Mathematics: Easy and *Hard. Why?*

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The Viewpoint *column offers readers of* The Mathematical Intelligencer *the opportunity to write about any issue of interest to the international mathematical community. Disagreement and controversy are welcome. The views and opinions expressed here, however, are exclusively those of the author, The publisher and editor-in-chief do not endorse them or accept responsibility for them. Articles for* Viewpoint *should be submitted to the Editor-in-Chief, Marjorie Senechal.*

Since the beginning of the economic crisis in 2008, and perhaps even earlier, student enrollment in mathematics and in mathematically heavily loaded subjects has shown continuing growth. Students who study these subjects expect to experience better chances in the job market because they have chosen subjects that are considered hard and where the money is. Most of these students were good at mathematics in high-school and they liked it. Many of them, however, were surprised at how difficult it was to proceed and to master new abstract ideas every week. Consequently, there are an increasing number of students who do not do what we want them to. In various places, both in Demark and abroad, I have seen how the *All Administrative University* (Ginsberg[1]) has increased the pressure to reduce the curricula in mathematics and to allow students to pass with a modest schematic training in some tools and without an introduction into a mathematician's way of thinking. I consider that a great folly. My suspicion is that many administrators simply have too little appreciation of a good mathematician's strong capacity for abstract thinking, and that in many job functions this capacity is more important than the superficial acquaintance with a finite number of tools embedded in fixed applications. Most mathematicians will probably agree with that. But even so, many of us succumb to the pressure. Moreover, it is a bit painful to see how difficult mathematical thinking is for many students. Here it may help to recall the working experience of research mathematicians who know, when honest, the hardness of mathematical work very well from their own struggles and frustrations, but they know also the gratifying feelings experienced when one has achieved something and understood, and finally all appears easy and natural.

Preface

When talking to students, colleagues, and administrators, we often deal with a wonderful mix of contrasting conundrums. In mathematics, we see the contrast of easy and hard, but also the visible and invisible, teacher and students, understanding versus proof, publication versus explanation, results versus processes, mathematical thinking versus mathematical tools, inspiring versus misleading, useful versus destructive, free and under restraint, and abstraction versus context. We see it and understand it based on our daily work. Others, the novices and the outsiders, see it as well, but do they understand it?

In my view, understanding of the two faces of mathematics, being easy and hard at the same time, may prevent some misunderstanding among our students, collaborators from other branches, and administrators. To upgrade the

Dedication In respectful memory of Ivor Grattan-Guinness (23 June 1941 to 12 December 2014).

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argumentation, I draw on the working experience of selected outstanding mathematicians by using the *trial court* format: to present strong evidence for one point of view (mathematics is hard) and then to counter it with evidence of the opposite.

Many of my witnesses are retired or passed away, some of them centuries ago. Is their testimony still valid when administrators[1] ask us to tune our teaching to the modern zap generation? Haven't the students changed, for example, since I was a beginner?

After a short autobiographical introduction, I address the question of the alleged change of mind in Part I before proceeding to my *trial court* in Part II. So, in Part I, I'm looking back to see whether the students' mind-set has changed. In Part II, I summarize why mathematics is, rightly, perceived as hard. After that I shall turn to the opposite position, that mathematics is easy and that it is a pity when mathematics is not perceived as easily accessible.

Autobiographical Introduction

The *logo* for this article (Figure 1) is from the front page of my latest monograph on the index theory of M. F. Atiyah and I. M. Singer,[2] which started with lectures I gave in 1971 in Allende's Chile. The President had given orders that there should be Further Education for all junior and senior high-school teachers in the country to mark the new era. So, some Chilean mathematicians invited me after I had recently finished a Ph.D. on that subject in Bonn. Perhaps overly optimistic, they asked me to give a series of elementary lectures to their teachers so that everybody could understand what modern mathematics was about and what the content of the Atiyah-Singer Index Theorem was. While I was giving these lectures, someone took notes, so that I came back to Europe in September 1971 with a 120-page manuscript. This was my first attempt at making a hard topic easy.

These 120 pages turned into a book of 769 pages in 2013. The publisher chose as the front page the famous Escher graphic of knights walking around a never-ending ascending path that never reaches higher levels: a good symbol of mathematical studies, easy and hard at the same time. You walk and walk, and you think you have made no progress. And yet, after a while, you think it's easy; you are on the same level now, with a better understanding, but you feel that that little was gained. This is probably the ambiguity of all learning, but it is especially so for mathematics. That is the *topic* of this paper.

General Meaning of Mathematical Working Experience

Here is an indication of why mathematical working experience has something to say to the general intellectual public.

Pulls and Pushes

Sometimes we who teach mathematics are told: *Make it easy* and *Don't lose a student!* However, our working experience shows that:

- mathematics is confusing and damned hard as long as one has not understood it, and when one has understood it, it is easy and clear; and
- mathematics is invisible for the students in their environment, unless they look a bit beneath the surface.[3]

All people have had their own personal experience: that they received *bruises* from mathematics. We got them ourselves as students. Later, as professionals, we also got bruises from working with mathematics. And for all people (laymen, students, and professionals) it can be difficult to recognize how mathematics works behind the scene in the real world. Of course, we may share the hope that, for doing, learning, applying, and teaching mathematics, we get something interesting out of the theories of communication and psychology; get some hints at how to make research, development, and teaching better. But do we?

Part I

Looking Back—Have the Students' Mind-Sets Changed?

Recall the functional administrators' claim: *The mind of the student has changed and your teaching is worthless unless you change yours, too.* That claim is supported by the general "wisdom" that we deal with a *browse-generation* or a *me-generation*, but it is misleading. In the sociology and neuropsychology literature, there is no evidence of such sudden, general, and deep changes regarding our students' or our own capacity for learning and teaching mathematics.[4]

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[1]Following the call of Benjamin Ginsberg (The Fall of the Faculty: The Rise of the All-Administrative University and Why It Matters, *Oxford University Press*, 2011, 288 pp., ISBN 9780199782444), I wish to support the resistance of mathematicians against administrators' uninformed and purely functional demands. We shall not obey but refuse the mentioned demands; and we shall further develop original, even risky ideas in our research and not just increase numerically our publication output to satisfy budget claims and funding agencies' priorities.

[2]Bleecker, David D., Booß-Bavnbek, Bernhelm. Index Theory—With Applications to Mathematics and Physics. *International Press*, Somerville, MA, 2013. xxii + 769 pp., ISBN: 978-1-57146-264-0. MR3113540.

