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Continuity of families of Calderón projections

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Abstract We consider a continuous family of linear elliptic differential operators over a smooth compact manifold with boundary. Using only elementary means (basic manipulations of closed subspaces in Banach spaces and the interpolation theorem for operators in Sobolev spaces), we prove that the images of the corresponding Calderón projections over the boundary form a continuous family in the relevant Sobolev spaces.

Keywords Calderón projection · Cauchy data spaces · Elliptic differential operators · Green's formula · Interpolation theorem · Manifolds with boundary · Parameter dependence

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1 Introduction

After

- stating our main result about the continuity of the Cauchy data spaces for varying coefficients of the underlying elliptic differential operator;
- we introduce our concepts and fix the somewhat intricate notations: mostly following FREY's beautiful PhD-thesis [21] for one single elliptic differential operator on a manifold with boundary,
  - we recall basic definitions and main results with emphasis on extensions to Sobolev spaces, domains, kernels, images, trace maps, regularity in Sobolev spaces, Cauchy data spaces,
  - give our version of the Frey-construction of the Calderón projector without invertible double and without unique continuation property (ucp) assumptions, and
  - illustrate our/FREY's constructions with some classical examples where the parameter dependence of the Calderón projector and the Cauchy data spaces can be determined explicitly;
- we summarize some elementary, but not widely known results both about curves of closed subspaces in Banach space, inspired by NEUBAUER's classical approach [29], and about the interpolation property of spaces and operators, closely following TARTAR's textbook [39, Lecture 21]; and
- we explain the place of our continuity result in the literature.

1.1 Our main result

Consider a family of linear elliptic differential operators of order $d \in \mathbb{N}$ with smooth coefficients

$$\big(P(b)\colon C^\infty(M;E) \longrightarrow C^\infty(M;F)\big)_{b\in B}\,,$$

where each operator acts on sections of a smooth Hermitian finite-dimensional vector bundle $E \to M$ to sections of a smooth Hermitian finite-dimensional vector bundle $F \to M$ over a smooth compact Riemannian manifold (M, g) of dimension $n \in \mathbb{N}$ with boundary $\varSigma$, with the parameter b running in a topological space B. For simplicity, we keep M, E and F fixed with its Riemannian and Hermitian metrics. We denote the restrictions of the bundles E, F to the boundary $\varSigma$ by E', F'.

We assume that the operator family is continuously varying in the parameter b. By that we mean that for all $s \geq 0$ the family of the extensions of $P(b)$ to bounded operators $P_s(b)$, $b \in B$, all on the Sobolev space $H^{d+s}(M;E)$

$$\big(P_s(b)\colon H^{d+s}(M;E) \longrightarrow H^s(M;F)\big)_{b\in B}$$

is continuously varying in b in the respective operator norms, i.e., we assume that for all $s \geq 0$ the mapping

$$P_s \colon B \longrightarrow \mathcal{B}(H^{d+s}(M;E), H^s(M;F)), \text{ given by } b \mapsto P_s(b)$$

is continuous, where $\mathcal{B}(X,Y)$ denotes the Banach space of bounded operators from a Banach space X to a Banach space Y, equipped with the operator norm.

As usual, we set

$$H^d_0(M;E) \;:=\; \overline{C^\infty_c(M \setminus \Sigma)}^{\|\cdot\|_{H^d(M)}} \tag{1}$$

and denote by $P_{\min}(b)\colon H^d_0(M;E) \to L^2(M;F)$ the extension of the operator $P(b)$ to the space $H^d_0(M;E)$. Correspondingly, we denote the extension of the *formal adjoint* operator $P^t(b)\colon C^\infty(M;F) \to C^\infty(M;E)$ of $P(b)$ (see, e.g., [24, Equation 4.3] in coordinates) to the space $H^d_0(M;F)$ by $P^t_{\min}(b)\colon H^d_0(M;F) \to L^2(M;E)$.

Theorem 1 (Main result) *Assume*

$$\dim\ker P^t_{\min}(b) = \text{constant} \;\text{ and }\; \dim\ker P_{\min}(b) = \text{constant}\,. \tag{2}$$

Then the family of the images $\big(\mathrm{im}\,(C_s(b))\big)_{b\in B}$ *of the Calderón projections*

$$C_s(b)\colon H^s(\Sigma;E'^d) \;\longrightarrow\; H^s(\Sigma;E'^d),\; b \in B,$$

makes a continuous family of closed subspaces for all $s \geq -\frac{d}{2}$.

Remark 1 Below in Section 1.3, Definition 1 and Proposition 4, we shall give a rigorous definition of the *Calderón projector* suitable for investigating the parameter dependence. Moreover, in Equation (21) we shall define the *Cauchy data space* $\Lambda_0(P(b)) \subset H^{-\frac{d}{2}}(\Sigma;E'^d)$ as the space of the homogenized Cauchy traces of the weak solutions u of $P(b)u = 0$ and introduce in Corollary 2, Equation (40) the Cauchy data spaces

$$\Lambda_{d+s}(P(b)) \;\subset\; H^{\frac{d}{2}+s}(\Sigma;E'^d) \quad \text{for } s \in [-d, \tfrac{1}{2}-d) \;\cup\; (-\tfrac{1}{2},\infty)$$

as the homogenized Cauchy traces of solutions u of $Pu = 0$ with u belonging to the intersection of the Sobolev space $H^{d+s}(M;E)$ and the domain of the maximal extensions.

According to Corollary 2, these Cauchy data spaces are precisely the images of the corresponding Calderón projections. Hence one can read Theorem 1 as the claim of a continuous variation of the Cauchy data spaces depending on the parameter b for each of the specified Sobolev orders s.

1.2 Trace maps on Sobolev spaces and natural extensions of elliptic differential operators over smooth compact manifolds with boundary

Without restricting the general validity of our results, we assume that

- our compact Riemannian manifold (M, g) *with* boundary is embedded in a compact smooth Riemannian manifold $(\tilde{M}, \tilde{g})$ of the same dimension n *without* boundary,
- our bundles E, F are extended to smooth Hermitian vector bundles $\widetilde{E}, \widetilde{F}$ over $\tilde{M}$, and
- for each $b \in B$, the operator $P(b)$ is extended to an elliptic differential operator $\widetilde{P}(b)$ over $\tilde{M}$ of the same order d from sections of $\widetilde{E}$ to sections of $\widetilde{F}$.

Sobolev spaces over $\tilde{M}$ and M. For our smooth manifold $M \subset \tilde{M}$ with boundary $\partial M =: \Sigma$, we set $M^0 := M \setminus \partial M$, and denote the space of sections with compact support in the interior by $C_c^\infty(M^0; \widetilde{E})$.

On $\tilde{M}$ with Riemannian metric $\tilde{g}$, we define the *Hodge-Laplace operator*

$$\Delta_0^{\tilde{M}} := \mathrm{d}^t\, \mathrm{d} \colon C_c^\infty(\tilde{M}; \mathbb{C}_{\tilde{M}}) \longrightarrow C_c^\infty(\tilde{M}; \mathbb{C}_{\tilde{M}}),$$

acting on functions (i.e., sections of the trivial bundle $\mathbb{C}_{\tilde{M}}$), where d^t denotes the formal adjoint of the exterior derivative

$$\mathrm{d} \colon C^\infty(\tilde{M}; \mathbb{C}_{\tilde{M}}) \longrightarrow C^\infty(\tilde{M}; \Lambda^1(\tilde{M})).$$

The operator $-\Delta_0^{\tilde{M}}$ is equal to the *Laplace-Beltrami operator* on the Riemann manifold $(\tilde{M}, \tilde{g})$. The *Friedrichs extension* $\Delta^{\tilde{M}}$ of the symmetric operator $\Delta_0^{\tilde{M}}$ is a positive self-adjoint operator on the Hilbert space $L^2(\tilde{M}; \mathbb{C}_{\tilde{M}})$ with dense domain, see the careful construction in [24, Theorem 12.24]. It gives rise to the *Sobolev spaces* on $\tilde{M}$

$$H^s(\tilde{M}) \;:=\; \mathscr{D}((\Delta^{\tilde{M}} + 1)^{s/2}), \quad s \geq 0, \tag{3}$$

where the right side denotes the domain of the fractional power of the elliptic operator $\Delta^{\tilde{M}} + 1$, densely defined in $L^2(\tilde{M}; \mathbb{C}_{\tilde{M}})$ as a closed operator and equipping the space with the graph norm. Since over closed manifolds there is no difference between minimal and maximal realizations, the domain is uniquely determined. For $\tilde{M} \subset \mathbb{R}^n$ open and $s \in \mathbb{N} \cup \{0\}$ we regain the usual Hilbert space

$$H^s(\tilde{M}) \;=\; \{u \in L^2(\tilde{M}) \mid D^\alpha u \in L^2(\tilde{M}) \text{ for } |\alpha| \leq s\}, \tag{4}$$

where the partial differentiation D^α with multiindex α is applied in the distribution sense and the scalar product and norm defined by

$$(u, v)_s \;:=\; \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_{L^2(M)} \quad \text{and} \quad \|u\|_s \;:=\; \sqrt{(u, u)_s}\,.$$

The corresponding Sobolev space on the compact submanifold M with boundary Σ may be defined as the quotient

$$H^s(M) \ := \ H^s(\tilde{M})/\left\{u \in H^s(\tilde{M}) \middle| \ u|_M = 0\right\}. \tag{5}$$

An important subspace is the function space

$$H_0^s(M;\mathbb{C}_M) := \overline{C_c^\infty(M^0)}^{\|\cdot\|_{H^s(M)}}.$$

For $s > 0$, we define the space $H^{-s}(\tilde{M})$ of distributions to be the so-called L^2-*dual* of $H^s(\tilde{M})$, i.e.,

$$H^{-s}(\tilde{M}) = \{u \in \mathscr{D}'(\tilde{M}) \big| \ \langle v, u\rangle \le \text{constant}\, \|v\|_{H^s(\tilde{M})}\}.$$

The above constructions can be generalized for sections of any bundle $\widetilde{E} \to \tilde{M}$ carrying an Hermitian structure $p \mapsto \langle .,.\rangle|_{\widetilde{E}_p}$ and an Hermitian connection. Let

$$\nabla^{\widetilde{E}} \colon C^\infty(\tilde{M};\widetilde{E}) \longrightarrow C^\infty(\tilde{M}; T^*\tilde{M} \otimes \widetilde{E}) \ \text{ and} \tag{6}$$

$$\nabla^{\widetilde{F}} \colon C^\infty(\tilde{M};\widetilde{F}) \longrightarrow C^\infty(\tilde{M}; T^*\tilde{M} \otimes \widetilde{F}) \tag{7}$$

be Hermitian connections, i.e., connections that are compatible with the Hermitian metrics on $\widetilde{E}$ and $\widetilde{F}$ respectively. To define Sobolev spaces of sections in vector bundles, one replaces the Laplacian $\mathrm{d}^t\,\mathrm{d}$ in the previous definition (3) by the *Bochner-Laplacian* $\nabla^t\nabla$.

