The Maslov index in symplectic Banach spaces

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Data (Vocabulary)

a) X complex vector space. (X, ω) **symplectic vector space** $:\Longleftrightarrow$ $\omega\colon X \times X \longrightarrow \mathbb{C}$ **symplectic form** on X, i.e., sesquilinear, skew-symmetric, and non-degenerate.

Natural concepts: λ^ω **annihilator, isotropic, co-isotropic, Lagrangian, Fredholm pair** for linear subspace $\lambda \subset X$ or pair (λ, μ).

b) $(X, |\cdot|)$ Banach space. $(X, |\cdot|, \omega)$ **(weak) symplectic Banach space** $:\Longleftrightarrow$ (X, ω) sympl. and ω bounded.

Natural concept: Curves in **Fredholm Lagrangian Grassmannian** $\mathcal{FL}(X)$.

Examples (Ode's and pde's)

Finite dim.: Initial and terminal values for first order linear Hamiltonian systems of rank m in $\mathbb{C}^m \oplus \mathbb{C}^m$ and correspondingly for regular linear (second order) Lagrangian systems of ode's.

Infinite dim.: Traces of solution space of symm. elliptic operator A over manifold M on hypersurfaces or boundary $\Sigma \subset M$ in $L^2(\Sigma)$ or, e.g. for first order ops., in $H^{1/2}(\Sigma)$. *Universal*: $\operatorname{dom}(A_{\max})/\operatorname{dom}(A_{\min})$.

Theorem (Goals and main results)

(i) The ***Maslov index*** $\mathrm{MAS}\colon \mathcal{FL}(X) \to \mathbb{Z}$ *is well-defined in symplectic Banach space* X*. It satisfies all natural properties.*

(ii) The ***spectral flow formula*** *holds. Data:*

- $(A(s))_{s\in[a,b]}$ *path of symmetric elliptic differential operators of order* $d > 0$ *on a compact smooth manifold with boundary.*
- *Assume that the dimension of solutions in* $\ker A(s)$ *with zero trace is constant.*
- $D(s)$ *path of well-posed self-adjoint boundary value conditions.*
- $\gamma(\ker A(s))$ *denotes the trace of* $\ker A(s)$*.*
- *Then we have*

$$\mathrm{SF}\{A(s)|_{D(s)}\} = -\,\mathrm{MAS}\{\gamma(D(s)), \gamma(\ker A(s))\}.$$

History (Pecularities)

(i) V.P. Maslov 1965, V. Arnold 1967: MAS non-trivial in finite dim.

(ii) A. Floer 1983: Morse index formula, revisited.

Challenges (Bad news in weak symplectic case)

1. $V \subset X$ proper closed linear subspace $\not\Longrightarrow V^{\omega\omega} = V$.
2. $(\lambda, \mu) \in \mathcal{FL}(X) \not\Longrightarrow \operatorname{ind}(\lambda, \mu) = 0$.
3. No Maslov cycle in general.
4. No Darboux's theorem to fix symplectic structure.
5. Difficult to prove $\pi_1(\mathcal{FL}(X)) \cong \mathbb{Z}$.
6. Bounded injective J defining $\omega(x, y) = J(x)(y)$ **not** invertible, i.g.
7. I. g., in weak symplectic Hilbert space **no** $J' \sim J$ s.t. $J'^2 = -I$.
8. $\exists$ strong symplectic Banach space (e.g., $X := \ell^p \oplus (\ell^p)^*$, $p \neq 2$) without symplectic splitting, i.e., **no** decomposition $X = X^+ \oplus X^-$ such that $\mp i\omega|_{X^\pm} > 0$, and $\omega(x, y) = 0$ for all $x \in X^+$ and $y \in X^-$.
9. No Maslov index via spectral flow of unitary generators.

Lemma (Good news — almost magical, purely algebraic technicalities)

a) [Annihilator confinement]
$\lambda \subset X$ and $\operatorname{codim}(\lambda) < \infty \Longrightarrow \dim \lambda^\omega \leq \dim X/\lambda$. *The equality holds if and only if* $\lambda = \lambda^{\omega\omega}$.

b) [Fredholm index never positive] $(\lambda, \mu) \in \mathcal{FL}(X)$ *and* λ, μ *isotropic* $\Longrightarrow \operatorname{ind}(\lambda, \mu) \leq 0$.

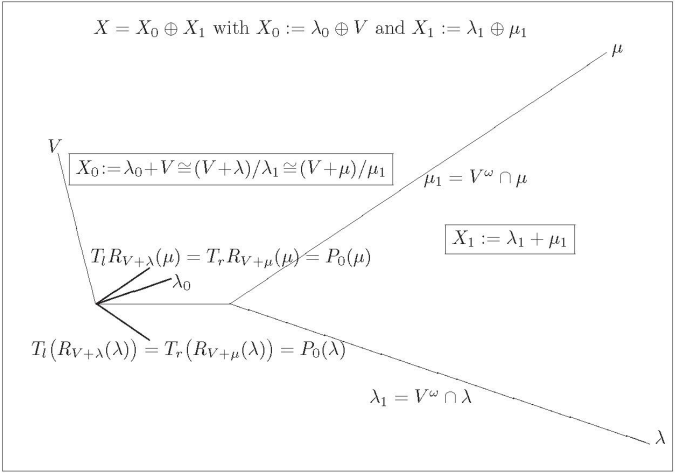
c) [Isotropic $\Rightarrow$ *Lagrangian under special circumstances] Like b) and* $\operatorname{ind}(\lambda, \mu) = 0$. *Then* λ, μ *Lagrangian.*

d) [Forced equalities, substitute for non-valid $V^{\omega\omega} = V$*]* $X_0, X_1 \subset X$ *with* $X = X_0 + X_1$ *and* $X_0 \subset X_1^\omega$
$\Longrightarrow X_0 = X_1^\omega$, $X_1 = X_0^\omega$, $X = X_0 \oplus X_1$, *and* X_0, X_1 *are symplectic.*

e) [Key observation] Pair of co-isotropic subspaces $(\lambda, \mu) \mapsto V, \lambda_0 \subset X$ *with* $X = V \oplus (\lambda + \mu)$ *and* $\lambda_0 := \lambda^\omega \cap \mu^\omega$. *Assume* $\dim \lambda_0 = \dim X/(\lambda + \mu) < \infty$.
$\Longrightarrow$ *Natural decomposition into sum of two symplectic spaces*
$X = X_0 \oplus X_1$, $X_0 := V + \lambda_0$, $\dim X_0 < \infty$, $X_1 := (V^\omega \cap \lambda) + (V^\omega \cap \mu)$.

Symplectic reduction to finite dimension

$W := V + \lambda$ closed co-isotropic; admits symplectic reduction $R_W(\lambda) := \big(\lambda + W^\omega) \cap W\big)/W^\omega \in \mathcal{L}(W/W^\omega)$:



Recently L. Wu and C. Zhu without symplectic reduction.

- X Banach; B top. space; $M, N \colon B \to \mathcal{S}(X) \in C^0$. Then $M + N \in C^0 \iff M \cap N \in C^0$.
- Continuous variation of basic op. & cont. var. of domain $\Longrightarrow$ cont. var. of *boundary value problem*: