,

$$0 = \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Every holomorphic (= complex differentiable) function f is twice complex differentiable and its derivative is given by

$$\begin{aligned}\frac{\partial f}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.\end{aligned}$$

In this way we have a holomorphic function $\phi := f'$ for each $u \in \operatorname{Ker}(\Delta)$. Since

$$\frac{\partial u}{\partial \nu} = \mathfrak{Re}\,(z^p)\,\frac{\partial u}{\partial x} + \mathfrak{Im}\,(z^p)\,\frac{\partial u}{\partial y} = \mathfrak{Re}\left(\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)z^p\right) = \mathfrak{Re}\,(\phi\,(z)\,z^p),$$

the boundary condition $\frac{\partial u}{\partial \nu} = 0$ $(\nu = z^p)$ then means that the real part $\mathfrak{Re}(\phi(z)z^p)$ vanishes for $|z| = 1$. For $p \geq 0$, $\phi(z)z^p$ is holomorphic as well as ϕ, and hence for $\phi(z) := \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, we have

$$\begin{aligned} &u \in \operatorname{Ker}\left(\Delta, \tfrac{\partial}{\partial \nu}\right) \text{ with } \nu = z^p,\ p \geq 0 \\ &\quad \Rightarrow \mathfrak{Re}\left(\phi(z)z^p\right) \in \operatorname{Ker}\left(\Delta, R\right) \text{ where } R\left(\cdot\right) = \left(\cdot\right)|_{\partial X}. \end{aligned}$$

Thus, we succeed in associating with the "oblique-angle" boundary-value problem for u a Dirichlet boundary-value problem for $\mathfrak{Re}\left(\phi(z)z^p\right)$, which has only the trivial solution by Example 2.2a. Since $\phi(z)z^p$ is holomorphic with $\mathfrak{Re}\left(\phi(z)z^p\right) = 0$, the partial derivatives of the imaginary part vanish, and so there is a constant $C \in \mathbb{R}$ such that $\phi(z)z^p = iC$ for all $z \in X$. If $p > 0$, then we have $C = 0$ (set $z = 0$). Hence $\phi = 0$, and (by the definition of ϕ) the function u is constant (i.e., $\dim \operatorname{Ker}\left(\Delta, \frac{\partial}{\partial \nu}\right) = 1$). If $p = 0$, then $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \phi(z) = iC$, and so $u(x,y) = -Cy + \widetilde{C}$, whence $\dim \operatorname{Ker}\left(\Delta, \frac{\partial}{\partial \nu}\right) = 2$ in this case.

We now come to the case $p < 0$, which curiously is not immediately reducible to the case $q > 0$ where $q := -p$. One might try to look for a solution by simply turning $\frac{\partial}{\partial \nu}$ around to $-\frac{\partial}{\partial \nu}$, but this is futile since the winding numbers of ν and $-\nu$ about 0 are the same. Besides, if $\nu_p\left(z\right) = z^p$, we do *not* have $\frac{\partial}{\partial \nu_{-p}} = -\frac{\partial}{\partial \nu_p}$. In order to reduce the boundary-value problem with $p < 0$ to the elementary Dirichlet problem, we must go through a more careful argument. Note that $\phi(z)z^p$ can have a pole at $z = 0$, whence $\mathfrak{Re}\left(\phi(z)z^p\right)$ is not necessarily harmonic. We write the holomorphic function $\phi(z)$ as

$$\phi(z) = \sum\nolimits_{j=0}^{q} a_j z^j + g\left(z\right) z^{q+1}$$

where $q := -p$ and g is holomorphic. We define a holomorphic function ψ by

$$\psi(z) := g\left(z\right) z + \sum\nolimits_{j=0}^{q-1} \overline{a}_j z^{q-j},$$

with $\psi(0) = 0$. Then one can write

$$\phi(z)z^p = a_q + \sum\nolimits_{j=0}^{q-1} \left(a_j z^{j-q} - \overline{a}_j z^{q-j}\right) + \psi(z)$$

The boundary condition $\frac{\partial u}{\partial \nu} = 0$ implies $\mathfrak{Re}\left(\phi(z)z^p\right) = 0$ for $|z| = 1$. By the above equation, we have $0 = \mathfrak{Re}\left(\phi(z)z^p\right) = \mathfrak{Re}(\psi(z) + a_q)$ for $|z| = 1$ since then $z^{-1} = \bar{z}$. Since ψ is holomorphic, we have again arrived at a Dirichlet boundary value problem; this time for the function $\mathfrak{Re}(\psi(z) + a_q)$. From Example 2.2a, it follows again that $\psi(z) + a_q$ is an imaginary constant, whence $\psi(z) = \psi(0) = 0$ and a_q is pure imaginary. We have

$$\phi(z) = \phi(z)z^p z^q = a_q z^q + \sum\nolimits_{j=0}^{q-1} \left(a_j z^j - \overline{a}_j z^{2q-j}\right)$$

for arbitrary $a_0, a_1, ..., a_{q-1} \in \mathbb{C}$ and $a_q \in i\mathbb{R}$. As a vector space over $\mathbb{R}$, the set

$$\left\{ \tfrac{\partial u}{\partial x} - i\tfrac{\partial u}{\partial y} \mid u \in \operatorname{Ker}\left(\Delta, \tfrac{\partial}{\partial \nu}\right) \right\}$$

has dimension $2q+1$; here we have restricted ourselves to real u, according to our convention above. Since u is uniquely determined by ϕ up to an additive constant, it follows that for $\nu = z^p$ and $p < 0$,

$$\dim \operatorname{Ker}\left(\Delta, \tfrac{\partial}{\partial\nu}\right) = 2q+2 = 2-2p$$

$\operatorname{Coker}\left(\Delta, \frac{\partial}{\partial\nu}\right)$: As Example 2.2b shows the equation $\Delta u = F$ has a solution for each $F \in C^\infty(X)$. In view of this we can show

$$\operatorname{Coker}\left(\Delta, \tfrac{\partial}{\partial\nu}\right) = \frac{C^\infty(X) \times C^\infty(\partial X)}{\operatorname{Im}\left(\Delta, \frac{\partial}{\partial\nu}\right)} \cong \frac{C^\infty(\partial X)}{\frac{\partial}{\partial\nu}\left(\operatorname{Ker}\Delta\right)}$$

as follows. We assign to each representative pair $(F,h) \in C^\infty(X) \times C^\infty(\partial X)$ the class of $h - \frac{\partial u}{\partial\nu} \in C^\infty(\partial X)$, where u is chosen so that $\Delta u = F$. This map is clearly well defined on the quotient space of pairs, and the inverse map is given by $h \mapsto (0,h)$. Hence, we have found a representation for $\operatorname{Coker}(\Delta, \frac{\partial}{\partial\nu})$ in terms of the "boundary functions" $\left\{\frac{\partial u}{\partial\nu} \mid u \in \operatorname{Ker}\Delta\right\}$, rather than the cumbersome pairs in $\operatorname{Im}\left(\Delta, \frac{\partial}{\partial\nu}\right)$. (This trick can always be applied for the boundary-value problems (P,R), when the operator P is surjective.)

We therefore investigate the existence of solutions of the equation $\Delta u = 0$ with the "inhomogeneous" boundary condition $\frac{\partial u}{\partial\nu} = h$, where h is a given C^∞ function on ∂X. According to the trick introduced in the first part of our proof, it is equivalent to ask for the existence of a holomorphic function ϕ with the boundary condition $\mathfrak{Re}\left(\phi(z)z^p\right) = h$, $|z| = 1$, i.e., for a solution of a Dirichlet problem for $\mathfrak{Re}\left(\phi(z)z^p\right)$. By Example 2.2b, there is a unique harmonic function which restricts to h on the boundary ∂X; hence, we have a (unique up to an additive imaginary constant) holomorphic function θ with $\mathfrak{Re}\theta(z) = h$ for $|z| = 1$.

In the case $p < 0$, the boundary problem for ϕ is always solvable; namely, set $\phi(z) := z^{-p}\theta(z)$. Hence, we have

$$\dim \operatorname{Coker}\left(\Delta, \tfrac{\partial}{\partial\nu}\right) = 0 \text{ for } \nu(z) = z^p \text{ and } p \leq 0.$$

For $p > 0$, we can construct a solution of the boundary-value problem in for ϕ from θ, if and only if there is a constant $C \in \mathbb{R}$, such that $(\theta(z) - iC)/z^p$ is holomorphic (i.e., the holomorphic function $\theta(z) - iC$ has a zero of order at least p at $z = 0$. Using the Cauchy Integral Formula, these conditions on the derivatives of θ at $z = 0$ correspond to conditions on line integrals around ∂X. In this way, we have $2p-1$ linear (real) equations that h must satisfy in order that the boundary-value problem have a solution. We summarize our results in the following table ($\nu(z) = z^p$) and Figure 6:

p	$\dim \operatorname{Ker}\left(\Delta, \frac{\partial}{\partial\nu}\right)$	$\dim \operatorname{Coker}\left(\Delta, \frac{\partial}{\partial\nu}\right)$	$\operatorname{index}\left(\Delta, \frac{\partial}{\partial\nu}\right)$
> 0	1	$2p-1$	$2-2p$
≤ 0	$2-2p$	0	$2-2p$

□

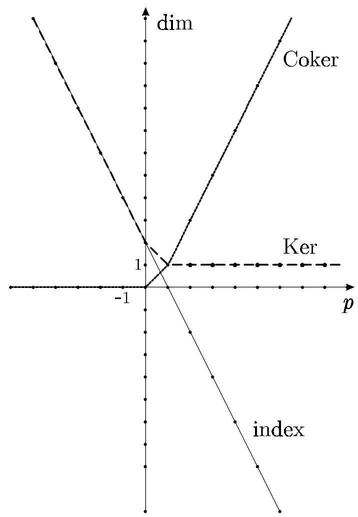

FIGURE 6. The dimensions of kernel, cokernel and the index of the Laplacian with boundary condition given by $\nu(z) := z^p$ for varying p

Note 1: We already noted in the proof the peculiarity that the case $p < 0$ cannot simply be played back to the case $p > 0$. This is reflected here in the asymmetry of the dimensions of kernel and cokernel and the index. It simply reflects the fact that there are "more" rational functions with prescribed poles than there are polynomials with "corresponding" zeros.

Note 2: In contrast to the Dirichlet Problem, which we could solve via integration by parts (i.e., via Stokes' Theorem), the above proof is function-theoretic in nature and cannot be used in higher dimensions. This is no loss in our special case, since the index of the "oblique-angle" boundary-value problem must vanish anyhow in higher dimensions for topological reasons; see [**Ho63**, p. 265 f.]. The actual mathematical challenge of the function-theoretic proof arises less from the restriction $\dim X = 2$ than from a certain arbitrariness, namely the tricks and devices of the definitions of the auxiliary functions ϕ, ψ, θ, by means of which the oblique-angle problem is reduced to the Dirichlet problem.

Note 3: The theory of ordinary differential equations easily conveys the impression that partial differential equations also possess a "general solution" in the form of a functional relation between the unknown function ("quantity") u, the independent variables x and some arbitrary constants or functions, and that every "particular solution" is obtained by substituting certain constants or functions f, h, etc. for the arbitrary constants and functions. (Corresponding to the higher degree of freedom in partial differential equations, we deal not only with constants of integration but with arbitrary functions.) The preceding calculations, regarding the boundary value problem of the Laplace operator, clearly indicate how limited this notion is which was conceived in the 18th century on the basis of geometric intuition and physical considerations. The classical recipe of first searching for general solutions and only at the end determining the arbitrary constants and functions fails. For example, the specific form of boundary conditions must enter the analysis to begin with.

EXAMPLE 2.7. Let $X := \{z = x + iy \mid |z| < 1\}$ be the unit disk and define an operator

$$T : C^\infty(X) \times C^\infty(X) \to C^\infty(X) \oplus C^\infty(X) \oplus C^\infty(\partial X) \text{ by}$$
$$T(u, v) := \left(\tfrac{\partial u}{\partial \bar{z}}, \tfrac{\partial v}{\partial z}, (u - v)\,|_{\partial X}\right)$$

where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ is complex differentiation and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ is the Cauchy-Riemann differential operator "formally adjoint" to $\frac{\partial}{\partial z}$. We show that $\dim(\operatorname{Ker} T) = 1$ and $\operatorname{Coker}(T) = \{0\}$ and hence that $\operatorname{index}(T) = 1$. Suppose that $(u, v) \in \operatorname{Ker} T$. Then $\frac{\partial u}{\partial \bar{z}} = 0$ and $\frac{\partial v}{\partial z} = 0$ in which case u and v are harmonic. Then since $(u - v)\,|_{\partial X} = 0$, we have $u = v$ on X. Now $\frac{\partial v}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} = 0 \Rightarrow v$ is holomorphic, and $v'(z) = \frac{\partial v}{\partial z} = 0 \Rightarrow v\ (= u)$ is constant. Thus, $\dim(\operatorname{Ker} T) = 1$. To show that $\operatorname{Coker}(T) = \{0\}$, or more precisely $(\operatorname{Im} T)^\perp = \{0\}$; see the footnote to Example 2.2, we choose arbitrary $f, g \in C^\infty(X)$ and $h \in C^\infty(\partial X)$ and prove that f, g and h must identically vanish, if

$$\int_X \left(\tfrac{\partial u}{\partial \bar{z}} f + \tfrac{\partial v}{\partial z} g\right) + \int_{\partial X} (u - v)h = 0 \text{ for all } u, v \in C^\infty(X). \tag{2.4}$$

Note that for $P, Q \in C^\infty(X)$

$$d\,(P dz + Q d\overline{z}) = \tfrac{\partial P}{\partial \bar{z}} d\overline{z} \wedge dz + \tfrac{\partial Q}{\partial z} dz \wedge d\overline{z} = \left(\tfrac{\partial P}{\partial \bar{z}} - \tfrac{\partial Q}{\partial z}\right) d\overline{z} \wedge dz$$
$$= \left(\tfrac{\partial P}{\partial \bar{z}} - \tfrac{\partial Q}{\partial z}\right) (dx - idy) \wedge (dx + idy) = 2i \left(\tfrac{\partial P}{\partial \bar{z}} - \tfrac{\partial Q}{\partial z}\right) dx \wedge dy$$

Thus we obtain the complex version of Stokes' Theorem,

$$\int_{\partial X} P dz + Q d\overline{z} = \int_X d\,(P dz + Q d\overline{z}) = 2i \int_X \left(\tfrac{\partial P}{\partial \bar{z}} - \tfrac{\partial Q}{\partial z}\right).$$

From this, we get

$$\int_X \tfrac{\partial u}{\partial \bar{z}} f = \int_X \tfrac{\partial}{\partial \bar{z}}(uf) - \int_X u \tfrac{\partial f}{\partial \bar{z}} = \frac{1}{2i} \int_{\partial X} uf\,dz - \int_X u \tfrac{\partial f}{\partial \bar{z}} \text{ and}$$
$$\int_X \tfrac{\partial v}{\partial z} g = \int_X \tfrac{\partial}{\partial z}(vg) - \int_X v \tfrac{\partial g}{\partial z} = \frac{-1}{2i} \int_{\partial X} vg d\bar{z} - \int_X v \tfrac{\partial g}{\partial z}.$$

Hence,

$$\int_X \left(\tfrac{\partial u}{\partial \bar{z}} f + \tfrac{\partial v}{\partial z} g\right) = -\int_X \left(u \tfrac{\partial f}{\partial \bar{z}} + v \tfrac{\partial g}{\partial z}\right) + \frac{1}{2i} \int_{\partial X} (uf\,dz - vg d\bar{z}).$$

Assuming (2.4), we have

$$\begin{aligned}0 &= \int_X \left(\tfrac{\partial u}{\partial \bar z} f + \tfrac{\partial v}{\partial z} g\right) + \int_{\partial X} (u-v)h \\ &= -\int_X \left(u \tfrac{\partial f}{\partial \bar z} + v \tfrac{\partial g}{\partial z}\right) + \frac{1}{2i}\int_{\partial X} (uf\,dz - vg\,d\bar z) + \int_{\partial X}(u-v)h\end{aligned}$$

By considering u and v with compact support inside the open disk, we deduce that $\frac{\partial f}{\partial \bar z} = 0$ and $\frac{\partial g}{\partial z} = 0$ (i.e., f and g are analytic and conjugate analytic respectively). Thus, (2.4) implies

$$0 = \frac{1}{2i}\int_{\partial X} (uf\,dz - vg\,d\bar z) + \int_{\partial X} (u-v)h,$$

for all $u, v \in C^\infty(X)$. Choosing $v = u$, we have

$$\begin{aligned}0 &= \frac{1}{2i}\int_{\partial X} u\,(f\,dz - g\,d\bar z) \ \text{ for all } u \Rightarrow f\,dz = g\,d\bar z \text{ on } \partial X \\ &\Rightarrow f\left(e^{i\theta}\right) ie^{i\theta} d\theta = -g\left(e^{i\theta}\right) ie^{-i\theta} d\theta \Rightarrow f\left(e^{i\theta}\right) e^{i\theta} = -g\left(e^{i\theta}\right) e^{-i\theta}.\end{aligned}$$

However, since f is analytic, the Fourier series of $f\left(e^{i\theta}\right) e^{i\theta}$ has a nonzero coefficient for $e^{im\theta}$ only when $m > 0$, and since g is conjugate analytic, $g\left(e^{i\theta}\right) e^{-i\theta}$ only has a nonzero coefficient for $e^{im\theta}$ only when $m < 0$. Thus, $f = g = 0$. Choosing $v = -u$, (2.4) then yields

$$0 = \int_{\partial X} (u-v)h = 2\int_{\partial X} uh \ \text{ for all } u \in C^\infty(X) \Rightarrow h = 0.$$

Remark 2.8. In engineering one calls a system of separate differential equations

$$\begin{aligned} Pu &= f \\ Qv &= g, \end{aligned}$$

which are related by a "transfer condition" $R(u, v) = h$, a "coupling problem"; when the domains of u and v are different, but have a common boundary (or boundary part) on which the transfer condition is defined, then we have a "transmission problem"; e.g., see [**Boo72**, p.7 ff]. Thus, we may think of T as an operator for a problem on the spherical surface $X \cup_{\partial X} X$ with different behavior on the upper and lower hemispheres, but with a fixed coupling along the equator.

2.2. The Index of Twisted Dirac Operators on Closed Manifolds. Recall that the index of a Fredholm operator is a measure of its asymmetry: it is defined by the difference between the dimension of the kernel (the null space) of the operator and the dimension of the kernel of the adjoint operator (= the codimension of the range). So, the index vanishes for self-adjoint Fredholm operators. For an elliptic differential or pseudo-differential operator on a closed manifold M, the index is finite and depends only on the homotopy class of the principal symbol σ of the operator over the cotangent sphere bundle S^*M. It follows (see [**LaMi**, p. 257]) that the index for elliptic *differential* operators always vanishes on *closed* odd-dimensional manifolds. On even-dimensional manifolds one has the Atiyah–Singer Index Theorem which expresses the index in explicit topological terms, involving the Todd class defined by the Riemannian structure of M and the Chern class defined by gluing two copies of a bundle over S^*M by σ.

The original approach in proving the Index Theorem in the work of Atiyah and Singer [**AtSi69**] is based on the following clever strategy. The invariance of the index under homotopy implies that the index (say, the *analytic index*) of an elliptic operator is stable under rather dramatic, but continuous, changes in its principal symbol while maintaining ellipticity. Moreover, the index is functorial with respect to certain operations, such as addition and composition. Thus, the indices of elliptic operators transform predictably under various global operations (or relations) such as direct sums, embedding and cobordism. Using K-theory, a topological invariant (say, the *topological index*) with the same transformation properties under these global operation is built from the symbol of the elliptic operator. It turns out that the global operations are sufficient to construct enough vector bundles and elliptic operators to deduce the Atiyah–Singer Index Theorem (i.e., *analytic index = topological index*). With the aid of Bott periodicity, it suffices to check that the two indices are the same in the trivial case where the base manifold is just a single point. A particularly nice exposition of this approach is found in E. Guentner's article [**Gu93**] following an argument of P. Baum.

Not long after this first proof (given in quite different variants), there emerged a fundamentally different means of proving the Atiyah–Singer Index Theorem, namely the *heat kernel method*. This is outlined here in the important case of the chiral half $\mathcal{D}^+$ of a twisted Dirac operator $\mathcal{D}$. (In the index theory of closed manifolds, one usually studies the index of a chiral half $\mathcal{D}^+$ instead of the total Dirac operator $\mathcal{D}$, since $\mathcal{D}$ is symmetric for compatible connections and then index $\mathcal{D} = 0$.) The heat kernel method had its origins in the late 1960s (e.g., in [**McK-Si**]) and was pioneered in the works [**Pa71**], [**Gi73**], [**ABP73**], etc.. In the final analysis, it is debatable as to whether this method is really much shorter or better. This depends on the background and tastes of the beholder. Geometers and analysts (as opposed to topologists) are likely to find the heat kernel method appealing, because K-theory, Bott periodicity and cobordism theory are avoided, not only for geometric operators which are expressible in terms of twisted Dirac operators, but also largely for more general elliptic pseudo-differential operators, as Melrose has done in [**Me93**]. Moreover, the heat method gives the index of a "geometric" elliptic differential operator naturally as the integral of a characteristic form (a polynomial of curvature forms) which is expressed solely in terms of the geometry of the operator itself (e.g., curvatures of metric tensors and connections). One does not destroy the geometry of the operator by taking advantage of the fact that it can be suitably deformed without altering the index. Rather, in the heat kernel approach, the invariance of the index under changes in the geometry of the operator is a consequence of the index formula itself rather than a means of proof. However, considerable analysis and effort are needed to obtain the heat kernel for $e^{-t\mathcal{D}^2}$ and to establish its asymptotic expansion as $t \to 0^+$. Also, it can be argued that in some respects the K-theoretical embedding/cobordism methods are more forceful and direct. Moreover, in [**LaMi**], we are cautioned that the index theorem for families (in its strong form) generally involves torsion elements in K-theory that are not detectable by cohomological means, and hence are not computable in terms of local densities produced by heat asymptotics. Nevertheless, when this difficulty does not arise, the K-theoretical expression for the topological index may be less appealing than the integral of a characteristic form, particularly for those who already understand and appreciate the geometrical formulation of characteristic classes. All disputes aside, the student who learns *both* approaches and formulations will be more accomplished (and probably older).

The classical geometric operators such as the Hirzebruch signature operator, the de Rham operator, the Dolbeaut operator and even the Yang-Mills operator can all be locally expressed in terms of chiral halves of twisted Dirac operators. Thus, we will focus on index theory for such operators. The index of any of these classical operators (and *their* twists) can then be obtained from the Local Index Theorem for twisted Dirac operators. This theorem supplies a well-defined n-form on M, whose integral is the index of the twisted Dirac operator. This n-form (or "index density") is expressed in terms of forms for characteristic classes which are polynomials in curvature forms. The Index Theorem thus obtained then becomes a formula that relates a global invariant quantity, namely the index of an operator, to the integral of a local quantity involving curvature. This is in the spirit of the Gauss-Bonnet Theorem which can be considered a special case.

Definition 2.9. Let M be an oriented Riemannian n-manifold ($n = 2m$ even) with metric h, and oriented orthonormal frame bundle FM. Assume that M has a spin structure $P \to FM$, where P is a principal $\operatorname{Spin}(n)$-bundle and the projection $P \to FM$ is a two-fold cover, equivariant with respect to $\operatorname{Spin}(n) \to \operatorname{SO}(n)$. Furthermore, let $E \to M$ be a Hermitian vector bundle with unitary connection ε. The *twisted Dirac operator* $\mathcal{D}$ associated with the above data is

$$\mathcal{D} := (1 \otimes \mathbf{c}) \circ \nabla : C^\infty (E \otimes \Sigma(M)) \to C^\infty (E \otimes \Sigma(M)) . \tag{2.5}$$

Here, $\Sigma(M)$ is the spin bundle over M associated to $P \to FM \to M$ via the spinor representation $\operatorname{Spin}(n) \to \operatorname{End}(\Sigma_n)$,

$$\mathbf{c} : C^\infty (\Sigma(M) \otimes TM^*) \to C^\infty (\Sigma(M))$$

is Clifford multiplication, and

$$\nabla : C^\infty (E \otimes \Sigma(M)) \to C^\infty (E \otimes \Sigma(M) \otimes TM^*)$$

is the covariant derivative determined by the connection ε and the spinorial lift to P of the Levi-Civita connection form, say θ, on FM.

Note that $\mathcal{D}$ here is a special case of an operator of Dirac type introduced in Subsection 1.3 with $\mathfrak{Cl}(M)$ acting on the second factor of $E \otimes \Sigma(M)$ which is playing the role of S. Let $\Sigma^{\pm}(M)$ denote the ± 1 eigenbundles of the complex Clifford volume element in $C^{\infty}(\mathfrak{Cl}(M))$, given at a point $x \in M$ by $i^m e_1 \cdots e_n$, where $e_1, \ldots, e_n$ is an oriented, orthonormal basis of $T_x M$. The $\Sigma^{\pm}(M)$ are the so-called chiral halves of $\Sigma(M) = \Sigma^{+}(M) \oplus \Sigma^{+}(M)$. Since

$$\begin{aligned}
&\nabla\left(C^{\infty}\left(E \otimes \Sigma^{\pm}(M)\right)\right) \subseteq C^{\infty}\left(E \otimes \Sigma^{\pm}(M) \otimes TM^{*}\right) \text{ and} \\
&(1 \otimes \mathbf{c})\left(C^{\infty}\left(E \otimes \Sigma^{\pm}(M) \otimes TM^{*}\right)\right) \subseteq C^{\infty}\left(E \otimes \Sigma^{\mp}(M)\right),
\end{aligned}$$

we have

$$\mathcal{D} = \mathcal{D}^{+} \oplus \mathcal{D}^{-}, \text{ where } \mathcal{D}^{\pm} : C^{\infty}\left(E \otimes \Sigma^{\pm}(M)\right) \to C^{\infty}\left(E \otimes \Sigma^{\mp}(M)\right).$$

The symbol of the first-order differential operator $\mathcal{D}$ is computed as follows. For $\phi \in C^{\infty}(M)$ with $\phi(x) = 0$ and $\psi \in C^{\infty}(E \otimes \Sigma(M))$, we have at x

$$\begin{aligned}
(1 \otimes \mathbf{c}) \circ \nabla(\phi\psi) &= (1 \otimes \mathbf{c}) \circ ((d\phi)\psi + \phi\nabla\psi) = (1 \otimes \mathbf{c}) \circ (d\phi)\psi \\
&= (1 \otimes \mathbf{c}(d\phi))\psi.
\end{aligned}$$

Thus, the symbol $\sigma(\mathcal{D}) : T_x M^{*} \to \operatorname{End}(\Sigma(M))$ at the covector $\xi_x \in T_x M^{*}$ is given by

$$\sigma(\mathcal{D})(\xi_x) = 1 \otimes \mathbf{c}(\xi_x) \in \operatorname{End}(E_x \otimes \Sigma_x).$$

For $\xi_x \neq 0$, $\sigma(\mathcal{D})(\xi_x)$ is an isomorphism, since

$$\sigma(\mathcal{D})(\xi_x) \circ \sigma(\mathcal{D})(\xi_x) = 1 \otimes \mathbf{c}(\xi_x)^2 = -|\xi_x|^2 \, \mathrm{I}.$$

Thus, $\mathcal{D}$ is an elliptic operator. Moreover, since $\sigma(\mathcal{D}^{+})$ and $\sigma(\mathcal{D}^{-})$ are restrictions of $\sigma(\mathcal{D})$, it follows that $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ are elliptic. It can be shown that $\mathcal{D}$ is formally self-adjoint, and $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ are formal adjoints of each other (see [**LaMi**]). We also have a pair of self-adjoint elliptic operators

$$\begin{aligned}
\mathcal{D}^2_{+} &:= \mathcal{D}^2 | C^{\infty}\left(E \otimes \Sigma^{+}(M)\right) = \mathcal{D}^{-} \circ \mathcal{D}^{+} \\
\mathcal{D}^2_{-} &:= \mathcal{D}^2 | C^{\infty}\left(E \otimes \Sigma^{-}(M)\right) = \mathcal{D}^{+} \circ \mathcal{D}^{-}.
\end{aligned}$$

For $\lambda \in \mathbb{C}$, let

$$V_{\lambda}\left(\mathcal{D}^2_{\pm}\right) := \left\{\psi \in C^{\infty}\left(E \otimes \Sigma^{\pm}(M)\right) \mid \mathcal{D}^2_{\pm}\psi = \lambda\psi\right\}.$$

From the general theory of formally self-adjoint, elliptic operators on compact manifolds, we know that

$$\operatorname{Spec}\left(\mathcal{D}^2_{\pm}\right) = \left\{\lambda \in \mathbb{C} \mid V_{\lambda}\left(\mathcal{D}^2_{\pm}\right) \neq \{0\}\right\}.$$

consists of the eigenvalues of $\mathcal{D}^2_{\pm}$ and is a discrete subset of $[0, \infty)$, the eigenspaces $V_{\lambda}\left(\mathcal{D}^2_{\pm}\right)$ are finite-dimensional, and an $L^2(E \otimes \Sigma^{\pm}(M))$-complete orthonormal set of vectors can be selected from the $V_{\lambda}\left(\mathcal{D}^2_{\pm}\right)$. Note that $\mathcal{D}^{+}\left(V_{\lambda}\left(\mathcal{D}^2_{+}\right)\right) \subseteq V_{\lambda}\left(\mathcal{D}^2_{-}\right)$, since for $\psi \in V_{\lambda}\left(\mathcal{D}^2_{+}\right)$

$$\begin{aligned}
\mathcal{D}^2_{-}\left(\mathcal{D}^{+}\psi\right) &= \left(\mathcal{D}^{+} \circ \mathcal{D}^{-}\right)\left(\mathcal{D}^{+}\psi\right) = \mathcal{D}^{+}\left(\left(\mathcal{D}^{-} \circ \mathcal{D}^{+}\right)(\psi)\right) \\
&= \mathcal{D}^{+}\left(\mathcal{D}^2_{+}(\psi)\right) = \mathcal{D}^{+}(\lambda\psi) = \lambda\mathcal{D}^{+}(\psi),
\end{aligned}$$

and similarly $\mathcal{D}^-\left(V_\lambda\left(\mathcal{D}_-^2\right)\right) \subseteq V_\lambda\left(\mathcal{D}_+^2\right)$. For $\lambda \neq 0$,

$$\mathcal{D}^\pm | V_\lambda\left(\mathcal{D}_\pm^2\right) : V_\lambda\left(\mathcal{D}_\pm^2\right) \to V_\lambda\left(\mathcal{D}_\mp^2\right)$$

is an isomorphism, since it has inverse $\frac{1}{\lambda}\mathcal{D}^\mp$. Thus the set of nonzero eigenvalues (and their multiplicities) of $\mathcal{D}_+^2$ coincides with that of $\mathcal{D}_-^2$. However, in general

$$\dim V_0\left(\mathcal{D}_+^2\right) - \dim V_0\left(\mathcal{D}_-^2\right) = \dim \operatorname{Ker}\left(\mathcal{D}_+^2\right) - \dim \operatorname{Ker}\left(\mathcal{D}_-^2\right) = \operatorname{index}\left(\mathcal{D}^+\right) \neq 0.$$

Since $\dim V_\lambda\left(\mathcal{D}_+^2\right) - \dim V_\lambda\left(\mathcal{D}_-^2\right) = 0$ for $\lambda \neq 0$, obviously

$$\begin{aligned} \operatorname{index}\left(\mathcal{D}^+\right) &= \dim V_0\left(\mathcal{D}_+^2\right) - \dim V_0\left(\mathcal{D}_-^2\right) \\ &= \sum\nolimits_{\lambda \in \operatorname{Spec}\left(\mathcal{D}_+^2\right)} e^{-t\lambda}\left(\dim V_\lambda\left(\mathcal{D}_+^2\right) - \dim V_\lambda\left(\mathcal{D}_-^2\right)\right). \end{aligned}$$

This may seem like a very inefficient way to write $\operatorname{index}\left(\mathcal{D}^+\right)$, but the point is that the sum can be expressed as the integral of the supertrace of the heat kernel for the spinorial heat equation $\frac{\partial \psi}{\partial t} = -\mathcal{D}^2\psi$, from which the Local Index Theorem (Theorem 2.13 below) for $\mathcal{D}^+$ will eventually follow. However, first the existence of the heat kernel needs to be established.

Let the *positive* eigenvalues of $\mathcal{D}_\pm^2$ be placed in a sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ where each eigenvalue is repeated according to its multiplicity. Let $u_1^\pm$, $u_2^\pm, \ldots$ be an L^2-orthonormal sequence in $C^\infty\left(E \otimes \Sigma^+(M)\right)$ with $\mathcal{D}_\pm^2\left(u_j^\pm\right) = \lambda_j u_j^\pm$ (i.e., $u_j^\pm \in V_{\lambda_j}\left(\mathcal{D}_\pm^2\right)$). We let $u_{0_1}^+, \ldots, u_{0_{n^+}}^+$ be an L^2-orthonormal basis of $\operatorname{Ker}\mathcal{D}_+^2 = \operatorname{Ker}\mathcal{D}_+$, and $u_{0_1}^-, \ldots, u_{0_{n^-}}^-$ be an L^2-orthonormal basis of $\operatorname{Ker}\mathcal{D}_-^2 = \operatorname{Ker}\mathcal{D}_-$. We can pull back the bundle $E \otimes \Sigma^\pm(M)$ via either of the projections $M \times M \times (0,\infty) \to M$ given by $\pi_1(x,y,t) := x$ and $\pi_2(x,y,t) := y$ and take the tensor product of the results to form a bundle

$$\mathcal{K}^\pm := \pi_1^*\left(E \otimes \Sigma^\pm(M)\right) \otimes \pi_2^*\left(E \otimes \Sigma^\pm(M)\right) \to M \times M \times (0,\infty).$$

Note that for $x \in M$, the Hermitian inner product $\langle\ ,\ \rangle_x$ on $\left(E \otimes \Sigma^\pm(M)\right)_x$ gives us a conjugate-linear map $\psi \mapsto \psi^*(\cdot) := \langle \cdot, \psi\rangle_x$ from $\left(E \otimes \Sigma^\pm(M)\right)_x$ to its dual $\left(E \otimes \Sigma^\pm(M)\right)_x^*$. Thus, we can (and do) make the identifications

$$\begin{aligned} &\pi_1^*\left(E \otimes \Sigma^\pm(M)\right) \otimes \pi_2^*\left(E \otimes \Sigma^\pm(M)\right) \cong \left(\pi_1^*\left(E \otimes \Sigma^\pm(M)\right)\right)^* \otimes \pi_2^*\left(E \otimes \Sigma^\pm(M)\right) \\ &\cong \operatorname{Hom}\left(\pi_1^*\left(E \otimes \Sigma^\pm(M)\right), \pi_2^*\left(E \otimes \Sigma^\pm(M)\right)\right). \end{aligned}$$

The full proof of the following Proposition 2.10 will be found in [**BlBo03**], but it is already contained in [**Gi95**] for readers of sufficient background.

Proposition 2.10. *For $t > t_0 > 0$, the series $k'^\pm$, defined by*

$$k'^\pm(x,y,t) := \sum_{j=1}^\infty e^{-\lambda_j t} u_j^\pm(x) \otimes u_j^\pm(y),$$

converges uniformly in $C^q(\mathcal{K}^{\pm}|M \times M \times (t_0, \infty))$ *for all* $q \geq 0$. *Hence* $k'^{\pm} \in C^{\infty}(\mathcal{K}^{\pm})$, *and (for* $t > 0$*)*

$$\frac{\partial}{\partial t} k'^{\pm}(x,y,t) = -\sum_{j=1}^{\infty} \lambda_j e^{-\lambda_j t} u_j^{\pm}(x) \otimes u_j^{\pm}(y) = -\mathcal{D}_{\pm}^2 k'^{\pm}(x,y,t)\,. \tag{2.6}$$

Definition 2.11. The *positive and negative twisted spinorial heat kernels* (or the *heat kernels for* $\mathcal{D}_{\pm}^2$) $k^{\pm} \in C^{\infty}(\mathcal{K}^{\pm})$ are given by

$$\begin{aligned} k^{\pm}(x,y,t) &:= \sum\nolimits_{i=1}^{n^{\pm}} u_{0_i}^{\pm}(x) \otimes u_{0_i}^{\pm}(y) + k'^{\pm}(x,y,t) \\ &= \sum\nolimits_{i=1}^{n^{\pm}} u_{0_i}^{\pm}(x) \otimes u_{0_i}^{\pm}(y) + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t} u_j^{\pm}(x) \otimes u_j^{\pm}(y) \text{ for } t > 0. \end{aligned}$$

The *total twisted spinorial heat kernel* (or the *heat kernel for* $\mathcal{D}^2$) is

$$k = \left(k^+, k^-\right) \in C^{\infty}\left(\mathcal{K}^+\right) \oplus C^{\infty}\left(\mathcal{K}^-\right) \cong C^{\infty}\left(\mathcal{K}^+ \oplus \mathcal{K}^-\right) \subseteq C^{\infty}(\mathcal{K}),$$
$$\text{where } \mathcal{K} := \mathcal{K}^+ \oplus \mathcal{K}^- = \operatorname{Hom}\left(\pi_1^*(E \otimes \Sigma(M)), \pi_2^*(E \otimes \Sigma(M))\right). \tag{2.7}$$

The terminology is justified in view of the following, whose proof is to be found in [**BlBo03**].

Proposition 2.12. *Let* $\psi_0^{\pm} \in C^{\infty}(E \otimes \Sigma^{\pm}(M))$ *and let*

$$\psi^{\pm}(x,t) = \int_M \left\langle k^{\pm}(x,y,t), \psi_0^{\pm}(y)\right\rangle_y \ \nu_y.$$

Then for $t > 0$, $\psi^{\pm}$ *solves the heat equation with initial spinor field* $\psi_0^{\pm}$:

$$\begin{aligned} &\frac{\partial \psi^{\pm}}{\partial t} = -\mathcal{D}_{\pm}^2 \psi \text{ and} \\ &\lim_{t \to 0^+} \psi^{\pm}(\cdot, t) = \psi_0^{\pm} \text{ in } C^q \text{ for all } q \geq 0. \end{aligned}$$

Moreover, for $\psi_0 \in C^{\infty}(E \otimes \Sigma(M))$ *and*

$$\psi(x,t) := \int_M \langle k(x,y,t), \psi_0 \rangle \ \nu_y,$$

we have $\dfrac{\partial \psi}{\partial t} = -\mathcal{D}^2 \psi$ *and* $\lim_{t \to 0^+} \psi(\cdot, t) = \psi_0(\cdot)$ *in* C^q *for all* $q \geq 0$.

For any finite dimensional Hermitian vector space $(V, \langle \cdot, \cdot \rangle)$ with orthonormal basis $e_1, \dots, e_N$, we have (for $v \in V$)

$$\begin{aligned} \operatorname{Tr}(v^* \otimes v) &= \sum\nolimits_{i=1}^{N} \langle (v^* \otimes v)(e_i), e_i \rangle = \sum\nolimits_{i=1}^{N} \langle v^*(e_i)\, v, e_i \rangle \\ &= \sum\nolimits_{i=1}^{N} \langle \langle e_i, v \rangle v, e_i \rangle = \sum\nolimits_{i=1}^{N} \langle e_i, v \rangle \langle v, e_i \rangle = \sum\nolimits_{i=1}^{N} |\langle e_i, v \rangle|^2 = |v|^2\,. \end{aligned}$$

In particular, $k^{\pm}(x,x,t) \in \operatorname{End}\left((E \otimes \Sigma^{\pm}(M))_x\right)$ and

$$\operatorname{Tr}\left(k^{\pm}(x,x,t)\right) = \sum\nolimits_{i=1}^{n^{\pm}} \left|u_{0_i}^{\pm}(x)\right|^2 + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t} \left|u_j^{\pm}(x)\right|^2.$$

Since this series converges uniformly and $\left\|u_{0_i}^{\pm}\right\|_{2,0} = \left\|u_j^{\pm}\right\|_{2,0} = 1$, we have

$$\int_M \operatorname{Tr}\left(k^{\pm}(x,x,t)\right)\ \nu_x = n^{\pm} + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t} < \infty.$$

For $t > 0$, we define the bounded operator $e^{-t\mathcal{D}_{\pm}^2} \in \operatorname{End}\left(L^2\left(E \otimes \Sigma^{\pm}(M)\right)\right)$ by

$$e^{-t\mathcal{D}_{\pm}^2}\left(\psi^{\pm}\right) = \sum\nolimits_{i=1}^{n^{\pm}} \left(u_{0_i}^{\pm}, \psi_0^{\pm}\right) u_{0_i}^{\pm} + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t}\left(u_j^{\pm}, \psi_0^{\pm}\right) u_j^{\pm}.$$

Note that $e^{-t\mathcal{D}_{\pm}^2}$ is of trace class, since

$$\operatorname{Tr}\left(e^{-t\mathcal{D}_{\pm}^2}\right) = n^{\pm} + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t} = \int_M \operatorname{Tr}\left(k^{\pm}(x,x,t)\right)\ \nu_x < \infty.$$

Now, we have

$$\begin{aligned} \operatorname{index}\left(\mathcal{D}^+\right) &= \dim V_0\left(\mathcal{D}_+^2\right) - \dim V_0\left(\mathcal{D}_-^2\right) \\ &= n^+ - n^- + \sum\nolimits_{j=1}^{\infty}\left(e^{-\lambda_j t} - e^{-\lambda_j t}\right) \\ &= n^+ + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t} - \left(n^- + \sum\nolimits_{j=1}^{\infty} e^{-\lambda_j t}\right) \\ &= \int_M \left(\operatorname{Tr}\left(k^+(x,x,t)\right) - \operatorname{Tr}\left(k^-(x,x,t)\right)\right)\ \nu_x. \end{aligned} \tag{2.8}$$

Since $\mathcal{D}^2 = \mathcal{D}_+^2 \oplus \mathcal{D}_-^2$, we also have the trace-class operator $e^{-t\mathcal{D}^2} \in \operatorname{End}\left(L^2\left(E \otimes \Sigma(M)\right)\right)$, whose trace is given by

$$\operatorname{Tr}\left(e^{-t\mathcal{D}^2}\right) = \int_M \operatorname{Tr}\left(k(x,x,t)\right)\ \nu_x = \int_M \left(\operatorname{Tr}\left(k^+(x,x,t)\right) + \operatorname{Tr}\left(k^-(x,x,t)\right)\right)\ \nu_x.$$

The *supertrace* of $k(x,x,t)$ is defined by

$$\operatorname{Str}\left(k(x,x,t)\right) := \operatorname{Tr}\left(k^+(x,x,t)\right) - \operatorname{Tr}\left(k^-(x,x,t)\right),$$

and in view of (2.8), we have

$$\operatorname{index}\left(\mathcal{D}^+\right) = \int_M \operatorname{Str}\left(k(x,x,t)\right)\ \nu_x. \tag{2.9}$$

The left side is independent of t and so the right side is also independent of t. The main task now is to determine the behavior of $\operatorname{Str}(k(x,x,t))$ as $t \to 0^+$. We suspect that for each $x \in M$, as $t \to 0^+$, $k(x,x,t)$ and $\operatorname{Str}(k(x,x,t))$ are influenced primarily by the geometry (e.g., curvature form Ω^θ of M with metric h and Levi-Civita connection θ, and the curvature Ω^ε of the unitary connection for E) near x, since the heat sources of points far from x are not felt very strongly at x for small t. Indeed, we will give an outline a proof of the following Local Index Formula, the full proof of which will appear in [**BlBo03**].

