Emergence of multiplicity of time scales in the modeling of climate, matter, life, and economy

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To the physicist Niels Boye Olsen, colleague, teacher, friend, on the occasion of his 80th birthday

Abstract. We address differences between characteristic times in climate change and show the universal emergence of multiple time scales in material sciences, biomedicine and economics.

Contents

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1. Introduction

To begin with, we recall challenges to transforming for sustainability related to multiple time scales in climate change and explain the structure of the paper.

1.1. Challenges to transforming for sustainability related to multiple time scales.

Transforming for sustainability and mitigating climate change hazards are challenged by two basic insights:

(1) Success of transforming and mitigation depends decisively on the *broad support by an informed public*. Necessary changes in production, trade and life style can hardly be implemented only by state regulation and command. The active participation of the citizens will be mandatory for efficiency and democracy.

(2) There are various *multiscale aspects* in climate change and sustainable development, e.g.,
   - the *instant accumulation* of greenhouse gases in the atmosphere with continuous changes of the radiation pattern, versus
   - the *time lag* in the development of secondary effects like the release of the green house gas methane from perm freezing areas and from presently stable methane hydrate crystals on ocean floors in a feedback loop and the unpredictable stirring of the oceans for storing CO_2 and uptake of heat in the various layers of the oceans.
   - The United Nations' *Sustainable development goals* of September 2015 [**1**], in particular SDG 13, Climate Action, may require long periods before dominant effects from a contribution can be recognized. Hence, long-term strategies are required against which the short- and medium-term actions can be measured.

It is well known that differences between characteristic time lengths provide difficulties in mathematical modeling, statistical sampling, and numerical simulation, as explained, e.g., in J.D. Logan's textbook [**2**]. Over the last three decades, many of these difficulties have been overcome by astonishing advances in multiscale modeling, computation and simulation, well documented in learned journals like SIAM's *Multiscale Modeling and Simulation* (MMS) and monographs like [**3, 4, 5, 6, 7**]. Impressive as these advances within the *scientific and design community* are, it seems to us that we have to worry about the lack of awareness of multiscale aspects in *public response* to the challenges of climate, health and economy, to name three fields of wide concern. In short, disregard of the multiscale aspects of a problem can become misleading in communicating threats and solutions, see our conclusions in Section 7.

1.2. Structure of this paper. This paper was initiated by public and scientific controversies around the multiscale problems of climate change modeling. While none of us is atmosphere physicist or active in other segments of climate modeling, we see related multiscale problems all around. In this multidisciplinary survey we show the universality of multiple time scales in mathematical modeling by taking a closer look at striking examples from our own research. In particular we show that multiple time scales can emerge from seemingly simple models, i.e., mathematical models where neither time lags nor characteristic differences between the length of oscillations are put into the form and coefficients of the equations beforehand. We show also that a relatively small variation of the coefficients can suppress — or support — possible time scale differences.

In Section 2, we recall basic mathematical concepts to explain the intricacies of multiscale modeling and simulation and the related challenges in the expert–public communication.

In Section 3, we explain the emergence of multiple time scales in climate modeling and give a toy model of the dynamic interaction of *Homo Sapiens* with the *Earth System*. We show that the most simple mathematical model for the emergence of multiple time scales arises in slow-fast processes of complex dynamics.

In Section 4, we summarize typical multiscale examples of material sciences and take a closer look at the emergence of multiple time scales arising in the computer simulation of the viscosity of soft materials.

In Section 5, we summarize typical multiscale examples of life sciences and take a closer look at the modeling and computer simulation of the production of blood and the development of some blood cancers and of the biphasic insulin secretion of pancreatic beta cells.

In Section 6, we summarize typical multiscale examples of the macrodynamics of capitalism and take a closer look at empirical data of a long-term Kondratiev-Schumpeter wave and a mathematical model of embedded Keynesian short-term business cycles.

In Section 7, we sketch a few possible conclusions for scientific, communicative and political challenges.

In each section, we shall provide *simple* explicit mathematical models of intricate systems with multiple time scales, e.g., a simple system of coupled ordinary differential equations and a simple slow-fast system in complex polynomial dynamics. In that way we hope to make people familiar with the *counter–intuitive* effects of two characteristic times.

2. Mathematical toy models with two characteristic time scales

2.1. Toy models with two characteristic time scales of the dynamic interaction of *Homo Sapiens* with the *Earth System*. We visualize the challenges by a three-compartmental toy model of the dynamic interaction of *Homo Sapiens* with the *Earth System*. Confronted with the serious tasks of transforming for sustainability, a technical discussion on multiple time scales may appear rather abstract. Who cares whether they exist or not, are universal or particular, inevitable or avertible? The following toy model shall illustrate how much we may compromise when we are not aware of possible multiple time scales.

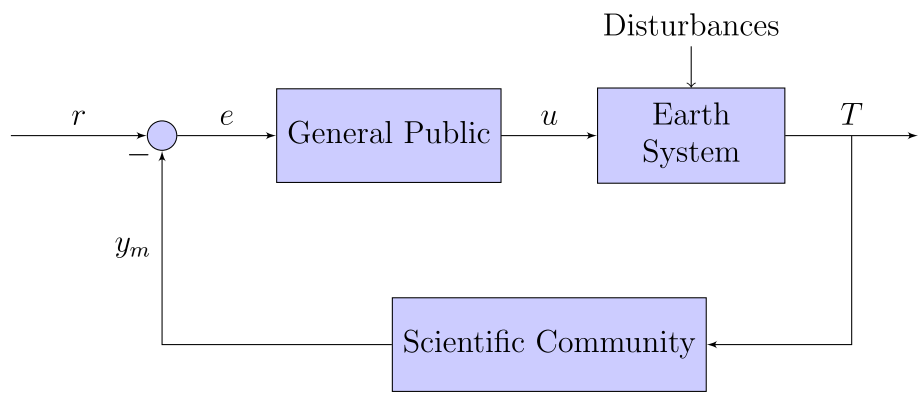

FIGURE 1. Compartment model of Homo Sapiens and the Earth System

2.2. The dynamic interaction of *Homo Sapiens* with the *Earth System*. Roughly speaking, transforming to sustainability involves three compartments, the *Earth System* and the two compartments of *Homo Sapiens*, the *General Public* and the *Scientific Community*. Changes of the compartment *Earth System* are subject both to human forcing u by the control compartment *General Public* and to natural *Disturbances*, resulting in an output variable, here named T like *Temperature*. Output is measured and analysed by the third compartment, the *Scientific Community*. The activity of the compartment *General Public* is essentially determined by the arrow e, indicating *Enlightenment* and *Politics*. It originates from a node, where there is an ongoing struggle between spontaneous, ideologically or commercially magnified influences r on the one side and the flow y_m on the other side, made by science based mathematical modeling and interpretations of scientific measurements and public experiences.

2.2.1. *A scenario of different awareness competences between the general public and the scientific community.* For our toy model we imagine that the general public and the scientific community have quite different awareness competences regarding the output variable $T :=$ average annual Earth temperature. Typically, "ordinary people" will feel dramatic changes of T, and the media and the education will often emphasize just that, in mathematical terms $T' = \frac{d}{dt}T$, the first derivative of T. One will be attentive to increasing rates of change, while decreasing rates of change will be perceived often as a relaxation even when T' remains positive. On the contrary, in mathematics and sciences, we would perceive a continuing increase of the output variable T as highly alarming even when the rates of increase should decrease.

Reason for concern is that a variety of feedback loops force our output variable onto radically different trajectories. We know, e.g., that the heat capacity of the atmosphere equals only the heat capacity of 10 meter of the oceans that have a mean depth of about 4000 meter. Therefore, there is room for much heat exchange between the atmosphere and the oceans. That heat exchange depends on the slow and not very well understood stirring in the oceans. So, we may argue that in the very long run the Earth System can handle the anthropogenic temperature forcing.

However, how long is the "very long run" and what about the other known feedback mechanisms? Recent measurements of the heat of the oceans have revealed that indeed about 93% of the radiative energy imbalance (due to anthropogene emissions) accumulates in the oceans as increased ocean heat content (OHC), see [**8**] reporting on the measurements of the recently completed worldwide grid of 3000 deep water temperature gauges and the perhaps controversial [**9**] using ocean warming outgassing of O_2 and CO_2, which can be isolated from the direct effects of anthropogenic emissions and CO_2 sinks, to independently estimate changes in OHC over time after 1991. That means inter alia, that for now we see only a 7% tip of the climate change problem on the Earth surface. We mentioned above the threat of giant methane release from the oceans and perm freezing areas when the output variable T passes a critical and presently still unknown value and the level of greenhouse gases in the atmosphere can no longer be controlled by human activity but will be governed by autonomous processes of heat forced release from natural sources.

2.2.2. *Different kinds of model credibility and uncertainty.* It is obvious that inaccurate or inaccessible data is one of the predominant sources of uncertainty about the future path of the output variable and the effect of different ways and levels of transforming for sustainability. In mathematical modeling and simulation, that uncertainty is called *aleatoric*. In principle it could be diminished by higher investments in sensor networks and research, as emphasized by J. Behrens [**10**, p. 286] — though hardly eliminated totally, as explained earlier in Subsection 3.2. Contrary to that, the model uncertainty described above is hard to reduce, since there can be many different characteristic time scales, e.g., one for forcing by radiation on the every day / annual scale and the other one for the various feed back loops on the scale of decades or centuries or millennia. There are just different regimes, and it is unclear beforehand which regime is dominant and for how long. This kind of uncertainties is called *epistemic*, since, as BEHRENS reminds us, "model uncertainty is inherent in the process of understanding nature by simplifying it to natural laws."

This is particularly true for climate change modeling, where the good old methodological demand of *falsification* is impossible to satisfy: There are simply no experiments nor observations of the decisive atmospheric, terrestrial and oceanic processes and their interaction on the relevant scales. All we have are only mathematical models with parameters of quite different origin. Some parameters are (i) well-established world constants, immediately derived from physical first principles; some are (ii) measurable in laboratory; but some are just (iii) estimates from fitting somehow available data series to chosen and uncertain systems of equations. With (i) and (ii), the situation is typical for multiscale modeling and simulation. The deviation comes with (iii).

To give an example: In 1996, the U.S.A. and Russia reached an agreement (informal, but since then honored by all nuclear powers, beside India, Pakistan and North Korea) to halt *all* nuclear test explosions. According to the testimony of the nuclear engineer M.F. HORSTEMEYER [**6**, Section 1.3, pp. 4f], they could do so because even the most abstract multiscale simulations of nuclear weapons and their equations and constants can be checked against previous large scale test observations, see also our comments below in Section 7.1. Astrophysicists like C. SAGAN and geologists and geophysicists have argued for the use of abundant planetary and paleontology data; until now, however, without substantial progress for Earth climate modeling.[1]

In the frame of this paper, our main mathematical conclusion is not necessarily to *trust* the existing mathematical models of climate change in lack of better alternative models, but

[1]Of course, looking back at earlier periods of Earth history may help visualizing the probable outcome of continuing excessive Anthropocene greenhouse gas emissions in the years 2030, 2050 or 2100, as K. BURKE and collaborators explain in [**11**, Abstract]: "Past Earth system states offer possible *model systems* (our emphasis) for the warming world of the coming decades. These include the climate states of the Early Eocene (ca. 50 Ma), the Mid–Pliocene (3.3–3.0 Ma), the Last Interglacial (129–116 ka), the Mid–Holocene (6 ka), preindustrial (ca. 1850 CE), and the 20th century." See also our Fig. 7. Inspite of all mandatory reservations against disseminating dystopian views in the scientific literature, it certainly is meritorious to draw parallels between past and future climates. We agree with F. LEHNER, a project scientist at the US National Center for Atmospheric Research, however, in his comment [**12**] to BURKE's study, that many uncertainties make it challenging to reconstruct and understand hydroclimate change, even over the last 1,000 years. As in our Table 1, to us the essential point is to be aware of the huge time scale difference between the supposed few years, be it 12, 30, or 100, to unleash mechanisms that can bring the Earth System back to mid–Pliocene temperatures, and the million years it may take the Earth System afterwards to get back to a climate similar to the present.

- to stick to the established wisdom, that we know the basic atmospheric, terrestrial and oceanic processes sufficiently well separately
- to point to the presence of multiple time scales and
- to argue for a corresponding *change of the mindsets* within the science community and the public.

2.3. The emergence of two characteristic time scales and typical results of understanding and non–understanding between science and the public. We summarize the communicative challenges of dealing with multiple characteristic time lengths of the Earth System (see Table 1), i.e.,

(1) dealing with situations that turn out as less frightening than predicted in the short time,
(2) guarding against situations that will kick off a chain reaction that makes further temperature rises unstoppable in the long run, and
(3) reversing the present imbalance between political intentions and actions, where real world data are ahead of real world intentions.

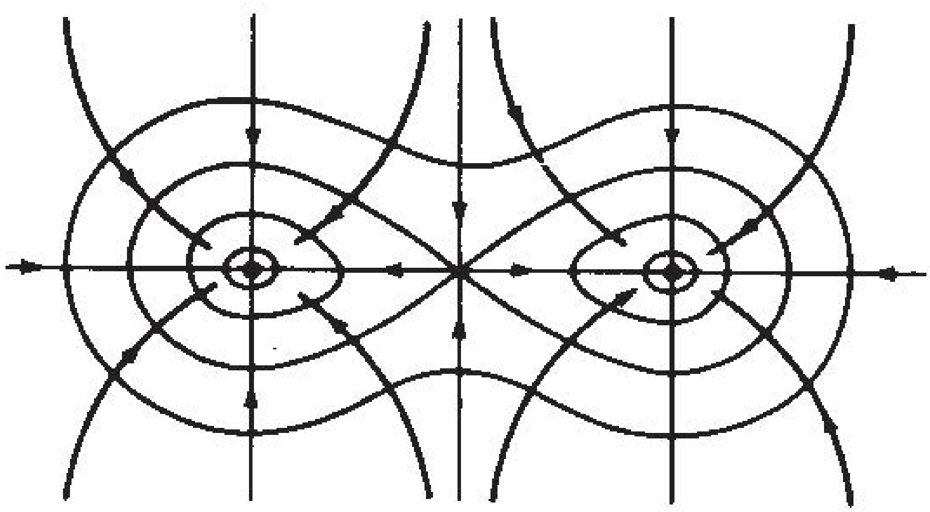

FIGURE 2. Dynamical system with the level curves of $V(T,\alpha) = T^2(T-1)^2 + \alpha^2$ and gradient lines of $(T', \alpha') = -\operatorname{grad} V(T,\alpha)$. Adapted from HIRSCH and SMALE [**13**, p. 203f]

A simple mathematical model in two dimensions illustrates aspects (1) and (2): In Fig 2, we depict the present time Earth System of Fig. 1 as a dynamical system in two variables, the global mean temperature T (horizontal axis, *abscissa*) and a universal control parameter α (vertical axis, *ordinate*). We imagine that the system is subjected to a bundle of geophysical laws, here depicted by level curves of a single aggregate potential $V(T,\alpha) = T^2(T-1)^2 + \alpha^2$. The level curve shaped like a reclining figure eight is $V^{-1}(\frac{1}{16})$. The drawn gradient lines represent typical trajectories (development paths in the time t) of the dynamical system $(T', \alpha') = -\operatorname{grad} V(T,\alpha)$ with a saddle point at $(\frac{1}{2}, 0)$ between two stable equilibria at $(0,0)$ and $(1,0)$.

Trajectories to the left of the threshold line $x = \frac{1}{2}$ tend toward $(0,0)$ (as $t \to +\infty$) in agreement with our short time expectations, namely that it should be possible to return to a benign climate equilibrium (at $(0,0)$) by transforming for sustainability. However, trajectories to the right tend toward $(1,0)$ which may indicate a hot bed far removed from present temperatures, subjected to strong internal feedback mechanisms and out of the range of human control.

