Math: Easy *and* Hard. Why?[1]

Bernhelm Booß-Bavnbek, Roskilde University, Denmark

Abstract. Math is easy and hard, visible and invisible, inspiring and misleading, useful and destructive, free and under restraint, and people love it or hate it. Why?

I confront encouraging findings on human development and language acquisition (mostly C.S. Peirce and N. Chomsky) with rather sceptical views (mostly P.J. Davis and Y. Manin) to explain why learning and teaching math is easy *and* hard at the same time; why we in mathematics struggle both for product (content, saving-bank of concepts and arguments) *and* process (methods, projects); why the meaning of mathematical understanding is contextually confined, but the triumph of mathematical experience is to become transitional and context free.

This paper is to support the resistance of mathematicians against administrators' purely functional demands. We shall refuse when they ask us to tune in our teaching to the modern zap generation; and we shall further develop original, even risky ideas in our research and not just increase numerically our publication output to satisfy budget claims and funding agencies' priorities.

PREFACE

1. Logo, topic, and approach

Logo and topic. The *logo* for this talk (Fig. 1 below) is from the front page of my recent monograph on the index theory of M.F. Atiyah and I.M. Singer[2] which started with lectures I gave in 1971 in Allende's Chile. The President had given orders that there should be a continued education for all junior and senior high school teachers in the country to mark the new era. So, some Chilean mathematicians invited me after I had recently finished a PhD on that subject in Bonn. They told me to give a series of elementary lectures to their teachers so that everybody could understand what modern mathematics was about and what the content was of the Atiyah-Singer Index Theorem. I gave these lectures; someone took notes during the lectures, so that I came back to Europe in September

[1] Farewell talk of Dec. 17, 2014 at Roskilde University, under the title Learning and Teaching Math: Easy and Hard – How? Reflections, Transcript.

[2] **Bleecker, David D.**; **Booß-Bavnbek, Bernhelm**. Index theory — with applications to mathematics and physics. *International Press, Somerville, MA,* 2013. xxii+769 pp. ISBN: 978-1-57146-264-0 MR3113540.

1971 with a manuscript of 120 pages. This was my first attempt at making a hard topic easy.

These 120 pages turned into a book of 769 pages last year, updated and further detailed so that one has a chance to understand. The publisher put as front page the famous Escher graphic of knights walking a never-ending ascending path that never reaches higher levels: a good symbol of mathematical studies, easy and hard at the same time. You walk and walk, think you have made no progress. After yet another while you think it's easy, you are on the same level now, with better understanding, but you feel it was nothing. This is probably the ambiguity of all learning, but it is specially so for mathematics. That is the *topic* of this talk.

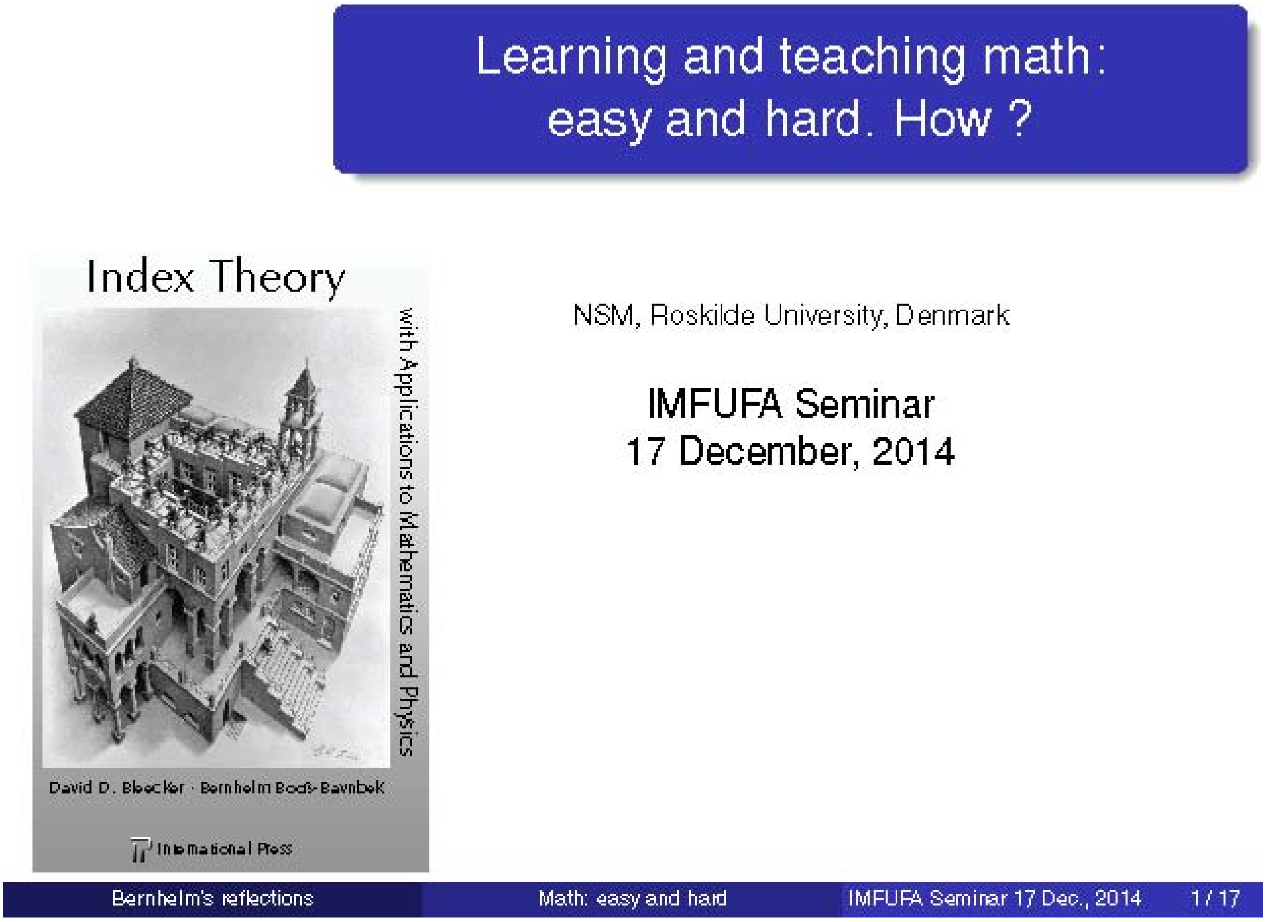

Figure 1. Logo for this talk

Approach. I call this *Reflections* contrary to a learned *Math lecture,* to emphasize the challenges of this talk now. Over the last 50 years I have given many math talks first for my professors and fellow students, then for my students in student-organized project groups and teacher-organized classes and in seminars for my math and physics colleagues. A math talk has a clear structure: You begin either with "Let H be a complex separable Hilbert space…" or with five minutes of how this or that problem has arisen and what the status is now. Then you proceed to your definitions, your claims, your theorem, and then you go to the proof; if time is not running out you also give an example and discuss further perspectives; but the topic gives the structure.

Contrary to math talks, reflections have no immanent structure; all depends on associations, slides are very useful since nobody can take notes anyway, and the order of presentation must be imposed artificially. Reflections are just a completely different style as compared with math talks. For math talks I need not discuss ever new orders of presentation with my wife as I had to do for this talk. Before a math talk I'm not nervous as I am now because in a math talk I believe to know everything relevant for my talk. I feel absolutely safe. In principal, in a math talk you know absolutely what you are doing. You are on the safe side. In that sense, a math talk is always easy.

For reflections, I have no canonical structure. Certainly, I have not read all of the references for the topic, even not necessarily the ones which are most important. Literally, I'm no expert in the field.

Dedication. I held this talk in the respectful memory of Ivor Grattan-Guinness (23 June 1941 – 12 December 2014).

2. Outline

Outline

1. Introduction
   - The meaning of math working experience
2. Looking back - who/what has changed?
   - Not the students: mind stability v. cataclysmic changes
   - Way of talking about our subject
     - Today less metaphysical exaggerations
     - Seminal changes of math content widely ignored
     - Administrative frame: Continuing dismantling of the public sector
3. Math is hard —
   - Students
   - LAGRANGE, GAUSS, PEIRCE, HIRZEBRUCH, SINGER, MANIN, ARNOLD, HÖRMANDER
4. Math is easy —
   - SCHOPENHAUER VS. GAUSS
   - CHOMSKY, ATIYAH, LAGRANGE, PEIRCE, GRAMSCI, FREIRE, NAUR, KIERKEGAARD

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Figure 2. Structure of these reflections

After the *Introduction* about the supposed general meaning of the working experience of mathematicians, I shall talk a short while about myself; looking back and telling you what has changed since I was a beginner. That will be *Part I* of this talk.

Preview of Part I. My first point is a message to the lower and higher ranks of the administrative body continuously asking for a *smoother adaption to a changing student generation*: I strongly resist this permanent talk about students having changed: "*Bernhelm, you have not noticed the changes in the mind of the students. We write the year 2014, not the year 1960 when you began studying math."* I feel I have to give an answer to these claims and complaints.

But something has changed. Happily, we have less exaggerations, vain promises today, and we are more realistic about the role of mathematics in applications as compared to the 1950s-1970s. On the other side, we are no longer speaking as frankly and proudly about the immense achievements of mathematics that happened in my own lifetime.

It seems to me, however, that the most severe of recent changes is the over-administration: The administrators set up strict regulations, organize a rush for grades, and show their contempt for the contemplative aspects of student and academic life.

Preview of Part II. To begin that Part, I shall summarize why math is, rightly, perceived as hard. I shall tell you about a student project of our Nat-Bachelor education a few years ago that gave me the idea and inspiration for this talk. Here you see my lack of structure. I cannot structure rigorously what I wish to explain. All I can give you is a list of these great white elephants, typically old men who said something important about math teaching and learning. I shall go through this list of witnesses.

After that I shall turn to the opposite position, that math is easy and that it is a scandal and the teachers' failure and proof of their unsufficient effort, if math is not perceived as easily accessible. To me, the heading *Math is hard* is with an exclamation, while I shall answer the possibly well-intended claim *Math is easy* with a question mark. This question is an old philosophical problem. Schopenhauer has made some remarkable comments; Gauss has answered in his remarkable way. Then I have once again an unstructured list of giants who have something interesting to say.

And that will be all.

INTRODUCTION

3. The general meaning of mathematical working experience

Here is an indication of why the mathematical working experience has something to say to the general intellectual public.

Pulls and pushes. On the top to the left side of the diagram (Fig. 3) you see the administrators who tell us all the time, *Make it easy! Remember, it must be easy! Don't lose a student. You are losing the students.* Etc. On the bottom of the left side we have the word of my colleague Mogens Niss repeated for years and years, that

- math is damned hard as long as one has not understood it, and when one has understood it, it is different; and
- math is invisible, unless one looks a bit beneath the surface.

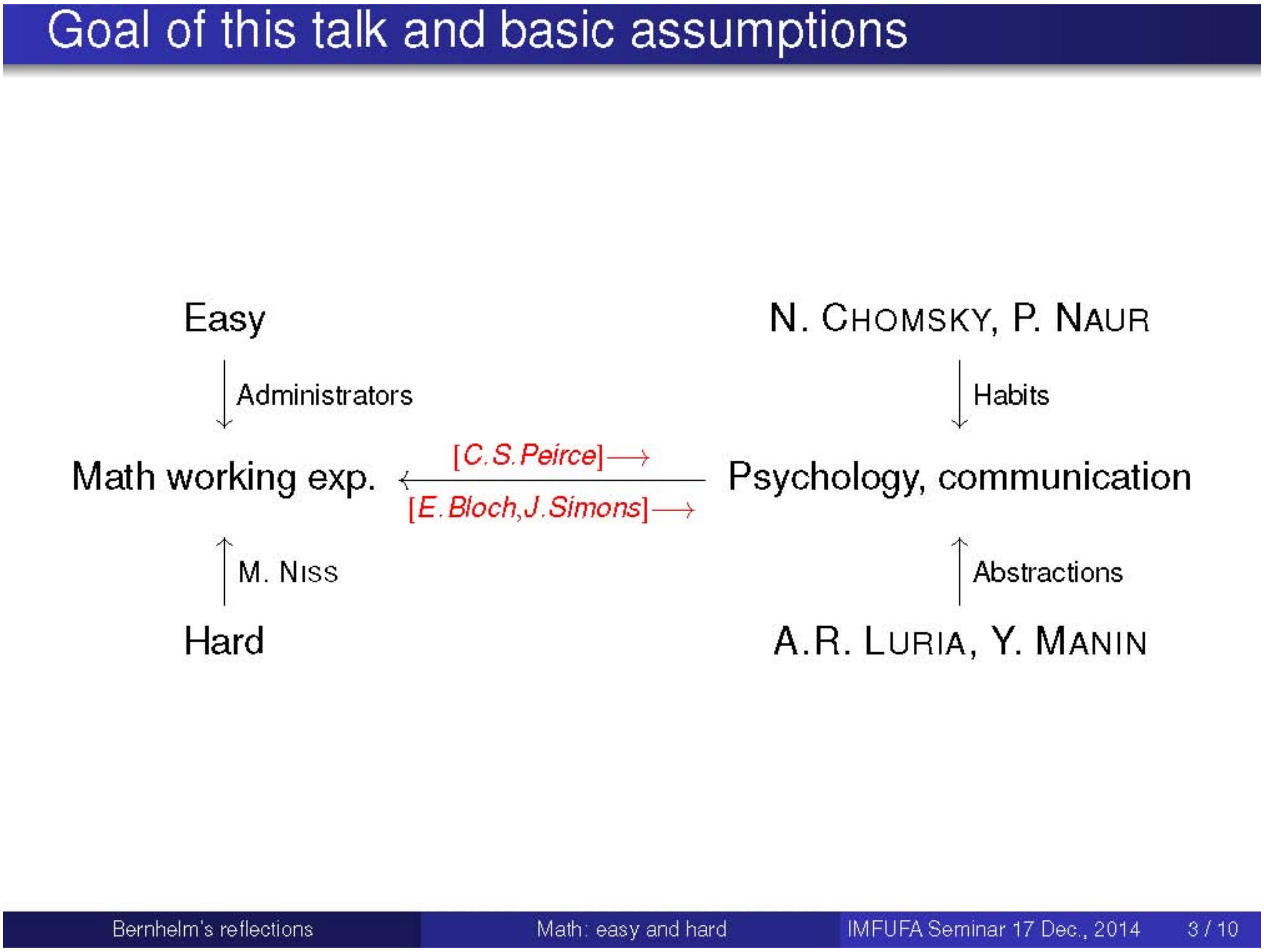

Figure 3. Goals and basic assumptions

These observations by Niss are my basic assumptions which I really don't want to discuss; all people have had their own personal experience: that they got *bruises* from mathematics. They got them. We got them when we were students. We also got bruises from working with mathematics. So, there is no discussion for me about that. This is one basic assumption.

The *invisibility* is my second assumption on the left bottom of the diagram which I'm not going to discuss here. So, my basic assumptions are the existence of various opposite pulls and pushes to math on the left side of the diagram.