Theorem 2.13 (The Local Index Theorem). *In the notation of Definitions 2.9 and 2.11, let $\mathcal{D} : C^\infty(E \otimes \Sigma(M)) \to C^\infty(E \otimes \Sigma(M))$ be a twisted Dirac operator and let $k \in C^\infty(\mathcal{K})$ be the heat kernel for $\mathcal{D}^2$. If Ω^ε is the curvature form* *of the unitary connection ε for E and Ω^θ is the curvature form of the Levi-Civita connection θ for (M,h) with volume element ν, then*

$$\lim_{t\to 0^+} \operatorname{Str}\left(k\left(x,x,t\right)\right) = \left\langle \operatorname{Tr}\left(e^{i\Omega^\varepsilon/2\pi}\right) \wedge \det\left(\frac{i\Omega^\theta/4\pi}{\sinh\left(i\Omega^\theta/4\pi\right)}\right)^{\frac{1}{2}}, \nu_x \right\rangle . \tag{2.10}$$

REMARK 2.14. As will be explained below, the right side is really the inner product, with volume form ν at x, of the canonical form $\mathbf{ch}\left(E,\varepsilon\right) \smile \widehat{\mathbf{A}}\left(M,\theta\right)$ (depending on the connections ε for E and the Levi-Civita connection θ for the metric h) which represents $\mathbf{ch}\left(E\right) \smile \widehat{\mathbf{A}}\left(M\right)$. As a consequence, we obtain the Index Theorem for twisted Dirac operators from the Local Index Formula in Corollary 2.15 below. Thus the Local Index Formula is stronger than the Index Theorem for twisted Dirac operators. Indeed, the Local Index Formula yields the Index Theorem for elliptic operators which are locally expressible as twisted Dirac operators or direct sums of such.

COROLLARY 2.15 (Index formula for twisted Dirac operators). *For an oriented Riemannian n-manifold M (n even) with spin structure, and a Hermitian vector bundle $E \to M$ with unitary connection, let $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$ be the twisted Dirac operator, with $\mathcal{D}^+ : C^\infty\left(E\otimes\Sigma^+\left(M\right)\right) \to C^\infty\left(E\otimes\Sigma^-\left(M\right)\right)$. We have*

$$\operatorname{index}\left(\mathcal{D}^+\right) = \left(\mathbf{ch}\left(E\right) \smile \widehat{\mathbf{A}}\left(M\right)\right)\left[M\right], \tag{2.11}$$

where $\mathbf{ch}\left(E\right)$ is the total Chern character class of E and $\widehat{\mathbf{A}}\left(M\right)$ is the total $\widehat{\mathbf{A}}$ class of M, both defined below. In particular, we obtain:

$$n = 2 \Rightarrow \quad \operatorname{index}\left(\mathcal{D}^+\right) = ch_1\left(E\right)\left[M\right] = c_1\left(E\right)\left[M\right] \ \text{ and}$$

$$n = 4 \Rightarrow \begin{cases} \operatorname{index}\left(\mathcal{D}^+\right) = \left(\mathbf{ch}\left(E\right) \smile \widehat{\mathbf{A}}\left(M\right)\right)\left[M\right] \\ = \left(-\dim E \cdot \frac{1}{24} p_1\left(TM\right) + ch_2\left(E\right)\right)\left[M\right] \\ = \left(-\frac{\dim E}{24} p_1\left(TM\right) + \frac{1}{2} c_1\left(E\right)^2 - c_2\left(E\right)\right)\left[M\right]. \end{cases}$$

PROOF. By (2.9), (2.10) and the above Remark, we have

$$\begin{aligned} \operatorname{index}\left(\mathcal{D}^+\right) &= \int_M \operatorname{Str}\left(k\left(x,x,t\right)\right)\nu_x \\ &= \lim_{t\to 0^+} \int_M \operatorname{Str}\left(k\left(x,x,t\right)\right)\nu_x = \int_M \lim_{t\to 0^+} \operatorname{Str}\left(k\left(x,x,t\right)\right)\nu_x \\ &= \int_M \left\langle \mathbf{ch}\left(E,\varepsilon\right)_x \smile \widehat{\mathbf{A}}\left(M,\theta\right)_x, \nu_x \right\rangle \nu_x = \left(\mathbf{ch}\left(E\right) \smile \widehat{\mathbf{A}}\left(M\right)\right)\left[M\right]. \end{aligned}$$

□

We now explain the meaning of the form

$$\operatorname{Tr}\left(e^{i\Omega^{\varepsilon}/2\pi}\right) \wedge \det\left(\frac{i\Omega^{\theta}/4\pi}{\sinh\left(i\Omega^{\theta}/4\pi\right)}\right)^{\frac{1}{2}}.$$

The first part $\operatorname{Tr}\left(e^{i\Omega^{\varepsilon}/2\pi}\right)$ is relatively easy. We have (recall $2m = \dim M$)

$$e^{i\Omega^{\varepsilon}/2\pi} := \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2\pi}\right)^{k} \Omega^{\varepsilon} \wedge \overset{k}{\cdots} \wedge \Omega^{\varepsilon} = \sum_{k=0}^{m} \frac{1}{k!}\left(\frac{i}{2\pi}\right)^{k} \Omega^{\varepsilon} \wedge \overset{k}{\cdots} \wedge \Omega^{\varepsilon}, \tag{2.12}$$

where $\Omega^{\varepsilon} \wedge \overset{k}{\cdots} \wedge \Omega^{\varepsilon} \in \Omega^{2k}\left(\operatorname{End}(E)\right)$. Also $\operatorname{Tr}\left(i^{k}\Omega^{\varepsilon} \wedge \overset{k}{\cdots} \wedge \Omega^{\varepsilon}\right) \in \Omega^{2k}(M)$ and

$$\operatorname{Tr}\left(e^{i\Omega^{\varepsilon}/2\pi}\right) \in \bigoplus_{k=1}^{m} \Omega^{2k}(M).$$

This (by one of many equivalent definitions) is a representative of the total Chern character $\mathbf{ch}(E) \in \bigoplus_{k=1}^{m} H^{2k}(M, \mathbb{Q})$. The curvature Ω^{θ} of the Levi-Civita connection θ for the metric h has values in the skew-symmetric endomorphisms of TM; i.e., $\Omega^{\theta} \in \Omega^{2}\left(\operatorname{End}(TM)\right)$. A skew-symmetric endomorphism of $\mathbb{R}^{2m}$, say $B \in \mathfrak{so}(n)$, has pure imaginary eigenvalues $\pm i r_k$, where $r_k \in \mathbb{R}$ $(1 \le k \le m)$. Thus, iB has real eigenvalues $\pm r_k$. Now $\frac{z/2}{\sinh(z/2)}$ is a power series in z with radius of convergence 2π. Thus, $\frac{isB/2}{\sinh(isB/2)}$ is defined for s sufficiently small and has eigenvalues $\frac{r_k s/2}{\sinh(r_k s/2)}$ each repeated twice. Hence

$$\det\left(\frac{isB/2}{\sinh(isB/2)}\right) = \prod_{k=1}^{m}\left(\frac{r_k s/2}{\sinh(r_k s/2)}\right)^{2} \text{ and}$$

$$\det\left(\frac{isB/2}{\sinh(isB/2)}\right)^{\frac{1}{2}} = \prod_{k=1}^{m}\frac{r_k s/2}{\sinh(r_k s/2)}.$$

The last product is a power series in s of the form

$$\prod_{k=1}^{m}\frac{r_k s/2}{\sinh(r_k s/2)} = \sum_{k=0}^{\infty} a_k\left(r_1^2, \dots, r_m^2\right) s^{2k}, \tag{2.13}$$

where the coefficient $a_k\left(r_1^2, \dots, r_m^2\right)$ is a homogeneous, symmetric polynomial in $r_1^2, \dots, r_m^2$ of degree k. One can always express any such a symmetric polynomial as a polynomial in the elementary symmetric polynomials $\sigma_1, \dots, \sigma_m$ in $r_1^2, \dots, r_m^2$, where

$$\sigma_1 = \sum\nolimits_{i=1}^{m} r_i^2,\ \sigma_2 = \sum\nolimits_{i<j}^{m} r_i^2 r_j^2,\ \sigma_2 = \sum\nolimits_{i<j<k}^{m} r_i^2 r_j^2 r_k^2, \dots.$$

These in turn may be expressed in terms of $\mathrm{SO}(n)$-invariant polynomials in the entries of $B \in \mathfrak{so}(n)$ via

$$\begin{aligned}\det(\lambda I - B) &= \prod_{j=1}^{m} (\lambda + ir_j)(\lambda - ir_j) = \prod_{j=1}^{m} \left(\lambda^2 + r_j^2\right) \\ &= \sum_{k=1}^{m} \sigma_k \left(r_1^2, \dots, r_m^2\right) \lambda^{2(m-k)}.\end{aligned}$$

On the other hand,

$$\det(\lambda I - B) = \sum_{k=1}^{m} \left(\frac{1}{(2k)!} \sum_{(i),(j)} \delta_{i_1 \cdots i_{2k}}^{j_1 \cdots j_{2k}} B^{i_1}_{\ j_1} \cdots B^{i_{2k}}_{\ j_{2k}} \right) \lambda^{2(m-k)}, \text{ and so}$$
$$\sigma_k \left(r_1^2, \dots, r_m^2\right) = \frac{1}{(2k)!} \sum_{(i),(j)} \delta_{i_1 \cdots i_{2k}}^{j_1 \cdots j_{2k}} B^{i_1}_{\ j_1} \cdots B^{i_{2k}}_{\ j_{2k}},$$

where $(i) = (i_1, \cdots, i_{2k})$ is an ordered $2k$-tuple of distinct elements of $\{1, \dots, 2m\}$ and (j) is a permutation of (i) with sign $\delta_{i_1 \cdots i_{2k}}^{j_1 \cdots j_{2k}}$. If we replace $B^i_{\ j}$ with the 2-form $\frac{1}{2\pi} \left(\Omega^\theta\right)^i_{\ j}$ relative to an orthonormal frame field, we obtain the Pontryagin forms

$$p_k \left(\Omega^\theta\right) := \frac{1}{(2\pi)^{2k} (2k)!} \sum_{(i),(j)} \delta_{i_1 \cdots i_{2k}}^{j_1 \cdots j_{2k}} \Omega^\theta_{i_1 j_1} \wedge \cdots \wedge \Omega^\theta_{i_{2k} j_{2k}},$$

which represent the Pontryagin classes of the $\mathrm{SO}(n)$ bundle FM. Note that $p_k \left(\Omega^\theta\right)$ is independent of the choice of framing by the ad-invariance of the polynomials σ_k. If we express the $a_k \left(r_1^2, \dots, r_m^2\right)$ as polynomials, say $\mathcal{A}_k (\sigma_1, \dots, \sigma_k)$, in the σ_j $(j \leq k)$, we can ultimately write

$$\det \left(\frac{isB/2}{\sinh(isB/2)} \right)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \mathcal{A}_k (\sigma_1, \dots, \sigma_k) s^{2k}.$$

Formally replacing B by $\frac{1}{2\pi}\Omega^\theta$, we finally have the reasonable definition

$$\det \left(\frac{i\Omega^\theta/4\pi}{\sinh(i\Omega^\theta/4\pi)} \right)^{\frac{1}{2}} := \sum_{k=0}^{\infty} \mathcal{A}_k \left(p_1 \left(\Omega^\theta\right), \dots, p_k \left(\Omega^\theta\right)\right),$$

where the $p_j \left(\Omega^\theta\right)$ are multiplied via wedge product when evaluating the terms in the sum; the order of multiplication does not matter since $p_j \left(\Omega^\theta\right)$ is of even degree $4j$. Also, since $\mathcal{A}_k \left(p_1 \left(\Omega^\theta\right), \dots, p_k \left(\Omega^\theta\right)\right)$ is a $4k$-form, there are only a finite number of nonzero terms in the infinite sum. Abbreviating $p_j \left(\Omega^\theta\right)$ simply by p_j, one finds

$$\det\left(\frac{i\Omega^\theta/4\pi}{\sinh\left(i\Omega^\theta/4\pi\right)}\right)^{\frac{1}{2}} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right) - \frac{1}{967\,680}\left(31p_1^3 - 44p_1p_2 + 16p_3\right) + \cdots . \tag{2.14}$$

This (by one definition) represents the total $\widehat{A}$-class of M, denoted by

$$\widehat{\mathbf{A}}\left(M\right) \in \bigoplus_{k=1}^{m} H^{2k}\left(M,\mathbb{Q}\right), \tag{2.15}$$

where actually $\widehat{\mathbf{A}}\left(M\right)$ has only nonzero components in $H^{2k}\left(M,\mathbb{Q}\right)$ when k is even (or $2k \equiv 0 \bmod 4$). In (2.10) the multi-degree forms (2.12) and (2.14) have been wedged, and the top ($2m$-degree) component (relative to the volume form) has been harvested.

We now turn to our outline of the proof of Theorem 2.13. The well-known heat kernel (or fundamental solution) for the ordinary heat equation $u_t = \Delta u$ in Euclidean space $\mathbb{R}^n$, is given by

$$e\left(x,y,t\right) = \left(4\pi t\right)^{-n/2}\exp\left(-\left|x-y\right|^2/4t\right). \tag{2.16}$$

Since $H\left(x,y,t\right)$ only depends on $r = |x-y|$ and t, it is convenient to write

$$e\left(x,y,t\right) = \mathcal{E}\left(r,t\right) := \left(4\pi t\right)^{-n/2}\exp\left(-r^2/4t\right). \tag{2.17}$$

We do not expect such a simple expression for the heat kernel $k = (k^+, k^-)$ of Definition 2.11. However, it can be shown that for $x, y \in M$ (of even dimension $n = 2m$) with $r = d(x,y) :=$ Riemannian distance from x to y sufficiently small, we have an asymptotic expansion as $t \to 0^+$ for $k\left(x,y,t\right)$ of the form

$$k\left(x,y,t\right) \sim H_Q\left(x,y,t\right) := \mathcal{E}\left(d(x,y),t\right)\sum\nolimits_{j=0}^{Q} h_j\left(x,y\right)t^j, \tag{2.18}$$

for any fixed integer $Q > m+4$, where

$$h_j\left(x,y\right) \in \operatorname{Hom}\Big(\left(E\otimes\Sigma\left(M\right)\right)_x, \left(E\otimes\Sigma\left(M\right)\right)_y\Big),\ j \in \{0,1,\ldots,Q\}.$$

The meaning of $k\left(x,y,t\right) \sim H_Q\left(x,y,t\right)$ is that for $d(x,y)$ and t sufficiently small,

$$\left|k\left(x,y,t\right) - H_Q\left(x,y,t\right)\right| \le C_Q\mathcal{E}\left(d(x,y),t\right)t^{Q+1} \le C_Q t^{Q-m+1},$$

where C_Q is a constant, independent of (x,y,t). We then have

$$k\left(x,x,t\right) \sim \left(4\pi t\right)^{-m}\sum_{j=0}^{Q} h_j\left(x,x\right)t^j = \left(4\pi\right)^{-m}\sum_{j=0}^{Q} h_j\left(x,x\right)t^{j-m}. \tag{2.19}$$

Using (2.9), i.e., $\int_M \mathrm{Str}\,(k\,(x,x,t))\;\nu_x = \mathrm{index}\,(\mathcal{D}^+)$ and (2.19), we deduce that

$$\int_M \mathrm{Str}\,(h_j\,(x,x))\;\nu_x = 0 \;\text{ for }\; j \in \{0,1,\ldots,m-1\}\,,\;\text{ while}$$
$$(4\pi)^{-m}\int_M \mathrm{Str}\,(h_m\,(x,x))\;\nu_x = \int_M \mathrm{Str}\,(k\,(x,x,t))\;\nu_x = \mathrm{index}\left(\mathcal{D}^+\right).$$

Thus, to prove the Local Index Formula, it suffices to show that

$$(4\pi)^{-m}\,\mathrm{Str}\,(h_m\,(x,x)) = \left\langle \mathrm{Tr}\left(e^{i\Omega^\varepsilon/2\pi}\right)\wedge\det\left(\frac{i\Omega^\theta/4\pi}{\sinh\left(i\Omega^\theta/4\pi\right)}\right)^{\frac{1}{2}},\nu_x\right\rangle.$$

While this may not be the intellectual equivalent of climbing Mount Everest, it is not for the faint of heart.

We choose a normal coordinate system $\left(y^1,\ldots,y^n\right)$ in a coordinate ball $\mathcal{B}$ centered at the fixed point $x\in M$, so that $\left(y^1,\ldots,y^n\right)=0$ at x. The coordinate fields $\partial_1 := \partial/\partial y^1,\ldots,\partial_n := \partial/\partial y^n$ are orthonormal at x, and for any fixed $y_0\in\mathcal{B}$ with coordinates $\left(y_0^1,\ldots,y_0^n\right)$, the curve $t\mapsto t\left(y_0^1,\ldots,y_0^n\right)$ is a geodesic through x. By parallel translating the frame $(\partial_1,\ldots,\partial_n)$ at x along these radial geodesics, we obtain an orthonormal frame field $(E_1,\ldots,E_n)$ on $\mathcal{B}$ which generally does not coincide with $(\partial_1,\ldots,\partial_n)$ at points $y\in\mathcal{B}$ other than at x. The framing $(E_1,\ldots,E_n)$ defines a particularly nice section $\mathcal{B}\to FM|B$ and we may lift this to a section $\mathcal{B}\to P|B$ of the spin structure, which enables us to view the space $C^\infty\left(\Sigma\left(M\right)|\mathcal{B}\right)$ of spinor fields on $\mathcal{B}$ as $C^\infty\left(\mathcal{B},\Sigma_n\right)$, i.e., functions on $\mathcal{B}$ with values in the fixed spinor representation vector space $\Sigma_n = \Sigma_n^+\oplus\Sigma_n^-$. By similar radial parallel translation (with respect to the connection ε) of an orthonormal basis of the twisting bundle fiber E_x, we can identify $C^\infty\left(E|\mathcal{B}\right)$ with $C^\infty\left(\mathcal{B},\mathbb{C}^N\right)$, where $N=\dim_{\mathbb{C}}E$. The coordinate expressions for the curvatures Ω^θ, Ω^ε and $\mathcal{D}^2$ are as simple as possible in this so-called radial gauge.

With the above identifications, we proceed as follows. For $0\le Q\in\mathbb{Z}$, let $\Psi_Q\in C^\infty\left(\mathcal{B}\times(0,\infty),\mathbb{C}^N\otimes\Sigma_{2m}\right)$ be of the form

$$\Psi_Q\,(y,t) := \mathcal{E}\,(r,t)\sum\nolimits_{k=0}^{Q} U_k\,(y)\,t^k,$$

where $U_k\in C^\infty\left(\mathcal{B},\mathbb{C}^N\otimes\Sigma_{2m}\right)$. If $U_0\,(0)\in\mathbb{C}^N\otimes\Sigma_{2m}$ is arbitrarily specified, we seek a formula for $U_k\,(y)$, $k=0,\ldots,Q$, such that

$$\left(\mathcal{D}^2+\partial_t\right)\Psi_Q\,(y,t) = \mathcal{E}\,(r,t)\,t^Q\mathcal{D}^2\,(U_Q)\,(y)\,, \tag{2.20}$$

where the square $\mathcal{D}^2$ of the Dirac operator $\mathcal{D}$ can be written (where "$\cdot$" is Clifford multiplication) as

$$\mathcal{D}^2\psi = -\Delta\psi + \tfrac{1}{2}\sum\nolimits_{j,k}\Omega^\varepsilon_{jk}E_j\cdot E_k\cdot\psi + \tfrac{1}{4}S\psi,$$

by virtue of the generalized Lichnerowicz formula (see [**LaMi**, p. 164]). It is convenient to define the 0-th order operator $\mathcal{F}$ on $C^{\infty}\left(\mathcal{B}, \mathbb{C}^N \otimes \Sigma_{2m}\right)$ via

$$\begin{aligned}&\mathcal{F}[\psi] := \tfrac{1}{2}\sum\nolimits_{j,k} \Omega^{\varepsilon}_{jk} E_j \cdot E_k \cdot \psi, \text{ so that}\\&\mathcal{D}^2 = -\Delta\psi + \left(\mathcal{F} + \tfrac{1}{4}S\right)[\psi].\end{aligned}$$

The desired formula for the $U_k(y)$ involves the operator A on $C^{\infty}\left(\mathcal{B}, \mathbb{C}^N \otimes \Sigma_{2m}\right)$ given by

$$A[\psi] := -h^{1/4}\mathcal{D}^2[h^{-1/4}\psi] = h^{1/4}\Delta[h^{-1/4}\psi] - \left(\mathcal{F} + \tfrac{1}{4}S\right)[\psi],$$

where $h^{1/4} := (\sqrt{\det h})^{1/2}$. For $s \in [0,1]$, let

$$A_s[\psi](y) := A[\psi](sy).$$

As is proved in [**Ble92**] or in the forthcoming [**BlBo03**], we have

PROPOSITION 2.16. *Let $U_0(0) \in \mathbb{C}^N \otimes \Sigma_{2m}$, and let $V_0 \in C^{\infty}\left(\mathcal{B}, \mathbb{C}^N \otimes \Sigma_{2m}\right)$ be the constant function $V_0(y) \equiv U_0(0)$. Then the $U_k(y)$ which satisfy (2.20) are given by*

$$U_k(y) = h(y)^{-1/4} V_k(y), \text{ where}$$

$$V_k(y) = \int_{I^k} \prod\nolimits_{i=0}^{k-1} (s_i)^i \left(A_{s_{k-1}} \circ \cdots \circ A_{s_0}[V_0]\right)(y)\, ds_0 \ldots ds_{k-1}, \tag{2.21}$$

and where $I^k = \{(s_0, \ldots s_k) : s_i \in [0,1], i \in \{0, \ldots, k-1\}\ \}$.

Note that $U_0(0) \in \mathbb{C}^N \otimes \Sigma_{2m}$ may be arbitrarily specified, and once $U_0(0)$ is chosen, the $U_m(y)$ are uniquely determined via (2.21). Let $h_k(y) \in \operatorname{End}\left(\mathbb{C}^N \otimes \Sigma_{2m}\right)$ be given by

$$h_k(y)(U_0(0)) := U_k(y) \tag{2.22}$$

(in particular, $h_0(0) = \mathrm{I} \in \operatorname{End}\left(\mathbb{C}^N \otimes \Sigma_{2m}\right)$), and

$$H_Q(0,y,t) := \mathcal{E}(r,t)\sum_{k=0}^{Q} h_k(y)\, t^k \in C^{\infty}\left(\mathcal{B}, \operatorname{End}\left(\mathbb{C}^N \otimes \Sigma_{2m}\right)\right).$$

We may regard $H_Q(0,y,t)$ as

$$H_Q(x,y,t) \in \operatorname{Hom}\left(E_x \otimes \Sigma(M)_x, E_y \otimes \Sigma(M)_y\right),$$

where we recall that $x \in M$ is the point about which we have chosen normal coordinates. For y sufficiently close to x, we set

$$H_Q(x,y,t) := \mathcal{E}(d(x,y), t)\sum_{k=0}^{Q} h_k(x,y)\, t^k. \tag{2.23}$$

Of course, one expects that $H_Q(x,y,t)$ provides the desired asymptotic expansion (2.18). Although this is very plausible, it is not at all easy to prove honestly. When proofs are attempted in the literature, often steps are skipped, hands are waved, and errors are made. An extremely careful (and hence nearly unbearable) proof will be provided in [**BlBo03**], but we must forgo this here. Thus, we will only state here without proof that

$$k(x,y,t) \sim H_Q(x,y,t) := \mathcal{E}(d(x,y),t) \sum_{j=0}^{Q} h_j(x,y)\, t^j$$

for $d(x,y)$ sufficiently small, where the h_j are given in (2.22).

Using normal coordinates $\left(y^1,\dots,y^{2m}\right) \in B(r_0,0)$ about $x \in M$ and the radial gauge, and selecting $V_0 \in \mathbb{C}^N \otimes \Sigma_{2m}$, by Proposition 2.16, we have

(2.24)

$$h_m(x,x)(V_0) = \int_{I^m} \prod_{i=0}^{m-1} (s_i)^i \left(\left(A_{s_{m-1}} \circ \dots \circ A_{s_0}\right)\left[\widetilde{V}_0\right]\right)(0)\, ds_0 \dots ds_{m-1},$$

where $\widetilde{V}_0 \in C^\infty\left(B(r_0,0), \mathbb{C}^N \otimes \Sigma_{2m}\right)$ is the constant extension of V_0. Recall that for $\psi \in C^\infty\left(B(r_0,0), \mathbb{C}^N \otimes \Sigma_{2m}\right)$, we have

$$A_s[\psi](y) := A[\psi](sy), \text{ where } A[\psi] := h^{1/4}\Delta[h^{-1/4}\psi] - \left(\mathcal{F} + \tfrac{1}{4}S\right)[\psi].$$

While the right side of (2.24) may seem unwieldy, there is substantial simplification due to facts that $\left(A_{s_{m-1}} \circ \dots \circ A_{s_0}\right)[\widetilde{V}_0](y)$ is evaluated at $y=0$ in (2.24). Also, if $\gamma^1,\dots,\gamma^n$ denote the so-called gamma matrices for Clifford multiplication by $\partial_1,\dots,\partial_n$, only those terms of $A_{s_{m-1}} \circ \dots \circ A_{s_0}[\widetilde{V}_0](0)$ which involve the product $\gamma_{n+1} := \gamma^1 \cdots \gamma^n$ will survive when the supertrace $\operatorname{Str}(h_m(x,x))$ is taken. As a consequence, we have the following simplification (essentially contained in [**Ble92**], or better yet, to appear in [**BlBo03**])

PROPOSITION 2.17. *Let $R_{klji}(0) = h\left(\Omega^\theta(\partial_i,\partial_j)\partial_l,\partial_k\right)$ denote the components of the Riemann curvature tensor of h at x, and let $\Omega^\varepsilon_{ij}(0) := \Omega^\varepsilon(\partial_i,\partial_j)$ at x. Set*

$$\widetilde{\theta}^1(\partial_j) := \tfrac{1}{8}\sum_{k,l,i} R_{klji}(0)\,\gamma^k\gamma^l y^i,$$

$$\mathcal{F}^0 := \tfrac{1}{2}\sum_{i,j} F_{ij} \otimes \gamma^i\gamma^j = \tfrac{1}{2}\sum_{i,j} \Omega^\varepsilon_{ij}(0) \otimes \gamma^i\gamma^j, \text{ and}$$

$$A^0 := \sum_i \left(\partial_i^2 + \widetilde{\theta}^1(\partial_i)^2\right) - \mathcal{F}_0. \tag{2.25}$$

For $V_0 \in \mathbb{C}^N \otimes \Sigma_{2m}$, define

(2.26)

$$h^0_m(0,0)(V_0) := \int_{I^k} \prod_{i=0}^{m-1} (s_i)^i \left(\left(A^0_{s_{m-1}} \circ \dots \circ A^0_{s_0}\right)\left[\widetilde{V}_0\right]\right)(0)\, ds_0 \dots ds_{m-1},$$

where $\widetilde{V}_0 \in C^\infty\left(B(r_0,0), \mathbb{C}^N \otimes \Sigma_{2m}\right)$ is the constant extension of $V_0 \in \mathbb{C}^N \otimes \Sigma_{2m}$. Then

$$\operatorname{Str}(h_m(0,0)) = \operatorname{Str}\left(h^0_m(0,0)\right). \tag{2.27}$$

In other words, in the computation of $\operatorname{Str}(h_m(0,0))$ given by (2.24), we may replace A by A^0.

This is a substantial simplification, not only in that A^0 is a second-order differential operator with coefficients which are at most quadratic in y, but it also shows that $\operatorname{Str}(h_m(x,x))$ only depends on the curvatures Ω^θ and Ω^ε at the point x. One might regard the gist of the Index Formula for twisted Dirac operator as exhibiting the global quantity $\operatorname{index}(\mathcal{D}^+)$ as the integral of a form which may be locally computed. From this perspective, Proposition 2.17 does the job. Also, knowing in advance that $\operatorname{index}(\mathcal{D}^+)$ is insensitive to perturbations in h and ε, one suspects that $\operatorname{Str}(h_m(x,x))\,\nu_x$ can be expressed in terms of the standard forms which represent characteristic classes for TM and E. The Local Index Formula confirms this. Moreover, for low values of m, say $m = 1$ or 2 (i.e., for 2 and 4-manifolds), one can directly compute $\operatorname{Str}\left(h_m^0(0,0)\right)$ using (2.26), and thereby verify Theorem 2.13 and hence obtain Corollary 2.15 rather easily. For readers who have no use for the Local Index Theorem beyond dimension 4, this is sufficient. It requires more effort to prove Theorem 2.13 for general m. For lack of space, we cannot go into the details of this here, but they can be found in [**Ble92**] or [**BlBo03**]. It is well worth mentioning that the appearance of the sinh function in the Local Index Formula has its roots in Mehler's formula for the heat kernel

(2.28)
$$e_a(x,y,t) = \frac{1}{\sqrt{4\pi\frac{\sinh(2at)}{2a}}}\exp\left(-\frac{1}{4\frac{\sinh(2at)}{2a}}\left(\cosh(2at)\left(x^2+y^2\right)-2xy\right)\right).$$

of the generalized 1-dimensional heat problem

$$\begin{aligned} &u_t = u_{xx} - a^2x^2u, \quad u(x,t)\in\mathbb{R},\ (y,t)\in\mathbb{R}\times(0,\infty), \\ &u(x,0) = f(x), \end{aligned}$$

where $0 \neq a \in \mathbb{R}$ is a given constant. A solution of this problem is given by

$$u(x,t) = \int_{-\infty}^{\infty} e_a(x,y,t)\, f(y)\, dy,$$

and this reduces to the usual formula as $a \to 0$. The nice idea of using Mehler's formula in a rigorous derivation of the Local Index Theorem appears to be due to Getzler in [**Get83**] and [**Get86**], although it was at least implicitly involved in earlier heuristic supersymmetric path integral arguments for the Index Theorem. In the same vein, further simplifications and details can be found in [**BeGeVe92**], [**Yu01**], and [**Ble92**], but the treatment to be found in [**BlBo03**] will be more self-contained and less demanding.

While the Local Index Theorem (Theorem 2.13) is stated for twisted Dirac operators, the same proof may be applied to obtain the index formulas for an elliptic operator, possibly on a *nonspin* manifold, which is only locally of the form of twisted Dirac operator $\mathcal{D}^+$. Indeed, if $\mathcal{A}$ is such an operator and k is the heat kernel for $\mathcal{A}^*\mathcal{A} \oplus \mathcal{A}\mathcal{A}^*$, then from the spectral resolution of $\mathcal{A}$, we can still deduce

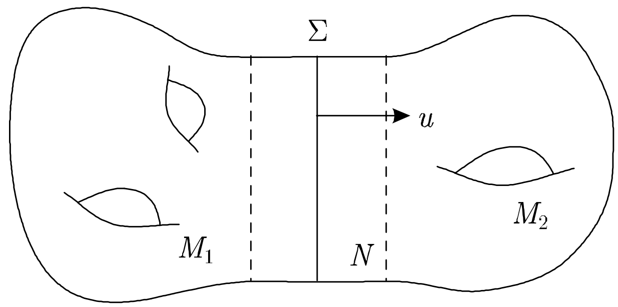

FIGURE 7. The partition of $M = M_1 \cup_\Sigma M_2$

from the asymptotic expansion of k that

$$\operatorname{index}(\mathcal{A}) = (4\pi)^{-m} \int_M \operatorname{Str}(h_m(x,x))\,\nu_x,$$

where the supertrace Str is defined in the natural way. The crucial observation is that since $\mathcal{A}$ is locally in the form of a twisted Dirac operator, we can compute $\operatorname{Str}(h_m(x,x))$ in exactly the same way (i.e., locally) as we have done. Since it is not easy to find first-order elliptic operators of geometrical significance which are not expressible in terms of locally twisted Dirac operators (or 0-th order perturbations thereof), the Local Index Formula for twisted Dirac operators is much more comprehensive than it would appear at first glance.

2.3. Dirac Type Operators on Manifolds with Boundary . We fix the notation and recall basic properties of operators of Dirac type on manifolds with boundary.

2.3.1. *The General Setting for Partitioned Manifolds.* Let M be a compact smooth Riemannian *partitioned* manifold

$$M = M_1 \cup_\Sigma M_2 := M_1 \cup M_2\ , \quad \text{where } M_1 \cap M_2 = \partial M_1 = \partial M_2 = \Sigma$$

and Σ a hypersurface (see Figure 7). We assume that $M \setminus \Sigma$ does not have a closed connected component (i.e. Σ intersects any connected component of M_1 and M_2). Let

$$\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$$

be an *operator of Dirac type* acting on sections of a Hermitian bundle S of Clifford modules over M, i.e. $\mathcal{D} = \mathbf{c} \circ \nabla$ where $\mathbf{c}$ denotes the Clifford multiplication and ∇ is a connection for S. Unlike the more general case introduced in Subsection 1.3, we assume here that ∇ is compatible with $\mathbf{c}$ in the sense that $\nabla \mathbf{c} = 0$. From this compatibility assumption it follows that $\mathcal{D}$ is symmetric and essentially self-adjoint over M.

For even $n = \dim M$, the splitting $\mathfrak{Cl}(M) = \mathfrak{Cl}^+(M) \oplus \mathfrak{Cl}^-(M)$ of the Clifford bundles induces a corresponding splitting of $S = S^+ \oplus S^-$ and a *chiral decomposition*

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- = (\mathcal{D}^+)^* \\ \mathcal{D}^+ & 0 \end{pmatrix}$$

of the *total* Dirac operator. The *chiral Dirac operators* $\mathcal{D}^\pm$ are elliptic but not symmetric, and for that reason they may have nontrivial indices which provide us with important topological and geometric invariants, as was found in the special case of twisted Dirac operators in Subsection 2.2.

Here we assume that all metric structures of M and S are product in a collar neighborhood $N = [-1,1] \times \Sigma$ of Σ. If u denotes the normal coordinate (running from M_1 to M_2), then

$$\mathcal{D}|_N = \sigma(\partial_u + \mathcal{B})\,, \text{ where } \sigma := \mathbf{c}\,(du)\,,\, \partial_u := \frac{\partial}{\partial u},$$

and $\mathcal{B}$ denotes the canonically associated Dirac operator over Σ, called the *tangential operator*. We have a similar product formula for the chiral Dirac operator. Here the point of the product structure is that then σ and $\mathcal{B}$ do not depend on the normal variable. Note that σ (Clifford multiplication by du) is a unitary mapping $L^2(\Sigma; S|_\Sigma) \to L^2(\Sigma; S|_\Sigma)$ with $\sigma^2 = -\mathrm{I}$ and $\sigma\mathcal{B} = -\mathcal{B}\sigma$. In the non-product case, there are certain ambiguities in defining a 'tangential operator' which we shall not discuss here (but see also Formula 1.12).

2.3.2. *Analysis tools: Green's Formula.* For notational economy, we set $X := M_2$. For greater generality, we consider the chiral Dirac operator

$$\mathcal{D}^+ : C^\infty(X; S^+) \to C^\infty(X; S^-)$$

and write $\mathcal{D}^-$ for its formally adjoint operator. The corresponding results follow at once for the total Dirac operator.

LEMMA 2.18. *Let $\langle .,. \rangle_\pm$ denote the scalar product in $L^2(X; S^\pm)$. Then we have*

$$\left\langle \mathcal{D}^+ f_+, f_- \right\rangle_- - \left\langle f_+, \mathcal{D}^- f_- \right\rangle_+ = -\int_\Sigma (\sigma\gamma_\infty f_+, \gamma_\infty f_-)\, d\,\mathrm{vol}_\Sigma$$

for any $f_\pm \in C^\infty(X; S^\pm)$.

Here

$$\gamma_\infty : C^\infty(X; S^\pm) \to C^\infty(\Sigma; S^\pm|_\Sigma) \tag{2.29}$$

denotes the restriction of a section to the boundary Σ.

2.3.3. *Cauchy Data Spaces and the Calderón Projection.* To explain the L^2 Cauchy data spaces we recall three additional, somewhat delicate and not widely known properties of operators of Dirac type on compact manifolds with boundary from [**BoWo93**]:

1. the invertible extension to the double;
2. the Poisson type operator and the Calderón projection; and
3. the twisted orthogonality of the Cauchy data spaces for chiral and total Dirac operators which gives the Lagrangian property in the symmetric case (i.e., for the total Dirac operator).

The idea and the properties of the Calderón projection were announced in Calderón [**Ca63**] and proved in Seeley [**Se66**] in great generality. In the following, we restrict ourselves to constructing the Calderón projection for operators of Dirac type (or, more generally, elliptic differential operators of first order) which simplifies the presentation substantially.

2.3.4. *Invertible Extension.* First we construct the *invertible double.* Clifford multiplication by the inward normal vector gives a natural clutching of S^+ over one copy of X with S^- over a second copy of X to a smooth bundle $\widetilde{S^+}$ over the closed double $\widetilde{X}$. The product forms of $\mathcal{D}^+$ and $\mathcal{D}^- = (\mathcal{D}^+)^*$ fit together over the boundary and provide a new operator of Dirac type, namely

$$\widetilde{\mathcal{D}^+} := \mathcal{D}^+ \cup \mathcal{D}^- : C^\infty\left(\widetilde{X}, \widetilde{S^+}\right) \to C^\infty\left(\widetilde{X}, \widetilde{S^-}\right). \tag{2.30}$$

Clearly $(\mathcal{D}^+ \cup \mathcal{D}^-)^* = \mathcal{D}^- \cup \mathcal{D}^+$, and so index $\widetilde{\mathcal{D}^+} = 0$. It turns out that $\widetilde{\mathcal{D}^+}$ is invertible with a pseudo-differential elliptic inverse $(\widetilde{\mathcal{D}^+})^{-1}$. Clearly, *local* solutions of the homogenous equation (here, solutions on *one* copy of X) do not extend to *global* solutions on $\widetilde{X}$ in general. As a matter of fact, the operator $\mathcal{D}^+$, even being the restriction of an invertible, locally defined differential operator, is not invertible in general, and we have $r^+(\widetilde{\mathcal{D}^+})^{-1}e^+\mathcal{D}^+ \neq \mathrm{I}$, where $e^+ : L^2(X;S^+) \to L^2(\widetilde{X};\widetilde{S^+})$ denotes the extension-by-zero operator and $r^+ : H^s\left(\widetilde{X};\widetilde{S^+}\right) \to H^s(X;S^+)$ the natural restriction operator for Sobolev spaces for s real. The precise decomposition $L^2(\widetilde{X};\widetilde{S^+}) = L^2(X;S^+) \times L^2(X;S^-)$ gives a different picture in case that a given operator T on one component can be extended to an invertible operator $\widetilde{T}$ on the whole space. This, indeed, would imply T invertible with $T^{-1} = r^+\left(\widetilde{T}\right)^{-1}e^+$. The L^2 extension of $\widetilde{\mathcal{D}^+}$, however, is not a precise extension of the L^2 extension of $\mathcal{D}^+$. Therefore, in our case, the L^2-argument breaks down.

Example 2.19. In the simplest possible two-dimensional case we consider the Cauchy-Riemann operator $\overline{\partial} : C^\infty(D^2) \to C^\infty(D^2)$ over the disc D^2, where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. In polar coordinates, this operator has the form $\frac{1}{2}e^{i\varphi}(\partial_r + \frac{i}{r}\partial_\varphi)$. Therefore, after some small smooth perturbations (and modulo the factor $\frac{1}{2}$), we assume that $\overline{\partial}$ has the following form in a certain collar neighborhood of the boundary:

$$\overline{\partial} = e^{i\varphi}(\partial_r + i\partial_\varphi)$$

Now we construct the invertible double of $\overline{\partial}$. By E^k, $k \in \mathbb{Z}$, we denote the bundle, which is obtained from two copies of $D^2 \times \mathbb{C}$ by the identification $(z,w) = (z, z^k w)$ near the equator. We obtain the bundle E^1 by gluing two halves of $D^2 \times \mathbb{C}$ by $\sigma(\varphi) = e^{i\varphi}$ and E^{-1} by gluing with the adjoint symbol. In such a way we obtain the operator

$$\widetilde{\overline{\partial}} := \overline{\partial} \cup (\overline{\partial})^* : C^\infty(S^2; E^1) \to C^\infty(S^2; E^{-1})$$

over the whole 2-sphere.

Let us analyze the situation more carefully. We fix $N := (-\varepsilon, +\varepsilon) \times S^1$, a bicollar neighborhood of the equator. The operator formally adjoint to $\overline{\partial}$ has the form

$$(\overline{\partial})^* = e^{-i\varphi}\left(-\partial_u + i\partial_\varphi + 1\right)$$

$(u = r - 1)$ in this cylinder. A section of E^1 is a couple (s_1, s_2) such that in N

$$s_2\,(u,\varphi) = e^{i\varphi} s_1\,(u,\varphi)\,.$$

The couple $\left(\overline{\partial} s_1, (\overline{\partial})^* s_2\right)$ is a smooth section of E^{-1}. To show this, we check that $(\overline{\partial})^* s_2 = e^{-i\varphi}\overline{\partial} s_1$. In the neighborhood N, we have

$$\begin{aligned}(\overline{\partial})^* s_2 &= (\overline{\partial})^*\left(e^{i\varphi} s_1\right) = e^{-i\varphi}\left(-\partial_u + i\partial_\varphi + 1\right)\left(e^{i\varphi} s_1\right)\\ &= \partial_u s_1 + i e^{-i\varphi}\partial_\varphi\left(e^{i\varphi} s_1\right) + s_1 = \left(\partial_u + i\partial_\varphi\right) s_1\\ &= e^{-i\varphi} e^{i\varphi}\left(\partial_u + i\partial_\varphi\right) s_1 = e^{-i\varphi}\left(\overline{\partial} s_1\right).\end{aligned}$$

Then the operator $\overline{\partial} \cup \overline{\partial}^*$ becomes injective and $\operatorname{index} \overline{\partial} \cup \overline{\partial}^* = 0$.