Natural domains of elliptic differential operators over compact manifolds with boundary. We write shorthand $E' := \widetilde{E}|_\Sigma = E|_\Sigma$ and $F' := \widetilde{F}|_\Sigma = F|_\Sigma$. We write $(\cdot,\cdot)_{\text{<space>}}$ for the L^2 inner products in the various spaces, and $d_g(\cdot,\cdot)$ for the arc length of a minimizing geodesic which yields locally a *distance function*, no matter whether $\tilde{M}$ and Σ are connected or not.

In a tubular neighbourhood of the boundary, say V, the function

$$V \ni p \mapsto x_1(p) \ := \ \begin{cases} d_g(p,\Sigma), & \text{if } p \in M,\\ -d_g(p,\Sigma), & \text{otherwise,}\end{cases}$$

is smooth and defines the *inward unit co-normal field* $\nu \in C^\infty(V; S(V))$ by

$$\nu \ := \ \mathrm{d}\,x_1 \ = \ \operatorname{grad} x_1 \colon V \to S(V), \tag{8}$$

where $S(V)$ denotes the unit sphere bundle in the cotangent vector bundle T^*V, see below. By Riemannian duality, we obtain the *inward unit normal tangential field* (the normalized tangent vectors of the minimizing geodesics), which we denote by ν' or $\frac{\partial}{\partial\nu}$.

We postpone the dependence of our elliptic differential operators of the parameter $b \in B$ and fix one operator $P := P(b)$ over the compact manifold M with boundary Σ. As emphasized before, we can assume that we are given its continuation $\widetilde{P}$ as an elliptic differential operator over the closed manifold $\tilde{M}$.

Recall that $\widetilde{P}$ can be expressed locally by a matrix of partial derivatives. Let $T^*\tilde{M}$ denote the *cotangent vector bundle* of $\tilde{M}$, $S(\tilde{M})$ the *unit sphere bundle* in $T^*\tilde{M}$ (relative to the Riemannian metric $\tilde{g}$), and $\pi : S(\tilde{M}) \to \tilde{M}$ the projection. Then associated with $\widetilde{P}$ there is a vector bundle homomorphism

$$\sigma(\widetilde{P})\colon \pi^*\widetilde{E} \to \pi^*\widetilde{F},$$

which is called the *principal symbol* of $\widetilde{P}$. In terms of local coordinates $\sigma(\widetilde{P})$ is obtained from $\widetilde{P}$ by replacing $\partial/\partial x_j$ by $i\xi_j$ in the highest order terms of $\widetilde{P}$ (here ξ_j is the jth coordinate in the cotangent bundle). $\widetilde{P}$ *elliptic* means that $\sigma(\widetilde{P})$ is an isomorphism.

We continue with a list of natural domains for P.

- $P_0 : C_c^\infty(M^0; E) \to C_c^\infty(M^0; F)$.
- $P_0^t : C_c^\infty(M^0; F) \to C_c^\infty(M^0; E)$ where P^t denotes the formally adjoint of P. Note that P^t is again elliptic.
- $P_{\min} := \overline{P_0}$, $P^t_{\min} := \overline{P^t_0}$.
- $P_{\max} := (P_0^t)^* = (P^t_{\min})^*$, i.e.,

$$\mathscr{D}(P_{\max}) = \{u \in L^2(M; E) | Pu \in L^2(M; E)\}.$$

- $P^t_{\max}$ is likewise defined.

$P_{\min}, P_{\max}$ are called *minimal and maximal extensions* of P_0. For a section $u \in \mathscr{D}(P_{\max})$, the "intermediate" derivatives $D^\alpha u$ (with $|\alpha| \le d$) need not exist as sections on M, even though Pu does so, see [24, p. 61 in Section 4.1].

Traces of Sobolev spaces over the boundary. Now we shall show the surjectivity of the trace map by a rather explicit construction of a continuous right inverse. For the proof of our Main Theorem in Section 2, we will need Frey's construction of the Calderón projector. Since that construction depends on a certain property that is true only for a particular choice of this right inverse, we repeat some computations of [21, Section 1.1].

Let $\gamma^j : C^\infty(M; E) \to C^\infty(\Sigma; E')$ be the trace map $\gamma^j u := (\nabla^E_{\nu'})^j u|_\Sigma$ of the jth jet in normal direction. Set

$$\rho^d \;:=\; \left(\gamma^0, ..., \gamma^{d-1}\right) : C^\infty(M; E) \longrightarrow C^\infty(\Sigma; E'^d), \tag{9}$$

yielding the array of $j - jets$ for $j = 0, \dots d-1$. Analogously, ∇^F gives rise to trace maps $\gamma^j : C^\infty(M; F) \to C^\infty(\Sigma; F')$. The corresponding maps for F will also be denoted by γ^j, ρ^d, respectively.

We recall *Green's Formula*, e.g., from Seeley [35, Equation 7], Trèves [40, Equation 5.41], Grubb [24, Proposition 11.3], or Frey [21, Proposition 1.1.2], with a description of the operator J in the error term:

Proposition 1 (Green's Formula for elliptic operators of order $d \geq 1$) *For each elliptic differential operator P of order d over M, there exists a (uniquely determined) differential operator*

$$J\colon C^\infty(\Sigma; E'^d) \longrightarrow C^\infty(\Sigma; F'^d),$$

such that for all $u \in C^\infty(M;E), v \in C^\infty(M;F)$ we have

$$(Pu, v)_{L^2(M,F)} - (u, P^t v)_{L^2(M,E)} \;=\; (J\rho^d u, \rho^d v)_{L^2(\Sigma;F'^d)}. \tag{10}$$

J is a matrix of differential operators J_{kj} of order $d-1-k-j$, $0 \leq k, j \leq d-1$, and $J_{kj} = 0$ if $k + j > d-1$ (J is upper skew-triangular). Moreover, for $j = d-1-k$ we have

$$J_{k,d-1-k} \;=\; \imath^d (-1)^{d-1-k} \sigma(P)(\nu).$$

Remark 2 For $d = 1, 2, 3$, we visualize the structure of the matrix J,

$$\begin{pmatrix} J_{00}^{[0]} \end{pmatrix}, \quad \begin{pmatrix} J_{00}^{[1]} & J_{01}^{[0]} \\ J_{10}^{[0]} & 0 \end{pmatrix}, \quad \begin{pmatrix} J_{00}^{[2]} & J_{01}^{[1]} & J_{02}^{[0]} \\ J_{10}^{[1]} & J_{11}^{[0]} & 0 \\ J_{20}^{[0]} & 0 & 0 \end{pmatrix}, \text{ etc.,}$$

where the mixed orders of the entries were marked by a superscript $[\langle order \rangle]$.

With [21, Theorem 1.1.4], we obtain a slight reformulation, sharpening, and generalization of the classical *Sobolev Trace Theorem* (see, e.g., [39, Lemma 16.1]):

Proposition 2 (Sobolev Trace Theorem)

1. $H^d(M;E) \subset \mathscr{D}(P_{\max})$ is dense.
2. We have continuous trace maps ρ^d (obtained by continuous extension):

$$(a)\;\; \rho^d\colon H^{d+s}(M;E) \longrightarrow \oplus_{j=0}^{d-1} H^{d+s-j-\frac{1}{2}}(\Sigma;E') \text{ for } s > -\frac{1}{2},$$
$$(b)\;\; \rho^d\colon \mathscr{D}(P_{\max}) \longrightarrow \oplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Sigma;E').$$

Moreover, the map (a) is surjective and has a continuous right-inverse η^d.
3. Green's formula

$$(Pu, v)_{L^2(M;F)} - (u, P^t v)_{L^2(M;E)} = (J\rho^d u, \rho^d v)_{L^2(\Sigma;F'^d)}$$

extends to $\mathscr{D}(P_{\max}) \times H^d(M;F)$, if the right hand side is interpreted as the L^2-dual pairing

$$\oplus_{j=0}^{d-1} H^{-d+j+\frac{1}{2}}(\Sigma;F') \times \oplus_{j=0}^{d-1} H^{d-j-\frac{1}{2}}(\Sigma;F') \to \mathbb{C}.$$

4. If $u \in \mathscr{D}(P_{\max})$, then $u \in H^d(M;E)$ if and only if

$$\rho^d u \;\in\; H^{d-\frac{1}{2}}(\Sigma;E') \oplus \cdots \oplus H^{\frac{1}{2}}(\Sigma;E').$$

5. If $u \in \mathscr{D}(P_{\max})$, then $u \in H_0^d(M;E)$ if and only if $\rho^d u = 0$.

Remark 3 Following Grubb [24, Section 9.1], we call the preceding operator family ρ^d with its domains in different Sobolev spaces by one name: the *Cauchy trace operator* associated with the order d.

It is well known that the trace operators do not extend to negative Sobolev spaces, even not to the whole $L^2(M;E)$ – naturally, however, to $\mathscr{D}(P_{\max})$. For the special case of half-spaces in $\mathbb{R}^n$, e.g., it is shown in [24, Remark 9.4] that the 0-trace map γ^0 makes sense on $H^s(\mathbb{R}^n_+)$ if and only if $s > \frac{1}{2}$. That makes the parts 2b, 3, and 4 of the preceding proposition particularly interesting.

Part 5 admits replacing the abstract definition of $H_0^s(M;E)$ in (1) for $s \in \mathbb{N}$ by a concrete check of the Cauchy boundary data of a given section. For the Euclidean case see [24, Theorem 9.6].

1.3 The construction of the Calderón projection

From [21, Section 2.3], we recall a variant of the construction of the Calderón projection C as a pseudodifferential projector operator of order 0, such that

$$C\colon H^{d+s-\frac{1}{2}}(\Sigma;E')\oplus\cdots\oplus H^{s+\frac{1}{2}}(\Sigma;E') \to H^{d+s-\frac{1}{2}}(\Sigma;E')\oplus\cdots\oplus H^{s+\frac{1}{2}}(\Sigma;E')$$

satisfies $\operatorname{im} C = \rho^d(\ker P)$, for suitable s (cf. Corollary 2).

Homogenization of Sobolev orders. For easier writing (and, hopefully, better understanding) we replace the preceding direct sum of Sobolev spaces of different order by a single Sobolev space of sections in a corresponding product bundle. Basically, we just revert the construction of the Sobolev chain out of one space and a chain of elliptic operators, given in Equation (3) (first for functions, and further-on generalized for sections over the closed big manifold $\tilde{M}$) by identifying Sobolev spaces of different order of sections over Σ via a scale of suitable elliptic operators over Σ.

