Math: Easy *and* Hard. Why?

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Abstract. Math is easy and hard, visible and invisible, inspiring and mis-leading, useful and destructive, free and under restraint, and people love it or hate it. Why?

I confront encouraging findings on human development and language acquisition with rather sceptical views to explain why learning and teaching math is easy *and* hard at the same time; why we in mathematics struggle both for product *and* process; why the meaning of mathematical understanding is contextually confined, but the triumph of mathematical experience is to become transitional and context free.

With this paper I wish to support the resistance of mathematicians against administrators' purely functional demands. We shall refuse when they ask us to tune in our teaching to the modern zap generation; and we shall further develop original, even risky ideas in our research and not just increase numerically our publication output to satisfy budget claims and funding agencies' priorities.

PREFACE

1. Logo and topic

The *logo* for this paper (Fig. 1 below) is from the front page of my recent monograph on the index theory of M.F. Atiyah and I.M. Singer[1] which started with lectures I gave in 1971 in Allende's Chile. The President had given orders that there should be Further Education for all junior and senior high school teachers in the country to mark the new era. So, some Chilean mathematicians invited me after I had recently finished a PhD on that subject in Bonn. They asked me to give a series of elementary lectures to their teachers so that everybody could understand what modern mathematics was about and what the content was of the Atiyah-Singer Index Theorem. While I was gaving these lectures, someone took notes, so that I came back to Europe in September 1971 with a manuscript of 120 pages. This was my first attempt at making a hard topic easy.

These 120 pages turned into a book of 769 pages in 2013. The publisher chose a front page of the famous Escher graphic of knights walking a never-ending ascending path that never reaches higher levels: a good symbol of mathematical studies, easy and hard at the same time. You walk and walk, think you have made no progress.

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[1] **Bleecker, David D.**; **Booß-Bavnbek, Bernhelm**. Index theory — with applications to mathematics and physics. *International Press, Somerville, MA,* 2013. xxii+769 pp. ISBN: 978-1-57146-264-0 MR3113540.

After yet another while you think it's easy, you are on the same level now, with a better understanding, but you feel it was nothing. This is probably the ambiguity of all learning, but it is specially so for mathematics. That is the *topic* of this paper.