2.3.5. *The Poisson Operator and the Calderón Projection.* Next we investigate the solution spaces and their traces at the boundary. For a total or chiral operator of Dirac type over a smooth compact manifold with boundary Σ and for any real s we define the *null space*

$$\operatorname{Ker}(\mathcal{D}^+, s) := \{ f \in H^s(X; S^+) \mid \mathcal{D}^+ f = 0 \text{ in } X \setminus \Sigma \}.$$

The null spaces consist of sections which are distributional for negative s; by elliptic regularity they are smooth in the interior; in particular they possess a smooth restriction on the hypersurface $\Sigma_\varepsilon = \{\varepsilon\} \times \Sigma$ parallel to the boundary Σ of X at a distance $\varepsilon > 0$. By a Riesz operator argument they can be shown to also possess a trace over the boundary. Of course, that trace is no longer smooth but belongs to $H^{s-\frac{1}{2}}(\Sigma; S^+|_\Sigma)$. More precisely, we have the following well-known *General Restriction Theorem* (for a proof see e.g. [**BoWo93**], Chapters 11 and 13):

Theorem 2.20. *(a) Let $s > \frac{1}{2}$. Then the restriction map γ_∞ of* (2.29) *extends to a bounded map*

$$\gamma_s : H^s(X; S^+) \to H^{s-\frac{1}{2}}(\Sigma; S^+|_\Sigma). \tag{2.31}$$

(b) For $s \le \frac{1}{2}$ *, the preceding reduction is no longer defined for arbitrary sections but only for solutions of the operator* $\mathcal{D}^+$*: let* $f \in \operatorname{Ker}(\mathcal{D}^+, s)$ *and let* $\gamma_{(\varepsilon)} f$ *denote the well-defined trace of* f *in* $C^\infty(\Sigma_\varepsilon; S^+|_\Sigma)$*. Then, as* $\varepsilon \to 0_+$*, the sections* $\gamma_{(\varepsilon)} f$ *converge to an element* $\gamma_s f \in H^{s-\frac{1}{2}}(\Sigma; S^+|_\Sigma)$*.*
(c) Let $\widetilde{\mathcal{D}^+}$ *denote the invertible double of* $\mathcal{D}^+$*,* r_+ *denote the restriction operator* $r_+ : H^s(\widetilde{X}; \widetilde{S}^+) \to H^s(X; S^+)$ *and let* γ_∞^* *be the dual of* γ_∞ *in the distributional sense. For any* $s \in \mathbb{R}$ *the mapping* (Poisson type operator)

$$\mathcal{K} := r_+ \left(\widetilde{\mathcal{D}^+}\right)^{-1} \gamma_\infty^* \sigma : C^\infty(\Sigma; S^+|_\Sigma) \to C^\infty(X; S^+)$$

extends to a continuous map $\mathcal{K}^{(s)} : H^{s-1/2}(\Sigma; S^+|_\Sigma) \to H^s(X; S^+)$ *with*

$$\operatorname{range} \mathcal{K}^{(s)} = \operatorname{Ker}(\mathcal{D}^+, s).$$

For $s = 0$, Theorem 2.20 can be reformulated in the following way:

Corollary 2.21. *For a constant* C *independent of* f*, we have*

$$\|\gamma(f)\|_{-\frac{1}{2}} \le C \left(\left\|\mathcal{D}^+ f\right\|_0 + \|f\|_0 \right) \text{ for all } f \in D_{\max}\left(\mathcal{D}^+\right),$$

where $D_{\max}(\mathcal{D}^+) := \left\{ f \in L^2(X; S) \mid \mathcal{D}^+ f \in L^2(X; S) \right\}$*. So, the restriction*

$$\gamma : D_{\max}\left(\mathcal{D}^+\right) \longrightarrow H^{-1/2}(\Sigma; S^+|_\Sigma)$$

is well defined and bounded.

Proofs of Theorem and Corollary can be found, e.g., in Booss–Bavnbek and Wojciechowski [**BoWo93**], Theorems 13.1 and 13.8 for our situation ($\mathcal{D}^+$ is of order 1); and in Hörmander [**Ho66**] in greater generality (Theorem 2.2.1 and the Estimate (2.2.8), p. 194).

The composition

$$\mathcal{P}(\mathcal{D}^+) := \gamma_\infty \circ \mathcal{K} : C^\infty(\Sigma; S^+|_\Sigma) \to C^\infty(\Sigma; S^+|_\Sigma) \tag{2.32}$$

is called the *(Szegö–)Calderón projection.* It is a pseudo-differential projection (idempotent, but in general not orthogonal). We denote by $\mathcal{P}(\mathcal{D}^+)^{(s)}$ its extension to the s-th Sobolev space over Σ. It has the following geometric meaning.

We now have three options of defining the corresponding *Cauchy data* (or *Hardy*) *spaces*:

Definition 2.22. For all real s we define

$$\Lambda(\mathcal{D}^+, s) := \gamma_s(\operatorname{Ker}(\mathcal{D}^+, s)),$$

$$\Lambda^{\text{clos}}(\mathcal{D}^+, s) := \overline{\gamma_\infty\{f \in C^\infty(X; S^+) \mid \mathcal{D}^+ f = 0 \text{ in } X \setminus \Sigma\}}^{H^{s-\frac{1}{2}}(\Sigma; S^+|_\Sigma)}, \text{ and}$$

$$\Lambda^{\text{Cald}}(\mathcal{D}^+, s) := \operatorname{range} \mathcal{P}(\mathcal{D}^+)^{(s-\frac{1}{2})}.$$

The range of a projection is closed; the inclusions of the Sobolev spaces are dense; and range $\mathcal{P}(\mathcal{D}^+) = \gamma_\infty\{f \in C^\infty(X; S^+) \mid \mathcal{D}^+ f = 0 \text{ in } X\backslash\Sigma\}$, as shown in [**BoWo93**]. So, the second and the third definition of the Cauchy data space coincide. Moreover, for $s > \frac{1}{2}$ one has $\Lambda(\mathcal{D}^+, s) = \Lambda^{\text{Cald}}(\mathcal{D}^+, s)$. This equality can be extended to the L^2 case ($s = \frac{1}{2}$, see also Theorem 2.31 below), and remains valid for any real s, as proved in Seeley, [**Se66**], Theorem 6. For $s \leq \frac{1}{2}$, the result is somewhat counter-intuitive (see also Example 2.24b in the following Subsection).

We have:

Proposition 2.23. *For all $s \in \mathbb{R}$*

$$\Lambda(\mathcal{D}^+, s) = \Lambda^{\text{clos}}(\mathcal{D}^+, s) = \Lambda^{\text{Cald}}(\mathcal{D}^+, s).$$

2.3.6. *Calderón and Atiyah–Patodi–Singer Projection* . The Calderón projection is closely related to another projection determined by the ‘tangential’ part of $\mathcal{D}^+$, described as follows. Let $\mathcal{B}$ denote the *tangential* symmetric elliptic differential operator over Σ in the product form

$$\mathcal{D}^+ = \sigma(\partial_u + \mathcal{B}) : C^\infty\left(N, S^+|N\right) \to C^\infty\left(N, S^-|N\right)$$

in a collar neighborhood N of Σ in X. It has discrete real eigenvalues and a complete system of L^2 orthonormal eigensections. Let $P_{\geq}(\mathcal{B})$ denote the spectral (Atiyah–Patodi–Singer) projection onto the subspace $L_+(\mathcal{B})$ of $L^2(\Sigma; S^+|_\Sigma)$ spanned by the eigensections corresponding to the nonnegative eigenvalues of $\mathcal{B}$. It is a pseudo-differential operator and its principal symbol p_+ is the projection onto the eigenspaces of the principal symbol $b(y,\zeta)$ of $\mathcal{B}$ corresponding to nonnegative eigenvalues. It turns out that p_+ coincides with the principal symbol of the Calderón projection.

We call the space of pseudo-differential projections with the same principal symbol p_+ the *Grassmannian* $\mathcal{G}\text{r}_{p_+}$ and equip it with the operator norm corresponding to $L^2(\Sigma; S^+|_\Sigma)$. It has countably many connected components; two projections P_1, P_2 belong to the same component, if and only if the *virtual codimension*

$$\mathbf{i}(P_2, P_1) := \text{index}\,\{P_2 P_1 : \text{range}\, P_1 \to \text{range}\, P_2\} \tag{2.33}$$

of P_2 in P_1 vanishes; the higher homotopy groups of each connected component are given by Bott periodicity.

Example 2.24. (a) For the Cauchy-Riemann operator on the disc $D^2 = \{|z| \leq 1\}$, the Cauchy data space is spanned by the eigenfunctions $e^{ik\theta}$ of the tangential operator ∂_θ over $S^1 = [0, 2\pi]/\{0, 2\pi\}$ for nonnegative k. So, the Calderón projection and the Atiyah–Patodi–Singer projection coincide in this case.

(b) Next we consider the cylinder $X^R = [0, R] \times \Sigma_0$ with $\mathcal{D}_R = \sigma(\partial_u + B)$. Here B denotes a symmetric elliptic differential operator of first order acting on sections of a bundle E over Σ_0 , and σ a unitary bundle endomorphism with $\sigma^2 = -\,\mathrm{I}$ and

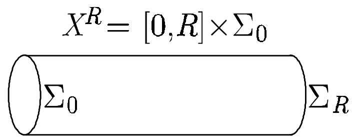

FIGURE 8. Cylinder of length R

$\sigma B = -B\sigma$. Let B be invertible (for the ease of presentation). Let $\{\varphi_k, \lambda_k\}$ denote B's spectral resolution of $L^2(\Sigma_0; E)$ with

$$\dots \lambda_{-k} \leq \dots \lambda_{-1} < 0 < \lambda_1 \leq \dots \lambda_k \leq \dots$$

Then

$$\begin{aligned} &B\varphi_k = \lambda_k \varphi_k && \text{for all } k \in \mathbb{Z}\setminus\{0\}\,, \\ &\lambda_{-k} = -\lambda_k\,,\ \sigma(\varphi_k) = \varphi_{-k}\,,\ \text{and } \sigma(\varphi_{-k}) = -\varphi_k && \text{for } k>0. \end{aligned} \tag{2.34}$$

We consider

$$\begin{aligned} f \in \operatorname{Ker}(\mathcal{D}_R, 0) &= \operatorname{span}\{e^{-\lambda_k u}\varphi_k\}_{k\in\mathbb{Z}\setminus\{0\}} \text{ in } L^2(X^R) \\ &= \operatorname{Ker}\mathcal{D}_{R\max} \text{ (kernel of maximal extension).} \end{aligned}$$

It can be written in the form

$$f(u,y) = f_>(u,y) + f_<(u,y),\ u \in [0,R]\,, \tag{2.35}$$

where

$$f_<(u,y) = \sum_{k<0} a_k e^{-\lambda_k u}\varphi_k(y) \text{ and } f_>(u,y) = \sum_{k>0} a_k e^{-\lambda_k u}\varphi_k(y).$$

Because of

$$\langle f, f\rangle_{L^2(X^R)} < +\infty \iff \langle f_<, f_<\rangle < +\infty \text{ and } \langle f_>, f_>\rangle < +\infty\,,$$

the coefficients a_k satisfy the conditions

$$\sum_{k<0} |a_k|^2 \frac{e^{-2\lambda_k R} - 1}{2|\lambda_k|} < +\infty \quad \text{or, equivalently,} \quad \sum_{k<0} |a_k|^2 \frac{e^{2|\lambda_k|R}}{|\lambda_k|} < +\infty \tag{2.36}$$

and

$$\sum_{k>0} |a_k|^2 \frac{1 - e^{-2\lambda_k R}}{2\lambda_k} < +\infty \quad \text{or, equivalently,} \quad \sum_{k>0} |a_k|^2/\lambda_k < +\infty\,. \tag{2.37}$$

We consider the Cauchy data space $\Lambda(\mathcal{D}_R, 0)$ consisting of all $\gamma(f)$ with $f \in \operatorname{Ker}(\mathcal{D}_R, 0)$. Here $\gamma(f)$ denotes the trace of f at the boundary

$$\Sigma = \partial X^R = -\Sigma_0 \sqcup \Sigma_R\,,$$

where Σ_R denotes a second copy of Σ_0. According to the spectral splitting $f = f_> + f_<$, we have

$$\gamma(f) = (s^0_<, s^R_<) + (s^0_>, s^R_>)$$

where

$$s^0_> = f_>(0),\ s^0_< = f_<(0),\ s^R_> = f_>(R),\ s^R_< = f_<(R).$$

Because of (2.36) and (2.37), we have

$$(s^0_<, s^R_>) \in C^\infty(\Sigma_0 \cup \Sigma_R) \text{ and } (s^0_>, s^R_<) \in H^{-1/2}(\Sigma_0 \cup \Sigma_R).$$

Recall that

$$\sum a_k \varphi_k \in H^s(\Sigma_0) \iff \sum |a_k|^2 |k|^{2s/(m-1)} < +\infty$$

and $|\lambda_k| \sim |k|^{\frac{1}{m-1}}$ for $k \to \pm\infty$, where $m-1$ denotes the dimension of Σ_0.

One notices that the estimate (2.36) for the coefficients of $s^0_<$ is stronger than the assertion that $\sum_{k<0} |a_k|^2 |\lambda_k|^N < +\infty$ for all natural N. Thus our estimates confirm that not every smooth section can appear as initial value over Σ_0 of a solution of $\mathcal{D}_R f = 0$ over the cylinder.

To sum up the example, the space $\Lambda(\mathcal{D}_R, 0)$ can be written as the graph of an unbounded, densely defined, closed operator $T : \operatorname{Dom} T \to H^{-\frac{1}{2}}(\Sigma_R)$, mapping $s^0_< + s^0_> =: s^0 \mapsto s^R := s^R_< + s^R_>$ with $\operatorname{Dom} T \subset H^{-\frac{1}{2}}(\Sigma_0)$. To obtain a closed subspace of $L^2(\Sigma)$ one takes the range $\Lambda(\mathcal{D}_R, \frac{1}{2})$ of the L^2 extension $\mathcal{P}(\mathcal{D}_R)^{(0)}$ of the Calderón projection. It coincides with $\Lambda(\mathcal{D}_R, 0) \cap L^2(\Sigma)$ by Proposition 2.23. In Theorem 2.31 we show without use of the pseudo-differential calculus why the intersection $\Lambda(\mathcal{D}_R, 0) \cap L^2(\Sigma)$ must be closed in $L^2(\Sigma)$. See [**ScWo00**, p. 1214] for another description of the Cauchy data space $\Lambda(\mathcal{D}_R, \frac{1}{2})$, namely as the graph of a unitary elliptic pseudo-differential operator of order 0.

Since $\Sigma = -\Sigma_0 \sqcup \Sigma_R$, the tangential operator takes the form $\mathcal{B} = B \oplus (-B)$ and we obtain from (2.34)

$$\operatorname{range} P_>(\mathcal{B})^{(0)} = L_+(\mathcal{B}) = \operatorname{span}_{L^2(\Sigma)}\{(\varphi_k, \sigma(\varphi_k))\}_{k>0}\,.$$

For comparison, we have in this example

$$\operatorname{range} \mathcal{P}(\mathcal{D}_R)^{(0)} = \Lambda(\mathcal{D}_R, \tfrac{1}{2}) = \operatorname{span}_{L^2(\Sigma)}\{(\varphi_k, e^{-\lambda_k R}\varphi_k)\}_{k>0}\,,$$

hence $L_+(\mathcal{B})$ and $\Lambda(\mathcal{D}_R, \frac{1}{2})$ are transversal subspaces of $L^2(\Sigma)$. On the semi-infinite cylinder $[0,\infty) \times \Sigma$, however, we have only one boundary component Σ_0. Hence

$$\operatorname{range} P_>(B)^{(0)} = \operatorname{span}_{L^2(\Sigma_0)}\{\varphi_k\}_{k>0} = \lim_{R\to\infty} \operatorname{range} \mathcal{P}(\mathcal{D}_R)^{(0)}.$$

One can generalize the preceding example: For any smooth compact manifold X with boundary Σ and any real $R \geq 0$, let X^R denote the stretched manifold

$$X^R := ([-R, 0] \times \Sigma) \cup_\Sigma X.$$

Assuming product structures with $\mathcal{D} = \sigma(\partial_u + \mathcal{B})$ near Σ gives a well-defined extension $\mathcal{D}_R$ of $\mathcal{D}$. Nicolaescu, [**Ni95**] proved that the Calderón projection and the Atiyah–Patodi–Singer projection coincide up to a finite-dimensional component in the *adiabatic limit* ($R \to +\infty$ in a suitable setting). Even for finite R and, in particular for $R = 0$, one has the following interesting result. It was first proved in

Scott, [**Sco95**] (see also Grubb, [**Gr99**] and Wojciechowski, [**DaKi99**], Appendix who both offered different proofs).

LEMMA 2.25. *For all $R \geq 0$, the difference $\mathcal{P}(\mathcal{D}_R) - P_{\geq}(\mathcal{B})$ is an operator with a smooth kernel.*

2.3.7. *Twisted Orthogonality of Cauchy Data Spaces.* Green's formula (in particular the Clifford multiplication σ in the case of Dirac type operators) provides a symplectic structure for $L^2(\Sigma; S|_\Sigma)$ for linear symmetric elliptic differential operators of first order on a compact smooth manifold X with boundary Σ. For elliptic systems of second-order differential equations, various interesting results have been obtained in the 1970s by exploiting the symplectic structure of corresponding spaces (see e.g. [**LaSnTu75**]). Restricting oneself to first-order systems, the geometry becomes very clear and it turns out that the Cauchy data space $\Lambda(\mathcal{D}, \frac{1}{2})$ is a Lagrangian subspace of $L^2(\Sigma; S|_\Sigma)$.

More generally, in [**BoWo93**] we described the orthogonal complement of the Cauchy data space of the chiral Dirac operator $\mathcal{D}^+$ by

$$\sigma^{-1}(\Lambda(\mathcal{D}^-, \tfrac{1}{2})) = (\Lambda(\mathcal{D}^+, \tfrac{1}{2}))^\perp . \tag{2.38}$$

We obtained a short exact sequence

$$0 \to \sigma^{-1}(\Lambda(\mathcal{D}^-, s)) \hookrightarrow H^{s-\frac{1}{2}}(\Sigma; S^+|_\Sigma) \stackrel{\mathcal{K}^{(s)}}{\to} \operatorname{Ker}(\mathcal{D}^+, s) \to 0.$$

For the total (symmetric) Dirac operator this means:

PROPOSITION 2.26. *The Cauchy data space $\Lambda(\mathcal{D}, \frac{1}{2})$ of the* total *Dirac operator is a Lagrangian subspace of the Hilbert space $L^2(\Sigma; S|_\Sigma)$ equipped with the symplectic form $\omega(\varphi, \psi) := (\sigma\varphi, \psi)$.*

2.3.8. *"Admissible" Boundary Value Problems.* We refer to [**BoWo93**, Chapter 18], [**BrLe99**], and [**Sc01**] for a rigorous definition and treatment of large classes of admissible boundary value problems defined by pseudo-differential projections. Prominent examples belong to the Grassmannian $\mathcal{G}\mathrm{r}_{p_+}$ (introduced above in Section 2.3.6 before Example 2.24). On even–dimensional manifolds, other prominent examples are local chiral projections and unitary modifications as explained in [**BoWo93**, p. 273].

For all admissible boundary conditions defined by a pseudo-differential projection R over Σ the following features are common (For simplicity, we suppress the distinction between total and chiral spinor bundle in this paragraph and denote the bundle by E):

1. We have an estimate

$$\|f\|_1 \leq C \left(\left\|\mathcal{D}^+ f\right\|_0 + \|f\|_0 + \|R \circ i \circ \gamma\,(f)\|_{\frac{1}{2}} \right) \text{ for } f \in H^1\,(X; E)\,. \tag{2.39}$$

2. Defining a domain by

$$D = \operatorname{Dom}(\mathcal{D}_R^+) := \left\{ f \in H^1\,(X; E) \mid R\,(f|_\Sigma) = 0 \right\}, \tag{2.40}$$

we obtain a closed Fredholm extension.

3. The restriction of $\mathrm{Dom}(\mathcal{D}_R^+)$ to the boundary makes a Fredholm pair with the Cauchy data space of $\mathcal{D}^+$.
4. The composition $R\mathcal{P}\mathcal{D}^+$ defines a Fredholm operator from the Cauchy data space to the range of R (see also Proposition 3.34).
5. The space $\mathrm{Ker}\left(\mathcal{D}_R^+\right)$ consists only of smooth sections.

Warning: A new feature of operators of Dirac type on manifolds with boundary is that the index of admissible boundary value problems can jump under continuous or even smooth deformation of the coefficients. E.g., this is the case for the Atiyah-Patodi-Singer boundary problem, as follows from the next subsection.

A total compatible (and so symmetric) Dirac operator $\mathcal{D}$ and an orthogonal admissible projection R define a self-adjoint extension $\mathcal{D}_D^+ = \mathcal{D}^{+*}|_D$, if the projection R defines the domain D and satisfies the symmetry condition $I - R = \sigma^{-1}R\sigma$. We denote by $\mathcal{G}\mathrm{r}^{\mathrm{sa}}_{p_+}$ or, shortly, $\mathcal{G}\mathrm{r}^{\mathrm{sa}}$ the subspace of $\mathcal{G}\mathrm{r}_{p_+}$ of orthogonal projections which satisfy the preceding condition and differ from the Atiyah–Patodi–Singer projection only by an infinitely smoothing operator.

Posing a suitable *well-posed* boundary value problem provides for a nicely spaced discrete spectrum near 0. Then, varying the coefficients of the differential operator and the imposed boundary condition suggests the use of the powerful topological concept of spectral flow. From (2.39) and a careful analysis of the corresponding parametrices we see in [**BoLePh01**, Section 3] under which conditions the curves of the induced self-adjoint L^2 extensions become continuous curves in $\mathcal{CF}^{\mathrm{sa}}\left(L^2(X;E)\right)$ in the gap topology so that their spectral flow is well-defined and truly homotopy invariant.

We summarize the main results. They depend strongly on the weak unique continuation property (either in the form of Section 1.4 or in the weaker form of (1.9) which is sufficient here) and the invertible extension (Section 2.3.4).

LEMMA 2.27. *For fixed $\mathcal{D}$ the mapping*

$$\mathcal{G}\mathrm{r}^{\mathrm{sa}}(\mathcal{D}) \ni P \longmapsto \mathcal{D}_P \in \mathcal{CF}^{\mathrm{sa}}\left(L^2(X;E)\right)$$

is continuous from the operator norm to the gap metric.

THEOREM 2.28. *Let X be a compact Riemannian manifold with boundary. Let $\{\mathcal{D}_s\}_{s\in M}$, M a metric space, be a family of compatible operators of Dirac type. We assume that in each local chart, the coefficients of $\mathcal{D}_s$ depend continuously on s. Then we have*
(a) The Poisson operator $K_s : L^2(\Sigma; E|_\Sigma) \to H^{1/2}(X;E)$ of $\mathcal{D}_s$ depends continuously on s in the operator norm.
(b) The Calderón projector $P_+(s) : L^2\left(\Sigma; E_{|\Sigma}\right) \to L^2\left(\Sigma; E_{|\Sigma}\right)$ of $\mathcal{D}_s$ depends continuously on s in the operator norm.
(c) The family

$$M \ni s \longmapsto (\mathcal{D}_s)_{P_+(s)} \in \mathcal{CF}^{\mathrm{sa}}\left(L^2(X;E)\right)$$

is continuous.
(d) Let $P_{t\{t\in Y\}}$ be a norm-continuous path of orthogonal projections in $L^2(X;E)$.

If

$$P_t \in \bigcap_{s \in M} \mathcal{G}\mathrm{r}^{\mathrm{sa}}\left(\mathcal{D}_s\right),\ t \in Y,$$

then

$$M \times Y \ni (s,t) \longmapsto \left(\mathcal{D}_s\right)_{P_t} \in \mathcal{CF}^{\mathrm{sa}}\left(L^2(X;E)\right)$$

is continuous.

Note that (b) is a pseudo-differential reformulation of the continuity of Cauchy data spaces which is valid in much greater generality (see our Section 1.2.2).

We close this Section with a recent result of [**Gr02**]:

LEMMA 2.29. *(G. Grubb 2002) Let $\mathcal{D}$ be an operator of Dirac type over a compact manifold with boundary. Let P be an orthogonal projection which defines an 'admissible' self–adjoint boundary condition for $\mathcal{D}$. Then there exists an invertible operator of Dirac type $\mathcal{B}'$ over the boundary such that $P = P_>(\mathcal{B}')$.*

Grubb's Lemma shows that the Atiyah–Patodi–Singer boundary projection is the most general admissible self–adjoint boundary condition, in the specified sense.

2.4. The Atiyah–Patodi–Singer Index Theorem. Let X be a compact, oriented Riemannian manifold with boundary $Y = \partial X$ with $\dim X = n = 2m$ even. Let $\mathcal{D} : C^\infty(X,S) \to C^\infty(X,S)$ be a compatible operator of Dirac type where $S \to X$ is a bundle of Clifford modules. Relative to the splitting $S = S^+ \oplus S^-$ into chiral halves, we have the operators $\mathcal{D}^+ : C^\infty(X,S^+) \to C^\infty(X,S^-)$ and $\mathcal{D}^- : C^\infty(X,S^-) \to C^\infty(X,S^+)$ which are formal adjoints on sections with support in $X \setminus Y$. We assume that all structures (e.g., Riemannian metric, Clifford module, connection) are products on some collared neighborhood $N \cong [-1,1] \times Y$ of Y. Then $\mathcal{D}^+|_N := \mathcal{D}^+ : C^\infty(N, S^+|N) \to C^\infty(N, S^-|N)$ has the form

$$\mathcal{D}^+|_N = \sigma(\partial_u + \mathcal{B}).$$

Here $u \in [-1,1]$ is the normal coordinate (i.e., $N = \{(u,y) \mid y \in Y,\ u \in [-1,1]\}$) with $\partial_u = \frac{\partial}{\partial u}$ the inward normal), $\sigma = \mathbf{c}(du)$ is the (unitary) Clifford multiplication by du with $\sigma(S^+|N) = S^-|N$, and

$$\mathcal{B} : C^\infty\left(Y, S^+|_Y\right) \to C^\infty\left(Y, S^+|_Y\right)$$

denotes the canonically associated (elliptic, self-adjoint) Dirac operator over Y, called the *tangential operator*. Note that due to the product structure, σ and $\mathcal{B}$ do not depend on u. Let $P_\geq(\mathcal{B})$ denote the spectral (Atiyah–Patodi–Singer) projection onto the subspace $L_+(\mathcal{B})$ of $L^2(Y, S^+|_{\partial X})$ spanned by the eigensections corresponding to the nonnegative eigenvalues of $\mathcal{B}$. Let

$$C^\infty\left(X, S^+; P_\geq\right) := \left\{\psi \in C^\infty\left(X, S^+\right) \mid P_\geq(\mathcal{B})\left(\psi|_Y\right) = 0\right\},\ \text{and}$$

$$\mathcal{D}^+_{P_\geq} := \mathcal{D}^+|_{C^\infty\left(X,S^+;P_\geq\right)} : C^\infty\left(X, S^+; P_\geq\right) \to C^\infty\left(X, S^-\right).$$

The eta function for $\mathcal{B}$ is defined by

$$\eta_{\mathcal{B}}(s) := \sum\nolimits_{\lambda \in \operatorname{spec}\mathcal{B}-\{0\}} (\operatorname{sign}\lambda)\, m_\lambda\, |\lambda|^{-s},$$

for $\Re(s)$ sufficiently large, where m_λ is the multiplicity of λ. Implicit in the following result (originating in [**AtPaSi75**]) is that $\eta_{\mathcal{B}}$ extends to a meromorphic function on all $\mathbb{C}$, which is holomorphic at $s=0$ so that $\eta_{\mathcal{B}}(0)$ is finite.

THEOREM 2.30 (Atiyah-Patodi-Singer Index Formula). *The above operator $\mathcal{D}^+_{P_\geq}$ has finite index given by*

$$\operatorname{index}\mathcal{D}^+_{P_\geq} = \int_X \left(\mathbf{ch}(S,\varepsilon)\wedge \widetilde{\mathbf{A}}(X,\theta)\right) - \frac{m_0+\eta_{\mathcal{B}}(0)}{2}.$$

where $m_0 = \dim(\operatorname{Ker}\mathcal{B})$, $\mathbf{ch}(S,\varepsilon)\in\Omega^*(X,\mathbb{R})$ is the total Chern character form of the complex vector bundle S with compatible, unitary connection ε, and $\widetilde{\mathbf{A}}(X,\theta)\in\Omega^*(X,\mathbb{R})$ is closely related to the total $\widehat{\mathbf{A}}(X,\theta)$ form relative to the Levi-Civita connection θ, namely $\widetilde{\mathbf{A}}(X,\theta)_{4k} = 2^{2k-m}\widehat{\mathbf{A}}(X,\theta)_{4k}$.

PROOF. (outline) The proof found in [**AtPaSi75**] or [**BoWo93**] is based on the heat kernel method for computing the index, but the process is less straightforward than in the closed case because of the boundary condition. The appropriate heat kernel is constructed by means of Duhamel's method. Namely, an exact kernel is obtained from an approximate one by an iterative process initiated by writing the error as the integral of a derivative of the convolution of the true and approximate kernel; see (2.50) below. The initial approximate heat kernel is obtained by patching together two heat kernels, denoted by $\mathcal{E}_c$ and $\mathcal{E}_d$. Here, $\mathcal{E}_c$ is a heat kernel for a Dirac operator over an infinite extension $[0,\infty)\times Y$ of the collared neighborhood $N=[0,1]\times Y$ of Y in X, for which the boundary condition $P_\geq(\mathcal{B})(\psi|_Y)=0$ is imposed. The heat kernel $\mathcal{E}_d$ is the usual one (without boundary conditions) for $e^{-t\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+}}$, where $\widetilde{\mathcal{D}^\pm}$ are chiral halves of the invertible Dirac operator (see (2.30)), namely

$$\widetilde{\mathcal{D}^\pm} := \mathcal{D}^\pm \cup \mathcal{D}^\mp : C^\infty\left(\widetilde{X},\widetilde{S^\pm}\right) \to C^\infty\left(\widetilde{X},\widetilde{S^\pm}\right)$$

over the double $\widetilde{X}$, a closed manifold without boundary.

We begin with the construction of $\mathcal{E}_c$. We let

$$D := \partial_u + \mathcal{B} : C^\infty\left(N, S^+|N\right) \to C^\infty\left(N, S^+|N\right),$$

which has formal adjoint $D^* := -\partial_u + \mathcal{B}$. Define

$\mathcal{D} := D|\operatorname{Dom}\mathcal{D}$, where

$\operatorname{Dom}\mathcal{D} := \left\{f\in H^1\left(\mathbb{R}_+\times Y, \pi^*\left(S^+|Y\right)\right) \mid P_\geq\left(f|_{\{0\}\times Y}\right)=0\right\}$, and

$\mathcal{D}^* = D^*|\operatorname{Dom}\mathcal{D}^*$, where

$\operatorname{Dom}\mathcal{D}^* := \left\{f\in H^1\left(\mathbb{R}_+\times Y, \pi^*\left(S^-|Y\right)\right) \mid P_<\left(f|_{\{0\}\times Y}\right)=0\right\}.$

We also have the Laplacians given by

$$\Delta_c := \mathcal{D}^*\mathcal{D}|\operatorname{Dom}\mathcal{D}^*\mathcal{D}, \text{ where}$$

$$\operatorname{Dom}\mathcal{D}^*\mathcal{D} := \left\{ \begin{array}{l} f \in H^1\left(\mathbb{R}_+ \times Y, \pi^*\left(S^+|Y\right)\right) | \\ P_{\geq}\left(f|_{\{0\}\times Y}\right) = 0,\ P_{<}\left(Df|_{\{0\}\times Y}\right) = 0 \end{array} \right\}, \text{ and}$$

$$\Delta_{c*} := \mathcal{D}\mathcal{D}^*|\operatorname{Dom}\mathcal{D}\mathcal{D}^*, \text{ where}$$

$$\operatorname{Dom}\mathcal{D}\mathcal{D}^* := \left\{ \begin{array}{l} f \in H^1\left(\mathbb{R}_+ \times Y, \pi^*\left(S^+|Y\right)\right) | \\ P_{<}\left(f|_{\{0\}\times Y}\right) = 0,\ P_{\geq}\left(D^*f|_{\{0\}\times Y}\right) = 0 \end{array} \right\}.$$

Let $\varphi_\lambda(y) \in C^\infty(Y, S^+|Y)$ be an eigensection of $\mathcal{B}$ with eigenvalue $\lambda \in \mathbb{R}$. Note that $g_\lambda(t; u, y) = f_\lambda(t, u)\varphi_\lambda(y)$ is a solution of the heat equation

$$\begin{aligned} 0 &= (\partial_t + \Delta_c) g_\lambda = \partial_t g_\lambda + (\partial_u + \mathcal{B})(-\partial_u + \mathcal{B}) g_\lambda = \left(\partial_t - \partial_u^2 + \mathcal{B}^2\right) g_\lambda \\ &= \left(\partial_t f_\lambda - \partial_u^2 f_\lambda + \lambda^2 f_\lambda\right)\varphi_\lambda(y), \end{aligned}$$

with $g_\lambda(t; \cdot, \cdot) \in \operatorname{Dom}\mathcal{D}^*\mathcal{D}$, when $f_\lambda : (0,\infty)\times[0,\infty) \to \mathbb{R}$ solves the heat problem

$$\begin{aligned} &\partial_t f_\lambda = \partial_u^2 f_\lambda - \lambda^2 f_\lambda \text{ with boundary condition} \\ &f_\lambda(t, 0) = 0 \text{ if } \lambda \geq 0 \\ &\partial_u f_\lambda(t,0) + \lambda f_\lambda(t,0) = Df_\lambda = 0 \text{ if } \lambda < 0. \end{aligned} \tag{2.41}$$

Recall that the *complementary error function* is

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2} d\xi.$$

Of use to us are the facts

$$\begin{aligned} &\operatorname{erfc}'(x) = \frac{-2}{\sqrt{\pi}} e^{-x^2} \text{ and} \\ &\operatorname{erfc}(x) \leq \frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{x + \sqrt{x^2 + \frac{4}{\pi}}} \leq e^{-x^2} \text{ for } x \geq 0. \end{aligned} \tag{2.42}$$

For $\lambda \geq 0$, the heat kernel for the problem (2.41) is (via the method of images)

$$e_\lambda(t; u, v) := \frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}\left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\right) \text{ for } u, v \geq 0 \text{ and } t > 0.$$

For $\lambda < 0$, and $u, v \geq 0$ and $t > 0$, the heat kernel is (using Laplace transforms)

$$e_\lambda(t; u, v) := \frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}\left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} + \lambda e^{\lambda(u+v)}\operatorname{erfc}\left(\frac{u+v}{2\sqrt{t}} - \lambda\sqrt{t}\right)\right).$$

The heat kernel for Δ_c (i.e., the kernel for $e^{-t\Delta_c}$) is then

$$\mathcal{E}_c(t; u, y; v, z) = \sum\nolimits_\lambda e_\lambda(t; u, v)\varphi_\lambda(y) \otimes \varphi_\lambda(z)^* \tag{2.43}$$

Here λ ranges over $\operatorname{spec}\mathcal{B}$ and (by a convenient abuse of notation) $\varphi_\lambda(y)\otimes\varphi_\lambda(z)^*$ is really a sum $\sum_k \varphi_{\lambda,k}(y) \otimes \varphi_{\lambda,k}(z)^*$ where $\{\varphi_{\lambda,1}, \ldots, \varphi_{\lambda,k}\}$ is an orthonormal eigenbasis of the eigenspace for λ, and $\left\{\varphi^*_{\lambda,1},\dots,\varphi^*_{\lambda,k}\right\}$ is the dual basis. Similarly, the heat kernel for Δ_{c*} is

$$\mathcal{E}_{c*}\left(t;u,y;v,z\right)=\sum\nolimits_{\lambda} e_{\lambda*}\left(t;u,v\right)\varphi_{\lambda}\left(y\right)\otimes\varphi_{\lambda}\left(z\right)^*, \tag{2.44}$$

where $e_{\lambda*}$ is the heat kernel for the problem

$$\begin{aligned}&\partial_t f_\lambda=\partial_u^2 f_\lambda-\lambda^2 f_\lambda \text{ with boundary condition}\\ &f_\lambda\left(t,0\right)=0 \text{ if } \lambda<0\\ &\partial_u f_\lambda\left(t,0\right)-\lambda f_\lambda\left(t,0\right)=D^* f_\lambda=0 \text{ if } \lambda\geq 0,\end{aligned} \tag{2.45}$$

where $f_\lambda:(0,\infty)\times[0,\infty)\to\mathbb{R}$. Thus, for $u,v\geq 0$ and $t>0$, we have

$$e_{\lambda*}\left(t;u,v\right):=\frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}\left(e^{-\frac{(u-v)^2}{4t}}-e^{-\frac{(u+v)^2}{4t}}\right) \text{ for } \lambda<0,$$

while for $\lambda\geq 0$ (note the sign change in passing from (2.41) to (2.45))

$$e_{\lambda*}\left(t;u,v\right):=\frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}\left(e^{-\frac{(u-v)^2}{4t}}+e^{-\frac{(u+v)^2}{4t}}-\lambda e^{\lambda(u+v)}\operatorname{erfc}\left(\tfrac{u+v}{2\sqrt{t}}+\lambda\sqrt{t}\right)\right).$$

Combining (2.43) and (2.44), we obtain the trace of kernel $\mathcal{E}_c-\mathcal{E}_{c*}$ for $e^{-t\Delta_c}-e^{-t\Delta_{c*}}$ evaluated at the point $(u,y;u,y)$ of the diagonal

$$\begin{aligned}&\mathcal{K}\left(t;u,y\right):=\operatorname{Tr}\sum\nolimits_{\lambda}\left(e_\lambda\left(t;u,u\right)-e_{*\lambda}\left(t;u,u\right)\right)\varphi_\lambda\left(y\right)\otimes\varphi^*_\lambda\left(y\right)\\ &=\sum_{\lambda\geq 0}\left(\frac{-e^{-\lambda^2 t}}{2\sqrt{\pi t}}e^{-u^2/t}-\frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}e^{-u^2/t}+\lambda e^{2\lambda u}\operatorname{erfc}\left(\tfrac{u}{\sqrt{t}}+\lambda\sqrt{t}\right)\right)\left|\varphi_\lambda\left(y\right)\right|^2\\ &+\sum_{\lambda<0}\left(\frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}e^{-u^2/t}+\lambda e^{-2\lambda u}\operatorname{erfc}\left(\tfrac{u}{\sqrt{t}}-\lambda\sqrt{t}\right)+\frac{e^{-\lambda^2 t}}{2\sqrt{\pi t}}e^{-u^2/t}\right)\left|\varphi_\lambda\left(y\right)\right|^2\\ &=\sum_{\lambda}\operatorname{sign}\left(\lambda\right)\left(\frac{-e^{-\lambda^2 t}e^{-u^2/t}}{\sqrt{\pi t}}+\left|\lambda\right|e^{2\left|\lambda\right|u}\operatorname{erfc}\left(\tfrac{u}{\sqrt{t}}+\left|\lambda\right|\sqrt{t}\right)\right)\left|\varphi_\lambda\left(y\right)\right|^2\\ &=\sum\nolimits_{\lambda}\operatorname{sign}\left(\lambda\right)\frac{\partial}{\partial u}\left(\tfrac{1}{2}e^{2\left|\lambda\right|u}\operatorname{erfc}\left(\tfrac{u}{\sqrt{t}}+\left|\lambda\right|\sqrt{t}\right)\right)\left|\varphi_\lambda\left(y\right)\right|^2.\end{aligned}$$

We used $\operatorname{erfc}'\left(x\right)=\frac{-2}{\sqrt{\pi}}e^{-x^2}$ to obtain the last equality, and we have set $\operatorname{sign}\left(0\right):=1$ for convenience. We compute (where the interchange of the sum and integral can be justified)

$$
\begin{aligned}
\mathcal{K}(t) &:= \int_0^\infty \int_Y \mathcal{K}(t;u,y)\,dydu \\
&= \int_0^\infty \int_Y \sum\nolimits_\lambda \operatorname{sign}(\lambda) \frac{\partial}{\partial u} \left(\tfrac{1}{2} e^{2|\lambda|u} \operatorname{erfc}\left(\tfrac{u}{\sqrt{t}} + |\lambda| \sqrt{t} \right) \right) |\varphi_\lambda(y)|^2 \,dydu \\
&= \sum_\lambda m_\lambda \operatorname{sign}(\lambda) \int_0^\infty \frac{\partial}{\partial u} \left(\tfrac{1}{2} e^{2|\lambda|u} \operatorname{erfc}\left(\tfrac{u}{\sqrt{t}} + |\lambda| \sqrt{t} \right) \right)\, du \\
&= \sum_\lambda m_\lambda \operatorname{sign}(\lambda)\ \tfrac{1}{2} e^{2|\lambda|u} \operatorname{erfc}\left(\tfrac{u}{\sqrt{t}} + |\lambda| \sqrt{t} \right) \Big|_{u=0}^\infty \\
&= -\sum_\lambda \frac{m_\lambda \operatorname{sign}\lambda}{2} \operatorname{erfc}\left(|\lambda| \sqrt{t} \right) = -\tfrac{1}{2} m_0 - \sum_{\lambda \neq 0} m_\lambda \frac{\operatorname{sign}\lambda}{2} \operatorname{erfc}\left(|\lambda| \sqrt{t} \right).
\end{aligned}
$$

Recall that m_λ is the multiplicity of λ. Thus,

$$
\mathcal{K}(t) + \tfrac{1}{2} m_0 = -\sum_{\lambda \neq 0} \frac{m_\lambda \operatorname{sign}\lambda}{2} \operatorname{erfc}\left(|\lambda| \sqrt{t} \right)
$$

For $|\lambda| > 0$, one verifies using integration by parts and substitution that

$$
\int_0^\infty \operatorname{erfc}\left(|\lambda| \sqrt{t} \right) t^{s-1} dt = \frac{|\lambda|^{-2s}}{s\sqrt{\pi}} \Gamma\left(s + \tfrac{1}{2} \right).
$$

Using this and the fact that $\mathcal{K}(t) + \frac{1}{2} h \leq C e^{-\alpha t}$ for constants C and $\alpha > 0$,

$$
\begin{aligned}
&\int_0^\infty \left(\mathcal{K}(t) + \tfrac{1}{2} m_0 \right) t^{s-1} dt \\
&= -\int_0^\infty \left(\sum\nolimits_{\lambda \neq 0} \frac{m_\lambda \operatorname{sign}\lambda}{2} \operatorname{erfc}\left(|\lambda| \sqrt{t} \right) \right) t^{s-1} dt \\
&= -\sum\nolimits_{\lambda \neq 0} \frac{m_\lambda \operatorname{sign}\lambda}{2} \int_0^\infty \operatorname{erfc}\left(|\lambda| \sqrt{t} \right) t^{s-1} dt \\
&= -\sum\nolimits_{\lambda \neq 0} \frac{m_\lambda \operatorname{sign}\lambda}{2} \frac{|\lambda|^{-2s}}{s\sqrt{\pi}} \Gamma\left(s + \tfrac{1}{2} \right) \\
&= -\frac{\Gamma\left(s + \frac{1}{2} \right)}{2s\sqrt{\pi}} \sum\nolimits_{\lambda \neq 0} m_\lambda \operatorname{sign}(\lambda) |\lambda|^{-2s} \\
&= -\frac{\Gamma\left(s + \frac{1}{2} \right)}{2s\sqrt{\pi}} \eta_B(2s).
\end{aligned}
$$