Following Calderón [17, Section 4.1, p. 76] and using the notation of [21, p. 26], we introduce a *homogenized* (*adjusted*) Cauchy data operator $\widetilde{\rho}^d$. We set $\Delta_\Sigma^{E'} := \left(\nabla_\Sigma^{E'}\right)^* \nabla_\Sigma^{E'}$, where $\nabla_\Sigma^{E'}$ denotes the restriction of $\nabla^{\widetilde{E}}$ (introduced in Equation (6)) to Σ. Then $\Phi := (\Delta_\Sigma + 1)^{1/2}$ is a pseudodifferential operator of order 1 which induces an isomorphism of Banach spaces

$$\Phi_{(s)}\colon H^s(\Sigma;E') \longrightarrow H^{s-1}(\Sigma;E') \text{ for all } s \in \mathbb{R}$$

and, in fact, generates the Sobolev scale $H^s(\Sigma;E')$, i.e., we may assume that the Sobolev norms in $H^s(\Sigma;E')$ are introduced by Φ^s for $s \geq 0$ and for negative s by duality, as explained above in Subsection 1.2 for the Sobolev spaces over the closed manifold $\tilde{M}$.

In order to achieve that all boundary data are of the same Sobolev order, we introduce the matrix

$$\Phi_d \quad := \quad \begin{pmatrix} \Phi^{\frac{d-1}{2}} & 0 & \cdots & 0 \\ 0 & \Phi^{\frac{d-3}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi^{\frac{-d+1}{2}} \end{pmatrix}. \tag{11}$$

It operates for $t_0, \ldots, t_{d-1} \in \mathbb{R}$ as

$$\begin{aligned} &\Phi_d \colon \oplus_{j=0}^{d-1} H^{t_j}(\Sigma; E) \longrightarrow \\ &H^{\frac{2t_0-(d-1)}{2}}(\Sigma; E') \oplus H^{\frac{2t_1-(d-3)}{2}}(\Sigma; E') \oplus \cdots \oplus H^{\frac{2t_{d-1}-(-d+1)}{2}}(\Sigma; E'). \end{aligned} \tag{12}$$

We apply Φ_d to the Cauchy trace data and set

$$\begin{aligned} &\tilde{\rho}^d := \Phi_d \circ \rho^d \colon H^{d+s}(M; E) \xrightarrow{\rho^d} H^{d+s-\frac{1}{2}}(\Sigma; E') \oplus \cdots \oplus H^{s+\frac{1}{2}}(\Sigma; E') \\ &\xrightarrow{\Phi_d} H^{\frac{d}{2}+s}(\Sigma; E') \oplus \cdots \oplus H^{\frac{d}{2}+s}(\Sigma; E') \;=\; H^{\frac{d}{2}+s}(\Sigma; E'^d) \text{ for } s > -\tfrac{1}{2} \end{aligned} \tag{13}$$

with operators

$$\begin{aligned} &\Phi^{\frac{d-1-2j}{2}} \colon H^{d+s-\frac{1}{2}-j}(\Sigma; E') \longrightarrow \\ &\qquad\qquad H^{d+s-\frac{1}{2}-j-(\frac{d-1-2j}{2})}(\Sigma; E') \;=\; H^{\frac{d}{2}+s}(\Sigma; E') \end{aligned}$$

for $j = 0, \ldots d-1$ and $s \in \mathbb{R}$. Similarly, we set $\widetilde{\eta}^d := \eta^d \circ \Phi_d^{-1}$. So, we obtain a condensed and adjusted Trace Theorem as a corollary to Proposition 2:

Corollary 1 (Homogenized trace map) *For $s > -\frac{1}{2}$, we have a continuous trace map*

$$\widetilde{\rho}^d \colon H^{d+s}(M; E) \;\longrightarrow\; H^{\frac{d}{2}+s}(\Sigma; E'^d). \tag{14}$$

By continuity it extends to $\widetilde{\rho}^d \colon \mathscr{D}(P_{\max}) \longrightarrow H^{-\frac{d}{2}}(\Sigma; E'^d)$.
For $\forall s \in \mathbb{R}$, we have an injective continuous operator

$$\widetilde{\eta}^d \colon H^{\frac{d}{2}+s}(\Sigma; E'^d) \;\longrightarrow\; H^{d+s}(\tilde{M}; \widetilde{E}), \tag{15}$$

such that $\widetilde{\rho}^d \circ \widetilde{\eta}^d = \mathrm{Id}$.

Remark 4 Similarly, we can replace the boundary operator J of Green's Formula (Proposition 1) by its adjusted version

$$\tilde{J} \;:=\; (\Phi_d^{(F')})^{-1} \circ J \circ (\Phi_d^{(E')})^{-1}. \tag{16}$$

Note that the formal inverse of $\varPhi_d$ is given by reversing the sequence of the operators in the diagonal:

$$(\varPhi_d)^{-1} = \begin{pmatrix} \varPhi^{\frac{d-1}{2}} & 0 & \cdots & 0 \\ 0 & \varPhi^{\frac{d-3}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varPhi^{\frac{-d+1}{2}} \end{pmatrix}^{-1} = \begin{pmatrix} \varPhi^{\frac{-d+1}{2}} & 0 & \cdots & 0 \\ 0 & \varPhi^{\frac{-d+3}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varPhi^{\frac{d-1}{2}} \end{pmatrix}.$$

The order of the operators $(\varPhi_d^{(E')})^{-1}$, J, and $(\varPhi_d^{(F')})^{-1}$, to be applied to a d–array $(g_1,\dots,g_d)\in H^{\frac{-d}{2}}(\Sigma,E'^d)$ of adjusted boundary values fit precisely, such that

$$\widetilde{J}\colon\ H^{\frac{-d}{2}}(\Sigma,E'^d)\ \longrightarrow\ H^{\frac{-d}{2}}(\Sigma,F'^d). \tag{17}$$

For $d=1$, this claim is trivial. For $d=2$, the homogenizing operators $(\varPhi_d^{(E')})^{-1}$ and $(\varPhi_d^{(F')})^{-1}$ are of the form $\begin{pmatrix} \varPhi^{-\frac12} & 0 \\ 0 & \varPhi^{\frac12}\end{pmatrix}$ and Green's boundary operator J of the form $\begin{pmatrix} J^1_{00} & J^0_{01} \\ J^0_{10} & 0\end{pmatrix}$, where the superscripts of the differential operators $J^1_{00}, J^0_{01}, J^0_{10}$ and the pseudodifferential operators $\varPhi^{-\frac12},\varPhi^{\frac12}$ give their order. So, for $g_1,g_2\in H^{\frac{-d}{2}}(\Sigma,E')$ we have

$$\begin{aligned}
\begin{pmatrix} \varPhi^{-\frac12} & 0 \\ 0 & \varPhi^{\frac12}\end{pmatrix}\begin{pmatrix} g_1 \\ g_2\end{pmatrix} &= \begin{pmatrix} \varPhi^{-\frac12}g_1 \\ \varPhi^{\frac12}g_2\end{pmatrix} \in H^{-\frac{d}{2}+\frac12}(\Sigma,E')\oplus H^{-\frac{d}{2}-\frac12}(\Sigma,E'),\\
\begin{pmatrix} J^1_{00} & J^0_{01} \\ J^0_{10} & 0\end{pmatrix}\begin{pmatrix} \varPhi^{-\frac12}g_1 \\ \varPhi^{\frac12}g_2\end{pmatrix} &= \begin{pmatrix} J^1_{00}\varPhi^{-\frac12}g_1 + J^0_{01}\varPhi^{\frac12}g_2 \\ J^0_{10}\varPhi^{-\frac12}g_1\end{pmatrix} \in \begin{matrix} H^{-\frac{d}{2}-\frac12}(\Sigma,F')\oplus \\ H^{-\frac{d}{2}+\frac12}(\Sigma,F')\end{matrix},\\
\Rightarrow \begin{pmatrix} \varPhi^{-\frac12} & 0 \\ 0 & \varPhi^{\frac12}\end{pmatrix}\begin{pmatrix} J^1_{00}\varPhi^{-\frac12}g_1 + J^0_{01}\varPhi^{\frac12}g_2 \\ J^0_{10}\varPhi^{-\frac12}g_1\end{pmatrix} &\in \begin{matrix} H^{-\frac{d}{2}}(\Sigma,F')\oplus \\ H^{-\frac{d}{2}}(\Sigma,F')\end{matrix} = H^{-\frac{d}{2}}(\Sigma,F'^d).
\end{aligned}$$

For any $d\ge 1$, we can confirm (17) by following the transformations in the preceding way.

Ingredients for FREY*'s Calderón projector definition by localization in collar neighbourhood.* There are various ways to construct the Calderón projector as a pseudodifferential operator $H^{-\frac{d}{2}}(\Sigma;E'^d)\to H^{-\frac{d}{2}}(\Sigma;E'^d)$ of order 0 that yields a projection (= a not necessarily self-adjoint idempotent) onto the Cauchy data space in the sense of Proposition 4, see our Section 1.6. Basically, they all use very similar ingredients leading to the following three tasks, corresponding to our intuition and with our notations:

(i) To come from $H^{-\frac{d}{2}}(\Sigma; E'^d)$ to the distribution space $H^{-d}(M;F)$ one uses the boundary operator $\widetilde{J}$ of Green's Formula, combined with the dual $(\widetilde{\rho}^d)^*$ of the adjusted Cauchy trace operator of order d for the bundle F. That's a classical device, see, e.g., Hörmander [25, Equation 17.3.9, elaborated in Chapter XX].

(ii) Then we need a pseudodifferential operator of order $-d$ serving as a quasi inverse of our elliptic operator P, kind of Poisson operator or fundamental solution or parametrix, to come back to $\mathscr{D}(P_{\max}) \subset H^0(M;E) = L^2(M;E)$. That is the delicate point since, over the open manifold M°, the operators P and $P_{\max}$ have infinite-dimensional kernel in dimensions ≥ 2. The standard way goes via the extension of P to an invertible operator on a closed extension of M or to a natural invertible double of the original operator with suitable boundary conditions. Here we take a different way, namely an extension in a tiny collar, see below. That's not a matter of taste but required to keep the parameter dependence under control for our proof of Theorem 1.

(iii) To end in the Cauchy data space, one needs a suitable shrinking operator before applying the adjusted Cauchy trace operator of the bundle E.

Let's do it. Closely following Frey's elaboration in [21, p. 16 and Section 2.3] of Hörmander [25, pp. 234ff, Equation 20.1.7, and Theorem 20.1.3], we simplify the calculations and keep them transparent by localizing them in a collar neighbourhood of the boundary $\Sigma = \partial M$.