[3]In the words of my Roskilde colleague Mogens Niss, Secretary-General of the International Commission on Mathematical Instruction (ICMI) from 1991–1998, we have to deal with "the combined obstacle of the invisibility of mathematics in society and the fact that mathematics is a difficult subject to learn, regardless of the approaches applied." (Niss, Mogens. Mathematics in Society. In: Biehler, I., et al. (eds.), Didactics of Mathematics as a Scientific Discipline. *Kluwer Academic Publishers,* Dordrecht, 1994, ISBN 0-306-47204-X, pp. 367-378.).

[4]For the systematic underestimation of dedicated students' capacity to protect themselves against the media dominance see McLuhan, Marshall; with Fiore, Quentin; produced by Jerome Agel. The Medium is the Massage: An Inventory of Effects. 1st Ed.: *Random House*, 1967; reissued by *Gingko Press*, 2001. ISBN 1-58423-070-3; see also Hurrelmann, Klaus. Selbstsozialisation oder Selbstorganisation? Ein sympathisierender, aber kritischer Kommentar. *Zeitschrift für Soziologie der Erziehung und Sozialisation*, 22/2 (2002), 155–166; see also Sutter, Tilmann. Medienanalyse und Medienkritik: Forschungsfelder einer konstruktivistischen Soziologie der Medien. *VS Verlag für Sozialwissenschaften*. Wiesbaden, 2010, ISBN 978-3-531-16910-1. Girls of 12 or 13 years who cannot resist the social pressure or their own wish to upload

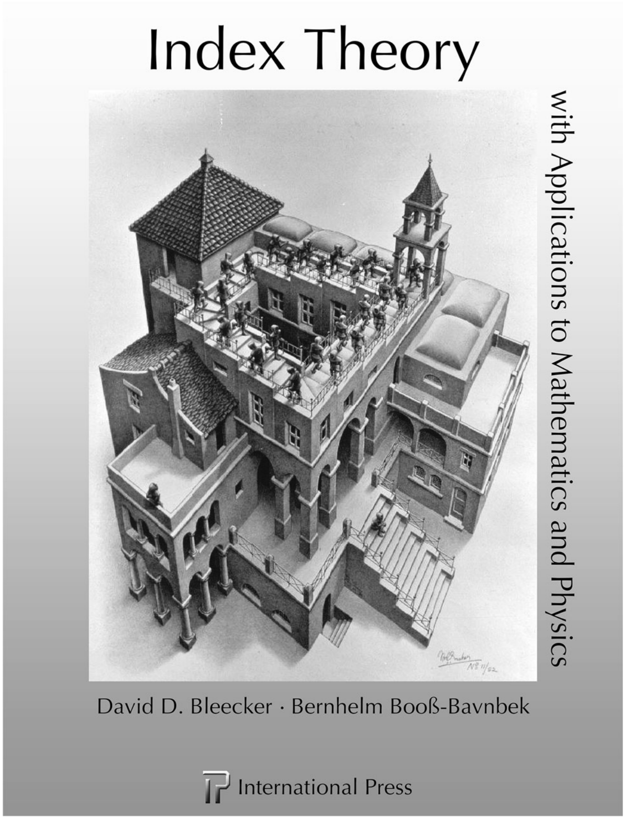

Figure 1. Cover art of my book with Bleecker: M. C. Escher, Ascending and Descending, 1960, Logo for this paper.

Evolutionary Evidence

Our ancestors have bred dogs for at least 20,000 to 30,000 years, perhaps for 500,000 years.[5] Most modern dogs do not look like wolves and each breed looks different. But the animal psychologists write in their books that we must expect 80% of the behavior of our dog to be lupine.[6] I do not know how they measure and quantify, but every dog owner can confirm: After 5000 generations of breeding adjustments, the basic behavior shows almost no change. There is an astonishing stability.

When talking about learning and teaching mathematics, we deal with the human mind: it is quite different from that of dogs, much more variable, namely much more able to adapt to new situations. But is there evolutionary evidence for deep changes in basic human behavior?

One answer is the Cro-Magnon aesthetics in La grotte de Lascaux, in Le tombe di Tarquinia, and in the works of Tiziano Vecellio, Paula Modersohn-Becker, and Jackson Pollock (see Figure 2). All of them provide strong evidence of the apparently indestructible stability of *human curiosity* and *imagination*, of *concentration* and *creativity*. This is exactly what we want from our students.

What Is the Supposed Momentous Historical Media-Generated Change of Consciousness About?

Can we easily discard all the present talk on ongoing media-generated changes of our own and our students' lifestyle and ways of thinking? The short answer is yes and no!

- No: social influences cannot be discarded. Otherwise, showing our personal example and delivering our teaching would be dispensable. And we don't want to believe that.
- Yes: to do mathematics, disturbing social influences must be eliminated or at least confined; learning mathematics requires full concentration and time, and a student will hardly make progress if he or she is not able to let himself or herself be absorbed by mathematics for hours, days, and years.

A. R. Luria

A. R. Luria lived from 1902 to 1977. In 1974, the Soviet neuropsychologist and developmental psychologist Alexander Romanovich Luria published a comprehensive empirical study about cognitive changes induced by social conditions—under the extreme social changes of the first years after the Bolshevist revolution.[7]

He indeed found remarkable differences. For example, people raised in larger and more urban places were good at taxonomic classification, that is, the use of *abstract categories* such as *tools* (assigning an ax, saw, shovel, quill, and a knitting needle to the same group), whereas people raised in remote areas were better at classifications regarding *practical situations* such as the work process of *sawing and chopping wood* or reconstructing a *meal* situation by grouping together objects such as a table, a tablecloth, a plate, a knife, a fork, bread, meat, and an apple. So, different cultural environments can induce different mind-sets.

Even so, Luria found something more, which now is relevant for doing mathematical research and developing applications, for learning and teaching mathematics, namely that all people can easily adapt to radical new ways

Footnote 4 continued
naked selfies to the Internet most probably must change later when they wish to study mathematics; see Politiken Digitalt. Unge sender hinanden afslørende nøgenbilleder i stort omfang. 3 April 2014.

[5]Larson, Greger; et al. Rethinking dog domestication by integrating genetics, archaeology, and biogeography. *Proc. Natl. Acad. Sci. U.S.A.* 5 June 5 2012; 109, 8878–8883.