On the right side we have the wide area of human psychology and communication. In the centre of the diagram is a black arrow pointing from the area of psychology and communication back to the math working experience. The black arrow symbolizes my lifelong hope that, for learning and teaching math, I get something interesting out of the theories of communication and psychology; get some hints at how to make teaching better. Many mathematicians share my hope: *If only I understood learners' psychology better!*

On the upper right is an input from Noam Chomsky and Peter Naur which I shall explain in more detail later in this talk. To me they can serve as brilliant advocates for the case that math is easy, natural, rooted in human habits: No reason to be afraid of it. On the bottom right we have the opposite influences due to the Russian psychologist Alexander Luria and the Russian mathematician Yuri Manin: They explain why we must accept that mathematical abstractions are hard and contrary to everyday judgements; with formal logic, abstractions, signs, and symbols.

So, the black arrow from the right to the left was my original plot for years. Of course, there are also the red arrows indicating new insight for psychology and communication that can be derived from the math working experience, namely handling something that is easy and hard at the same time and where the frontiers are difficult to line out between the areas that are easy and those that are hard. I have borrowed the idea of the red arrows from the US-American philosopher Charles Sanders Peirce. It was our Roskilde colleague Peder Voetmann Christiansen who introduced me to Peirce's immensely rich world of thinking. I learned of

- Peirce's *desperate hope*[3] of using the working experiences of mathematicians and physicists to substantiate a philosophy of sustainable behaviour; and
- Peirce's *claim to philosophy*, namely to explain how meaningful communication between humans is possible, i.e., to investigate the reason and the conditions for the possibility of inter-subjective communication between humans (Klaus Oehler's

[3] **Peirce, Charles S.** The Architecture of Theories (unpublished notes for Monist paper no. 1 of same title, MS 956; NAT 126-135, around 1890). German translation of excerpts in **Peirce, Charles S.** Naturordnung und Zeichenprozess. Schriften über Semiotik und Naturphilosophie. Mit einem Vorwort von Ilya Prigogine. Herausgegeben und eingeleitet von Helmut Pape. *Suhrkamp Taschenbuch. Wissenschaft, Frankfurt a. Main*, 1991. ISBN 3-518-28512-2. „Das Ziel einer Theorie ist, etwas verständlich zu machen. Das Ziel der Philosophie ist, alles verständlich zu machen. Die Philosophie postuliert somit, dass die Naturvorgänge verständlich sind. *Postuliert* sage ich, nicht: nimmt an. Es mag sich anders verhalten; aber nur soweit es sich so verhält, kann die Philosophie ihren Zweck erfüllen. Sie ist daher gehalten, sich *nach* dieser Annahme *zu richten*, sei sie nun wahr oder nicht. Sie ist eine verzweifelte Hoffnung. Aber soweit der Naturprozess verständlich ist, ist der Naturprozess mit dem Vernunftprozess identisch. Es muss praktisch angenommen werden, dass das Gesetz des Seins und das Gesetz des Denkens eins sind." (p. 133)

re-formulation[4,5]). Here the point is that we have many examples of successful inter-subjective communication without understanding why it works; and that the math working experience may help to understand successes.

To illustrate the read arrows, the ones going from left to right in the diagram, I refer to some people in quite different contexts who demonstrate the usefulness of the afore-mentioned math working experience in psychology and communication:

1. **Inevitable conflict between *Easy* and *Hard*.** In 1935, the exiled German Marxist philosopher Ernst Bloch (1885--1977) recognized the core problem of doing math, namely the inevitable conflict between *Easy* and *Hard* in forms of political communication: *The Nazis talk fraudulently, but to humans; the communists' talking is completely true, but only on things*.[6]
2. **Intimate relation between mathematical thinking and common sense**. To emphasize the relation between mathematical thinking and common sense I refer to the US-American mathematician James H. Simons. In 1966, he proved a breaking new result about multidimensional varieties. Later he left mathematics. In 2014 he became one of the 100 richest people in the world and invited speaker for the *Einstein Public Lecture in Mathematics* of the American Mathematical Society and the Mathematical Sciences Research Institute. His recipe for getting rich: *Mathematics + Common Sense + Good Luck*.[7]
3. **Need for sensitivity and judgement**. Peace researchers and peace activists express the basic math working challenge when they pray like late-medieval monks: *O my Lord, give me the strength to fight where I can change, the patience to abstain from hopeless fights, and the wisdom to distinguish between the two situations*.

So much about the general meaning of math working experience.

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[4] **Peirce, Charles S.** How to make our ideas clear (1878). German introduction, translation, and comments by Klaus Oehler. Klostermann Texte Philosophie 1968. 3d ed. 1985, *Vittorio Klostermann, Frankfurt am Main*. ISBN 3-465-01650-5. "Nach Peirce ist es Aufgabe der Philosophie, Klarheit über Methoden und Bedingungen der Erkenntnis hervorzubringen und den Zusammenhang von Wissenschaft, Geschichte, Leben und Handeln des Menschen aufzuklären und so zur Meisterung der Zukunft der Menschenwelt beizutragen, indem sie die Menschen dazu anhält, ihr Bewusstsein zu entwickeln und auszubilden für die Bedingungen, ohne die Fortschritt, Wachstum und ein harmonisches Zusammenleben nicht mehr möglich sind." (p. 24)

[5] **Oehler, Klaus**. Sachen und Zeichen. Zur Philosophie des Pragmatismus.*Vittorio Klostermann, Frankfurt a. Main*, 1995. "Unter den durch die Natur- und Geschichtswissenschaften veränderten Bedingungen des philosophischen Denkens stellte sich in der zweiten Hälfte des neunzehnten Jahrhunderts die alte Frage nach dem Grund der Möglichkeit der intersubjektiven Verständigung der Menschen neu, und alte Antworten verloren ihre Gültigkeit. ... Für Peirce stellte sich das Problem in Gestalt der Frage: Wie ist Kommunikation überhaupt möglich? Anders als Kant näherte er sich dem Problem so, dass er die Untersuchung der Formen des Denkens bewusst als eine Untersuchung des Gebrauchs der Sprache durchführte und schließlich als erster aus der Erkenntnis, dass alle unsere Äußerungen Zeichen sind, den Schluss zog, dass Kommunikation nur in den Rahmen einer allgemeinen Theorie der Zeichen erklärbar und darstellbar ist. ... Seine wichtigste Neuerung auf dem Gebiet der Zeichentheorie ... war, dass er Zeichen als gesellschaftlich genormte Vermittlungsinstanzen deutete, durch die etwas, zum Beispiel ein Wort oder eine Gebärde, auf etwas anderes, das Bezeichnete, sich bezieht, und diese Beziehung von einem Dritten, dem Rezipienten oder Interpreten des Zeichens, verstanden wird, der selber wiederum als ein Zeichen interpretierbar ist, und so weiter ins Unendliche eines Interpretationszusammenhanges, der unsere Welt ist." (pp. 34-35)

[6] **Bloch, Ernst.** Erbschaft dieser Zeit (Zürich, 1935), reprinted in *Gesamtausgabe Bd. 4, Suhrkamp, Frankfurt am Main* 1985. "Nazis sprechen betrügend, aber zu Menschen, die Kommunisten völlig wahr, aber nur von Sachen." (p. 153)

[7] https://www.msri.org/general_events/20847#description (accessed January 13, 2015).

PART I

4. Looking back – has the students’ mind-set changed?

I told you already of the administrators’ claim: *The mind of the students has changed and your teaching is worthless unless you change, too.* That claim is supported by the general “wisdom” that we deal with a browse-generation or a me-generation.

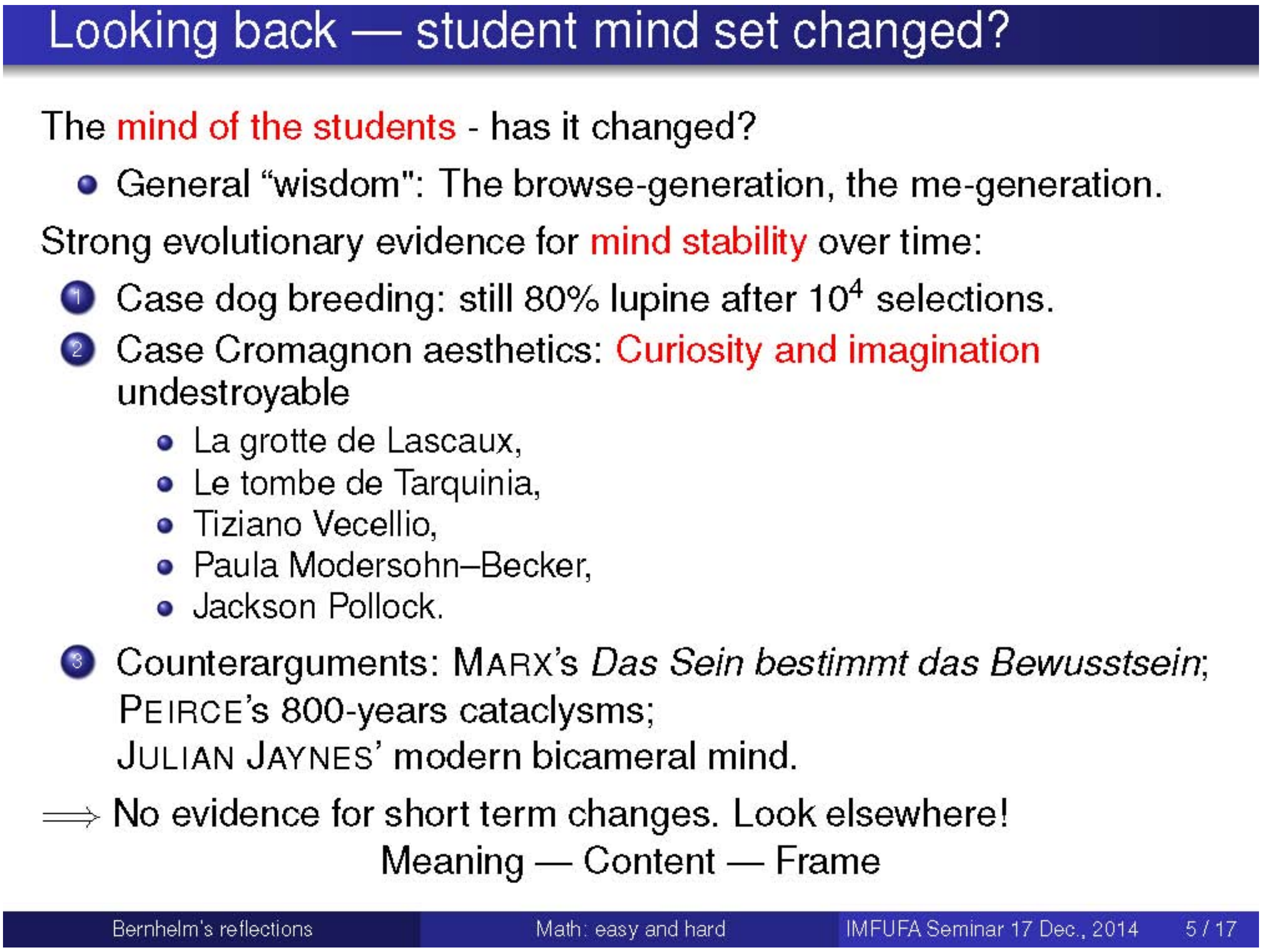

Figure 4. Quick review: Has the student mind set changed?

My message to you is that the administrators’ claim is misleading; in the sociology and neuropsychology literature, there is no evidence of such sudden and general deep changes regarding our students’ or our own capacity of doing and learning mathematics (see Fig. 4 for a quick review).

Evolutionary evidence. My wife is a dog breeder. Our ancestors have bred dogs for at least 20.000-30.000 years, perhaps for 500.000 years.[8] Of course, modern dogs do not look like wolves and each breed looks different. But the animal psychologists write in their books that we must expect 80% of the behaviour of our dog to be lupine.[9] I do not know how they measure and quantify, but everyone who has dogs can confirm: After 5.000 breeding selections the basic behaviour has almost not changed. There is an astonishing stability.

Figure 5. Long-time stability regarding creativity, sensitivity, imagination, and persistence

[8] A devastating account of the uncertainties and contradictions in present knowledge and scientific and pseudo-scientific claims about dog domestication can be found in **Larson, Greger; et al.** Rethinking dog domestication by integrating genetics, archaeology, and biogeography. *Proc Natl Acad Sci U S A.* Jun 5, 2012; 109(23), 8878–8883.

[9] **Svartberg, Kenth; Forkman, Björn**. Personality traits in the domestic dog (Canis familiaris). *Applied Animal Behaviour Science* **79** (2002), 133–155: "The finding of a major behavioural dimension in different groups of dog breeds, together with comparable results previously found for wolves (Canis lupus), suggests that the dimension is evolutionarily stable and has survived the varied selection pressures encountered during domestication." See also **McBride, Anne**. The human–dog relationship. In: Robinson, I. (Ed.), The Waltham Book of Human–Animal Interaction: Benefits and Responsibilities of Pet Ownership*. Pergamon, Oxford*, 1995. ISBN 978-1483234748, pp. 99–112.; and **Tami, Gabriela; Gallagher, Anne**. Description of the behaviour of domestic dog (Canis familiaris) by experienced and inexperienced people. *Applied Animal Behaviour Science* **120** (2009), 159–169.

When talking about human learning and teaching, the human mind is quite different, much more variable, namely much more able to adapt to new situations. But is there evolutionary evidence for deep changes in basic human behaviour?

You get one answer when you look at the Cro-Magnon aesthetics. You will notice a strong evidence for the apparently indestructible stability of *human curiosity* and *imagination*, of *creativity* and *concentration*, of all that we want from our students. Look at La grotte de Lascaux, Le tombe di Tarquinia, Tiziano Vecellio's Charles Vth Emperor, Paula Modersohn-Becker, and Jackson Pollock.

Apparently, what we love in our students, their curiosity, sensitivity, imagination, constructive power, and persistence has been socially stable over long, long periods.

Counterarguments. There are some counterarguments:

Karl Marx (1818-1883). In his *Zur Kritik der politischen Ökonomie, Vorwort* of 1859 (MEW13, pp. 8-9), Marx emphasized that *Das Sein bestimmt das Bewusstsein* – i.e., the way of being, the social circumstances are determining for your consciousness, and not the opposite way around as all religion and many philosophers and psychologists will tell you.