We attach a tiny neck to the manifold M by choosing a sufficiently small positive real number ε, such that

$$N \;:=\; \{p \in \tilde{M} \mid p \in M \text{ or } d_{\tilde{g}}(p,\Sigma) \leq \varepsilon\}$$

is a smooth manifold with smooth boundary which we denote by ∂N. For the open submanifold N° and the bundle $\widetilde{E}|_{N^\circ}$ (shortly written as E), we recall the common notation

$$\begin{aligned} H^s_{\mathrm{loc}}(N^\circ;E) &:= \{u \in \mathcal{D}'(N^\circ;E) \mid \chi u \in H^s(N^\circ;E) \text{ for all } \chi \in C^\infty_c(N^\circ;E)\},\\ H^s_{\mathrm{comp}}(N^\circ;E) &:= \{u \in H^s(N^\circ;E) \mid \operatorname{supp} u \text{ compact}\}, \end{aligned}$$

where $\mathcal{D}'(N^\circ;E)$ denotes the set of distributional sections.

Task (i) Consider the adjusted trace map (here for the bundle E, but in our application for the bundle F, correspondingly)

$$\widetilde{\rho}^d_N \colon H^d_{\mathrm{loc}}(N^\circ;E) \;\longrightarrow\; H^{d/2}(\Sigma;E'^d),$$

whose dual

$$(\widetilde{\rho}^d_N)^* \colon H^{-d/2}(\Sigma;E'^d) \;\longrightarrow\; H^{-d}_{\mathrm{comp}}(N^\circ;E)$$

is given by

$$\langle u, (\widetilde{\rho}^d_N)^* g\rangle \;=\; \langle \widetilde{\rho}^d_N u, g\rangle \quad \text{for } u \in H^d_{\mathrm{loc}}(N^\circ;E) \text{ and } g \in H^{-d/2}(\Sigma;E'^d).$$

Task (ii) For $P^N : C^\infty(N;E) \to C^\infty(N;F)$, consider the *Cauchy data space*

$$\Lambda_0(P^N) \ := \ \{h \in W(P^N) |\ \exists u \in \ker P^N_{\max} \text{ with } \widetilde{\rho}^d u = h\},$$

where $W(P^N)$ denotes the image of $\widetilde{\rho}^d \colon \mathcal{D}(P^N_{\max}) \to H^{-\frac{d}{2}}(\partial N; E^d|_{\partial N})$ of Corollary 1, i.e., the space of boundary values of weak solutions to P^N (= sections belonging to the maximal domain of P^N, not necessarily vanishing under the operation of P^N). It can be identified with

$$\mathscr{D}(P^N_{\max})/\mathscr{D}(P^N_{\min}),$$

and can be mapped into the orthogonal complement of the graph $\mathrm{Gr}(P^N_{min})$ in

$$L^2(N;E) \oplus L^2(N;F),$$

i.e.,

$$\begin{aligned} W(P^N) \ &\longrightarrow \ L^2(N;E) \oplus L^2(N;F), \\ u + \mathscr{D}(P^N_{\min}) \ &\mapsto \ (\mathrm{Id} - pr^{ort}_{\mathrm{Gr}(P^N_{\min})})(u, P^N_{\max} u). \end{aligned}$$

Here $pr_{\mathrm{Gr}(P^N_{\min})}$ denotes the orthogonal projection of $L^2(N;E) \oplus L^2(N;F)$ onto $\mathrm{Gr}(P^N_{min})$. In the following, let us denote by $\widetilde{\rho}^d_{\partial N}$ the trace map

$$\widetilde{\rho}^d_{\partial N} \colon \mathscr{D}(P^N_{\max}) \ \longrightarrow \ H^{-d/2}(\partial N; E^d|_{\partial N}).$$

Let $\Lambda_0(P^N)^\perp$ denote the orthogonal complement of the Cauchy data space in $W(P^N) \subset L^2(N;E) \oplus L^2(N;F)$. Since

$$(\Lambda_0(P^N), \Lambda_0(P^N)^\perp)$$

is a Fredholm pair in $W(P^N)$, it follows in the usual way ([7], [13] or [21, Theorem 1.3.4.(iii)]) that

$$P^N_0 : \{u \in \mathscr{D}(P^N_{\max}) |\ \widetilde{\rho}^d_{\partial N} u \perp \Lambda_0(P^N)\} \to L^2(N;F)$$

is a Fredholm realization of P^N. The operator

$$\widetilde{P}^N_0 \colon \mathscr{D}(P^N_0) \cap (\ker P^N_0)^\perp \to \operatorname{im} P^N_0$$

has a continuous inverse which, composed with the orthogonal projection onto $\operatorname{im} P^N_0$, yields a bounded operator

$$Q^N : L^2(N;F) \to H^d(\tilde{M}; \widetilde{E}). \tag{18}$$

Denote by pr_1, pr_2 the orthogonal projections onto the finite-dimensional spaces $\ker P^N_0$, $(\operatorname{im} P^N_0)^\perp$, respectively. We have

$$\begin{aligned} Q^N P^N_0 u \ &= \ (\mathrm{Id} - pr_1) u \ \text{ and} \\ P^N Q^N v \ &= \ (\mathrm{Id} - pr_2) v \ \text{ for } u \in \mathscr{D}(P^N_0), \ v \in L^2(N;F). \end{aligned}$$

If ∂N is sufficiently "close" to $\partial M = \Sigma$, we have

$$\ker P^N_{\min} \ = \ \ker P_{\min} \quad \text{and} \quad \ker P^{N,t}_{\min} \ = \ \ker P^t_{\min}.$$

It follows that

$$\widetilde{\rho}^d \circ pr_1 \ = 0 \ \text{ and } \ \widetilde{\rho}^d \circ pr_2 \ = \ 0.$$

From that we obtain

Proposition 3 (Frey [21], Proposition 2.3.1) *The restriction*

$$Q^N \colon C^\infty_c(N^\circ; F) \ \longrightarrow \ C^\infty(N^\circ; E) \tag{19}$$

of the bounded operator of (18) *is a pseudodifferential parametrix for* P^N.

Note that the natural extension

$$Q^N \colon H^{-d}_{\text{comp}}(N^\circ; F) \ \longrightarrow \ L^2_{\text{loc}}(N; E) \tag{20}$$

will be denoted by Q^N, as well.

Task (iii) Denote by $r_+ \colon L^2_{\text{loc}}(N; E) \to L^2(M; E)$ the restriction operator to M.

Definition of the Calderón projector on $H^{-\frac{d}{2}}(\Sigma; E'^d)$ and illustration for the Laplacian. Having provided the required ingredients, we can now define

Definition 1 The operator

$$C_+ \colon H^{\frac{-d}{2}}(\Sigma; E'^d) \to H^{\frac{-d}{2}}(\Sigma; E'^d), \ C_+ h \ := \ -\widetilde{\rho}^d r_+ Q^N (\widetilde{\rho}^d_N)^* \widetilde{J} h$$

is called the *Calderón projection.*

Basic properties of the Calderón projector and illustration for the Cauchy-Riemann operator over the unit disc. We summarize the two characterizing properties of the Calderón projector in Proposition 4 and derive the relation between Cauchy data spaces and images of the Calderón projector for the scale of Calderón projectors in Sobolev spaces of suitable order in Corollary 2.

Proposition 4 *(cf. [21, Theorem 2.3.5])*

(i) C_+ is a projection onto the Cauchy data space $\Lambda_0(P) \subset H^{-d/2}(\Sigma; E'^d)$, defined by

$$\Lambda_0(P) \ := \ \{\widetilde{\rho}^d u | u \in \mathscr{D}(P_{\max}), P_{\max} u = 0\}. \tag{21}$$

(ii) C_+ is a 0-th order pseudodifferential operator which extends to a scale $\{C_\sigma \colon H^\sigma(\Sigma; E'^d) \to H^\sigma(\Sigma; E'^d)\}$ for $\sigma \in \mathbb{R}$.

Because the Calderón projection is a 0-th order pseudodifferential operator and $C^2 = C$, we have

Corollary 2

$$H^{\frac{d}{2}+s}(\Sigma;E'^{d})\cap \operatorname{im}(C(P)\colon H^{-d/2}(\Sigma;E'^{d})\to H^{-d/2}(\Sigma;E'^{d})) \quad (22)$$
$$= \operatorname{im}(C(P)\colon H^{\frac{d}{2}+s}(\Sigma;E'^{d})\to H^{\frac{d}{2}+s}(\Sigma;E'^{d})), \quad \textit{for } s\geq -d.$$

When $s>-\frac{1}{2}$ *or* $-d\leq s<\frac{1}{2}-d$

$$\operatorname{im}(C(P)\colon H^{\frac{d}{2}+s}(\Sigma;E'^{d})\to H^{\frac{d}{2}+s}(\Sigma;E'^{d}))$$
$$=\{\widetilde{\rho}^{d}u|u\in H^{d+s}(M;E)\cap \mathscr{D}(P_{\max}), Pu=0\}\cap H^{\frac{d}{2}+s}(\Gamma;E'^{d}). \quad (23)$$

Proof Since $C(P)$ is a projection on $H^{\frac{d}{2}+s}(\Sigma;E'^{d})$, i.e., $C(P)^2=C(P)$, and $H^{\frac{d}{2}+s}(\Sigma;E'^{d})\subseteq H^{-d/2}(\Sigma;E'^{d})$, for $s\geq -d$, (22) follows.

When $s>-\frac{1}{2}$, on one hand, by the trace theorem, $\widetilde{\rho}^{d}u\in H^{\frac{d}{2}+s}(\Sigma;E'^{d})$.

On the other hand, assume that

$$u\in \mathscr{D}(P_{\max}),\ Pu=0 \text{ and } \widetilde{\rho}^{d}u\in H^{\frac{d}{2}+s}(\Sigma;E'^{d}).$$

Then, using the analogue of $\widetilde{\eta}^{d}$ for the complement $\tilde{M}\setminus M$, we can extend u to some $\widetilde{u}$ such that $\widetilde{u}|_{\tilde{M}\setminus M}\in H^{d+s}_{loc}(\overline{\tilde{M}\setminus M};E)$ and

$$\widetilde{\rho}^{d}r_{-}\widetilde{u}=\widetilde{\rho}^{d}r_{+}\widetilde{u}=\widetilde{\rho}^{d}u, \quad (24)$$

where we denote by $r_{\pm}$ the restriction onto M, $\tilde{M}\setminus M$, resp. Since $\tilde{M}$ is a smooth manifold without boundary, using Green's formula for M and $\tilde{M}\setminus M$, we have, for $v\in C^{\infty}_{c}(\tilde{M};\widetilde{E})$

$$(\widetilde{u},P^{t}(1+\Delta)^{-\frac{s}{2}}v)_{L^2(\tilde{M};\widetilde{E})}$$
$$=((1+\Delta)^{\frac{s}{2}}Pr_{+}\widetilde{u},v)_{L^2(M;F)}+((1+\Delta)^{\frac{s}{2}}Pr_{-}\widetilde{u},v)_{L^2(\tilde{M}\setminus M;F)}$$
$$=((1+\Delta)^{\frac{s}{2}}Pu,r_{+}v)_{L^2(M;F)}+((1+\Delta)^{\frac{s}{2}}Pr_{-}\widetilde{u},v)_{L^2(\tilde{M}\setminus M;F)}. \quad (25)$$

From $P\colon H^{d+s}(\tilde{M};\widetilde{E})\to H^{s}(\tilde{M};\widetilde{F})$ and $r_{-}\widetilde{u}\in H^{d+s}(\overline{\tilde{M}\setminus M};E)$, we have $(1+\Delta)^{\frac{s}{2}}Pr_{-}\widetilde{u}\in L^2(\tilde{M}\setminus M;F)$. Together with (25), we have $(1+\Delta)^{\frac{s}{2}}P\widetilde{u}\in L^2(\tilde{M};\widetilde{F})$, thus $P\widetilde{u}\in H^{\frac{s}{2}}(\tilde{M};\widetilde{F})$. Since $P^{\tilde{M}}$ is elliptic, $\widetilde{u}\in H^{s+d}_{loc}(\tilde{M};E)$. It follows that $u\in H^{d+s}(M;E)$.