[6]Svartberg, Kenth; Forkman, Björn. Personality traits in the domestic dog (Canis familiaris). *Applied Animal Behaviour Science* 79 (2002), 133–155; McBride, Anne. The human–dog relationship. In: Robinson, I. (Ed.), The Waltham Book of Human–Animal Interaction: Benefits and Responsibilities of Pet Ownership. *Pergamon*, Oxford, 1995, ISBN 978-1483234748, pp. 99–112; Tami, Gabriela; Gallagher, Anne. Description of the behaviour of domestic dog (Canis familiaris) by experienced and inexperienced people. *Applied Animal Behaviour Science* 120 (2009), 159–169.

[7]Luria, Alexander R. The Cognitive Development: Its Cultural and Social Foundations. *Harvard University Press,* 1976 (Translation from the Russian original of 1974), ISBN 0-674-13731-0. Here pp. v and, in particular, pp. 64 and 164. Similarly, but from a religious (Roman Catholic) perspective, the essay: Ong, Walter J. Interfaces of the Word: Studies in the Evolution of Consciousness and Culture. *Ithaca, N. Y. Cornell University Press,* 1977, 352 pp., ISBN 0-8014-1105-x.

Figure 2. Art evidence of long-time mind stability makes students' alleged short-term mind changes highly improbable. (All reproductions from Wiki, public domain.)

of thinking in new environments. The only precondition is that the new environments are presented in a peaceful way, notwithstanding how dramatic and even painful the changes and the challenges may be.

Preliminary Conclusion

There is no evidence of short-term changes of the students' mind-sets. Neither the media nor twelve years of possibly poor schooling can squander the natural mathematical curiosity of an attentive audience in young minds.

Part II

Mathematics IS Hard—How and Why?

The Vest Trick

Some time ago, I was consultant/supervisor for a third-semester project. In the semester opening I presented myself doing the entertaining vest trick[8] to illustrate what a topologist (my specialty) is doing (Figure 3), namely to think about questions such as *How is it possible that I can remove a vest under my jacket? Has it something to do with the uneven number of my heads or the even number of my arms* etc.? The students asked *Is this mathematics*? We discussed it. I had to admit that a similar problem in two dimensions is well understood. It is the question of whether a closed curve in the punctured plane (a plane in which one point, e.g., the origin, is removed) is contractible. The question can be easily decided by calculating the winding number of the curve; and there are various and seemingly very different methods for doing that (calculating a path integral, solving a differential equation, by combinatorics, etc.). The curve is contractible if and only if the winding number vanishes. But what can we do with the three-dimensional case?

[8]Nicely discussed in Stewart, Ian. Mathematical Recreations—The Topological Dressmaker. *Scientific American*, July 1993. A related topological puzzle is *The Dirac String Problem*, thoroughly explained in Hansen, Vagn Lundsgaard. *Braids and Coverings: Selected Topics. With appendices by Lars Gæde and Hugh R. Morton*. *London Mathematical Society* Student Texts, 18. *Cambridge University Press,* Cambridge, 1989, x + 191 pp., ISBN: 0-521-38479-6, pp. 39–45.

Figure 3. The author demonstrating the vest trick: exploring counterintuitive 3D. Photograph courtesy Poul Erik Nikander Thomsen, Roskilde University.

Those were *not* the questions that the students were most interested in. *Neither* were they surprised that I could not tell them a solution at once. What triggered their curiosity and resulted in a full semester's work was their own question, namely, if can one tell from the outside whether a mathematical problem is hard or easy. More precisely: *How can it be that many problems in mathematics are easy to formulate but difficult to solve*? They investigated that question by looking carefully at various historical examples (long and fascinating stories). Anyway, in this way they gave me the idea for this Part II of my article: What were the answers of the giants of mathematics?

J.-L. Lagrange

J.-L. Lagrange lived from 1736 to 1813. To me, Giuseppe Lodovico (Luigi) Lagrangia (Joseph-Louis Lagrange) is one of the most important figures in the history of mathematics. He was extremely successful in introducing radically new and often highly abstract concepts, making mathematical ideas clear and comprehensible even to a non-genius. Otherwise these ideas would have remained the domain of the intuition of outstanding people. He replaced Euclid's polygons and Descartes' curves by homotopies; his Second Letter to Euler, of 12 August, 1755,[9] is the birth certificate of deformation theory and differential topology; and he replaced the Eulerian mechanics that attempted to follow the ever–more-confusing *visible* orbits of single pieces by his mechanics of the more easily tangible underlying *invisible* potentials.

When he died on 10 April 1813, there was an official obituary (by Delambre)[10] and a "Supplement" by a person designated *G*. The supplement dealt with his last days and the thoughts he expressed shortly before his death. Nobody knows who *G*. was. There are some speculations, which were investigated and reported in a paper by the mathematics historian Ivor Grattan-Guinness.[11] He argues that *G*.'s Supplement is believable.

The following quotes by Lagrange are from Grattan-Guinness's paper. Until his death, according to these documents, Lagrange felt sorry for his students that they had to read his textbooks, Lagrange's masterpieces in mechanics, which are so much more elaborate, intricate, and harder

[9]Bleecker & Booß-Bavnbek, l.c., pp. 65f, Latin with English translation.

[10]Delambre, Jean-Baptiste Joseph. Notice sur la vie et les ouvrages de M. le Comte J.-L. Lagrange. In : Œuvres de Lagrange. *Gauthier-Villars*, 1867 (1, pp. ix–li). https://fr.wikisource.org/wiki/Notice_sur_la_vie_et_les_ouvrages_de_M._le_Comte_J.-L._Lagrange.

[11]Grattan-Guinness, Ivor. A Paris Curiosity, 1814: Delambre's Obituary of Lagrange, and Its "Supplement." In: *Mathemata*, pp. 493–510, Boethius Texte Abh. Gesch. Exakt. Wissensch., XII, *Steiner, Wiesbaden,* 1985. MR0799763.

than all previous mechanics treatises. Of course, Lagrange was right: His books were dispensable for the calculation of simple mechanical systems—but indispensable for making complex mechanical systems transparent, understandable, and calculable for the human brain. He felt "…sorry for the young geometers who have such thorns to swallow. If I had to start again, I would not study: These large in-4° would make me too scared." He proposed instead a one-volume reprint of original works of the calculus by Fermat, Leibniz, l'Hôpital, and especially John Bernoulli's lectures on the integral calculus, together with another volume comprising items by Euler and d'Alembert.

