Geometric analysis and spectral synthesis

Case study of explorative paths in mathematical research: The Maslov index in symplectic Banach spaces

Mem. Amer. Math. Soc. no. 1201 **252**/2 (March 2018), 1-118

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RUC, IMFUFA seminar, 25 April, 2018

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Outline

1. Goals and the mathematics-historical background
   - Geometric analysis
   - Decomposition of space/manifolds/surfaces & recomposing
   - Perturbation theory
   - Spectral spectacles
2. Elementary examples of underlying concepts
   - The 1D set–up: first order linear Hamiltonian system
   - Linear function spaces, subspaces, and operators
3. Recent main results and highlights
   - Counterexamples v. working knowledge
   - General spectral flow formula
   - Maslov index: finite or ∞-dimensional?
   - Variational results
4. Epistemological conclusions
   - Say something typical about mathematical research

Key experience 1, Index: $A : X \longrightarrow Y$ linear, X, Y finite-dimensional linear spaces.

Definition (Index)

$\operatorname{ind} A := \dim\ker A - \operatorname{codim}\operatorname{im} A$.

Theorem (Fundamental Theorem of Linear Algebra)

$\dim\ker A + \dim\operatorname{im} A = \dim X$, *i.e.,* $\operatorname{ind} A = \dim X - \dim Y$.

Meaning: $X = Y$ Euclidean or Hermitian, $T \in \operatorname{End}(X)$. Then *symmetry* $\dim\ker A^* = \dim\ker A$, i.e., $\mu = \mu^*$ multiplicity of 0-eigenvalue (zero mode).

Symmetry breaks down in ∞-dim. spaces. Ex: shift operator; integral equations; certain partial diff. eqs.: Fredholm operators. *Index has geometric meaning*!

Key experience 2, Principal symbol:

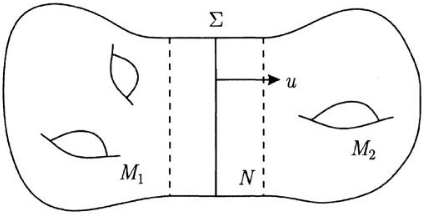
- ODEs $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $t \in \mathbb{R}$, HILBERT V. HEISENBERG
- PDEs $A(x)(p) = \sum_{\|\alpha\| \leq k} a_\alpha(p) D^\alpha(x)|_p$, $p \in \mathbb{R}^n$

- L. EULER, H. POINCARÉ, E. NOETHER: Triangulation, simplicial homology, chain of groups
- P. HEEGAARD, F. WALDHAUSEN: Splitting; skeleton decomposition and reconstruction by gluing; suspension and desuspension
- J. MILNOR ET AL.: Cutting and pasting; connected sum $X \# Y$; surgery $\partial(D^{p+1} \times S^{q-1}) \sim S^p \times S^{q-1} \sim \partial(S^p \times D^q)$
- S. NOVIKOV: *Exact additivity* of Euler characteristic and Hirzebruch signature

B^3 AND K.P. WOJCIECHOWSKI: *Error term* for addition of certain other invariants under decomposition

Decomposition of space, manifolds, surfaces II

$M = M_1 \cup_\Sigma M_2$ closed partitioned Riemannian manifold, $E \to M$ Clifford bundle, $A\colon C^\infty(M;E) \to C^\infty(M;E)$ compatible total Dirac type operator with chiral components $A = \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix}$.



CD *Cauchy data spaces* = traces at boundary/hypersurface of kernel of linear elliptic operator of first order

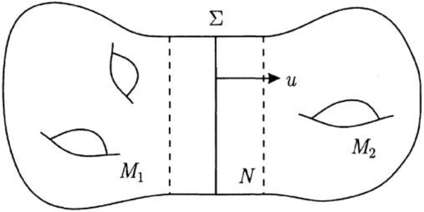
Then $A_+ : C^\infty(M;E_+) \to C^\infty(M;E_-)$ Fredholm operator and $\bigl(\mathrm{CD}(A_+|_{M_1}), \mathrm{CD}(A_+|_{M_2})\bigr)$ Fredholm pair, and

Theorem (Bojarski Induction Conjecture)

$\mathrm{ind}(A_+) = \mathrm{ind}\bigl(\mathrm{CD}(A_+|_{M_1}), \mathrm{CD}(A_+|_{M_2})\bigr)$ *[*BBB & WOJCIECHOWSKI *1986]*

Perturbation result

$M = M_1 \cup_\Sigma M_2$ closed partitioned Riemannian manifold, $E \to M$ Clifford bundle, $\left(A_s\colon C^\infty(M;E) \to C^\infty(M;E)\right)_{s\in[0,1]}$ curve of compatible total Dirac type operators (so, self-adjoint).