When $-d\leq s<\frac{1}{2}-d$, consider the (modified) trace map

$$\widetilde{\rho}^{d}_{N}\colon H^{-s}_{\text{loc}}(N^{\circ};E)\to H^{-s-\frac{d}{2}}(\Gamma;E'^{d}),$$

whose dual,

$$(\widetilde{\rho}^{d}_{N})^{*}\colon H^{s+\frac{d}{2}}(\Gamma;E'^{d})\to H^{-s}_{\text{loc}}(N^{\circ};E)$$

is given by

$$\langle u,(\widetilde{\rho}^{d}_{N})^{*}h\rangle=\langle \widetilde{\rho}^{d}_{N}u,h\rangle,\ u\in H^{-s}_{\text{loc}}(N^{\circ};E), h\in H^{s+\frac{d}{2}}(\Gamma;E'^{d}).$$

Then we have $C_{+}h=-\widetilde{\rho}^{d}u$, where $u:=-r_{+}Q^{N}(\widetilde{\rho}^{d}_{N})^{*}\widetilde{J}h\in H^{d+s}(M;E)$. In fact we have

$$H^{s+\frac{d}{2}}(\Gamma;E'^{d})\xrightarrow{\widetilde{J}}H^{s+\frac{d}{2}}(\Gamma;F'^{d})\xrightarrow{(\widetilde{\rho}^{d}_{N})^{*}}$$
$$H^{s}(N;F)\xrightarrow{Q^{N}}H^{s+d}(N;E)\xrightarrow{r_{+}}H^{s+d}(M;E).$$

1.4 Examples and illustrations

We illustrate our/FREY's constructions with some classical examples where the parameter dependence of the Calderón projector and the Cauchy data spaces can be determined explicitly.

Example 1 (1D examples) From COURANT and HILBERT [19, Section V.1.3a, p. 278f], we recall the standard form of a differential equation of second order on the interval. So, Let $M := [x_0, x_1]$, $x_0, x_1 \in \mathbb{R}$ and $x_0 < x_1$ with the boundary $\Sigma = -\{x_0\} \cup \{x_1\}$ and $\int_\Sigma \varphi = \varphi(x_1) - \varphi(x_0)$ for $\varphi\colon \Sigma \to \mathbb{C}$. We set

Warning: in [CH] real

$$Pu \;:=\; (au')' \;-\; bu' \;+\; (cu)' \;\; - du \quad \text{for } u \in C^\infty(M; \mathbb{C}_M)$$

with $a, b, c, d \in C^1(M, \mathbb{C})$. Here f' denotes the derivative $\frac{df}{dx}$ for $f \in C^1(M, \mathbb{C})$. Then the formal adjoint P^t takes the form

$$P^t v \;=\; (\overline{a} v') \;+\; (\overline{b} v)' \;-\; \overline{c} v' \;-\; \overline{d} v \quad \text{for } v \in C^\infty(M; \mathbb{C}_M),$$

where complex conjugation is marked in the usual way. The Cauchy trace operator associated with the order $d = 2$ is

$$\rho^d \colon u \;\mapsto\; \big(u(x_0), u(x_1)\big) \;\oplus\; \big(u'(x_0), u'(x_1)\big),$$

and we obtain Green's formula

$$\int_{x_0}^{x_1} \big(Pu\,\overline{v} \;-\; u\,\overline{P^t v}\big)\,dx \;=$$

$$\Big\langle \mathcal{J}_0 \begin{pmatrix} u(x_0) \\ u'(x_0) \end{pmatrix}, \begin{pmatrix} v(x_0) \\ v(x_0) \end{pmatrix} \Big\rangle \;+\; \Big\langle \mathcal{J}_1 \begin{pmatrix} u(x_1) \\ u'(x_1) \end{pmatrix}, \begin{pmatrix} v(x_1) \\ v(x_1) \end{pmatrix} \Big\rangle$$

with (ROUGHLY, make precise) $\mathcal{J}_0 = -\begin{pmatrix} c-b & a \\ -a & 0 \end{pmatrix}$ and $\mathcal{J}_1 = \begin{pmatrix} c-b & a \\ -a & 0 \end{pmatrix}$. THAT'S ALL A BIT COMPLICATED, BUT NOW COMES THE WORK, NAMELY TO DETERMINE THE PARAMETRIX OF P AND, FINALLY, THE Calderón PROJECTOR.

THEN IT WOULD BECOME CLEAR HOW $C_+(a, b, c, d)$ DEPENDS ON THE BASE POINT $(a, b, c, d) \in B := C^1(M, \mathbb{C}^4)$.

...

...

Remark 5 Our preceding calculations fit nicely with GRUBB's [24, Example 11.1]

$$C_+(\alpha) \;=\; \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\alpha} \\ -\frac{\alpha}{2} & \frac{1}{2} \end{pmatrix}$$

for the operator $Pu := -u'' + \alpha^2 u$ on the complete manifold $M := [0, \infty[$, where $\alpha > 0$.

We want instead of C_+ an orthogonal projection onto the Cauthy data space. The orthogonal projection onto $\operatorname{im} C_+$, which we denote by C_+^{ort} here, is given by the well-known formula (cf. [11, Lemma 12.8])

$$C_+^{ort} := C_+ C_+^* (C_+ C_+^* + (\operatorname{Id} - C_+^*)(\operatorname{Id} - C_+))^{-1}. \tag{26}$$

Since $C_+ C_+^* + (\operatorname{Id} - C_+^*)(\operatorname{Id} - C_+)$ is elliptic we infer that C_+^{ort} is still a classical pseudodifferential projector of order 0.

So by (26),

$$C_+^{ort}(\alpha) \;=\; \frac{1}{\alpha^2+1}\begin{pmatrix} 1 & -\alpha \\ -\alpha & \alpha^2 \end{pmatrix}.$$

$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. According to the definition and property of Fourier transformation and Fubini Theorem, we have

$$\begin{aligned} Qf &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{\xi^2+\alpha^2}\hat{f}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^i(x-y)\xi}{\xi^2+\alpha^2} f(y)dyd\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^i(x-y)\xi}{\xi^2+\alpha^2} d\xi f(y)dy \end{aligned}$$

Since $\rho^2 u = \begin{pmatrix} u(0) \\ \frac{du}{dx}(0) \end{pmatrix}$, and using the residues we have for $x > 0$

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{\xi^2+\alpha^2} &= \frac{\pi}{\alpha}e^{-x\alpha}, \\ \int_{-\infty}^{+\infty} \xi\frac{e^{ix\xi}}{\xi^2+\alpha^2} &= \pi i e^{-x\xi}. \end{aligned}$$

So

$$\begin{aligned} Q(\rho^2)^* Jg(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^i(x-y)\xi}{\xi^2+\alpha^2} d\xi \overline{(\rho^2)^* J\bar{g}} dy \\ &= \rho^2 \int_{-\infty}^{+\infty} \frac{e^i(x-y)\xi}{\xi^2+\alpha^2} d\xi \cdot Jg \\ &= \frac{1}{2\alpha}e^{-x\alpha}g_1 - \frac{1}{2}e^{-x\alpha}g_0, \end{aligned}$$

where $g = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$.

Now we consider the case $M =]0,1[$. Remember we always use the inward normal vector field on the boundary, on the boundary $\{x_0 = 0\}$, we have $\rho^2 u =$

$(u(0), u'(0))$, while on the boundary $\{x_1 = 1\}$, we have $\rho^2 u = (u(1), -u'(1))$, and we obtain Green's formula

$$\int_{x_0}^{x_1} \big(Pu\,\overline{v} \;-\; u\,\overline{P^t v}\big)\,dx \;=\; \Big\langle \mathcal{J}_0 \begin{pmatrix} u(x_0) \\ u'(x_0) \end{pmatrix}, \begin{pmatrix} v(x_0) \\ v(x_0) \end{pmatrix} \Big\rangle \;+\; \Big\langle \mathcal{J}_0 \begin{pmatrix} u(x_1) \\ -u'(x_1) \end{pmatrix}, \begin{pmatrix} v(x_1) \\ -v(x_1) \end{pmatrix} \Big\rangle$$

with $\mathcal{J}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For the boundary value $h = \big(u_0, v_0\big) \;\oplus\; \big(u_1, v_1\big)$, we have

$$Q(\rho^2)^* Jh = \frac{1}{2\alpha} e^{-x\alpha} v_0 - \frac{e^{-x\alpha}}{2} u_0 + \frac{e^{(x-1)\alpha}}{2\alpha} v_1 - \frac{1}{2} e^{(x-1)\alpha} u_1.$$

So the Calderón projector is

$$C_+(\alpha) \begin{pmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\alpha} & \frac{1}{2} e^{-\alpha} & -\frac{1}{2\alpha} e^{-\alpha} \\ -\frac{\alpha}{2} & \frac{1}{2} & \frac{\alpha}{2} e^{-\alpha} & -\frac{1}{2} e^{-\alpha} \\ \frac{1}{2} e^{-\alpha} & -\frac{1}{2\alpha} e^{-\alpha} & \frac{1}{2} & -\frac{1}{2\alpha} \\ \frac{\alpha}{2} e^{-\alpha} & -\frac{1}{2} e^{-\alpha} & -\frac{\alpha}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \end{pmatrix},$$

and

$$C_+^{ort}(\alpha) = \frac{1}{a^2 - b^2} \begin{pmatrix} ac_{11} - bc_{13} & ac_{12} & -bc_{11} + ac_{13} & -bc_{12} \\ ac_{12} & ac_{22} - bc_{24} & -bc_{12} & -bc_{22} + ac_{24} \\ ac_{13} - bc_{11} & -bc_{12} & -bc_{13} + ac_{11} & ac_{12} \\ -bc_{12} & ac_{24} - bc_{22} & ac_{12} & ac_{22} - bc_{24} \end{pmatrix}, \tag{27}$$

where

$$\begin{aligned} a &= (\frac{\alpha}{2} + \frac{1}{2\alpha})^2 (1 + e^{-2\alpha}), \;\; b = (\frac{1}{2\alpha^2} - \frac{\alpha^2}{2}) e^{-\alpha}, \\ c_{11} &= (\frac{1}{4} + \frac{1}{4\alpha^2})(1 + e^{-2\alpha}), \;\; c_{12} = (\frac{\alpha}{4} + \frac{1}{4\alpha})(-1 + e^{-2\alpha}), \\ c_{13} &= (\frac{1}{2} + \frac{1}{2\alpha^2}) e^{-\alpha}, \;\; c_{22} = (\frac{\alpha^2}{4} + \frac{1}{4})(1 + e^{-2\alpha}), \\ c_{24} &= (-\frac{\alpha^2}{2} - \frac{1}{2}) e^{-\alpha}. \end{aligned}$$

Now when $\alpha = 0$, that is, for the operator $Pu := -u''$, the parametrix is

$$Qf(x) = -\frac{1}{2} \int_{-\infty}^{+\infty} |y - x| f(y) dy.$$

When $M = [0, +\infty[$,

$$\begin{aligned} Q(\rho^2)^* Jg &= -\frac{1}{2} \int_{-\infty}^{+\infty} |y - x| \overline{(\rho^2)^* J\overline{g}} dy dy \\ &= -\frac{1}{2}(x v_0 + u_0), \end{aligned}$$

where $g = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$. So $C_+(0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, but it is not a projection.