Seemingly, Marx points in another direction, contrary to my claim of student mind-set stability. However, what Marx was addressing were the big secular changes from feudalism to capitalism or even from older societies to more modern ones with radically changed means of production and property and dominance relations; and within society different classes that have completely different life situations[10]. It would be a misreading of Marx to conclude *Now we have globalization and cell phones; the youngsters have a completely different life situation than a few decades ago. Can you imagine: You had no cell phone when you were a kid, but they have! Their mind must have changed!*

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[10] I better quote in context: "The totality of these relations of production constitutes the economic structure of society, the real foundation, on which arises a legal and political superstructure and to which correspond definite forms of social consciousness. The mode of production of material life conditions the general process of social, political and intellectual life. It is not the consciousness of men that determines their existence, but their social existence that determines their consciousness. At a certain stage of development, the material productive forces of society come into conflict with the existing relations of production or -- this merely expresses the same thing in legal terms -- with the property relations within the framework of which they have operated hitherto. From forms of development of the productive forces these relations turn into their fetters. Then begins an era of social revolution. The changes in the economic foundation lead sooner or later to the transformation of the whole immense superstructure. In studying such transformations it is always necessary to distinguish between the material transformation of the economic conditions of production, which can be determined with the precision of natural science, and the legal, political, religious, artistic or philosophic -- in short, ideological forms in which men become conscious of this conflict and fight it out. Just as one does not judge an individual by what he thinks about himself, so one cannot judge such a period of transformation by its consciousness, but, on the contrary, this consciousness must be explained from the contradictions of material life, from the conflict existing between the social forces of production and the relations of production."
[http://www.marxists.org/archive/marx/works/download/Marx_Contribution_to_the_Critique_of_Political_Economy.pdf, p. 4, accessed January 19, 2015]. German original at http://www.mlwerke.de/me/me13/me13_007.htm

The common exaggeration of the role of gadgets and mass communication is unfounded: 100 years ago, children in the Danish harbour of Elsinore were following much more rapid innovations in ship building and propulsion than the now ongoing slow and mostly marginal innovations in electronic toys, all modelled on the only technological break-through of the last 70 years, namely the remote control of a TV set. Every day, ships were running into the harbour from all parts of the world for shortly afterwards leaving for their next turn. They confronted children with a much more diverse globalization than electronic media can do now. Certainly, the insecurity of the parents' job situation, the narrow flats and the permanent noise of the neighbour's children calling for sharing the next street game were more decisive obstructions against intellectual and academic contemplation than the superficial consumerism of electronic offers.

Charles S. Peirce (1839-1914). Another, possibly more serious but rather speculative counterargument is due to Peirce. He had a weird love for mysticism. He counted years and found that every 800 years back the public mind was completely different.

He wrote this around 1900 and told the reader that around 1100, with the confrontation between Christian and Moslem beliefs, the way of thinking, the scholastic thinking, was radically different from our way of thinking; and that we cannot really understand how people were thinking around 1100. He considered the 800 years cycle a biological constant of the human brain; and indeed, around the year 300 of our time you can see with Constantine the Great another radical break when the whole mind-set of classical antiquity was abolished and replaced by the Christian state religion of late antiquity. A completely new mind-set had appeared. The next 800 years back you are in 500 BCE with Solon in Athens and the Mediterranean openness to long-distance commerce and mind games of democratic rule, scenic theatre and a greater authority of math competence than that of religion. And so on, though with ever more fading evidence of the differences in assumed mind-sets for every 800 years further back.

Julian Jaynes (1920-1997). A third counterargument against my claim of strong student mind-set stability can be derived from the work of the neurologist Julian Jaynes in the 1970s. He claims that the triumph of the modern bicameral mind is a relatively new event. Roughly summarizing him, he refers to Homer and other written sources of early Mediterranean antiquity that report about how people could hear inner voices of their imagined gods and ancestors.

Jaynes explains this by an early lack of a physical membrane or psychological barrier between the two cameras of the brain, contrary to the bicameral brain of almost all presently living humans.[11] If we are willing to follow Jaynes, we can postulate that there has been a radical change of human mind-sets beginning around 3.000 years ago. Once again, this is a completely different time scale.

What is the supposed ongoing secular media-generated change of consciousness about? It seems a bit like guessing when Peirce and Jaynes write about dramatic changes of human capacities and consciousness. Their claims are based only on

[11] **Jaynes, Julian**. The origin of consciousness in the breakdown of the bicameral mind. *Houghton Mifflin, Boston*, 1976. Numerous reprints. ISBN 0-618-05707-2.

highly selective evidence, respectively, exaggeration of single phenomena. Can we equally easily discard all the present talking on ongoing dramatic media-generated changes of our students' life style and ways of thinking?

The short answer is yes and no!

- No: social influences cannot be discarded. Otherwise, showing our personal example and delivering our teaching would be dispensable. And that we don't wish to believe.
- Yes: for doing math, disturbing social influences must be eliminated or at least confined; learning math requires time and full concentration, and a student will hardly make progress, if he or she is not able to let her be absorbed by mathematics for hours, days and years.

A. Luria (1902–1977). In 1974, the Soviet neuropsychologist and developmental psychologist Alexander Romanovich Luria published a comprehensive empirical study about cognitive changes induced by social conditions – under the extreme social changes of the first years after the Bolshevist revolution:

> The history of this book is somewhat unusual. All of its observational material was collected in 1931-32, during the Soviet Union's most radical restructuring: the elimination of illiteracy, the transition to a collectivist economy, and the readjustment of life to new socialist principles. This period offered a unique opportunity to observe how decisively all these reforms effected not only a broadening of outlook but also radical changes in the structure of cognitive processes. The Marxist-Leninist thesis that all fundamental human cognitive activities take shape in a matrix of social history and form the products of sociohistorical development was amplified by L. S. Vygotsky to serve as the basis of a great deal of Soviet psychological research. None of the investigations, however, was sufficiently complete or comprehensive to verify these assumptions directly. The experimental program described in this book was conceived in response to this situation, and at Vygotsky's suggestion.[12]

He found indeed remarkable differences, e.g., that people grown up in larger and more urban places were good in taxonomic classification, i.e., the use of *abstract categories* like *tools* while people grown up in remote places were better in classifications regarding *practical situations*, like the work process of *sawing and chopping wood*.[13] Luria himself concluded, that *the basic categories of human mental life can be understood as products of social history — they are subject to change when the basic forms of social practice are altered and thus are social in nature*.[14]

Hence, for doing, learning, and teaching math, Luria's message is that all people can easily adapt to radical new ways of thinking in new environments. The only precondition is

[12] **Luria, Alexander R**. The cognitive development: its cultural and social foundations. *Harvard University Press.* 1976 (Translation from the Russian original of 1974). ISBN 0-674-13731-0. Here p. v.

[13] For details of Luria's careful design of the observational studies see, e.g., p. 64.

[14] l.c., p. 164. Similarly, but from a religious (Roman catholic) perspective the essay **Ong, Walter J**. Interfaces of the word : studies in the evolution of consciousness and culture. *Ithaca, N. Y. Cornell University Press,* 1977. - 352 s. ISBN 0-8014-1105-x.

that the new environments are presented in a peaceful way, however dramatic and even painful the changes and the challenges may be.

- When we agree with McLuhan[15] that not so much the content of the media but their very presence has an impact on the mind, there is only one consequence, namely to tell the students they only can make progress with their studies when they protect themselves against the media dominance. Sorry, there is no third way.
- When we agree with Hurrelmann[16] that a student today - and the professor - meet dramatic increased possibilities (and demands) to design their own personality[17], there is only one consequence, namely to decide whether one wishes that *Selbstsozialisation* or prefers a path of cultural-historical socialization to ones subject.
- When we agree with Sutter's distinction of the two levels of media competence[18], control of media devices and understanding the offers of media of all kind, then we must applaud and support the students in their efforts on both levels.

Preliminary conclusion. There is no evidence of short term changes of the students' mind-set. We must look somewhere else, we must find other places of change when we wish to understand how and why the learning and teaching of mathematics has changed or is changing, almost by necessity.

The places I shall have a closer look at in the following are
- how we speak of the meaning of mathematics,
- changes in the content of mathematics, and
- how the institutional frame of learning and teaching math has changed in critical directions.

5. Looking back – vain promises?

In 1976, Klaus Krickeberg and I edited an anthology *Mathematisierung der Einzelwissenschaften – On the mathematization of the single sciences* with the publisher Birkhäuser in Basel, now bought by Springer.[19] The acceptance by the publisher was on the recommendation of Alexander Ostrowski.

[15] **McLuhan, Marshall; with Fiore, Quentin; produced by Jerome Agel**. The Medium is the Massage: An Inventory of Effects. 1st Ed.: *Random House* 1967; reissued by *Gingko Press*, 2001. ISBN 1-58423-070-3.

[16] **Hurrelmann, Klaus**. Selbstsozialisation oder Selbstorganisation? Ein sympathisierender, aber kritischer Kommentar. *Zeitschrift für Soziologie der Erziehung und Sozialisation*, **22**/2 (2002), 155-166.

[17] One of the new phenomena is that girls of 12 or 13 years can't resist the social press or their own wish to upload naked selfies to the internet; see **Politiken Digitalt**. Unge sender hinanden afslørende nøgenbilleder i stort omfang. 3 April, 2014.

[18] **Sutter, Tilmann**. Medienanalyse und Medienkritik: Forschungsfelder einer konstruktivistischen Soziologie der Medien. *VS Verlag für Sozialwissenschaften. Wiesbaden.* 2010. ISBN 978-3-531-16910-1.

[19] Mathematisierung der Einzelwissenschaften. (German) Herausgegeben von Bernhelm Booss und Klaus Krickeberg. Interdisciplinary Systems Research, No. 24. *Birkhäuser Verlag, Basel-Stuttgart,* 1976. 363 pp. ISBN: 3-7643-0867-2 MR0479789.

Recently, a former Bielefeld colleague of mine sent me a note with quotes from a new paper that referred to the old book in quite positive terms.[20] So, I looked again in the book, and was shocked: With hindsight, we got it all wrong.

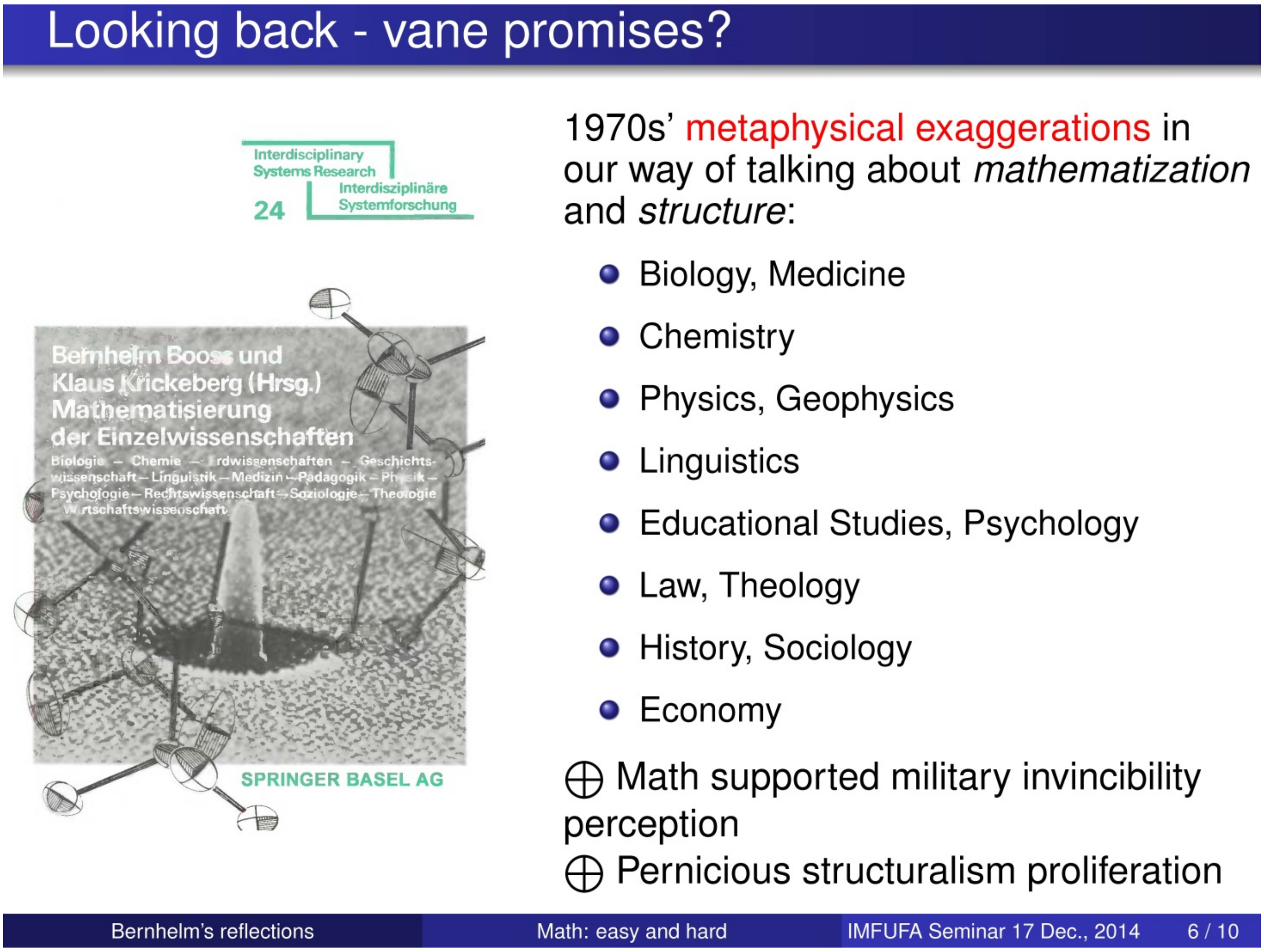

Figure 6. Vane promises of the past

Biology. Let's begin with our chapter on *biology*, drafted by two eminent Bielefeld biologists. Of course, much thinking in biology of that time was under the influence of the then only 20 years old discovery of DNA as carrier of heritable information. There were fantastic ideas about the vanishing walls between biology and medicine: *Now we can understand the cell. For biochemistry and molecular biology, we have superb laboratory equipment. The only thing missing is the mathematics required to melt together observations, problems, and concepts.*

It did not go like that. To the best of my knowledge, there was only one breaking new discovery in pharmacy since the seminal (and possibly soon outdated) discovery of antibiotic by Alexander Fleming and besides the design of nice psychosomatic pills like

[20] **Nickel, Gregor**. Mathematik und Mathematisierung der Wissenschaften. Ethische Erwägungen. In: Jochen Berendes (Hrsg.) Autonomie durch Verantwortung. Impulse für die Ethik in den Wissenschaften. *mentis Verlag. Paderborn.* 2007. ISBN 978-3897855496, pp. 319-346.

Viagra. I mean the re-discovery of the benign health effects of water, both by oral rehydration (preferably by balanced salt solution with certain sugars) against diarrhoea and cold water bathes against skin burn bruises – both contrary to what was common medical wisdom in my school years and both therapies inhibited by their cheap availability.