The worst thing happening in this simple model is that we become accustomed to the effective short range chances of optimistic human stewardship, advocated eloquently, e.g., by H. Rosling in [**14**], but failing to notice the proximity of the threshold and so foolishly thoughtless glide over the threshold into a region where processes take over that work on much larger time scales and therefore will in the beginning not be easy to notice, but which will be irreversible for long Earth periods and harmful or even fatal for human kind.

Admitted, this is only a fancied simple two–dimensional model to illustrate the emergence of two characteristic time scales. It is a toy model like our scheme in Fig. 1, designed from an educational point of view. The sad fact is, however, that comprehensive geophysical investigations of climate trajectories yield results that qualitatively remind of Fig. 2, see Fig. 3 from the recent [**15**]. We quote:

> Currently, the Earth System is on a Hothouse Earth pathway driven by human emissions of greenhouse gases and biosphere degradation toward a planetary threshold at $\sim 2\,^\circ$C, beyond which the system follows an essentially irreversible pathway driven by intrinsic biogeophysical feedbacks. The other pathway leads to Stabilized Earth, a pathway of Earth System stewardship guided by human-created feedbacks to a quasistable, human-maintained basin of attraction.
>
> "Stability" (vertical axis) is defined here as the inverse of the potential energy of the system. Systems in a highly stable state (deep valley) have low potential energy, and considerable energy is required to move them out of this stable state. Systems in an unstable state (top of a hill) have high potential energy, and they require only a little additional energy to push them off the hill and down toward a valley of lower potential energy.

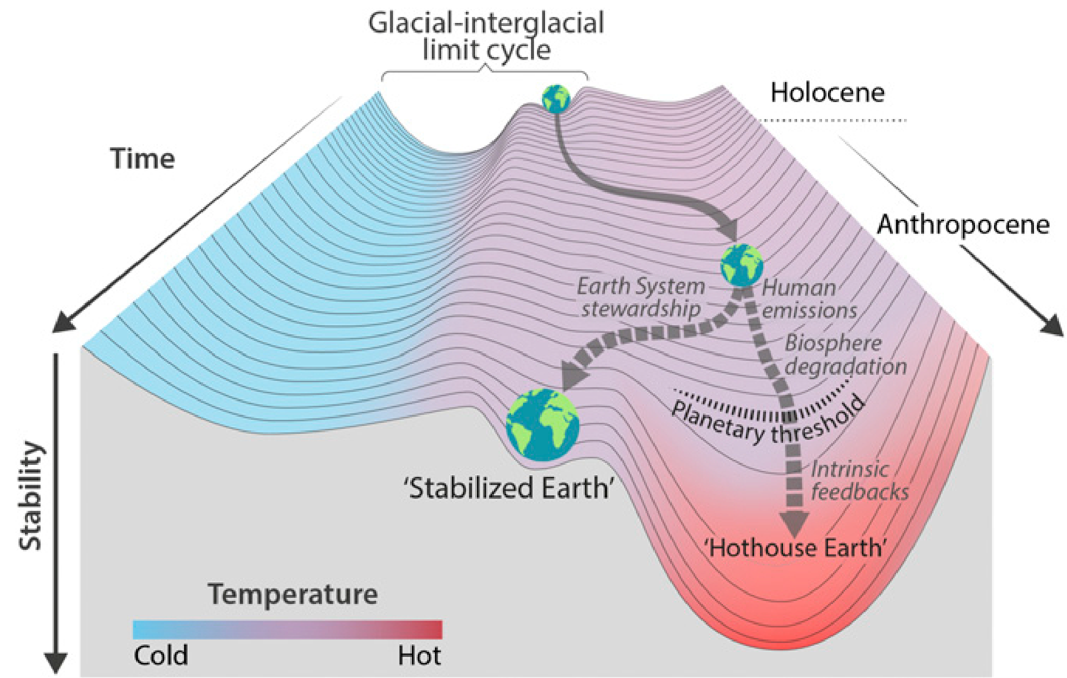

FIGURE 3. Stability landscape showing the pathway of the Earth System out of the Holocene and thus, out of the glacial-interglacial limit cycle and to its present position in the hotter Anthropocene. The fork in the road in Fig. 2 is shown here as the two divergent pathways of the Earth System in the future (broken arrows). From STEFFEN and co-authors [**15**, Fig. 2]

A simple mathematical toy-model similar to the system described by [**15**] can illustrate the time-scale issues at hand. We will here consider a one-dimensional system of temperature, T, subject to a bistable potential V, with a fourth-order polynomial relation such that $V(T) = T^4 - T^2 - \alpha T$, where α is a parameter dependent on the level of anthropogenic emission of greenhouse gasses. For $\alpha = 0$, this system has two minima of equal stability, but different temperature, T_1 and T_2 with $T_1 < T_2$. This is illustrated in the left panel of Fig. 4 as a full blue line. For larger α, the stability in the two minima starts changing such that the high-temperature minimum grows more stable (increasing barrier height for returning to the lower temperature equilibrium), while the low-temperature minimum becomes unstable (decreasing barrier height for crossing over to the higher temperature equilibrium) and eventually smooths out and disappears, shown as dash-dotted red line in the figure.

In this toy-model, a simulation with stochasticity can illustrate a possible development of the earth-temperature, and the corresponding stability. The temperature of the earth is considered to move in accordance with Langevin dynamics, with small stochastic jumps in the time-derivative of the temperature dependent on the current gradient of the stability, see, e.g., [**16**, Ex. 8.26]. With such a system, the temperature of the Earth will be likely to stay close to a local minima, although small changes may occur on short time-scales. In the right-hand panel of figure 4, an example is shown where the initial temperature is close to the low-temperature stability minimum. At a point in time, denoted *Change* in the figure, the α parameter starts increasing linearly with time (In the figure, the trajectory color changes with the value of α). This change signifies a change in the temperature-stability landscape caused by large-scale human intervention in the earth system. In the simulation, increasing α beyond a tipping point eventually leads the Earth moving from the low-temperature "well" into the high-temperature "well".

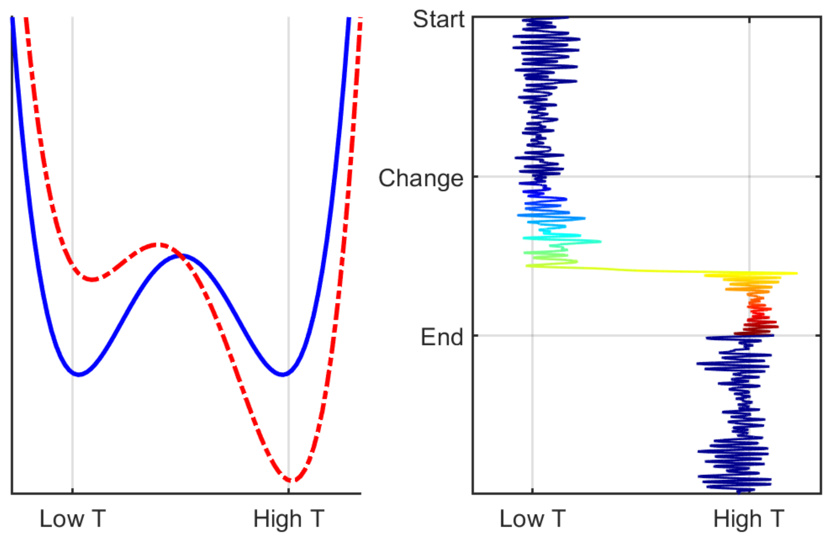

FIGURE 4. Illustration of stability dependence on temperature. The left-hand panel displays the high stability potential for $\alpha = 0$ (blue curve) and a modified potential (red dotted curve) where T_{Low} lost its stability (see text). The right-hand panel displays a typical Langevin simulation with α being zero initially. At a given time (noted *change*) the parameter α starts increasing with time. Note that in this model, after falling in the well at T_{High}, even reversing the potential from red to blue would not permit returning to T_{Low} from T_{High}.

The significance of the trajectory of the earth temperature in Fig. 4 is in the sudden change in temperature. While near the low-temperature minimum, oscillations on a short time-scale are somewhat insignificant, and even at the onset of α increasing with time, there seems to be no particular cause for alarm. However, suddenly the "bump" in the middle of the stability-potential is crossed, and the temperature continues to rise uncontrollably, until the high-temperature minima is reached.

Additionally, reversing the human-caused issues, e.g., setting $\alpha = 0$, does not allow for a steady return to the original situation. Indeed, to escape the high-temperature "well", α would have become negative, to cross the "bump" going from right to left. Returning would take not just halting the human-caused increase of α, but an active effort to produce the opposite effect resulting in a negative α.

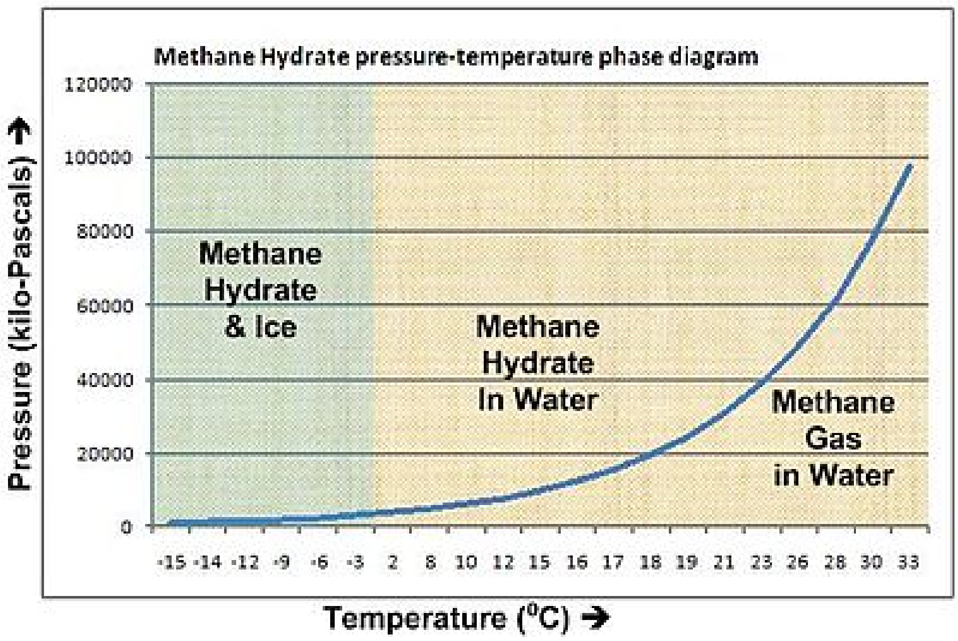

FIGURE 5. Methane clathrate is released as gas into the surrounding water column or soils when ambient temperature increases. Public domain, from Wikipedia contributors, *Clathrate gun hypothesis*. Diagram based on data from [**18**].

3. Multiple time scales in climate change modeling

3.1. Historical insight on the stability landscape of Earth. The Earth was formed 4.5 billion years ago. Many details of Earth history are lost due to plate tectonics. However, there exists geological evidence that the climate has been significantly different from how it has been in recent geological periods where the Earth is predominantly in an *ice age* basin (blue basin on Fig. 3) in the stability landscape. The climate fluctuated into shorter *warm*

No.	t	Name	Description	Size [years]	Effect
1.	t_{civ}	Civilisation time	Holocene, phase of relatively stable postglacial climate	10^4	Development of civilisation
2.	t_{rad}	Human forced radiation pattern changes	Impact on the radiative energy balance of the Earth due to Anthropocene greenhouse gas emission	instant	Slow, but continuing rise of Earth temperature with yearly variation
3.	t_{adapt}	Adaptation time	Transforming for sustainability, prevention, mitigation,	$10^1 - 10^2$	Establishing consensus between people, government, stakeholders; changing mindsets and habits
4.	t_{sec}	Secondary effects	Release of methane from melting perm freezing areas and oceanic methane clathrates	$10^1 - 10^2$	Presumably rapid rise of Earth temperature beyond (not yet precisely known) threshold
5.	$t_{\text{e-folding}}$	E-folding time scale of CO_2	Time for an atmospheric CO_2 concentration to decrease to a proportion of e^{-1}, $\sim 37\%$, of it's original	$5 \times 10^1 - 10^2$	Misleading expectation that fossil fuel CO_2 in the atmosphere was to diminish according to *linear kinetics*
6.	t_{mean}	Mean lifetime of CO_2	Time of the elevated CO_2 concentration of the atmosphere according to carbon cycle models	$10^4 - 10^5$	Leftover CO_2 in the atmosphere after ocean invasion interacts with the land biosphere
7.	t_{reverse}	Reverse time	Returning to present climate equilibrium orbit	10^6	Swinging back by renewed organic and oceanic uptake

TABLE 1. Characteristic time scales of climate change: Anthropogenic emissions occur on top of an active natural carbon cycle that circulates carbon among the reservoirs of the atmosphere, ocean, and terrestrial biosphere on timescales from sub-daily to millennial, while reversing the *clathrate gun* (see below) and exchanges with geologic reservoirs occur at longer timescales, [**17**].

periods where humans developed civilization eventually (red basin on Fig. 3). With the dramatic influence that humans have exerted on the climate, it is

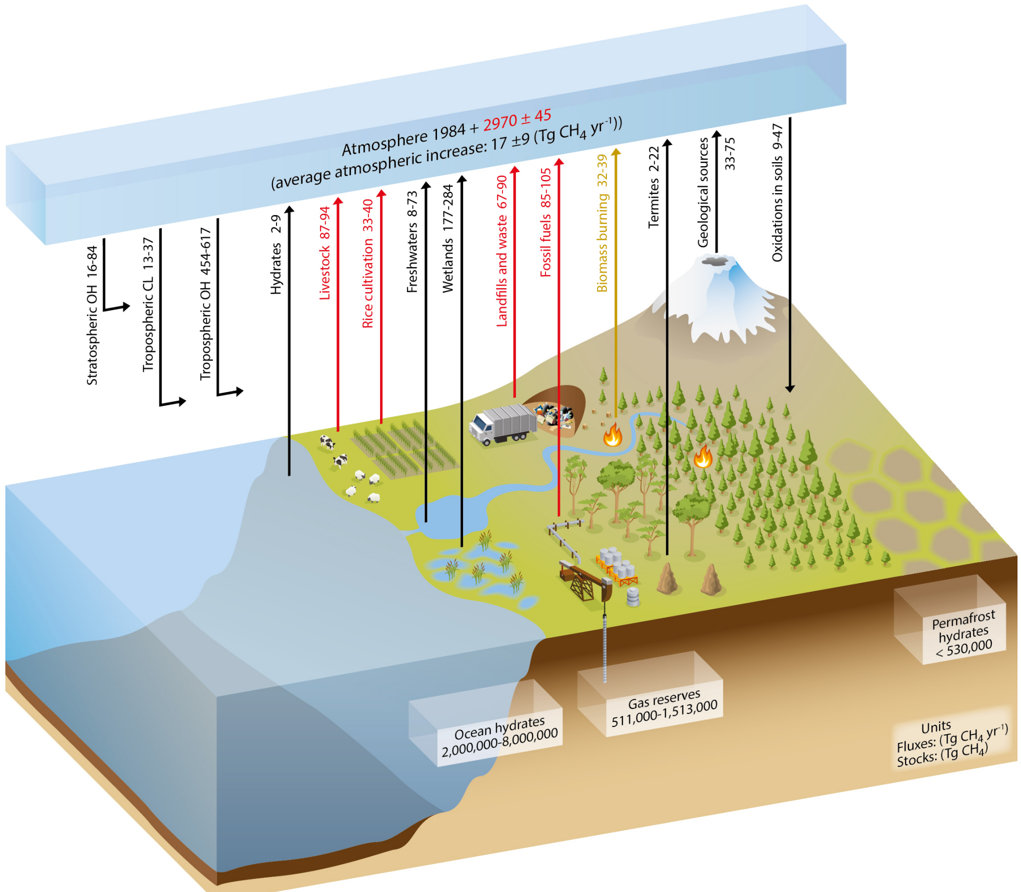

FIGURE 6. Global methane stocks and fluxes estimated for the 2000–2009 time period. Black arrows denote fluxes that are not directly caused by human activities since 1750, red arrows anthropogenic fluxes, and the light brown arrow a combined natural plus anthropogenic flux. From the Intergovernmental Panel on Climate Change . . . .

instructive to ask: how different can the climate of Earth be? By briefly discussing two examples from geology we can learn that the Earth can have widely different climates reflecting several basins in the stability landscape.