When $M =]0,1[$, $x \in M$,

$$\begin{aligned} Q(\rho^2)^* Jh(x) &= -\frac{1}{2}\int_{-\infty}^{+\infty} |y-x|\overline{(\rho^2)^* J\bar{h}} \\ &= -\frac{1}{2}(xv_0 + u_0 + (1-x)v_1 + u_1). \end{aligned}$$

So $C_+(0) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$,

$$C_+^{ort}(0) = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} & \frac{2}{5} \end{pmatrix}. \tag{28}$$

So for the case $M =]0,1[$, by computation $\lim_{\alpha>0,\alpha\to 0} C_+^{ort}(\alpha) = C_+^{ort}(0)$. In fact, we can use the power series of exponential function to write every term in the matrix $C_+^{ort}(\alpha)$ (27) as a ratio of two power series of α, actually, it can be reduced to this form

$$\frac{a_2\alpha^2 + a_3\alpha^3 + \cdots}{b_2\alpha^2 + b_3\alpha^3 + \cdots},$$

where $a_i, b_i \in \mathbb{R}$, $i \in \mathbb{N}$, $i \geq 2$, then when $\alpha \to 0$, the ratio $\frac{a_2}{b_2}$ is just the corresponding term in $C_+^{ort}(0)$ (28).

Example 2 (2D examples of second order) Similarly, we can derive the Calderón projection for a general elliptic differential equation of second order over a domain $M \subset \mathbb{R}^2$. Following [19, Section V.1.3b, p. 279f], we set

$$Pu \ := \ (pu_x)_x \ + \ (pu_y)_y \ - \ qu \tag{29}$$

etc. ...

etc. ...

etc. ...

For $p = 1$ and $q = 0$ in (29), we obtain the Laplacian and can compare our approach with the classical formulas of potential theory.

Example 3 (Reconsidering potential theory) Following Seeley [35], we illustrate Frey's construction of the Calderón projector for the Laplacian and its single and double layer potentials.

IT WOULD BE NICE TO ADD A POTENTIAL, i.e., $q \neq 0$, AND THEN TO INVESTIGATE THE PARAMETER DEPENDENCE FOR VARYING q.

We denote by $K_+ h := r_+ Q^N (\widetilde{\rho}_N^d)^* \widetilde{J} h$, thus K_+ is a multiple layer potential with d "layers". In case $P = -\sum_{i=1}^n \frac{\partial}{\partial x_i^2}$, K_+ is the familiar combination of single and double layer potentials whereby a harmonic function is represented in terms of its Cauchy data on the boundary.

$$K_+(g_0, g_1)(x) = \int_\Sigma \left(\frac{\partial \mathcal{E}}{\partial \nu_{y'}} (x - y') \Phi_{y'}^{-\frac{1}{2}} g_0(y') ds_{y'} - \mathcal{E}(x - y') \Phi_{y'}^{\frac{1}{2}} g_1(y') \right) ds_{y'}. \ (x \in M).$$

For harmonic u, we have the representation (cf. [22])

$$u(x) = \int_\Sigma \left(-\frac{\partial \mathcal{E}}{\partial \nu} (x - y) u(y) + \mathcal{E}(x - y) \frac{\partial u}{\partial \nu_y}(y) \right) ds_y. \ (y \in M).$$

Where ds is the $n-1$-dimensional area element in the boundary Σ,

$$\mathcal{E}(x - y) = \mathcal{E}(|x - y|) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x - y|^{2-n}, & n > 2 \\ \frac{1}{2\pi} \log |x - y|, & n = 2, \end{cases}$$

is the fundamental solution of Laplace's equation. $\omega_n := \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of unit ball in $\mathbb{R}^n$. The Newtonian potential of f is

$$-Qf(x) = \int_N \mathcal{E}(x - y) f(y) dy.$$

We have

$$C_+(g_0, g_1) = -(\Phi^{\frac{1}{2}} \gamma^0 E^0 K_+(g_0, g_1), \Phi^{-\frac{1}{2}} \gamma^1 K_+(g_0, g_1)).$$

According to the properties of single and double layer potentials (following Eskin [20]), we have

$$\begin{aligned} \Phi^{\frac{1}{2}} \mathcal{E}^0 K_+(g_0, g_1)(x') = &-\frac{1}{2} g_0(x') + \int_\Sigma \Phi_{x'}^{\frac{1}{2}} \Phi_{y'}^{-\frac{1}{2}} \frac{\partial \mathcal{E}}{\partial \nu_{y'}} (x' - y') g_0(y') ds_{y'} \\ &- \int_\Sigma \Phi_{x'}^{\frac{1}{2}} \Phi_{y'}^{\frac{1}{2}} \mathcal{E}(x' - y') g_1(y') ds_{y'}. \ \ (x' \in \Sigma) \end{aligned}$$

$$\begin{aligned} \Phi^{-\frac{1}{2}} \mathcal{E}^1 K_+(g_0, g_1)(x') = &-\frac{1}{2} g_1(x') - \int_\Sigma \Phi_{x'}^{-\frac{1}{2}} \Phi_{y'}^{\frac{1}{2}} \frac{\partial \mathcal{E}}{\partial \nu_{x'}} (x' - y') g_1(y') ds_{y'} \\ &+ \int_\Sigma \Phi_{x'}^{-\frac{1}{2}} \Phi_{y'}^{-\frac{1}{2}} \frac{\partial^2 \mathcal{E}}{\partial \nu_{x'} \partial \nu_{y'}} (x' - y') g_0(y') ds_{y'}. \ \ (x' \in \Sigma) \end{aligned}$$

For $-\Delta$, we have Green's formula as in (10),

$$\int_\Omega -\Delta u v dx - \int_\Omega u(-\Delta v) dx = \int_\Sigma (J_0 \rho^2 u, \rho^2 v) ds, \tag{30}$$

where $J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\rho^2 u = \begin{pmatrix} u \\ \frac{\partial u}{\partial \nu} \end{pmatrix}$, ν is the unit inward normal to Σ.

As in (13), $\Phi_2 = \begin{pmatrix} \Phi^{\frac{1}{2}} & 0 \\ 0 & \Phi^{-\frac{1}{2}} \end{pmatrix}$, $\tilde{\rho}^2 = \Phi_2 \rho^2 = \begin{pmatrix} \Phi^{\frac{1}{2}} \gamma^0 \\ \Phi^{-\frac{1}{2}} \gamma^0 \frac{\partial}{\partial \nu} \end{pmatrix}$.

Denote by $B_0 = \frac{1}{2} \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}$, from [21, Chapter 4.2], the boundary condition B_0 differs from the true Calderón projector by a pseudodifferential operator of order -1 thus by a compact operator.

Proposition 5 *(cf.[20, Theorem 17.1]) Let $P(\xi)$ be any arbitrary nonzero polynomial of degree m. Then there exists $E \in \mathscr{D}'(\mathbb{R}^n)$ such that*

$$P(-i\frac{\partial}{\partial x})E = \delta. \tag{31}$$

So for any partial differential operator with constant coefficients, we can construct a fundamental solution in $\mathscr{D}'(\mathbb{R}^n)$, i.e., E satisfies (31).

If f is a distribution with a compact support, then

$$u = E * f$$

gives a particular solution to

$$P(-i\frac{\partial}{\partial x})u = f.$$

From [35] we take

Example 4 (Calderón projection for Cauchy-Riemann operator on unit disk) ONCE AGAIN, IT WOULD BE NICE TO ADD A PARAMETER, AND THEN TO INVESTIGATE THE PARAMETER DEPENDENCE.

...

Consider the operator $P = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$. Then $Pu = 0$ means u is holomorphic. Let D denote the disk $\{x^2 + y^2 < 1\}$, and S^1 = boundary of D. Let

$$H_+ = \{u \in L^2(D); u\text{'s first derivatives} \in L^2(D), Pu = 0\}. \tag{32}$$

$$H_- = \{u \in L^2_{loc}(\mathbb{R}^2 \setminus D); u\text{'s first derivatives} \in L^2_{loc}(\mathbb{R}^2 \setminus D), Pu = 0\}. \tag{33}$$

$$H^{\frac{1}{2}}(S^1) = \{f = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}; \Sigma_{n \in \mathbb{Z}}(1 + |n|)|a_n|^2 < \infty\}. \tag{34}$$

We define

$$Q\colon H^{\frac{1}{2}}(S^1) \to H_+ \tag{35}$$

$$\sum_{n=-\infty}^{+\infty} a_n e^{in\theta} \mapsto \sum_{n=0}^{+\infty} a_n z^n. \tag{36}$$

$$C^+ : H^{\frac{1}{2}}(S^1) \to H^{\frac{1}{2}}(S^1) \tag{37}$$

$$\sum_{n=-\infty}^{+\infty} a_n e^{in\theta} \mapsto \sum_{n=0}^{+\infty} a_n e^{in\theta}, \tag{38}$$

the C^+ is just the Calderón projection for the operator P on the unit disk.