Three years ago, in our Springer monograph *BetaSys*, we had to refute all these expectations for a single case, namely the insulin secretion from pancreatic beta-cells, where a comprehensive understanding still is missing however wanted for treatment, therapy and cure of diabetes.[21]

Chemistry. In *chemistry* it was the same with exaggerated expectations of the potential role of mathematics. 40 years ago, it was the high time of stereo-chemistry. People thought: *When we have a 3D model of a molecule, suddenly all the chemical and biological properties of the molecule will be released, just from a mathematical analysis of the 3D-formula. Please, mathematicians, help us a bit with the necessary data mining and modelling to organize the stereo chemistry so that we can calculate all the properties of a compound we are interested in - on the back of an envelope or a mainframe computer.*

It didn't turn out that way. Seen from numerical mathematics, however, it is remarkable that finally computer simulations in chemistry have achieved the status of a third methodology, besides experiment and theory – after decades when computer simulations in material sciences were considered either as fancied and most often misleading images of reality or dispensable confirmation of what was known in advance.[22]

Physics. In *physics*, it was the time of the standard model of electrical, strong and weak interactions of particles. One had one striking example, namely how that standard model had led to the discovery of the omega particle. Now, 50 years later, we have perhaps a second example, the discovery (or confirmation) of the Higgs particle exactly in the way that was predicted. OK, that was an achievement mainly of engineering art and lavish funds, but also a bit a result of a mathematical model.

However, the great physics discoveries of new properties in material sciences of the past half century, like the high-temperature superconductivity, were not supported essentially by mathematical models but rather by a return to the patience and endurance of the alchemists with almost endless series of repeated tests until they found that a kind of insolation material shows superconductivity when cooled down relatively modestly.

Our writing about the role of mathematics in physics was misleading because it was oriented and dominated by the standard model. Only when I came to Roskilde I learned from my physics colleagues that the main force of math in physics is seldom a total model for the phenomena one is interested in, but more typical the more modest support of model-based measurements, i.e., the design of a transducer that, e.g., translates the

[21] **Booß-Bavnbek, Bernhelm; Klösgen, Beate; Larsen, Jesper; Pociot, Flemming; Renström, Eric** (eds). BetaSys: Systems biology of regulated exocytosis in pancreatic β-cells. *Springer, New York*, 2011, 558 pp. ISBN 978 1 4419 6955 2.

[22] **Schlick, Tamar**. The 2013 Nobel Prize in Chemistry celebrates computations in chemistry and biology. *SIAM News* **46**/10 (December 2013), 4 pp.

shear modules of thermo-elasticity, one is interested in for understanding soft materials, into easier measurable electric currents and voltages.

The sin of the cycle model of modelling. The experimental physicist colleagues in Roskilde vaccinated me against what the biologists of my book called the cycle model. In the cycle model you begin with a segment of reality. Then you approximate reality as best you can by some formalism. Then you work around with the formalism. Then you compare the achieved results with reality. And then, eventually, you begin a new cycle. Terrible! Meaningless and, actually, misleading. Our fault, and the stupid cycle model is since then apparently not removable from the math educational literature.[23]

Geophysics. The only field where we were right was geophysics, more precisely the field of numerical weather prediction. At that time, we had already very nice pressure charts and temperature measurements. We could predict wind directions, temperatures in different layers, and precipitation, all, though, under the assumption of perseverance, i.e., assuming only homogeneous movements of high and low pressure regions, but no changes in their strength. And we could use our experience; find comparable weather situations in our archives, and make better founded, but in general not really more reliable predictions on that bases.

On the other side, the basic equations for air movements, connecting hydrodynamics with thermodynamics, were given in 1904 by Vilhelm Bjerknes[24], and the numerical schemes to solve them developed by L.F. Richardson during World War I. On the initiative of John von Neumann, numerical weather prediction became the third agenda of the evolving mainframes, after their triumphs in cryptology and the design of the hydrogen bomb. During the 1950s-70s, it was in vain. As predicted by Richardson, the complexity of the calculations was way beyond all machine capacities. So, in principle, there was nothing special in our optimistic evaluation of chances of a break-through in meteorology, besides our expectation that new ideas in the design of numerical algorithms would make the calculations performable. And exactly that happened, in combination with the continuing growth of the machine power.

This morning I heard the weather prediction of the Danish Institute of Meteorology for the next seven days. For Christmas Eve, they predicted that there will be no snow anywhere in Denmark and nowhere a ground temperature below 4°C. 50 years ago, such a prediction would have been ridiculous. Now the odds are 1:10. Still on average every 10th seven-day prediction is qualitatively misleading due to lack of some relevant initial data and due to the mathematical and physical fact that some of the equations are not well-posed. But 9 out of 10 hold. [It turned out that the preceding prediction held only until the early evening of Christmas Eve. Then the temperature dropped rapidly. Later in that night we got 2-3 cm snow in Copenhagen. So, be careful with betting on weather predictions.]

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[23] Of course, it is more demanding to argue for a more realistic, problem oriented description of the semiotic interplay between the signs of reality, the signs of codified experience and the capacity and interest of the sign interpreter in a pragmatic set-up à la Peirce (see above).

[24] **Bjerknes, Vilhelm.** Das Problem der Wettervorhersage, betrachtet vom Standpunkte der Mechanik und der Physik. *Meteorologische Zeitschrift* no. 21 (1904), 1-7.

Linguistics. Educational Studies. Psychology. How embarrassing what we had promised, e.g., of Artificial Intelligence (AI). Nothing of the early promises was delivered in the 50 years since then: no automatic language translation, as [Google translate] can witness, and no automatic pattern recognition as witnessed by the inverse Turing test *Completely Automated Public Turing test to tell Computers and Humans Apart* (*CAPTCHA*), widely applied to prevent hacking of internet pages.

Law and Theology. **History. Sociology**. The only contribution I'm not ashamed of was in sociology. I received a letter from Niklas Luhmann:

> "Lieber Herr Booß ... Ich stelle mir jetzt für die Zusammenarbeit zwischen Mathematikern und Soziologen folgendes Modell vor: Der Soziologe berichtet dem Mathematiker über die weichen Stellen in seiner Theorie, von denen er erkennen kann, dass größere Begriffsschärfe, größere Kontrollierbarkeit der Konsistenz und vor allem größere Eliminierungseffekte einen analytischen Gewinn erbringen würden. Der Mathematiker müsste dann prüfen, ob er zumindest Suchhinweise, wenn nicht gar Modelle oder Erfahrungen mit Modellkonstruktionen zur Verfügung stellen kann. Meine Befürchtung ist, dass genuin soziologische Theorieüberlegungen, die nicht vorweg im Blick auf mathematische Modelle oder statistische Methoden der Datenanalyse konzipiert sind, ihre Abstraktionsgewinne mit einer Unschärfe bezahlen müssen, die für den Mathematiker nichts mehr besagt."

Luhmann was the only person who could see that we were getting astray. His reservations were not in the way of the Frankfurter School of Horkheimer, Adorno and Habermas. For my mathematization study, I had not contacted these sociologists who were politically on my side, on the Left, because they were not oriented, not informed about the existence of a rich world of mathematical concepts. It would have been wonderful to discuss with Adorno a whole night about music, because he understood much about it; but it would have been meaningless to talk with him about the actual or potential role of mathematics in and for sociology. I never tried, neither with Habermas.

But I tried with Luhmann, because his system thinking was bubbling with mathematical metaphors and he apparently liked talking with a mathematician. He warned me against the usual way of logicism that associates *Abstraktionsgewinne* (= gains on the level of abstraction) with *Schärfe* (distinguishing sharpness), while Luhmann associated them in his provocative manner with *Unschärfe* (lack of distinguishing sharpness).

Economy. I need not tell you how mathematically challenging – and meaningless many constructions were and are that were invented

- either for macro-economic decisions and predictions, in the absence of relevant state power to control key parameters,
- or for the design of ever more intricate instruments of the financial markets, in the absence of transparency regarding their impact on whole economies.

A brilliant thinker, Reinhard Selten, wrote the report for our book. Later he was IMFUFA guest and winner of the so-called (by Alfred Nobel not wanted) Nobel Prize in Economy, jointly with Harzany and Nash, and rightly so for both their mathematical discovery and their empirical investigations of the predominance of irrationality in economic decisions. It is already in our book, but easy to overlook behind the unfounded general optimism of our chapter on economy.

How could this happen? These exaggerations, these vain expectations, how could they happen? I have two explanations.

A. *Science optimism.* It was only 25 years after World War II. In a Western view, the war was not won by the Red Army, but decided by
   1. big British computers against Nazi German Enigma-cryptology and
   2. British and later US-American nuclear energy against Hiroshima and Nagasaki of Imperial Japan, and the victory of the Allied seemed temporarily only threatened by
   3. German jet propulsion for fighter planes and ballistic missiles (Wunderwaffe V2).

This Triad of math supported, actually math induced military technologies, computers, nuclear power, and jet propulsion, were not abolished after the end of World War II, but, on the contrary, flourished and went through series of civil triumphs. Born were the idea of unlimited energy resources; the idea of unlimited computer power by program-controlled machines; and the idea of unlimited mass tourism. The Triad became the symbol of science optimism and the widely cultivated belief in social progress through science and mathematization. This science optimism became also deeply rooted in the evolving Soviet societies, with continuing promises of surpassing hardships of life, not by increase of individual or local responsibility but by the blessings of technological progress.

The science and mathematization optimism became suddenly overruled, not so much by limitations appearing in the capacity of mathematical modelling and simulation, but rather by becoming aware of unpredicted consequences of the previously acclaimed introduction of ever new chemical substances in the environment. Typically, the unpredicted consequences were malign ones, like in pollution or in invalidating side effects of pharmacological substances intended for better health. From a scientific point of view, the greatest scandal was the 120 years belated discovery of benign side effects of aspirin (acetylsalicylic acid), namely to thin the blood of elderly persons to prevent blood coagulations in brain or heart.

B. Structuralism proliferation. That is my second explanation. It is shocking to read what 40-50 years ago very intelligent people wrote about the almost automatic, divine power of mathematics by freely available structural analysis and easily conductible generalization. One example:

> "Skulle man ... kort sige, hvad der er karakteristisk for den `matematiske tænkemåde', kunne man gøre det ved at påstå, at den består i at generalisere. Både på et meget primitivt niveau --- begrebet variabel er vel det første eksempel, man møder på et generaliseringstræk --- og på overordnede niveauer, hvor generalisering træder i værk over for komplekser og relationer, samt som sidste fase, hvor generaliseringen virker på totale strukturer..." (Dansk matematisk landsmøde, 1971, Working paper)

The mystery, the access key to and the underlying strength of the high climbing paths of math development and the wide fields of math application were erroneously attributed to generalization.

I had the luck to be educated at Bonn University, at that time one of the world centres of mathematics. Our bible was Hirzebruch's Habilitationsschrift[25] that became the manifest for studying thoroughly the generic cases and the underlying geometric situation.[26] There, in the early 1960s, the worst word was generalization; the worst one could say of some mathematical achievements was *GAN – Generalized Abstract Nonsense*. Of course, specialization and generalization are inherent in all goal-oriented activity, be it practical or theoretical. How to balance specialization and generalization is a deep question that goes beyond the goal of this talk.[27]

Here only this: In mathematics (and sciences and other subjects like law) we are proud of our competence to deal with *general specialities*, i.e., with very special and deep achievements of wide relevance and applicability, while we, e.g., admire journalists and other effective communicators and social network builders who are competent in handling *special generalities*. Such people know whom to find and to ask a special question, often without having own judgement in the matter under consideration.

At that time, in textbooks and math educational circles, there still was the quasi-religious adoration of generalization (mostly OECD financed, and with view to the role of future math education in the system confrontation between socialism and capitalism – but that is another story). That were the last outrunning wriggles from the great wave towards structure in the topology and geometry of the 1920s, personalized by the genius of Emmy Noether and her three Noether boys, Pjotr Alexandrov, Wolfgang Krull (later my teacher in Bonn), and Bertel van der Waerden. Roughly speaking, Noether's discovery was that the homology group of, e.g., a topological manifold

- carries more information than the knowledge of all the Betti numbers, and, at the same time,
- is more versatile for algebraic constructions, embracing, e.g., large commutative diagrams of groups and their homomorphisms.

From Noether it was only a little step to the French Bourbaki circle and their heroic effort to apply Noether's philosophy to the whole of mathematics.

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[25] **Hirzebruch, Friedrich.** Neue topologische Methoden in der algebraischen Geometrie. (German) Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 9. *Springer-Verlag, Berlin-Göttingen-Heidelberg,* 1956. viii+165 pp. MR0082174, often reprinted and translated.

[26] What I call the *Hirzebruch style* as opposed to generalizations and structuralism, is called the *Russian way* in **Arnold, Vladimir I.** Polymathematics: is mathematics a single science or a set of arts? *Mathematics: frontiers and perspectives,* 403--416, *Amer. Math. Soc., Providence, RI,* 2000. MR1754788: "The Russian way to formulate problems is to mention the first nontrivial case (in a way that no one would be able to simplify it). The French way is to formulate it in the most general form making impossible any further generalisation." [p. 407]

[27] On one of our joint train trips between Bonn and Bielefeld Hirzebruch admitted how relieved he was about Grothendieck's proof of what is now called the Riemann-Roch-Hirzebruch Theorem (RRH). He admitted that his own proof of RRH left room for doubts, because it was built on a combination of thorough analysis of *generic cases* with *arithmetic virtuosity*, while Grothendieck's proof finally explained why the RRH *should* be right.

Already in the course of the 1950s the limitations of the Bourbaki program became visible and soon after the most advanced segments of the math community were engaged in a counter-wave with focus on

1. generic examples,
2. single invariants,
3. special dimensions, and
4. new methods or new applications of known methods.

We were

- no longer interested in general properties of the n-sphere S^n, for all natural n, but, e.g., in the 2-sphere, the 3-sphere, the 4-sphere, or the 7-sphere;
- no longer interested in general partial differential equations, but rather in particular types, e.g., elliptic ones or geometrically defined operators like the Laplace operator and the Dirac operator;
- no longer interested in general solutions but rather in decisive characteristic properties and single invariants, previously mal-considered in the early days of Noether's structuralism: e.g., the gap of the spectrum around zero, or the index (chirality) of zero modes; etc., etc.

The predominance of structuralism was over; the unfounded expectations upon generalization became discarded from main stream mathematics. For the general intellectual public, however, the seeming consistence of the Bourbaki program was sufficiently seductive to make people believe in almost automatic progress by structuralism, mathematization, and generalization. Happily, it is over.

6. Looking back – seminal changes of mathematical concepts

In Fig. 7 I have a list of 15 turning points in the history of mathematics with radically new concepts, each of far-reaching philosophical implication and having changed our world view (provided by P.J. Davis).

To begin with the first item, we recall the challenging philosophical question of existence brought up by the stupefying discovery that $\sqrt{2}$ cannot be written as a fraction of natural numbers. Next on the list, you have Euclid with the new concepts of axioms and idealization. Etc., etc.

When I went through the list it struck me that two of the listed 15 turning points came up in my time, and that the syllabus press dictated by poor superiors does not permit to disseminate such a list among our students and make them as proud as we are that they carry great heritages and live themselves in the middle of a time with two mathematics based new world-visions emerging.