Delambre had quoted Lagrange, "If I had had a fortune, I would probably not have made my profession [état] in mathematics." *G.* supplemented this by recalling an occasion when Lagrange had met "a young man devoting himself to the exact sciences with much ardour," and on asking him "Do you have a fortune?" and receiving a negative answer had replied: "so much the worse, sir. The lack of fortune and of the existence it can give in the world, is a constant stimulus which nothing can replace, and without which one cannot bring to hard tasks all the necessary progress [suite]." Lagrange knew what he was talking about. His father had been rich after an advantageous marriage, but he had lost all in risky businesses. To Lagrange this was not deplorable. Because he knew how hard mathematics can be—and he knew that its hardness is widely recognized and rewarded with a quiet and studious life of doing something difficult that other people cannot do.

When Lagrange was teaching, his "researching intelligence" (*G.*) could cause sudden lapses in conversation. *G.* described the effect on his lectures at the École Polytechnique:

> "Who has not seen him suddenly interrupt himself thus in the lectures which he gave at the École Polytechnique, appearing sometimes embarrassed like a beginner, leaving the blackboard and coming to sit down opposite the audience, while teachers and students, confused on the seats [bans] expected in a respectful silence that he would have led his thought back from the spaces that it had gone to travel through."

To Lagrange, all mathematics was hard, also when it was seemingly easy for the student and would reveal its hardness only for the expert. So, the main goal of a mathematician's life was to think of how to make mathematics easier and more accessible, sometimes at the cost of introducing further abstract, and more elaborate concepts.

In essence and in my reformulation: Mathematics can be made easy and comprehensible only by accepting and enduring its hardness. Students are exposed to the cultural clash immanent in abstractions, formalism, and symbol processing. Teachers must help them to experience that clash as a positive step, like the processes of adolescence or seeking work abroad, and not as a series of defeats. For sure, it does not help with well-intended lies or self-deception about easy access to mathematical abstractions as demanded by the new caste of administrators. Acquiring mathematical experience is nothing that falls from heaven or comes from playing on the ground. It requires work, concentration, exercises, and endurance: Ὁ μὴ δαρεὶς ἄνθρωπος οὐ παιδεύεται ("The non-flayed human will not be educated," Menander, c. 341/342– c. 290 BCE, disseminated by J. W. Goethe as a motto over his autobiography *Dichtung und Wahrheit*), or less draconic, Ohne Fleiß kein Preis ("No pain, no gain," after Hesiod, thought by scholars to have been active between 750 and 650 BCE).

The mathematicians I admire most are very close to Lagrange's position in continuing a lifelong interest in teaching mathematics and insisting that the essence of mathematics, triggering curiosity and creativity and its true place in applications, is that it is hard, and that it becomes dispensable and replaceable by engineering arts and econometric analyses, etc., when it becomes easy.

I. M. Singer

I. M. Singer (born 1924) can rightly take pride in his achievements, among others, the Index Theorems, which brought him the Abel Prize in 2004 jointly with M. F. Atiyah. When afterward he was asked what he would do next, he did not hesitate: "Now I want to use more sophisticated mathematics not yet available to physics."[12] Clearly, to Singer, the role of mathematics is to handle extremely hard problems.

Part of the story is that this same man, during all his recent years (he is now 93 years old), participates at MIT in teaching beginning mathematics and, as he says with great intellectual satisfaction, nursing and watching the emerging mathematics understanding of young students:

> A while back, I decided to be a TA in the freshman calculus course. I think I was motivated to do so because I had been too far removed from undergraduates. Making contact with freshmen again was a wonderful experience… Teaching does integrate with my other work. I'm inclined to understand rather than solve. For me, doing research means understanding something nobody has understood, and then telling others about it. What makes me a good teacher is empathy. I can put myself in the position of a student and know what they don't understand. If I know them well enough, I can explain what they don't understand in terms they can comprehend.[13]

V. I. Arnold

Some attribute to V. I. Arnold (1937–2010), and his former students, the most decisive advances in the mathematical understanding of dynamical systems since the seminal work of H. Poincaré more than 100 years ago. When he was asked about the situation of mathematics in Russia after the fall of the Soviet Union, he deplored it in his

[12]Singer, I. M. Transcript of May 12, 2010, MIT150 interview, http://mit150.mit.edu/infinite-history/isadore-singer, accessed May 21, 2015.
[13]L.c.

sarcastic way: "Well, it's terrible. Now the professors are cleverer and know more than the students."

How sad. Indeed, teaching and learning mathematics is only interesting when the teacher in each meeting with the students, say of one hour, gets at least one new mathematical idea. Otherwise it does not work with our goal, namely to socialize a new generation of mathematics students to the way of mathematical thinking.[14] The hour would have been lost—or could have been left to an electronic instruction device—with the same default result.

In an article[15] tracing the history of his own research, Arnold showed how apparently unrelated subjects are linked by a kind of mycelium from which theorems pop up like mushrooms (see Figure 4). Continuing his lifelong battle against formalism and Bourbakism, he distinguishes the ease of communicating formal theorems from the hardship of explaining the underlying ideas in the following parable:

> "When you are collecting mushrooms, you only see the mushroom itself. But if you are a mycologist, you know that the real mushroom is in the earth. There's an enormous thing down there, and you just see the fruit, the body that you eat.
>
> "In mathematics, the upper part of the mushroom corresponds to theorems that you see, but you don't see the things which are below, that is: *problems, conjectures, mistakes, ideas,* and so on.
>
> "You might have several unrelated mushrooms being unable to see what their relation is unless you know what is behind. And that's what I am now trying to describe. This is difficult, because to study the visible part of the mathematical mushroom you use the left half of the brain, the logic, while for the other part the left brain has no role at all, since this part is highly illogical. It is hence difficult to communicate it to others."

E. Artin

In the same vein, Artin (1898–1962) wrote in his famous Bourbaki review of 1953:

> "We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt he must always fail. Mathematics is logical to be sure; each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that its perception should be instantaneous. We all have experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications."[16]

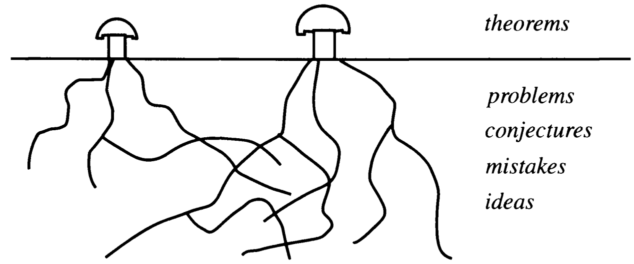

Figure 4. The mathematical mushroom, according to V. I. Arnold (2004). Reprinted with permission of the Mathematical Association of America.