A. FLOER

Then all $\left(\mathrm{CD}(A_s|_{M_1}), \mathrm{CD}(A_s|_{M_2})\right)$ Fredholm pairs of *Lagrangian subspaces*, and

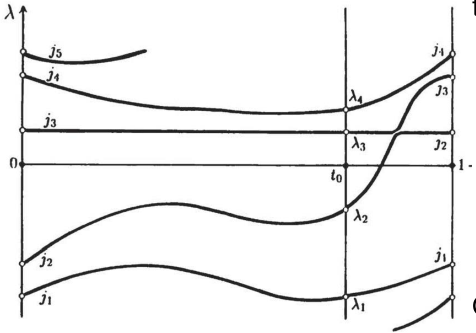
Theorem (A. FLOER, T. YOSHIDA, L. NICOLAESCU)

$\mathrm{SF}(A_s)_{s\in[0,1]} = \mathrm{MAS}\left(\mathrm{CD}(A_s|_{M_1}), \mathrm{CD}(A_s|_{M_2})\right)_{s\in[0,1]}$ *[*NICOLAESCU *1995]*

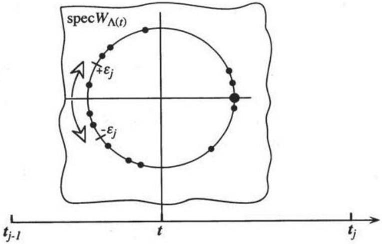
Challenge: Investigate the validity!

Spectral flow and Maslov index: rough idea

Roughly speaking the spectral flow $\mathrm{SF}\{A_t\}_{0\le t\le 1}$ is the net number of eigenvalues of A_t that pass from $-$ to $+$ while the parameter t of the family is running from 0 to 1.



The Maslov index $\mathrm{MAS}\{\mathrm{CD}(A_t|_{M_1}), \mathrm{CD}(A_t|_{M_2})\}_{0\le t\le 1}$ is the net number of eigenvalues of $W_t := U_t V_t^{-1}$ of the unitary generators U_t, V_t of the Lagrangians that pass through -1 from above.



OBS: $\forall \Lambda_t$ Lagrangian $\dot{\exists} U_t$ unitary s.t. $\Lambda_t = \mathcal{G}(U_t)$ in suitable splitting of $L^2(\Sigma)$

For loops: SF ∞-dimensional; MAS non-trivial even in finite dimension!

Spectral spectacles

Spectral synthesis: Mostly we sense/measure eigenvalues. Task: reconstruction of source!

- *Linear algebra*: X complex Euclidean space, $\dim X = n$, $A \in \mathrm{End}(X)$ self-adjoint. Then $\mathrm{spec}(A) \subset \mathbb{R}$ and $\exists e_1, \dots, e_n$ orthonormal basis of X of eigenvectors.
  So, $\mathrm{spec}(A) \mapsto \sim\ A$.
- *Can you hear the shape of a drum*? [M. KAC, 1966]
  Size yes, else not always.
- *Many components* of $\mathrm{spec}(A)$: discrete; continuous; residual $: \iff \lambda Id - A$ not injective; injective, dense range, but not surjective; else.
- X complex separable Hilbert space, $A \in \mathrm{End}(X)$ *closed, self-adjoint, Fredholm*. Then $\mathrm{spec}(A)$ discrete close to 0.
- *Pragmatic focus*: ind and SF here, else det...

1D set-up: First order linear Hamiltonian system

Data: Time $t \in [0, T]$, variational parameter $s \in [0, 1]$, *Hamiltonian*

$$(A_s x)(t) := -j_{s,t}\frac{d}{dt}x(t) - b_{s,t}x(t) - \frac{1}{2}\left(\frac{d}{dt}j_{s,t}\right)x(t)$$

with $x \in H^1([0, T], \mathbb{C}^m) = \overline{C^1([0, T], \mathbb{C}^m)}$, $j_{s,t}, b_{s,t} \in C^0\big([0, 1] \times [0, T], \mathrm{gl}(m, \mathbb{C})\big)$, $j^*_{s,t} = -j_{s,t}$ invertible, $b^*_{s,t} = b_{s,t}$, $j_{s,t} \in C^1$ in t for fixed s.

Boundary conditions: W_s continuous curve of subspaces of $\mathbb{C}^m \oplus \mathbb{C}^m$.