Example A. Snowball Earth: Imagine an Earth where ice glaciers stretch all the way from the poles to the equator. The hypothesis known as *Snowball Earth* suggests that the Earth was in a cycle where ice covered Earth several times around 700 million years ago. The albedo effect (reflection of light on the ice sheet) of ice is a feedback mechanism that can stabilize the Earth in such a cold state. However, carbon dioxide has possibly also played a crucial role, and interestingly the period ends with the Cambrian explosion where multicellular life appears in abundance. This explosion of life indicates an increase of carbon dioxide production, taking the Earth away from the cold state.

Example B. Warm Earth: At 55 million years ago to the *Paleocene-Eocene Thermal Maximum* occurred. At this time we would experience the opposite climate on Earth, where palm trees and crocodiles could be found in the Arctic. In this period the global temperature was 8 degree warmer than today. For reference, the *Medieval Warm Period* (ca. year 1100) and the *Little Ice Age* (ca. year 1600) only represent temperature anomalies of about 0.2 degrees. Thus, these variations in historical times are small compared to the discussed scenarios. The hypothetical *Hothouse Earth* proposed by STEFFEN and co-authors [**15**] is a scenario where the collapse of the Earth's ecosystem caused by human emission of carbon dioxide results in a warm Earth. One relevant feedback mechanism for this Hothouse Earth basin in the stability landscape is methane clathrates discussed in the following section.

3.2. Multiscale climate modeling — the example of the clathrate gun hypothesis. To give an example, we recall the emergence of multiple time scales in climate change modeling. In the scientific literature it is claimed that we are in danger of passing tipping points, beyond which human efforts to prevent catastrophic climate change will become useless because of feedback loops. See the comprehensive 2007– and 2013–reports [**19, 20**] of the *Intergovernmental Panel on Climate Change* (IPCC) on the physical science basis and the *Clathrate Gun Hypothesis*, see Fig. 5 of the run-away of methan dissolution when, e.g., the expected sea level increase cannot keep up with the expected temperature increase, and Fig. 6 with estimates of the global methane stocks and annual fluxes. We quote the chemist J.S. AVERY:

> If we look at the distant future, by far the most dangerous feedback loop involves methane hydrates or methane clathrates (a partly frozen slushy mix of methane gas and ice, usually found in sediments, i.e., crystalline water–based solids physically resembling ice, in which the host molecule is water and the guest molecule is methane; their detailed formation and decomposition mechanisms are still not well understood, see [**21, 22**]). When organic matter is carried into the oceans by rivers, it decays to form methane. The methane then combines with water to form hydrate crystals, which are stable at the temperatures and pressures which currently exist on ocean floors. However, if the temperature rises, the crystals become unstable, and methane gas bubbles up to the surface.

> Methane is a greenhouse gas which is 70 times as potent as CO_2.
>
> The worrying thing about the methane hydrate deposits on ocean floors is the enormous amount of carbon involved: roughly 10,000 gigatons. To put this huge amount into perspective, we can remember that the total amount of carbon in world CO_2 emissions since 1751 has only been 337 gigatons. A runaway, exponentially increasing, feedback loop involving methane hydrates could lead to one of the great geological extinction events that have periodically wiped out most of the animals and plants then living. From [**23**, Section 4.6].

We refer also to the disturbing news about jumps in trajectories of the Earth System in [**15**] by W. Steffen and collaborators.

3.3. Multiscale computational methods in climate modeling. Leaving the multiscale problems in climate *modeling* aside, we shall give just one example of the intricacies of multiscale *computational methods* in climate modeling from every-day simulation experience in atmospheric science, following the geophysicist R. Klein in [**24**, p. 1002 and p. 1004]. Roughly speaking, we distinguish between diabatic and adiabatic temperature changes. Diabatic changes are very slow and at small rate — but irreversible and so decisive for temperature changes in the long run, while adiabatic changes, being more frequent and at larger rates, are reversible and so negligible in the long run. Hence, there are two different characteristic times, t_d for diabatic changes and t_a for adiabatic changes with $t_d \gg t_a$. Truncation errors from the discretization of the adiabatic processes become an irreversible character and can dominate in the long run over the essential diabatic changes. Worst of all, we can not do without simulating the adiabatic processes: they are needed to calculate the diabatic changes.

3.4. Smoothing–out multiscale problems in climate modeling generates over– and underestimation of the risk of unwanted impacts of greenhouse gas emission. To sum up we emphasize a few common misconceptions in climate modeling when disregarding the emergence of multiple time scales.

3.4.1. *Multi–model based projections.* To solve multiscale problems a natural first–order approximation is the decomposition of the problem in a multitude of submodels, each with its own characteristic scale, and then patching the results by a averaging process. The Danish meteorologists Madsen, Langen, Boberg and Christensen point to the systematic failure of that way of dealing with multiscale problems in *multi-model based projections*: In [**26**],

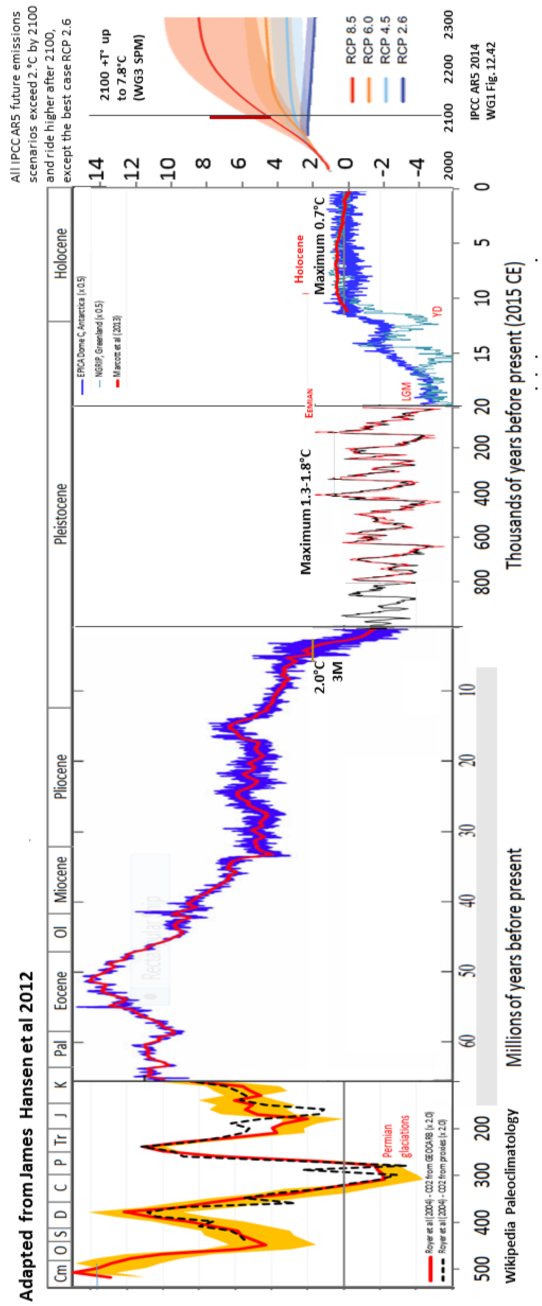

FIGURE 7. Earth temperature with multiple time scales, adapted from [**25**], now public domain wiki

they show an inflated uncertainty in multimodel–based regional climate projections. Roughly speaking, the complexity of atmospheric physics does not permit precise global and longterm climate simulations. Therefore, regional longterm projections are typically based on patching multiple models together to obtain the geographical distribution of the multimodel mean results. Trivially, that procedure runs on its way in the probabilistic intricacies of taking means of non-comparable magnitudes. *Consequently*, as the Danes write, *the risk of unwanted impacts may be overestimated (or underestimated, our insertion) at larger scales as climate change impacts will never be realized as the worst (or best) case everywhere*.

3.4.2. *Unfounded linearizations.* A related common failure of dealing with multiscale problems is approximating the underlying equations by *linearization*. That is, e.g., the case in thoughtless use of the concept of Global Warming Potential (GWP). In [**19**, Section 2.10], the comprehensive 2007-IPCC report on the physical science basis of climate change compares the anticipated climate change impact of a compound i (say methane or an aerosol) with the anticipated climate change impact of the reference substance $r := CO_2$ by setting

$$GWP_i := \frac{\int_0^{TH} RF_i(t)\, dt}{\int_0^{TH} RF_r(t)\, dt} = \frac{\int_0^{TH} a_i \cdot [C_i(t)]\, dt}{\int_0^{TH} a_r \cdot [C_r(t)]\, dt}, \tag{3.1}$$

where TH denotes the choice of a time horizon, important for evaluating differences in the degradation/ocean- and land-depositing processes; RF_i and RF_r the global mean radiative forcing of components i, r with $RF_i < 0$ for i aerosol; a_i and a_r the radiative forcing per unit mass increase in atmospheric abundance of components i, r (radiative efficiency), and $[C_i(t)]$ and $[C_r(t)]$ the time-dependent abundance of the components. Note that the radiative efficiency is considered as being scale independent, i.e., the pattern of absorption and scattering is considered as fixed and so the radiative forcing as linear in the concentrations.

Equation (3.1) may be useful to tune multi-component abatement strategies by providing numerical values for the trade-off between emissions of different forcing agents, in particular after the minor corrections made in the more recent comprehensive 2013-IPCC report on the physical science basis of climate change [**20**, Section TS.3.8 and Section 8.7]. However, the nominator and denominator of (3.1) itself are anticipated; they are fancied and do not yield appropriate impact functions but would be gravely misleading: Clearly, the greenhouse effect of a thin layer of CO_2 molecules can be both calculated and measured in laboratory, contrary to the greenhouse effect of the 700 km thick troposphere, stratosphere, mesosphere, and thermosphere layers. Any assumption of approximatively proportionality must lead astray due to the non–linear radiative interaction.

One has to worry that there is no longer a characteristic time or a characteristic temperature difference to observe. We may have already fabricated the irrevocable preconditions for a hot bed path towards large-scale climate changes following a set pattern and being yet beyond a tipping point.

3.4.3. *Ill–posed problems.* A third source of uncertainty originates from the well–known property of *ill–posedness* of the initial conditions, boundary conditions and coefficients of atmospheric equations as discussed in [**19**, Paragraph 11.10.1.2].

3.4.4. *Discarding secondary effects and slipping across thresholds.* In our context, the worst common failure is focusing on processes only in one of the multiple time scales and neglecting processes dominated by concurring and immanent other relevant characteristic time scales like *secondary effects*, as discussed in our Section 3.2, or *slipping involuntarily and inadvertently* into another orbit, as sketched in the following Section 2.3 and exemplified further below in Section 4 in material sciences, in Section 5 in biomedicine and in Section 6 in economics.

4. Multiple time scales of matter — Viscosity of soft materials

4.1. Classes of multiscale problems. Most problems in material sciences have multiple time scales. A chemical reaction, for example, may begin slowly and the concentration changes little over a long time; then, the reaction may suddenly go to completion with a large change in concentration over a short time. There are two time scales involved in such a process. Another example occurs in fluid flow, where the processes of heat diffusion, advection, and possible chemical reaction all have different scales. The processes at different time scales are governed by physical laws of different character, see Fig. 8.

The mathematicians W. E and B. Engquist distinguish between two classes of multiscale problems, see [**3**, Section 1.4.3] and [**27**, p. 1068f]. *Type A problems* are problems with localized defects around which microscopic models have to be used; elsewhere one can use some macroscopic models. As example they mention the crack propagation in solids. *Type B problems* are those for which the microscale model is needed everywhere either as a supplement to or as the replacement of the macroscale model. This occurs, for example, when the microscale model is needed to supply the missing constutitive relation in the macroscale model. Below in Section 5 we shall present a type B model, that describes the insulin secretion of a glucose stimulated beta cell by a macroscale model in the time range of minutes and the space range of μm, but depends on a microscale model of the electrical input of Ca^{++} oscillations in microdomains in the time range of seconds and the space range of nm.

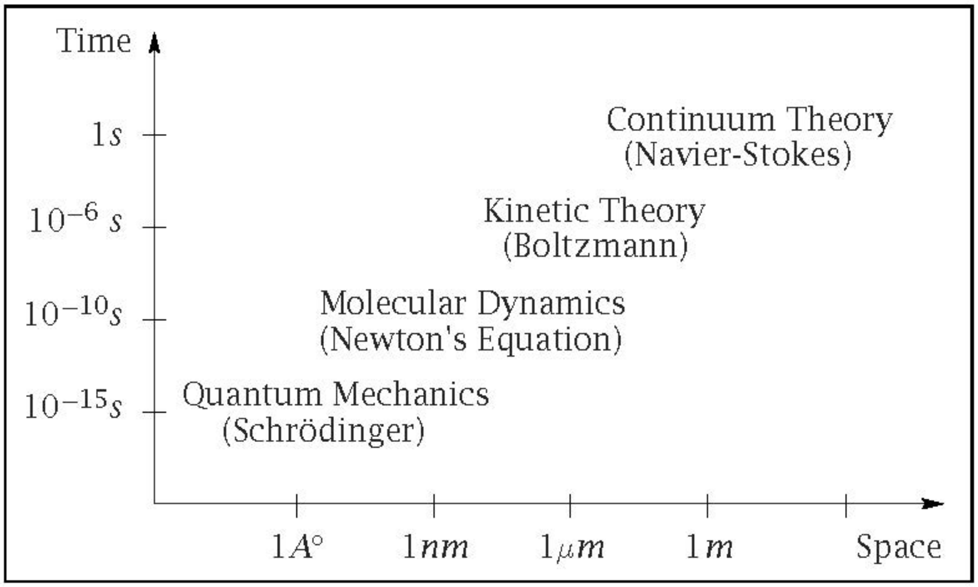

FIGURE 8. Commonly used models of physics at different scales. From E and ENGQUIST [**27**, Fig. 1]

In this section we present a kind of *type C multiscale model*, where there is no multiplicity of models and scales to begin with and the multiple time scales emerge in the course of the model application.

4.2. The emergence of multiplicity of time-scales in liquid dynamics. When liquids are cooled, dynamics may become dramatically slow. As an example, Fig. 9 shows results of measurements an silicon oil (chosen since it is chemically stable resulting in reproducible measurements). Specifically, the measured quantity is the *dielectric relaxation time*. In layman terms, it tells how fast the molecules rotate. Ignoring interactions between molecules, a *back of the envelope* calculation suggests that the rotational time should in the order of 1 picosecond [ps]. However, the measured times are significantly slower due to collective dynamics. The sluggish dynamics reflect the emergence of a slow time-scale in the system [**28, 29, 30, 31, 32**]. Below we will look into answering the question "Why are cold liquids dynamics surprisingly slow?" by studying a model liquid.