Suppose that we have an asymptotic expansion

$$
\begin{aligned}
&\mathcal{K}(t) \sim \sum\nolimits_{k=-n+1}^{N} a_k t^{k/2} \text{ as } t \to 0^+;\ \text{i.e.,} \\
&\mathcal{K}(t) - \sum\nolimits_{k=-n+1}^{N} a_k t^{k/2} = \mathrm{O}(t^{\frac{1}{2}(N+1)}) \text{ as } t \to 0^+.
\end{aligned}
$$

In (2.57) below, such an asymptotic expansion will eventually be produced (as was done in [**BoWo93**, p. 239f]) from the known asymptotic expansion (see [**Gi95**, p. 68]) of the heat kernel on a closed manifold, namely the double of M. Since $\Re(s) > -\frac{N+1}{2} \Rightarrow \Re\left(\frac{1}{2}(N+1)+s-1\right) > -1$, we then have that

$$f_1(s) := \int_0^1 \left(\mathcal{K}(t) - \sum\nolimits_{k=-n+1}^{N} a_k t^{k/2}\right) t^{s-1} dt = \int_0^1 \mathrm{O}\left(t^{\frac{1}{2}(N+1)+s-1}\right) ds$$

is holomorphic for $\Re(s) > -\frac{N+1}{2}$. We also have the entire function

$$f_\infty(s) := \int_1^\infty \left(\mathcal{K}(t) + \tfrac{1}{2}h\right) t^{s-1} dt$$

We claim that

$$-\frac{\Gamma\left(s+\frac{1}{2}\right)}{2s\sqrt{\pi}} \eta_{\mathcal{B}}(2s) = \frac{m_0}{2s} + \sum\nolimits_{k=-n+1}^{N} \frac{a_k}{s+\frac{1}{2}k} + f_1(s) + f_\infty(s).$$

Indeed, we have

$$\begin{aligned}
&-\frac{\Gamma\left(s+\frac{1}{2}\right)}{2s\sqrt{\pi}} \eta_{\mathcal{B}}(2s) \\
&= \int_0^\infty \left(\mathcal{K}(t) + \tfrac{1}{2}h\right) t^{s-1} dt = \int_0^1 \left(\mathcal{K}(t) + \tfrac{1}{2}h\right) t^{s-1} dt + f_\infty(s) \\
&= \int_0^1 \left(\tfrac{1}{2}m_0 + \sum\nolimits_{k=-n+1}^{N} a_k t^{k/2}\right) t^{s-1} dt + f_1(s) + f_\infty(s) \\
&= \int_0^1 \tfrac{1}{2}m_0 t^{s-1} dt + \sum\nolimits_{k=-n+1}^{N} a_k \int_0^1 t^{\left(s+\frac{1}{2}k\right)-1} dt + f_1(s) + f_\infty(s) \\
&= \tfrac{1}{2}m_0 \left(\left.\frac{t^s}{s}\right|_{t=0}^{t=1}\right) + \sum\nolimits_{k=-n+1}^{N} a_k \left.\frac{t^{s+\frac{1}{2}k}}{s+\frac{1}{2}k}\right|_{t=0}^{t=1} + f_1(s) + f_\infty(s) \\
&= \frac{m_0}{2s} + \sum\nolimits_{k=-n+1}^{N} \frac{a_k}{s+\frac{1}{2}k} + f_1(s) + f_\infty(s).
\end{aligned}$$

Thus, where $\theta_N(s) = f_1(s) + f_\infty(s)$ is holomorphic for $\Re(s) > -\frac{N+1}{2}$,

$$\eta_{\mathcal{B}}(2s) = -\frac{2s\sqrt{\pi}}{\Gamma\left(s+\frac{1}{2}\right)} \left(\frac{m_0}{2s} + \sum\nolimits_{k=-n+1}^{N} \frac{a_k}{\frac{1}{2}k+s} + \theta_N(s)\right). \tag{2.46}$$

The heat kernels, say $\mathcal{E}_d$ for $\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+}$ and $\mathcal{E}_{d*}$ for $\widetilde{\mathcal{D}^+}\widetilde{\mathcal{D}^-}$, over the double $\widetilde{X}$ are more familiar. For $t > 0$,

$$\operatorname{Tr} e^{-t\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+}} - \operatorname{Tr} e^{-t\widetilde{\mathcal{D}^+}\widetilde{\mathcal{D}^-}} = \int_{\widetilde{X}} \operatorname{Tr}\left(\mathcal{E}_d(t;x,x) - \mathcal{E}_{d*}(t;x,x)\right) dx,$$

and there is the asymptotic expansion

$$F(t;x) := \operatorname{Tr}\left(\mathcal{E}_d(t;x,x) - \mathcal{E}_{d*}(t;x,x)\right) \sim \sum\nolimits_{k\geq -n} \alpha_k(x)\, t^{k/2}.$$

For $0 < a < b < 1$, let $\rho_{(a,b)} : [0,1] \to [0,1]$ be C^∞ and increasing with

$$\rho_{(a,b)}(u) = \begin{cases} 0 & \text{for } u \le a \\ 1 & \text{for } u \ge b. \end{cases}$$

Thinking of u as the normal coordinate function on $N = [0,1] \times Y \subset X$, and extending by the constant values 0 and 1, we can (and do) regard $\rho_{(a,b)}$ as a function on X. Let

$$\begin{aligned} Q(t;x,x') &:= \varphi_c(x)\,\mathcal{E}_c(t;x,x')\,\psi_c(x') + \varphi_d(x)\,\mathcal{E}_d(t;x,x')\,\psi_d(x') \\ &:= \left(1-\rho_{(5/7,6/7)}(x)\right)\mathcal{E}_c(t;x,x')\left(1-\rho_{(3/7,4/7)}(x')\right) \\ &\quad + \rho_{(1/7,2/7)}(x)\,\mathcal{E}_d(t;x,x')\,\rho_{(3/7,4/7)}(x'). \end{aligned}$$

We have that $\psi_c + \psi_d = 1$ and $\{\psi_c, \psi_d\}$ is a partition of unity for the cover $\left\{u^{-1}\left(\frac{3}{7},\infty\right), u^{-1}\left[0,\frac{4}{7}\right)\right\}$ of X. Moreover, for $j = c, d$

$$\varphi_j|\operatorname{supp}\psi_j = 1 \text{ and } \operatorname{dist}\left(\operatorname{supp}\partial_u^k\varphi_j, \operatorname{supp}\psi_j\right) \ge \tfrac{1}{7} \text{ (for } k \ge 1). \tag{2.47}$$

Of course, we do not expect $Q(t;x,x')$ to equal the exact kernel for $\partial_t + \mathcal{D}^*\mathcal{D}$ throughout X. However, note that for $x, x' \in [0,1/7) \times Y$, $Q(t;x,x') = \mathcal{E}_c(t;x,x')$, so that $Q(t;x,x')$ meets the APS boundary condition. Let $x' \in X$ be fixed. For $x \in X \setminus [0,6/7) \times Y$, we have $Q(t;x,x') = \mathcal{E}_d(t;x,x')\,\psi_d(x')$, while for $x \in [0,1/7) \times Y$, $Q(t;x,x') = \mathcal{E}_c(t;x,x')\,\psi_c(x')$. Thus, for $x \in X \setminus \left(\left[\frac{1}{7},\frac{6}{7}\right] \times Y\right)$, $-(\partial_t + \mathcal{D}^*\mathcal{D})\,Q(\cdot;\cdot,x')|_{(t;x,x')} = 0$. Moreover, for $d(x,x') < 1/7$,

$$\begin{aligned} &\varphi_c(x)\,\psi_c(x') + \varphi_d(x)\,\psi_d(x') \\ &= \left(1-\rho_{(5/7,6/7)}(x)\right)\left(1-\rho_{(3/7,4/7)}(x')\right) + \rho_{(1/7,2/7)}(x)\,\rho_{(3/7,4/7)}(x') \\ &= 1 - \rho_{(5/7,6/7)}(x) - \rho_{(3/7,4/7)}(x') \\ &+ \rho_{(5/7,6/7)}(x)\,\rho_{(3/7,4/7)}(x') + \rho_{(1/7,2/7)}(x)\,\rho_{(3/7,4/7)}(x') \\ &= 1 - \rho_{(5/7,6/7)}(x) - \rho_{(3/7,4/7)}(x') + \rho_{(5/7,6/7)}(x) + \rho_{(3/7,4/7)}(x') = 1 \end{aligned}$$

Note that $d(x,x') < 1/7 \Rightarrow \rho_{(5/7,6/7)}(x)\,\rho_{(3/7,4/7)}(x') = \rho_{(5/7,6/7)}(x)$, etc.. Because of this, we expect that

$$\begin{aligned} &\lim_{t\to 0^+} Q(t;x,x') \\ &= \lim_{t\to 0^+}\left(\varphi_c(x)\,\mathcal{E}_c(t;x,x')\,\psi_c(x') + \varphi_d(x)\,\mathcal{E}_d(t;x,x')\,\psi_d(x')\right) \\ &= \varphi_c(x)\,\delta(x,x')\,\psi_c(x') + \varphi_d(x)\,\delta(x,x')\,\psi_d(x') \\ &= \left(\varphi_c(x)\,\psi_c(x') + \varphi_d(x)\,\psi_d(x')\right)\delta(x,x') = \delta(x,x'). \end{aligned}$$

Thus, on the operator level, we expect that $\lim_{t\to 0^+} Q_t = \mathrm{I}$; i.e.,

$$\lim_{t\to 0^+} Q_t(f)(x) = \lim_{t\to 0^+}\int_X Q(t;x,x')\,f(x')\,dx' = \int_X \delta(x,x')\,f(x')\,dx' = f(x).$$

The extent to which $Q(\cdot;\cdot,x')$ fails to satisfy the heat equation in $\left[\frac{1}{7},\frac{6}{7}\right] \times Y$ is given by

$$C(t;x,x') := -(\partial_t + \mathcal{D}^*\mathcal{D})\,Q(\cdot;\cdot,x')|_{(t;x,x')}.$$

For $x' \in X$ fixed and $x = (u, y) \in N = [0,1] \times Y$, we have

$$\begin{aligned}
(2.48)\quad & -C\left(t;x,x'\right) \\
&= \left(\partial_t + \mathcal{D}^*\mathcal{D}\right) Q\left(t;x,x'\right) \\
&= \left(\partial_t + \mathcal{D}^*\mathcal{D}\right) \left(\sum\nolimits_{j\in\{c,d\}} \varphi_j\left(x\right) \mathcal{E}_j\left(t;x,x'\right) \psi_j\left(x'\right)\right) \\
&= \sum_{j\in\{c,d\}} \partial_u^2\left(\varphi_j\left(x\right)\right) \mathcal{E}_j\left(t;x,x'\right) \psi_j\left(x'\right) + 2\partial_u\left(\varphi_j\left(x\right)\right) \partial_u\mathcal{E}_j\left(t;x,x'\right) \psi_j\left(x'\right)
\end{aligned}$$

For $d\left(x,x'\right) < \frac{1}{7}$, we have $C\left(t;x,x'\right) = 0$, since (2.47) implies that $\psi_c\left(x'\right) = 0$ when $\partial_u^k\left(\varphi_j\left(x\right)\right) \neq 0$ and $d\left(x,x'\right) < \frac{1}{7}$. Using this and the estimates

$$\begin{aligned}
\mathcal{E}_j\left(t;x,x'\right) &\leq At^{-n/2}e^{-Bd\left(x,x'\right)^2/t} \text{ and} \\
\partial_u\mathcal{E}_j\left(t;x,x'\right) &\leq At^{-(n+1)/2}e^{-Bd\left(x,x'\right)^2/t}
\end{aligned}$$

for positive constants A and B, it follows from (2.48) that

$$\begin{aligned}
\left|C\left(t;x,x'\right)\right| &\leq c_1 t^{-(n+1)/2}e^{-Bd\left(x,x'\right)^2/t} \\
(2.49)\quad &\leq c_1 t^{-(n+1)/2}e^{-B7^{-2}/t} = c_1 e^{-c_2/t}, \text{ for } 0 < t < T_0 < \infty,
\end{aligned}$$

for positive constants c_1 and c_2.

Let $\mathcal{E}\left(t;x,x'\right)$ be the exact heat kernel for $\mathcal{D}^*\mathcal{D}$ and let the corresponding operator be $\mathcal{E}\left(t\right) = \exp\left(-t\mathcal{D}^*\mathcal{D}\right)$. On the operator level, we have

$$\begin{aligned}
\mathcal{E}\left(t\right) - Q\left(t\right) &= \mathcal{E}\left(t\right) Q\left(0\right) - \mathcal{E}\left(0\right) Q\left(t\right) \\
&= \int_0^t \tfrac{d}{ds}\left(\mathcal{E}\left(s\right) Q\left(t-s\right)\right) ds \\
&= \int_0^t \tfrac{d\mathcal{E}}{ds} Q\left(t-s\right) + \mathcal{E}\left(s\right) \tfrac{d}{ds} Q\left(t-s\right)\ ds \\
&= \int_0^t -\mathcal{D}^*\mathcal{D}\mathcal{E}\left(s\right) Q\left(t-s\right) + \mathcal{E}\left(s\right) \tfrac{d}{ds} Q\left(t-s\right)\ ds \\
&= \int_0^t \mathcal{E}\left(s\right) \left(-\mathcal{D}^*\mathcal{D} - \tfrac{d}{d(t-s)}\right) Q\left(t-s\right)\ ds \\
(2.50)\quad &= \int_0^t \mathcal{E}\left(s\right) C\left(t-s\right)\ ds.
\end{aligned}$$

Hence on the kernel level,

$$\begin{aligned}
\mathcal{E}\left(t;x,x'\right) &= Q\left(t;x,x'\right) + \int_0^t \int_X \mathcal{E}\left(s;x,z\right) C\left(t-s;z,x'\right)\ dzds \\
(2.51)\quad &= Q\left(t;x,x'\right) + \left(\mathcal{E} * C\right)\left(t;x,x'\right),
\end{aligned}$$

where the convolution operation is defined by

$$(\alpha * \beta)(t; x, x') := \int_0^t \int_X \alpha(s; x, z)\, \beta(t - s; z, x')\, dz ds.$$

Note that (2.51) can be written as

$$\mathcal{E}(\mathrm{I} - * C) = \mathcal{E} - \mathcal{E} * C = Q.$$

Thus, at least formally,

$$\begin{aligned} \mathcal{E} &= Q(\mathrm{I} - * C(t))^{-1} = Q\left(\mathrm{I} + \sum\nolimits_{k=1}^{\infty} (*C)^k\right) \\ &= Q + Q * \sum\nolimits_{k=1}^{\infty} C_k = Q + Q * \mathcal{C}, \text{ where} \end{aligned} \tag{2.52}$$

$$C_1 := C = -(\partial_t + \mathcal{D}^* \mathcal{D})\, Q,\ C_{k+1} := C_1 * C_k, \text{ and } \mathcal{C} := \sum\nolimits_{k=1}^{\infty} C_k.$$

The proof of the validity of the Levi sum (2.52) for $\mathcal{E}$ rests on the estimate (2.49) and we refer to [**BoWo93**, Ch. 22] for the details. The above constructions can be also be applied to obtain a kernel $\mathcal{E}_*(t; x, x')$ for $\exp(-t\mathcal{D}\mathcal{D}^*)$ starting from

$$Q_*(t; x, x') := \varphi_c(x)\, \mathcal{E}_{c*}(t; x, x')\, \psi_c(x') + \varphi_d(x)\, \mathcal{E}_{d*}(t; x, x')\, \psi_d(x').$$

Recall that for $(u, y) \in [0, \infty) \times Y$,

$$\begin{aligned} &\mathcal{K}(t; u, y) := \mathrm{Tr}(\mathcal{E}_c(t; (u, y), (u, y)) - \mathcal{E}_{c*}(t; (u, y), (u, y))), \text{ and} \\ &\mathcal{K}(t) := \int_0^{\infty} \int_Y \mathcal{K}(t; u, y)\; dy du. \end{aligned} \tag{2.53}$$

For $x \in \widetilde{X}$, let

$$F(t; x) := \mathrm{Tr}(\mathcal{E}_d(t; x, x) - \mathcal{E}_{d*}(t; x, x)).$$

Now,

$$\operatorname{index} \mathcal{D}^+_{P_{\geq}} = \mathrm{Tr}\left(e^{-t\mathcal{D}^*\mathcal{D}} - e^{-t\mathcal{D}\mathcal{D}^*}\right) = \mathrm{Tr}(\mathcal{E}(t) - \mathcal{E}_*(t)).$$

We use the following notation for functions which are "exponentially close" as $t \to 0^+$:

$$f(t) \sim_{\exp} g(t) \quad \Leftrightarrow |f(t) - g(t)| \leq a e^{-b/t} \text{ for } t \in (0, \varepsilon) \tag{2.54}$$

for some positive constants a, b, ε. We claim (but must omit important details here – see [**BoWo93**, Ch. 22]) that

$$\mathrm{Tr}(\mathcal{E}(t) - \mathcal{E}_*(t)) \sim_{\exp} \mathrm{Tr}(Q(t) - Q_*(t)).$$

Since $\mathcal{E} = Q + Q * \mathcal{C}$ (resp., $\mathcal{E}_* = Q_* + Q_* * \mathcal{C}_*$), this result ultimately rests on (2.49) (resp., the corresponding result for C_*). Since $\varphi_c \psi_c = \psi_c$ and $\varphi_d \psi_d = \psi_d$,

$$\begin{aligned}
&\operatorname{Tr}\left(Q\left(t\right) - Q_*\left(t\right)\right) = \int_X \operatorname{Tr} Q\left(t; x, x\right)\, dx \\
&= \int_X \operatorname{Tr} \begin{pmatrix} \varphi_c\left(x\right)\left(\mathcal{E}_c - \mathcal{E}_{c*}\right)\left(t; x, x\right)\psi_c\left(x\right) \\ +\varphi_d\left(x\right)\left(\mathcal{E}_d - \mathcal{E}_{c*}\right)\left(t; x, x\right)\psi_d\left(x\right) \end{pmatrix} dx \\
&= \int_0^1 \int_Y \mathcal{K}\left(t; u, y\right)\psi_c\left(u\right)\, dydu + \int_X F\left(t; x\right)\psi_d\left(x\right)\, dx.
\end{aligned}$$

Let

$$\mathcal{K}_a\left(t\right) := \int_0^a \int_Y \mathcal{K}\left(t; u, y\right) dydu.$$

Note that

$$\begin{aligned}
\mathcal{K}\left(t\right) - \mathcal{K}_a\left(t\right) &= \int_a^\infty \int_Y \mathcal{K}\left(t; u, y\right) dydu \\
&= \sum\nolimits_\lambda m_\lambda \operatorname{sign}\left(\lambda\right) \tfrac{1}{2} e^{2|\lambda| u} \operatorname{erfc}\left(\tfrac{u}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\Big|_{u=a}^{\infty} \\
&= -\tfrac{1}{2}\sum\nolimits_\lambda m_\lambda \operatorname{sign}\left(\lambda\right) e^{2|\lambda| a} \operatorname{erfc}\left(\tfrac{a}{\sqrt{t}} + |\lambda|\sqrt{t}\right).
\end{aligned}$$

Thus (using (2.42)) for some constant $c > 0$

$$\begin{aligned}
\left|\mathcal{K}\left(t\right) - \mathcal{K}_a\left(t\right)\right| &\leq \sum\nolimits_\lambda m_\lambda e^{2|\lambda| a} \exp\left(-\left(\tfrac{a}{\sqrt{t}} + |\lambda|\sqrt{t}\right)^2\right) \\
&\leq \tfrac{1}{2} e^{-a^2/t} \sum\nolimits_\lambda m_\lambda e^{-|\lambda|^2 t} \sim_{t\to 0^+} c e^{-a^2/t} t^{-n/2},
\end{aligned}$$

which says that $\mathcal{K}\left(t\right) \sim_{\exp} \mathcal{K}_a\left(t\right)$. Hence,

$$\begin{aligned}
&\operatorname{index} \mathcal{D}^+_{P_\geq} = \operatorname{Tr}\left(\mathcal{E}\left(t\right) - \mathcal{E}_*\left(t\right)\right) \sim_{\exp} \operatorname{Tr}\left(Q\left(t\right) - Q_*\left(t\right)\right) \\
&= \int_0^1 \int_Y \mathcal{K}\left(t; u, y\right)\psi_c\left(u\right)\, dydu + \int_X F\left(t; x\right)\psi_d\left(x\right)\, dx \\
&\sim_{\exp} \left(\int_0^\infty \int_Y \mathcal{K}\left(t; u, y\right)\, dydu + \int_X F\left(t; x\right)\psi_d\left(x\right)\, dx\right) \\
&= \mathcal{K}\left(t\right) + \int_X F\left(t; x\right)\, dx.
\end{aligned} \tag{2.55}$$

The last equality follows from (2.53) and the fact that

$$F\left(t; x\right) = \operatorname{Tr}\left(\mathcal{E}_d\left(t; x, x\right) - \mathcal{E}_{d*}\left(t; x, x\right)\right) = 0 \text{ for } x \in N = [0,1] \times Y, \tag{2.56}$$

which is seen as follows. Since $\sigma^* = -\sigma$,

$$\begin{aligned}
&\widetilde{\mathcal{D}^+}|_N = \sigma(\partial_u + \mathcal{B}) : C^\infty\left(N, \widetilde{S^+}\right) \to C^\infty\left(N, \widetilde{S^-}\right) \text{ implies} \\
&\widetilde{\mathcal{D}^-}|_N = \left(\sigma(\partial_u + \mathcal{B})\right)^* = (\partial_u - \mathcal{B})\sigma : C^\infty\left(N, \widetilde{S^-}\right) \to C^\infty\left(N, \widetilde{S^+}\right),
\end{aligned}$$

Thus, over N,

$$\begin{aligned}\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+} &= (\partial_u - \mathcal{B})\sigma\sigma(\partial_u + \mathcal{B}) = \left(-\partial_u^2 + \mathcal{B}^2\right)\Big|C^\infty\left(N, \widetilde{S^+}\right) \text{ and}\\ \widetilde{\mathcal{D}^+}\widetilde{\mathcal{D}^-} &= \sigma(\partial_u + \mathcal{B})(\partial_u - \mathcal{B})\sigma = \sigma^2(\partial_u - \mathcal{B})(\partial_u + \mathcal{B})\\ &= \left(-\partial_u^2 + \mathcal{B}^2\right)\Big|C^\infty\left(N, \widetilde{S^-}\right).\end{aligned}$$

Since $\left[\left(-\partial_u^2 + \mathcal{B}^2\right), \sigma\right] = 0$, σ maps the eigenspaces of $\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+}$ pointwise isometrically (over N) onto those of $\widetilde{\mathcal{D}^+}\widetilde{\mathcal{D}^-}$. Hence, we obtain (2.56) upon taking the trace of the eigensection expansion of $\mathcal{E}_d(t; x, x) - \mathcal{E}_{d*}(t; x, x)$ for $x \in N$. From (2.55), we obtain

$$\begin{aligned}\mathcal{K}(t) &= \operatorname{index}\mathcal{D}^+_{P_\geq} - \int_X F(t; x)\, dx\\ &\sim \operatorname{index}\mathcal{D}^+_{P_\geq} - \int_X \sum\nolimits_{k\geq -n} \alpha_k(x)\, t^{k/2}\, dx\\ &= \operatorname{index}\mathcal{D}^+_{P_\geq} - \sum\nolimits_{k\geq -n} A_k t^{k/2} \text{ (for } A_k := \int_X \alpha_k(x)\, dx),\end{aligned} \tag{2.57}$$

where $F(t; x) \sim \sum_{k\geq -n} \alpha_k(x)\, t^{k/2}$ is the asymptotic expansion of the trace $F(t; x)$ of the kernel for $e^{-t\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+}} - e^{-t\widetilde{\mathcal{D}^+}\widetilde{\mathcal{D}^-}}$, which is known for elliptic differential operators (e.g., $\widetilde{\mathcal{D}^-}\widetilde{\mathcal{D}^+}$ and $\widetilde{\mathcal{D}^+}\widetilde{\mathcal{D}^-}$) over closed manifolds such as $\widetilde{X}$ (see [**Gi95**, p. 68]). Thus, according to (2.46) with $a_k = -A_k$ for $k \neq 0$ and $a_0 = \operatorname{index}\mathcal{D}^+_{P_\geq} - A_0$,

$$\eta_{\mathcal{B}}(2s) = -\frac{2s\sqrt{\pi}}{\Gamma\left(s + \frac{1}{2}\right)}\left(\frac{m_0}{2s} + \sum\nolimits_{k=-n+1}^{N} \frac{a_k}{\frac{1}{2}k + s} + \theta_N(s)\right)$$

$$\eta_{\mathcal{B}}(2s) = -\frac{2s\sqrt{\pi}}{\Gamma\left(s + \frac{1}{2}\right)}\left(\frac{m_0 + 2\operatorname{index}\mathcal{D}^+_{P_\geq}}{2s} - \sum\nolimits_{k=-n+1}^{N} \frac{A_k}{\frac{1}{2}k + s} + \theta_N(s)\right).$$

Setting $s = 0$, yields $\eta_{\mathcal{B}}(0) = -\left(m_0 + 2\operatorname{index}\mathcal{D}^+_{P_\geq} - 2A_0\right)$ or

$$\operatorname{index}\mathcal{D}^+_{P_\geq} = A_0 - \tfrac{1}{2}\left(m_0 + \eta_{\mathcal{B}}(0)\right).$$

□

2.5. Symplectic Geometry of Cauchy Data Spaces. As we have seen in our Section 1.2.2 there exists a concept of Cauchy data spaces which solely is based on the concepts of minimal and maximal domain and which is more elementary and more general than the definitions provided in Section 2.3.5 which are based on pseudo-differential calculus.

In this Section we stay in the real category and do not assume product structure near $\Sigma = \partial X$ unless otherwise stated. The operator $\mathcal{D}$ need not be of Dirac type. We only assume that it is a linear, elliptic, symmetric, differential operator of first order.

2.5.1. *The Natural Cauchy Data Space.* Let $\mathcal{D}_0$ denote the restriction of $\mathcal{D}$ to the space $C_0^\infty(X;S)$ of smooth sections with support in the interior of X. As mentioned above, there is no natural choice of the order of the Sobolev spaces for the boundary reduction. Therefore, a systematic treatment of the boundary reduction may begin with the minimal closed extension $\mathcal{D}_{\min} := \overline{\mathcal{D}_0}$ and the adjoint $\mathcal{D}_{\max} := (\mathcal{D}_0)^*$ of $\mathcal{D}_0$. Clearly, $\mathcal{D}_{\max}$ is the maximal closed extension. This gives

$$D_{\min} := \operatorname{Dom}(\mathcal{D}_{\min}) = \overline{C_0^\infty(X;S)}^{\mathcal{G}} = \overline{C_0^\infty(X;S)}^{H^1(X;S)}$$

and

$$\begin{aligned} D_{\max} &:= \operatorname{Dom}(\mathcal{D}_{\max}) \\ &= \{u \in L^2(X;S) \mid \mathcal{D}u \in L^2(X;S) \text{ in the sense of distributions}\}. \end{aligned}$$

Here, the superscript $\mathcal{G}$ means the closure in the graph norm which coincides with the first Sobolev norm on $C_0^\infty(X;S)$. We form the space β of *natural boundary values* with the *natural trace map* γ as described in Section 1.2.

There we defined also the *natural Cauchy data space* $\Lambda(\mathcal{D}) := \gamma(\operatorname{Ker}\mathcal{D}_{\max})$ under the assumption that $\mathcal{D}$ has a self-adjoint L^2 extension with a compact resolvent. Such an extension always exists. Take for instance $\mathcal{D}_{\mathcal{P}(\mathcal{D})}$, the operator $\mathcal{D}$ with domain

$$\operatorname{Dom}_{\mathcal{P}(\mathcal{D})} := \{f \in H^1(X;S) \mid \mathcal{P}(\mathcal{D})^{(\frac{1}{2})}(f|_\Sigma) = 0\},$$

where $\mathcal{P}(\mathcal{D})$ denotes the Calderón projection defined in (2.32).

Clearly, $D_{\max}$ and $D_{\min}$ are $C^\infty(X)$ modules, and so the space β is a $C^\infty(\Sigma)$ module. This shows that β is *local* in the following sense: If Σ decomposes into r connected components $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$, then β decomposes into

$$\beta = \bigoplus_{j=1}^{r} \beta_j ,$$

where

$$\beta_j := \gamma\left(\{f \in D_{\max} \mid \operatorname{supp} f \subset N_j\}\right)$$

with a suitable collared neighborhood N_j of Σ_j . Note that each β_j is a closed symplectic subspace of β and therefore a symplectic Hilbert space.

By Theorem 2.20a and, alternatively and in greater generality, by Hörmander [**Ho66**] (Theorem 2.2.1 and the Estimate (2.2.8), p. 194), the space β is naturally embedded in the distribution space $H^{-\frac{1}{2}}(\Sigma; S|_\Sigma)$. Under this embedding we have $\Lambda(\mathcal{D}) = \Lambda(\mathcal{D}, 0)$, where the last space was defined in Definition 2.22.

If the metrics are product close to Σ, we can give a more precise description of the embedding of β, namely as a *graded* space of distributions. Let $\{\varphi_k, \Lambda_k\}$ be a spectral resolution of $L^2(\Sigma)$ by eigensections of $\mathcal{B}$. (Here and in the following we do not mention the bundle S). Once again, for simplicity, we assume $\operatorname{Ker}\mathcal{B} = \{0\}$ and have $\mathcal{B}\varphi_k = \lambda_k\varphi_k$ for all $k \in \mathbb{Z} \setminus \{0\}$, and $\lambda_{-k} = -\lambda_k$, $\sigma(\varphi_k) = \varphi_{-k}$, and

$\sigma(\varphi_{-k}) = -\varphi_k$ for $k > 0$. In [**BoFu99**], Proposition 7.15 (see also [**BrLe99**] for a more general setting) it was shown that

$$\beta = \beta_- \oplus \beta_+ \quad \text{with}$$

$$\beta_- := \overline{[\{\varphi_k\}_{k<0}]}^{H^{\frac{1}{2}}(\Sigma)} \text{ and } \beta_+ := \overline{[\{\varphi_k\}_{k>0}]}^{H^{-\frac{1}{2}}(\Sigma)} . \tag{2.58}$$

Then β_- and β_+ are Lagrangian and transversal subspaces of β.

2.5.2. *Criss–cross Reduction.* Let us define two Lagrangian and transversal subspaces $L_\pm$ of $L^2(\Sigma)$ in a similar way, namely by the closure in $L^2(\Sigma)$ of the linear span of the eigensections with negative, resp. with positive eigenvalue. Then L_+ is dense in β_+, and β_- is dense in L_-. This anti–symmetric relation may explain some of the well–observed delicacies of dealing with spectral invariants of continuous families of Dirac operators.

Moreover, $\gamma(D_{\rm aps}) = \beta_-$, where

$$D_{\rm aps} := \{f \in H^1(X) \mid P_>(f|_\Sigma) = 0\} \tag{2.59}$$

denotes the domain corresponding to the Atiyah–Patodi–Singer boundary condition. Note that a series $\sum_{k<0} c_k\varphi_k$ may converge to an element $\varphi \in L^2(\Sigma)$ without converging in $H^{\frac{1}{2}}(\Sigma)$. Therefore such $\varphi \in L_-$ can not appear as trace at the boundary of any $f \in D_{\max}$.

Recall Proposition 1.27 and note that $(\beta_-, \Lambda(\mathcal{D}))$ is a Fredholm pair.

This can all be achieved without the symbolic calculus of pseudo-differential operators. Therefore one may ask how the preceding approach to Cauchy data spaces and boundary value problems via the maximal domain and our symplectic space β is related to the approach via the Calderón projection, which we reviewed in the preceding section. How can results from the distributional theory be translated into L^2 results?

To relate the two approaches we recall a fairly general symplectic 'Criss–Cross' Reduction Theorem from [**BoFuOt01**] (Theorem 1.2). Let β and L be symplectic Hilbert spaces with symplectic forms ω_β and ω_L, respectively. Let

$$\beta = \beta_- \dot{+} \beta_+ \qquad \text{and} \qquad L = L_- \dot{+} L_+$$

be direct sum decompositions by transversal (not necessarily orthogonal) pairs of Lagrangian subspaces. We assume that there exist continuous, injective mappings

$$i_- : \beta_- \longrightarrow L_- \qquad \text{and} \qquad i_+ : L_+ \longrightarrow \beta_+$$

with dense images and which are compatible with the symplectic structures, i.e.

$$\omega_L(i_-(x), a) = \omega_\beta(x, i_+(a)) \quad \text{for all } a \in L_+ \text{ and } x \in \beta_- .$$

Let $\mu \in \mathcal{FL}_{\beta_-}(\beta)$, e.g. $\mu = (\mu \cap \beta_-) \dot{+} \nu$ with a suitable closed ν. Let us define (see also Figure 9)

$$\tau(\mu) := i_-(\mu \cap \beta_-) + \operatorname{graph}(\varphi_\mu) , \tag{2.60}$$

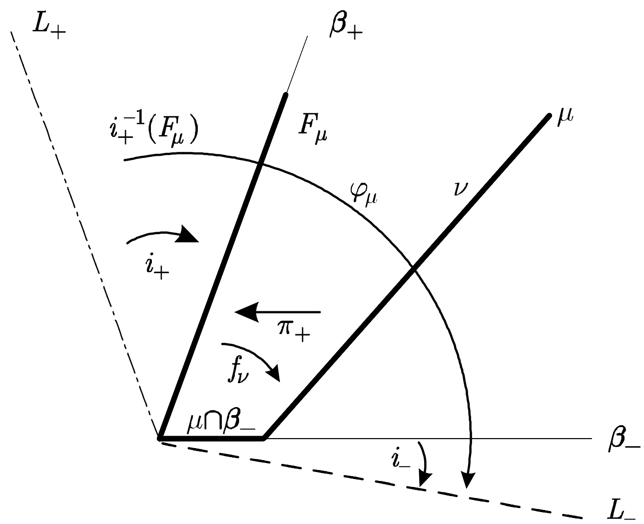

FIGURE 9. The mapping $\tau : \mathcal{FL}_{\beta_-}(\beta) \to \mathcal{FL}_{L_-}(L)$

where

$$\begin{array}{cccc} \varphi_\mu : & i_+^{-1}(F_\mu) & \longrightarrow & L_- \\ & x & \mapsto & i_- \circ f_\nu \circ i_+(x) \end{array} .$$

Here F_μ denotes the image of μ under the projection π_+ from μ to β_+ along β_- and $f_\nu : F_\mu \longrightarrow \beta_-$ denotes the uniquely determined bounded operator which yields ν as its graph. Then:

THEOREM 2.31. *The mapping (2.60) defines a continuous mapping*

$$\tau : \mathcal{FL}_{\beta_-}(\beta) \longrightarrow \mathcal{FL}_{\mathbf{L}_-}(L)$$

which maps the Maslov cycle $\mathcal{M}_{\beta_-}(\beta)$ of β_- into the Maslov cycle $\mathcal{M}_{L_-}(L)$ of L_- and preserves the Maslov index

$$\mathbf{mas}(\{\mu_s\}_{s\in[0,1]}, \beta_-) = \mathbf{mas}(\{\tau(\mu_s)\}_{s\in[0,1]}, L_-)$$

for any continuous curve $[0,1] \ni s \mapsto \mu_s \in \mathcal{FL}_{\beta_-}(\beta)$.

In the product case, the 'Criss–Cross' Reduction Theorem implies for our two types of Cauchy data that all results proved in the theory of natural boundary values (β theory) remain valid in the L^2 theory. In particular we have:

COROLLARY 2.32. *The $L^2(\Sigma)$ part $\Lambda(\mathcal{D}) \cap L^2(\Sigma)$ of the natural Cauchy data space $\Lambda(\mathcal{D})$ is closed in $L^2(\Sigma)$. Actually, it is a Lagrangian subspace of $L^2(\Sigma)$ and it forms a Fredholm pair with the component L_- , defined at the beginning of this subsection.*

2.6. Non-Additivity of the Index.

2.6.1. *The Bojarski Conjecture.* The *Bojarski Conjecture* gives quite a different description of the index of an elliptic operator over a closed partitioned manifold $M = M_1 \cup_\Sigma M_2$. It relates the 'quantum' quantity index with a 'classical' quantity, the Fredholm intersection index of the Cauchy data spaces from both sides of the hypersurface Σ. It was suggested in [**Bo79**] and proved in [**BoWo93**] for operators of Dirac type.

PROPOSITION 2.33. *Let M be a partitioned manifold as before and let $\Lambda(\mathcal{D}_j^+, \frac{1}{2})$ denote the L^2 Cauchy data spaces, $j = 1, 2$ (see Definition 2.22). Then*

$$\operatorname{index} \mathcal{D}^+ = \operatorname{index}\left(\Lambda\left(\mathcal{D}_1^+, \tfrac{1}{2}\right), \Lambda\left(\mathcal{D}_2^+, \tfrac{1}{2}\right)\right).$$

Recall that

$$\begin{aligned} \operatorname{index}\left(\Lambda\left(\mathcal{D}_1^+, \tfrac{1}{2}\right), \Lambda\left(\mathcal{D}_2^+, \tfrac{1}{2}\right)\right) &:= \dim\left(\Lambda\left(\mathcal{D}_1^+, \tfrac{1}{2}\right) \cap \Lambda\left(\mathcal{D}_2^+, \tfrac{1}{2}\right)\right) \\ &\quad - \dim\left(\frac{L^2\left(\Sigma; S|_\Sigma\right)}{\Lambda\left(\mathcal{D}_1^+, \tfrac{1}{2}\right) + \Lambda\left(\mathcal{D}_2^+, \tfrac{1}{2}\right)}\right). \end{aligned}$$

It is equal to $\mathbf{i}(\mathrm{I} - \mathcal{P}(\mathcal{D}_2^+), \mathcal{P}(\mathcal{D}_1^+))$, where $\mathcal{P}(\mathcal{D}_j^+)$ denotes the corresponding Calderón projections, and $\mathbf{i}(\cdot, \cdot)$ was defined in (2.33).

The proof of the Proposition depends on the unique continuation property for Dirac operators and the Lagrangian property of the Cauchy data spaces, more precisely the chiral twisting property (2.38) .

2.6.2. *Generalizations for Global Boundary Conditions.* On a smooth compact manifold X with boundary Σ, the solution spaces $\operatorname{Ker}(\mathcal{D}, s)$ depend on the order s of differentiability and they are infinite-dimensional. To obtain a finite index one must apply suitable boundary conditions (see [**BoWo93**] for local and global boundary conditions for operators of Dirac type). In this report, we restrict ourselves to boundary conditions of Atiyah–Patodi–Singer type (i.e., $P \in \mathcal{G}\mathrm{r}(\mathcal{D})$), and consider the extension

$$\mathcal{D}_P : \operatorname{Dom}(\mathcal{D}_P) \longrightarrow L^2(X; S) \tag{2.61}$$

of $\mathcal{D}$ defined by the domain

$$\operatorname{Dom}(\mathcal{D}_P) := \{f \in H^1(X; S) \mid P^{(0)}(f|_\Sigma) = 0\}. \tag{2.62}$$

It is a closed operator in $L^2(X; S)$ with finite-dimensional kernel and cokernel. We have an explicit formula for the adjoint operator

$$(\mathcal{D}_P)^* = \mathcal{D}_{\sigma(\mathrm{I} - P)\sigma^*}. \tag{2.63}$$

In agreement with Proposition 1.27b, the preceding equation shows that an extension $\mathcal{D}_P$ is self-adjoint, if and only if $\operatorname{Ker} P^{(0)}$ is a Lagrangian subspace of the symplectic Hilbert space $L^2(\Sigma; S|_\Sigma)$.

Let us recall the *Boundary Reduction Formula for the Index* of (global) elliptic boundary value problems discussed in [**BoWo85**] (inspired by [**Se69**], see [**BoWo93**] for a detailed proof for Dirac operators). Like the Bojarski Conjecture, the point of the formula is that it gives an expression for the index in terms of the geometry of the Cauchy data in the symplectic space of all (here L^2) boundary data.

Proposition 2.34.

$$\operatorname{index} \mathcal{D}_P = \operatorname{index}\left\{ P\mathcal{P}(\mathcal{D}) : \Lambda(\mathcal{D}, \tfrac{1}{2}) \to \operatorname{range}(P^{(0)})\right\}$$

2.6.3. *Pasting Formulas.* We shall close this section by mentioning a slight modification of the Bojarski Conjecture/Theorem, namely a non-additivity formula for the splitting of the index over partitioned manifolds.

From the Atiyah–Singer Index Theorem (here the expression of the index on the closed manifold M by an integral of the *index density*) and the Atiyah–Patodi–Singer Index Theorem (Theorem 2.30), we obtain at once

$$\operatorname{index} \mathcal{D} = \operatorname{index}(\mathcal{D}_1)_{P_\leq} + \operatorname{index}(\mathcal{D}_2)_{P_\geq} - \dim \operatorname{Ker}(\mathcal{B}).$$

Then an Agranovic–Dynin type correction formula (based on Proposition 2.34) yields:

Theorem 2.35. *Let P_j be projections belonging to $\mathcal{G}\mathrm{r}(\mathcal{D}_j)$, $j = 1, 2$. Then*

$$\operatorname{index} \mathcal{D} = \operatorname{index}(\mathcal{D}_1)_{P_1} + \operatorname{index}(\mathcal{D}_2)_{P_2} - \mathbf{i}\,(P_2, \mathrm{I} - P_1)$$

It turns out that the boundary correction term $\mathbf{i}(P_2, \mathrm{I} - P_1)$ equals the index of the operator $\sigma(\partial_t + \mathcal{B})$ on the cylinder $[0,1] \times \Sigma$ with the boundary conditions P_1 at $t = 0$ and P_2 at $t = 1$. A direct proof of Theorem 2.35 can be derived from Proposition 2.34 by elementary operations with the virtual indices $\mathbf{i}\,(P_1, \mathcal{P}(\mathcal{D}_1))$ and $\mathbf{i}\,(P_2, \mathcal{P}(\mathcal{D}_2))$; see, e.g., [**DaZh96**] in a more general setting.