Denote by

$$(Hu)(z) \ := \ \frac{1}{\pi\sqrt{-1}} P.V. \int_{S^1} \frac{u(s)}{s-z} ds, \ (\forall z \in S^1),$$

where $P.V.$ means principal value integral, that is

$$P.V. \int_{S^1} \frac{u(s)}{s-z} ds \ := \ \lim_{\varepsilon 0} \int_{|s-z| \geq \varepsilon, s \in S^1} \frac{u(s)}{s-z} ds.$$

Then the singular integral operator $H \in \mathfrak{L}(L^2(S^1))$. The Fourier multiplier is $m(H) = \operatorname{sign} \xi$ and H is self-adjoint, i.e., $H = H^*$. Finally, we have $C^+ u = \frac{1}{2}(\mathrm{Id} + H)u$, for $u \in L^2(S^1)$. The Fourier multiplier is $m(C^+) = \frac{1}{2}(1 + \operatorname{sign} \xi)$ and C^+ is an orthogonal projection operator on the boundary, i.e., $C_+^2 = C_+$, $C_+^* = C_+$.

1.5 Technical tools: continuity of curves of closed subspaces in Banach space and the interpolation property

The proof of our main theorem depends essentially on the following quite basic functional analytic tools.

Continuity of curves of closed subspaces in Banach space. We recall from our [12, Appendix A3], based on NEUBAUER's elementary, but deeply original [29]:

Proposition 6 *Let X be a Banach space and let $(M(b))_{b \in B}$, $(N(b))_{b \in B}$ be two continuous families of closed subspaces of X, where B is a parameter space. Assume that $M(b) + N(b)$ is closed for all $b \in B$. (a) Then $M(b) \cap N(b)$ is continuous if and only if $M(b) + N(b)$ is continuous.*
(b) Assume that $\dim(M(b) \cap N(b)) \equiv$ constant or $\dim(X/(M(b) + N(b)) \equiv$ constant, then the families $(M(b) \cap N(b))_{b \in B}$ and $(M(b) + N(b))_{b \in B}$ are continuous.

Interpolation of the Calderón projector between Sobolev spaces. Interpolation theory can be applied easily for intermediate Sobolev spaces between two given Sobolev spaces to establish estimates for the operator norm of an intermediate operator, see CALDERÓN's classic announcement [15, Theorem 2] of great generality, worked out in [16] and in [27, Theorem 5.1] by J.–L. LIONS and MAGENES. See also the more recent lecture notes [39, Definition 21.5 and Lemma 21.6] by TARTAR.

We give a slimmed-down version of interpolation theory for intermediate spaces, not striving for greatest generality.

Definition 2 (Interpolation property) Let $\mathbb{E}_0$ and $\mathbb{E}_1$ be normed spaces with $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$ continuously embedded and dense.
(a) An *intermediate space* between $\mathbb{E}_0$ and $\mathbb{E}_1$ is any normed space $\mathbb{E}$ such that $\mathbb{E}_1 \subset \mathbb{E} \subset \mathbb{E}_0$ (with continuous embeddings).
(b) An *interpolation space* between $\mathbb{E}_0$ and $\mathbb{E}_1$ is any intermediate space $\mathbb{E}$ such that every linear mapping from $\mathbb{E}_0$ into itself which is continuous from $\mathbb{E}_0$ into itself and from $\mathbb{E}_1$ into itself is automatically continuous from $\mathbb{E}$ into itself. It is said to be of *exponent* θ (with $0 < \theta < 1$), if there exists a constant c_1 such that one has

$$\|A\|_{\mathcal{B}(\mathbb{E},\mathbb{E})} \;\leq\; c_1 \, \|A\|^{1-\theta}_{\mathcal{B}(\mathbb{E}_0,\mathbb{E}_0)} \, \|A\|^{\theta}_{\mathcal{B}(\mathbb{E}_1,\mathbb{E}_1)} \quad \text{for all } A \in \mathcal{B}(\mathbb{E}_0,\mathbb{E}_0). \tag{39}$$

Here we denote the normed algebra of bounded endomorphisms of a normed space X by $\mathcal{B}(X,X)$.

For the chain of Sobolev spaces over our closed manifold Y and $s_0 < s_1$ we set $\mathbb{E}_0 := H^{s_0}(Y;E'^d)$ and $\mathbb{E}_1 := H^{s_1}(Y;E'^d)$. In our application at the end of the proof of our main theorem later below, we shall choose $s_0 := -\frac{d}{2}$ and $s_1 = \frac{d}{2}$. We exploit that the Sobolev spaces are Hilbert (or Hilbertable) spaces and admit a self–adjoint positive isometry $\Lambda\colon \mathbb{E}_1 \to \mathbb{E}_0$ which is a closed densely defined operator in $\mathbb{E}_0$ with domain $\mathcal{D}(\Lambda)$.

Proposition 7 (Interpolation between Sobolev spaces) *For each* $t \in]s_0, s_1[$ *the Sobolev space* $H^t(Y;E'^d)$ *is an interpolation space between* $\mathbb{E}_0 := H^{s_0}(Y;E'^d)$ *and* $\mathbb{E}_1 := H^{s_1}(Y;E'^d)$ *of exponent*

$$\theta(t) \;=\; \frac{t-s_0}{s_1-s_0}\,.$$

More precisely, we have for all $\theta \in]0,1[$ *and corresponding* $t \in]s_0,s_1[$*:*

1. *Identifying interpolation spaces between Sobolev spaces, [27, Definition 2.1 and Theorem 7.1]:* $H^t(Y;E'^d) \;=\; [\mathbb{E}_1,\mathbb{E}_0]_{1-\theta} \;:=\; \mathcal{D}(\Lambda^\theta)$ *with* $\|u\|_{[\mathbb{E}_1,\mathbb{E}_0]_{1-\theta}} \;:=\; \text{graph norm } (\Lambda^\theta u) \;=\; \left(\|u\|^2_{\mathbb{E}_0} + \|\Lambda^\theta u\|^2_{\mathbb{E}_0}\right)^{1/2}$.
2. *Interpolation property of Sobolev norms, [27, Proposition 2.3]: There exists a constant* c *such that* $\|u\|_{[\mathbb{E}_1,\mathbb{E}_0]_{1-\theta}} \;\leq\; c\,\|u\|^{\theta}_{\mathbb{E}_0}\,\|u\|^{1-\theta}_{\mathbb{E}_1}$ *for all* $u \in \mathbb{E}_1$.
3. *Interpolation theorem, [27, Theorem 5.1]: See Equation* (39)*.*
4. *Continuous parameter dependence: See Equation* (41)*.*

Remark 6 Statements (1), (2) are immediate from the definition of Sobolev spaces; for (2) see also [24, Theorem 7.22] with GRUBB's four–lines proof in the Euclidean case based on the Hölder Inequality. Statement (3) is deeper and its proof uses analytic functions with values in Banach spaces. Statement (4) can be derived from (3).

1.6 Predecessors and cross references

We sketch the origins and the establishment of the Calderón projector, its wide application fields, and the previous approaches and partial results regarding its dependence of a parameter.

Origins and making of the Calderón projector. In the early 1960s, Singer's and Atiyah's stooping success with the Index Theorem for elliptic operators (see, e.g. [2], [4], [23], and [32]) directed the attention of many analysts and geometers to *integro-differential* or *Calderón–Zygmund operators*, now well-established as *pseudodifferential operators*. The main interest was in the algebraic properties, providing a sufficiently large class of operators admitting parametrices (inverse operators modulo lower order or compact operators), continuous deformations and homotopy invariances, all mandatory for index calculations and Fredholm operator theory.

A second interest aroused from Calderón's surprising observation (see [14] and the beautiful exposition in [17, Section 4.1]) that for an elliptic differential operator over a smooth compact manifold with boundary, one can construct a pseudodifferential projection on the space of sections over the boundary such that the space of all traces at the boundary of the original operator's null space (its kernel) is the range of the projection. Later it was called the *Calderón projector* and its range the *Cauchy data space* of the given elliptic operator.

The first comprehensive proof of Calderón's observation was given by Seeley in [33, Theorem 2], [35], and [36]. It is based on the construction of an invertible double over the closed double of the original compact manifold with boundary with an inserted collar and with a careful analysis of the solution spaces close to the boundary. Seeley's approach was reproduced in many places, see, e.g., Grubb's textbook [24, Section 11.1]. Our [11, Chapters 8-9 and 11-13] with Wojciechowski give all details for Dirac type operators, where the construction becomes rather simple due to symmetry properties which, in particular, yield the unique continuation property of solutions of the homogenous equation $Pu = 0$.

Due to the many choices, Seeley's approach is not suitable for families of elliptic differential operators. In [8], jointly with Lesch, we worked out an alternative construction for elliptic operators of first order, based also on the concept of an invertible double, but without choices and therefore applicable to families. From that construction, however, we could not derive the continuous parameter dependence of the Calderón projector directly, but only via a demanding analysis of sectorial projections, see below our Paragraph on "Previous results on the continuity of families of Calderón projectors".

A third ingenious construction was given by Birman and Solomyak in [3], based on the concept of *elliptic towers* and slightly simplified for first order operators in [10, Section 5]. In its spirit, their way is rather axiomatic and highly promising. It is less explicit, though, and does not invite to estimates of norms for families of Calderón projectors.

In this paper we follow a fourth way, found by Frey in [21, Section 2.3]. It is inspired by a radically different approach due to Hörmander in [25, Theorem 20.1.3]. In its original version, it yielded only an approximative projection onto the Cauchy data spaces (i.e., modulo lower order operators, but with the correct principal symbol), while Frey's modification yields a *precise* Calderón projector, actually for linear elliptic differential operators of any order $d \geq 1$.

Application fields. The wide application of Calderón projectors and Cauchy data spaces is due to its role in the reduction of well-posed elliptic boundary value problems to an integro–differential problem over the boundary. Topologically, that is a kind of desuspension from a problem over a manifold with boundary to a corresponding problem over the boundary, i.e., a reduction of the dimension of a problem by one, at the cost of the higher complexity of the induced pseudodifferential operator over the boundary.

In that way, the Calderón projectors and their images, the Cauchy data spaces, play a role

1. for proving the delicate *Hirzebruch-Thom Cobordism Theorem* (decisive for the first proof of the Atiyah-Singer Index Theorem) in an elementary way, see [33, Theorem 3], [11, Chapter 21], and [8, Section 6] in chronological order;
2. for formulating and proving the *Bojarski Conjecture* for the index of elliptic differential operators over partitioned manifolds, [11, Chapter 24];
3. as projections onto *Lagrangian* subspaces in naturally *symplectic* section spaces over the boundary for elliptic and *symmetric* differential operators over manifolds with boundary. That interesting property of Cauchy data spaces was proved in our [10, Proposition 3.2], [11, Corollary 12.6] and further elaborated by McDuff and Salamon in [28, Exercise 2.17 = Exercise 2.1.16 in 3. ed.], in our [7, Proposition 3.5], and by Brüning and Lesch in [13];
4. as "reference points" in *Agranovič–Dynin type correction formulae* for the index, see [1], [11, Chapter 21], [38, Theorem 2] and Morchio's and Strocchi's view upon quantum field theory and *chiral anomaly* in our [9];
5. in yielding a Fredholm determinant as correction term in the *Scott–Wojciechowski Formula* of the ζ-regularized determinant of Dirac type operators over compact manifolds with boundary (see Scott's and Wojciechowski's [34] or our lecture notes [5]);
6. in the *Yoshida-Nicolaescu* (Morse=Maslov) *Formulas* expressing the *spectral flow* of a curve of self–adjoint elliptic boundary value problems by the *Maslov index* of a corresponding curve of Fredholm pairs of corresponding images of the Calderón projections and the projections defining the boundary conditions (for a review see our [12, Section 4.1]);
7. in inverse problems, e.g., in *Electrical Impedance Tomography* which consists in determining the electrical properties of a medium by making voltage and current measurements at the boundary of the medium. In the mathematical literature this is also known as *Calderón's problem* from Calderón's pioneer contribution [18], where he linked the study of the Cauchy data spaces to inverse problems, see Uhlmann [41] for a survey, also explaining the role of the Calderón projection and Cauchy data spaces for rigorous approaches to cloaking and invisibility.