From a reductionist viewpoint one should apply Quantum Mechanics to understand dynamics of liquids (lower left corner on Fig. 8). In that theoretical framework, the state of the liquid is described by a multiparticle wave-function $|\Psi\rangle$. This mathematical object contains information about all here relevant subatomic particles — electrons, neutrons and protons. For a given Hamiltonian $\widehat{\mathcal{H}}$, the time propagation of the wave-function can in principle be computed

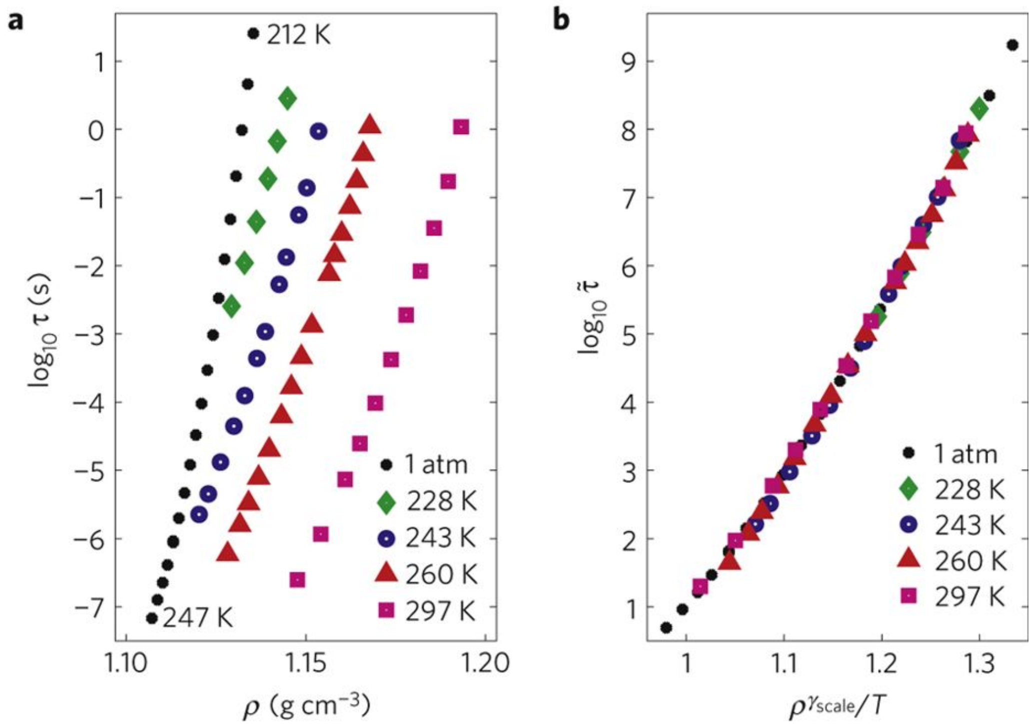

FIGURE 9. (a) Measurements of the *dielectric relaxation time* τ, as a function of mass density ρ (note in the text we use the number density, the molar mass gives the proportionality between the two densities), of the silicone oil DC704. The measured time τ is related to the rotation time of molecules. Measurements are done along isotherms of four different choices of the temperature T and along the atmospheric pressure isobar (1 atm). The relaxation time at ambient pressure (black dots) changes more than eight orders over a narrow temperature span of only 35 degrees. (b) The *reduced* (i.e., dimensionless) relaxation times $\tilde{\tau} := \tau\rho^{1/3}\sqrt{m/T}$, where m denotes the mass of the molecule. Interestingly, the different $\tilde{\tau}$ collapse into a single master curve when plotted against $\Gamma = \rho^{\gamma_{\rm scale}}/T$, where $\gamma_{\rm scale} = 6$ is a scaling exponent. This indicates a hidden scale invariance that collapses the phase diagram from two variables (ρ and T) into a single (Γ). In the text, we investigate a model where particles interact via inverse power-laws: r^{-n} where r denotes the distance between two pairs. This model also posses this scale invariance with $\gamma_{\rm scale} = n/d$ where d is the dimension of space. Thus, we only have to investigate a single state variable. We choose temperature.

by solving the equations of motions suggested by ERWIN SCHRÖDINGER in 1925: $i|\dot{\Psi}\rangle = \widehat{\mathcal{H}}|\Psi\rangle$ (here the dot referrer to a time derivative). Unfortunately, only a few textbook examples like the harmonic oscillator or a particle in an infinite square well can be solved analytically. Today, computers are routinely used to solve the equations of motions of atomic systems on the picoseconds time scale using clever approximations. This timescale is long enough to understand liquid dynamics at high temperatures, however, longer time-scales are needed to understand the dynamics of cold liquids. Thus, we cannot hope to solve the Schrödinger equation explicitly. Instead we will address the question using a classical potential that approximates the true Quantum Mechanical energy surface and dynamics.

We investigate a classical Hamiltonian using one of the numerical integration methods generally referred to as *Molecular Dynamics* [**33**]: Consider N particles on a d-dimensional torus (i.e. a periodic d-dimensional box) of volume $V = L^d$, where L is the side length. For simplicity we study a $d = 2$ dimensional liquid. Let the $d \times N$ dimensional collective coordinate be $\mathbf{R} := \{\mathbf{r}_1, \mathbf{r}_1, \ldots, \mathbf{r}_N\}$, so the potential energy function is $U(\mathbf{R})$ (defined in the paragraph below). The (classical) Hamiltonian $\mathcal{H}$ is the sum of the potential and the kinetic energy: $\mathcal{H}(\mathbf{R}, \dot{\mathbf{R}}) = U(\mathbf{R}) + K(\dot{\mathbf{R}})$, where $K(\dot{\mathbf{R}}) := \frac{1}{2}\sum_i^N m_i|\mathbf{v}_i|^2$, m_i denotes the mass and $\mathbf{v}_i := \dot{\mathbf{r}}_i$ denotes the velocity of particle i.

The dynamics of the system is computed numerically by solving Newtons classical equations of motion using a leap-frog algorithm: If Δt is a time step and $\mathbf{F}_i = -\nabla_i U$ is the force on particle i, then the next velocity and position in a adjacent timestep is found as

$$\mathbf{v}^{(i)}_{t+\Delta t/2} \;=\; \mathbf{v}^{(i)}_{t-\Delta t/2} + \mathbf{F}_i\Delta t/m_i \text{ and } \mathbf{r}^{(i)}_{t+\Delta t} \;=\; \mathbf{r}^{(i)}_t + \mathbf{v}^{(i)}_{t+\Delta t/2}\Delta t,$$

correspondingly. This integration scheme is symplectic and the same trajectory is generate if time is reversed. Thus, there is no systematic drift of the total energy (except from numerical truncation of floating points), contrary to the popular fourth order Runge-Kutta (RK4) integration scheme.

The *kinetic temperature* of a system is

$$T \;:=\; \frac{1}{k_B}\langle\sum_i^N (m_i|\mathbf{v}_i|^2)/N_f\rangle,$$

where $\langle\cdots\rangle$ is a time average, k_B denotes the Boltzmann constant and $N_f := dN - d$ is the number of degrees of freedom in the system (the removal of d degrees of freedom accounts for the fixed total momentum). The temperature is determined by the initial positions and velocities of the particles. Alternatively we can control the temperature by coupling our system to a heat bath with some temperature as done with a Langevin thermostat: Imagine a thin gas (the heat bath) interacting weakly with particles in the system. The particles in the gas will apply a drag force, and random kicks to particles of the system. In the above mentioned algorithm we can model this by computing the force on particle i as

$$\mathbf{F}_i = -\nabla_i U + \gamma \bar{v}_i + \sqrt{2\gamma k_B T}\mathcal{R}(t),$$

where γ determines the coupling with the heat bath, $\bar{v}_i$ denotes the velocity of the particle, and $\mathcal{R}(t)$ is a delta-correlated Gaussian process with zero-mean: $\langle \mathcal{R}(t)\mathcal{R}(t')\rangle = \delta(t-t')$ with the distribution $P(\mathcal{R}) := \exp(-\mathcal{R}^2/2)/\sqrt{2\pi}$.

We need to define a potential energy function, $U(\mathbf{R})$, of a model liquid not prone to crystallization (since we are interested in the liquid state). Inspired by the Kob-Andersen binary inverse power-law (KABIP) model presented in [**34**] we propose a model where the potential energy function is a sum of inverse power laws in the pair distances: $U(\mathbf{R}) = \sum_{i>j}^{N} \varepsilon u(|\mathbf{r}_j - \mathbf{r}_i|/\sigma\sigma_{ij})$, where the pair energy function for a given dimensionless pair diameter σ_{ij} is $u(r) = r^{-18} - 1.5^{-18}$ for $r < 1.5$ and zero otherwise (ε and σ are discussed below). The truncation of the potential at 1.5 makes computations faster, since forces only have to be computed between neighbors. Fortunately, since $1.5^{-18} \ll 1$ the truncation does not influence results much at the investigated temperatures. The interaction parameters between different types of pairs are $\sigma_{AA} = 1.1$ and $\sigma_{AB} = \sigma_{BB} = 0.9$. The parameters ε and σ set an energy- and a length scale, respectively. All particles are given the same mass $m_i = m$. Results are presented in units derived from ε, σ, m, and the Boltzmann constant k_B.

We shall investigate a system of $N = 1600$ particles at number density $\rho = N/V = \sigma^{-3}$ consisting to 70% of the larger A particles and to 30% of the smaller B particles (i.e. the system size is $L = 40\sigma$). The model is implemented into the RUMD software package [**35**] that utilizes graphic cards for fast computations. The code also uses a "neighbor list" and a "cell list" resulting in an algorithm where the computational time only scales as the number of particles N (not N^2 as it would be expected if the force on particle i depends on the positions of all other particles). We refer reader to a standard textbook on Molecular Dynamics for more details: see [**33**].

Below we will investigate the dynamics as a function of the temperature with a focus on low-temperature dynamics, however, let us first do a *back of the envelope* calculation where we use an mean-field approximation, i.e. our calculation will only evolve averages such as the mean density of particles (ρ) and the temperature. A characteristic time is given as the average time it takes two particles to encounter. Let us assume that this is when a particle has traveled 10% of an inter-particle distance $l_0 = 0.1\sqrt[3]{1/\rho}$. The average velocity is $v_0 = \sqrt{dk_BT/m}$ (assuming the system is large, $N_f = N$). Thus, the inter-particle collision time is expected to be in the order of $t_0 = l_0/v_0 = 0.1\sqrt[3]{1/\rho}\sqrt{m/dk_BT}$. For short times, $t \ll t_0$, the particles are expected to move ballistically: $\mathbf{r}_i(t) = \mathbf{r}_i(0) + \mathbf{v}_i(0)t$. For long-times, $t \gg t_0$, particles will have many encounters and the movement becomes diffusive: $\langle|\mathbf{r}_i(t)-\mathbf{r}_i(0)|^2\rangle = 2dDt$, where $R^2 \equiv \langle|\mathbf{r}_i(t) - \mathbf{r}_i(0)|^2\rangle$ is the mean squared displacement and D the diffusion constant.

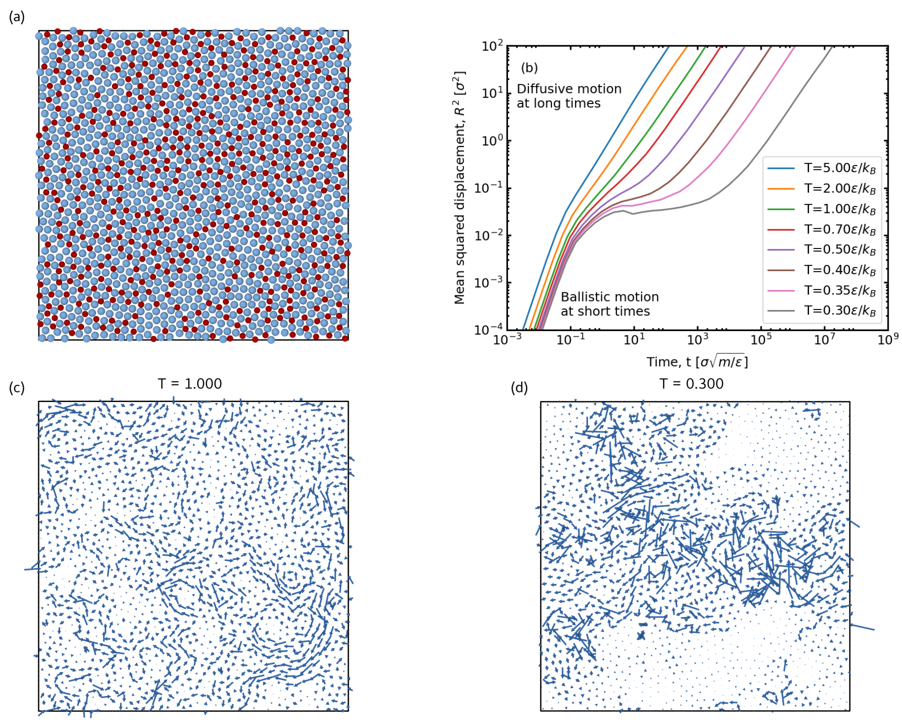

FIGURE 10. Model of a two dimensional model of a liquid consisting of large A particles (blue) and smaller B particles (red). (a) Representative configuration at $T = 0.3\varepsilon/k_B$. (b) The mean squared displacement, R^2 at a range of temperatures in units of. A slow time-scale emerges at low temperatures signaled by the appearance of a caging plateau in the mean squared displacement. (c) Displacement vectors at temperature $T = \varepsilon/k_B$ and (d) $T = 0.3\varepsilon/k_B$, where the mean squared displacement is $R^2 = \sigma^2$ for the A particles.

For our model at temperature $T = \varepsilon/k_B$ the characteristic time is $t_0 = 0.1\sigma\sqrt{m/2\varepsilon} \simeq 0.07\sigma\sqrt{m/\varepsilon}$. Values for a molecular liquid, like the silicone oil DC704 (Fig. 9), are in the order of $\varepsilon \simeq 1$ kcal/mol, $\sigma \simeq 1$ nm, $m \simeq 100$ u resulting in the time-scale $t_0 \simeq 0.3$ ps. Chemical details of a particular molecule change ε, σ and m resulting in changes of t_0 within approximately one order of magnitude. Thus, the fact that the actual relaxation time measured (as exemplified on Fig. 9) is many orders of magnitude slower signals the emergence of a slower timescale.

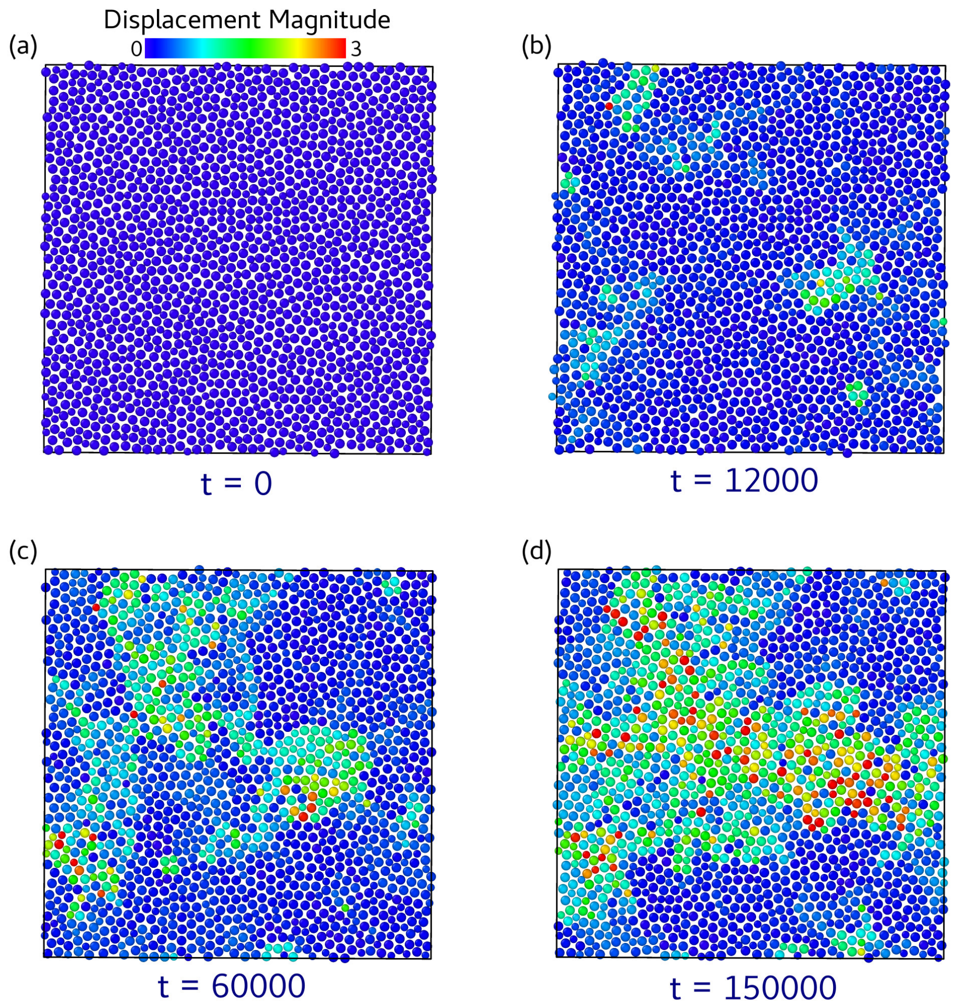

FIGURE 11. The magnitude of particles' displacement after a time-interval of $t = 0$ (a), $t = 12000\sigma\sqrt{m/\varepsilon}$ (b), $t = 60000\sigma\sqrt{m/\varepsilon}$ (c) and $t = 150000\sigma\sqrt{m/\varepsilon}$ (c) at temperature $T = 0.3\varepsilon/k_B$. The dynamics are spatially heterogeneous, where a few particles move far. Clusters of particles move together in an avalanche, that in turn triggers avalanches in the vicinity.