Remark 2.36. (*a*) In this section we have not always distinguished between the *total* and the *chiral* Dirac operator because all the discussed index formulas are valid in both cases.
(*b*) Important index formulas for (global) elliptic boundary value problems for operators of Dirac type can also be obtained without analyzing the concept and the geometry of the Cauchy data spaces (see e.g. the celebrated Atiyah–Patodi–Singer Index Theorem 2.30 or [**Sc01**] for a recent survey of index formulas where there is no mention of the Calderón projection). The basic reason is that the index is an invariant represented by a local density inside the manifold plus a correction term which lives on the boundary and may be local or non-local as well. However, these formulas do not explain the simple origin of the index or the spectral flow, namely that *all* index information is naturally coded by the geometry of the Cauchy data spaces. To us it seems necessary to use the Calderón projection in order to understand (not calculate) the index of an elliptic boundary problem and the reason for the locality or non-locality.

2.7. Pasting of Spectral Flow.

2.7.1. *Spectral Flow and the Maslov Index.* Let $\{\mathcal{D}_t\}_{t\in[0,1]}$ be a continuous family of (from now on always *total*) Dirac operators with the same principal symbol and the same domain D. To begin with, we do not distinguish between the case of a closed manifold (when D is just the first Sobolev space and all operators are essentially self-adjoint) and the case of a manifold with boundary (when D is specified by the choice of a suitable boundary value condition).

We consider the *spectral flow* $\mathrm{SF}\,\{\mathcal{D}_{t,D}\}$ (see Section 1.1.4). We want a pasting formula for the spectral flow. To achieve that, one replaces the spectral flow of a continuous one-parameter family of self-adjoint Fredholm operators, which is a 'quantum' quantity, by the *Maslov index* of a corresponding path of Lagrangian Fredholm pairs, which is a 'quasi-classical' quantity. The idea is due to Floer and was worked out subsequently by Yoshida in dimension 3, by Nicolaescu in all odd dimensions, and pushed further by Cappell, Lee and Miller, Daniel and Kirk, and many other authors. For a survey, see [**BoFu98**], [**BoFu99**], [**DaKi99**], [**KiLe00**].

In this review we give two spectral flow formulas of that type. To begin with, we consider the case of a manifold with boundary. Then weak UCP for Dirac type operators (established in Section 1.4) implies weak inner UCP in the sense of (1.9), if the manifold is connected. Actually, it would suffice that there is no connected component without boundary. Let us fix the space β for the family. By Proposition 1.27c the corresponding family $\{\Lambda(\mathcal{D}_t)\}$ of natural Cauchy data spaces is continuous. Applying the *General Boundary Reduction Formula* for the spectral flow (Theorem 1.28) gives a family version of the Bojarski conjecture (our Proposition 2.33):

THEOREM 2.37. *The spectral flow of the family $\{\mathcal{D}_{t,D}\}$ and the Maslov index $\mathbf{mas}\,(\{\Lambda(\mathcal{D}_t)\},\gamma(D))$ are well-defined and we have*

$$\mathrm{SF}\,\{\mathcal{D}_{t,D}\} = \mathbf{mas}\,(\{\Lambda(\mathcal{D}_t)\},\gamma(D))\,. \tag{2.64}$$

We have various corollaries for the spectral flow on closed manifolds with fixed hypersurface (see [**BoFu99**]). Note that a partitioned manifold $M = M_1 \cup_\Sigma M_2$ can be considered as a manifold $M^\# = M_1 \sqcup M_2$ with boundary $\partial M^\# = -\Sigma \sqcup \Sigma$. Here $\sqcup$ denotes taking the disjoint union. Then any elliptic operator $\mathcal{D}$ over M defines an operator $\mathcal{D}^\#$ over $M^\#$ with natural Fredholm extension $\mathcal{D}^\#_D$ by fixing the domain

$$D := \{(f_1,f_2)\in H^1(M_1)\times H^1(M_2)\mid (f_1|_\Sigma, f_2|_\Sigma)\in\Delta\},$$

where Δ denotes the diagonal of $L^2\,(-\Sigma;S|_\Sigma)\times L^2\,(\Sigma;S|_\Sigma)$. By elliptic regularity we have

$$\operatorname{Ker}\mathcal{D}^\#_D = \operatorname{Ker}\mathcal{D} \text{ and } \mathrm{SF}\left\{\mathcal{D}^\#_{t,D}\right\} = \mathrm{SF}\,\{\mathcal{D}_t\}\,.$$

For product structures near Σ, one can apply Theorem 2.31 and obtain an L^2 version of the preceding Theorem which gives a new proof and a slight generalization of the Yoshida–Nicolaescu Formula (for details see [**BoFuOt01**], Section 3). For safety reasons, we assume that $\mathcal{D}_t$ is a zero-order perturbation of $\mathcal{D}_0$, induced by a continuous change of the defining connection, or alternatively, $\mathcal{D}_t := \mathcal{D}_0 + C_t$ where C_t is a self-adjoint bundle morphism.

THEOREM 2.38.

$$\mathrm{SF}\{\mathcal{D}_t\} = \mathbf{mas}\left(\left\{\Lambda_t^1 \cap L^2(-\Sigma) \dot{+} \Lambda_t^2 \cap L^2(\Sigma)\right\}, \Delta\right) =: \mathbf{mas}\left(\{\Lambda_t^1 \cap L^2(-\Sigma)\}, \{\Lambda_t^2 \cap L^2(\Sigma)\}\right),$$

where the Cauchy data spaces Λ_t^j *are taken on each side* $j = 1, 2$ *of the hypersurface* Σ.

REMARK 2.1. Here it is not compelling to use the symplectic geometry of the Cauchy data spaces (see Remark 2.36b). Actually, deep gluing formulas can and have been obtained for the spectral flow by coding relevant information not in the full infinite-dimensional Cauchy data spaces but in families of Lagrangian subspaces of suitable finite-dimensional symplectic spaces, like the kernel of the tangential operator (see [**CaLeMi96**] and [**CaLeMi00**]).

2.7.2. *Correction Formula for the Spectral Flow.* Let D, D' with $D_{\min} < D, D' < D_{\max}$ be two domains such that both $\{\mathcal{D}_{t,D}\}$ and $\{\mathcal{D}_{t,D'}\}$ become families of self-adjoint Fredholm operators. We assume that D and D' differ only by a finite dimension, more precisely that

$$\dim \frac{\gamma(D)}{\gamma(D) \cap \gamma(D')} = \dim \frac{\gamma(D')}{\gamma(D) \cap \gamma(D')} < +\infty\,. \tag{2.65}$$

Then we find from Theorem 2.37 (for details see [**BoFu99**], Theorem 6.5):

$$\begin{aligned} &\mathbf{sf}\,\{\mathcal{D}_{t,D}\} - \mathbf{sf}\,\{\mathcal{D}_{t,D'}\} \\ &= \mathbf{mas}\,(\{\Lambda\,(\mathcal{D}_t)\}\,, \gamma(D')) - \mathbf{mas}\,(\{\Lambda\,(\mathcal{D}_t)\}\,, \gamma(D)) \\ &= \sigma_{\mathrm{H\ddot{o}r}}\,(\Lambda(\mathcal{D}_0), \Lambda(\mathcal{D}_1); \gamma(D'), \gamma(D)) \end{aligned} \tag{2.66}$$

(see Remark 1.26b). The assumption (2.65) is rather restrictive. The pair of domains, for instance, defined by the Atiyah–Patodi–Singer projection and the Calderón projection, may not always satisfy this condition. For the present proof, however, it seems indispensable.

3. The Eta Invariant

As mentioned in the Introduction, a systematic functional analytical frame is missing for the eta invariant in contrast to the index and the spectral flow (see, however, [**CoMo95**] for an ambitious approach to establish analogues of Sobolev spaces, pseudo-differential operators, and zeta and eta functions in the context of noncommutative spectral geometry). Basically, however, the concept of the eta invariant of a (total and compatible, hence self-adjoint) Dirac operator is rather an immediate generalization of the index. Instead of measuring the chiral asymmetry of the zero eigenvalues we now measure the asymmetry of the *whole* spectrum.

Let $\mathcal{D}$ be an operator of Dirac type; i.e., roughly speaking, an operator with a real discrete spectrum which is nicely spaced without finite accumulation points and with an infinite number of eigenvalues on both sides of the real line. In close analogy with the definition of the zeta-function for essentially positive elliptic operators like the Laplacian, we set

$$\eta_{\mathcal{D}}(s) := \sum_{\lambda \in \operatorname{spec}(\mathcal{D}) \setminus \{0\}} \operatorname{sign}(\lambda)\, \lambda^{-s}\,.$$

Clearly, the formal sum $\eta_{\mathcal{D}}(s)$ is well defined for complex s with $\Re(s)$ sufficiently large, and it vanishes for a symmetric spectrum (i.e., if for each $\lambda \in \operatorname{spec}(\mathcal{D})$ also $-\lambda \in \operatorname{spec}(\mathcal{D})$).

In Subsections 2.2 and 2.4, we expressed the index by the difference of the traces of two related heat operators. Similarly, we also have a heat kernel expression for the eta function. Let $\mathcal{D}$ be any self-adjoint operator with compact resolvent and let $\{\lambda_k\}$ denote its eigenvalues ordered so that $\cdots \leq \lambda_{k-1} \leq \lambda_k \leq \lambda_{k+1} \leq \cdots$, each repeated according its multiplicity. Formally, we have

$$\begin{aligned}
(3.1) \qquad \eta_{\mathcal{D}}(s) \cdot \Gamma(\frac{s+1}{2}) &= \sum\nolimits_{\lambda_k \neq 0} \operatorname{sign}(\lambda_k) \cdot |\lambda_k|^{-s} \cdot \int_0^\infty r^{\frac{s-1}{2}} e^{-r}\, dr \\
&= \sum\nolimits_{\lambda_k \neq 0} \lambda_k (\lambda_k^2)^{-\frac{s+1}{2}} \int_0^\infty (t\lambda_k^2)^{\frac{s-1}{2}} e^{-t\lambda_k^2}\, d(t\lambda_k^2) \\
&= \sum\nolimits_{\lambda_k \neq 0} \int_0^\infty t^{\frac{s-1}{2}} \lambda_k e^{-t\lambda_k^2}\, dt = \int_0^\infty t^{\frac{s-1}{2}} \operatorname{Tr} e^{-t\mathcal{D}^2}\, dt.
\end{aligned}$$

For comparison we give the corresponding formula for the zeta function of the (positive) Dirac Laplacian $\mathcal{D}^2$:

$$(3.2) \qquad \zeta_{\mathcal{D}^2}(s) := \operatorname{Tr}(\mathcal{D}^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} e^{-t\mathcal{D}^2}\, dt.$$

For the zeta function, we must assume that $\mathcal{D}$ has no vanishing eigenvalues (i.e. $\mathcal{D}^2$ is positive). Otherwise the integral on the right side is divergent. (The situation, however, can be cured by subtracting the orthogonal projection onto the kernel of $\mathcal{D}^2$ from the heat operator before taking the trace.) For the eta function, on the contrary, it clearly does not matter whether there are 0–eigenvalues and whether the summation is over all or only over the nonvanishing eigenvalues.

The derivation of (3.1) and (3.2) is completely elementary for $\Re(s) > \frac{1+\dim X}{2}$, resp. $\Re(s) > \frac{\dim X}{2}$, where X denotes the underlying manifold. It follows at once that $\eta(s)$ (and $\zeta(s)$) admit a meromorphic extension to the whole complex plane. However, it is not clear at all how to characterize the operators for which the eta function (and the zeta function) have a finite value at $s = 0$, the *eta invariant* (resp. the *zeta invariant*).

Historically, the eta invariant appeared for the first time in the 1970s as an error term showing up in the index formula for the APS spectral boundary value problem of a Dirac operator $\mathcal{D}$ on a compact manifold X with smooth boundary Σ (see our Subsection 2.4). More precisely, what arose was the eta invariant of the tangential operator (i.e., the induced Dirac operator over the closed manifold Σ). Even in that case it was hard to establish the existence and finiteness of the eta invariant.

Strictly speaking, one can define the eta invariant as the constant term in the Laurent expansion of the eta function around the point $s = 0$. For various applications this suffices. Many practical calculations, however, are much facilitated when we know the regularity *a priori*.

Basically, there are three different approaches to establish it: the original proof by Atiyah, Patodi, and Singer, worked out in [**BoWo93**, Corollary 22.9] and summarized in our Subsection 2.4; it is based on an assumption about the existence of a suitable asymptotic expansion for the corresponding heat kernel on the infinite cylinder $\mathbb{R}_+ \times \Sigma$. An intrinsic proof can be found in Gilkey [**Gi95**, Section 3.8]. It does not exploit that Σ bounds X, but requires strong topological means.

For a compatible (!) Dirac operator over a closed manifold Bismut and Freed [**BiFr86**] have shown that the eta function is actually a holomorphic function of s for $\Re(s) > -2$. They used the heat kernel representation (3.1) which implies that the eta invariant, when it exists, can be expressed as

$$\eta_{\mathcal{D}}(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \cdot \operatorname{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2}\right) dt.$$

It follows from (3.1) that the estimate

$$|\operatorname{Tr} \mathcal{D}e^{-t\mathcal{D}^2}| < c\sqrt{t}$$

implies the regularity of the eta function at $s = 0$. In fact, Bismut and Freed proved a sharper result, using nontrivial results from stochastic analysis. Inspired by calculations presented in Bismut and Cheeger [**BiCh89**, Section 3], Wojciechowski gave a purely analytic reformulation of the details of their proof. This provides a third and completely elementary way of proving the regularity of the eta function at $s = 0$. The essential steps are:

THEOREM 3.1. *Let $\mathcal{D} : C^\infty(\Sigma; E) \to C^\infty(\Sigma; E)$ denote a compatible Dirac operator over a closed manifold Σ of odd dimension m. Let $\mathrm{e}(t; x, x')$ denote the integral kernel of the heat operator $\mathrm{e}^{-t\mathcal{D}^2}$. Then there exists a positive constant C such that*

$$|\operatorname{Tr}(\mathcal{D}_x e(t; x, x'))|_{x=x'}| < C\sqrt{t}$$

for all $x \in \Sigma$ and $0 < t < 1$.

We recall the definition of the 'local' η function.

DEFINITION 3.2. Let $\{f_k;\lambda_k\}_{k\in\mathbb{Z}}$ be a discrete spectral resolution of $\mathcal{D}$. Then we define

$$\begin{aligned}\eta_{\mathcal{D}}(s;x) &:= \sum_{\lambda_k\neq 0} \operatorname{sign}(\lambda_k)|\lambda_k|^{-s}\left\langle f_k(x), f_k(x)\right\rangle_{E_x}\\ &= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\left(\sum_{\lambda_k\neq 0}\lambda_k e^{-t\lambda_k^2}\left\langle f_k(x), f_k(x)\right\rangle\right)dt\\ &= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\operatorname{Tr}\mathcal{D}\,\mathrm{e}(t;x,x)dt.\end{aligned}$$

COROLLARY 3.3. *Under the assumptions of the preceding theorem the 'local' η function $\eta_{\mathcal{D}}(s;x)$ is holomorphic in the half plane $\Re(s)\geq -2$ for any $x\in\Sigma$.*

In the decade or so following 1975, it was generally believed that the existence of a finite eta invariant was a very special feature of operators of Dirac type on closed manifolds which are boundaries, and then, more generally, of Dirac type operators on all closed manifolds. Only after the seminal paper by Douglas and Wojciechowski [**DoWo91**] was it gradually realized that globally elliptic self-adjoint boundary value problems for operators of Dirac type also have a finite eta invariant. Once again, there are quite different approaches to obtain that result.

The work by Wojciechowski and collaborators is based on the Duhamel Principle and provides complete asymptotic expansions of the heat kernels for the self-adjoint Fredholm extension $\mathcal{D}_P$ where P belongs to the smooth self-adjoint Grassmannian. We recall from our review of the Atiyah–Patodi–Singer Index Theorem that the Duhamel Principle allows one to study the interior contribution and the boundary contribution separately and identify the singularities caused by the boundary contribution. It seems that Wojciechowski's method is only applicable if the metric structures close to the boundary are product.

Based on joint work with Seeley, Grubb [**Gr99**] also obtained nice asymptotic expansions of the trace of the heat kernels in (3.1) and (3.2). Contrary to the qualitative arguments of the Duhamel–Wojciechowski approach, Grubb's approach requires the explicit computation of certain coefficients in the expansion. Some of them have to vanish to guarantee the desired regularity.

For a slightly larger class of self-adjoint Fredholm extensions, Brüning and Lesch [**BrLe99a**] also studied the eta invariant. However, the authors had to deal with the residue of the eta function at $s=0$ which is not present in the Duhamel–Wojciechowski approach.

In spite of the differences between the various approaches and types of results it seems that one consequence can be drawn immediately, namely there must be a deeper meaning of the eta invariant beyond its role as error term in the APS index formula. Happily, such a meaning was found by Singer in [**Si85**] for the eta invariant on closed manifolds and successively generalized by Wojciechowski and collaborators for manifolds with boundary, namely the identification of the eta invariant as the phase of the zeta function regularized determinant. We shall explain this now, and postpone our main topic, the pasting formulas for the eta invariant on partitioned manifolds.

3.1. Functional Integrals and Spectral Asymmetry. Several important quantities in quantum mechanics and quantum field theory are expressed in terms of *quadratic functionals* and *functional integrals*. The concept of the determinant for Dirac operators arises naturally when one wants to evaluate the corresponding path integrals. As Itzykson and Zuber report in the chapter on *Functional Methods* of their monograph [**ItZu80**]: "The path integral formalism of Feynman and Kac provides a unified view of quantum mechanics, field theory, and statistical models. The original suggestion of an alternative presentation of quantum mechanical amplitudes in terms of path integrals stems from the work of Dirac (1933) and was brilliantly elaborated by Feynman in the 1940s. This work was first regarded with some suspicion due to the difficult mathematics required to give it a decent status. In the 1970s it has, however, proved to be the most flexible tool in suggesting new developments in field theory and therefore deserves a thorough presentation."

We shall restrict our discussion to the easiest variant of that complex matter by focusing on the partition function of a quadratic functional given by the Euclidean action of a *Dirac operator* which is assumed to be elliptic with imaginary time due to Wick rotation and coupled to continuously varying vector potentials (sources, fields, connections), for the ease of presentation in vacuum. We refer to Bertlmann, [**Be96**] and Schwarz, [**Sch93**] for an introduction to the quantum theoretic language for mathematicians and for a more extensive treatment of general aspects of quadratic functionals and functional integrals involving the relations to the Lagrangian and Hamiltonian formalism.

There are various alternative notions of "path integral" around, some more sophisticated than others. A mathematically rigorous formulation of the concept of "path integrals", as physicists typically "understand" it in quantum field theory is flimsy at best. For fields on Minkowski space, they are not mathematical integrals at all, because no measure is defined. For fields on Euclidean space (with imaginary time), one can construct genuine measures in limited settings which are not entirely realistic. Even then there is the issue of continuing the integrals back to real time which is generally ill-defined on a curved space-time. Many physicists do not care about such matters. Indeed, such physicists use path integrals primarily as compact generators of recipes to produce Feynman integrals which when regularized and renormalized yield coefficients in a formal power series in coupling constants for physical quantities of interest. However, no one has ever proved that these series converge, even for quantum electrodynamics (QED). Indeed the consensus of those who care is that these are only asymptotic series. Adding first few terms of these renormalized perturbation series yields remarkable 11-decimal point agreement with experiment, and QED is thereby hailed as a huge success. For many mathematicians and a few physicists, the great tragedy is that these recipes work so well without a genuine mathematical foundation. Although path integrals may not make precise sense per se, not only do they generate successful recipes in physics, but they can motivate fruitful ideas and precise concepts. In mathematics, path integrals have motivated the ζ-regularized determinant for the (Euclidean) Dirac operator, as a mathematically genuine canonical object, independent of particular choices made for regularization, which can be precisely calculated in principle.

A special feature of Dirac operators is that their determinants involve a *phase*, the imaginary part of the determinant's logarithm. As we will see now, this is a consequence of the fact that, unlike second-order semi-bounded Laplacians, first-order Dirac operators have an infinite number of both positive and negative eigenvalues. Then the phase of the determinant reflects the spectral asymmetry of the corresponding Dirac operator.

The simplest path integral we meet in quantum field theory takes the form of the *partition function* and can be written formally as the integral

$$Z(\beta) := \int_\Gamma e^{-\beta S(\omega)}\, d\omega\,, \tag{3.3}$$

where $d\omega$ denotes functional integration over the space $\Gamma := \Gamma(M;E)$ of sections of a Euclidean vector bundle E over a Riemannian manifold M.

In quantum theoretic language, M is space or space-time; a $\omega \in \Gamma$ is a position function of a particle or a spinor field. The scaling parameter β is a real or complex parameter, most often $\beta = 1$. The functional S is a quadratic real-valued functional on Γ defined by $S(\omega) := \langle \omega, T\omega\rangle$ with a fixed linear symmetric operator $T : \Gamma \to \Gamma$. Typically $T := \mathcal{D}$ is a Dirac operator and $S(\omega)$ is the action $S(\omega) = \int_M \langle \omega, \mathcal{D}\omega\rangle$.

Mathematically speaking, the integral (3.3) is an oscillating integral like the *Gaussian integral*. It is ill-defined in general because

(I) as it stands, it is meaningless when $\dim \Gamma(M;E) = +\infty$ (i.e., when $\dim M \geq 1$); and,
(II) even when $\dim \Gamma(M;E) < \infty$ (i.e. when $\dim M = 0$ and M consists of a finite set of points), the integral $Z(\beta)$ diverges unless $\beta S(\omega)$ is positive and nondegenerate.

Nevertheless, these expressions have been used and construed in quantum field theory. As a matter of fact, reconsidering the physicists' use and interpretation of these mathematically ill-defined quantities, one can describe certain formal manipulations which lead to normalizing and evaluating $Z(\beta)$ in a mathematically precise way.

We begin with a few calculations in *Case II*, inspired by Adams and Sen, [**AdSe96**], to show how *spectral asymmetry* is naturally entering into the calculations even in the finite-dimensional case and how this suggests a definition of the determinant in the infinite-dimensional case for the Dirac operator.

Then, let $\dim\Gamma = d < \infty$ and, for a symmetric endomorphism T, let

$$S(\omega) := \langle \omega, T\omega\rangle \ \text{ for all } \omega \in \Gamma.$$

Case 1. We assume that S positive and nondegenerate, i.e. T is strictly positive, say $\operatorname{spec} T = \{\lambda_1, \dots \lambda_d\}$ with all $\lambda_j > 0$. This is the classical case. We choose an orthonormal system of eigenvectors $(e_1, \dots, e_d)$ of T as basis for Γ. We have $S(\omega) = \sum \lambda_j x_j^2$ for $\omega = \sum x_j e_j$ and, for real $\beta > 0$, we get

$$\begin{aligned}
Z(\beta) &= \int_G e^{-\beta S(\omega)}\, d\omega = \int_{\mathbb{R}^d} dx_1 \dots dx_d\, e^{-\beta \sum \lambda_j x_j^2}\\
&= \int_{-\infty}^{\infty} e^{-\beta\lambda_1 x_1^2} dx_1 \int_{-\infty}^{\infty} e^{-\beta\lambda_2 x_2^2} dx_2 \cdots \int_{-\infty}^{\infty} e^{-\beta\lambda_d x_d^2} dx_d\\
&= \sqrt{\frac{\pi}{\beta\lambda_1}}\sqrt{\frac{\pi}{\beta\lambda_2}}\cdots\sqrt{\frac{\pi}{\beta\lambda_d}} = \pi^{d/2}\cdot\beta^{-d/2}\cdot(\det T)^{-1/2}\,.
\end{aligned}$$

In that way the determinant appears when evaluating the simplest quadratic integral.

Case 2. If the functional S is positive and degenerate, $T \geq 0$, the partition function is given by

$$Z(\beta) = \pi^{\zeta/2}\cdot\beta^{-\zeta/2}\cdot(\det \omega T)^{-1/2}\cdot \operatorname{Vol}(\operatorname{Ker} T),$$

where $\zeta := \dim\Gamma - \dim\operatorname{Ker} T$ and $\widetilde{T} := T|_{(\operatorname{Ker} T)^\perp}$, but, of course $\operatorname{Vol}(\operatorname{Ker} T) = \infty$. For approaches to "renormalizing" this quantity in quantum chromodynamics, we refer to [**AdSe96**], [**BMSW97**], [**Sch93**]. One approach customary in physics is to take $\pi^{\zeta/2}\beta^{-\zeta/2}(\det\widetilde{T})^{-1/2}$ as the definition of the integral by setting the factor $\operatorname{Vol}(\operatorname{Ker} T)$ equal to 1.

Case 3. Now we assume that the functional S is nondegenerate, i.e. T invertible, but S is neither positive nor negative. We decompose $\Gamma = \Gamma_+ \times \Gamma_-$ and $T = T_+ \oplus T_-$ with $T_+, -T_-$ strictly positive on $\Gamma_\pm$. Formally, we obtain

$$\begin{aligned}
Z(\beta) &= \left(\int_{\Gamma_+} d\omega_+ e^{-\beta\langle\omega_+, T_+\omega\rangle}\right)\left(\int_{\Gamma_-} d\omega_- e^{-(-\beta)\langle\omega_-, -T_-\omega\rangle}\right)\\
&= \pi^{d_+/2}\beta^{-d_+/2}(\det T_+)^{-1/2}\,\pi^{d_-/2}(-\beta)^{-d_-/2}(\det -T_-)^{-1/2}\\
&= \pi^{\zeta/2}\beta^{-d_+/2}(-\beta)^{-d_-/2}(\det|T|)^{-1/2}
\end{aligned}$$

where $d_\pm := \dim\Gamma_\pm$, hence $\zeta = d_+ + d_-$ and $|T| := \sqrt{\widetilde{T}^2} = T_+ \oplus -T_-$.

Case 4. In the preceding formula, the term $(\beta)^{-d_+/2}(-\beta)^{-d_-/2}$ is undefined for $\beta \in \mathbb{R}_\pm$. We shall replace it by a more intelligible term for $\beta = 1$ by first expanding $Z(\beta)$ in the upper complex half plane and then formally setting $\beta = 1$. More precisely, let $\beta \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \Im z > 0\}$ and write $\beta = |\beta| e^{i\theta}$ with $\theta \in [0,\pi]$, hence $-\beta = |\beta| e^{i(\theta-\pi)}$ with $\theta - \pi \in [-\pi, 0]$. We set $\beta^a := |\beta|^a e^{i\theta a}$ and get

$$\begin{aligned}
\beta^{-d_+/2}(-\beta)^{-d_-/2} &= (|\beta| e^{i\theta})^{-d_+/2}(|\beta| e^{i(\theta-\pi)})^{-d_-/2}\\
&= |\beta|^{-\zeta/2}\, e^{-i\frac{d_+}{2}\theta}\, e^{-i\frac{d_-}{2}\theta}\, e^{i\pi\frac{d_-}{2}}\,.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
-\frac{d_+}{2}\theta-\frac{d_-}{2}\theta+\pi\frac{d_-}{2} &= -\frac{\theta}{2}\left(d_++d_-\right)+\frac{\pi}{2}\left(\frac{d_-}{2}+\frac{d_+}{2}+\frac{d_-}{2}-\frac{d_+}{2}\right)\\
&= -\frac{\theta}{2}\zeta+\frac{\pi}{4}\left(\zeta-\eta\right)=-\frac{\pi}{4}\left(\frac{2\theta\zeta}{\pi}+(\eta-\zeta)\right),
\end{aligned}
$$

where $\zeta := d_+ + d_-$ is the finite-dimensional equivalent of the ζ invariant, counting the eigenvalues, and $\eta := d_+ - d_-$ the finite-dimensional equivalent of the η invariant, measuring the spectral asymmetry of T. We obtain

$$
Z(\beta)=\pi^{\zeta/2}\left|\beta\right|^{-\zeta/2}e^{-i\frac{\pi}{4}\left(\frac{2\zeta\theta}{\pi}+(\eta-\zeta)\right)}\left(\det|T|\right)^{-1/2}
$$

and, formally, for $\beta = 1$, i.e. $\theta = 0$,

$$
Z(1)=\pi^{\zeta/2}\underbrace{e^{-i\frac{\pi}{4}(\eta-\zeta)}\left(\det|T|\right)^{-1/2}}_{=:\det T}. \tag{3.4}
$$

Equation (3.4) suggests a nonstandard definition of the determinant for the infinite-dimensional case.

REMARK 3.4. (a) The methods and results of this section also apply to real-valued quadratic functionals on complex vector spaces. Since the integration in (3.3) in this case is over the real vector space underlying Γ, the expressions for the partition functions in this case become the square of those above.
(b) In the preceding calculations we worked with ordinary commuting numbers and functions. The resulting Gaussian integrals are also called *bosonic* integrals. If we consider *fermionic* integrals, we work with Grassmannian variables and obtain the determinant not in the denominator but in the nominator (see e.g. [**BeGeVe92**] or [**Be96**]).

3.2. The ζ–Determinant for Operators of Infinite Rank. Once again, our point of departure is finite-dimensional linear algebra. Let $T : \mathbb{C}^d \to \mathbb{C}^d$ be an invertible, positive operator with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_d$. We have the equality

$$
\begin{aligned}
\det T=\prod\lambda_j &= \exp\{\sum \ln\lambda_j e^{-s\ln\lambda_j}|_{s=0}\}\\
&= \exp(-\frac{d}{ds}(\sum\lambda_j^{-s})|_{s=0}) = e^{-\frac{d}{ds}\zeta_T(s)|_{s=0}},
\end{aligned}
$$

where $\zeta_T(s) := \sum_{j=1}^{d}\lambda_j^{-s}$.

We show that the preceding formula generalizes naturally, when T is replaced by a *positive definite self-adjoint elliptic* operator L (for the ease of presentation, of second order, like the Laplacian) acting on sections of a Hermitian vector bundle over a closed manifold M of dimension m. Then L has a discrete spectrum $\operatorname{spec} L = \{\lambda_j\}_{j\in\mathbb{N}}$ with $0 < \lambda_1 \leq \lambda_2 \leq \dots$, satisfying the asymptotic formula $\lambda_n \sim Cn^{m/2}$

for a constant $C > 0$ depending on L (see e.g. [**Gi95**], Lemma 1.6.3). We extend $\zeta_L(s) := \sum_{j=1}^{\infty} \lambda_j^{-s}$ in the complex plane by

$$\zeta_L(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\ \operatorname{Tr} e^{-tL}\, dt$$

with $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$. Note that e^{-tL} is the heat operator transforming any initial section f_0 into a section f_t satisfying the heat equation $\frac{\partial}{\partial t} f + Lf = 0$. Clearly $\operatorname{Tr} e^{-tL} = \sum e^{-t\lambda_j}$.

One shows that the original definition of $\zeta_L(s)$ yields a holomorphic function for $\Re(s)$ large and that its preceding extension is meromorphic in the entire complex plane with simple poles only. The point $s = 0$ is a regular point and $\zeta_L(s)$ is a holomorphic function at $s = 0$. From the asymptotic expansion of $\Gamma(s) \sim \frac{1}{s} + \gamma + sh(s)$ close to $s = 0$ with the Euler number γ and a suitable holomorphic function h we obtain an explicit formula

$$\zeta_L'(0) \sim \int_0^\infty \frac{1}{t} \operatorname{Tr} e^{-tL}\, dt - \gamma \zeta_L(0)\,.$$

This is explained in great detail, e.g., in [**Wo99**]. Therefore, Ray and Singer in [**RaSi71**] could introduce $\det_z(L)$ by defining:

$$\det{}_z L := e^{-\frac{d}{ds}\zeta_L(s)|_{s=0}} = e^{-\zeta_L'(0)}\,.$$

The preceding definition does not apply immediately to the main hero here, the Dirac operator $\mathcal{D}$ which has infinitely many positive λ_j and negative eigenvalues $-\mu_j$. Clearly by the preceding argument

$$\det{}_z \mathcal{D}^2 = e^{-\zeta'_{\mathcal{D}^2}} \qquad \text{and} \qquad \det{}_z |\mathcal{D}| = e^{-\zeta'_{|\mathcal{D}|}} = e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}}\,.$$

For the Dirac operator we set

$$\ln \det \mathcal{D} := -\tfrac{d}{ds}\zeta_{\mathcal{D}}(s)|_{s=0}$$

with, choosing [2] the branch $(-1)^{-s} = e^{i\pi s}$,

$$\begin{aligned}
\zeta_{\mathcal{D}}(s) &= \sum \lambda_j^{-s} + \sum (-1)^{-s} \mu_j^{-s} = \sum \lambda_j^{-s} + e^{i\pi s} \sum \mu_j^{-s} \\
&= \frac{\sum \lambda_j^{-s} + \sum \mu_j^{-s}}{2} + \frac{\sum \lambda_j^{-s} - \sum \mu_j^{-s}}{2} \\
&\qquad + e^{i\pi s} \left\{ \frac{\sum \lambda_j^{-s} + \sum \mu_j^{-s}}{2} - \frac{\sum \lambda_j^{-s} - \sum \mu_j^{-s}}{2} \right\} \\
&= \tfrac{1}{2} \left\{ \zeta_{\mathcal{D}^2}(\frac{s}{2}) + \eta_{\mathcal{D}}(s) \right\} + \tfrac{1}{2} e^{i\pi s} \left\{ \zeta_{\mathcal{D}^2}(\frac{s}{2}) - \eta_{\mathcal{D}}(s) \right\},
\end{aligned}$$

[2] Choosing the alternative representation, namely $(-1)^{-s} = e^{-i\pi s}$ yields the opposite sign of the phase of the determinant which may appear to be more natural for some quantum field models and also in view of (3.4). However, we follow the more common convention introduced by Singer in [Si85, p. 331] when defining the determinant of operators of Dirac type.

where $\eta_{\mathcal{D}}(s) := \sum \lambda_j^{-s} - \sum \mu_j^{-s}$. Later we will show that $\eta_{\mathcal{D}}(s)$ is a holomorphic function of s for $\Re(s)$ large with a meromorphic extension to the whole complex plane which is holomorphic in the neighborhood of $s = 0$. We obtain

$$\zeta'_{\mathcal{D}}(s) = \tfrac{1}{4}\zeta'_{\mathcal{D}^2}(\tfrac{s}{2}) + \tfrac{1}{2}\eta'_{\mathcal{D}}(s) + \tfrac{1}{2}i\pi e^{i\pi s}\{\zeta_{\mathcal{D}^2}(\tfrac{s}{2}) - \eta_{\mathcal{D}}(s)\} \\ + \tfrac{1}{2}e^{i\pi s}\{\tfrac{1}{2}\zeta'_{\mathcal{D}^2}(\tfrac{s}{2}) - \eta'_{\mathcal{D}}(s)\}.$$

It follows that

$$\zeta'_{\mathcal{D}}(0) = \tfrac{1}{2}\zeta'_{\mathcal{D}^2}(0) + \tfrac{i\pi}{2}\left\{\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0)\right\}$$

and

$$\begin{aligned}\det{}_z\mathcal{D} &= e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)}\, e^{-\frac{i\pi}{2}\left\{\zeta_{\mathcal{D}^2}(0)-\eta_{\mathcal{D}}(0)\right\}} = e^{-\frac{i\pi}{2}\left\{\zeta_{|\mathcal{D}|}(0)-\eta_{\mathcal{D}}(0)\right\}}\, e^{-\zeta'_{|\mathcal{D}|}(0)} \\ &= e^{-\frac{i\pi}{2}\left\{\zeta_{|\mathcal{D}|}(0)-\eta_{\mathcal{D}}(0)\right\}}\det{}_z|\mathcal{D}|\,.\end{aligned}$$

So, the Dirac operator's 'partition function' in the sense of (3.3) becomes

$$Z(1) = \pi^{\zeta_{|\mathcal{D}|}(0)}(\det{}_\zeta\mathcal{D})^{-\frac{1}{2}}\,.$$

3.3. Spectral Invariants of Different 'Sensitivity'. In the preceding formulas four spectral invariants of the Dirac operator $\mathcal{D}$ enter:

3.3.1. *The Index.* First recall that the index of arbitrary elliptic operators on closed manifolds and the spectral flow of 1-parameter families of self-adjoint elliptic operators are topological invariants and so stable under small variation of the coefficients and, by definition, solely depending on the multiplicity of the eigenvalue 0. In the theory of bounded or closed (not necessarily bounded) Fredholm operators and bounded or not necessarily bounded self-adjoint Fredholm operators and the related K and K^{-1} theory, we have a powerful functional analytical and topological frame for discussing these invariants. Moreover, index and spectral flow are local invariants, i.e., can be expressed by an integral where the integrand is locally expressed by the coefficients of the operator(s). Consequently, we have simple, precise pasting formulas for index and spectral flow on partitioned manifolds where the error term is localized along the separating hypersurface.

On manifolds with boundary the Calderón projection and its range, the Cauchy data space, change continuously when we vary the Dirac operator as shown in Section 2.3.8, exploiting the unique continuation property of operators of Dirac type. The same is not true for the Atiyah–Patodi–Singer projection: it can jump from one connected component of the Grassmannian to another component under small changes of the Dirac operator. Correspondingly, the index of the APS boundary problem can jump under small variation of the coefficients (i.e., of the defining connection or the underlying Riemannian or Clifford structure).

Regarding parity of the manifold, the index density (of all elliptic differential operators) vanishes on odd-dimensional manifolds for symmetry reason. Then, in the closed case the index vanishes, and on manifolds with boundary the APS Index Theorem takes the simple form $\operatorname{index}\mathcal{D}_{P_{\geq}} = -\dim\operatorname{Ker}B^{+}$ where we have the total Dirac operator on the left and a chiral component of the induced tangential operator on the right. Once again, the formula shows the instability of the index under small changes of the Dirac operator. This is no contradiction to the stability of the index on the spaces $\mathcal{F}$, respectively $\mathcal{CF}$, discussed in Section 1.1.3 because the graph norm distance between two APS realizations $\mathcal{D}_{P_\geq}$ and $\mathcal{D}'_{P_\geq}$ can remain bounded away from zero when $\mathcal{D}$ runs to $\mathcal{D}'$. This is the case if and only if the dimension of the kernel of the tangential operator changes under the deformation.

3.3.2. *The ζ–invariant.* Similarly, $\zeta_{\mathcal{D}^2}(0)$ is also local; i.e., it is given by the integral $\int_M \alpha(x)dx$, where $\alpha(x)$ denotes a certain coefficient in the heat kernel expansion and is locally expressed by the coefficients of $\mathcal{D}$. In particular, $\zeta_{\mathcal{D}^2}(0)$ remains unchanged for small changes of the spectrum. Actually, $\zeta_L(0)$ vanishes when L is the square of a self-adjoint elliptic operator on a closed manifold of odd dimension. It can be defined (and it vanishes, see [**PaWo02a**, Appendix]) for a large class of squares of operators of Dirac type with globally elliptic boundary conditions on compact, smooth manifolds (of odd dimension) with boundary. So, there are no nontrivial pasting formulas at all in such cases.

3.3.3. *The η–invariant.* Unlike the index and spectral flow on closed manifolds, we have neither an established functional analytical nor a topological frame for discussing $\eta_{\mathcal{D}}(0)$, nor is it given by an integral of a locally defined expression. On the circle, e.g., consider the operator

$$\mathcal{D}_a := -i\frac{d}{dx} + a = e^{-ixa}\mathcal{D}_0 e^{ixa}, \tag{3.5}$$

so that $\mathcal{D}_a$ and $\mathcal{D}_0$ have the same total symbol (i.e., coincide locally), but $\eta_{\mathcal{D}_a}(0) = -2a$ depends on a.

The η–invariant depends, however, only on finitely many terms of the symbol of the resolvent $(\mathcal{D}-\lambda)^{-1}$ and the real part (in $\mathbb{R}/\mathbb{Z}$) will not change when one changes or removes a finite number of eigenvalues. The integer part changes according to the net sign change occurring under removing or modifying eigenvalues. Moreover, the first derivative of the eta invariant of a smooth 1-parameter family of Dirac type operators is local, namely the spectral flow, as noted in our Introduction. This leads again to precise (though not so simple) pasting formulas for the eta invariant on partitioned manifolds.

In even dimensions, the eta invariant vanishes on any closed manifold Σ for any Dirac type operator which is the tangential operator of a Dirac type operator on a suitable manifold which has Σ as its boundary because of the induced precise symmetry of the spectrum due to the anti-commutativity of the tangential operator with Clifford multiplication. For the study of eta of boundary value problems on even-dimensional manifolds see [**KlWo96**].

3.3.4. *The Modulus of the Determinant.* The number $\zeta'_{\mathcal{D}^2}(0)$ is the most delicate of the invariants involved: It is neither a local invariant, nor does it depend only on the total symbol of the Dirac operator. Even small changes of the eigenvalues will change the ζ' invariant and hence the determinant. Moreover, no precise pasting formulas are obtained but only adiabatic ones (i.e., by inserting a long cylinder around the separating hypersurface (see [**PaWo02a**], [**PaWo02b**], [**PaWo02c**]).

Without proof we present the main results by Wojciechowski and collaborators, based on [**Wo99**] where the ζ–function regularized determinant was established for pseudo-differential boundary value conditions belonging to the smooth, self-adjoint Grassmannian. The first is a boundary correction formula, proved in [**ScWo00**] (see also the recent [**Sco02**]):

THEOREM 3.5. *(Scott, Wojciechowski). Let $\mathcal{D}$ be a Dirac operator over an odd–dimensional compact manifold M with boundary Σ and let $P \in \mathcal{G}\mathrm{r}^{\mathrm{sa}}(\mathcal{D})$. Then the range of the Calderón projection $\mathcal{P}(\mathcal{D})$ (the Cauchy data space $\Lambda(\mathcal{D}, \frac{1}{2})$) and the range of P can be written as the graphs of unitary, elliptic operators of order 0, K, resp. T which differ from the operator $(B^+B^-)^{-1/2}B^+ : C^\infty(\Sigma; S^+|_\Sigma) \to C^\infty(\Sigma; S^-|_\Sigma)$ by a smoothing operator. Moreover,*

$$\det{}_\zeta \mathcal{D}_P = \det{}_\zeta \mathcal{D}_{\mathcal{P}(\mathcal{D})} \cdot \det{}_{\mathrm{Fr}} \frac{1}{2}(\mathrm{I} + KT^{-1}) . \tag{3.6}$$

The second result, in most simple form, is found in [**PaWo00**]:

THEOREM 3.6. *(Park, Wojciechowski) Let $R \in \mathbb{R}$ be positive, let M^R denote the stretched partitioned manifold $M^R = M_1 \cup_\Sigma [-R, 0] \times \Sigma \cup_\Sigma [0, R] \times \Sigma \cup_\Sigma M_2$, and let $\mathcal{D}_R$, $\mathcal{D}_{1,R}$, $\mathcal{D}_{2,R}$ denote the corresponding Dirac operators. We assume that the tangential operator $\mathcal{B}$ is invertible. Then*

$$\lim_{R\to\infty} \frac{\det{}_\zeta \mathcal{D}_R^2}{\Big(\det{}_\zeta (\mathcal{D}_{1,R})^2_{\mathrm{I}-P_>}\Big) \cdot \Big(\det{}_\zeta (\mathcal{D}_{2,R})^2_{P_>}\Big)} \;=\; 2^{-\zeta_{\mathcal{B}^2}(0)} .$$

Although Felix Klein in [**Kl27**] rated the determinant as *simplest example of an invariant*, today we must give an inverse rating. For the present authors, it is not the invariants that are stable under the largest transformation groups which deserve the highest interest, but rather (according to Dirac's approach to elementary particle physics) the *finest* invariants which exhibit *anomalies* under small perturbations. Correspondingly, the determinant and its amplitude are the most subtle and the most fascinating objects of our study. They are much more difficult to comprehend than the η-invariant 3.3.2; and the η-invariant is much more difficult to comprehend than the index.