F. Hirzebruch

Since the 1950s, Hirzebruch (1927–2012) was the outstanding figure of mathematics in Western Germany. He was the natural candidate as director of a Max-Planck Institute (MPI) in mathematics, and he became the director of the first MPI in mathematics in Bonn in the 1980s. But for decades there had been no MPI for mathematics.

Shortly after the end of World War II, many MPIs were founded and lavishly financed to bring the sciences in the Federal Republic of Germany rapidly back to a top international level after their decline and demolition during the Nazi period.

Once I asked Hirzebruch why the Bonn MPI for mathematics came so late? He told me frankly that at least one reason was a controversy between him and the Board of the Max-Planck Gesellschaft (MPG).

Contrary to the MPI tradition of teaching-free research, Hirzebruch had insisted that research in mathematics without teaching is meaningless, that, as a rule, new mathematical results are too hard to be digested at a distance; that they will falter rapidly when they are not forwarded instantaneously to new generations in interpersonal communication; that most young students need the contact and the role model of a successful researcher to overcome the hardships of acquiring mathematics. Consequently, there should only be very few permanent positions for the directors and support staff, whereas the main human resources should consist of university teachers on leave as guest researchers for midterm stays. It took him several decades to reach the MPG's acceptance for this deviating status of mathematics research, that is, that it is meaningless without the umbilical cord to teaching and that all members of the MPI for mathematics had to have an association with teaching.

Y. I. Manin

Like Artin and Hirzebruch, Manin (born 1937) is a magician who can create a world of deep interrelated concepts and

[14]In essence, this goal is expressed in S. Eilenberg's sententious teaching advice: "Mathematics is a performance art, but one whose only audience is fellow performers." Cited from Bass, Hyman. Mathematics and Teaching. *Notices Amer. Math. Soc.* 62/6 (June/July 2015), 630–636.

[15]Arnold, Vladimir I. From Hilbert's superposition problem to dynamical systems. *The American Mathematical Monthly* 111, No. 7 (Aug.–Sep., 2004), 608–624, preview at https://www.jstor.org/stable/4145164?seq=1#page_scan_tab_contents. Reprinted in: Mathematical Events of the Twentieth Century, pp. 19–47, *Springer*, Berlin, 2006. MR2182777.

[16]Artin, Emil. *Éléments de mathématique.* by N. Bourbaki. Book II, *Algebra.* Chaps. I–VII. Book review. *Bull. AMS* 59/5 (1953), 474–479, here p. 475.

results for his audience within 60 min, so that most people in his audience have a strong feeling of having understood a lot, of being almost able to walk on water. Of course, when you go home and begin to work your way through your notes, your feeling will change, and you will feel stupid and discouraged: there are too many things you can't understand in detail, and in mathematics that means that you don't understand.

Manin himself commented in his textbook on mathematical logic[17] that mathematical abstractions are hard to grasp; that thinking in symbols, although extremely effective in many contexts and indispensable in some, is deeply against human nature. He explains that concept very carefully in his book and partially with references to facts based on some observations made by the psychologist A. Luria on patients with brain injuries. Some of the patients had preserved a sound judgment of the situation in hospital, for example, of differences between various doctors and nurses in their competences and engagement, but they lost the ability to think in relations: is an elephant bigger than a fly or a fly bigger than an elephant? His claim: *Abstraction is in essence more difficult for human nature than making judgments on personal relationships,* etc.

Note that fully, consciously, and explicitly, Manin's insight or claim is directed against the traditional claims of all logicists and many adepts of mathematization who consider the process of abstraction and formalization as a process of simplification and clarification.

C. S. Peirce

C. S. Peirce (1839–1914) had an anthropological message that our concepts, also our scientific concepts, have evolved in human praxis during more than 100,000 years of experience with the various contexts humans have had throughout time.

The good side of the message is that in most situations, common sense and scientific, mathematics-based arguments need not contradict each other. The bad side of the message is, with a view toward the emerging quantum mechanics at Peirce's time, we have a problem when dealing with phenomena in an artificial environment that our mind for thousands of years has not been accustomed to. Then we must transgress common sense because it will for the most part be systematically misleading.[18]

C. F. Gauss and L. Hörmander

C. F. Gauss (1777–1855) and L. Hörmander (1931–2012) were masters in standard formulations when they reviewed the work of other mathematicians: *Incomprehensible—wrong—I did it a long time ago.*

To me, such typical referee reports prove that reading mathematics papers is always hard, even for the greatest mathematics geniuses. Correspondingly, we have in mathematics two very different exclamations of agreement, *it's trivial* and *it's clear*. The first is pejorative: don't waste my time with your boring stuff; the second is highest acclamation: aha, now I see; this is really hard stuff you are telling me!

H. Cramér

H. Cramér (1893–1985), in his monumental monograph *Mathematical Methods of Statistics* of 1945, proved that the Chi square test statistic, that is, the sum of relative errors between observed and expected magnitudes with f degrees of freedom, is distributed like the corresponding Chi square distribution with f degrees of freedom. For $f = 1$, it is the classical result by Karl Pearson of 1900, and its proof is reproduced in most textbooks of mathematical statistics. For applications in material sciences, biology, and medicine, Cramér's theorem is applied. Perhaps it is the most applied mathematical theorem of the 20th century. But to my best knowledge, its proof has never (!) been reproduced. You can only find it in Cramér's textbook.[19] It is lengthy and not very inspiring. It is laborious—and boring. The main idea is much clearer for $f = 1$ than in the general case.

Such is mathematics that it has theorems that are easy to apply but hard to understand and, in practice, perhaps understood only by the author of the theorem and a handful of readers of the original publication.

P. J. Davis

In a public talk in Roskilde, P. J. Davis (born 1923) gave a similar example when he confessed that he never had completed his checks of the proof of the *principal axis theorem* on block-diagonalization of normal matrices in linear algebra in spite of the fact that this theorem was a central tool in many of his works on effective numerical methods.

In a paper, Davis expanded on his view that we must live with some imperfections also in mathematics, that some basic tasks in numerical analysis are too hard to admit a rigorous approach.[20] Among his examples he refers to the concept of *numerical stability* in iterations, when, contrary to the toy examples of elementary classes in numerical analysis, no estimates are available about

[17]Manin, Yuri I. A Course in Mathematical Logic for Mathematicians. Second edition. Chapters I–VIII translated from the Russian by Neal Koblitz. With new chapters by Boris Zilber and the author. Graduate Texts in Mathematics, 53, *Springer,* New York, 2010. Xviii + 384 pp. ISBN: 978-1-4419-0614-4 MR2562767.