Figure 10a shows a representative configuration from the molecular dynamics of the simple model of a liquid, and Fig. 10b shows the mean squared displacement, R^2, at a range of temperatures. At high temperatures ($T > \varepsilon/k_B$), we have a simple behavior from ballistic to diffusive motion. At low temperatures ($T < \varepsilon/k_B$), a slow time-scale emerges. Molecules stay caged for a long time, seen as a plateau in R^2 in Fig. 10b. This emergence of a slow time-scale is related to dynamical heterogeneity. Figure 10c shows the displacement vectors of particles when $R^2 = \sigma^2$ at $T = \varepsilon/k_B$. This can be compared with the displacement vectors at low temperatures shown on Fig. 10d. At low temperatures, about half of the particles have moved a lot, while the remaining have not moved. Moreover, at low temperatures flow events occur in sudden avalanches where many particles move collectively. Figure 11 show the magnitude of the displacements at different times. After a time-interval of $t = 12000\sigma\sqrt{m/\varepsilon}$ (Fig. 11b), a few regions of particles have moved in a localized cluster. The regions of mobility facilitate dynamics in nearby regions (Fig. 11c), explaining the dynamical heterogeneity (Figs. 10d and 11d).

Interestingly, the liquid dynamics at low temperatures have some parallels to the climate of Earth. Think of the one coordinate in the $\mathbf{R}$ vector as the temperature somewhere, and the remainder as other parameters that are important for the Earth system. In both cases we will see that the temperature is fluctuating around some local fixed point. However, at some time in the development an *avalanche* will occur, and the observed parameter will change a lot in a short time. In both cases, the origin is a strong feed-back coupling to the remaining part of parameter space (this is not the case in high-temperature dynamics, where remaining particles/parameters can be treated as a mean-field).

5. Multiple time scales of life — Exemplified in cell physiology

In the life sciences, multi-scale processes are the norm. Spatial scales vary over as much as 15 decades of magnitude as we progress from processes involving genes, proteins, cells, organs, organisms, communities, and ecosystems; time scales vary from times that it takes for protein to fold to times for evolution to occur. Several scales can occur in the same problem.

In this section we shall address the emergence of multiple time scales in two fields of human cell physiology, the production of blood in the human body with new light on the rise of malignancies in slow-fast processes, and the insulin secretion of pancreatic beta cells, where the recognition of two different characteristic times plays a role in diabetes diagnosis and therapy. Like in many other physiological problems, in cancer and diabetes research multiple spatial and time scales are intertwined and the theoretical and application challenges *push* the research towards nano geometry, see [**36, 37**]. Moreover, advances on the technological side *pull* towards multi scale analysis: even for a single cell there has a wide range of observational means become available, with length scales from Å in electron microscopy to μm in confocal fluorescence microscopy and multifocal multiphoton imaging, and a corresponding wide range of time scales.

5.1. Multiscale models of the production of blood in the human body. An illustrative example is the production of blood in the human body and how slow processes in an otherwise fast system can lead to haematopoitic malignacies, such as leukemia or myeloproliferative neoplasms (MPNs). As the majority of the cells that constitute the blood have a lifespan in the order of days, and reconstitution following loss of blood is of a similar magnitude, MPNs develop on a much greater timescale, estimated as about a decade [**38**]. Understanding exactly how these malignancies arise and develop on the slow timescale is key to efficient treatment.

Some blood cancers such as the MPN malignancies are believed to emerge from mutations in the haematopoeitic stem cells, located in the bone marrow. A random mutation in one such stem cell causes it to be dysfunctional, e.g. to produce an excess of a particular sub-type of cell or to be non-reactive to signals limiting its reproduction. As the stem cell self-renews, a fitness advantage from the mutation can lead to the mutation-type becoming the dominant type of stem cell.

[**38**] describes a system of six coupled ordinary differential equations, in an effort to model the blood producing system of the human body, and the development of MPNs. The model combines the behavior of the stem cells with a feedback from the blood through an abstract measure of inflammation of the immune system. Since MPNs are rare, the random mutation of stem cells are expected to occur on a timescale greater than the average human life-time. As such, the rate of mutation can be considered as effectively zero, and instead a single mutated cell is added initially. Figure 12 displays how the blood cell count develops in the model. For the count of various types of blood cells, there is typically a threshold, which, when exceeded, is grounds for further investigations of the patient. While above the threshold the risks of complications such as Thrombosis is also expected to be greatly increased. The figure displays an estimate of this threshold as a black dotted line at around 143% [**39, 40**]. Interestingly the model predicts that this threshold is not exceeded until about 17.5 years after the initial mutation of a stem cell. In addition, this is the same point at which the mutated blood cells constitute about half of the total blood cell count. Even though the blood production is capable of fast reconstitution after blood loss and although the main constituents of the blood have a fast turnover-rate, the time-scale of the development of MPNs is much slower. Yet once a stage is reached were the disease is immediately noticeable, the mutated cells make up such a large part of the blood that treatment must lead to drastic changes to negate the harm done within the two-decade long progression of the disease. Discovering the presence of the mutations is improbable since many short-term factors affect the blood cells counts, and, apart from specialized clinical assays, only long-term monitoring of blood cell counts can be expected to warn of mutated stem

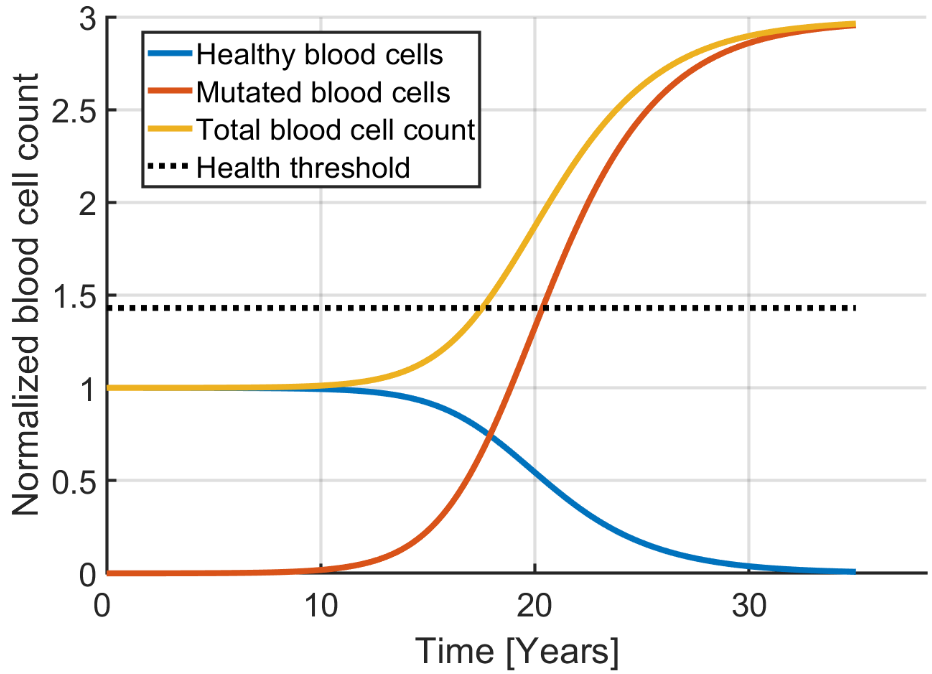

FIGURE 12. Development of blood cell count in the model of [**38**] following mutation of a single stem cell. The dotted boundaries are estimates of the interval within which an individual would be considered healthy and where the risk of complications is low.

cells. As such, diagnosis of MPNs before complications arise is rare, partly due to the difference in time-scale of the development.

5.2. Regulated exocytosis in pancreatic β–cells. Another illustrative example of the emergence of multiple time scales in cell physiology is the biphasic insulin secretion of pancreatic β-cells.

5.2.1. *Discovery of biphasic insulin secretion.* Precisely 50 years ago, the biochemist G.M GRODSKY and coworkers demonstrated in [**42**], that glucose-induced insulin secretion in response to a step increase in blood glucose concentrations follows a biphasic time course consisting of a rapid and transient first phase followed by a slowly developing and sustained second phase, see Fig. 13.

In some aspects, it reminds on the ominous feature of two characteristic times in climate change: by constant forcing,

***to begin with*:** nothing extraordinary is observable;
***then for a short while*:** consequences become

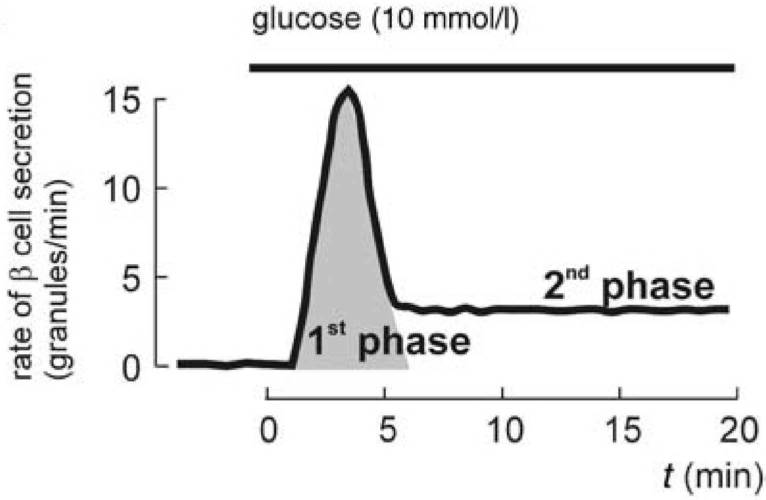

FIGURE 13. Schematic of insulin secretion elicited by an increase of the extracellular glucose concentration to ≥ 10 mmol/l. Insulin secretion is triggered with a delay of ~ 1 min (the time needed for glucose to be metabolised) and then follows a biphasic time course. From [**41**, Fig. 2A, p. 1031]

first: ever more observable and
further-on: go less-and-less noted in spite of continuing forcing and continuing and now substantial aggregation of consequences
until: (in the climate change situation, a tipping point is reached and feed-back mechanisms take over).

As emphasized in P. RORSMAN's and E. RENSTRÖM's review [**41**] and our monograph [**43**], understanding the reason for the biphasic feature of (normal) insulin secretion is wanted for better diagnosis and treatment of diabetes mellitus type 1 and type 2.

5.2.2. *Mathematical models of the exocytosis cascade.* Mathematically, it is easy to repeat the biphasic feature in a black box model, alone with two compartments, i.e., just two coupled differential equations with suitably tuned coefficients. See GRODSKY et al. in [**44**] and the follow-up literature, though all these two-compartments models with coefficients that cannot be interpreted in physiological biomedical, biochemical or bio-electrical terms nor measured independently.

The mathematician A. SHERMAN and collaborators provided in [**45**] a six compartment model with coefficients that partly can be assigned biomedical empirical evidence to. The authors introduce the following system of 8 coupled ordinary differential equations to model the exocyosis cascade (EC) with insulin vesicles aggregated in 6 pools $N_j, j = 1, \dots 6$ of different maturity and vicinity to the cell membrane and the two variables N_R, N_F of re-supply and secretion:

$$\begin{aligned}
\dot{N}_1 &= -[3k_1C_{md}(t) + r_{-1}]N_1 + k_1N_2 + r_1N_5, \\
\dot{N}_2 &= 3k_1C_{md}(t)N_1 - [2k_1C_{md}(t) + k_{-1}]N_2 + 2k_{-1}N_3, \\
\dot{N}_3 &= 2k_1C_{md}(t)N_2 - [k_1C_{md}(t) + 2k_{-1}]N_3 + 3k_1N_4, \\
\dot{N}_4 &= k_1C_{md}(t)N_3 - [3k_{-1} + u_1]N_4, \\
\dot{N}_5 &= r_1N_1 - [r_1 + r_{-2}]N_5 + r_2N_6, \\
\dot{N}_6 &= r_3 + r_{-2}N_5 - [r_{-3} + r_2]N_6, \\
\dot{N}_F &= u_1N_4 - u_2N_F, \\
\dot{N}_R &= u_2N_F - u_3N_R.
\end{aligned} \tag{5.1}$$

For the parameters see Tabel 2

Parameter	Value	Parameter	Value
k_1	$20\ \mu M^{-1}s^{-1}$	k_{-1}	$100\ s^{-1}$
r_1	$0.6\ s^{-1}$	r_{-1}	$1.0\ s^{-1}$
r_2^0	$0.006\ s^{-1}$	r_{-2}	$0.001\ s^{-1}$
r_3^0	$1.205\ s^{-1}$	r_{-3}	$0.0001\ s^{-1}$
u_1	$2000\ s^{-1}$	u_2	$3.0\ s^{-1}$
u_3	$0.02\ s^{-1}$	K_p	$2.3\ \mu M$

TABLE 2. Kinetic parameters of the EC model at the resting state

In [**46**], we have given a radically different approach. We model the morphology and dynamics of the making of the fusion pore or porosome as a cup-shaped lipoprotein structure (a dimple or pit) on the cytosol side of the plasma membrane and describe the formation of the dimple by a free boundary problem. We discuss the various forces acting and analyse the magnetic character of the wandering electromagnetic field wave produced by intracellular spatially distributed pulsating (and well-observed) release and binding of Ca^{++} ions anteceding the bilayer membrane vesicle fusion of exocytosis. Our approach explains the energy efficiency of the dimple formation prior to hemifusion and fusion pore and the observed flickering in secretion. It provides a frame to relate characteristic time lengths of exocytosis to the frequency, amplitude and direction of propagation of the underlying electromagnetic field wave.