3.4. Pasting Formulas for the Eta–Invariant - Outlines. In the rest of this review, i.e. over the next 33 pages, we shall prove a strikingly simple (to state) additivity property of the η-invariant. We fix the assumptions and the notation.

3.4.1. *Assumptions and Notation.* (a) Let M be an odd–dimensional closed *partitioned* Riemannian manifold $M = M_1 \cup_\Sigma M_2$ with M_1, M_2 compact manifolds with common boundary Σ. Let S be a bundle of Clifford modules over M.
(b) To begin with we assume that $\mathcal{D}$ is a *compatible* (= true) Dirac operator over M. Thus, in particular, $\mathcal{D}$ is symmetric and has a unique self-adjoint extension in $L^2(M; S)$.

(c) We assume that there exists a bicollared cylindrical neighborhood (*a neck*) $N \simeq (-1,1) \times \Sigma$ of the separating hypersurface Σ, such that the Riemannian structure on M and the Hermitian structure on S are product in N; i.e., they do not depend on the normal coordinate u, when restricted to $\Sigma_u = \{u\} \times \Sigma$. Our convention for the orientation of the coordinate u is that it runs from M_1 to M_2; i.e., $M_1 \cap N = (-1,0] \times \Sigma$ and $N \cap M_2 = [0,1) \times \Sigma$. Then the operator $\mathcal{D}$ takes the following form on N:

$$\mathcal{D}|_N = \sigma(\partial_u + B), \tag{3.7}$$

where the principal symbol in u–direction $\sigma : S|_\Sigma \to S|_\Sigma$ is a unitary bundle isomorphism (Clifford multiplication by the normal vector du) and the tangential operator $B : C^\infty(\Sigma; S|_\Sigma) \to C^\infty(\Sigma; S|_\Sigma)$ is the corresponding Dirac operator on Σ. Note that σ and B do not depend on the normal coordinate u in N and they satisfy the following identities

$$\sigma^2 = -\mathrm{I}\ ,\ \sigma^* = -\sigma\ ,\ \sigma \cdot B = -B \cdot \sigma\ ,\ B^* = B. \tag{3.8}$$

Hence, σ is a skew-adjoint involution and S, the bundle of spinors, decomposes in N into $\pm i$–eigenspaces of σ, $S|_N = S^+ \oplus S^-$. It follows that (3.7) leads to the following representation of the operator $\mathcal{D}$ in N

$$\mathcal{D}|_N = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \left(\partial_u + \begin{pmatrix} 0 & B_- = B_+^* \\ B_+ & 0 \end{pmatrix} \right),$$

where $B_+ : C^\infty(\Sigma; S^+) \to C^\infty(\Sigma; S^-)$ maps the spinors of positive chirality into the spinors of negative chirality.

(d) To begin with we consider only the case of $\operatorname{Ker} B = \{0\}$. That implies that B is an *invertible* operator. More precisely, there exists a pseudo-differential elliptic operator L of order -1 such that $BL = \mathrm{I}_S = LB$ (see, for instance, [**BoWo93**], Proposition 9.5).

(e) For real $R > 0$ we study the closed *stretched* manifold M^R which we obtain from M by inserting a cylinder of length $2R$, i.e. replacing the collar N by the cylinder $(-2R-1, +1) \times \Sigma$

$$M^R = M_1 \cup ([-2R, 0] \times \Sigma) \cup M_2\,.$$

We extend the bundle S to the stretched manifold M^R in a natural way. The extended bundle will be also denoted by S. The Riemannian structure on M and the Hermitian structure on S are product on N. Hence we can extend them to smooth metrics on M^R in a natural way and, at the end, we can extend the operator $\mathcal{D}$ to an operator $\mathcal{D}^R$ on M^R by using formula (3.7). Then M^R splits into two manifolds with boundary: $M^R = M_1^R \cup M_2^R$ with $M_1^R = M_1 \cup ((-2R, -R] \times \Sigma)$, $M_2^R = ([-R, 0) \times \Sigma) \cup M_2$, and $\partial M_1 = \partial M_2^R = \{-R\} \times \Sigma$. Consequently, the operator $\mathcal{D}^R$ splits into $\mathcal{D}^R = \mathcal{D}_1^R \cup \mathcal{D}_2^R$. We shall impose spectral boundary conditions to obtain self-adjoint operators $\mathcal{D}_{1,P_<}$, $\mathcal{D}_{1,P_<}^R$, $\mathcal{D}_{2,P_>}$, and $\mathcal{D}_{2,P_>}^R$ in the corresponding L^2 spaces on the parts (see (3.9)).

(f) We also introduce the complete, noncompact Riemannian manifold *with cylindrical end*

$$M_2^\infty := ((-\infty, 0] \times \Sigma) \cup M_2$$

by gluing the half–cylinder $(-\infty, 0] \times \Sigma$ to the boundary Σ of M_2. Clearly, the Dirac operator $\mathcal{D}$ extends also to $C^\infty(M_2^\infty, S)$.

REMARK 3.7. Our presentation is somewhat simplified by our assumption (b) that $\mathcal{D}$ is compatible and assumption (d) that the tangential operator B is invertible. Both assumptions can be lifted. This is done in the literature; see [**Wo95**] and [**Wo99**].

We recall the following ideas in the big scheme from Section 2.5 of this review. Let $P_>$ (respectively $P_<$) denote the spectral projection of B onto the subspace of $L^2(\Sigma; S|_\Sigma)$ spanned by the eigensections corresponding to the positive (respectively negative) eigenvalues. Then $P_>$ is a self-adjoint elliptic boundary condition for the operator $\mathcal{D}_2 = \mathcal{D}|_{M_2}$ (see [**BoWo93**], Proposition 20.3). This means that the operator $\mathcal{D}_{2,P_>}$ defined by

$$\begin{aligned} \mathcal{D}_{2,P_>} &= \mathcal{D}|_{M_2} \\ \mathrm{Dom}(\mathcal{D}_{2,P_>}) &= \{s \in H^1(M_2; S|_{M_2}) \mid P_>(s|_\Sigma) = 0\} \end{aligned} \tag{3.9}$$

is an unbounded self-adjoint operator in $L^2(M_2; S|_{M_2})$ with compact resolvent. In particular,

$$\mathcal{D}_{2,P_>} : \mathrm{Dom}(\mathcal{D}_{2,P_>}) \to L^2(M_2; S|_{M_2})$$

is a Fredholm operator with discrete real spectrum and the kernel of $\mathcal{D}_{2,P_>}$ consists of smooth sections of $S|_{M_2}$.

As mentioned before, the eta function of $\mathcal{D}_{2,P_>}$ is well defined and enjoys all properties of the eta function of the Dirac operator defined on a closed manifold. In particular, $\eta_{\mathcal{D}_{2,P_>}}(0)$, the eta invariant of $\mathcal{D}_{2,P_>}$, is well defined. Similarly, $P_<$ is a self-adjoint boundary condition for the operator $\mathcal{D}|_{M_1}$, and we define the operator $\mathcal{D}_{1,P_<}$ using a formula corresponding to (3.9). *To keep track of the various manifolds, operators, and integral kernels we refer to the following table where we have collected the major notations.*

manifolds	operators	integral kernels
$M = M_1 \cup_\Sigma M_2$	$e^{-t\mathcal{D}^2},\ \mathcal{D}e^{-t\mathcal{D}^2}$	$\mathcal{E}(t; x, x')$
$M^R = M_1^R \cup_\Sigma M_2^R$	$e^{-t(\mathcal{D}^R)^2},\ \mathcal{D}^R e^{-t(\mathcal{D}^R)^2}$	$\mathcal{E}^R(t; x, x')$
M_2	$e^{-t\mathcal{D}^2_{2,P_>}},\ \mathcal{D}_2 e^{-t\mathcal{D}^2_{2,P_>}}$	$\mathcal{E}_2(t; x, x')$
$M_2^R = ([-R, 0] \times \Sigma) \cup M_2$	$e^{-t(\mathcal{D}^R_{2,P_>})^2},\ \mathcal{D}_2^R e^{-t(\mathcal{D}^R_{2,P_>})^2}$	$\mathcal{E}_2^R(t; x, x')$
$M_2^\infty = ((-\infty, 0] \times \Sigma) \cup M_2$	$e^{-t(\mathcal{D}_2^\infty)^2},\ \mathcal{D}_2^\infty e^{-t(\mathcal{D}_2^\infty)^2}$	$\mathcal{E}_2^\infty(t; x, x')$
$\Sigma^\infty_{\mathrm{cyl}} = (-\infty, +\infty) \times \Sigma$	$e^{-tD^2_{\mathrm{cyl}}},\ D_{\mathrm{cyl}} e^{-tD^2_{\mathrm{cyl}}}$	$\mathcal{E}_{\mathrm{cyl}}(t; x, x')$
$\Sigma^\infty_{\mathrm{cyl}/2} = [0, +\infty) \times \Sigma$	$e^{-tD^2_{\mathrm{aps}}},\ D_{\mathrm{aps}} e^{-tD^2_{\mathrm{aps}}}$	$\mathcal{E}_{\mathrm{aps}}(t; x, x')$

In addition, on M_2^R we have the operator $Q_2^R(t)$ with kernel $Q_2^R(t;x,x')$ and

$$C^R(t) = \big((\mathcal{D}^R_{2,P_>})^2 + \tfrac{d}{dt}\big)\, Q_2^R(t) \text{ with kernel } C^R(t;x,x').$$

3.4.2. *The Gluing Formulas.* The most basic results for pasting η are the following theorem on the adiabatic limits of the η invariants and its additivity corollary:

THEOREM 3.8. *Attaching a cylinder of length $R>0$ at the boundary of the manifold M_2, we can approximate the eta invariant of the spectral boundary condition on the prolonged manifold M_2^R by the corresponding integral of the 'local' eta function of the closed stretched manifold M^R:*

$$\lim_{R\to\infty}\left\{\eta_{\mathcal{D}^R_{2,P_>}}(0) - \int_{M_2^R} \eta_{\mathcal{D}^R}(0;x)\,dx\right\} \equiv 0 \bmod \mathbb{Z}.$$

COROLLARY 3.9. $\eta_{\mathcal{D}}(0) \equiv \eta_{\mathcal{D}_{1,P_<}}(0) + \eta_{\mathcal{D}_{2,P_>}}(0) \bmod \mathbb{Z}$.

REMARK 3.10. (a) With hindsight, it is not surprising that modulo the integers the preceding additivity formula for the η-invariant on a partitioned manifold is precise. An intuitive argument runs as follows. "Almost all" eigensections and eigenvalues of the operator $\mathcal{D}$ on the closed partitioned manifold $M = M_1 \cup M_2$ can be traced back, either to eigensections $\psi_{1,k}$ and eigenvalues $\mu_{1,k}$ of the spectral boundary problem $\mathcal{D}_{1,P_<}$ on the part M_1, or to eigensections $\psi_{2,\ell}$ and eigenvalues $\mu_{2,\ell}$ of the spectral boundary problem $\mathcal{D}_{2,P_>}$ on the part M_2. While we have no explicit exact correspondence, due to the product form of the Dirac operator in a neighborhood of the separating hypersurface, eigensections on one part M_1 or M_2 of the manifold M can be extended to smooth sections on the whole of M. These are not true eigensections of $\mathcal{D}$, but they have a relative error which is rapidly decreasing as $R\to\infty$ when we attach cylinders of length R to the part manifolds or, equivalently, insert a cylinder of length $2R$ in M. There is also a residual set $\{\mu_{0,j}\}$ of eigenvalues of $\mathcal{D}$ which can neither be traced back to eigenvalues of $\mathcal{D}_{1,P_<}$ nor to those of $\mathcal{D}_{2,P_>}$. These eigenvalues can, however, be traced back to the kernel of the Dirac operators $\mathcal{D}_1^\infty$ and $\mathcal{D}_2^\infty$ on the part manifolds M_1^∞ and M_2^∞ with cylindrical ends. Because of Fredholm properties the residual set is finite and, hence (as noticed in Section 3.3) can be discarded for calculating the eta invariant modulo $\mathbb{Z}$.
Therefore, no R (i.e., no prolongation of the bicollar neighborhood N) enters the *formula*. Nevertheless, our *arguments* rely on an adiabatic argument to separate the spectrum of $\mathcal{D}$ into its three parts

$$\operatorname{spec}\mathcal{D} \sim \{\mu_{0,j}\} \cup \{\mu_{1,k}\} \cup \{\mu_{2,\ell}\}. \tag{3.10}$$

For the most part, however, we need not make all arguments explicit on the level of the single eigenvalue. It suffices to work on the level of the eta invariant for the following reason. Unlike the index, the eta invariant cannot be described by a local formula, as explained in Section 3.3. Nevertheless, it can be described by an integral over the manifold. The integrand, however, is not defined in local terms solely. In particular, when writing the eta function in integral form and decomposing the η integral

$$\eta_{\mathcal{D}}(s) = \int_M \eta_{\mathcal{D}}(s; x, x)\, dx = \int_{M_1} \eta_{\mathcal{D}}(s; x_1, x_1)\, dx_1 + \int_{M_2} \eta_{\mathcal{D}}(s; x_2, x_2)\, dx_2$$

there is no geometrical interpretation of the integrals on the right over the two parts of the manifold. This is very unfortunate. But for sufficiently large R, the integrals become intelligible and can be read as the η invariants of $\mathcal{D}^R_{1,P_<}$ and $\mathcal{D}^R_{2,P_>}$. That is the meaning of the adiabatic limit.

(b) Theorem 3.40 can be generalized to larger classes of boundary conditions by variational argument (see [**LeWo96**]) yielding the general gluing formula (in $\mathbb{R}/\mathbb{Z}$)

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1,P_1}}(0) + \eta_{\mathcal{D}_{2,P_2}}(0) + \eta^N_{P_1, \mathrm{I} - P_2}(0),$$

where $\eta^N_{P_1,\mathrm{I}-P_2}(0)$ denotes the η–invariant on the cylinder N, see also [**BrLe99**], [**DaFr94**], [**MaMe95**], and the recent review [**PaWo02e**]. G. Grubb's new result of [**Gr02**], mentioned in our Introduction, may open an alternative route for proving the general gluing formula. The integer jump was calculated in [**KiLe00**] yielding (among other formulas)

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1,\mathrm{I}-P}}(0) + \eta_{\mathcal{D}_{2,P}}(0) + 2\,\mathrm{SF}\{\mathcal{D}_{1,\mathrm{I}-P_t}\} + 2\,\mathrm{SF}\{\mathcal{D}_{2,P_t}\},$$

where $\{P_t\}$ is a smooth curve in the Grassmannian from P to the Calderón projection $\mathcal{P}(\mathcal{D}_2)$.

(c) An interesting feature of [**KiLe00**] is that the pasting formula is derived from the Scott–Wojciechowski Comparison Formula (Theorem 3.5). This is quite analogous to the existence of two completely different proofs of the pasting formula for the index (Theorem 2.35), where also the derivation from the boundary reduction formula is much, much shorter than arguing via the Atiyah-Singer Index Theorem and the Atiyah-Patodi-Singer Index Theorem plus the Agranovic-Dynin Formula. However, for this review we prefer the long way because of the many interesting insights about the gluing on the eigenvalue level which can be gained.

3.4.3. *Plan of the Proof.* Let $\mathcal{E}^R_2(t)$ denote the integral kernel of the operator $\mathcal{D}^R_2 e^{-t(\mathcal{D}^R_{2,P_>})^2}$ defined on the manifold $M^R_2 = ([-R, 0] \times \Sigma) \cup M_2$. Without proof, we mentioned before that the eta invariant of the self-adjoint operator $\mathcal{D}^R_{2,P_>}$ is well defined and we have

$$\eta_{\mathcal{D}^R_{2,P_>}}(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dt}{\sqrt{t}} \int_{M^R_2} \mathrm{Tr}\, \mathcal{E}^R_2(t; x, x)\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M^R_2} \mathrm{Tr}\, \mathcal{E}^R_2(t; x, x)\, dx \tag{3.11}$$

$$+ \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^\infty \frac{dt}{\sqrt{t}} \int_{M^R_2} \mathrm{Tr}\, \mathcal{E}^R_2(t; x, x)\, dx. \tag{3.12}$$

We first deal with the integral of (3.11) and show that it splits into an interior contribution and a cylinder contribution as $R \to \infty$. This will be done in Subsection 3.5 by first in paragraph 3.5.1 applying the Duhamel method which we introduced before in the proof of Theorem 2.30 (pp. 63ff). By Lemma 3.11, Lemma 3.12, Proposition 3.13, and Corollary 3.14, we can replace the heat kernel $\mathcal{E}_2^R$ of the Atiyah-Patodi-Singer boundary problem on the prolonged manifold M_2^R with boundary Σ by an artificially glued integral kernel Q_2^R which, near the boundary, is equal to the heat kernel of the APS problem on the half-infinite cylinder and in the interior equal to the heat kernel on the stretched closed manifold. We can do it in such a way that the original small-t integral (i.e., the integral from 0 to $\sqrt{R}$ in (3.11)) can be approximated by the new integral up to an error of order $O(e^{-cR})$. Then we shall show in Lemma 3.15 and Lemma 3.16 (paragraph 3.5.2) that the cylinder contribution is traceless. More precisely, we obtain that the trace $\operatorname{Tr} Q_2^R(T;x,x)$ can be replaced pointwise (for $x \in M_2^R$) by the trace $\operatorname{Tr} \mathcal{E}^R(t;x,x)$ of the integral kernel of the operator $\mathcal{D}^R e^{-t(\mathcal{D}^R)^2}$ which is defined on the stretched manifold M^R. Consequently, the small-t *chopped* η-invariant of the closed stretched manifold M^R coincides with the sum of the small-t chopped η-invariants of the APS problems on the two prolonged manifolds M_1^R and M_2^R up to $O\left(e^{-cR}\right)$.

Then we will show that the integral of (3.12) vanishes as $R \to \infty$. This is in Lemma 3.28 (p. 114f) a direct consequence of Theorem 3.17 which states that the eigenvalues of $\mathcal{D}^R_{2,P_>}$ are uniformly bounded away from 0. Theorem 3.17 is of independent interest. The proof is a long story stretching over Subsections 3.6 and 3.7. First, in paragraph 3.6.1 we shall consider the operator D_{cyl} on the infinite cylinder $\Sigma^\infty_{\mathrm{cyl}}$ in Definition 3.19 and Lemma 3.20. We obtain that D_{cyl} has no eigenvalues in the interval $(-\lambda_1, \lambda_1)$ where λ_1 denotes the smallest positive eigenvalue of the tangential operator B on the manifold Σ. Next, in paragraph 3.6.2 we investigate the operator $\mathcal{D}_2^\infty$ on the manifold M_2^∞ with infinite cylindrical end in Lemma 3.21, Lemma 3.22, Lemma 3.23, and Proposition 3.24. Proposition 3.24 states that $\mathcal{D}_2^\infty$ has only finitely many eigenvalues in the aforementioned interval $(-\lambda_1, \lambda_1)$, each of finite multiplicity. Its proof is somewhat delicate and involves Lemma 3.25 and Corollary 3.26. Then, in Subsection 3.7 the proof of Theorem 3.17 follows with Lemma 3.27.

By then we will have that the sum of the η-invariants of the APS boundary problems on the prolonged manifolds M_1^R and M_2^R can be replaced by the small-t chopped η-invariant on the stretched closed manifold M^R with an error exponentially vanishing as $R \to \infty$. We then would like to repeat the preceding chain of arguments around Theorem 3.17 to show that the large-t chopped η-invariant of the operator $\mathcal{D}^R$ on M^R also vanishes. However, this can be done only in $\mathbb{R}/\mathbb{Z}$ because in general the eigenvalues of $\mathcal{D}^R$ are not bounded away from 0. So we shall present a different chain of arguments in Subsection 3.8. The main technical result is Theorem 3.29, once again of independent interest. It describes a partition of the eigenvalues into two subsets, the exponentially small ones and the eigenvalues bounded away from 0. The key for our arguments is a gluing construction (Definition 3.30). Then we first show by Lemma 3.31, Lemma 3.32 and Proposition 3.33 that $\mathcal{D}^R$ has at least $q := \dim \operatorname{Ker} \mathcal{D}_1^\infty + \dim \operatorname{Ker} \mathcal{D}_2^\infty$ exponentially small eigenvalues belonging to eigensections which we can approximate by pasting together L^2 solutions. Then we shall show in Lemma 3.34, Theorem 3.35, Lemma 3.36 and Proposition 3.37 that this makes the list of eigenvalues approaching 0 as $R \to \infty$ complete.

It follows in Lemma 3.38 (at the beginning of the closing Subsection 3.7) that the large-t chopped η-invariant on M^R vanishes asymptotically up to an integer error. This establishes Theorem 3.8. To arrive at Corollary 3.9, we must get rid of the adiabatic limit. This is a simple consequence of the locality of the derivative in R-direction of the η-invariants on the stretched part manifolds with APS boundary condition and on the stretched closed manifold (Proposition 3.39).

3.5. The Adiabatic Additivity of the Small-t Chopped η-invariant.

3.5.1. *Applying Duhamel's Method to the Small-t Chopped η-invariant.* The simplest construction of a parametrix for $\mathcal{E}_2^R(t)$ (i.e., of an approximate heat kernel) is the following: we glue the kernel $\mathcal{E}$ of the operator $\mathcal{D}e^{-t\mathcal{D}^2}$ (given on the whole, closed manifold M) and the kernel $\mathcal{E}_{\text{aps}}^\infty$ of the L^2 extension of the operator $\sigma(\partial_u + B)e^{-t(\sigma(\partial_u+B))^2}$, given on the semi-infinite cylinder $[-R,\infty)\times\Sigma$ and subject to the Atiyah–Patodi–Singer boundary condition at the end $u=-R$. In that construction the gluing happens on the neck $N=[0,1)\times\Sigma$ with suitable cutoff functions.

Locally, the heat kernel is always of the form $(4\pi t)^{-m/2}e^{c_1 t}e^{-|x-x'|^2/4t}$. By Duhamel's Principle we get after gluing a similar global result for the kernel $\mathrm{e}_2^R(t;x,x')$ of the operator $e^{-t(\mathcal{D}^R_{2,P_>})^2}$ and, putting a factor $t^{-1/2}$ in front, for the kernel of the combined operator $\mathcal{D}e^{-t(\mathcal{D}^R_{2,P_>})^2}$ (e.g., see Gilkey [**Gi95**], Lemma 1.9.1). That yields two crucial estimates:

LEMMA 3.11. *There exist positive reals c_1, c_2, and c_3 which do not depend on R, such that for all $x,x'\in M_2^R$ and any $t>0$ and $R>0$,*

$$|\mathrm{e}_2^R(t;x,x')| \le c_1 \cdot t^{-\frac{m}{2}} \cdot e^{c_2 t} \cdot e^{-c_3\frac{d^2(x,x')}{t}}, \tag{3.13}$$

$$|\mathcal{E}_2^R(t;x,x')| \le c_1 t^{-\frac{1+m}{2}} \cdot e^{c_2 t} \cdot e^{-c_3\frac{d^2(x,x')}{t}}. \tag{3.14}$$

Here $d(x,x')$ denotes the geodesic distance.

Notice that exactly the same type of estimate is also valid for the kernel $\mathcal{E}^R(t;x,x')$ on the stretched closed manifold M^R and for the kernel $\mathcal{E}_{\text{aps}}^\infty(t;x,x')$ on the infinite cylinder. For details see also [**BoWo93**], Theorem 22.14. There, however, the term $e^{c_2 t}$ was suppressed in the final formula because the emphasis was on small time asymptotics.

As mentioned before, as $R\to\infty$, we want to separate the contribution to the kernel $\mathcal{E}_2^R$ which comes from the cylinder and the contribution from the interior by a gluing process. Unfortunately, the inequality 3.14 does not suffice to show that the contribution to the eta invariant, more precisely to the integral (3.11), which comes from the 'error' term vanishes with $R \to \infty$. Therefore, we introduce a different parametrix for the kernel $\mathcal{E}_2^R$.

Instead of gluing over the fixed neck $N = [0,1) \times \Sigma$, we glue over a segment N^R of growing length of the attached cylinder, say $N^R := (-\frac{4}{7}R, -\frac{3}{7}R) \times \Sigma$ (the reason for choosing these ratios will be clear soon). Thus, we choose a smooth partition of unity $\{\chi_{\mathrm{aps}}, \chi_{\mathrm{int}}\}$ on M_2^R suitable for the covering $\{U_{\mathrm{aps}}, U_{\mathrm{int}}\}$ with $U_{\mathrm{aps}} := [-R, -\frac{3}{7}R) \times \Sigma$ and $U_{\mathrm{int}} := \big((-\frac{4}{7}R, 0] \times \Sigma\big) \cup M_2$, hence $U_{\mathrm{aps}} \cap U_{\mathrm{int}} = N^R$. Moreover, we choose nonnegative smooth cutoff functions $\{\psi_{\mathrm{aps}}, \psi_{\mathrm{int}}\}$ such that

$$\psi_j \equiv 1 \text{ on } \{x \in M_2^R \mid \operatorname{dist}(x, \operatorname{supp} \chi_j) < \frac{1}{7}R\} \text{ and}$$
$$\psi_j \equiv 0 \text{ on } \{x \in M_2^R \mid \operatorname{dist}(x, \operatorname{supp} \chi_j) \geq \frac{2}{7}R\}$$

for $j \in \{\text{aps,int}\}$. We notice

$$\operatorname{dist}(\operatorname{supp} \psi_j', \operatorname{supp} \chi_j) = \operatorname{dist}(\operatorname{supp} \psi_j'', \operatorname{supp} \chi_j) \geq \frac{1}{7}R\,. \tag{3.15}$$

Moreover, we may assume that

$$\left|\frac{\partial^k \psi_j}{\partial u^k}\right| \leq c_0/R$$

for all k, where c_0 is a certain positive constant.

For any parameter $t > 0$, we define an operator $Q_2^R(t)$ on $C^\infty(M_2^R; S)$ with a smooth kernel, given by

$$Q_2^R(t;x,x') := \psi_{\mathrm{aps}}(x)\mathcal{E}_{\mathrm{aps}}^\infty(t;x,x')\chi_{\mathrm{aps}}(x') + \psi_{\mathrm{int}}(x)\mathcal{E}^R(t;x,x')\chi_{\mathrm{int}}(x'). \tag{3.16}$$

Recall that $\mathcal{E}^R$ denotes the kernel of the operator $\mathcal{D}^R e^{-t(\mathcal{D}^R)^2}$, given on the stretched closed manifold M^R. Notice that, by construction, $Q_2^R(t)$ maps $L^2(M_2^R; S)$ into the domain of the operator $\mathcal{D}_{2,P_>}^R$.

Then, for $x' \in U_{\mathrm{aps}}$ with $\chi_{\mathrm{aps}}(x') = 1$, we have by definition:

$$Q_2^R(t;x,x') = \begin{cases} \mathcal{E}_{\mathrm{aps}}^\infty(t;x,x') & \text{if } d(x, \operatorname{supp} \chi_{\mathrm{aps}}) < \frac{1}{7}R, \text{ and} \\ 0 & \text{if } d(x, \operatorname{supp} \chi_{\mathrm{aps}}) \geq \frac{2}{7}R. \end{cases} \tag{3.17}$$

Correspondingly, we have for $x' \in U_{\mathrm{int}}$ with $\chi_{\mathrm{int}}(x') = 1$,

$$Q_2^R(t;x,x') = \begin{cases} \mathcal{E}^R(t;x,x') & \text{if } d(x, \operatorname{supp} \chi_{\mathrm{int}}) < \frac{1}{7}R, \text{ and} \\ 0 & \text{if } d(x, \operatorname{supp} \chi_{\mathrm{int}}) \geq \frac{2}{7}R. \end{cases} \tag{3.18}$$

For fixed $t > 0$, we determine the difference between the precise kernel $\mathcal{E}_2^R(t;x,x')$ and the approximate one $Q_2^R(t;x,x')$. Let $C^R(t)$ denote the operator $\Big((\mathcal{D}_{2,P_>}^R)^2 + \frac{d}{dt}\Big) \circ Q_2^R(t)$ and $C^R(t;x,x')$ its kernel. By definition, we have $\Big((\mathcal{D}_{2,P_>}^R)^2 + \frac{d}{dt}\Big) \circ \mathcal{E}_2^R(t) = 0$. Thus, $C^R(t)$ 'measures' the error we make when replacing the precise kernel $\mathcal{E}_2^R(t;x,x')$ by the glued, approximate one.

More precisely, we have by Duhamel's Formula

$$\mathcal{E}_2^R(t;x,x') - Q_2^R(t;x,x') = -\int_0^t ds \int_{M_2^R} dz\, \mathcal{E}_2^R(s;x,z) C^R(t-s;z,x')$$

with

$$\begin{aligned}
C^R(t-s;z,x') &= \left((\mathcal{D}^R_{2,(z)})^2 + \mathcal{D}\right) Q_2^R(t-s;z,x')\\
&= \left((\mathcal{D}^R_{2\,(z)})^2 - \tfrac{d}{ds}\right) Q_2^R(t-s;z,x')\\
&= \psi''_{\mathrm{aps}}(z)\mathcal{E}^R_{\mathrm{aps}}(t-s;z,x')\chi_{\mathrm{aps}}(x') + 2\psi'_{\mathrm{aps}}(z)\frac{\partial}{\partial u}(\mathcal{E}^R_{\mathrm{aps}}(t-s;z,x'))\chi_{\mathrm{aps}}(x')\\
&\qquad + \psi_{\mathrm{aps}}(z)\underbrace{\left(\mathcal{D}^2_{(z)} - \tfrac{d}{ds}\right)\mathcal{E}^R_{\mathrm{aps}}(t-s;z,x')}_{=0}\chi_{\mathrm{aps}}(x')\\
&\quad + \psi''_{\mathrm{int}}(z)\mathcal{E}^R(t-s;z,x')\chi_{\mathrm{int}}(x') + 2\psi'_{\mathrm{int}}(z)\frac{\partial}{\partial u}(\mathcal{E}^R(t-s;z,x'))\chi_{\mathrm{int}}(x')\\
&\qquad + \psi_{\mathrm{int}}(z)\underbrace{\left((\mathcal{D}^R_{(z)})^2 - \tfrac{d}{ds}\right)\mathcal{E}^R(t-s;z,x')}_{=0}\chi_{\mathrm{int}}(x').
\end{aligned}$$

Here, $\mathcal{D}_{(z)}$ denotes the operator $\mathcal{D}$ acting on the z variable; and in the partial derivative $\frac{\partial}{\partial u}$ the letter u denotes the normal coordinate of the variable z.

As stated in (3.15), the supports of χ_j and ψ'_j (and, equally, ψ''_j) are disjoint and separated from each other by a distance $R/7$ in the normal variable for $j \in \{\mathrm{aps},\mathrm{int}\}$. Then the error term $C^R(t-s;z,x')$ vanishes both for the distance in the normal variable $d(z,x') < R/7$ and, actually, whenever z or x' are outside the segment $[-\frac{6}{7}R, \frac{1}{7}R] \times \Sigma$.

Let z and x' be on the cylinder and $|u-v| > R/7$ where u and v denote their normal coordinates. We investigate the error term $C^R(t-s;z,x')$ which consists of six summands. Two of them vanish as we have pointed out above. The remaining four summands involve the kernels $\mathcal{E}^\infty_{\mathrm{aps}}(t-s;z,x')$ on the infinite cylinder $[-R,\infty)\times\Sigma$ and $\mathcal{E}^R(t-s;z,x')$ on the stretched closed manifold M^R. We shall use that both kernels can be estimated according to inequality (3.14).

We estimate the first summand

$$\begin{aligned}
|\psi''_{\mathrm{aps}}(z)\mathcal{E}^\infty_{\mathrm{aps}}(t-s;z,x')\chi_{\mathrm{aps}}(x')| &\le \frac{c_0}{R}c_1(t-s)^{-\frac{1+m}{2}}e^{c_2 t}e^{-c_3\frac{d^2(z,x')}{t-s}}\\
&\le c_1' e^{c_2' t} e^{-c_3' R^2/t}.
\end{aligned}$$

Here we have used $t \ge s \ge 0$ and

$$(t-s)^{-(1+m)/2}e^{-c_2\frac{d^2(z,x')}{(t-s)}} \le c t^{-(1+m)/2} e^{-c_2\frac{d^2(z,x')}{t}} \le \tilde{c} e^{-c_2\frac{d^2(z,x')}{2t}}.$$

Similarly we estimate the second summand

$$2\left|\psi'_{\rm aps}(z)\frac{\partial}{\partial u}\mathcal{E}^{\infty}_{\rm aps}(t-s;z,x')\chi_{\rm aps}(x')\right| \\ \leq \frac{c_0}{R}c_1\frac{(t-s)^{-\frac{1+m}{2}}}{\sqrt{t}}e^{c_2t}e^{-c_3\frac{d^2(z,x')}{t-s}} \leq c_1'e^{c_2't}e^{-c_3'R^2/t}\,,$$

where the factor $1/\sqrt{t}$ comes from the differentiation of the kernel as explained before. The third and fourth summands, involving the kernel $\mathcal{E}^R$ of the closed stretched manifold M^R, are treated in exactly the same way. Altogether we have proved

Lemma 3.12. *The error kernel $C^R(t;u,v)$ vanishes for $u\notin[-\frac{6}{7}R,-\frac{1}{7}R]$. Moreover, $C^R(t;u,v)$ vanishes whenever $|u-v|\leq R/7$. For arbitrary $x,x'\in M_2^R$ we have the estimate*

$$|C^R(t;x,x')|\leq c_1e^{c_2t}e^{-c_3R^2/t}$$

with constants c_1,c_2,c_3 independent of x,x',t,R.

We consider the pointwise error

$$\mathcal{E}_2^R(t;x,x)-Q_2^R(t;x,x)=\int_0^t ds\int_{M_2^R}dz\;\mathcal{E}_2^R(s;x,z)C^R(t-s;z,x).$$

We obtain the following proposition as a consequence of the preceding lemma.

Proposition 3.13. *For all $x\in M_2^R$ and all $t>0$ we have*

$$\operatorname{Tr}\mathcal{E}_2^R(t;x,x)-\operatorname{Tr}Q_2^R(t;x,x)=\operatorname{Tr}\left(\mathcal{E}_2^R(t;x,x)-Q_2^R(t;x,x)\right).$$

Moreover, there exist positive constants c_1,c_2,c_3, independent of R, such that the 'error' term satisfies the inequality

$$\left|\mathcal{E}_2^R(t;x,x)-Q_2^R(t;x,x)\right|\leq c_1\cdot e^{c_2t}\cdot e^{-c_3(R^2/t)}\,.$$

PROOF. We estimate the error term

$$
\begin{aligned}
&|\mathcal{E}_2^R(t;x,x) - Q_2^R(t;x,x)| \\
&\qquad \leq \int_0^t ds \int_{M_2^R} dz \, |\mathcal{E}_2^R(s;x,z) C^R(t-s;z,x)| \\
&\qquad \leq c_1^2 e^{c_2 t} \cdot \int_0^t ds \int_{M_2^R} dz \left\{ s^{-\frac{d+1}{2}} \cdot e^{-c_3 \frac{d^2(x,z)}{s}} \right\} \cdot e^{-c_3 \frac{d^2(x,z)}{t-s}} \\
&\qquad \leq c_1^2 e^{c_2 t} \cdot \int_0^t ds \int_{\operatorname{supp}_z C^R(t-s;z,x)} dz \, e^{-c_4 \frac{d^2(x,z)}{s}} \cdot e^{-c_3 \frac{d^2(x,z)}{t-s}} \\
&\qquad \leq c_1^2 e^{c_2 t} \cdot \int_0^t ds \int_{\operatorname{supp}_z C^R(t-s;z,x)} dz \, e^{-c_5 \frac{t \cdot R^2}{s(t-s)}} \\
&\qquad \leq c_1^2 e^{c_2 t} \cdot cR \cdot \int_0^t ds \, e^{-c_5 \frac{t \cdot R^2}{s(t-s)}} \leq c_1^2 e^{c_2 t} \cdot 2cR \cdot \int_0^{t/2} ds \, e^{-c_5 \frac{t \cdot R^2}{s(t/2)}} \\
&\qquad = c_1^2 e^{c_2 t} \cdot 2cR \cdot \int_0^{t/2} ds \, e^{-2c_5 \frac{R^2}{s}} .
\end{aligned}
$$

Here we have used that $\operatorname{Vol}(\operatorname{supp}_z C^R(t-s;z,x)) \sim \operatorname{Vol}(\Sigma) \cdot R$ according to Lemma 3.12. We investigate the last integral.

$$
\begin{aligned}
\int_0^t e^{-\frac{c}{s}} \, ds &= -\int_0^t \frac{s^2}{c} \cdot e^{-\frac{c}{s}} \cdot (-\frac{c}{s^2}) \, ds \\
&\qquad < -\int_0^t \frac{t^2}{c} \cdot e^{-\frac{c}{s}} \cdot (-\frac{c}{s^2}) \, ds = -\frac{t^2}{c} \int_\infty^{\frac{c}{t}} e^{-r} \, dr = \frac{t^2}{c} e^{-\frac{c}{t}} .
\end{aligned}
$$

Thus we have

$$
|\mathcal{E}_2^R(t;x,x) - Q_2^R(t;x,x)| \leq c_1^2 e^{c_2 t} \cdot 2cR \cdot \frac{t^2}{c_6 R^2} e^{-\frac{c_6 R^2}{t}} \leq c_7 e^{c_2 t} \cdot e^{-c_8 (R^2/t)} .
$$

□

The preceding proposition shows that, for t smaller than $\sqrt{R}$, the trace $\operatorname{Tr} \mathcal{E}_2^R(t;x,x)$ of the kernel of the operator $\mathcal{D}_2^R e^{-t(\mathcal{D}_{2,P_>}^R)^2}$ approaches the trace $\operatorname{Tr} Q_2^R(t;x,x)$ of the approximative kernel pointwise as $R \to \infty$. In particular, we have:

COROLLARY 3.14. *The following equality holds, as $R \to \infty$,*

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}_2^R(t;x,x) \, dx \\
&\qquad = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} Q_2^R(t;x,x) \, dx + \mathrm{O}(e^{-cR}).
\end{aligned}
$$

Proof. We have

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}_2^R(t;x,x)\,dx \\
&\qquad\qquad = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} Q_2^R(t;x,x)\,dx \\
&\qquad\qquad\quad + \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr}\left(\mathcal{E}_2^R(t;x,x) - Q_2^R(t;x,x)\right)\,dx,
\end{aligned}$$

and we have to show that the second summand on the right side is $\mathrm{O}(e^{-cR})$ as $R\to\infty$. We estimate

$$\begin{aligned}
&\left|\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr}\left(\mathcal{E}_2^R(t;x,x) - Q_2^R(t;x,x)\right)\,dx\right| \\
&\leq \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} |\mathcal{E}_2^R(t;x,x) - Q_2^R(t;x,x)|\,dx \\
&\leq \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} c_1\cdot e^{c_2 t}\cdot e^{-c_3(R^2/t)}\,dx \\
&\leq \frac{c_1 \operatorname{Vol}(M_2^R)}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{e^{c_2 t}\cdot e^{-c_3(R^2/t)}}{\sqrt{t}}\,dt \\
&\leq c_4 R \int_0^{\sqrt{R}} e^{c_2\sqrt{R}}\cdot e^{-c_5 R^{3/2}}\,dt \leq c_4 R^{3/2}\cdot e^{-c_6 R} \leq c_7\cdot e^{-c_8 R}\,.
\end{aligned}$$

□

3.5.2. *Discarding the Cylinder Contributions.* Corollary 3.14 shows that the essential part of the local eta function of the spectral boundary condition on the half manifold with attached cylinder of length R, i.e., the 'small–time' integral from 0 to $\sqrt{R}$ can be replaced, as $R\to\infty$, by the corresponding integral over the trace $\operatorname{Tr} Q_2^R(t;x,x)$ of the approximate kernel, constructed in (3.16). Now we show that $\operatorname{Tr} Q_2^R(t;x,x)$ can be replaced pointwise (for $x\in M_2^R$) by the trace $\operatorname{Tr}\mathcal{E}^R(t;x,x)$ of the kernel of the operator $\mathcal{D}^R e^{-t(\mathcal{D}^R)^2}$ which is defined on the stretched closed manifold M^R.

Consider the Dirac operator

$$\sigma(\partial_u + B) : C^\infty([0,\infty)\times\Sigma;S)\to C^\infty([0,\infty)\times\Sigma;S),$$

on the semi-infinite cylinder with the domain

$$\{s\in C_0^\infty([0,\infty)\times\Sigma;S)\mid P_>(s|_{\{0\}\times\Sigma})=0\}.$$

It has a unique self-adjoint extension which we denote by D_{aps}. Recall that the integral kernel $\mathcal{E}_{\mathrm{aps}}^\infty$ of the operator $D_{\mathrm{aps}} e^{-t(D_{\mathrm{aps}})^2}$ enters in the definition of the approximative kernel Q_2^R as given in (3.16). We show that $\mathcal{E}_{\mathrm{aps}}^{\infty}(t;x,x)$ is traceless for all $x \in [0,\infty) \times \Sigma$. Then

$$\mathrm{Tr}\, Q_2^R(t;x,x) = \mathrm{Tr}\, \mathcal{E}^R(t;x,x) \quad \text{for all } x \in M_2^R, \tag{3.19}$$

follows.

To prove that a product TV is traceless, the following easy result can be used.

LEMMA 3.15. *Let σ be unitary with $\sigma^2 = -\mathrm{I}$. We consider an operator V of trace class which is 'even', i.e. it commutes with σ. Moreover, T is odd, i.e. it anticommutes with σ. Then*

$$\mathrm{Tr}(TV) = 0.$$

PROOF. We have, by unitary equivalence,

$$\mathrm{Tr}(TV) = \mathrm{Tr}(-\sigma(TV)\sigma) = \mathrm{Tr}(-\sigma T\sigma V) = \mathrm{Tr}(\sigma^2 TV) = \mathrm{Tr}(-TV).$$

□

LEMMA 3.16. *Let $\chi : [0,\infty) \to \mathbb{R}$ be a smooth function with compact support and $t > 0$. Then the trace of the operator $\chi \cdot D_{\mathrm{aps}} e^{-t(D_{\mathrm{aps}})^2}$ vanishes. In particular,*

$$\int_\Sigma \mathrm{Tr}\, \mathcal{E}_{\mathrm{aps}}^{\infty}(t;u,y;u,y)\, dy = 0$$

for all $u \in [0,\infty)$.