[18]Peirce, Charles Sanders. The Architecture of Theories. *The Monist*, 1891 (*CP* 6.7–25, 31–34). Reprinted in Philosophical writings by Peirce—Selected and edited with an introduction by Justus Buchler. *Dover Publications*, New York, 1955, ISBN 0-486-20217-8, pp. 315–323, here p. 317: ''Thus it is that, our minds having been formed... under the influence of phenomena governed by the laws of mechanics, certain conceptions entering into those laws become implanted in our minds, so that we readily guess at what the laws are. Without such a natural prompting, having to search blindfold for a law which would suit the phenomena, our chance of finding it would be as one to infinity. The further physical studies depart from phenomena which have directly influenced the growth of the mind, the less we can expect to find the laws which govern them 'simple,' that is, composed of a few conceptions natural to our minds.''

[19]Cramér, Harald. Mathematical Methods of Statistics. Princeton Mathematical Series, vol. 9. *Princeton University Press*, Princeton, N. J., 1946. Xvi + 575, pp. MR0016588, here chapter 29.

[20]Davis, Philip J. The Relevance of the Irrelevant Beginning, *Science Open Research,* 2014, 5 pp., DOI: 10.14293/A2199-1006.01.SOR-MATH.6G464.v1.

the achieved precision of an approximate result. Nevertheless, we have to stop the iterations at some point. For solving systems of differential equations, a common stop rule is when the results become unchanged under further iteration or refinement of the underlying discretization. Then a result seems to become stable and reliable, while we have examples in which numerical stability can be achieved far from the true result. To comfort our mathematical fears and uncertitude, Davis usually cites Richard Hamming (1915–1998) for having said, "I would never fly with a plane where the construction depends on the difference between the Riemann and Lebesgue integral."[21]

Davis points to another symptom of the difficulty of doing mathematics, namely our almost unlimited *freedom* to add or to remove assumptions, although sharply *restrained* by logical demands regarding the formulation and consistency, and even more sharply restrained by respectful regard for the history of a topic and which examples or expansions might be considered meaningful and which might not be.

Warning 1

From a technological point of view, *hard and presently unsolved problems* are wonderful and highly applicable, like the present lack of efficient algorithms to factorize a given product of two large prime numbers into its two components, or other presently unsolved problems regarding elliptic curves that keep the common public keys in cryptology relatively safe!

Warning 2

For numerical algorithms in the analysis of dynamical systems and of combinatorial tasks, for example, in graph theory, mathematicians try to give *asymptotic estimates* about the complexity (i.e., the expected time necessary for a solution) of a problem. By definition, the problems that are hardest to solve are the so-called NP-complete problems such as the traveling-salesman problem. For practical purposes, the perfect organization of just-in-time delivery for retail store chains shows that one never should become blocked in the search for practical solutions by seemingly insurmountable asymptotic estimates.

Piet Hein

Piet Hein (1905–1996) wrote the following verse:

> Problems worthy
> of attack
> prove their worth
> by hitting back.

Mathematics IS Easy—Really?

In the previous section "Mathematics is Hard—How and Why?," I tried to explain why learning and doing mathematics is hard, by necessity. But what about the many people, pupils, students, teachers, and researchers, who love to spend many hours thinking about a mathematical problem; some early in the morning when one is fresh, some late in the night when one is not disturbed, some on their desk and some while jogging or walking their dogs? And what about the rich treasures of investigations, suggestions, and predictions of how doing mathematics can be made easier and more accessible?

I comment on the most outspoken positions.

A. Schopenhauer

In his treatise *Die Welt als Wille und Vorstellung* (*The World as Will and Representation*), philosopher Arthur Schopenhauer (1818)—or rather a philosopher-poet like the many other German philosopher-poets Hegel, Nietzsche, Heidegger with their love for extensive formulations—released the following torrent of words[22] against the mathematicians' arrogance and stupidity making mathematics, according to Schopenhauer, unnecessarily hard and nonintelligible, and that Euclid's classical arguments were monstrous and dispensable:

> …mathematical knowledge *that* something is the case is the same thing as knowledge of *why* it is the case, even though the Euclidean method separates these two completely, letting us know only the former, not the latter. But, in Aristotle's splendid words from the *Posterior Analytics*, I, 27: 'A science is more exact and more excellent if it tells us simultaneously *what* something is and *why* it is, not *what* it is and *why* it is separately.' In physics we are satisfied only when our recognition *that* something is the case is united with our recognition of *why* it is, so the fact that the mercury in a Torricelli tube is 28 inches high is a poor kind of knowledge if we do not add that it is held at this height to counterbalance the atmosphere. So why should we be satisfied in mathematics with the following occult quality of the circle: the fact that the segments of any two intersecting chords always contain equal rectangles? Euclid certainly demonstrates it in the 35th proposition of the third book, but why it is so remains in doubt. Similarly, Pythagoras' theorem tells us about an occult quality of the right-angled triangle: Euclid's stilted (stelzbeinig), indeed underhand (hinterlistig), proof leaves us without an explanation of why, while the following simple and well-known figure (Figure 5) yields more insight into the matter in one glance than that proof, and also gives us a strong inner conviction of the necessity of this property and of its dependence on the right angle:

[21]Hamming, Richard W. Mathematics on a Distant Planet. *Amer. Math. Monthly*, 105 (1998), no. 7, 640–650. http://www.ams.org/mathscinet-getitem?mr=1633089. The full quote is "for more than 40 years I have claimed that if whether an airplane would fly or not depended on whether some function that arose in its design was Lebesgue but not Riemann integrable, then I would not fly in it. Would you? Does Nature recognize the difference? I doubt it!" [p. 644]. Certainly, Hamming's insistence on robustness in applications is a relief. However, it is a fact that certain highly applicable concepts, such as the Hilbert space L^2 of equivalence classes of measurable, square-integrable functions, can only be established by embracing all Lebesgue integrable functions to obtain the indispensable completeness.

[22]Schopenhauer, Arthur. Die Welt als Wille und Vorstellung. *Werke in 5 Bänden, hrsg. von* L. Lütkehaus. *Haffmans, Zürich*, 1991, vol. 1, §15, p. 119. English translation in: The world as will and representation; translated and edited by Judith Norman, Alistair Welchman, Christopher Janaway; with an introduction by Christopher Janaway. *The Cambridge Edition of the Works of Schopenhauer. Cambridge University Press,* Cambridge and New York, 2010, p. 98.

