The Maslov index in symplectic Banach spaces

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Foundational and explorative aspects of mathematical research

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Data (Vocabulary)

- X complex Banach space
- $\omega\colon X \times X \longrightarrow \mathbb{C}$ symplectic form on X, i.e., sesquilinear, skew-symmetric, non-degenerate, and bounded
- $L \subset X$ Lagrangian subspace $\iff L^\omega = L$
- (L, M) Fredholm pair of Lagrangian subspaces
- Curve $(L(s), M(s))_{s\in[0,1]}$ in Fredholm Lagrangian Grassmannian $\mathcal{FL}(X)$

Example (Classical mechanics)

$\mathcal{H} : f \mapsto -j\frac{d}{dt}f + bf - \frac{1}{2}(\frac{d}{dt}j)f$ first order linear Hamiltonian system of rank m, $j^*(t) = -j(t)$, $b^*(t) = b(t)$, $t \in [0, T]$
X:=$C^m \oplus \mathbb{C}^m$ initial and terminal values
$$\omega(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}) := \left\langle \begin{pmatrix} -j(0) & 0 \\ 0 & j(T) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\rangle$$
W boundary condition, $\mathcal{G}(\Gamma(T))$ Lagrangian $\subset X$, Γ fundamental sol.

Example (PDEs)

M manifold with boundary Σ, A symm. elliptic differential operator
$X := L^2(\Sigma)$ sections over Σ symplectic Hilbert space
$\gamma(D)$ b.c. and $\gamma(\ker A)$ Lagrangian subspaces

Theorem (Goals and main results)

(i) The Maslov index $\mathrm{MAS}\colon \mathcal{C}\left(\mathcal{FL}(X)\right) \to \mathbb{Z}$ *is well-defined in symplectic Banach space X. It satisfies all natural properties.*
(ii) The spectral flow formula holds. Data:

- $(A(s))_{s\in[a,b]}$ *path of symmetric elliptic differential operators on M*
- $\dim\{\ker A(s)$ *with zero trace*$\}$ *constant*
- *$D(s)$ path of well-posed self-adjoint b.c.*
- *Then we have* $\mathrm{SF}\{A(s)|_{D(s)}\} = -\,\mathrm{MAS}\{\gamma(D(s)), \gamma(\ker A(s))\}$