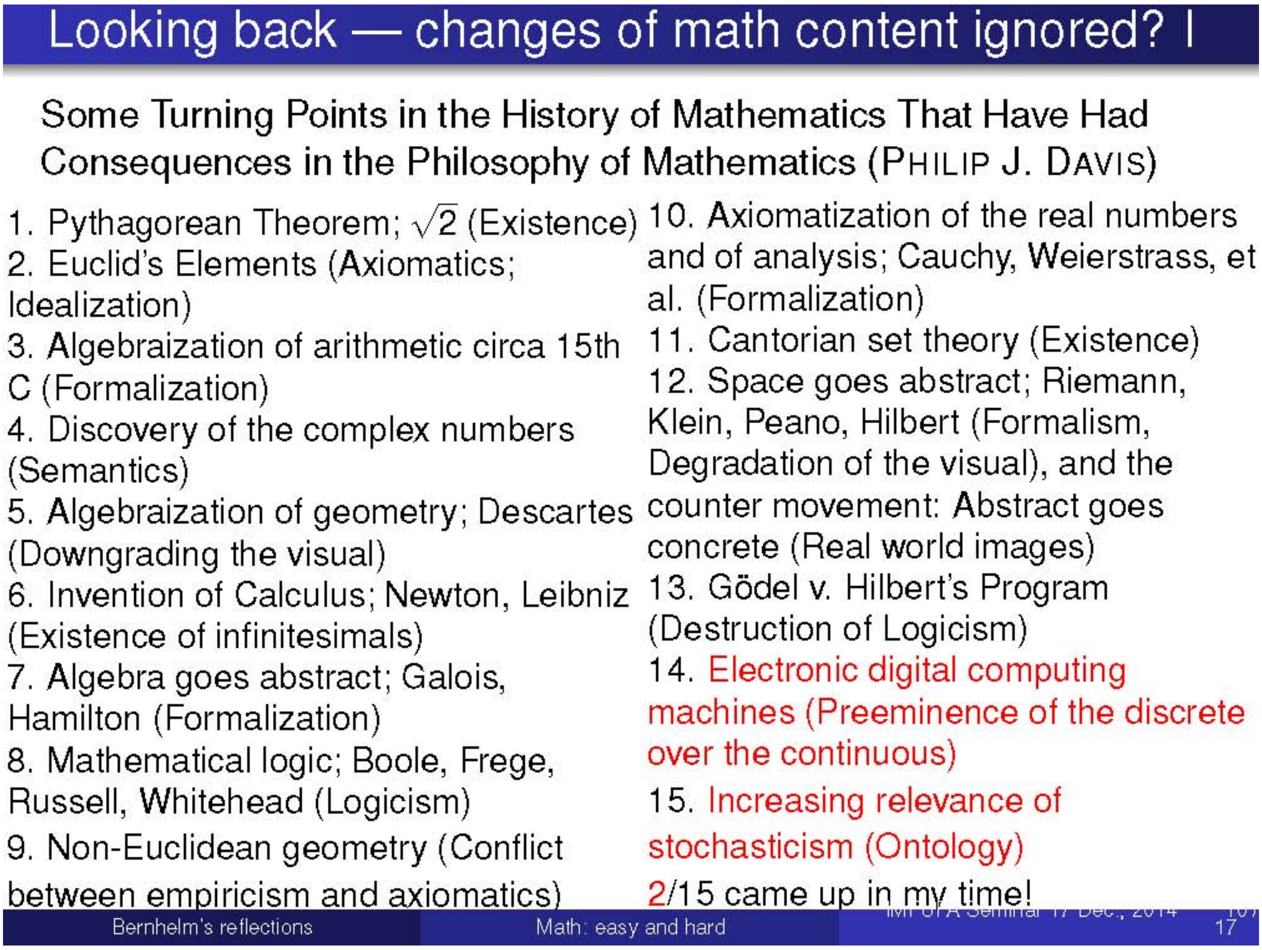

Figure 7. P.J. Davis' list of turning points in the history of mathematics

A student can make the same discovery of the fantastic math time she or he is living in, when she looks at the fabulous list of the Great Seven Math Riddles with their mountain climbing exercises, no matter how meaningless these sportive extremes should have been or are for the development of mathematics and how easy these problems could be replaced by other equally difficult and not less superfluous challenges (as Gauss commented upon the prize of the French Academy for proving or disproving Fermat's Last Theorem[28]).

Why are we not telling the students, that they are privileged to live in the golden age of mathematics, where something has changed and a lot has been achieved: not in the sense of these wrong promises of easy and universal applicability of math. We are far away from any fulfilment of that. But in another sense, namely regarding the emergence of new mathematical concepts of far reaching consequences and the complete solution of

[28] On March 21, 1816, **in** a letter to Olbers, Gauss wrote about the recent mathematical contest of the Paris Académie des Sciences on Fermat's last theorem: "… I confess that Fermat's theorem as an isolated proposition has very little interest for me, because I could easily lay down a multitude of such propositions, which one could neither prove nor disprove." Quoted from **Ribenboim, Paulo**. 13 lectures on Fermat's last theorem. *Springer-Verlag, New York-Heidelberg,* 1979. xvi+302 pp. (1 plate). ISBN: 0-387-90432-8 MR0551363, p. 3.

old riddles really something great and enriching is ongoing! Under the eyes of our students, and, hopefully, they can later tell their grandchildren that they have taken part in these seminal changes!

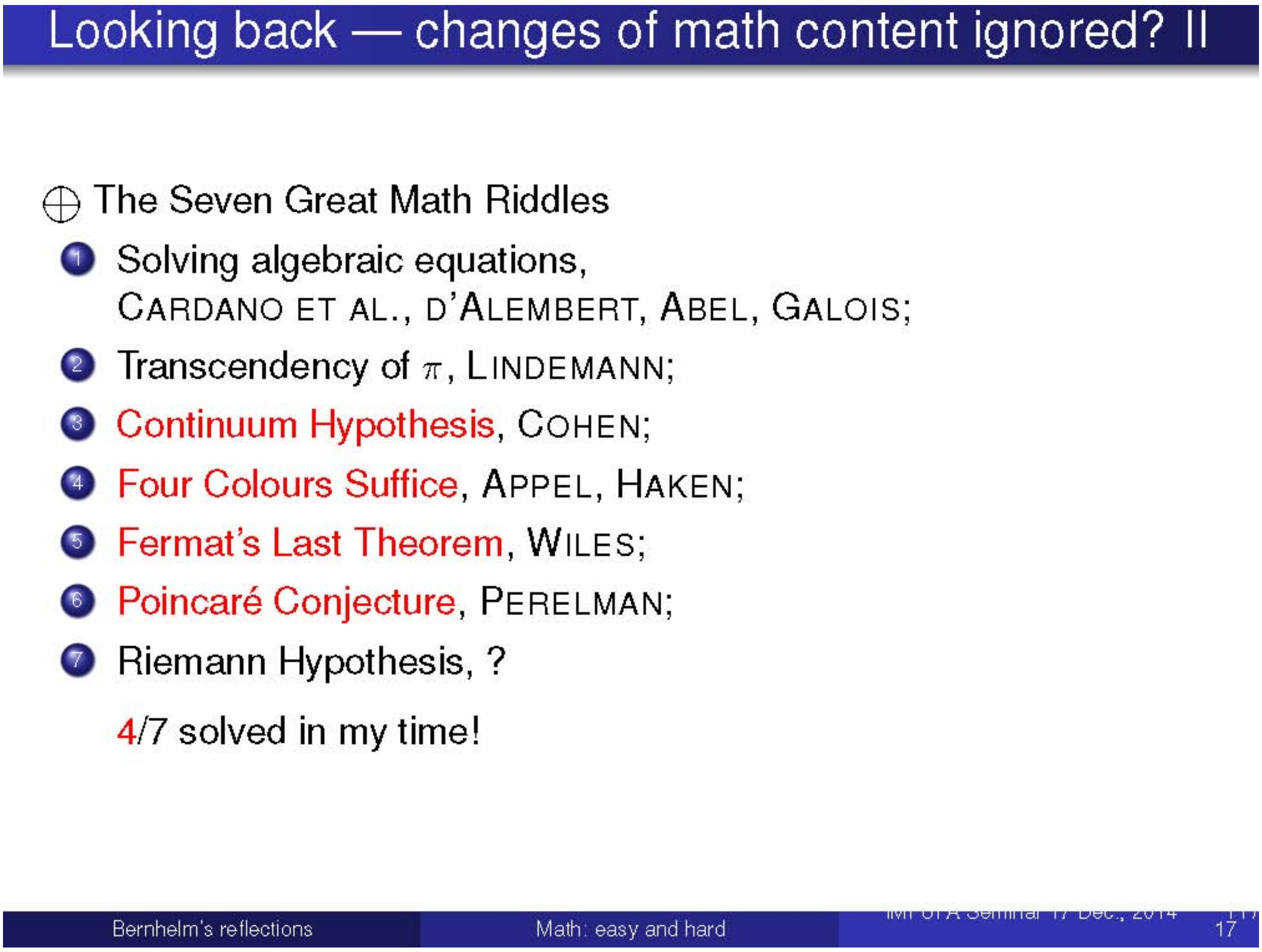

Figure 8. Should we follow the media and emphasize sportive math extremes?

7. Looking back – changes of the administrative frame: Continuing dismantling of the public sector

I quit my employment at an institution of higher education with some bitterness and some relief:

- bitterness about the ongoing loss of social capital and continuing dismantling of the public sector;
- relief that I shall not be subdued under the fashionable modern over-administration and exposed any longer to regulations and procedures that are imposed upon university teachers and students by well-meaning but uninformed leaders on all levels who are forced to take or implement decisions beyond their capacity of understanding the consequences.

To understand the enormous changes of the administrative frame of higher education and scientific research and the accompanying loss of social capital, I shall recall some material conditions of the life of a young mathematician 50 years ago (see Fig. 9).

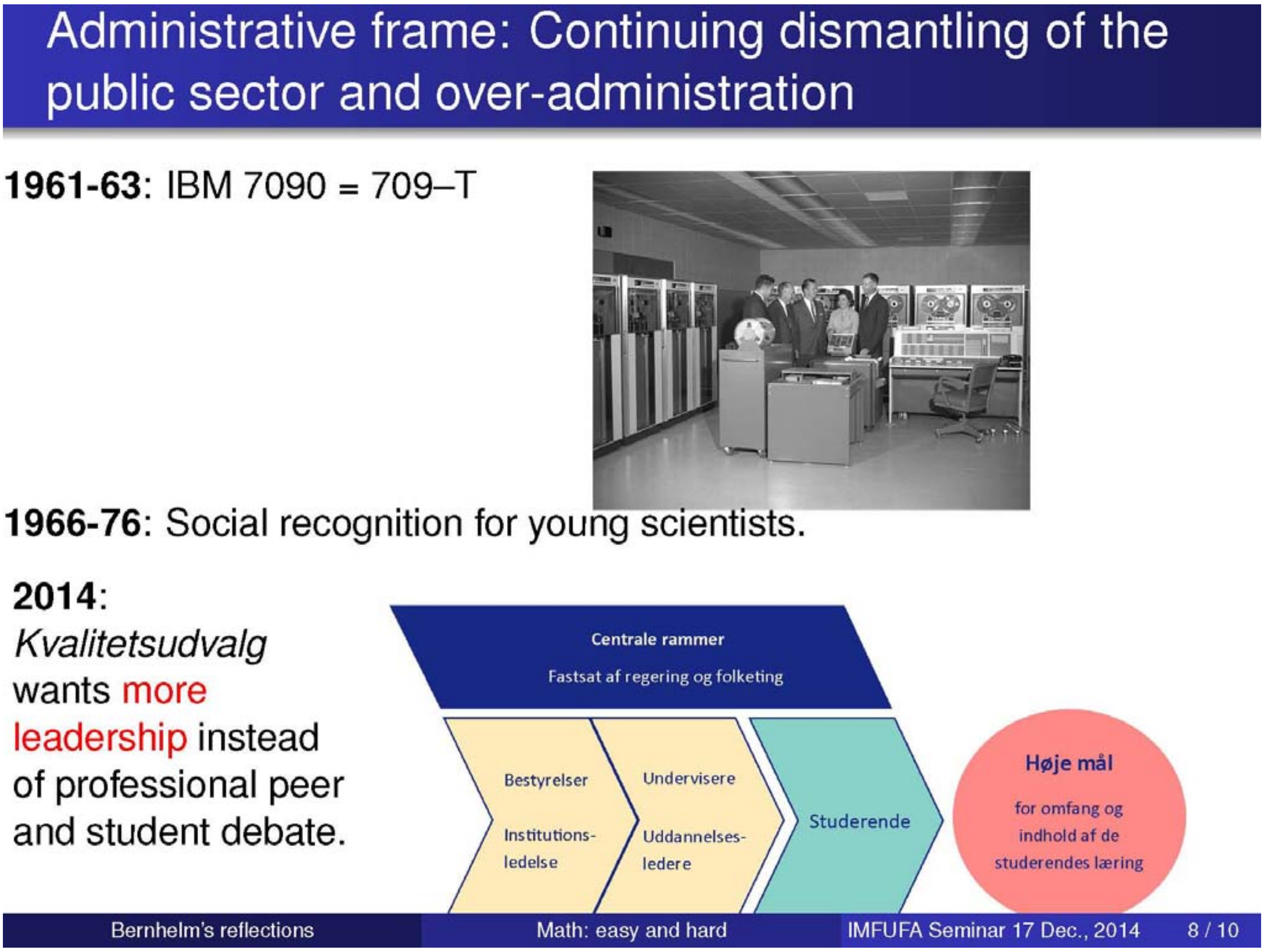

Figure 9. Loss of social capital 1961-2014

The public was in front. In 1962/63, my third year of studying math, the first two IBM Seventy 9 transistorized (709 T, colloquial 7090) were installed in Western Germany. These mainframes were not ordered for the military, neither bought by financial or industrial giants like Deutsche Bank, Siemens or Mercedes, they were bought by the *Rheinisch-Westfälisches Institut für Instrumentelle Mathematik* in Bonn and the *Deutsches Rechenzentrum* in Darmstadt, two academic institutions. One year before, the first IBM 7090 had been installed at the NASA centre at Ames in the U.S.A. The rent for such a mainframe was \$ 63.500 a month (equal to \$ 520.000 in 2014), the total price of the German acquisition was 2 x 1,8 x 10^{10} Deutsche Mark (of 1962), plus a fully acclimatized hall of 340 m^2 ground area for one single IBM 7090 and the required periphery (card punchers and readers etc.). These were huge expenses that were given to universities. For us today it is incredible, but it was true. It was not the military, nor the industry, nor the financial sector, but the public that had these two machines first.

Social recognition for young scientists. Looking back at the late 1960s and early 1970s, the social recognition of young scientists strikes me. Of course, after the exams some of my pals went to industry, not all with the best grades. Among the best ones, however, hardly any went to industry. The most challenging positions, the most attractive and best paid jobs for young scientists were in the public, at universities, in sectorial research institutions and in central public administration.