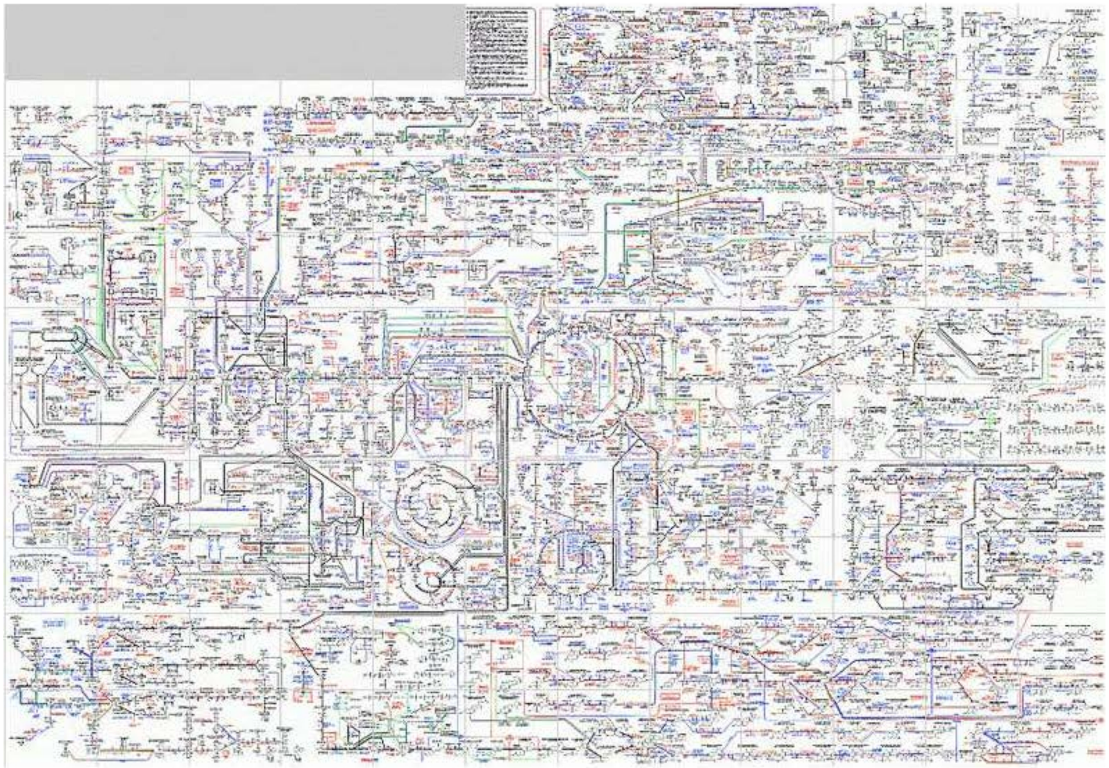

FIGURE 14. Metabolic pathways in a single cell, courtesy by JOHN PEZZUTO

Like in climate modeling, however, it is only a model and an imperfect one. So, again like in climate modeling, we must distinguish between the aspects of the model, that are based on first principles and/or undeniable empirical evidence — and the assumptions and approximations which are required where conclusive evidence still is missing. Then one can only hope that our mathematical and physical assumptions and conclusions are reasonable as long we don't have a comprehensive experimental programme to verify or falsify them. Clearly one would wish to have perfect models, both for climate change and for cell physiology that are sufficiently detailed to guide action in prevention, attenuation, treatment and cure. Whether a perfect model, be it for climate change or cell physiology is ever obtainable, is not clear with view upon the immense complexity of the agents involved, see Fig. 14.

Therefore, it seems us so important in all otherwise well–founded and well–intended approximations and refinements not to loose the sight upon the possible emergence of multiple time scales.

6. Multiple time scales of economy — Short and long waves

Multiscale modeling and simulation is most elaborated in material sciences, in integrated computational engineering of solids, and in the simulation of soft materials, as shown in Section 4. It is also well–developed in modern mathematical physiology with the simultaneous treatment of multiple spatial and temporal scales, as shown in Section 5. In economics, multiscale modeling and simulation date back about 150-200 years.

Economic activities are spread in space and time. Space scales are ranging from the small distances between workplace and home, to the possibly large distances between birthplace and workplace, and between places of production and use or consumption of parts and services. Time scales range from the seconds in financial decisions over working hour, day, week or month, to sowing and harvesting periods; process time for a particular manufactured good; lead time from ordering to delivery; terms for employing workforce, amortization of fixed assets, borrowing, investing, and accounting; and time frames for input-output tables and planning horizons in enterprises or on national level. In principle, all these different time scales can be inserted in a variety of economic multiscale models, serving different goals — better or less successful.

Most of these discrete variables can adequately be approximated by continuous variables. Even so, as in the preceding sections, our question is when and how *different characteristic time scales* emerge, no matter if the models are in discrete or continuous time. On the way to answering that question, we indicate how multiscale *thinking* continues to reveal new economic phenomena and to explore new concepts and innovative computational paradigms. We emphasize *thinking* and *data gathering* contrary to *modeling* and *simulation*: clearly, in the world of real economics one hardly finds convincing non–trivial predictions based on mathematical modeling and numerical simulation as witnessed, e.g., by the disasters of 1928/2008 and at other turning points. As in climate change modeling and simulation, understanding the basic lows, familiarity with the empirical evidence and adapting the mindset to the multiscale character of the very problems may be more important than proliferation and trust in seemingly precise predictions. Therefore, in this section our emphasis is on multiscale data gathering and analysis.

In Section 6.1 we introduce to the search for characteristic times in the development of society, culture and economics, first in Paragraph 6.1.1 by reviewing general concepts of periods in history and then in Paragraph 6.1.2 by recalling the rise of interest for multiscale problems among outstanding economic thinkers of the last 100 years.

In Section 6.2 we address the emergence of multiple time scales in economic cycles and show how neglecting these multiscale aspects must lead astray. In Paragraphs 6.2.1 we explain the suggestive power of short term fluctuations that are easy to detect, to model and to simulate. In Paragraph 6.2.2 we show that the more decisive longterm cyclic behavior is not easily approachable. Alone the detection requires advanced numerical, geometrical and statistical tools like penalized splines, and credible modeling and simulation seem impossible due to the unknown magnitude and influence of background variables.

In Section 6.3 we shall shortly mention basic multiscale spatio–temporal problems.

6.1. Time scales in society. The efforts of integrating microeconomics into the macroeconomics and macrodynamics of capitalism yield many mathematical challenges. For our survey we have chosen the emergence of two different characteristic times in economic cycles. As

- in *climate modeling*, where the small characteristic time of direct effects of radiative forcing is immediate while the giant characteristic time of reversing secondary effects only is accessible by theoretical means,
- in data of *economic growth* only the small-scale fluctuations of business cycles are immediately sensible at a time scale of 6-8 years and mathematically easy to model, estimate and simulate, while the in many aspects more decisive long-term cycles on a time scale of 40-60 years require theoretical means to become visible.

6.1.1. *Cyclic changes in society, economics, and culture.* The US-American physicist–philosopher and semiotics–logician C.S. PEIRCE argued in [**47**, Evolutionary Love, pp. 361–374] for a characteristic length of ca. 800 years for distinguishable phases in the slow evolution of rationality through teaching, disseminating, also geographically, canonizing, disputing, testing, challenging, falsifications, running out of steam and giving way to new mindsets and new cultures. He identified dramatic breaks in Greece around 400 BCE, in Byzantium around 400 CE, the predominance of scholasticism since 1200 CE (and radically new thoughts based on his semiotics and "pragmaticism" emerging around our present time).

The German historian and philosopher of history O. SPENGLER whose interests included mathematics, speculated independently in [**48**] in the same direction with approximately the same times scales as PEIRCE, pointing to shared features in rise, culmination and decline of different epochs in history.

The US-American artists and historians W. STRAUSS and N. HOWE reformulated SPENGLER's approach in biologistical terms in their *Generational Theory* [**49**]. Roughly speaking, they take an average life to be 80 years, and consisting of four periods of $\sim$ 20 years: Childhood, Young adult, Midlife, Elderhood. A generation is an aggregate of people born every $\sim$ 20 years. They describe a four–stage eternally repeating cycle of social or mood eras of approximately 20 years' length which they call *turnings*: The High, The Awakening, The Unraveling and The Crisis. Hence, in that setting each generation experiences four turnings. Etc.

While the preceding approaches to the emergence of multiple time scales in society and culture, inspiring as they may be, are hard to verify in a rigorous manner, there are many more hard data and easier quantifiable multiple time scales in economic theory: more precisely, in the description and analyzes of macro-economic cycles in capitalist economies we can follow the emergence of multiple time scales.

6.1.2. *Economic cycles.* One feature has attracted the interest of many thinkers, namely the ups and downs in economy. It seems that these oscillations are a characteristic feature of *capitalism*, with a mass of people free to sell their work and to buy consumer goods, and classes of property or capital owners free to determine the direction of their land or capital use for making profits. Both freedoms can though be restricted by state regulation and unions and other alliances within parts of the working masses and groups of entrepreneurs.

In history, other economic systems have also had their crises due to weather, demographic factors (diseases, migration, birth surplus) and war. For the misery of crises in capitalism, it was common to blame demographic factors as well, the unrestrained reproduction of the working masses, as the English cleric T.R. Malthus (1766–1834) put it — until the investigations by K. Marx (1818–1883) gave an explanation for the cyclic occurrence of crises in capitalism solely due to the internal logic of its functioning, see, in particular, [**50**, Chapters 22–25].

Since then, a handful of mostly Austrian–British–American giants of economic thinking, clever mathematical modelers and profound empirical analysts, have investigated the macrodynamics of capitalism more closely, finally arriving at a clear view upon the emergence of *two characteristic times* in the continuing evolution of capitalism.

- J.M. Keynes (1883–1946) confirmed and explained in [**51**] the phenomenon of *business cycles* as a *macroeconomic* generalization of the trivial pork and cattle cycles that describe the phenomenon of cyclical fluctuations of supply and prices in livestock markets, see Section 6.2.1.
- J.A. Schumpeter (1883–1950) described in the monumental [**52**] the superposition of waves of different length in the evolution of capitalism. He showed how technological innovations had driven the longterm waves, baptized by him as *Kondratieff waves* in honor of the Soviet economist N. Kondratiev who earlier had observed waves of different length in global capitalist economy. As emphasized in [**53**,

p. 126], "the central point of [Schumpeter's] whole life work [is]: that capitalism can only be understood as an evolutionary process of continuous innovation and 'creative destruction'", see also [**54**]. Not surprising for aficionados of multiscale modeling and multiscale data collection, the view upon the process of creative destruction would be blurred by neglecting the multiscale aspects of real economies and freezing all but a few variables. Worse, illusions would be generated about the possibility to draw policy conclusions from such naive data and simplistic models.

- A.F. Burns (1904–1987), as researcher, as consultant for various US-governments, and as Chairman of the Federal Reserve for eight years, tried to extract concrete advice for economic regulations from Schumpeter's work on economic growth and technological innovation in the aftermath of The Great Depression (1928-1940), and to eliminate Keynes' and Schumpeter's accompanying criticism of capitalism, see [**55**] and the seminal joint work [**56**] with his mentor (open to socialist ideas) W.C. Mitchell (1874–1948).
- R.M. Goodwin (1913–1996), as a mathematician, selected the Great Ratios (see also Table 3):
  - The *employment rate* on the external labor market is defined as $e := L/L^s$, where L denotes the employment and L^s the labor supply. Note that e is positively correlated or for simplicity identified with the rate u of capacity utilization of firms.
  - The *share* of wages in national income is defined as $v := \frac{WL}{pY}$, where W, L, p, Y denote the nominal wages, the employment, the price level, and the aggregate income.

  These secularly trendless magnitudes turned out to be essential for describing, modeling and simulating the economic cycles in a rigorous mathematical and socio-politically meaningful way by combining basic thoughts of Marx, Keynes and Schumpeter. Goodwin announced his approach in [**57**] and elaborated it in a series of contributions, partly reprinted in [**58**].
- M. Friedman (1912–2006) put monetary politics into the frame of long waves, thereby addressing old questions of the labor movement regarding the interrelations between employment, wages and inflation, see [**59**]. It seems, though, as explained in [**60**] "that a Marxian reinterpretation of the baseline Monetarist model of inflation, stagflation, and disinflation may be more to the point from a factual viewpoint than Friedman's initial and later attempts to explain these phenomena against the background of a Walrasian (i.e., neoclassical) understanding of the working of the economy".

Symbol	Description
C	Aggregate planned consumption
I	Aggregate planned investment
K	Capital stock
L, L^{sppl}	Employment, labor supply
W	Nominal wages
Y	Aggregate income (= supply, i.g., with time lag)
c	Marginal propensity to consume
$e := L/L^{\mathrm{sppl}}$	Rate of employment
p	Price level
$s := 1 - c$	Savings rate
u	Rate of capacity utilization of firms
$v := \frac{WL}{pY}$	Share of wages
$\nu := K/Y$	Capital coefficient
C_t, I_t, Y_t etc.	Discrete time values of C, I, Y
Y_0, p_0 etc.	Steady state values (or short–run equilibrium values)

TABLE 3. Standard notations for modeling Keynesian trade waves in discrete time, from [**60**, pp. xiii-xiv].

6.2. The emergence of two different characteristic times in economic cycles.

6.2.1. *Keynes' business cycles.* The elegance of KEYNES' thoughts about short term business cycles, his intimate relations with the empirical and political side of economics[2] and the operationality and transparency of supporting mathematical expressions and arguments may have blocked partially for an early awareness of the economic longterm cycles.

Before addressing the longterm cycles (here called *Schumpeterian*), we shall present a few basic facts about the short-term cycles (here called *Keynesian*).

[2] "Our criticism of the accepted classical theory of economics has consisted not so much in finding logical flaws in its analysis as in pointing out that its tacit assumptions are seldom or never satisfied, with the result that it cannot solve the economic problems of the actual world." [**51**, Chapter 24]

Following the main stream of economic literature, we give a *discrete* time Keynesian (or Hicksian) Trade Cycle Model, as discussed and simplified in [**60**, Section 3.7]. For a mathematically more elegant model in continuous time in the tradition of LOTKA–VOLTERRA environmental models we refer to [**61**].

A. The ground model of C-I-Y–interaction. As always in modeling and simulation, *the first task* is the choice of a few basic variables and of simple approximative relations between them. Using the notations of Table 3, the multiplier–accelerator interaction between consumption, investment and income can be modeled by the following three equations

$$C_t = c\,Y_{t-1}\,, \tag{6.1}$$

$$I_t = \nu\,(Y_{t-1} - Y_{t-2}) \text{ and} \tag{6.2}$$

$$Y_t = C_t + I_t + A\,. \tag{6.3}$$

Hence, we have a consumption (and so savings) function which is lagged by one period. Furthermore, investors are here purely looking backward by using the last observed change in sales instead of the currently expected one for their investment decision. Equation (6.3) finally describes goods–market equilibrium with an additional term A, which is assumed to be positive and stands for *autonomous demand*.

Plugging (6.1) and (6.2) in (6.3) yields a non-homogeneous difference equation of order two with constant coefficients

$$Y_t - (c+\nu)Y_{t-1} + \nu Y_{t-2} - A = 0.$$

In the standardized variable $Z := Y - \frac{1}{s}\,A$, i.e., by subtracting the steady state solution $Y_0 = \frac{1}{s}A$ with $s := 1 - c$ from the variable Y, the preceding equation takes the homogeneous form

$$P(Z) \overset{!}{\equiv} 0 \quad \text{for} \quad P(Z)(t) := Z_t - (c+\nu)Z_{t-1} + \nu Z_{t-2}. \tag{6.4}$$

The textbook solution of (6.4) (see NÖRLUND's classic [**62**, X.5, Paragraph 156] of 1924 for linear difference equations of arbitrary positive order with constant coefficients) is obtained by first noting that the coefficient of the highest order term Z_t is 1 and of the lowest order term Z_{t-2} is not vanishing, hence any solution of (6.4) must be of the form $Z(t) = \lambda^t\,w(t)$ for a suitable $\lambda \in \mathbb{C}$ and a suitable function w. Then $P(Z) \equiv 0$ implies $w \equiv 0$ and λ a root of the characteristic equation $\lambda^2 - (c+\nu)\lambda + \nu = 0$. We focus on the case $c < 2\sqrt{\nu} - \nu$ yielding a pair of complex conjugate $\lambda_{1,2} = |\lambda|(\cos\theta + i\sin\theta)$ and, finally, a real cyclic solution to (6.4)

$$Z(t) = |\lambda|^t\,(\delta\,\cos(\theta t - \varepsilon))\,, \tag{6.5}$$

where ε, δ are given by initial conditions (values that describe the displacement from the equilibrium $Z = 0$ if such a displacement occurs). Let us further assume that $|\lambda| > 1$ to avoid the implosive behavior of the economy, but so accepting explosive behavior, i.e., cyclic fluctuations with explosively increasing amplitudes and constant period of $\tau = 2\pi/\theta$. To sum up, that will occur when $c < 2\sqrt{\nu} - \nu$ and $\nu > 1$ hold true simultaneously, that is for accelerator coefficients that are larger than one and marginal propensities to consume which are sufficiently low.