Symplectic structure: For all $(x_1, x_2), (y_1, y_2) \in \mathbb{C}^{2m}$ set $\omega : \mathbb{C}^{2m} \times \mathbb{C}^{2m} \to \mathbb{C}$: $\omega(\cdots) := -\langle j_{s,0}x_1, y_1\rangle + \langle j_{s,T}x_2, y_2\rangle$. OBS: ω is *sesquilinear*, *bounded*, *skew-symmetric*, and *non-degenerate*. Then

1. The graphs $\{\mathcal{G}(\Gamma_s(T))\}$ of the *fundamental solutions* $\{\Gamma_s(T)\}$ form a curve of *Lagrangian subspaces* of $\mathbb{C}^{2m}$.
2. Let $\{W_s\}$ be a curve of Lagrangian subspaces. Then the boundary value problems $\{A_{s,W_s}\}$ form a curve of *closed self-adjoint Fredholm operators* in $L^2([0, T], \mathbb{C}^m)$, and
3. We have $\mathrm{SF}\{A_{s,W_s}\} = \mathrm{MAS}\{\mathcal{G}(\Gamma_s(T)), W_s\}$.

Linear function spaces, subspaces, and operators

Some distinctions and clarifications:

- Linear spaces, Normed spaces, Banach spaces, separable Hilbert spaces, Euclidean spaces.
- L^2-spaces, Sobolev spaces, Quotient spaces: $\mathcal{D}_{\max}/\mathcal{D}_{\min}$
- Closed subspaces, Fredholm pairs of closed subspaces
- Linear operators, Bounded operators, Closed operators, Fredholm operators, Symmetric operators, Self-adjoint operators
- Differential and integral operators: $A := \frac{d}{dt}$ not continuous in $L^2([-\pi,\pi])$: $\|A(f_n)\| = n\|f_n\| = n$ for $f_n(t) := \frac{1}{2\pi}e^{int}$ with $\|f_n\| = 1$ for all f_n, $n = 1, 2, \cdots$.
- Strong and weak symplectic spaces:

  Weak: $\omega(x,y) = (Jx, y)$, $J : X \to X'$ linear injective.
  Strong: $\iff$ J invertible (if X Hilbert space, then strong $\iff$ $J^2 = -\mathrm{Id}$)

Counterexamples v. working knowledge

Data: $A(s)\colon C_0^\infty(M;E) \to C_0^\infty(M;E)$, $s \in [0,1]$ curve of symmetric elliptic first order differential operators, M compact manifold, $\partial M = \Sigma$.
What fixed? $H^1(M;E)$ and $H^{1/2}(\Sigma; E|_\Sigma) \cong H^1(M;E)/H_0^1(M;E)$.

On $L^2(\Sigma; E|_\Sigma)$ strong $\omega(s)_{\mathrm{Green}}(x,y) := -\langle J(s)x, y\rangle_{L^2}$.
On $H^{1/2}(\Sigma; E|_\Sigma)$ induced weak $\omega(s)(x,y) := \omega(s)_{\mathrm{Green}}(x,y)$
$= -\langle J'(s)x, y\rangle_{H^{1/2}}$ with compact $J'(s) = (\mathrm{Id} + |B|)^{-1/2} J(s)$, B formally self-adjoint elliptic of first order on Σ.

Obstructions:

- $J'(s)^2 \neq -\mathrm{Id}$, so $H^{1/2} \neq \ker(J'(s) - i\,\mathrm{Id}) \oplus \ker(J'(s) + i\,\mathrm{Id})$.
- In general, $\lambda^{\omega(s)\omega(s)} \supset\neq \lambda$ for closed linear subspace λ.
- I.g., $\operatorname{ind}(\lambda,\mu) \neq 0$ for $(\lambda,\mu) \in \mathcal{FL}$.
- I.g., $\mathcal{L}$ not contractible and $\pi_1(\mathcal{FL}_0(X,\lambda)) \stackrel{?}{=} \mathbb{Z}$ for $\lambda \in \mathcal{L}$; valid for strong symplectic Hilbert space.

A naturally looking General spectral flow formula

Theorem (BBB, C. ZHU, *Memoirs Am. Math. Soc.* 2018)

$\mathrm{SF}\{A(s)_{\mathcal{D}(s)}\}_{0\le s\le 1} = \mathrm{MAS}\{\mathrm{CD}\big(A(s)\big), \gamma\big(\mathcal{D}(s)\big)\}_{0\le s\le 1}$, *admitting*

- *smooth variation of operator $A(s)$ and*
- *continuous variation of Fredholm domain $\mathcal{D}(s)$, and*
- *demanding constant «ghosts'» dimensions (or weak inner UCP).*