After the "Diplom" which is perhaps comparable to the Danish candidate degree, I came, e.g., to a traffic planning agency. It was not a private company but a public institution working for and paid by the *Landesverkehrsplan Nordrhein-Westfalen*. There was the money to hire people, to rent buildings, to buy equipment – and to give us the necessary time to make something of value both for the concrete project and for our own careers. Compared with normal salaries, the salary for research associates was quite high. The social security was high. At sectorial research institutions, usually you were employed as *Angestellte* with lasting, in-terminated work contracts.

In the same way, immediately after my PhD in Bonn I had in Bielefeld as a young director of the *Centre of Mathematization* my own secretary and a staff of student help. Doing efficient work as a young scientist, without a secretary? Nobody could imagine that. It was just an expression of the above analysed and possibly exaggerated science optimism of that time; but it was also a sign of social capital, around sciences and mathematics.
Of course, all that had its price.

These privileges, high salary, wide freedom for own research and other optimal working conditions were given under an implicit expectation that you would represent the goals of the institution and the sponsoring government around the clock. Twice I lost my job solely for political reasons in the FRG, the Federal Republic of (Western) Germany, the first time in March 1968 for having collaborated with the *Sozialistische Deutsche Studentenbund (SDS)* in compiling a red booklet on controversial circumstances at Bonn University under the title *Die repressive Universität*[29]; the second time in 1976 in Bielefeld for having served as a member at the board of the (legal) Communist Party for the heavily industrialized Ruhr-Westfalen district and having run for that party as candidate in state parliament elections. Never mind, it was a great time! At least I was refunded my first class train ticket when I had to show up for hearings in the science ministry of the state in Düsseldorf to defend my political association.

[29] It was followed up by criminal prosecution for *Rädelsführer* (ringleader who starts and leads a disturbance, a conspiracy, or a criminal gang). The threat of punishment was 4 years of *Zuchthaus* (at that time a special prison for those convicted of serious crimes). I was saved by the general amnesty for student unrest. The case is documented by my defence attorney in **Kaul, Friedrich K.** In Robe und Krawatte – Vor Gerichten der BRD. *Verlag Das Neue Berlin*. Berlin 1972, here the chapter *Bonner Studenten 1969*, pp. 5-56, available at http://milne.ruc.dk/~Booss/Math_Easy-and-hard_Presentation/1969_F.K.Kaul_BonnerStudenten.pdf.

And now? That all has changed. Nobody would any longer be afraid of a mathematician gathering 0,8% votes of the electorate to overthrow a government. But also the confidence has gone that students and professors could best steer universities in shared responsibility.

Three weeks ago I attended a 1-day conference called by the Expert Committee on *Quality in Higher Education in Denmark: New Ways - a higher education system for the future.* The committee's chairman explained the committee's approach by the here reproduced scheme (Fig. 9): To the right, in the red ball you read the empty formula "High goals (emphasized) for the extent and content of the students' learning". Friendly enough, there is a block in green to the left of the red ball with the label "students". So, they have not forgotten that higher education has something to do with students. However, clearly the largest area is taken by the administrative blocks of regulators, comprising government and parliament, university boards, department heads, study leaders and, not totally forgotten but marginalized, university teachers. The message is *More leadership, more administration*! instead of professional peer and student debate. Pfui Teufel! I need not tell you how glad I became on that conference about my retirement and how much we all must share Kolmogorov's concern, as quoted, ironized, but shared by V.I. Arnold:

> According to Kolmogorov, Hilbert was seriously worried by what would happen to the *Mathematische Annalen* cover in 500 years: he thought that the names of the former Editors would fill up all the space.
>
> Kolmogorov objected to Hilbert that our culture would probably not survive for such a long period: the united bureaucrats of all countries will soon be able to stop all kind of creativity, making further mathematical discoveries impossible, as are geographical discoveries today.[30]

[30] **Arnold, Vladimir I.** Polymathematics, l.c.

PART II

8. Math is hard – How and why?

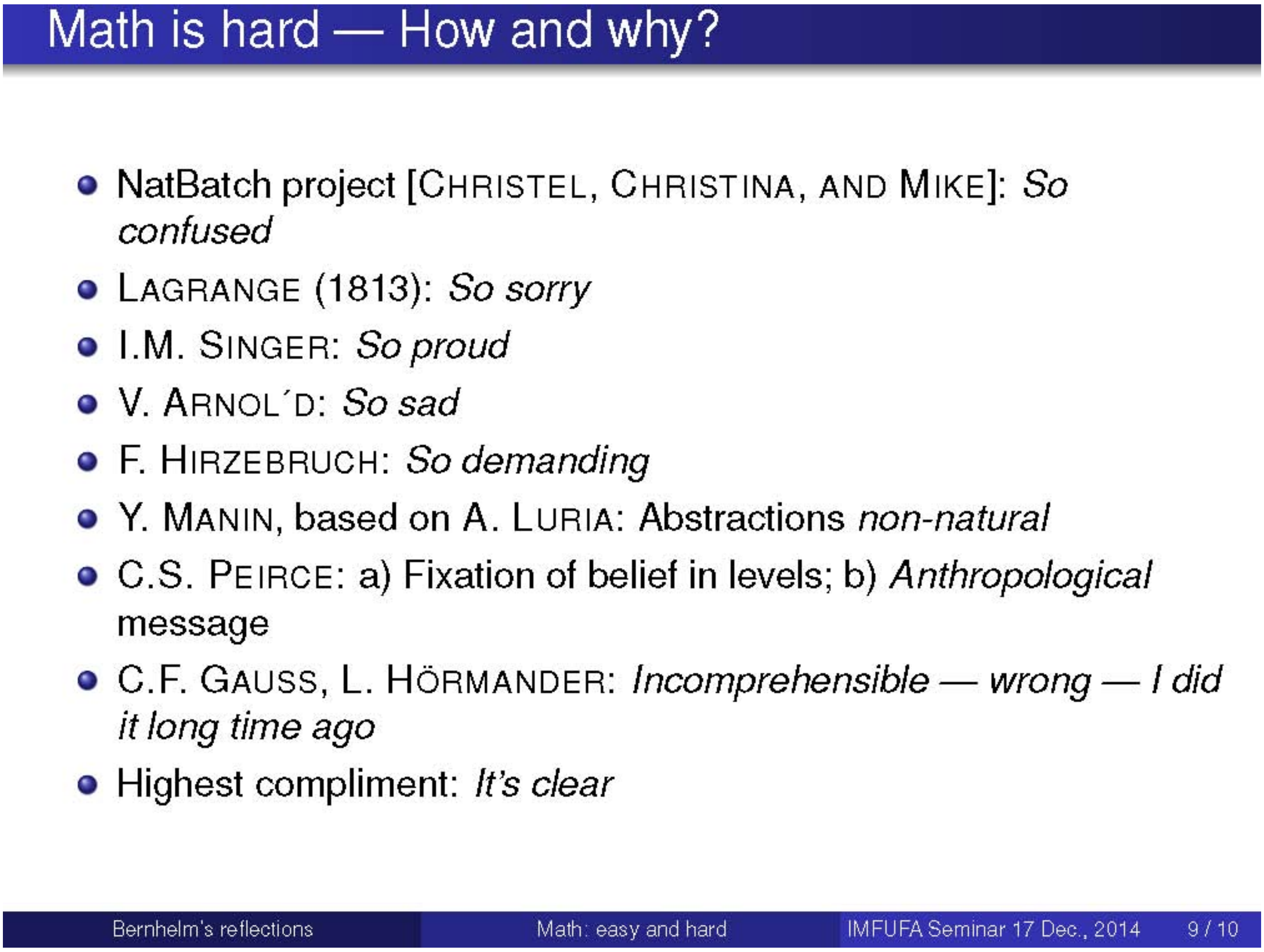

Figure 10. Why math is hard: key witnesses

The vest trick. Some time ago, I was consultant/supervisor of a third semester project. In the semester opening I presented myself with the entertaining vest trick, to illustrate what a topologist (my speciality) is doing, namely to think about questions like *How is it possible that I can remove a vest under my jacket*? *Has it something to do with the uneven number of my heads or the even number of my arms etc.*? The students asked *Is this mathematics*? We discussed it. I had to admit that a similar problem in two dimensions is well understood. It is the question whether a closed curve in the punctured plane (a plane where one point, e.g., the origin is removed) is contractible. The question can be easily decided by calculating the winding number of the curve; and there are various and seemingly very different methods to do that (calculating a path integral, solving a differential equation, by combinatorics, etc.). The curve is contractible if and only if the winding number vanishes.

Figure 11. The vest trick: exploring counter-intuitive 3D

But what can we do with the three-dimensional case? How can we argue, when my body and the jacket are removed from 3-space and the question arises whether the surface formed by the vest between body and jacket is contractible in the amputated space? When we go from two-dimensional space to three-dimensional space, all our intuition is lost and we have only two methods back, to try or to calculate. But calculate what and calculate how?

Those were *not* the questions, which the students were most interested in. They were *neither* surprised over that I could not tell them a solution at once. What so triggered their curiosity and gave a full semester's work was their own question, namely can one tell from the outside whether a mathematical problem is hard or easy. More precisely: *How can it be that many problems in mathematics are easy to formulate but difficult to solve*? They investigated that question by looking carefully at various historical examples (long and fascinating stories). Anyway, in this way they gave me the idea for this talk.

J.-L. Lagrange (1736-1813). To me, Giuseppe Lodovico (Luigi) Lagrangia (Joseph-Louis Lagrange) is one of the most important figures in the history of mathematics. He was extremely successful in introducing radically new and often highly abstract concepts to make mathematical ideas clear and comprehensible also to a non-genius which otherwise would have remained the domain of the intuition of outstanding people. He replaced Euclid's polygons and Descartes's curves by homotopies; to me, his Second letter to Euler, of 12 August, 1755, is the birth certificate of deformation theory and differential topology; and he replaced the Eulerian mechanic that intended following the ever more confusing visible orbits of single pieces by his mechanic of the easier capable underlying invisible potentials.

When he died on the 10 April 1813, there were an official obituary (by Delambre) and a 'Supplement' by a *G.* The supplement dealt with his last time and the thoughts he expressed shortly before his death. Nobody knows who the G. was. There are some speculations which were investigated in a paper by the math historian Ivor Grattan-Guinness.[31] He argues that G.'s Supplement is believable.

The following quotes of Lagrange are from Grattan-Guinness' paper. Until his death, according to these documents, Lagrange war so sorry for his students, that they had to read his textbooks, Lagrange's masterpieces in Mechanics that are so much more elaborated, intricate, and harder than all previous mechanics treatises. Of course, Lagrange was right: His books were dispensable for the calculation of simple mechanical systems - but indispensable for making complex mechanical systems transparent for the human brain and understandable and calculable. He felt "sorry for the young geometers who have such thorns to swallow. If I had to start again, I would not study: These large in-4° would make me too scared". He proposed instead a one-volume reprint of original works of the calculus by Fermat, Leibniz, l'Hôpital and especially John Bernoulli's lectures on the integral calculus, together with another volume comprising items by Euler and d'Alembert.

Already Delambre quoted Lagrange for "If I had had a fortune, I would probably not have made my profession [état] in mathematics." G. supplemented by recalling an occasion when Lagrange had met 'a young man devoting himself to the exact sciences with much ardour', and upon asking him 'Do you have a fortune?' and receiving a negative answer had replied: 'so much the worse, sir. The lack of fortune and of the existence it can give in the world, is a constant stimulus which nothing can replace, and without which one cannot bring to hard tasks all the necessary progress [suite]'.

When Lagrange was teaching, his 'researching intelligence' (G.) could cause sudden lapses in conversation. G. described the effect on his lectures at the École Polytechnique:

> Who has not seen him suddenly interrupt himself thus in the lectures which he gave at the École Polytechnique, appearing sometimes embarrassed like a beginner, leaving the blackboard and coming to sit down opposite the audience, while teachers and students, confused on the

[31] **Grattan-Guinness, Ivor**. A Paris curiosity, 1814: Delambre's obituary of Lagrange, and its "supplement''. *Mathemata,* 493--510, Boethius Texte Abh. Gesch. Exakt. Wissensch., XII, *Steiner, Wiesbaden,* 1985. MR0799763.

> seats [bans] expected in a respectful silence that he would have led his thought back from the spaces that it had gone to travel through.

To Lagrange, all mathematics was hard, also when it was seemingly easy for the student and would relieve its hardness only for the expert. So, the main goal of a mathematician's life was to think how to make math easier and more accessible, sometimes at the cost of introducing further abstract and more elaborated concepts.

In essence and in my reformulation: Math can be made easy and comprehensible only by accepting and enduring its hardness. Students are exposed to the cultural clash immanent in abstractions, formalism and symbol processing. Teachers must help them to experience that clash as a positive step like processes of adolescence or seeking work abroad, and not as a series of defeats. For sure, it doesn't help with well-intended lies or self-deception about easy access to mathematical abstractions as demanded by the new caste of administrators. Acquiring mathematical experience is nothing that falls from heaven or comes from playing on the ground. It requires work, concentration, exercises, and endurance: Ὁ μὴ δαρεὶς ἄνθρωπος οὐ παιδεύεται (The non-flayed human will not be educated, Menander, c. 341/42– c. 290 BCE, disseminated by J.W. Goethe as motto over his autobiography *Dichtung und Wahrheit*), or less draconic, Ohne Fleiss kein Preis (Without hard working no praise, after Hesiod, thought by scholars to have been active between 750 and 650 BCE).

All mathematicians I admire most are very close to Lagrange's position in continuing a life-long interest in teaching math and insisting that the essence of math, triggering curiosity and creativity and its true place in applications is that it is hard, and that it becomes dispensable and replaceable by engineering arts and econometric analyses etc. when it becomes easy.

I.M. Singer (*1924). Rightly, he can be proud of his achievements, among others the Index Theorem, which brought him the Abel Prize in 2004. When afterwards he was asked what to do next, he did not hesitate: *Now I must do some hard mathematics, because the only meaning of mathematics is to do something the engineer cannot do, others cannot do. I shall now for the rest of my life concentrate on the hardest to me still incomprehensible parts of mathematics. That is the role of mathematics, to deal with extremely hard problems.*

Part of the story is that in all recent years the same man, now 90 years old, participates at MIT in the math teaching of beginners, and as he says with great intellectual satisfaction, nursing and watching the emerging math understanding of young students: *It is fascinating to see when a student understands something, and sometimes even more fascinating when a student misses a point. This is deeply fascinating even in elementary calculus classes.*

V.I. Arnold (1937-2010). Some attribute to him and his former students the most decisive advances in the mathematical understanding of dynamical systems since the seminal work of H. Poincaré more than 100 years ago. When he was asked about the situation of math in Russia after the fall of the Soviet Union he deplored in his sarcastic way: *Well, it's terrible. Now the professors are cleverer and know more than the students.*

How sad. Indeed, teaching and learning math is only interesting when the teacher in each meeting with the students, say of one hour, gets at least one new mathematical idea. Otherwise it doesn't work with our goal, namely to socialize a new generation of math students to the way of mathematical thinking. The hour would have been lost – or could have been left to an electronic instruction device - with the same default result.

In an article[32] tracing the history of his own research, Arnold showed how apparently unrelated subjects are linked by a kind of mycelium from which theorems pop up like mushrooms. Continuing his life-long battle against formalism and Bourbakism, he distinguishes the easiness of communicating formal theorems from the hardship of explaining the underlying ideas in the following parable:

> When you are collecting mushrooms, you only see the mushroom itself. But if you are a mycologist, you know that the real mushroom is in the earth. There's an enormous thing down there, and you just see the fruit, the body that you eat.
>
> In mathematics, the upper part of the mushroom corresponds to theorems that you see, but you don't see the things which are below, that is: *problems, conjectures, mistakes, ideas,* and so on.
>
> You might have several unrelated mushrooms being unable to see what their relation is unless you know what is behind. And that's what I am now trying to describe. This is difficult, because to study the visible part of the mathematical mushroom you use the left half of the brain, the logic, while for the other part the left brain has no role at all, since this part is highly illogical. It is hence difficult to communicate it to others.

F. Hirzebruch (1927-2012). Since the 1950s he was the outstanding figure of mathematics in Western Germany. He was the natural candidate as director of a Max-Planck Institute (MPI) in mathematics, and he became the director of the first MPI in mathematics in Bonn in the 1980s. But for decades there was no MPI for mathematics. Shortly after the end of World War II, many MPIs were founded and lavishly financed to bring the sciences in the Federal Republic of Germany rapidly back to international top level after the decline and demolition during Nazi time.

Once I asked Hirzebruch why the Bonn MPI for mathematics came so late? He frankly told me that at least one reason was a controversy between him and the Board of the Max-Planck Gesellschaft (MPG).

Contrary to the MPI tradition of teaching-free research, Hirzebruch had insisted that research in mathematics without teaching is meaningless, that, as a rule, new

[32] **Arnold, Vladimir I.** From Hilbert's superposition problem to dynamical systems. *Mathematical events of the twentieth century,* 19--47, *Springer, Berlin,* 2006. MR2182777.

mathematical results are too hard to be digested at the distance; that they will falter rapidly when they are not forwarded instantaneously to new generations in interpersonal communication; that most young students need the contact and the role model of a successful researcher to overcome the hardships of acquiring math. Consequently, there should only be very few permanent positions for the directors and support staff, while the main human resources should consist of university teachers on leave as guest researchers for midterm stays. It took him several decades to reach the MPG's acceptance for this deviating status of math research, that it is meaningless without the umbilical cord to teaching and that all members of the MPI for math had to have an association with teaching.

Y.I. Manin (*1937). Like Hirzebruch, Manin is a magician who can create a world of deep interrelated concepts and results to his audience within 60 minutes, and so that most people in his audience have a strong feeling of having understood a lot, of being almost able to walk on water. Of course, when you go home and begin to work your way through your notes, your feeling will change and you will feel stupid and discouraged: too many things you can't understand in detail, and that means in math that you don't understand.

Manin himself commented that in his textbook on Mathematical Logic[33], namely that mathematical abstractions are hard to grasp; that thinking in symbols, while extremely effective in many contexts and indispensable in some, is deeply against the human nature. He explains that very carefully in his book and partially with references to facts based on some observations made by the psychologist A. Luria on patients with brain injuries. Some of the patients had preserved a sound judgement of the situation in hospital, e.g., of differences between various doctors and nurses in their competences and engagement, but lost the ability to think in relations: is an elephant bigger than a fly or a fly bigger than an elephant? His claim: *Abstraction is in essence more difficult for the human nature than doing judgements on personal relationships etc.*

Note that fully, and consciously and explicitly, Manin's insight or claim is directed against the traditional claims of all logicists and many adepts of mathematization who consider the process of abstraction and formalization as a process of simplification and clarification.

C.S. Peirce (1839-1914). He pointed out that formal logic, while being the base of all communication in mathematics (and elsewhere), does not suffice alone in handling doubts in mathematics (and elsewhere), that there is no single method to follow alone, but that four levels of *Fixation of belief* are distinguishable and intimately interwoven[34]:

1. The method of tenacity --- contrary to social impulse, but deadly necessary when you work your way through a long mathematical proof.

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[33] **Manin, Yuri I.** A course in mathematical logic for mathematicians. Second edition. Chapters I–VIII translated from the Russian by Neal Koblitz. With new chapters by Boris Zilber and the author. Graduate Texts in Mathematics, 53. *Springer, New York,* 2010. xviii+384 pp. ISBN: 978-1-4419-0614-4 MR2562767

[34] **Peirce, Charles Sanders**. The Fixation of Belief. *Popular Science Monthly* **12** (1877), 1-15. Available at: http://www.peirce.org/writings/p107.html or in my re-wording at http://milne.ruc.dk/~booss/Mamocalc/MaMoTeach_bbb/Peirce_on_the_Philosophy_of_Modeling.pdf

2. The method of authority --- contrary to widening one's view, but temporarily appropriate when you proceed by trusting a textbook author or a key reference without checking all details a new.
3. The method of doubt and discussion --- makes inquiry a matter of fashion and taste and is mostly discarded in mathematics and ridiculed by Peirce; there is though a tradition of *phenomenology* and *hermeneutics* with roots in theology and literature studies with emphasis on the interpretation of texts and related shifts and doubts. Inspired by G. Galilei's parole of *reading the book of nature,* it was elaborated towards math and sciences by E. Husserl and G.-C. Rota.
4. The method of science --- subjected to the pragmatic maxim, i.e., public, reproducible truth on reality, whether on properties of our brain and our way of thinking, as in mathematics, or on some real objects and their relations, as in sciences.

Moreover, Peirce had an anthropological message that our concepts, also our scientific concepts have evolved in human praxis of more than 100.000 years in experience with the various contexts humans have had over time.

The good side of the message is, that in most situations common sense and scientific, mathematics based arguments need not contradict each other. The bad side of the message is, with view to the emerging quantum mechanics at Peirce's time, that we have a problem when dealing with phenomena in an artificial environment that our mind has not been accustomed to for thousands of years. Then we must transgress common sense because it will for the most part be systematically misleading.

My next witnesses are

C.F. Gauss (1777-1855) and **L. Hörmander** (1931-2012). They were masters in standard formulations when they reviewed the work of other mathematicians:

Incomprehensible --- wrong --- I did it a long time ago.

To me, such typical referee reports prove that reading math papers is always hard, also for the greatest math geniuses.[35] Correspondingly, we have in mathematics two very different exclamations of agreement, *it's trivial* and *it's clear*. The first is pejorative: don't waste my time with your boring stuff; the second is highest acclamation: aha, now I see! This is really hard stuff you are telling me!

My last witnesses on the hardness of math are

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[35] Once I was also subjected to such a devastating treatment by an anonymous referee who was by 100% chance Hörmander. He rightly pointed out a couple of minor errors that he as the referee, strangely enough, did not correct himself. More important and showing his mathematical high class, he noticed that one formula looked *interesting*. It was K.P. Wojciechowski's and my discovery of the Lagrangian property of Cauchy data spaces of elliptic differential operators on compact manifolds with smooth boundary. For us the discovery was accidental. Only the referee's genius recognized the far reaching relevance. His hint became the basis of my mathematical work in the last 20 years or so. It led me to 10-15 separate papers and bestowed Wojciechowski and me with many references in the literature to that discovery by other mathematicians.

H. Cramér (1893 – 1985) and **P.J. Davis** (*1923). In his monumental monograph *Mathematical Methods of Statistics* of 1945, Cramér proved that the chi-square test statistic, i.e., the sum of relative errors between observed and expected magnitudes with f degrees of freedom, is distributed like the corresponding chi-square distribution with f degrees of freedom. For $f = 1$, it is the classical result by Karl Pearson of 1900, the proof of it is reproduced in most textbooks of mathematical statistics. For applications in material sciences, biology and medicine, Cramér's theorem is applied. Perhaps it is the most applied mathematical theorem of the 20th century. But to my best knowledge its proof has never (!) been reproduced. You can only find it in Cramér's textbook[36]. It is lengthy and not very inspiring. It is laborious – and boring. The main idea is much clearer for $f = 1$ than in the general case.

Such is mathematics that it has theorems that are easy to apply but hard to understand and, in practice, perhaps understood only by the author of the theorem and a handful readers of the original publication.

In a public talk in Roskilde, Davis gave a similar example when he confessed that he never had completed his checks of the proof of the *principal axis theorem* on block-diagonalization of normal matrices in linear algebra in spite of the fact that that theorem was a central tool in many of his works on effective numerical methods.

In a recent paper, Davis expanded on his view that we must live with some imperfections also in mathematics, that some basic tasks in numerical analysis are too hard to admit a rigorous approach.[37] Among his examples he refers to the concept of *numerical stability* in iterations, when, contrary to the toy examples of elementary classes in numerical analysis, no estimates are available about the achieved precision of an approximate result. Nevertheless, we have to stop the iterations at some point. For solving systems of differential equations, a common stop rule is when the results become unchanged under further iteration or refinement of the underlying discretization. Then a result seems to become stable and reliable, while we have examples where numerical stability can be achieved far from the true result. To comfort our mathematical fears and incertitude Davis usually cites Richard Hamming (1915-1998) for having said *I would never fly with a plane where the construction depends on the difference between Riemann and Lebesgue integral.*[38]

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[36] **Cramér, Harald**. Mathematical Methods of Statistics. Princeton Mathematical Series, vol. 9. *Princeton University Press, Princeton, N. J.,* 1946. xvi+575 pp. MR0016588, here chapter 29.

[37] **Davis, Philip J.** The relevance of the irrelevant beginning, *ScienceOpen Research*, 2014, 5 pp, DOI: 10.14293/A2199-1006.01.SOR-MATH.6G464.v1.

[38] **Hamming, Richard W.** Mathematics on a distant planet. *Amer. Math. Monthly* **105** (1998), no. 7, 640--650. MR1633089, The full quote is "for more than 40 years I have claimed that if whether an airplane would fly or not depended on whether some function that arose in its design was Lebesgue but not Riemann integrable, then I would not fly in it. Would you? Does Nature recognize the difference? I doubt it!" [p. 644]. Certainly, Hamming's insistence on robustness in applications is a relief. However, it is a fact that certain highly applicable concepts like the Hilbert space L^2 of equivalence classes of measurable, square-integrable functions can only be established by embracing all Lebesgue integrable functions to obtain the indispensable completeness.

Davis points to another symptom of the difficulty of doing math, namely our almost unlimited *freedom* to add or to remove assumptions that is though sharply *restrained* by logical demands regarding the formulation and consistency and even more sharply restrained by respectful regards to the history of a topic and which examples or expansions might be considered meaningful and which not.

Warning 1. In this talk, I emphasize limitations in a mathematician's or a math student's daily practice; I elaborate on the hardness of math, on the difficulty of reaching reliable results, and on the absence, on the lack of fully trustworthy working methods in various parts of mathematics and of the use of mathematics. To be honest, I do it with more pride than shame: In math we are mostly proud and, in general, less ashamed when we can point to a particularly hard problem nobody could solve so far. The whole idea of public keys in cryptology is based on the present lack of efficient algorithms to factorize a given product of two large prime numbers into its two components, or on other presently unsolved problems, e.g. with elliptic curves. So, from a technological point of view, *hard and presently unsolved problems* are wonderful and highly applicable!

Warning 2. For numerical algorithms in the analysis of dynamical systems and of combinatorial tasks, e.g., in graph theory, mathematicians try to give *asymptotic estimates* about the complexity (i.e., the expected time necessary for a solution) of a problem. By definition, the problems that are hardest to solve are the so-called NP-complete problems like the travelling salesman problem. The for practical purposes perfect organization of just-in-time delivery for retail chains shows that one never should become blocked in search for practical solutions by asymptotic seemingly insurmountable estimates.

To sum up:

Piet Hein (1905-1996).

Problems worthy
of attack
prove their worth
by hitting back.

9. Math is easy – Really?

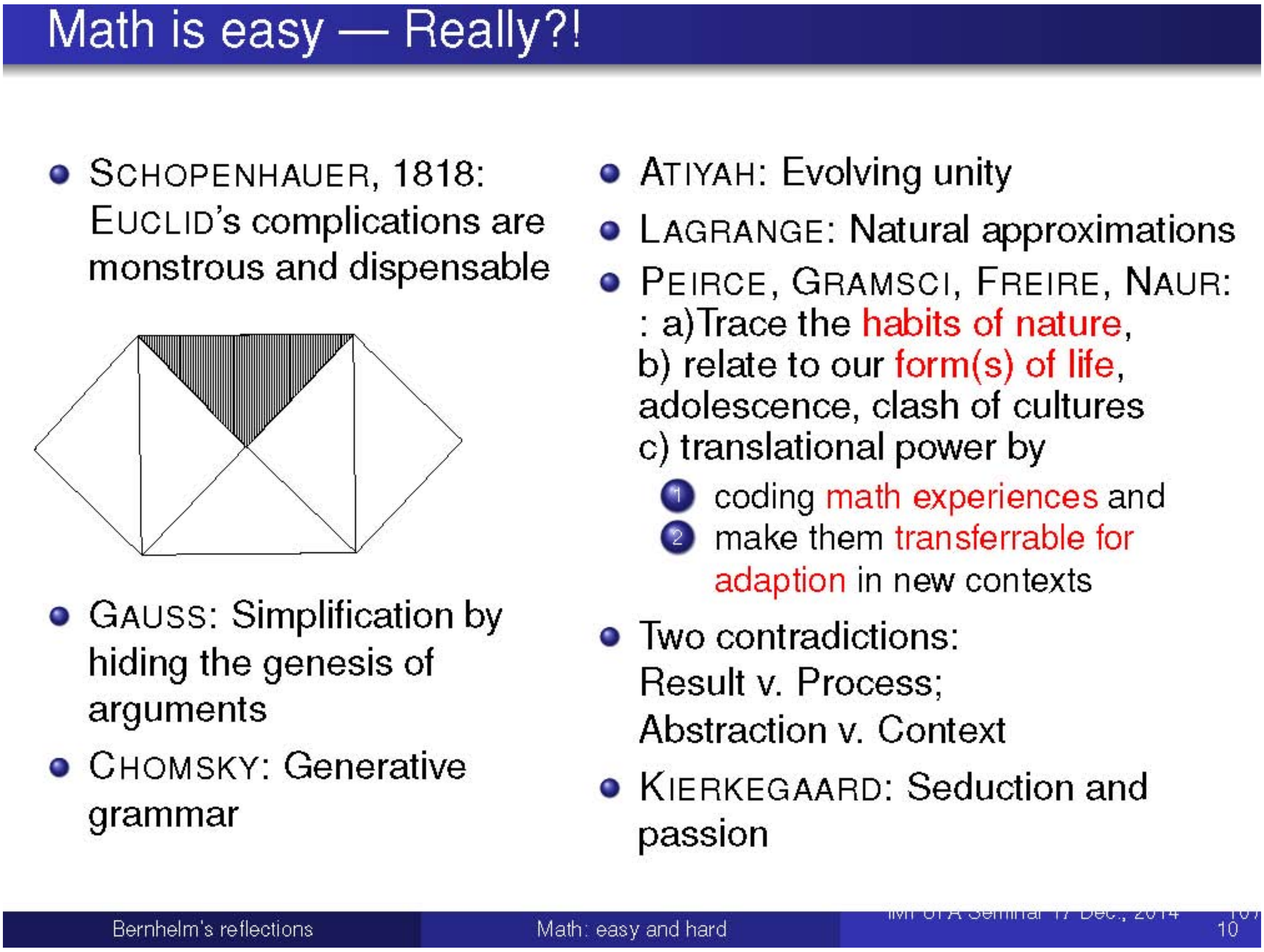

Figure 12. Classical and modern arguments, why math should be easy!?