PROOF. Clearly, $D_{\mathrm{aps}}^2 = (\sigma(\partial_u + B))^2 = -\partial_u^2 + B^2$ is even, hence also the power series $e^{-t(D_{\mathrm{aps}})^2}$ is even. On the other hand, as with B, σB is also odd. So,

$$\mathrm{Tr}\left(\chi \cdot \sigma B e^{-t(D_{\mathrm{aps}})^2}\right) = 0.$$

To show that

$$\mathrm{Tr}\left((\chi \cdot \sigma \partial_u e^{-t(D_{\mathrm{aps}})^2}\right) = 0,$$

we need a slightly more specific argument: Let $\mathrm{e}_{\mathrm{aps}}^{\infty}$ denote the heat kernel of the operator D_{aps}. For $u,v \in [0,\infty)$ and $y,z \in \Sigma$ it has the following form (see e.g. [**BoWo93**], Formulae 22.33 and 22.35):

$$\mathrm{e}_{\mathrm{aps}}(t;u,y;v,z) = \sum_{k\in\mathbb{Z}} e_k(t;u,v)\varphi_k(y) \otimes \varphi_k^*(z)$$

for an orthonormal system $\{\varphi_k\}$ of eigensections of B. Hence,

$$\sigma\partial_u\, \mathrm{e}_{\mathrm{aps}}(t;u,y;v,z) = \sum_{k\in\mathbb{Z}} e_k'(t;u,v)\sigma\varphi_k(y) \otimes \varphi_k^*(z).$$

But $\langle \sigma\varphi_k; \varphi_k \rangle = 0$ on Σ since σ is skew-adjoint. □

3.6. Asymptotic Vanishing of Large-t Chopped η-invariant on Stretched Part Manifold. So far we have found

$$\eta_{\mathcal{D}^R_{2,P_>}}(0) = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}^R(t;x,x)\,dx + \mathrm{O}(e^{-cR}) + \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}_2^R(t;x,x)\,dx$$

as $R \to \infty$. To prove Theorem 3.8, we still have to show

$$\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}_2^R(t;x,x)\,dx = \mathrm{O}(e^{-cR}) \text{ and} \tag{3.20}$$

$$\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}^R(t;x,x)\,dx = \mathrm{O}(e^{-cR}), \tag{3.21}$$

as $R \to \infty$. Recall that $\mathcal{E}_2^R(t;x,x')$ denotes the kernel of the operator $\mathcal{D}_2^R e^{-t(\mathcal{D}^R_{2,P_>})^2}$ on the compact manifold M_2^R with boundary $\{-R\} \times \Sigma$, and $\mathcal{E}^R(t;x,x')$ the kernel of the operator $\mathcal{D}^R e^{-t(\mathcal{D}^R)^2}$ on the closed stretched manifold M^R.

In the following we show (3.20), i.e. that we can neglect the contribution to the eta invariant of $\mathcal{D}^R_{2,P_>}$ which comes from the large t asymptotic of $\mathcal{E}_2^R(t;x,x')$. The key to that is that the eigenvalue of $\mathcal{D}^R_{2,P_>}$ with the smallest absolute value is uniformly bounded away from zero.

Theorem 3.17. *Let $\mu_0(R)$ denote the smallest (in absolute value) nonvanishing eigenvalue of the operator $\mathcal{D}^R_{2,P_>}$ on the manifold M_2^R. Let us assume, as always in this section, that $\operatorname{Ker} B = \{0\}$. Then there exists a positive constant c_0, which does not depend on R such that $\mu_0(R) > c_0$ for R sufficiently large.*

Remark 3.18. As we will discover, this result indicates that the behavior of the small eigenvalues on M_2^R differs from that on the stretched, closed manifold M^R. On the manifold with boundary M_2^R with the attached cylinder of length R, the eigenvalues are bounded away from 0 when $R \to \infty$ due to the spectral boundary condition. That is the statement of Theorem 3.17 which we are going to prove in the next two sections. However, on M^R, the set of eigenvalues splits into one set of eigenvalues becoming exponentially small and another one of eigenvalues being uniformly bounded away from 0 as $R \to \infty$. This we are going to show further below. Roughly speaking, the reason for the different behavior is that on M_2^R the *eigensections* must satisfy the spectral boundary condition. Therefore they are exponentially decreasing on the cylinder, and the *eigenvalues* are bounded away from 0. But on M^R we have to cope with *eigensections* on a closed manifold which need not decrease, but require part of the *eigenvalues* to decrease exponentially (for details see Theorem 3.29 below).

3.6.1. *The Cylindrical Dirac Operator.* To prove Theorem 3.17 we first recall

a few properties of the cylindrical Dirac operator $D_{\mathrm{cyl}} := \sigma(\partial_u + B)$ on the infinite cylinder $\Sigma^\infty_{\mathrm{cyl}} := (-\infty, +\infty) \times \Sigma$. A special feature of the cylindrical manifold $\Sigma^\infty_{\mathrm{cyl}}$ is that we may apply the theory of *Sobolev spaces* exactly as in the case of $\mathbb{R}^m$. The point is that we can choose a covering of the open manifold $\Sigma^\infty_{\mathrm{cyl}}$ by a *finite* number of coordinate charts. We can also choose a finite trivialization of the bundle $S|_{\Sigma^\infty_{\mathrm{cyl}}}$. Let $\{U_\iota, \kappa_\iota\}_{\iota=1}^K$ be such a trivialization, where $\kappa_\iota : S|_{U_\iota} \to V_\iota \times \mathbb{C}^N$ is a bundle isomorphism and V_ι an open (possibly non–compact) subset of $\mathbb{R}^m$. Let $\{f_\iota\}$ be a corresponding partition of unity. We assume that for any ι the derivatives of the function f_ι are bounded.

Definition 3.19. We say that a section (or distribution) s of the bundle S over $\Sigma^\infty_{\mathrm{cyl}}$ belongs to the p-th *Sobolev space* $\mathcal{H}^p(\Sigma^\infty_{\mathrm{cyl}}; S)$, $p \in \mathbb{R}$, if and only if $f_\iota \cdot s$ belongs to the Sobolev space $\mathcal{H}^p(\mathbb{R}^m; \mathbb{C}^N)$ for any ι. We define the p-th *Sobolev norm*

$$\|s\|_p := \sum\nolimits_{\iota=1}^K \left\| (\mathrm{I} + \Delta_\iota)^{p/2} (f_\iota \cdot s) \right\|_{L^2(\mathbb{R}^m)} ,$$

where Δ_ι denotes the Laplacian on the trivial bundle $V_\iota \times \mathbb{C}^N \subset \mathbb{R}^m \times \mathbb{C}^N$.

Lemma 3.20. *(a) For the unique self-adjoint L^2 extension of D_{cyl} (denoted by the same symbol) we have*

$$\mathrm{Dom}(D_{\mathrm{cyl}}) = \mathcal{H}^1(\Sigma^\infty_{\mathrm{cyl}}; S).$$

(b) Let λ_1 denote the smallest positive eigenvalue of the operator B on the manifold Σ. Then we have

$$\left\langle (D_{\mathrm{cyl}})^2 s; s \right\rangle \geq \lambda_1^2 \|s\|^2 \tag{3.22}$$

for all $s \in \mathrm{Dom}(D_{\mathrm{cyl}})$, and for any $\mu \in (-\lambda_1, +\lambda_1)$ the operator

$$D_{\mathrm{cyl}} - \mu : \mathcal{H}^1(\Sigma^\infty_{\mathrm{cyl}}; S) \to L^2(\Sigma^\infty_{\mathrm{cyl}}; S)$$

is an isomorphism of Hilbert spaces.
(c) Let $\mathcal{R}_{\mathrm{cyl}}(\mu)$ denote the inverse of the operator $D_{\mathrm{cyl}} - \mu$. Then the family $\{\mathcal{R}_{\mathrm{cyl}}(\mu)\}_{\mu \in (-\lambda_1, \lambda_1)}$ is a smooth family of elliptic pseudo-differential operators of order -1.

Proof. (a) follows immediately from the corresponding result on the model manifold $\mathbb{R}^m$. To prove (b) we consider a spectral resolution $\{\varphi_k, \lambda_k\}_{k \in \mathbb{Z} \setminus 0}$ of $L^2(\Sigma; S)$ generated by the tangential operator B. Because of (3.8), we have $\lambda_{-k} = -\lambda_k$. We can assume $\varphi_{-k} = \sigma \varphi_k$ for $k \in \mathbb{N}$. We consider a section s belonging to the dense subspace $C_0^\infty(\Sigma^\infty_{\mathrm{cyl}}; S)$ of $\mathrm{Dom}(D_{\mathrm{cyl}})$, and expand it in terms of the preceding spectral resolution

$$s(u, y) = \sum\nolimits_{k \in \mathbb{Z} \setminus \{0\}} f_k(u) \varphi_k(y).$$

Since $(D_{\mathrm{cyl}})^2 = -\partial_u^2 + B^2$, we obtain

$$(D_{\mathrm{cyl}})^2 s = \sum\nolimits_k (\lambda_k^2 f_k - f_k'')\varphi_k,$$

hence

$$\begin{aligned}\left\langle (D_{\mathrm{cyl}})^2 s; s\right\rangle &= \sum\nolimits_k \int_{-\infty}^{\infty} (\lambda_k^2 f_k(u) - f_k''(u))\bar{f}_k(u)du \\ &\geq \lambda_1^2 \left\|s\right\|^2 - \sum\nolimits_k \int_{-\infty}^{\infty} f_k''(u)\bar{f}_k(u)du \\ &= \lambda_1^2 \left\|s\right\|^2 + \sum\nolimits_k \int_{-\infty}^{\infty} f_k'(u)\bar{f}_k'(u)du \geq \lambda_1^2 \left\|s\right\|^2 .\end{aligned}$$

It follows that $(D_{\mathrm{cyl}})^2$ (and therefore D_{cyl}) has bounded inverse in $L^2(\Sigma^\infty_{\mathrm{cyl}}; S)$ and, more generally, that $(D_{\mathrm{cyl}})^2 - \mu$ is invertible for $\mu \in (-\lambda_1, \lambda_1)$. To prove (c) we apply the symbolic calculus and construct a parametrix S for the operator D_{cyl}; i.e., S is an elliptic pseudo-differential operator of order -1 such that $SD_{\mathrm{cyl}} = \mathrm{I} + T$, where T is a smoothing operator. Thus $D_{\mathrm{cyl}}^{-1} = S - TD_{\mathrm{cyl}}^{-1}$. The operator TD_{cyl}^{-1} is a smoothing operator, hence D_{cyl}^{-1} is an elliptic pseudo-differential operator of order -1. The same argument can be applied to the resolvent $\mathcal{R}_{\mathrm{cyl}}(\mu) = (D_{\mathrm{cyl}} - \mu)^{-1}$ for arbitrary $\mu \in (-\lambda_1, \lambda_1)$. The smoothness of the family follows by standard calculation. □

3.6.2. *The Part Manifolds with Half-infinite Cylindrical Ends.* To prove Theorem 3.17 we need to refine the preceding results on the infinite cylinder $\Sigma^\infty_{\mathrm{cyl}}$ to the Dirac operator, naturally extended to the manifold $M_2^\infty = ((-\infty, 0] \times \Sigma) \cup M_2$ with cylindrical end. Let $C_0^\infty(M_2^\infty; S)$ denote the space of compactly supported smooth sections of S over M_2^∞. Then

$$\mathcal{D}_2^\infty|_{C_0^\infty(M_2^\infty;S)} : C_0^\infty(M_2^\infty; S) \to L^2(M_2^\infty; S) \tag{3.23}$$

is symmetric. Moreover, we have

Lemma 3.21. *Let $s \in C^\infty(M_2^\infty; S)$ be an eigensection of $\mathcal{D}_2^\infty$. Then there exist $C, c > 0$ such that, on $(-\infty, 0] \times \Sigma$, one has $|s(u, y)| \leq Ce^{cu}$.*

Proof. Let $\{\varphi_k, \lambda_k\}_{k \in \mathbb{Z}\setminus 0}$ be a spectral resolution of the tangential operator B. Because of (3.8) we have $\lambda_{-k} = -\lambda_k$ and we can assume that $\varphi_{-k} = \sigma\varphi_k$ for $k \in \mathbb{N}$. Then

$$\left\{\varphi_k^\pm = \tfrac{1}{\sqrt{2}}\left(\varphi_k \pm \sigma\varphi_k\right), \pm\lambda_k\right\}_{k\in\mathbb{N}} \tag{3.24}$$

is a spectral resolution of the composed operator σB on Σ. Notice that we have

$$\sigma\varphi_k^+ = -\varphi_k^- \quad \text{and} \quad \sigma\varphi_k^- = \varphi_k^+ . \tag{3.25}$$

Let $s \in C^\infty(M_2^\infty; S)$ and

$$\mathcal{D}^\infty_{L^2}\psi = \mu\psi \tag{3.26}$$

with $\mu \in \mathbb{R}$. We expand $s|_{(-\infty,0]\times\Sigma}$ in terms of the spectral resolution of σB just constructed:

$$s(u,y) = \sum_{k=1}^{\infty} f_k(u)\varphi_k^+(y) + g_k(u)\varphi_k^-(y)\,.$$

Because of (3.24), (3.25), and (3.26) the coefficients f_k, g_k must satisfy the system of ordinary differential equations

$$\begin{pmatrix} \lambda_k & \partial/\partial u \\ -\partial/\partial u & -\lambda_k \end{pmatrix} \begin{pmatrix} f_k \\ g_k \end{pmatrix} = \mu \begin{pmatrix} f_k \\ g_k \end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix} f_k' \\ g_k' \end{pmatrix} = \mathbf{A} \begin{pmatrix} f_k \\ g_k \end{pmatrix} \qquad \text{with } \mathbf{A} := \begin{pmatrix} 0 & -(\mu+\lambda_k) \\ \mu-\lambda_k & 0 \end{pmatrix}.$$

Since $s \in L^2$, of the eigenvalues $\pm\sqrt{\lambda_k^2-\mu^2}$ of $\mathbf{A}$**,** only those which are on the positive real line enter in the construction of s by solving the preceding differential equation. In particular, all coefficients f_k, g_k must vanish identically for $\lambda_k \le \mu$. Thus,

(3.27)

$$s(u,y) = \sum_{\lambda_k>\mu} a_k \left(\exp\left(\sqrt{\lambda_k^2-\mu^2}\,u\right)\,\varphi_k^+ - \frac{\lambda_k-\mu}{\sqrt{\lambda_k^2-\mu^2}} \exp\left(\sqrt{\lambda_k^2-\mu^2}\,u\right)\,\varphi_k^- \right)$$

and, in particular,

$$|s(u,y)| \le C \exp\left(\sqrt{\lambda_{k_0}^2-\mu^2}\,\frac{u}{2}\right), \qquad u<0,$$

for some constant C, where λ_{k_0} denotes the smallest positive eigenvalue of B with $\lambda_{k_0} > |\mu|$. □

In spectral theory we are looking for self-adjoint L^2 extensions of a symmetric operator. We recall: on a *closed* manifold, the Dirac operator is *essentially self-adjoint*; i.e. its minimal closed extension is self-adjoint (and therefore there do not exist other self-adjoint extensions) and it is a Fredholm operator. On a compact manifold with boundary, the situation is much more complicated. There is a huge variety of dense domains to which the Dirac operator can be extended such that it becomes self-adjoint; and there is a smaller, but still large variety where the extension of the Dirac operator becomes self-adjoint *and* Fredholm (see Section 1.2.2 above and Booß-Bavnbek and Furutani [**BoFu98**]); a special type of self-adjoint and Fredholm domains are the domains specified by the boundary conditions belonging to the Grassmannian of all self-adjoint generalized Atiyah–Patodi–Singer projections.

Now we shall show that the situation on manifolds with (infinite) cylindrical ends resembles the situation on closed manifolds.

We recall the following simple lemma (see also Reed and Simon [**ReSi72**], Theorem VIII.3, Corollary, p. 257).

Lemma 3.22. *Let A be a densely defined symmetric operator in a separable complex Hilbert space $\mathcal{H}$. We assume that* $\operatorname{range}(A+i\mathrm{I})$ *is dense in $\mathcal{H}$. Then A is essentially self-adjoint.*

Proof. Since A is symmetric, the operator $A+i\,\mathrm{I}$ is injective and the operator $(A+i\,\mathrm{I})^{-1}$ is well defined and bounded on the dense subspace $\operatorname{range}(A+i\,\mathrm{I})$ of $\mathcal{H}$. Then the closure R_i of $(A+i\,\mathrm{I})^{-1}$ has the whole space $\mathcal{H}$ as domain and R_i is bounded and injective. Now a standard argument of functional analysis (see e.g. Pedersen [**Pe89**], Proposition 5.1.7) says that the inverse R_i^{-1} of a densely defined, closed, and injective operator R_i has the same properties. Thus our R_i^{-1} is closed; and by construction it is the minimal closed extension of $A+i\,\mathrm{I}$. Therefore, $R_i^{-1}-i\,\mathrm{I}$ is symmetric and the minimal closed extension of A, hence self-adjoint and equal A^*. □

We apply the lemma for $\mathcal{H} = L^2(M_2^\infty;S)$ and take for A the operator of (3.23). To prove that the range $(\mathcal{D}_2^\infty + i\mathrm{I})(C_0^\infty(M_2^\infty;S))$ is dense in $L^2(M_2^\infty;S)$ we consider a section $s \in L^2(M_2^\infty;S)$ which is orthogonal to $(\mathcal{D}_2^\infty + i\mathrm{I})(C_0^\infty(M_2^\infty;S))$; i.e. the distribution $(\mathcal{D}_2^\infty - i\mathrm{I})s$ vanishes when applied to any test function, hence

$$(\mathcal{D}_2^\infty - i\mathrm{I})s = 0. \tag{3.28}$$

Since $\mathcal{D}_2^\infty - i$ is elliptic, by elliptic regularity s is smooth at all interior points, that is for our complete manifold in all points. On the cylinder $(-\infty,0]\times\Sigma$ we expand s in terms of the eigensections of the composed operator σB on Σ. It follows that s satisfies an estimate of the form

$$|s(u,y)| \le Ce^{cu}, \qquad (u,y)\in(-\infty,0]\times\Sigma, \tag{3.29}$$

for some constants $C,c>0$ (according to Lemma 3.21). On the manifold M_2^R with cylindrical end of finite length R we apply Green's formula and get

$$\left\langle \mathcal{D}_2^R s^R; s^R\right\rangle - \left\langle s^R; \mathcal{D}_2^R s^R\right\rangle = -\int_{\{-R\}\times\Sigma} (\sigma s|_{\{-R\}\times\Sigma}\,dy, s|_{\{-R\}\times\Sigma}), \tag{3.30}$$

where s^R denotes the restriction of s to the manifold M_2^R with boundary $\{-R\}\times\Sigma$. For $R\to\infty$, the right side of (3.30) vanishes; and the left side becomes $2i\,|s|^2$ by (3.28). Hence $s=0$. Thus we have proved

Lemma 3.23. *The operator* (3.23) *is essentially self-adjoint.*

We denote the (unique) self-adjoint L^2 extension by the same symbol $\mathcal{D}_2^\infty$, and we define the Sobolev spaces on the manifold M_2^∞ as in Definition 3.19. Once again, the point is that manifolds with cylindrical ends, even if they are not compact but only complete, are like the infinite cylinder sufficiently simple to be covered by a finite system of local charts. Clearly

$$\operatorname{Dom}(\mathcal{D}_2^\infty) = \mathcal{H}^1(M_2^\infty;S) \text{ and } \mathcal{D}_2^\infty : \mathcal{H}^1(M_2^\infty;S) \to L^2(M_2^\infty;S)$$

is bounded. There are, however, substantial differences between the properties of the simple Dirac operator D_{cyl} on the infinite cylinder and the Dirac operator $\mathcal{D}_2^\infty$

on the manifold with cylindrical end. For instance, from B the discreteness of the spectrum and the regularity at 0 (i.e., 0 is not an eigenvalue) are passed on to D_{cyl}, but not to $\mathcal{D}_2^\infty$. Yet we can prove the following result:

PROPOSITION 3.24. *The operator*

$$\mathcal{D}_2^\infty : \mathrm{Dom}(\mathcal{D}_2^\infty) = \mathcal{H}^1(M_2^\infty; S) \to L^2(M_2^\infty; S)$$

is a Fredholm operator and its spectrum in the interval $(-\lambda_1, \lambda_1)$ consists of finitely many eigenvalues of finite multiplicity. Here λ_1 denotes the smallest positive eigenvalue of B.

NOTE . Actually, using more advanced methods one can show that the essential spectrum of $\mathcal{D}_2^\infty$ is equal to $(-\infty, -\lambda_1] \cup [\lambda_1, \infty)$ (see for instance Müller [**Mu94**], Section 4).

Before proving the proposition we shall collect various criteria for the compactness of a bounded operator between Sobolev spaces on an open manifold. Let X be a complete (not necessarily compact) Riemannian manifold with a fixed Hermitian bundle. Recall the three cornerstones of the Sobolev analysis of Dirac operators for X *closed*.

Rellich Lemma: The inclusion $\mathcal{H}^1(X) \subset L^2(X)$ is compact.

Compact Resolvent: To each Dirac operator $\mathcal{D}$ we have a parametrix $\mathcal{R}$ which is an elliptic pseudo-differential operator of order -1 with principal symbol equal to the inverse of the principal symbol of $\mathcal{D}$. So $\mathcal{R}$ is a bounded operator from $L^2(X)$ to $\mathcal{H}^1(X)$, and hence compact in $L^2(X)$. In particular, for μ in the resolvent set the resolvent $(\mathcal{D} - \mu\mathrm{I})^{-1}$ is compact as operator in $L^2(X)$.

Smoothing Operator: Any integral operator over X with smooth kernel is a smoothing operator, i.e. it maps distributional sections of arbitrary low order into smooth sections. Moreover, it is of trace class and thus compact.

In the *general* case, i.e. for not necessarily compact X, the Rellich Lemma remains valid for sections with compact support. A compact resolvent is not attainable, hence the essential spectrum appears. Operators with smooth kernel remain smoothing operators, but in general they are no longer of trace class nor compact. We recall:

LEMMA 3.25. *Let X be a complete (not necessarily compact) Riemannian manifold with fixed Hermitian bundle. Let K be a compact subset of X.*
(a) The injection $\mathcal{H}^1(X) \subset L^2(X)$ defines a compact operator when restricted to sections with support in K. In particular, for any cutoff function χ with support in K and any bounded operator $\mathcal{R} : L^2(X) \to \mathcal{H}^1(X)$ the operator $\chi\mathcal{R}$ is compact in $L^2(X)$.
(b) Let $T : L^2(X) \to L^2(X)$ be an integral operator with a kernel $k(x,y) \in L^2(X^2)$. Then the operator T is a bounded, compact operator (in fact it is of Hilbert–Schmidt class).

(c) Let $T : L^2(X) \to L^2(X)$ be a bounded compact operator and $\mathcal{H}'$ a closed subspace of $L^2(X)$, e.g. $\mathcal{H}' := L^2(X')$ where X' is a submanifold of X of codimension 0. Assume that $T(\mathcal{H}') \subset \mathcal{H}'$. Then $T|_{\mathcal{H}'}$ is compact as operator from $\mathcal{H}'$ to $\mathcal{H}'$.

PROOF. (a) follows immediately from the local Rellich Lemma. (b) is the famous Hilbert–Schmidt Lemma. Also (c) is well known, see e.g. Hörmander [**Ho85**, Proposition 19.1.13] where (c) is proved within the category of trace class operators. $\square$

In general an integral operator T with smooth kernel is not compact even if either $\operatorname{supp}_x k(x,x')$ or $\operatorname{supp}_{x'} k(x,x')$ are contained in a compact subset $K \subset X$. Consider for instance on $\Sigma^\infty_{\mathrm{cyl}} = (-\infty, +\infty) \times \Sigma$ an integral operator T with a smooth kernel of the form

$$k(x,x') = \chi(x) d(x,x'),$$

where $d(x,x')$ denotes the distance and χ is a function with support in a ball of radius 1 (and equal 1 in a smaller ball). Then T is not a compact operator on $L^2(\Sigma^\infty_{\mathrm{cyl}})$: choose a sequence $\{s_n\}$ of L^2 functions of norm 1 and with $\operatorname{supp} s_n$ contained in a ball of radius 1 such that $d(\operatorname{supp}\chi, \operatorname{supp} s_n) = n$. Then for any n we have $|Ts_n| > Cn$. Thus T is not compact, in fact not even bounded.

For the bounded resolvent (see Lemma 3.20)

$$\mathcal{R}_{\mathrm{cyl}} : L^2(\Sigma^\infty_{\mathrm{cyl}}; S) \to \mathcal{H}^1(\Sigma^\infty_{\mathrm{cyl}}; S)$$

we have, however, the following corollary to the preceding lemma. It provides an example of a compact integral operator on an open manifold with a smooth kernel which is compactly supported only in one variable.

COROLLARY 3.26. *Let χ and ψ be smooth cutoff functions on $\Sigma^\infty_{\mathrm{cyl}}$ with support contained in the half-cylinder $(-\infty, 0) \times \Sigma$. Let $\operatorname{supp}\chi$ be compact. Then the operators $\chi\mathcal{R}_{\mathrm{cyl}}\psi$ and $\psi\mathcal{R}_{\mathrm{cyl}}\chi$ are compact in $L^2(\Sigma^\infty_{\mathrm{cyl}}; S)$.*

PROOF. The operator $\chi\mathcal{R}_{\mathrm{cyl}}\psi$ is compact according to the preceding lemma, claim (a). Its adjoint operator is $\psi\mathcal{R}_{\mathrm{cyl}}\chi$, since $\mathcal{R}_{\mathrm{cyl}}$ is self-adjoint. Thus it is also compact (even if its range is *not* compactly supported). $\square$

3.7. The Estimate of the Lowest Nontrivial Eigenvalue. In this section we prove Theorem 3.17. Recall that the tangential operator B is assumed to be nonsingular and that λ_1 denotes the smallest positive eigenvalue of B. So far, we have established that

I: the operator D_{cyl} on the infinite cylinder $\Sigma^\infty_{\mathrm{cyl}}$ has no eigenvalues in the interval $(-\lambda_1, +\lambda_1)$, and

II: the operator $\mathcal{D}^\infty_2$ on the manifold M^∞_2 with infinite cylindrical end has only finitely many eigenvalues in the interval $(-\lambda_1, +\lambda_1)$, each of finite multiplicity.

We have to show that

III: the nonvanishing eigenvalues of $(\mathcal{D}_2^R)_{P_>}$ are bounded away from 0 by a bound independent of R.

Proof of Theorem 3.17 The idea of the proof is the following. We define a positive constant μ_1 independent of R. Then let R be a positive real (more precisely $R > R_0$ for a suitable positive R_0), and $s \in L^2(M_2^R; S)$ any eigensection with eigenvalue $\mu \in (-\lambda_1/\sqrt{2}, +\lambda_1/\sqrt{2})$, i.e.

$$s \in \operatorname{Dom}(\mathcal{D}_2^R)_{P_>} \text{ i.e. } P_>(s|_{\{-R\}\times\Sigma}) = 0 \text{ and } \mathcal{D}_2^R s = \mu s.$$

Then we show that $\mu^2 > \mu_1/2$ for a certain real $\mu_1 > 0$ which is independent of R and s. A natural choice of μ_1 is

$$\mu_1 = \min\left\{ \frac{|\mathcal{D}_2^\infty \Psi|^2}{|\Psi|^2} \,\middle|\, \Psi \in \mathcal{H}^1(M_2^\infty; S) \text{ and } \Psi \perp \operatorname{Ker} \mathcal{D}_2^\infty \right\}. \tag{3.31}$$

Note that by II above (Proposition 3.24), $\operatorname{Ker}\mathcal{D}_2^\infty$ is of finite dimension. We shall define a certain extension $s^\infty \in \mathcal{H}^1(M_2^\infty; S)$ of s.

The reasoning would be easy, if we could extend s to an eigensection of $\mathcal{D}_2^\infty$ on all of M_2^∞. Then it would follow at once that the discrete part of the spectrum of $\mathcal{D}_2^\infty$ is not empty, μ belongs to it, $\sqrt{\mu_1}$ is the smallest eigenvalue > 0, and hence we would have $\mu^2 > \mu_1/2$ as desired. In general, such a convenient extension of the given eigensection s cannot be achieved. But due to the spectral boundary condition satisfied by s in the hypersurface $\{-R\} \times \Sigma$, the eigensection s over M_2^R can be continuously extended by a section over $(-\infty, -R] \times \Sigma$ on which the Dirac operator vanishes. By construction, both the enlargement α of the L^2 norm of s by the chosen extension and the cosine, say β, of the angle between s^∞ and $\operatorname{Ker}\mathcal{D}_2^\infty$ can be estimated independently of the specific choice of s and μ. It turns out that they both decrease exponentially with growing R.

Let $\{s_1, \dots, s_q\}$ be an orthonormal basis of $\operatorname{Ker}\mathcal{D}_2^\infty$ and set

$$\widetilde{s} := s^\infty - \sum_{j=1}^q \langle s^\infty; s_j \rangle \, s_j \, .$$

Clearly, the section $\widetilde{s}$ belongs to $\mathcal{H}^1(M_2^\infty; S)$ and is orthogonal to $\operatorname{Ker}\mathcal{D}_2^\infty$. Hence, on the one hand,

$$\frac{|\mathcal{D}_2^\infty \widetilde{s}|^2}{|\widetilde{s}|^2} \geq \mu_1 \, . \tag{3.32}$$

On the other hand, we have by construction

$$|\mathcal{D}_2^\infty \widetilde{s}|^2 = |\mathcal{D}_2^\infty s^\infty|^2 = \left|\mathcal{D}_2^R s\right|^2_{M_2^R} = \mu^2 \, .$$

Finally, we shall prove that

$$|\widetilde{s}| \to 1 \text{ as } R \to \infty. \tag{3.33}$$

Then the estimate

$$\mu^2 > \frac{\mu_1}{2} \tag{3.34}$$

follows for sufficiently large R. Since we have assumed that $\mu^2 < \frac{\lambda_1^2}{2}$, we have also $\mu_1 < \lambda_1^2$, hence μ_1 belongs to the discrete part of the spectrum of $(\mathcal{D}_2^\infty)^2$ and, by the *Min–Max Principle* (e.g., see [**ReSi78**]), must be its smallest eigenvalue > 0.

Thus, to prove the Theorem 3.17 we are left with the task of first constructing a suitable extension $\widetilde{s}$ of s and then proving (3.33).

We expand $s|_{[-R,0]\times\Sigma}$ in terms of a spectral resolution

$$\{\varphi_k, \lambda_k; \sigma\varphi_k, -\lambda_k\}_{k\in\mathbb{N}}$$

of $L^2(\Sigma; S)$ generated by B:

$$s(u,y) = \sum_{k=1}^{\infty} f_k(u)\varphi_k(y) + g_k(u)\sigma\varphi_k(y)\,.$$

Since $\sigma(\partial_u + B)s|_{[-R,0]\times\Sigma} = 0$, the coefficients must satisfy the system of ordinary differential equations

$$\begin{pmatrix} f_k' \\ g_k' \end{pmatrix} = \mathbf{A}_k \begin{pmatrix} f_k \\ g_k \end{pmatrix} \qquad \text{with } \mathbf{A}_k := \begin{pmatrix} -\lambda_k & \mu \\ -\mu & \lambda_k \end{pmatrix}. \tag{3.35}$$

Moreover, since $P_> s|_{\{-R\}\times\Sigma} = 0$ we have

$$f_k(-R) = 0 \quad \text{ for any } k \geq 1. \tag{3.36}$$

Thus, for each k the pair (f_k, g_k) is uniquely determined up to a constant a_k. More explicitly, since the eigenvalues of $\mathbf{A}_k$ are $\pm(\lambda_k^2 - \mu^2)^{1/2}$, a suitable choice of the eigenvectors of $\mathbf{A}_k$ gives

$$f_k(u) = a_k \frac{\mu}{\sqrt{\lambda_k^2 - \mu^2}} \sinh \sqrt{\lambda_k^2 - \mu^2}(R+u) \quad \text{and}$$

$$g_k(u) = a_k \left(\cosh \sqrt{\lambda_k^2 - \mu^2}(R+u) + \frac{\lambda_k}{\sqrt{\lambda_k^2 - \mu^2}} \sinh(\lambda_k^2 - \mu^2)^{1/2}(R+u) \right).$$

We assume $|s|_{L^2} = 1$. Then we have, with $v := (\lambda_k^2 - \mu^2)^{1/2}(R+u)$:

$$
\begin{aligned}
1 &\geq \int_{[-R,0]\times\Sigma} |s(u,y)|^2\, du dy = \sum_{k=1}^{\infty} \int_{-R}^{0} \left(|f_k(u)|^2 + |g_k(u)|^2\right) du \\
&= \sum_{k=1}^{\infty} |a_k|^2 \frac{1}{(\lambda_k^2-\mu^2)^{1/2}} \int_0^{(\lambda_k^2-\mu^2)^{1/2}R} \Bigg(\frac{\mu^2}{\lambda_k^2-\mu^2}\cdot \sinh^2 v \\
&\qquad\qquad + \cosh^2 v + 2\frac{\lambda_k}{(\lambda_k^2-\mu^2)^{1/2}} \cdot \cosh v \cdot \sinh v + \frac{\lambda_k^2}{\lambda_k^2-\mu^2}\cdot \sinh^2 v \Bigg) dv \\
&= \sum_{k=1}^{\infty} |a_k|^2 \Bigg\{ -\frac{\lambda_k^2}{\lambda_k^2-\mu^2}\cdot R + (1/4)\cdot \frac{\mu^2}{(\lambda_k^2-\mu)^{3/2}} \cdot \sinh(2(\lambda_k^2-\mu^2)^{1/2}R) \\
&\quad + (1/4)\cdot \left(1 + \frac{\lambda_k^2}{\lambda_k^2-\mu^2}\right)\cdot (\lambda_k^2-\mu^2)^{1/2}\cdot \sinh(2(\lambda_k^2-\mu^2)^{1/2}R) \\
&\quad + \frac{\lambda_k^2}{\lambda_k^2-\mu^2}\cdot \cosh^2((\lambda_k^2-\mu^2)^{1/2}R) \Bigg\}.
\end{aligned}
$$

Since $\lambda_k^2 \geq \lambda_1^2 > 2\mu^2$ we have $2(\lambda_k^2-\mu^2)^{1/2} > \sqrt{2}\lambda_k > \lambda_k$. Moreover, we have for all $k \geq 1$

$$
\begin{aligned}
-\frac{\lambda_k^2}{\lambda_k^2-\mu^2}\cdot R &+ \frac{\lambda_k^2}{\lambda_k^2-\mu^2}\cdot \frac{(\lambda_k^2-\mu^2)^{1/2}}{4}\cdot \sinh(2(\lambda_k^2-\mu^2)^{1/2}R) \\
&> -R + \frac{(\lambda_1^2-\mu^2)^{1/2}}{4}\cdot \sinh(2(\lambda_1^2-\mu^2)^{1/2}R) \\
&> -R + \frac{\sqrt{2}}{8}\lambda_1 \cdot \sinh(\sqrt{2}\lambda_1 R) > 0,
\end{aligned}
$$

if $R \geq R_0$ for some positive R_0 which depends only on λ_1 and not on μ, s and k.

Thus, for any k the sum in the braces can be estimated in the following way:

$$
\{\dots\} > \frac{\lambda_k^2}{\lambda_k^2-\mu^2}\cdot \cosh^2((\lambda_k^2-\mu^2)^{1/2}R) > \frac{1}{4}e^{2(\lambda_k^2-\mu^2)^{1/2}R} > \frac{1}{4}e^{\lambda_k R}.
$$

Hence, we have

$$
\sum_{k=1}^{\infty} |a_k|^2 \cdot e^{\lambda_k R} \leq 4\,. \tag{3.37}
$$

Note that the preceding estimate does not depend on R (provided that $R > R_0$), k or the specific choice of s, and that R_0 only depends on λ_1.

According to (3.37) the absolute value of the coefficients a_k is rapidly decreasing in such a way that, in particular, we can extend the eigensection s of $\mathcal{D}^R_{2,P_>}$, given on M_2^R to a continuous section on M_2^∞ by the formula

$$
s^\infty(x) := \begin{cases} s(x) & \text{for } x \in M_2^R \\ \sum_{k=1}^{\infty} a_k e^{\lambda_k(R+u)}\sigma\varphi_k(y) & \text{for } x = (u,y) \in (-\infty,-r]\times\Sigma. \end{cases}
$$

By construction, s^∞ is smooth on $M_2^\infty \setminus (\{-R\} \times \Sigma)$ and belongs to the Sobolev space $\mathcal{H}^1(M_2^\infty; S)$. It follows from (3.37) that

$$|s^\infty|^2_{L^2} = |s|^2_{L^2} + \sum_{k=1}^\infty |a_k|^2 \int_{-\infty}^{-R} e^{2\lambda_k(R+u)}\, du = 1 + \sum_{k=1}^\infty |a_k|^2 \frac{1}{2\lambda_k}$$

$$\leq 1 + \frac{1}{2\lambda_1} \cdot \sum_{k=1}^\infty |a_k|^2 \leq 1 + \frac{1}{2\lambda_1} \cdot \left(\sum\nolimits_{k=1}^\infty |a_k|^2 \cdot e^{\lambda_k R}\right) \cdot e^{-\lambda_1 R}$$

$$\leq 1 + \frac{2}{\lambda_1} \cdot e^{-\lambda_1 R}. \tag{3.38}$$

Next, let $\Psi \in \operatorname{Ker} \mathcal{D}_2^\infty$ and assume that $|\Psi| = 1$. By (3.27), on $(-\infty, 0] \times \Sigma$ the section Ψ has the form

$$\Psi((u,y)) = \sum_{k=1}^\infty b_k e^{\lambda_k u} G(y) \varphi_k(y) \tag{3.39}$$

with

$$\sum_{k=1}^\infty \int_{-\infty}^0 |b_k|^2 e^{2\lambda_k u}\, du = \sum_{k=1}^\infty \frac{1}{2\lambda_k} \cdot |b_k|^2 < +\infty.$$

Set $l := \Psi|_{M_2^R}$. Then l satisfies the equations

$$\mathcal{D}_2^R l = 0 \text{ and } P_>(l|_{\{-R\}\times\Sigma}) = 0.$$

Hence, l belongs to $\operatorname{Ker} \mathcal{D}^R_{2,P_>}$. This implies the following equality:

$$\int_{M_2^R} \langle s^\infty(x); \Psi(x)\rangle\, dx = \frac{1}{\mu} \cdot \int_{M_2^R} \left\langle \mathcal{D}_2^R s^\infty(x); l(x)\right\rangle dx$$

$$= \frac{1}{\mu} \cdot \int_{M_2^R} \left\langle s^\infty(x); \mathcal{D}_2^R l(x)\right\rangle dx - \frac{1}{\mu} \cdot \int_\Sigma \langle \sigma s^\infty(-R,y); l(-R,y)\rangle\, dx.$$

On the other hand,

$$\int_{(-\infty,-r]\times\Sigma} \langle s^\infty(x); \Psi(x)\rangle\, dx = \sum_{k>0} \frac{a_k \overline{b_k}}{2\mu} \cdot e^{-\lambda_k R} \leq C_1 e^{-\lambda_1 R}.$$

Therefore

$$|\langle s^\infty; \Psi\rangle| \leq C_1 e^{-\lambda_1 R}. \tag{3.40}$$

Hence, we have proved

LEMMA 3.27. *Any eigensection $s \in \mathcal{H}^1(M_2^R; S)$ of $\mathcal{D}^R_{2,P_>}$ with eigenvalue $\mu \in (-\lambda_1/\sqrt{2}, \lambda_1/\sqrt{2})$ can be extended to a continuous section s^∞ on M_2^∞ which is smooth on $M_2^\infty \setminus (\{-R\} \times \Sigma)$ and belongs to the first Sobolev space $\mathcal{H}^1(M_2^\infty; S)$. Moreover, the enlargement of the norm of s by the extension and the cosine of the angle between s^∞ and $\operatorname{Ker} \mathcal{D}_2^\infty$ are exponentially decreasing by formulae* (3.38) *and* (3.40).

The final step in proving the Theorem 3.17 follows at once from the preceding lemma. We recall: by definition of μ_1 we have $|\mathcal{D}_2^\infty \widetilde{s}|^2 / |\widetilde{s}|^2 \geq \mu_1$ and by construction of $\widetilde{s}$ we have $|\mathcal{D}_2^\infty \widetilde{s}|^2 = \mu^2$. Thus, we have $\mu^2 \geq |\widetilde{s}|^2 \cdot \mu_1$. To get the desired bound $\mu^2 > \mu_1/2$, it remains to show that $|\widetilde{s}|^2 > 1/2$ for sufficiently large R.