Note that this solution Z of (6.5) and the corresponding solution Y of (6.2.1) only provide so far an explosive cycle around a stationary level of national income like a forced oscillator with one characteristic time solely, the period τ, totally neglecting the multiscale character of an economy.

B. Introducing exogenous growth — the unrestricted model. For the emergence of a second characteristic time, *the next task* is to allow steady growth in such an approach, e.g., by adding a trend component

$$A_t \;=\; (1+g)^t A, \tag{6.6}$$

that is to assume that autonomous expenditures grow with a constant rate g over time. In [**60**, pp. 83f], Flaschel gives an elegant explanation of how to derive a general cyclic solution of the system given by equations (6.1), (6.2), (6.3) and (6.6) from the solution (6.5), just exploiting the linearity of the relations:

$$Y(t) \;=\; (1+g)^t \frac{(1+g)^2}{(1+g)(s+g) - \nu g} A \;+\; |\lambda_1|^t \left(\delta \cos(\theta t - \varepsilon)\right), \tag{6.7}$$

where θ is determined by $\lambda_1/|\lambda_1| = \cos\theta + i \sin\theta$, (i.e., $\cos\theta = \frac{c+\nu}{2\sqrt{\nu}}$) and where δ and ε are the two initial conditions needed to supply a unique solution for our inhomogeneous difference equation of order 2.

To sum up, (6.7) gives the circular flow of income of the unlimited multiplieraccelerator model of *C*-*I*-*Y*–interaction with exogenous growth. A possible result of this goodsmarket interaction is provided in Fig. 15a. This figure shows the case of a cyclically explosive multiplier-accelerator interaction, which comes about for accelerator coefficients $\nu > 1$ and marginal propensities to consume c that are sufficiently low.

C. Damping the oscillations in the restricted model — still futile? As Figure 15a immediately shows, there is a *third task*, namely adding extra forces to this model to keep its dynamics within reasonable bounds. The simplest way of doing this is to add ceilings and floors like in modeling the damped elastic spring. This will give rise to a damped type of behavior, where the ceiling, the floor, or both delimiters can be operative (Fig. 15b).

Flaschel has the following disenchanted comments in [**60**, p. 85]: "These simulations in particular show ... that ceilings (and floors) are only very briefly

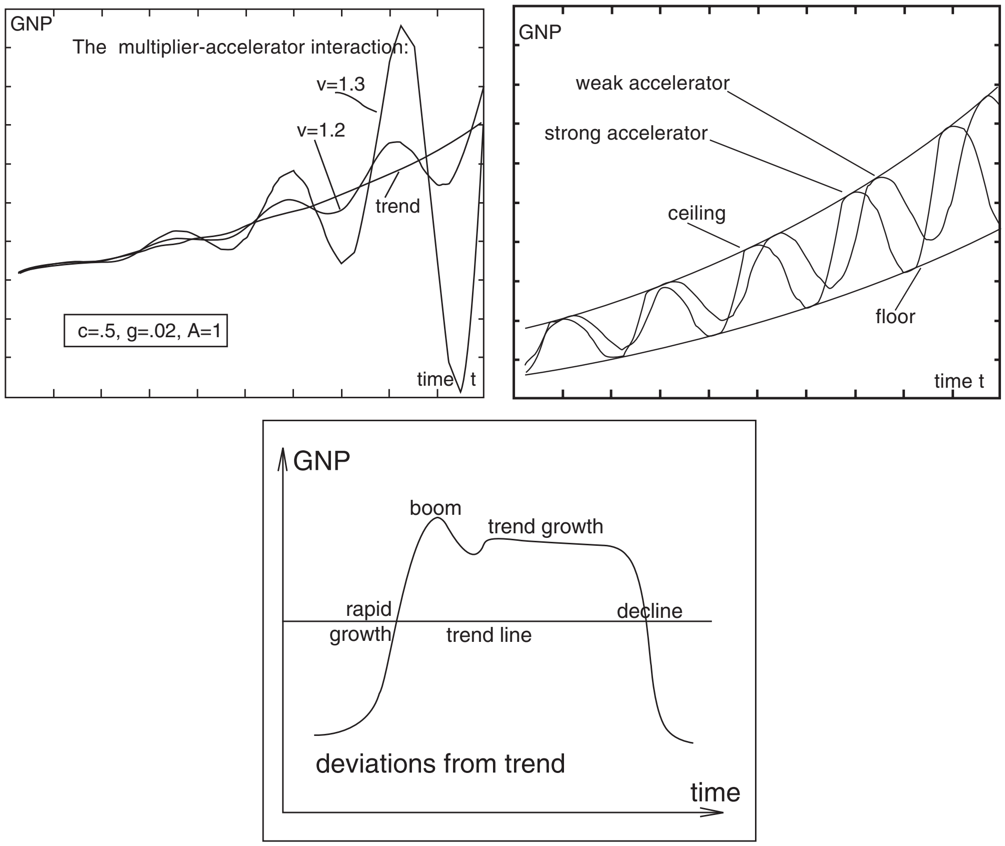

Figure 15. a) The dynamics of the solution (6.7) of the unrestricted multiplier–accelerator model for typical parameters. b) The dynamics of the damped solutions of the restricted multiplier–accelerator model. c) The stylized business cycle of the 1950s and 1960s. All figures from [**60**, pp. 85f].

operative during each cycle... The cycle ... therefore bears no close resemblance to the form of the business cycle that was (and to some extend still is) believed to be typical (at least) for the fifties and the sixties of (the 20th) century," referring to our Fig. 15c.

Flaschel points to several other shortcomings of such simple modeling and advocates the incorporation of prices and inflation to gain realism. Later on, in the same monograph (p. 275) he criticizes the choice of certain assumptions in mainstream economic modeling as, axiomatically seen, to be wrong "so that complicated additional constructions (epicycles) become necessary to reconcile this approach with the facts". He cites consentingly a question once raised by J. Fuhrer, the Executive Vice President & Senior Policy Advisor of the Federal Reserve Bank of Boston regarding monetary models:

Are we adding "epicycles" to a dead model?

We shall not judge, but rather immediately proceed to the *empirical* side. We will see clearly that short-term and long-term fluctuations appear simultaneously in the *data*, no matter how ambivalent or even unreliable and implausible the *mathematical modeling* and the *numerical simulations* may appear.

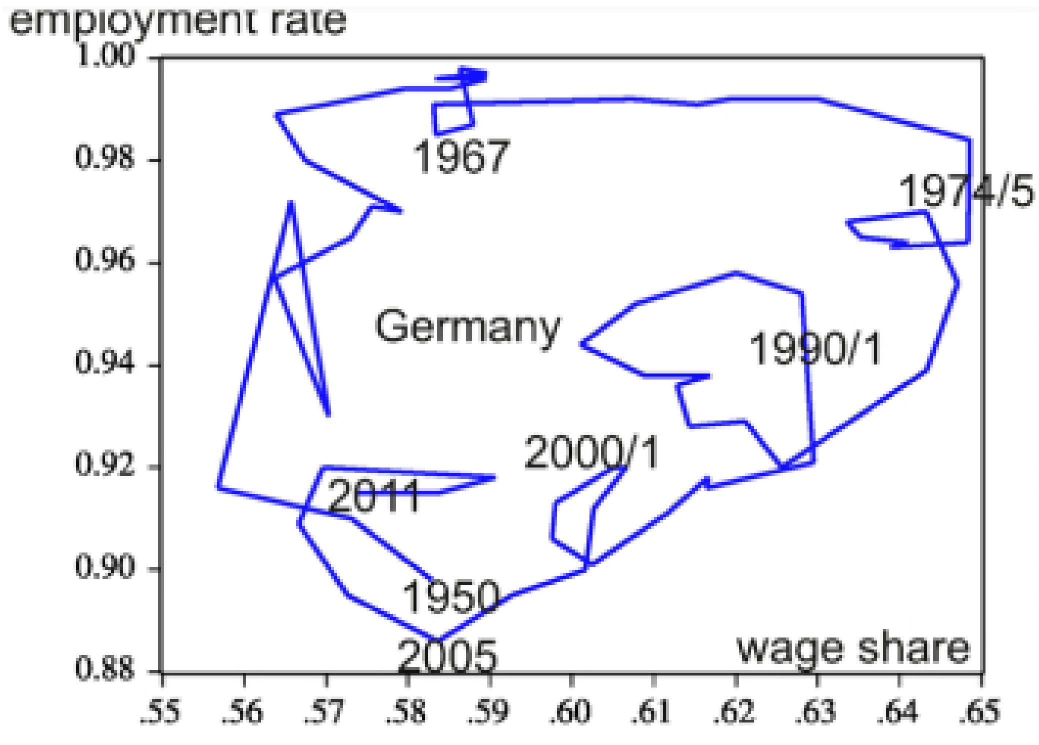

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FIGURE 16. Exploring the postwar economy 1950-2011 in Germany with bivariate loops using penalized spline regression. The main loop is the ca. 60-years long SCHUMPETER–wave with six ca. eight-years KEYNES–business cycles embedded. Reprinted by courtesy of P. FLASCHEL, Bielefeld, based on data provided by C. GROTH, Copenhagen

6.2.2. *Schumpeter–cycles.* As emphasized before, long waves are not always visible in economic data:

Short–phase waves: Production data like GDP will mostly disclose only a longterm growth trend with *short-term oscillations* of the type of KEYNES' business cycles.

Long–phase waves: Happily, the coordinate change to GOODWIN's great ratios $e := L/L^{\mathrm{sppl}}, v := \frac{WL}{pV}$ (see Tbl. 3) shows the evolution of the employment rate of workers in the sphere of production $e(t)$ and its consequences for the income distribution between capital and labor $v(t)$ over *longer periods*.

To explore the emergence of two characteristic time scales in these two variables, we refer to [**63**]: they assume that $y(t) := (e(t), v(t))$ follows a *long term trend* $c(t) = (c_1(t), c_2(t))$ and a *business cycle fluctuation* $g(t) = (g_1(t), g_2(t))$, and model $y(t) = c(t) + g(t) + \varepsilon(t)$ with an error term $\varepsilon(t)$. In that way, the data can point to two essential aspects of economic crises in capitalism that otherwise are nowadays disguised behind GDP data, namely

(1) the periodic *creative destruction* of the means and processes of agricultural and industrial production and of rendering services, and
(2) the damping potential of democratic regulation.

It seems that these two aspects (or *mechanisms*) are operative both in short–phase and long–phase waves.

D. The unregulated economy at MARX' time. Roughly speaking, the unregulated capitalism at MARX' time led to deep economic crises every 6–10 years. There was no theoretical need to distinguish long waves and short waves: the short waves were sufficiently brutal not only for the working population but also for the land and capital owners to provide some creative destruction in an apparent manner.

E. The continuously regulated economy under strong democratic influences. That changed with the foundation and growing power of trade unions of working people that gave

Long–phase waves: much larger time span for the essential ups and downs, pictured in the long cycles,

Short–phase waves: while it supported some crisis management to get over the short cycles.

The paramount example is provided by the postwar economy 1950-2011 in Germany (first Western Germany and West Berlin, then, since 1990, the unified Germany). After the break-up of the biggest German capital concentrations by the victory powers of World War II, some substantial rights were granted to the working people and there was an agreement between the major political parties to respect these rights and to hinder excessive influence of the capital side. Moreover, in collective negotiations between unions and management, there were often an invisible third social partner present, namely the social rights granted in the Eastern part of Germany.

Our Fig. 16 was found by exploring the German data with *bivariate loops* using *penalized spline regression*, as explained in [**63**]. It is based on data provided by C. GROTH and J.B. MADSEN, Copenhagen, see also [**64**]. Of course, the raw data for such a long period give a cloud which can be difficult to interpret in the presence of multiple time scales, as emphasized above in our Introduction in Section 1.1. Following the mentioned stochastic geometry approach, one arrives at a *wage share* $\times$ *employment rate* diagram showing a main loop of the ca. 60-years long–wave of the KONDRATIEV–SCHUMPETER–GOODWIN–FRIEDMAN (you name it) clockwise oriented cycle with six ca. eight-years KEYNES–HICKSian also clockwise oriented business cycles embedded.

F. An economy with rapid shifts between regulation and de–regulation. In [**63**], KAUERMANN, TEUBER and FLASCHEL found that the corresponding time series of the US economy for the period 1955–2005 yield a different Fig. 17: The long-phase wave in (c) seems to dipsy-doodle and lacks any remembrance of a cycle, presumably due to the state interventions of price and wage-stops of the NIXON administration and the abolition of union influence by the REAGAN administration. Probably for the same reason, the KEYNES business cycles in (d) show strong erratic fluctuations. Of course, one could force the data numerically to yield a curve resembling a cycle, see Fig. 18. Globally, the cycle form is misleading for the US economy when compared with the statistically more reliable Fig. 17.c. However, we can read the swivels of the US economy easier in the numerically forced plot.

We must conclude, that the fluctuations of a real economy reveal the emergence of at least two different and each important time scales, as seen above in material sciences and cell physiology and claimed to be decisive in understanding climate change. Like always in multiscale modeling, there can be strong arguments for focusing on short term oscillations and what to do about them, as well as on a secular trend and what to do about it. Focus on claims from one side are stifling discussion and can lead astray. We borrow the wording from ANDREW SMITHERS, the London based founder of Smithers & Co., which provides economics-based asset allocation advice to over 100 fund management companies. In a letter to Financial Times [**65**], he commented on raising concerns about an evolving secular stagnation (e.g., in [**66, 67, 68**]) and warned: "claims of secular stagnation stifle serious discussion and thus inhibit the chances of improved policies". Note that the concerns regarding climate change expressed above in Sections 2 and 3 are mostly about neglecting the secondary and, to be feared, secular effects of greenhouse gas emissions in the scientific literature and public perception due to – understandable – focus on the Paris accord and other short range estimates of immediate consequences of radiative forcing for the next few decades. SMITHERS' message is that regarding the global economy the challenges of dealing with emerging multiple time scales may be reversed.