In the previous Section 8, I tried to explain why learning and doing math is hard, by necessity. But what about the many people, pupils, students, teachers, researchers, who love to spend many hours thinking about a mathematical problem; some early in the morning when one is fresh, some late in the night when one is not disturbed, some on their desk and some while jogging or walking their dogs? And what about the rich treasures of investigations, suggestions and predictions how doing math can and in the long run will be made easier and more accessible?
Let me comment upon the most outspoken positions.

A. Schopenhauer (1818). In his treatise *Die Welt als Wille und Vorstellung* (*The World as Will and Representation*), the philosopher – or rather a philosopher-poet like the many other German philosopher-poets Hegel, Nietzsche, Heidegger with their love for extensive formulations – Arthur Schopenhauer released the following torrent of words against the mathematicians' arrogance and stupidity making math, according to Schopenhauer unnecessarily hard and non-intelligible, and that Euclid's classical arguments were monstrous and dispensable:

> … mathematical knowledge *that* something is the case is the same thing as knowledge of *why* it is the case, even though the Euclidean method separates these two completely, letting us know only the former, not the latter. But, in Aristotle's splendid words from the *Posterior Analytics*, I, 27: 'A science is more exact and more excellent if it tells us simultaneously *what* something is and *why* it is, not *what* it is and *why* it is separately.' In physics we are satisfied only when our recognition *that* something is the case is united with our recognition of *why* it is, so the fact that the mercury in a Torricelli tube is 28 inches high is a poor kind of knowledge if we do not add that it is held at this height to counterbalance the atmosphere. So why should we be satisfied in mathematics with the following occult quality of the circle: the fact that the segments of any two intersecting chords always contain equal rectangles? Euclid certainly demonstrates it in the 35th proposition of the third book, but why it is so remains in doubt. Similarly, Pythagoras' theorem tells us about an occult quality of the right-angled triangle: Euclid's stilted (stelzbeinig), indeed underhand (hinterlistig), proof leaves us without an explanation of why, while the following simple and well-known figure yields more insight into the matter in one glance than that proof, and also gives us a strong inner conviction of the necessity of this property and of its dependence on the right angle:

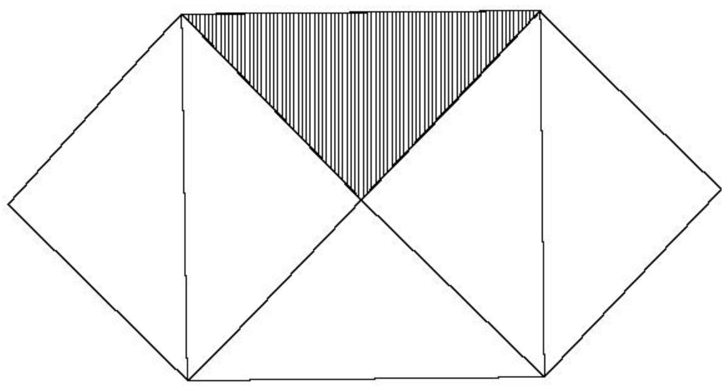

Figure 13. Schopenhauer's fantasied "simplification"

As often when outsiders comment on math it strikes me how little they understand of the clou of a mathematical achievement. So also Schopenhauer: The clou of Pythagoras' Theorem is its validity for *all* rectangular triangles in the plane, i.e., even when the sides at the right angle are unequal. By the way, that's until today the most typical application of the theorem in construction: To check whether the walls in a room or house are rectangular, a carpenter would mark a 3-meter (or yards) point upward in a corner, a 4-meter (or yards) point along a wall on the floor, and then check whether the straight line between the two marks is exactly $5 = \sqrt{(3^2 + 4^2)}$ meters (or yards).

One would expect an error term; but no, Pythagoras claims and Euclid proves that the error term vanishes even when we deform the rectangular triangle, within the class of rectangular triangles. Later generations proved that Pythagoras' theorem remains basically valid even for non-rectangular triangles, incorporating an error term coming from the cosine of the included angle, and for rectangular triangles on a sphere, incorporating a curvature error term coming from the sphere's radius.

So, for a mathematician the Pythagoras' Theorem is an approximation theorem, that you can change something with controlled effects, sometimes with zero effect, sometimes with nonvanishing, but calculable effects.

Of course, Schopenhauer is right: math can be much easier when we remove the clous and reduce it to trivialities. Actually, we can answer Schopenhauer, that math would become even easier, when we reduce it to the empty set. So far Schopenhauer only shows his lack of understanding.

However, rightly he points to the difference between checking a proof, line by line, contrary to grasping the reason for the validity of a claim. Every mathematician has experienced it: that we still don't understand a given proof after we have checked it step-by-step. Hence, in modern textbooks and for papers in learned journals, authors are praised when they explain the underlying idea of a proof before the reproduction of the proof in its details.

C.F. Gauss (1777-1855). His reply to Schopenhauer was: *On the contrary! Mathematics is so difficult that we never should tell the reader how we got the idea. In most cases it will be either impossible or distracting to make the idea explicit. To make results accessible we shall hide all complications we had to meet and overcome, and keep silent about the wrong tracks we went when searching and finding the proof. What counts in mathematics is only a presentation of the purified final form*.[39] For 200 years, Gauss perception of simplicity through hiding the birth bangs and presenting only the sleek version has dominated the publication culture of mathematics. Fortunately it has been on the retreat along with the retreat of Bourbakism.

M.F. Atiyah (*1929). While he personally has contributed to the creation of many new mathematical theories, concepts, and methods, he insists that math is getting easier and more transparent by the emergence of any new mathematical achievement. He compares math with a ware house*. Looking for a box of nails in a small country shop and finding the right ones can be harder than looking around in a big specialized department store like Bauhaus.* Clearly, it is easier to find your way around in a big, well-organized modern department store than an old fashioned pop store. Making math more complex opens many new crossroads and makes search and communication easier. Such is the argument.[40]

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[39] A typical example is provided by Gauss' first proof of the Fundamental Theorem of Algebra of 1799, **Gauss, Carl Friedrich.** Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*. Helmstedt: C. G. Fleckeisen*. 1799 (tr. New proof of the theorem that every integral rational algebraic function of one variable can be resolved into real factors of the first or second degree). German translation in: **Netto, Eugen** (ed.): Die vier Gauss'schen Beweise für die Zerlegung ganzer algebraischer Funktionen in reelle Factoren ersten oder zweiten Grades (1799–1849), *Ostwald's Klassiker der Exakten Wissenschaften Nr. 14, Wilhelm Engelmann, Leipzig* 1890 , pp. 3-36, 83 (figures). Accessible at University of Toronto, https://archive.org/details/dieviergaussscheOOgausuoft. Contrary to d'Alembert's predecessor proof of 1746, Gauss keeps this proof deliberately in purely real terms. However, one can easily trace the underlying complex constructions in the real presentation.

[40] **Atiyah, Michael F.** Trends in pure mathematics. In: *Proc. of the 3rd Internat. Congress on Mathematical Education (Karlsruhe 1976).* 1979, pp. 61-74. Reprinted in Collected works vol. 1, pp. 261—276. MR0951896.

Atiyah's optimistic claim is based on his view of ever clearer emerging unity of mathematics. In a recent paper that unity belief was refuted by Davis and me as a myth.[41]

J.-L. Lagrange (1736-1813). This is the same Lagrange who appeared before as witness of the inevitable hardness of math. Now I call him as witness for the ease and simplicity of mathematical physics. Regarding the celestial bodies of our Solar system, he noticed that the planets are moving on almost circular orbits and most comets on very eccentric elliptic orbits. He concluded that *Nature favours planetary approximations by grouping heavenly bodies according to very small and enormous eccentricity*.[42] Indeed, for each of the two extreme cases we have specific and very powerful expansions, which would fail in the middle range. Modern astrophysics teaches us, however, that this is a very special property of our solar system due to the dominance of the two gas planets Jupiter and Saturn which, at the whole, make our system so surprisingly stabile. Other solar systems in the Milky Way seem to provide for more challenging mathematical problems.

We may expand Lagrange's argument for *nature provided simplicity* to large parts of mathematical physics where we, e.g., not have to deal with very general differential equations with arbitrarily varying coefficients but with geometrically defined operators with strong inherent symmetries like the Laplace or the Dirac operator that are, moreover, often controlled by potentials and other background fields. Therefore large parts of mathematical physics that are based on first principles and geometry are mathematically easier and more accessible than some parts of biology that are less mathematized, based on ad-hoc assumptions and so tangled up in non-controllable generalizations.

More and more trustworthy arguments for explaining why doing math can appear easy and almost natural, from time to time. Until now, in this Section, I discarded common, partly ingenious suggestions and beliefs why and how doing math can become easier. I shall now turn to considerations that are also controversial, but definitely not to be discarded by me. It seems to me that they have the potential to explain why and how learning and doing math can appear personally satisfactory, natural and, from time to time even easy for some people in lucky moments and periods of their life.

I have written about the following quite different approaches separately and extensively[43] and shall be brief in this talk.

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[41] **Booß-Bavnbek, Bernhelm; Davis, Philip J.** Unity and Disunity in Mathematics. *Newsletter ofthe European Mathematical Sociey* No. 87 (March 2013), 28-31.

[42] Here is the full quote of Lagrange given in his above cited obituary by the anonymous G., offering a Lagrange type witticism: "It seems that nature had disposed these orbits [of the heavenly bodies] specially so that one may calculate them. Thus the [sic] eccentricity of the planets is very small, and that of the comets is enormous. Without this disparity [,] so favourable to approximations, and if these constants [of the orbits] were of an average magnitude*, goodbye geometers*; one could do nothing."

[43] **Booß-Bavnbek, Bernhelm**. On the difficulties of acquiring mathematical experience, *EM TEIA – Revista de Educação Matemática e Tecnológica Iberoamericana* **5** - número 1 (2014), 1-24. Also: http://milne.ruc.dk/~Booss/Math_Easy-and-hard_Presentation/2014_BBB_EMTEIA.pdf

N. Chomsky (*1928). His message, or at least the message disseminated by his student Pinker[44], is that *Math is easy. Every child has solved the greatest math exercise of her or his life at the age of two, when it forms the generative grammar of the child's mother tongue and assembles the patterns and basic structures out of single words.* I'd better add that some of Chomsky's claims are controversial, in particular his biologistic assumption of special genetic grammar traits of the human race that are not confirmed by molecular geneticists.

C.S. Peirce (1839-1914)[45], **A. Gramsci** (1891-1937)[46], **P. Freire** (1921-1997)[47], **P. Naur** (*1928)[48]. To me, these four names stand both for

- deep insight into the complexity of human thinking and communication, and for
- demystification of feeling, learning, and doing by relating it to human *habits* and *forms of life*.

Their teaching for the topic of this talk can be roughly summarized in the following short formula:

1. Trace the *habits of nature*;
2. relate our feeling, thinking, and doing to our *form(s) of life*, take the risks and jumps of adolescence and accept the related clash of cultures;
3. for mathematics, exploit the translational power (and handle the 2 contradictions below) by
    a. *coding math experiences* and
    b. make them *transferrable for adaption* in new contexts.

Two contradictions. All math learning and teaching has to live with and to handle the two following contradictions:

A. Result v. process. We need to teach *results*, not only *processes*, not only ways of thinking; one needs results in sciences and mathematics.
B. Context v. abstraction. Students learn best in *context*, when they can see meaning and embedding in context; however, the power of mathematics is that it can be solved from the context; that is the true power of *abstractions* that we can transport experiences from one context to another one.

We cannot discard or bridge these two contradictions firmly. We must tell the math education administrators, that doing, learning, teaching math is difficult and requires time for the body and peace for the mind. We cannot deliver what they want, an easier, faster and more accessible teaching in the sense they want. Our only hope is:

S. Kierkegaard (1813 -1855). In *Enten-eller*, he explained the two to him most difficult situations in life, the love for another human and the love for God[49]. I don't agree fully with Kierkegaard, neither with the first situation where I have some personal experience,

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[44] **Pinker, Steven.** The language instinct. *New York: William Morrow*, 1994.
[45] l.c.
[46] **Gramsci, Antonio**. Selections from the Prison Notebooks. *New York: International,* 1971.
[47] **Freire, Paolo.** Pedagogy of the oppressed. *New York: Herder and Herder*, 1972.
[48] **Naur, Peter.** Computing: a human activity. *New York: ACM/Addison-Wesley*, 1992.
[49] **Kierkegaard, Søren**. Either/Or. Volume I. *Princeton: Princeton University*, 1959. See *The Immediate Stages of the Erotic or the Musical Erotic*, pp. 43-134, in particular pp. 62, 93, 114.

nor with the second, where I'm blank. Anyway, Kierkegaard emphasizes that both of these two situations require deep feelings: Let yourself be *seduced* and develop the *passion*!

Afterword. This is what math doing, learning, and teaching is good for when it is successful. On some occasions you'd better lie and follow the love advice of Elias Canetti (1905- 1994): *Don't tell me who you are. I want to adore you.* So, you need not tell the students the full truth[50] every day, e.g., about the

- destructive sides of math supported technology; about
- math induced inhuman formatting of social organisation; and
- the deformations of the mind by naïve belief in logic and modelling.

[50] In **Hardy, G. H.** A mathematician's apology. With a foreword by C. P. Snow. Reprint of the 1967 edition. Canto. *Cambridge University Press, Cambridge,* 1992. 153 pp. ISBN: 0-521-42706-1 MR1148590, p. 33, n .16, Hardy ponders about his 1915 quote: "a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life." See also **Arnold, Vladimir I.** Polymathematics, l.c., p. 403, paraphrasing Hardy: "All mathematics is divided into three parts: cryptography (paid by CIA, KGB and the like), Hydrodynamics (supported by manufacturers of atomic submarines) and celestial mechanics (financed by military and other institutions dealing with missiles, such as NASA)." and the anthology Mathematics and war. Papers from the International Meeting held in Karlskrona, August 29–31, 2002. Edited by Bernhelm Booß-Bavnbek and Jens Høyrup. *Birkhäuser Verlag, Basel,* 2003. viii+416 S. ISBN: 3-7643-1634-9 MR2033623, free download at http://www.springer.com/gp/book/9783764316341#otherversion=9783034880930.