Often when outsiders comment on mathematics it strikes me how little they understand of the crux of a mathematical achievement. So also Schopenhauer: The crux of Pythagoras' Theorem is its validity for *all* right triangles in the plane, that is, even when the sides at the right angle are unequal. By the way, that's until today the most typical application of the theorem in construction: To check whether the walls in a room or a house are rectangular, a carpenter would mark a 3-meter (or yard) point upward in a corner, a 4-meter (or yard) point along a wall on the floor, and then check whether the straight line between the two marks is exactly $5 = \sqrt{3^2 + 4^2}$ meters (or yards).

One would expect an error term; but no, Pythagoras claims and Euclid proves that the error term vanishes even when we deform the right triangle, within the class of right triangles. Later generations proved that Pythagoras' theorem remains basically valid even for non-right triangles, incorporating an error term coming from the cosine of the included angle, and for right triangles on a sphere, incorporating a curvature error term coming from the sphere's radius. So, for a mathematician, the Pythagoras' Theorem is an approximation theorem, in which you can change something with controlled effects, sometimes with zero effect, sometimes with nonvanishing, but calculable effects.

Of course, mathematics can be much easier when we remove the key points and reduce it to trivialities. Actually, we can answer Schopenhauer, that mathematics would become even easier when we reduce it to the empty set. So far, Schopenhauer only shows his lack of understanding.

However, he rightly points to the difference between *checking a proof, line by line*, as opposed to *grasping the reason for the validity of a claim*. Every mathematician has experienced it: that we still do not understand a given proof after we have checked it step-by-step. Hence, in modern textbooks and for papers in learned journals, authors are praised when they explain the underlying idea of a proof before the reproduction of the proof in its details.

C. F. Gauss

Gauss' (1777–1855) reply to Schopenhauer was: "On the contrary! Mathematics is so difficult that we never should tell the reader how we got the idea. In most cases it will be either impossible or distracting to make the idea explicit. To make results accessible we shall hide all complications we had to meet and overcome, and keep silent about the wrong tracks we went when searching and finding the proof. What counts in mathematics is only a presentation of the purified final form."[23] For 200 years, Gauss' perception of simplicity through hiding the birth pangs and presenting only the sleek version has dominated the publication culture of mathematics. Fortunately, it has been on the retreat along with the retreat of Bourbakism.

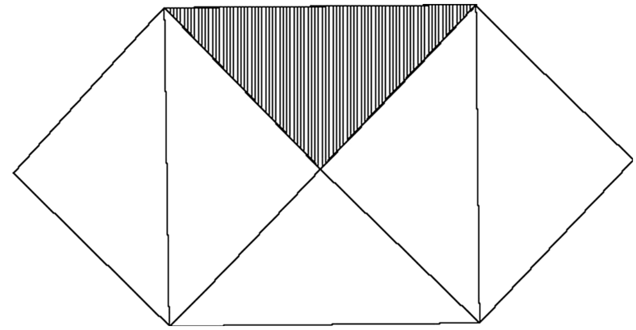

Figure 5. Schopenhauer's fantasied "simplification." Reprinted from the second German edition of Die Welt als Wille und Vorstellung, Leipzig 1844, vol. 1, book 1, §15, public domain.

M. F. Atiyah

Although Atiyah (born 1929) has personally contributed to the creation of many new mathematical theories, concepts, and methods, he insists that mathematics is becoming easier and more transparent by the emergence of each new mathematical achievement. He compares mathematics with a warehouse: "Looking for a box of nails in a small country shop and finding the right ones can be harder than looking around in a big specialized department store like Bauhaus." Clearly, it is easier to find your way around in a big, well-organized modern department store than an old-fashioned mom-and-pop store. Making mathematics more complex opens many new crossroads and makes search and communication easier. Such is the argument.[24]

Atiyah's optimistic claim is based on his view of an ever clearer emerging unity of mathematics. In a paper, Davis and I refuted that unity belief as a myth.[25]

[23] A typical example is provided by Gauss' first proof of the Fundamental Theorem of Algebra of 1799. Gauss, Carl Friedrich. Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse. *Helmstedt: C. G. Fleckeisen*. 1799 (tr. New proof of the theorem that every integral rational algebraic function of one variable can be resolved into real factors of the first or second degree). German translation in: Netto, Eugen (ed.): Die vier Gauss'schen Beweise für die Zerlegung ganzer algebraischer Funktionen in reelle Factoren ersten oder zweiten Grades (1799–1849), *Ostwald's Klassiker der Exakten Wissenschaften Nr. 14, Wilhelm Engelmann, Leipzig*, 1890, pp. 3–36, 83 (figures). Accessible at University of Toronto, https://archive.org/details/dieviergausssche00gausuoft. Contrary to d'Alembert's proof of 1746, Gauss keeps this proof deliberately in purely real terms. However, one can easily trace the underlying complex constructions in the real presentation.

[24] Atiyah, Michael F. Trends in Pure Mathematics. In: *Proc. of the 3rd Internat. Congress on Mathematical Education (Karlsruhe 1976).* 1979, pp. 61–74. Reprinted in Collected Works, vol. 1, pp. 261–276. MR0951896. In H. Bass, l.c., p. 633, a similar thought is elaborated under the heading *compression*, which means "a process by which certain fundamental mathematical concepts or structures are characterized and named and so cognitively rescaled so that they become, for the expertly initiated, as mentally manipulable as counting numbers is for a child.".

[25] Booß-Bavnbek, Bernhelm; Davis, Philip J. Unity and Disunity in Mathematics. *Newsletter of the European Mathematical Society*, No. 87 (March 2013), 28–31. Reprinted and extended in Davis, Philip J. Unity and Disunity, And Other Mathematical Essays. *American Mathematical Society*, Providence, RI, 2015. pp. ix + 149, ISBN: 978-1-4704-2023-9, MR3495468.