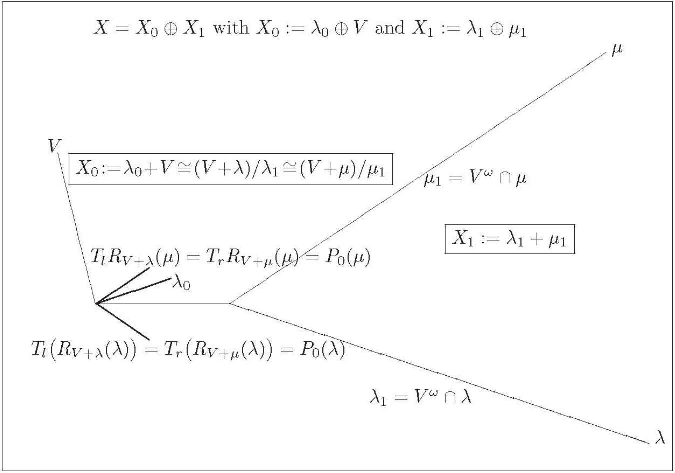
Here $\gamma\colon \mathcal{D}_{\max} \longrightarrow \mathcal{D}_{\max}/\mathcal{D}_{\min}$ natural projection and $\gamma(D) := \big(D\cap\mathcal{D}_{\max} + \mathcal{D}_{\min}\big)/\mathcal{D}_{\min}$ for any linear subspace $D \subset L^2(M;E)$.

NOTE 1: Similarly, general validity of Nicolaescu splitting theorem for weak inner UCP.

NOTE 2: Result purely *foundational*: fulfilling clear concrete vision of 1993. Proof (via several surprising key lemmata) typically *explorative*: tools unexpected!

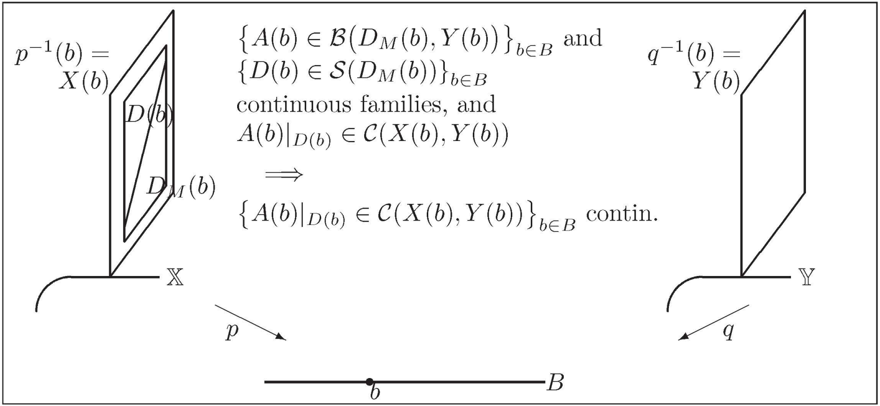
Symplectic reduction to finite dimension

(X,ω) symplectic vector space, $(\lambda,\mu)\in\mathcal{FL}_0(X,\omega)\implies$
Natural decomposition $X=X_0\oplus X_1$, $X_0:=V+\lambda_0$, $\dim X_0<\infty$, $W:=V+\lambda$ closed co-isotropic; admits symplectic reduction $R_W(\lambda):=(\lambda+W^\omega)\cap W)/W^\omega\in\mathcal{L}(W/W^\omega)$:



Variational results, finally confirming expectation

- X Banach; B top. space; $M, N \colon B \to \mathcal{S}(X) \in C^0$. Then $M + N \in C^0 \iff M \cap N \in C^0$.
- Continuous variation of basic op. & cont. var. of domain $\Longrightarrow$ cont. var. of *boundary value problem*:



Something typical about mathematical research I

1. Mathematical research is*a riddle wrapped in a mystery inside an enigma*, paraphrasing W. CHURCHILL, 1 Oct. 1939, on Russia
   - Similar M. PROUST on the search of a happy life
   - *Rational choices and conscious forming of a path* more volatile than *obeying of and adapting to hard laws of reality*
2. Views upon the dark
   - F. DYSON: {Math $\Rightarrow$ concentration $\Rightarrow$ being absorbed} $\Longrightarrow$ {*Theorizing* of math. research meaningless}.
   - J. TITS: We are tapping at the walls in a *dark room* and then we break through.
   - R. KADISON: Watching a *pub brawl*, we get into the fight, hit and support, get some blows, but seek to stay with the winners.
   - K.P. WOJCIECHOWSKI: We are not scavengers; we rather are *tigers*, mostly sleepy or striving around and exploring our territory; and then we jump and hit.
   - R. SEELEY: There are only a few *designers*; all we others are like *plumbers*.

Something typical about mathematical research II

***Perhaps there is a key* — Crossroads, distinctions, and interaction**

3. H. WEYL: *In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics*.
    - Structure, arithmetic, algebra v. geometry
    - H. RADEMACHER and O. TOEPLITZ: *Von Zahlen und Figuren*
    - Y. MANIN: *Von Zahlen und Figuren, 2002*
4. Semiotic:
    - Syntax v. semantic
    - D. HOFSTADTER: *Gödel, Escher, Bach: An Eternal Golden Braid*
5. M. OTTE: Foundational v. explorative
    - Clarification of *concepts* v. focus on *counter-intuitive*
    - Reflecting: some *metaphysical reality* (unity and order) v. *possibilities* of human activity
    - Antiquity v. modernity (renaissance)