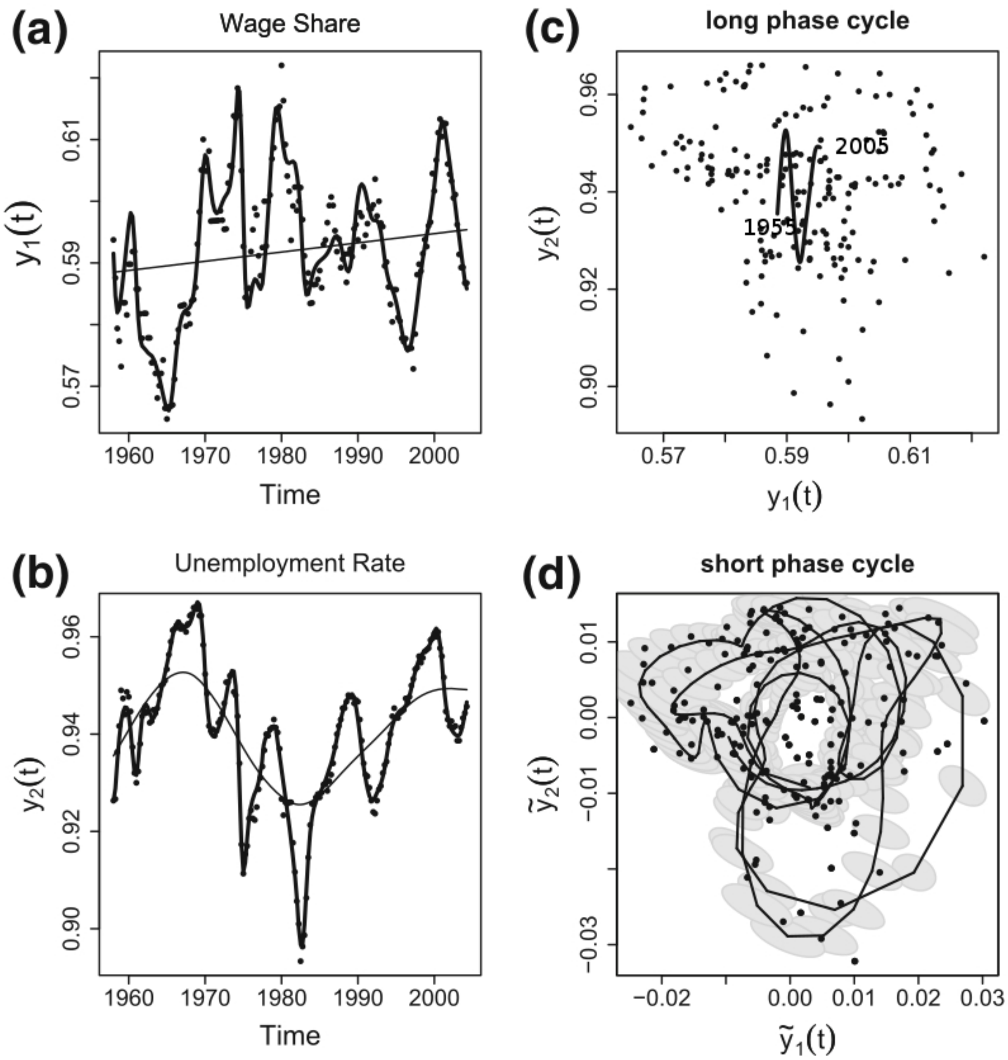

FIGURE 17. Exploring the US economy 1955-2005 with bivariate loops using penalized spline regression. Plot **a** and **b** show the time series wage share $y_1(t)$ and employment rate $y_2(t)$, with long–phase estimate (solid line) and final estimation (bold solid line). Plot **c** shows the observation cloud with the estimated zigzag line instead of an expected long–phase cycle and **d** shows the de-trended time series with estimated short–phase business cycle (solid line) and its confidence region (grey shaded area). Reprinted from [**63**, Fig. 4]

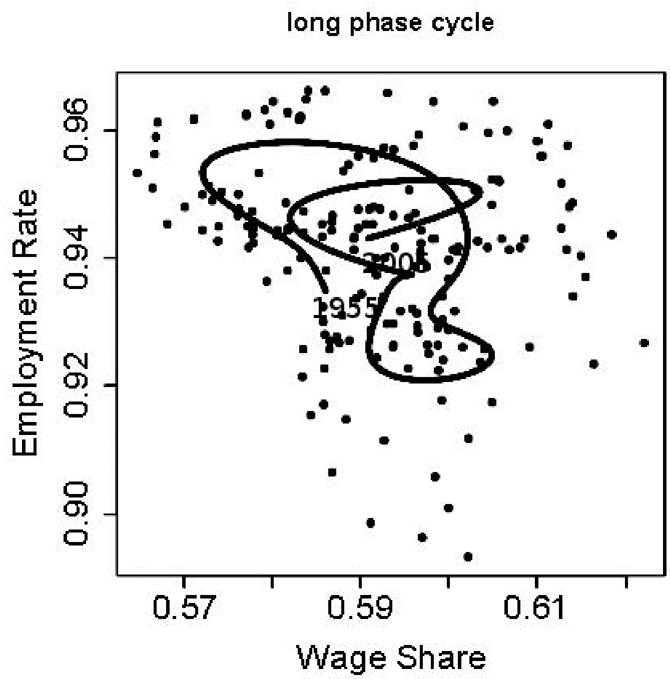

FIGURE 18. The plot shows the observations with a numerically forced long phase cycle (solid line). Courtesy by P. FLASCHEL

6.3. Multiscale spatio–temporal problems in economics. At the end of this section we turn to the intricacies of multiple time scales in location modeling with its own multiscale challenges.

Multiscale spatial economy is the subject of mathematical location theory. It had its golden years in the middle of the 19th century during the railroad revolution of transport, and was revived by the analysis of trade patterns and location of economic activity during the ongoing globalization of the economy:

> "Patterns of trade and location have always been key issues in the economic debate. What are the effects of free trade and globalization? What are the driving forces behind worldwide urbanization?" (From the citation of awarding *The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel* 2008 to P. KRUGMAN)

A multi–disciplinary example of high numerical complexity is to uncover the spatio–temporal patterns and dynamics of urban waste generation and management in a metropolitan area like Shanghai by a high spatial resolution material stock and flow analysis, see [**69**].

In the context of this survey, the essential point of the multiscale problems in location theory can be formulated via the emergence of multiple time scales: In the epoch of the building of railroads, e.g., the characteristic length for economic activity was approximately $s = 50$ km and the typical velocity $v = 50$ km/h, yielding a characteristic time $t_{\text{trade}} = s/v$ of one hour, while the characteristic time t_{build} for building a new connection typically was larger by a factor 1.7×10^5, namely around two years. In nucleus, that is the way multiscale problems arouse in location theory.

For the history and the technical details of multiscale modeling of the spatial allocation of resources we refer to [**70**], see also [**71, 72**].

7. Discussion and conclusions

It is well-known that multiscale problems present challenges not only to our modeling and simulation skill set but also to our communication skills. In [**6**, Preface], M. Horstemeyer tells the following illuminating story with regard to Integrated Computational Materials Engineering (ICME) and the use of multiscale modeling in engineering design:

> While working at Sandia National Laboratories (laboratories with the mission to maintain the reliability and surety of US American nuclear weapon systems, *added by the authors*) in the mid-1990s, there was a meeting of an engineering mechanic, physicist, and materials scientist, and they were talking about stress. At the end of the meeting, they had all agreed that they understood each other's position. After the meeting, I interviewed each person separate from the others and asked what he or she thought about when the stress discussion came about. The physicist talked about pressure, pressure, pressure. The materials scientist talked about strain, strain, strain. And the engineering mechanics researcher talked about second-rank tensor, second-rank tensor, second-rank tensor. They had thought that they communicated, but they really did not because the paradigm of each one's discipline skewed his or her semantical communication. This is often the case for interdisciplinary researchers, so one has to be careful when discussing multiscale modeling or history modeling from process to performance using the ICME tools with others who were trained under a different paradigm.
>
> Because of these different paradigms, I decided shortly after those interviews to perform simulations at all the different

> length scales and to try to understand the pertinent cause-effect relationships with the hope that I could understand the bridging concepts...

When *experts* with different background have difficulties catching each other's characteristic spatial or time scales, it seems appropriate to close our survey with a few observations on challenges of communicating multiple time scales to a *wider audience*.

7.1. Communicating the emergence of multiple time scales.

7.1.1. *The public disregard.* Regarding multi-scale problems we register a wide gap between

(A) the mathematical proficiency in modeling and simulation of multiscale systems, and
(B) the public disregard even of the most elementary multiscale aspects, like the emergence of multiple time scales.

Claim (A) is evident from

- the enormous literature, both
  - learned journals,
  - specialized textbooks as [**3, 4, 5, 6, 7, 73**] and 120 other textbooks and scientific monographs with the entry *multiscale* in the title, according to zbMATH `https://zbmath.org/?q=ti%3Amultiscale+%26+dt%3Ab&p=1`, and
  - a variety of survey articles like [**27, 74**];
- the new paradigm of modeling (often required by the intricacies of the numerical simulation of multiscale problems), namely doing the mathematical modeling, i.e., the choice of the relevant equations on the different scales, in parallel with designing the numerical algorithms;[3]
- the multiscale modeling and simulation for design verification and validation purposes of nuclear weapons that permitted the *Treaty on*

[3] "...in much of computational mathematics, we are used to taking for granted that the models are given, they are the ultimate truth, and our task is to provide methods to analyze and solve them. This shields us from the frontiers of science where phenomena are analyzed and models are formulated.

Multiscale, multiphysics modeling brings in a new paradigm. Here the problems are given, and a variety of mathematical models at different levels of detail can be considered. The right equation is selected during the process of computation according to the accuracy needs. This brings mathematical analysis and computation closer to the actual scientific and engineering problems. It may no longer be necessary to wait for scientists to develop simplified equations before computational modeling can be done. This is an exciting new opportunity for computational science and for applied mathematics. It will bring applied mathematics closer to other fields of mathematics, as, for example, mathematical physics and probability theory. It will also bring these fields closer to the frontiers of science." E and Engquist in [**27**, Outlook, p. 1069] *the Limitation of Underground Nuclear Weapon Tests*, also known as the *Threshold Test Ban Treaty* (TTBT) to enter into force in 1990, see what MARK HORSTEMEYER recalls in [**6**, Section 1.3, pp. 4f] about that side of the history and reliability of multiscale modeling and simulation.

Claim (B) is evident from a large study [**75**] of 2016. Assessing student perceptions and comprehension of climate change in Portuguese higher education institutions and surveying studies from other countries, the authors found

(1) a lack of interrelation between the common attention to Climate Change (CC) in general terms and the personal or political attitudes of the respondents;
(2) a different, but mostly low level of physical understanding;
(3) absence of any feeling for dynamics and characteristic time scales, e.g., when the respondents were mixing the presence, the near future, and further developments.

So much for the academic youth. For quite another cohort, investors on the financial markets, we can derive a similar disregard of the multiple time scales of climate change and transforming for sustainability. In [**76**], KATE ALLEN, a capital markets correspondent for the *Financial Times* reports:

> Sales of green bonds are stuttering after several years of rapid growth. In the three months to the end of September 2018, issuers around the world sold \$31.6bn of green-labelled debt, according to research by credit rating agency Moody's. That is 30 per cent lower than the tally for the second quarter, and 18 per cent down on the \$38.5bn sold in the same quarter of 2017.

Moody's had originally forecast that green bond sales in 2018 would hit \$250bn, a considerable increase from last year's record \$163bn.

In the annual *BP Statistical Review of World Energy*, S. DALE, British Petrol's chief economist gives a related picture of the growth of world coal consumption in 2017 by 1% after annual decrease since 2013, a corresponding increase by 1.5% of CO_2 emissions from energy consumption, and the lack of almost any improvement in the power sector fuel mix over the past 20 years. "The share of coal in the power sector in 1998 was 38% — exactly the same as in 2017", states DALE in [**77**]; and the price per ton of CO_2 is on the order of 1\$, while it should be 25\$ per ton of CO_2 *now* to keep the temperature increase below $2\frac{1}{2}$°C in 2050 according to [**78**, p. 316].

In [**79**], chief economics commentator at the Financial Times, M. WOLF, searched for an explanation of that "shameful" behavior of investors. Not surprisingly, he too found a disregard of the multiple time scales at the heart of the problem:

> In all, we need to shift the world on to a different investment and growth path right now. This is more technically possible than we used to think. But it is politically highly challenging. Above all, climate change involves huge distributional issues between rich countries and poor ones, between countries that caused the problem and those that did not, between countries that matter for the solution and those that do not and, *not least, between people today, who make the decisions, and people tomorrow, who suffer the results* (emphasized by the authors). The natural tendencies are either to do nothing, while insisting there is no problem, or to agree there is a problem, while merely pretending to act. It is not clear which form of obfuscation is worse.[4]

7.1.2. *Can scientists reach the public?* To communicate the emergence of multiple time scales, we may draw on experts in science communication. The British study [**84**] distinguishes between three phases associated with the development of science communication (with somewhat unlucky acronyms):

- **SL:** scientific literacy;
- **PUS:** public understanding of science; and,
- **PEST:** public engagement with science and technology (the current challenge).

Similarly the recent [**85**, Introduction, p. 1-4].

Another study [**86**, pp. 106f] of 2002 recalls the continuous flow of firm recommendations for public consultation to become an integral part of doing science – not an optional add-on. They comment in rather sharp wording: "This may seem a bit heretical in lands where science policy is still in the hands of the science mafia, and the game is how to limit and exclude rather than to engage, listen and learn. But there is more than a grain of commonsense in it." They quote from the British Council's useful report on the democratization of science a list of essential preconditions, among them: *independent advice and research*; and *initiatives to forecast, recognize and resolve conflict*. Surely, that are valuable guidelines for communicating the emergence of multiple time scales in climate change and transforming for sustainability, combined with the advice of [**87**]: There is an obligation for scientists to communicate their research to the rest of society, to inform people about scientific advances, and

[4]This is not the place to discuss whether this is a clash between economic rationality (more precisely, capitalists' profit orientation) and environmental, science based arguments, as proponents of the Climate Justice Movement may argue, see, e.g., [**80, 81, 82**] — or just, as, e.g., M. Porter, a Harvard authority on competitive strategy, or W. Nordhaus, the 2017 Nobel laureate in economic theory, may claim with Wolf, a common disregard of shared values in multiple time scales, see [**83**] and [**78**, Ch. 26, "Prisoners of the present"].

to ultimately engage them in a two-way dialog so that the general public does not just understand what science is doing, but that they also have a say in what is being done.

7.1.3. *Urgent tasks for the IPCC.* Clearly, one has to be grateful to the IPCC that it opts in The IPCC Special Report on 1.5°C [**88**] for a rigorous interpretation of the 1.5°C limit on global warming. We deplore though, that the report solely describes the means to reach that goal and the necessary adaptations but underplays the alarming fact that global warming is accelerating and that there are no signs for a decrease in global carbon emissions (see [**89**]). Based on our insight into the emergence of multiple time scales we fully agree with the comment [**90**] in *Nature* that

> "Policymakers should ask the IPCC for another special report, this time on the rates of climate change over the next 25 years... Researchers should improve climate models to describe the next 25 years in more detail, including the latest data on the state of the oceans and atmosphere, as well as natural cycles. They should do more to quantify the odds and impacts of extreme events. The evidence will be hard to muster, but it will be more useful in assessing real climate dangers and responses."

7.2. Possible conclusions for scientific, communicative and political challenges. Transforming for sustainability and mitigating climate change hazards depends decisively on the broad support of an informed public. In our contribution, we point to the multiple time scales in climate change and sustainable development. Greenhouse gases accumulate rapidly in the atmosphere with immediate changes in the radiation pattern while secondary effects develop slowly like the release of methane from the oceans and perm freezing areas and, in the opposite direction, the binding and storage of CO_2 in the oceans. Such huge differences between characteristic time lengths provide not only difficulties in mathematical modeling, statistical sampling, and numerical simulation but can become misleading in communicating threats and solutions. Disregard of multiple time scales of a problem can either induce overestimation in the short run and underestimation in the end, i.e., the ominous cry wolf effect, and/or underestimation in the short run and overestimation in the end, resulting in fatalistic forfeiting or preposterous activism.

In multiscale modeling and simulation, the general wisdom (quoted occasionally by W. E) is

> "For most of the problems we are facing in science and engineering, the theoretical challenges lie in mathematics and algorithms."

Regarding the emergence of multiple time scales in climate change, an even better advice might be to recall what we have learned as children, namely to take care of our resources and not to throw our garbage in the environment, visible or invisible, — and to follow that teaching rigorously.

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