J.-L. Lagrange

This is the same Lagrange (1736–1813) who appeared previously as witness of the inevitable hardness of mathematics. Now I call him as witness for the ease and simplicity of mathematical physics. Regarding the celestial bodies of our solar system, he noticed that the planets are moving on almost circular orbits and most comets on very eccentric elliptic orbits. He concluded that "nature favors planetary approximations by grouping heavenly bodies according to very small and enormous eccentricity."[26] Indeed, for each of the two extreme cases we have specific and very powerful expansions, which would fail in the middle range. Modern astrophysics teaches us, however, that this is a very special property of our solar system because of the dominance of the two gas planets Jupiter and Saturn, which, on the whole, make our system so surprisingly stable. Other solar systems in the Milky Way seem to provide for more challenging mathematical problems.

We may expand Lagrange's argument for *nature provided simplicity* to large parts of mathematical physics where we, for example, do not have to deal with very general differential equations with arbitrarily varying coefficients but with geometrically defined operators with strong inherent symmetries such as the Laplace or the Dirac operator that are, moreover, often controlled by potentials and other background fields. Therefore, large parts of mathematical physics that are based on first principles and geometry are mathematically easier and more accessible than some parts of biology that are less mathematized, based on ad hoc assumptions and so tangled up in noncontrollable generalizations.

More, and More Trustworthy Arguments

Until now, in this Section, I discarded common suggestions and beliefs about why and how doing mathematics can become easier, no matter how brilliant they are. I shall now turn to considerations that are also controversial, but definitely not to be discarded by me. It seems to me that they have the potential to explain why and how learning and doing mathematics can appear personally satisfactory, natural, and, from time to time, even easy for some people in lucky moments and periods of their lives.

I have written about the following quite different approaches separately and extensively[27] and I shall be brief in this article.

N. Chomsky

Based on Chomsky's (born 1928) linguistic research, his message, or at least the message disseminated by his student Pinker,[28] is: "Every child has solved the greatest mathematics exercise of her or his life at the age of two, when it forms the generative grammar of the child's mother tongue and assembles the patterns and basic structures out of single words." Hence, we may conclude: "Basically, mathematics is easy. Everyone has done it; everyone can do it." I had better add that some of Chomsky's claims are controversial, in particular his biologistic assumptions of special genetic grammar traits of the human race that are not confirmed by molecular geneticists.

C. S. Peirce and P. Naur

To me, the names Peirce (1839–1914)[29] and Naur (1928–2016)[30] both stand for

- deep insight into the complexity of human thinking and communication, and for
- demystification of feeling, learning, and doing by relating it to human *habits* and *forms of life*.

Their teaching for our topic can be roughly summarized in the following short formula:

1. Trace the *habits of nature*;
2. relate our feeling, thinking, and doing to our *form(s) of life*, take the risks and jumps of adolescence, and accept the related clash of cultures;
3. for mathematics, exploit the translational power (and handle the two contradictions that follow) by *coding mathematics experiences* and make them *transferrable for adaption* in new contexts.

Two Contradictions

All mathematics learning and teaching has to live with, and to handle, the two following contradictions:

A. Result v. process. We need to teach *results*, not only *processes*, not only ways of thinking; one needs results in sciences and mathematics.

B. Context v. abstraction. Students learn best in *context*, when they can see meaning and embedding in context; however, the power of mathematics is that it can be separated from the context; the true power of *abstractions* is that we can transport experiences from one context to another one.

We cannot discard or bridge these two contradictions firmly. We cannot deliver what the mathematics education administrators want, an easier, faster, and more accessible teaching in the sense that they want. We must tell them that doing, learning, and teaching mathematics is difficult and requires time for the body, peace for the mind, and passion.

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[26]Here is the full quote of Lagrange given in his obituary cited earlier by the anonymous *G.*, offering a Lagrange-type witticism:'' It seems that nature had disposed these orbits [of the heavenly bodies] specially so that one may calculate them. Thus the [sic] eccentricity of the planets is very small, and that of the comets is enormous. Without this disparity [,] so favourable to approximations, and if these constants [of the orbits] were of an average magnitude, *goodbye geometers*; one could do nothing.''.

[27]Booß-Bavnbek, Bernhelm. On the difficulties of acquiring mathematical experience, *EM TEIA—Revista de Educação Matemática e Tecnológica Iberoamericana* 5—número 1 (2014), 1–24. Also: http://thiele.ruc.dk/~Booss/Math_Easy-and-hard_Presentation/2014_BBB_EMTEIA.pdf.

[28]Pinker, Steven. The Language Instinct. *William Morrow*, New York, 1994.

[29]l.c.

[30]Naur, Peter. Computing: a Human Activity. *ACM/Addison-Wesley*, New York, 1992.

S. Kierkegaard

In *Enten-eller*, Kierkegaard (1813–1855) explained the two most difficult situations in life for him, the love for another human and the love for God.[31] I don't agree fully with Kierkegaard, neither with the first situation where I have some personal experience, nor with the second, where I'm blank. Anyway, Kierkegaard emphasizes that both of these two situations require deep feelings: Let yourself be *seduced* and develop the *passion*!

Afterword

Mathematics doing, learning, and teaching is rewarding when it is successful. On some occasions you'd better lie and follow the love advice of Elias Canetti (1905–1994): "Don't tell me who you are. I want to adore you." So you need not tell the students the full truth[32] every day, for example, about

- the destructive sides of mathematics-supported technology;
- the mathematics-induced inhuman formatting of social organization; and
- the deformations of the mind by the naïve belief in logic and modeling.

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[31]Kierkegaard, Søren. Either/Or. Volume I. *Princeton University Press*, Princeton, 1959. See *The Immediate Stages of the Erotic or the Musical Erotic*, pp. 43–134, in particular pp. 62, 93, 114.

[32]In Hardy, G. H. A Mathematician's Apology. With a foreword by C. P. Snow. Reprint of the 1967 edition. Canto. *Cambridge University Press*, Cambridge, 1992, 153 pp. ISBN: 0-521-42706-1, MR1148590, p. 33, n. 16, Hardy ponders his 1915 quote:''a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life.'' See also Arnold, Vladimir I. Polymathematics, l.c., p. 403, paraphrasing Hardy:''All mathematics is divided into three parts: cryptography (paid by CIA, KGB, and the like), hydrodynamics (supported by manufacturers of atomic submarines), and celestial mechanics (financed by military and other institutions dealing with missiles, such as NASA),'' and the anthology Mathematics and War. Papers from the International Meeting held in Karlskrona, August 29–31, 2002. Edited by Bernhelm Booß-Bavnbek and Jens Høyrup. *Birkhäuser Verlag*, Basel, 2003. Viii + 416 S. ISBN: 3-7643-1634-9, MR2033623, free download at http://www.springer.com/gp/book/9783764316341#otherversion=9783034880930.