Previous results on the continuity of families of Calderón projectors. Perturbations of the Calderón projector were investigated in the literature in two directions. In a series of papers, WOJCIECHOWSKI with collaborators determined the homotopy type of the *Grassmannian* of all pseudodifferential operators with the principal symbol of the Calderón projector in the symmetric and in the not necessarily symmetric case, explained in detail in [11, Chapter 15], see also PRESSLEY's and SEGAL's considerations in [37].

In this paper, we are addressing a different type of perturbation, to prove that the Calderón projectors and the Cauchy data spaces depend continuously on the underlying elliptic differential operator.[1]

- For operators of Dirac type, LIVIU NICOLAESCU solved that problem in [30,31].
- For perturbations by operators of lower order, the problem was solved in our [7], jointly with KENRO FURUTANI, and
- in a similar way for curves of elliptic differential operators with symmetric principal symbol with MATTHIAS LESCH in our beforementioned paper [8].
- From that paper, jointly with GUOYUAN CHEN, we could derive the continuous variation of the Calderón projector for arbitrary continuous curves of elliptic differential operators of first order in [6]. Alas, the proof was very lengthy.

The present proof is not only a generalization of our previous result to higher order operators, but first of all, we hope, a readable, comprehensible and useful presentation.

2 Proof of our main theorem

To prove Theorem 1, we shall establish the following partial results:

1. For all $b \in B$ and $s > \frac{1}{2}$, we obtain $\ker P_{s,m}(b) = \ker P_{\min}(b)$ by elliptic regularity in M°, where $P_{s,m}(b) := P_s(b)|_{H_o^{d+s}(M;E)}$.
2. The curve $\big(\ker P_{\min}(b)\big)_{b\in B}$ is continuous in $H^{d+s}(M;E)$.
3. The curve $\big(\widetilde{\rho}^d(\ker P_s(b))\big)_{b\in B}$ is continuous in $H^{d/2+s}(\Sigma;E'^d)$ for all $s \geq 0$.
4. For $s = -d$, we obtain that the curve $\big(\widetilde{\rho}^d(\ker P_{-d}(b))\big)_{b\in B}$ is continuous in $H^{-3d/2}(\Sigma;E'^d)$.
5. For the two cases (3) and (4) we have $\operatorname{im} C_s(b) = \widetilde{\rho}^d(\ker P_s(b))$.
6. Then, by interpolation, (3) and (4) imply that the curve $\big(\operatorname{im} C_s(b)\big)_{b\in B}$ is continuous in all intermediate Sobolev spaces.

Let us go to the technicalities.

Proof Given a continuous family of elliptic differential operators $P(b)$ of order d, that is,

$$P(b)\colon H^{d+s}(M,E) \to H^s(M,F),$$

[1] Apparently, for inverse problems like no. 7 in the preceding list, the continuity of the Calderón projector is less critical than the quite different problem of the stability of the reconstruction procedure.

such that the closed set

$$\operatorname{Gr} P(b) \subset H^{d+s}(M, E) \times H^s(M, F)$$

is continuous with respect to $b \in B$, when $s \geq 0$.

$H^d(M; E) \subset \mathscr{D}(P_{\max}(b)) \subset L^2(M; E)$.

Case 1 When $s \geq 0$, by Corollary 2, we have

$$\operatorname{im} C(b) = \tilde{\rho}^d(\ker P(b)). \tag{40}$$

$$H^{d+s}(M; E) \subset H^d(M; E) \subset \mathscr{D}(P_{\max}(b)) \subset L^2(M; E),$$

and we consider linear bounded operators

$$P(b) \colon H^{d+s}(M; E) \to L^2(M; F),$$

$$\ker P = \{u | Pu = 0\} \cong \ker P \times \{0\} = \operatorname{Gr} P \cap (H^{d+s}(M; E) \times \{0\}).$$
$$\operatorname{Gr}(P(b)) + H^{d+s}(M; E) \times \{0\} = H^{d+s}(M; E) \times \operatorname{im} P(b),$$

$$\begin{aligned} H^{d+s}(M; E) \times L^2(M; F)/H^{d+s}(M; E) \times \operatorname{im} P(b) &\cong L^2(M; F)/\operatorname{im} P(b) \\ &\cong \operatorname{im} P(b)^{\perp} \\ &= \ker P^*(b) = \ker P^t_{\min}(b). \end{aligned}$$

So by Proposition 6b, when $\dim \ker P^t_{\min}(b) =$ constant, $\ker P(b)$ is continuous about $b \in I$.

Since

$$\ker(\widetilde{\rho}^d \colon \ker P(b) \to \widetilde{\rho}^d(\ker P(b))) = \ker P_{\min}(b),$$

so when $\dim \ker P_{\min}(b) =$ constant, by [12, Corollary A.3.15] and the continuity property of $\ker P(b)$, we have $\tilde{\rho}^d(\ker P(b))$ is continuous. By (40), we have the continuity property of $\operatorname{im} C(b)$ about $b \in I$.

Case 2 When $-d \leq s \leq 0$, we use interpolation theory.

$$P(b) \colon H^{d+s}(M; E) \cap \mathscr{D}(P_{\max}(b)) \to L^2(M; F).$$

When $s = -d$, we claim that,

$$P(b) \colon \mathscr{D}(P_{\max}(b)) \to L^2(M; F)$$

is a continuous family of closed operators, i.e., the family of closed subspaces

$$\operatorname{Gr}(P_{\max}(b)) \subseteq L^2(M; E) \times L^2(M; F),$$

is continuous with respect to $b \in B$.

In fact, $P_{\min}(b) \colon H^d_0(M; E) \to L^2(M; F)$ is a bounded linear operator and continuous with respect to $b \in B$. As an unbounded closed operator,

$$P_{\min}(b) \colon L^2(M; E) \supseteq H^d_0(M; E) \to L^2(M; F),$$

the graph norm of $P_{\min}(b)$ is continuous with respect to $b \in B$, that is

$$\{(u, P(b)u) | u \in H_0^d(M;E)\} \subseteq L^2(M;E) \times L^2(M;F)$$

is continuous with respect to $b \in B$. Since $P_{\max} = (P^t_{\min})^*$, $\operatorname{Gr} P_{\max} = (J \operatorname{Gr} P^t_{min})^{\perp}$, where $J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ is a strong symplectic structure, we may claim

$$\begin{aligned} \ker P_{\max} &= \{u \in \mathscr{D}(P_{\max}) \in | Pu = 0\} \\ &\cong \ker P_{\max} \times \{0\} = \operatorname{Gr} P_{\max} \cap (L^2(M;E) \times \{0\}). \end{aligned}$$

$$\operatorname{Gr}(P_{\max}) + L^2(M;E) \times \{0\} = L^2(M;E) \times \operatorname{im} P_{\max},$$

$$L^2(M;E) \times L^2(M;F) / L^2(M;E) \times \operatorname{im} P_{\max} \cong \operatorname{im} P_{\max}^{\perp} = \ker P^*_{\max} = \ker P^t_{\min}.$$

By the assumption $\dim \ker P^t_{\min} = \text{constant}$ and Proposition 6b, we have that $\ker P(b)$ is continuous with respect to b, then $\operatorname{im} C(b) = \widetilde{\rho}^d(\ker P(b))$ is continuous with respect to b.

Since

$$\ker(\widetilde{\rho}^d \colon \ker P(b) \to \widetilde{\rho}^d(\ker P(b))) = \ker P_{\min}(b),$$

when $\dim \ker P_{\min}(b) = \text{constant}$, by [12, Corollary A.3.15] and the continuity of $\ker P(b)$, we have $\widetilde{\rho}^d(\ker P(b))$ is continuous. Then when $s = -d$, $\operatorname{im} C(b) = \widetilde{\rho}^d(\ker P(b))$ is continuous with respect to b.

For the Calderón projection C, we have the gap

$$\hat{\delta}(\operatorname{im}(C(b_1)), \operatorname{im}(C(b_2))) = \|C(b_1) - C(b_2)\|,$$

where $C(b_1)$, $C(b_2)$ are the orthogonal projections on $\operatorname{im}(C(b_1))$, $\operatorname{im}(C(b_2))$, respectively. (See [26, §IV.2.1] for the definition of $\hat{\delta}$ and the above relation.) So $\operatorname{im}(C(b))$ is continuous if and only if $C(b)$ is continuous with respect to b.

Fix parameter $b \in B$, $C(b) \colon H^t(\Sigma, E'^d) \to H^t(\Sigma, E'^d)$ is a bounded linear operator, we denote $C_t = C|_{H^t(\Sigma, E'^d)}$.

We have proved that

$$C(b) \colon H^{\frac{d}{2}}(\Sigma, E'^d) \to H^{\frac{d}{2}}(\Sigma, E'^d)$$

and

$$C(b) \colon H^{-\frac{d}{2}}(\Sigma, E'^d) \to H^{-\frac{d}{2}}(\Sigma, E'^d)$$

are continuous with respect to $b \in B$. By interpolation theory, summarized above in the second paragraph of Subsection 1.5,

$$[H^{-\frac{d}{2}}(\Sigma, E'^d), H^{\frac{d}{2}}(\Sigma, E'^d)]_\theta = H^{-\frac{d}{2} + \theta d}(\Sigma, E'^d),$$

for $0 < \theta < 1$, and we obtain

$$\|C_t(b_1) - C_t(b_2)\| \leq c_2 \|C_{s_0}(b_1) - C_{s_0}(b_2)\|^{\frac{s_1 - t}{s_1 - s_0}} \|C_{s_1}(b_1) - C_{s_1}(b_2)\|^{\frac{t - s_0}{s_1 - s_0}}, \quad (41)$$

where $s_0 \leq t \leq s_1$. We use the situation $s_0 = -\frac{d}{2}$, $s_1 = \frac{d}{2}$.

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