Since $\widetilde{s}$ is the orthogonal projection of s^∞ onto $(\operatorname{Ker} \mathcal{D}_2^\infty)^\perp$ and the basis $\{s_1, \dots, s_q\}$ of $\operatorname{Ker} \mathcal{D}_2^\infty$ is orthonormal, we have

$$|\widetilde{s}|^2 = |s^\infty|^2 - \sum_{j=1}^q |\langle s^\infty; s_j \rangle|^2 \leq 1 + \frac{2}{\lambda_1} e^{-\lambda_1 R} - \sum_{j=1}^q |\langle s^\infty; s_j \rangle|^2 .$$

Thus, Theorem 3.17 follows from

$$||\widetilde{s}|^2 - 1| \leq \frac{2}{\lambda_1} e^{-\lambda_1 R} + q C_1 e^{-2\lambda_1 R} \leq C_2 e^{-C_3 R}.$$

We finish this section by proving the asymptotic estimate (3.20). Recall that $\mathcal{E}_2^R(t; x, x')$ denotes the kernel of the operator $\mathcal{D}_{2,P_>}^R e^{-t(\mathcal{D}_{2,P_>}^R)^2}$, where $\mathcal{D}_{2,P_>}^R$ denotes the Dirac operator over the manifold M_2^R with the spectral boundary condition at the boundary $\{-R\} \times \Sigma$. Then we have:

Lemma 3.28.

$$\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^\infty \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \mathcal{E}_2^R(t; x, x) \, dx = \mathrm{O}(e^{-cR}).$$

Proof. For any eigenvalue $\mu \neq 0$ of $\mathcal{D}_{2,P_>}^R$ and $R > 0$ sufficiently large ($R \cdot c_0^2 \geq 1$, where c_0 denotes the lower uniform bound for μ^2 of Theorem 3.17) we have

$$\begin{aligned} (3.41) \quad |\int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} \mu e^{-t\mu^2} \, dt| &\leq \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} |\mu| e^{-t\mu^2} \, dt = \int_{|\mu| R^{1/4}}^\infty e^{-\tau^2} \, d\tau \\ &\leq \int_{|\mu| R^{1/4}}^\infty \tau e^{-\tau^2} \, d\tau = \left[-\frac{1}{2} e^{-\tau^2} \right]_{|\mu| R^{1/4}}^\infty \frac{1}{2} e^{-\mu^2 \sqrt{R}} , \end{aligned}$$

which gives

$$\begin{aligned} |\int_{\sqrt{R}}^\infty \frac{dt}{\sqrt{t}} \int_{M_2^R} \operatorname{Tr} \left(\mathcal{D}_{2,P_>}^R e^{-t(\mathcal{D}_{2,P_>}^R)^2} \right)| &\leq \int_{\sqrt{R}}^\infty \frac{dt}{\sqrt{t}} \int_{M_2^R} \sum_{\mu \neq 0} |\mu| e^{-t\mu^2} \, dt \\ &\leq \frac{1}{2} \cdot \sum_{\mu \neq 0} e^{-\mu^2 \sqrt{R}} = \frac{1}{2} \cdot \sum_{\mu \neq 0} e^{-(\sqrt{R} - 1)\mu^2} \cdot e^{-\mu^2} \\ &\leq C_1 \cdot e^{-\sqrt{R} \mu_0^2} \operatorname{Tr} \left((e^{-\left(\mathcal{D}_{2,P_>}^R\right)^2} \right) \leq C_2 \cdot e^{-\sqrt{R} \mu_0^2} \operatorname{Vol}(M_2^R) \\ &\leq C_3 \cdot e^{-\sqrt{R} \mu_0^2} \leq C_3 \cdot e^{-C_4 \sqrt{R}} . \end{aligned}$$

Here we have exploited that the heat kernel $\mathrm{e}_2^R(t;x,x')$ of the operator $\mathcal{D}^R_{2,P_>}$ can be estimated by

$$|\mathrm{e}_2^R(t;x,x')| \le c_1 \cdot t^{-\frac{m}{2}} \cdot e^{c_2 t} \cdot e^{-c_3 \frac{d^2(x,x')}{t}}$$

according to (3.13). Thus,

$$|\mathrm{Tr}\left(e^{-\left(\mathcal{D}^R_{2,P_>}\right)^2}\right)| \le \int_{M_2^R} |\mathrm{Tr}\, \mathrm{e}_2^R(1;x,x)|\, dx \le c_1 \cdot e^{c_2} \cdot \int_{M_2^R} dx. \tag{3.42}$$

□

3.8. The Spectrum on the Closed Stretched Manifold. Thus far, we have proved the asymptotic equation

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M_2^R} \mathrm{Tr}\, \mathcal{E}^R(t;x,x)\, dx + \mathrm{O}(e^{-cR}) = \eta_{\mathcal{D}^R_{2,P_>}}(0)$$

as $R \to \infty$. It follows that

$$\lim_{R\to\infty} \eta_R = \lim_{R\to\infty} \left(\eta_{\mathcal{D}^R_{1,P_<}}(0) + \eta_{\mathcal{D}^R_{2,P_>}}(0)\right),$$

where

$$\eta_R := \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{M^R} \mathrm{Tr}\, \mathcal{E}^R(t;x,x)\, dx.$$

To prove Theorem 3.8, we still have to show (3.21), i.e., that we can extend the integration from $\sqrt{R}$ to infinity:

$$\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{dt}{\sqrt{t}} \int_{M_2^R} \mathrm{Tr}\, \mathcal{E}^R(t;x,x)\, dx = \mathrm{O}(e^{-cR}) \text{ as } R \to \infty.$$

Recall that $\mathcal{E}^R(t;x,x')$ denotes the kernel of the operator $\mathcal{D}^R e^{-t(\mathcal{D}^R)^2}$ on the closed stretched manifold M^R.

Formally, our *task* of proving the preceding estimate is reminiscent of our previous task of proving the corresponding estimate for the kernel $\mathcal{E}_2^R(t;x,x')$ of the operator $\mathcal{D}_2^R e^{-t(\mathcal{D}^R_{2,P_>})^2}$ (see Lemma 3.28). Both integrals are over the same prolonged compact manifold M_2^R with boundary $\{-R\}\times\Sigma$. However, the methods we can apply are different: In the previous case, we had a uniform positive bound for the absolute value of the smallest nonvanishing eigenvalue of the boundary value problem $\mathcal{D}^R_{2,P_>}$ for sufficiently large R.

As mentioned above in Remark 3.18, such a bound does not exist for the Dirac operator $\mathcal{D}^R$ on the closed stretched manifold M^R. Moreover, for the spectral boundary condition we shall show

$$\dim \mathrm{Ker}\, \mathcal{D}^R_{2,P_>} = \dim \mathrm{Ker}\, \mathcal{D}_{2,P_>} \text{ and } \eta_{\mathcal{D}^R_{2,P_>}}(0) = \eta_{\mathcal{D}_{2,P_>}}(0)$$

for any R (see Proposition 3.39 below). For $\mathcal{D}^R$, on the contrary, the dimension of the kernel can change and, thus, $\eta_{\mathcal{D}^R}$ can admit an integer jump in value as

$R \to \infty$. This is due to the presence of 'small' eigenvalues created by L^2 solutions of the operators $\mathcal{D}_1^\infty$ and $\mathcal{D}_2^\infty$ on the half-manifolds with cylindrical ends. We use a straightforward analysis of small eigenvalues inspired by the proof of Theorem 3.17 to prove the following result

THEOREM 3.29. *There exist $R_0 > 0$ and positive constants a_1, a_2, and a_3, such that for any $R > R_0$, the eigenvalue μ of the operator $\mathcal{D}^R$ is either bounded away from 0 with $a_1 < |\mu|$, or is exponentially small $|\mu| < a_2 e^{-a_3 R}$. Let $\mathcal{W}^R$ denote the subspace of $L^2(M^R; S)$ spanned by the eigensections of $\mathcal{D}^R$ corresponding to the exponentially small eigenvalues. Then* $\dim \mathcal{W}^R = q$, *where* $q = \dim(\operatorname{Ker} \mathcal{D}_1^\infty) + \dim(\operatorname{Ker} \mathcal{D}_2^\infty)$.

Recall from Proposition 3.24 that the operator $\mathcal{D}_j^\infty$, acting on the first Sobolev space $\mathcal{H}^1(M_j^\infty; S)$, is an (unbounded) self-adjoint Fredholm operator in $L^2(M_j^\infty; S)$ which has a discrete spectrum in the interval $(-\lambda_1, +\lambda_1)$ where λ_1 denotes the smallest positive eigenvalue of the tangential operator B. Thus, the space $\operatorname{Ker} \mathcal{D}_j^\infty$ of L^2 solutions is of finite dimension.

To prove the theorem we first investigate the small eigenvalues of the operator $\mathcal{D}^R$ and the pasting of L^2 solutions. Let $R > 0$. We reparametrize the normal coordinate u such that $M_1^R = M_1 \cup ((-R, 0] \times \Sigma)$ and $M_2^R = ([0, R) \times \Sigma) \cup M_2$, and introduce the subspace $\mathcal{V}^R \subset L^2(M^R; S)$ spanned by L^2 solutions of the operators $\mathcal{D}_j^\infty$. We choose an auxiliary smooth real function $f^R = f_1^R \cup f_2^R$ on M^R with $f^R = 1$ outside the cylinder $[-R, R] \times \Sigma$, and where f^R is a function of the normal variable u on the cylinder. Moreover, we assume $f^R(-u) = f^R(u)$ (i.e., $f_1^R(-u) = f_2^R(u)$), and that f_2^R is an increasing function of u with

$$f_2^R(u) = \begin{cases} 0 & \text{for } 0 \le u \le \frac{R}{4} \\ 1 & \text{for } \frac{R}{2} \le u \le R. \end{cases}$$

We also assume that there exists a constant $\gamma > 0$ such that $|\frac{\partial^p f_2^R}{\partial u^p}(u)| < \gamma R^{-p}$. If $s_j \in C^\infty(M_j^\infty; S)$, we define $s_1 \cup_{f^R} s_2$ by the formula

$$\left(s_1 \cup_{f^R} s_2\right)(x) := \begin{cases} f_1^R(x) s_1(x) & \text{for } x \in M_1^R \\ f_2^R(x) s_2(x) & \text{for } x \in M_2^R. \end{cases}$$

Clearly, we have

$$\begin{aligned} s_1 \cup_{f^R} s_2 &= s_1 \cup_{f^R} 0 + 0 \cup_{f^R} s_2 \\ \mathcal{D}^R(s_1 \cup_{f^R} s_2) &= (\mathcal{D}_1^\infty s_1) \cup_{f^R} (\mathcal{D}_2^\infty s_2) + s_1 \cup_{g^R} s_2 \text{ and} \\ \left|s_1 \cup_{f^R} s_2\right|^2 &= \left|s_1 \cup_{f^R} 0\right|^2 + \left|0 \cup_{f^R} s_2\right|^2, \end{aligned} \tag{3.43}$$

where $g^R := g_1^R \cup g_2^R$ with $g_j^R(u, y) = \sigma(y) \frac{\partial f_j^R}{\partial u}(u, y)$ and $|\cdot|$ denotes the L^2 norm on the manifold M^R.

DEFINITION 3.30. The subspace $\mathcal{V}^R \subset C^\infty(M^R; S)$ is defined by

$$\mathcal{V}^R := \operatorname{span}\{s_1 \cup_{f^R} s_2 \mid s_j \in \operatorname{Ker} \mathcal{D}_j^\infty\}.$$

Let $\{s_{1,1},\dots,s_{1,q_1}\}$ be a basis of $\operatorname{Ker}\mathcal{D}_1^\infty$ and $\{s_{2,1},\dots,s_{2,q_2}\}$ a basis of $\operatorname{Ker}\mathcal{D}_2^\infty$. Then the $q=q_1+q_2$ sections $\{s_{1,\nu_1}\cup_{f^R}0\}\cup\{0\cup_{f^R}s_{2,\nu_2}\}$ form a basis of $\mathcal{V}^R$. We want to show that $\mathcal{V}^R$ approximates the space $\mathcal{W}^R$ of eigensections of $\mathcal{D}^R$ corresponding to the 'small' eigenvalues, for R sufficiently large. We begin with an elementary result:

Lemma 3.31. *There exists R_0, such that for any $R>R_0$ and any $s\in\mathcal{V}^R$, the following estimate holds*

$$\left|\mathcal{D}^R s\right| \le e^{-\lambda_1 R}\,|s|\,.$$

Proof. It suffices to prove the estimate for basis sections of $\mathcal{V}^R$. Thus, let $s=s_1\cup_{f^R}0$ with $s_1\in\operatorname{Ker}\mathcal{D}_1^\infty$. By (3.43) we have

$$\mathcal{D}^R s(x)=\begin{cases}0 & \text{for } x\in M_1\cup M_2\\ \sigma(y)\frac{\partial f_1^R}{\partial u}(u,y)\cdot s_1(u,y) & \text{for } x=(u,y)\in[-R,R]\times\Sigma.\end{cases}\tag{3.44}$$

Here f_1^R is continued in a trivial way on the whole cylinder $[-R,R]\times\Sigma$. Now, s_1 is a L^2 solution of $\mathcal{D}_1^\infty$, hence $s_1(u,y)=\sum_k c_k e^{-(R+u)\lambda_k}\varphi_k(y)$ on this cylinder where $\{\varphi_k,\lambda_k;\sigma\varphi_k,-\lambda_k\}_{k\in\mathbb{N}}$ is, as above, a spectral resolution of $L^2(\Sigma;S)$ for B. We estimate the norm of $\mathcal{D}^R s$:

$$\begin{aligned}
\left|\mathcal{D}^R s\right|^2 &= \left|\frac{\partial f_1^R}{\partial u}\cdot s_1\right|^2\\
&=\sum_k\int_{-\frac{R}{2}}^{-\frac{R}{4}}\int_\Sigma\left(\frac{\partial f_1^R}{\partial u}\right)^2\cdot|c_k|^2\cdot e^{-2(R+u)\lambda_k}(\varphi_k(y);\varphi_k(y))\,dy\,du\\
&=\int_{-\frac{R}{2}}^{-\frac{R}{4}}\left(\frac{\partial f_1^R}{\partial u}\right)^2\cdot\sum_k|c_k|^2\cdot e^{-2(R+u)\lambda_k}\cdot 1\cdot du\\
&\le\frac{\gamma^2}{R^2}\cdot\sum\nolimits_k\left(|c_k|^2\cdot\int_{-\frac{R}{2}}^{-\frac{R}{4}}e^{-2(R+u)\lambda_k}\,du\right)\\
&=\frac{\gamma^2}{R^2}\cdot\sum\nolimits_k\left(|c_k|^2\cdot\int_{R\lambda_k}^{\frac{3}{2}R\lambda_k}e^{-v}\,\frac{dv}{2\lambda_k}\right)\\
&\le\frac{\gamma^2}{R^2}\cdot\sum\nolimits_k|c_k|^2\cdot\frac{e^{-R\lambda_k}-e^{-\frac{3}{2}R\lambda_k}}{2\lambda_k}\\
&\le\frac{\gamma^2}{R^2}\cdot\sum\nolimits_k\frac{e^{-R\lambda_k}}{2\lambda_k}|c_k|^2\le\frac{\gamma^2}{R^2}e^{-R\lambda_1}\cdot\sum_k\frac{|c_k|^2}{2\lambda_k}.
\end{aligned}$$

On the other hand, we have the elementary inequality

$$\begin{aligned}
|s|^2=\left|s_1\cup_{f^R}0\right|^2 &\ge\int_{-R}^{-R+1}\int_\Sigma|s_1(u,y)|^2\,dy\,du\\
&=\sum|c_k|^2\cdot\frac{1-e^{-2\lambda_k}}{2\lambda_k}\ge d\cdot\sum\frac{|c_k|^2}{2\lambda_k},
\end{aligned}$$

with $0 < d \le 1 - e^{-2\lambda_1}$. Thus, we have the following estimate for any $s \in \mathcal{V}^R$ of the form $s_1 \cup_{f^R} 0$ and for sufficiently large R

$$\left|\mathcal{D}^R s\right|^2 \le \frac{\gamma^2}{R^2 d} e^{-R\lambda_1} \cdot d \cdot \sum_k \frac{|c_k|^2}{2\lambda_k} \le \frac{\gamma^2}{R^2 d} e^{-R\lambda_1} \cdot |s|^2 \le e^{-R\lambda_1} \cdot |s|^2 .$$

For $s = 0 \cup_{f^R} s_2$, we estimate the norm of $\mathcal{D}_R s$ in the same way, in view of the fact that s_2 has the form $s_2(u,y) = \sum_k d_k e^{(u+R)\lambda_k} \sigma(y) \varphi_k(y)$ on the cylinder. □

Let $\{\lambda_k; \psi_k\}$ denote a spectral decomposition of the space $L^2(M^R; S)$ generated by the operator $\mathcal{D}^R$. For $a > 0$, let P_a denote the orthogonal projection onto the space $\mathcal{H}_a := \operatorname{span}\{\psi_k \mid |\lambda_k| > a\}$.

LEMMA 3.32. *For sufficiently large R, we have the estimate*

$$|(\mathrm{I} - P_{e^{-R\lambda_1/4}})\, s| \le e^{-R\lambda_1/2} \cdot |s|\, , \textit{ for all } s \in \mathcal{V}^R.$$

PROOF. We represent s as the series $s = \sum_k a_k \psi_k$. We have

$$\begin{aligned} |(\mathrm{I} - P_{e^{-R\lambda_1/4}})\, s| &= \sum_{\lambda_k^2 > e^{-R\lambda_1/2}} a_k^2 \le \sum_{\lambda_k^2 > e^{-R\lambda_1/2}} e^{\frac{R\lambda_1}{2}} \cdot \lambda_k^2 a_k^2 \\ &\le \sum\nolimits_k e^{R\lambda_1/2} \cdot \lambda_k^2 a_k^2 = e^{R\lambda_1/2} \left|\mathcal{D}^R s\right|^2 \\ &\le e^{R\lambda_1/2} e^{-R\lambda_1} |s|^2 = e^{-R\lambda_1/2} |s|^2 . \end{aligned}$$

□

PROPOSITION 3.33. *The spectral projection $P_{e^{-R\lambda_1/4}}$ restricted to the subspace $\mathcal{V}^R$ is an injection. In particular, $\mathcal{D}^R$ has at least q eigenvalues ρ such that $|\rho| \le e^{-R\lambda_1/4}$, where q is the sum of the dimensions of the spaces $\operatorname{Ker} \mathcal{D}_j^\infty$ of L^2 solutions of the operators $\mathcal{D}_1^\infty$ and $\mathcal{D}_2^\infty$.*

PROOF. Let $s \in \mathcal{V}^R$, and assume that $P_{e^{-R\lambda_1/4}}(s) = 0$. We have

$$|s| = |(\mathrm{I} - P_{e^{-R\lambda_1/4}})\, s| \le e^{-\frac{R\lambda_1}{2}} \cdot |s| \le \frac{1}{2} |s|\, ,$$

for R sufficiently large. □

The proposition shows that the operator $\mathcal{D}^R$ has at least q exponentially small eigenvalues with corresponding eigensections, which we can approximate by pasting together L^2 solutions. Now we will show that this makes the list of eigenvalues approaching 0 as $R \to +\infty$ complete.

Let ψ be an eigensection of $\mathcal{D}^R$ corresponding to an eigenvalue μ, where $|\mu| < \lambda_1$. As in the proof of Theorem 3.17, we expand $\psi|_{[-R,R]\times\Sigma}$ in terms of a spectral resolution

$$\{\varphi_k, \lambda_k; \sigma\varphi_k, -\lambda_k\}_{k\in\mathbb{N}}$$

of $L^2(\Sigma; S)$ generated by B:

$$\psi(u,y) = \sum_{k=1}^{\infty} f_k(u)\varphi_k(y) + g_k(u)\sigma\varphi_k\,,$$

where the coefficients satisfy the system of ordinary differential equations of (3.35)

$$\begin{pmatrix} f_k' \\ g_k' \end{pmatrix} = \mathbf{A}_k \begin{pmatrix} f_k \\ g_k \end{pmatrix} \qquad \text{with } \mathbf{A}_k := \begin{pmatrix} -\lambda_k & \mu \\ -\mu & \lambda_k \end{pmatrix}.$$

For the eigenvalues $\pm\sqrt{\lambda_k^2 - \mu^2}$ of $\mathbf{A}_k$ and the eigenvectors

$$\begin{pmatrix} \lambda_k + \sqrt{\lambda_k^2 - \mu^2} \\ \mu \end{pmatrix} \text{ and } \begin{pmatrix} \mu \\ \lambda_k + \sqrt{\lambda_k^2 - \mu^2} \end{pmatrix},$$

we get a natural splitting of $\psi(u,y)$ in the form $\psi(u,y) = \psi_+(u,y) + \psi_-(u,y)$ with

$$\psi_+(u,y) = \sum_k a_k e^{-\sqrt{\lambda_k^2-\mu^2}\,u}\left\{\left(\lambda_k + \sqrt{\lambda_k^2-\mu^2}\right)\varphi_k(y) + \mu\sigma(y)\varphi_k(y)\right\}, \text{ and}$$

$$\psi_-(u,y) = \sum_k b_k e^{\sqrt{\lambda_k^2-\mu^2}\,u}\left\{\mu\varphi_k(y) + \left(\lambda_k + \sqrt{\lambda_k^2-\mu^2}\right)\sigma(y)\varphi_k(y)\right\}.$$

Then we have the following estimate of the L^2 norm of ψ in the y direction on the cylinder:

LEMMA 3.34. *Assume that $|\psi| = 1$. There exist positive constants c_1, c_2 such that $\left|\psi_{|\{u\}\times\Sigma}\right| \le c_1 e^{-c_2 R}$ for $-\frac{3}{4}R \le u \le \frac{3}{4}R$.*

PROOF. We have

$$\begin{aligned}
&\left|\psi_{|\{-R+r\}\times\Sigma}\right|^2 \\
&\le e^{-2r\sqrt{\lambda_k^2-\mu^2}} \cdot \left|\sum\nolimits_k a_k e^{-R\sqrt{\lambda_k^2-\mu^2}}\left\{\left(\lambda_k + \sqrt{\lambda_k^2-\mu^2}\right) f_k + \mu\sigma f_k\right\}\right|^2 \\
&= e^{-2r\sqrt{\lambda_k^2-\mu^2}} \cdot \left|\psi_{|\{-R\}\times\Sigma}\right|^2 .
\end{aligned}$$

In the same way we get

$$\left|\psi_{|\{R-r\}\times\Sigma}\right|^2 \le e^{-2r\sqrt{\lambda_k^2-\mu^2}} \cdot \left|\psi_{|\{R\}\times\Sigma}\right|^2 .$$

Let us observe that, in fact, the argument used here proves that

$$\begin{aligned}
&\left|\psi_{+|\{r\}\times\Sigma}\right| \le e^{-(r-s)\sqrt{\lambda_k^2-\mu^2}} \cdot \left|\psi_{+|\{s\}\times\Sigma}\right|, \text{ and} \\
&\left|\psi_{-|\{s\}\times\Sigma}\right| \le e^{-(r-s)\sqrt{\lambda_k^2-\mu^2}} \cdot \left|\psi_{-|\{r\}\times\Sigma}\right|,
\end{aligned}$$

for any $-R < s < r < R$. We also have another elementary inequality

$$\left|\psi_{|\{r\}\times\Sigma}\right|^2 \ge \left|\psi_{+|\{r\}\times\Sigma}\right|^2 - 2\cdot\left|\psi_{+|\{r\}\times\Sigma}\right| \cdot \left|\psi_{-|\{r\}\times\Sigma}\right| .$$

This helps estimate the L^2 norm of $\psi_\pm$ in the y direction. We have

$$\begin{aligned}
|\psi|^2 &\geq \int_{-R}^{-R+1} \left|\psi_{|\{u\}\times\Sigma}\right|^2 du \\
&\geq \int_{-R}^{-R+1} \left(\left|\psi_{+|\{u\}\times\Sigma}\right|^2 - 2\left|\psi_{+|\{u\}\times\Sigma}\right| \left|\psi_{-|\{u\}\times\Sigma}\right|\right) du \\
&\geq \left|\psi_{+|\{-R\}\times\Sigma}\right|^2 \\
&\quad - 2\int_{-R}^{-R+1} \left|\psi_{+|\{-R\}\times\Sigma}\right| e^{-2R\sqrt{\lambda_k^2-\mu^2}} \left|\psi_{-|\{R\}\times\Sigma}\right| du \\
&\geq \left|\psi_{+|\{-R\}\times\Sigma}\right|^2 - 2e^{-2R\sqrt{\lambda_k^2-\mu^2}} \left|\psi_{+|\{-R\}\times\Sigma}\right| \left|\psi_{-|\{R\}\times\Sigma}\right| .
\end{aligned}$$

In the same way we obtain

$$|\psi|^2 \geq \left|\psi_{-|\{R\}\times\Sigma}\right|^2 - 2e^{-2R\sqrt{\lambda_k^2-\mu^2}} \left|\psi_{+|\{-R\}\times\Sigma}\right| \left|\psi_{-|\{R\}\times\Sigma}\right| .$$

We add the last two inequalities and use

$$2\left|\psi_{+|\{-R\}\times\Sigma}\right| \;\left|\psi_{-|\{R\}\times\Sigma}\right| \leq \left|\psi_{+|\{-R\}\times\Sigma}\right|^2 + \left|\psi_{-|\{R\}\times\Sigma}\right|^2$$

to obtain

$$2|\psi|^2 \geq \left(1 - e^{-2R\sqrt{\lambda_k^2-\mu^2}}\right)\left(\left|\psi_{+|\{-R\}\times\Sigma}\right|^2 + \left|\psi_{-|\{R\}\times\Sigma}\right|^2\right).$$

This gives us the inequality we need, namely

$$\left|\psi_{\pm|\{\mp R\}\times\Sigma}\right|^2 \leq 4|\psi|^2 .$$

Now we finish the proof of the lemma.

$$\begin{aligned}
\left|\psi_{|\{u\}\times\Sigma}\right| &= \left|\psi_{+|\{u\}\times\Sigma} + \psi_{-|\{u\}\times\Sigma}\right| \\
&\leq e^{-(u+R)\sqrt{\lambda_k^2-\mu^2}} \left|\psi_{+|\{-R\}\times\Sigma}\right| + e^{-(R-u)\sqrt{\lambda_k^2-\mu^2}} \left|\psi_{-|\{R\}\times\Sigma}\right| \\
&\leq 2\left(e^{-(u+R)\sqrt{\lambda_k^2-\mu^2}} + e^{-(R-u)\sqrt{\lambda_k^2-\mu^2}}\right)|\psi| \leq c_1 e^{-c_2 R},
\end{aligned}$$

for certain positive constants c_1, c_2 when $-\frac{3}{4}R \leq u \leq \frac{3}{4}R$. □

We are ready to state the technical main result of this section.

THEOREM 3.35. *Let ψ denote an eigensection of the operator $\mathcal{D}^R$ corresponding to an eigenvalue μ, where $|\mu| < \lambda_1$. Assume that ψ is orthogonal to the subspace $P_{e^{-R\lambda_1/4}}\mathcal{V}^R \subset L^2(M^R;S)$. Then there exists a positive constant c, such that $|\mu| > c$.*

To prove the theorem we may assume that $|\psi| = 1$. We begin with an elementary consequence of Lemma 3.32.

LEMMA 3.36. *For any $s \in \mathcal{V}^R$ we have*

$$|\langle\psi; s\rangle| \leq e^{-R\lambda_1/2}|s| .$$

Proof. We have

$$
\begin{aligned}
|\langle\psi; s\rangle| &= |\langle\psi; P_{e^{-R\lambda_1/4}}(s) + (s - P_{e^{-R\lambda_1/4}}(s))\rangle| = |\langle\psi; P_{e^{-R\lambda_1/4}}(s)\rangle| \\
&\le |\psi|\ |P_{e^{-R\lambda_1/4}}(s)| \le e^{-R\lambda_1/2}\,|s|\,.
\end{aligned}
$$

□

We want to compare ψ with the eigensections on the corresponding manifolds with cylindrical ends. We use ψ to construct a suitable section on $M_2^\infty = ((-\infty, R] \times \Sigma) \cup M_2$ (Note the reparametrization compared with the convention chosen in the beginning of this chapter). Let $h : M_2^\infty \to \mathbb{R}$ be a smooth increasing function such that h is equal to 1 on M_2 and h is a function of the normal variable on the cylinder, equal to 0 for $u \le \frac{1}{2}R$, and equal to 1 for $\frac{3}{4}R \le u$. We also assume, as usual, that $|\frac{\partial^p h}{\partial u^p}| \le \gamma R^{-p}$ for a certain constant $\gamma > 0$. We define

$$
\psi_2^\infty(x) := \begin{cases} h(x)\psi(x) & \text{for } x \in M_2^R \\ 0 & \text{for } x \in (-\infty, 0] \times \Sigma. \end{cases}
$$

Proposition 3.37. *There exist positive constants c_1, c_2, such that*

$$
|\langle\psi_2^\infty; s\rangle| \le c_1 e^{-c_2 R}\,|s|
$$

for any $s \in \operatorname{Ker} \mathcal{D}_2^\infty$.

Proof. For a suitable cutoff function f_2^R we have

$$
\begin{aligned}
|\langle\psi_2^\infty; s\rangle| &= \left|\int_{M_2^\infty} (\psi_2^\infty(x); s(x))\,dx\right| = \left|\int_{M_2^R} \big(h(x)\psi(x); f_2^R(x)s(x)\big)\,dx\right| \\
&\le \left|\int_{M_2^R} \big(\psi(x); f_2^R(x)s(x)\big)\,dx\right| + \left|\int_{M_2^R} \big((1-h(x))\psi(x); f_2^R(x)s(x)\big)\,dx\right|.
\end{aligned}
$$

We use Lemma 3.36 to estimate the first summand:

$$
\begin{aligned}
\left|\int_{M_2^R} \big(\psi(x); f_2^R(x)s(x)\big)\,dx\right| &= \left|\int_{M_2^R} \big(\psi(x); \big(0 \cup_{f^R} s\big)(x)\big)\,dx\right| \\
&= |\big\langle\psi; 0 \cup_{f^R} s\big\rangle| \le e^{-R\lambda_1/2}\ |s|\,.
\end{aligned}
$$

We use Lemma 3.34 to estimate the second summand:

$$\begin{aligned}
&\left|\int_{M_2^R} \big((1-h(x))\psi(x); f_2^R(x)s(x)\big)\ dx\right| \\
&\leq \int_{M_2^R} \left|\big((1-h(x))\psi(x); f_2^R(x)s(x)\big)\right|\ dx \\
&\leq \int_{M_2^R} |((1-h(x))\psi(x))|\ \left|f_2^R(x)s(x)\right|\ dx \\
&\leq \left(\int_{M_2^R} |((1-h(x))\psi(x))|^2\ dx\right)^{\frac{1}{2}} |s| \\
&\leq \left(\int_0^{\frac{3}{4}R} \left|\psi_{|\{u\}\times\Sigma}\right|^2\right)^{\frac{1}{2}} |s|\ du \\
&\leq \left(c_1^2 e^{-2c_2R}\frac{3}{4}R\right)^{\frac{1}{2}} |s| \leq c_3 e^{-c_4R}\,|s|\,.
\end{aligned}$$

□

Proof of Theorem 3.35. Now we estimate μ^2 from below by following the proof of Theorem 3.17. We choose $\{s_k\}_{k=1}^{q_2}$ an orthonormal basis of the kernel of the operator $\mathcal{D}_2^\infty$. Let us define

$$\widetilde{\psi} := \psi_2^\infty - \sum_{k=1}^{q_2} \langle \psi_2^\infty, s_k\rangle\, s_k.$$

Then $\widetilde{\psi}$ is orthogonal to $\operatorname{Ker}\mathcal{D}_2^\infty$, and it follows from Proposition 3.37 that

$$\left|\widetilde{\psi}\right| \geq \tfrac{1}{3}\,|\psi_2^\infty| > \kappa > 0,$$

for R large enough, where κ is independent of R, of the specific choice of the eigensection ψ, and of the cutoff function h. Let μ_1^2 denote the smallest nonzero eigenvalue of the operator $(\mathcal{D}_2^\infty)^2$. Once again, it follows from the Min-Max Principle, that $\left\langle (\mathcal{D}_2^\infty)^2\,\widetilde{\psi}; \widetilde{\psi}\right\rangle \geq \mu^2\kappa^2$. We have

$$\begin{aligned}
\mu^2 = \left\langle \left(\mathcal{D}^R\right)^2\psi;\psi\right\rangle &\geq \int_{M^R} \left|\mathcal{D}^R\psi\,(x)\right|^2\ dx \\
&= \int_{M^R} \left|\mathcal{D}^R\left(h(x)\psi\,(x) + (1-h(x))\,\psi\,(x)\right)\right|^2\ dx \\
&\geq \int_{M^R} \left|\mathcal{D}^R h(x)\psi\,(x)\right|^2 dx - \int_{M^R} \left|\mathcal{D}^R((1-h(x))\,\psi\,(x))\right|^2\ dx
\end{aligned}$$

It is not difficult to estimate the first term from below. We have

$$\int_{M_2^\infty} \left|\left(\mathcal{D}_2^\infty \psi_2^\infty\right)(x)\right|^2 \, dx = \left\langle \left(\mathcal{D}_2^\infty\right)^2 \psi_2^\infty ; \psi_2^\infty \right\rangle = \left\langle \left(\mathcal{D}_2^\infty\right)^2 \widetilde{\psi} ; \widetilde{\psi} \right\rangle \geq \mu_1^2 \kappa^2$$

We estimate the second term as follows:

$$\int_{M^R} \left|\mathcal{D}^R (1 - h(x)\psi(x))\right|^2 \, dx = \int_{M_2^R} \left|(1-h(x))\left(\mathcal{D}^R \psi\right)(x)) - \sigma(x) \tfrac{\partial h}{\partial u}(x) \psi(x)\right|^2 \, dx$$

$$\leq \int_{M_2^R} \left(\left|\mu\left(1-h(x)\right)\psi(x)\right)\right|^2 + 2\left|\mu\left(1-h(x)\right)\psi(x)\right)\right| + \left|\sigma(x) \tfrac{\partial h}{\partial u}(x)\psi(x)\right|^2 \right) \, dx$$

Now we use Lemma 3.34 successively to estimate each summand on the right side by $c_1 e^{-c_2 R}$. This gives us

$$\int_{M_2^R} \left|\mathcal{D}^R\left(\left(1-h\right)\psi\right)(x)\right|^2 \, dx \leq c_3 e^{-c_4 R},$$

and finally we have $\mu^2 \geq \mu_1^2 \kappa^2 - c_3 e^{-c_4 R} \geq \frac{1}{2}\mu_1^2\kappa^2$ for R large enough. □

Theorem 3.29 is an easy consequence of Theorem 3.35.

3.9. The Additivity for Spectral Boundary Conditions. We finish the proof of Theorem 3.8. We still have to show equation (3.21), i.e.

Lemma 3.38. *We have* $\eta^R = \mathrm{O}(e^{-cR}) \bmod \mathbb{Z}$ *where*

$$\eta^R := \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\mathcal{D}^R e^{-t\mathcal{D}_R^2}\right) dt.$$

Proof. It follows from Theorem 3.29 that we have 'exponentially small' eigenvalues corresponding to the eigensections from the subspace $\mathcal{W}^R$ and the eigenvalues μ bounded away from 0, with $|\mu| \geq a_1$, corresponding to the eigensections from the orthogonal complement of $\mathcal{W}^R$. First we show that we can neglect the contribution due to the eigenvalues that are bounded away from 0. We are precisely in the same situation as with the large t asymptotic of the corresponding integral for the Atiyah–Patodi–Singer boundary problem on the half manifold with the cylinder attached. Literally, we can repeat the proof of Lemma 3.28 by replacing $\mathcal{D}^R_{2,P_>}$ by $\mathcal{D}^R$ and the uniform bound for the smallest positive eigenvalue of $\mathcal{D}^R_{2,P_>}$ by our present bound a_1. Thus, we have

$$\left|\int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\mathcal{D}^R e^{-t\mathcal{D}_R^2}|_{(\mathcal{W}^R)^\perp}\right) dt\right| \leq \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} \left\{\sum\nolimits_{|\mu|\geq a_1} |\mu| e^{-t\mu^2}\right\} dt$$

$$\leq \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} \left\{\sum\nolimits_{|\mu|\geq a_1} e^{-(t-1)\mu^2}\right\} dt \leq \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} \left\{\sum\nolimits_{|\mu|\geq a_1} e^{-\mu^2}\right\} e^{-(t-2)a_1^2} \, dt$$

$$\leq e^{2a_1^2} \operatorname{Tr}\left(e^{-t\mathcal{D}_R^2}\right) \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} e^{-ta_1^2} \, dt = e^{2a_1^2} \operatorname{Tr}\left(e^{-t\mathcal{D}_R^2}\right) \frac{1}{a_1} \int_{\sqrt{R}}^\infty \frac{1}{\sqrt{t}} e^{-ta_1^2} \, a_1 \, dt$$

$$\leq \frac{e^{2a_1^2}}{2a_1} \operatorname{Tr}\left(e^{-t\mathcal{D}_R^2}\right) e^{-a_1^2\sqrt{R}}.$$

For the last inequality see (3.41). A standard estimate on the heat kernel of the operator $\mathcal{D}^R$ gives (as in (3.42)) the inequality $\operatorname{Tr}\left(e^{-t\mathcal{D}_R^2}\right) \leq b_3 \cdot \operatorname{Vol}(M^R) \leq b_4 R$, which implies that

$$\left|\int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\mathcal{D}^R e^{-t\mathcal{D}_R^2}|_{(\mathcal{W}^R)^\perp}\right) dt\right| \leq b_5 e^{-b_6\sqrt{R}}. \tag{3.45}$$

That proves that the contribution from the large eigenvalues disappears as $R \to \infty$. The essential part of η^R comes from the subspace $\mathcal{W}^R$:

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}\left(\mathcal{D}^R e^{-t\mathcal{D}_R^2}|_{\mathcal{W}^R}\right) dt = \\ &\sum_{|\mu|<a_1} \frac{1}{\sqrt{\pi}} \int_{\sqrt{R}}^{\infty} \frac{1}{\sqrt{t}} \mu e^{-t\mu^2}\, dt = \sum_{|\mu|<a_1} \operatorname{sign}(\mu) \frac{2}{\sqrt{\pi}} \int_{|\mu| R^{1/4}}^{\infty} e^{-v^2}\, dv. \end{aligned} \tag{3.46}$$

It follows from Theorem 3.29 that $\lim_{R\to\infty} |\mu|\, R^{1/4} = 0$. Thus, the right side of (3.46) is equal to

$$\operatorname{sign}_R(\mathcal{D}) := \sum_{|\mu|<a_1} \operatorname{sign}(\mu) \tag{3.47}$$

plus the smooth error term which is rapidly decreasing as $R \to \infty$. □

Thus we have proved Theorem 3.8. In particular, we have proved

$$\lim_{R\to\infty} \eta_{\mathcal{D}^R}(0) \equiv \lim_{R\to\infty} \left\{\eta_{\mathcal{D}^R_{1,P_<}}(0) + \eta_{\mathcal{D}^R_{2,P_>}}(0)\right\} \bmod \mathbb{Z}.$$

To establish the true additivity assertion of Corollary 3.9, we show that the preceding η invariants do not depend on R modulo integers.

Proposition 3.39. (W. Müller.) *The eta invariant $\eta_{\mathcal{D}^R_{2,P_>}}(0) \in \mathbb{R}/\mathbb{Z}$ is independent of the cylinder length R.*

Proof. Near to the boundary of M_2^R we parametrize the normal coordinate $u \in [-R, 1)$ with the boundary at $u = -R$. First we show that $\dim \operatorname{Ker} \mathcal{D}^R_{2,P_>}$ is independent of R. Let $s \in \operatorname{Ker} \mathcal{D}^R_{2,P_>}$, namely

$$s \in C^\infty(M_2^R; S), \quad \mathcal{D}_2^R s = 0, \text{ and } P_>(s|_{\{-R\}\times\Sigma}) = 0. \tag{3.48}$$

As in equation (3.27) (and in equation (3.39) of the proof of Theorem 3.17) we may expand $s|_{[-R,0]\times\Sigma}$ in terms of the eigensections of the tangential operator B:

$$s(u,y) = \sum_{k=1}^{\infty} e^{\lambda_k u} \sigma(y) \varphi_k(y).$$

Let $R' > R$. Then s can be extended in the obvious way to $\widetilde{s} \in \operatorname{Ker} \mathcal{D}^{R'}_{2,P_>}$, and the map $s \mapsto \widetilde{s}$ defines an isomorphism of $\operatorname{Ker} \mathcal{D}^R_{2,P_>}$ onto $\operatorname{Ker} \mathcal{D}^{R'}_{2,P_>}$. Next, observe that there exists a smooth family of diffeomorphisms $f_R : [0,1) \to [-R,1)$ which have the following cutoff properties

$$f_R(u) = \begin{cases} u & \text{for } \frac{2}{3} < u < 1 \\ u+R & \text{for } 0 \le u < \frac{1}{3}. \end{cases}$$

Let $\psi_R : [0,1) \times \Sigma \to [-R,1) \times \Sigma$ be defined by $\psi_R(u,y) := (f_R(u), y)$, and extend ψ_R to a diffeomorphism $\psi_R : M_2 \to M_2^R$ in the canonical way, i.e., ψ_R becomes the identity on $M_2 \setminus ((0,1) \times \Sigma)$. There is also a bundle isomorphism which covers ψ_R. This induces an isomorphism $\psi_R^* : C^\infty(M_2^R; S) \to C^\infty(M_2; S)$. Let $\widetilde{\mathcal{D}_2^R} := \psi_R^* \circ \mathcal{D}_2^R \circ (\psi_R^*)^{-1}$. Then $\{\widetilde{\mathcal{D}_2^R}\}_R$ is a family of Dirac operators on M_2, and $\widetilde{\mathcal{D}_2^R} = \sigma(\partial_u + B)$ near Σ. We pick the self-adjoint L^2 extension defined by $\operatorname{Dom} \widetilde{\mathcal{D}_{2,P_>}^R} := \psi_R^* \left(\operatorname{Dom} \mathcal{D}_{2,P_>}^R \right)$. Hence,

$$\eta_{\widetilde{\mathcal{D}_{2,P_>}^R}}(s) = \eta_{\mathcal{D}_{2,P_>}^R}(s) \quad \text{and} \quad \operatorname{Ker} \widetilde{\mathcal{D}_{2,P_>}^R} = \psi_R^* \operatorname{Ker} \mathcal{D}_{2,P_>}^R\,.$$

In particular, $\dim \widetilde{\mathcal{D}_{2,P_>}^R}$ is constant, and we apply variational calculus to get

$$\frac{d}{dR}\left(\eta_{\mathcal{D}_{2,P_>}^R}(0) \right) = -\frac{2}{\sqrt{\pi}} c_m(R),$$

where $c_m(R)$ is the coefficient of $t^{-1/2}$ in the asymptotic expansion of

$$\operatorname{Tr} \dot{A}_R e^{-tA_R^2} \sim \sum_{j=0}^{\infty} c_j(R) t^{(j-m-1)/2}$$

with $A_R := \widetilde{\mathcal{D}_{2,P_>}^R}$ and $m := \dim M$. Now let S_R^1 denote the circle of radius $2R$. We lift the Clifford bundle from Σ to the torus $\mathbf{T}_R := S_R^1 \times \Sigma$. We define the action of $\widehat{D^R} : C^\infty(\mathbf{T}_R, S) \to C^\infty(\mathbf{T}_R, S)$ by $\widehat{D^R} = \sigma(\partial_u + B)$. Since $c_m(R)$ is locally computable, it follows in the same way as above that

$$\frac{d}{dR}\left(\eta_{\widehat{D^R}}(0) \right) = -\frac{2}{\sqrt{\pi}} c_m(R).$$

But a direct computation shows that the spectrum of $\widehat{D^R}$ is symmetric. Hence $\eta_{\widehat{D^R}}(s) = 0$ and, therefore, $c_m(R) = 0$. □

In the same way we show that $\eta_{\mathcal{D}^R}$ is independent of R. This proves the additivity assertion of Corollary 3.9. In fact, we have proved a little bit more:

THEOREM 3.40. *The following formula holds for R large enough*

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1,P_<}}(0) + \eta_{\mathcal{D}_{2,P_>}}(0) + \operatorname{sign}_R(\mathcal{D}),$$

where $\operatorname{sign}_R(\mathcal{D}) := \sum_{|\mu| < a_1} \operatorname{sign}(\mu)$*; see (3.47).*

Theorem 3.40 has an immediate corollary which describes the case in which our additivity formula holds in $\mathbb{R}$, not just in $\mathbb{R}/\mathbb{Z}$.

COROLLARY 3.41. *If* $\operatorname{Ker}\mathcal{D}_1^\infty = \{0\} = \operatorname{Ker}\mathcal{D}_2^\infty$, *then*

$$\eta_{\mathcal{D}}(0) = \eta_{\mathcal{D}_{1,P_<}}(0) + \eta_{\mathcal{D}_{2,P_>}}(0).$$

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