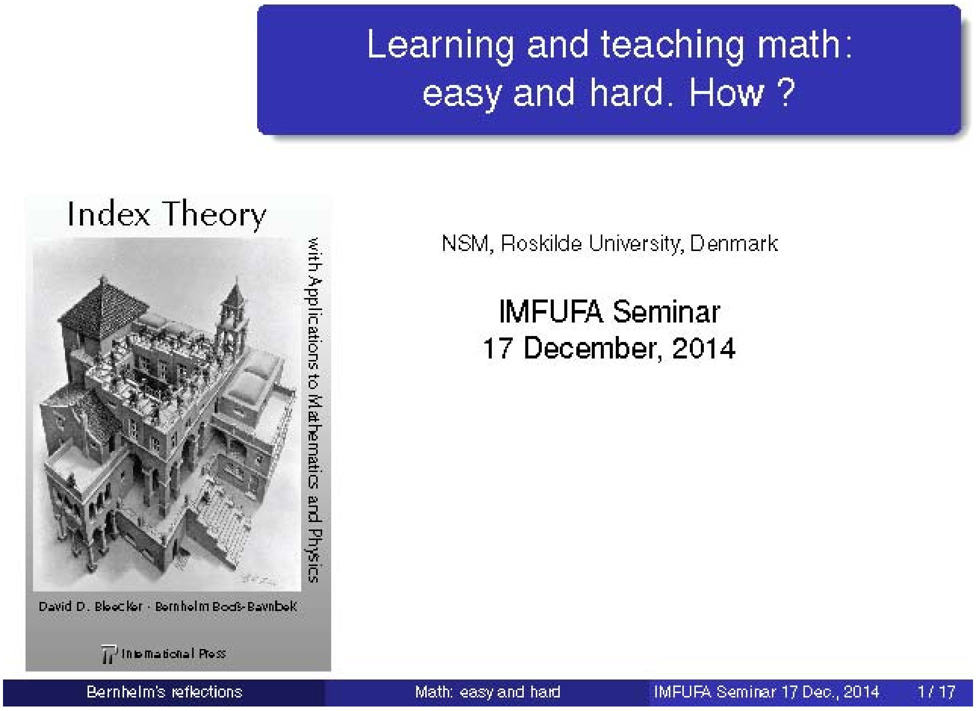

Figure 1. Logo for this paper

Dedication. In respectful memory of Ivor Grattan-Guinness (23 June 1941 – 12 December 2014).

2. Outline

This paper is in two parts. In Part I, I'm looking back to see whether the students' mind-set has changed since I was a beginner. In Part II, I shall summarize why math is, rightly, perceived of as hard. After that I shall turn to the opposite position, that math is easy and that it is a pity, when math is not perceived of as easily accessible.

3. The general meaning of mathematical working experience

Here is an indication of why the mathematical working experience has something to say to the general intellectual public.

Pulls and pushes. Our administrators tell us all the time, *Make it easy! Remember, it must be easy! Don't lose a student. You are losing the students.* Etc. However, our working experience is, that

- math is confusing and damned hard as long as one has not understood it, and when one has understood it, it is easy and clear; and
- math is invisible for the students in their environment, unless they look a bit beneath the surface.

All people have had their own personal experience: that they got *bruises* from mathematics. We got them when we were students. Later, as professionals, we also got bruises from working with mathematics. And for all people (laymen, students, and professionals) it can be difficult to recognize how mathematics works behind the scene in the real world. Of course, we may share the hope that, for doing, learning, applying, and teaching math, we get something interesting out of the theories of communication and psychology; get some hints at how to make research, development, and teaching better. But do we?

PART I

4. Looking back – have the students' mind-sets changed?

Recall the functional administrators' claim: *The mind of the students has changed and your teaching is worthless unless you change it, too.* That claim is supported by the general "wisdom" that we deal with a *browse-generation* or a *me-generation*, but it is misleading; in the sociology and neuropsychology literature, there is no evidence of such sudden and general, deep changes regarding our students' or our own capacity of learning and teaching mathematics.[2]

Evolutionary evidence. Our ancestors have bred dogs for at least 20.000-30.000 years, perhaps for 500.000 years.[3] Most modern dogs do not look like wolves and each breed looks different. But the animal psychologists write in their books that we must expect 80% of the behaviour of our dog to be lupine.[4] I do not know how they measure and quantify, but every dog owner can confirm: After 5.000 generations of breeding adjustments the basic behaviour has almost not changed. There is an astonishing stability.

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[2] For the systematic underestimation of dedicated students' capacity to protect themselves against the media dominance see **McLuhan, Marshall; with Fiore, Quentin; produced by Jerome Agel**. The medium is the massage: An inventory of effects. 1st Ed.: *Random House* 1967; reissued by *Gingko Press*, 2001. ISBN 1-58423-070-3; **Hurrelmann, Klaus**. Selbstsozialisation oder Selbstorganisation? Ein sympathisierender, aber kritischer Kommentar. *Zeitschrift für Soziologie der Erziehung und Sozialisation*, **22**/2 (2002), 155-166; **Sutter, Tilmann**. Medienanalyse und Medienkritik: Forschungsfelder einer konstruktivistischen Soziologie der Medien. *VS Verlag für Sozialwissenschaften. Wiesbaden.* 2010. ISBN 978-3-531-16910-1. Girls of 12 or 13 years who can't resist the social press or their own wish to upload naked selfies to the internet most probably must change later when they wish to study mathematics; see **Politiken Digitalt**. Unge sender hinanden afslørende nøgenbilleder i stort omfang. 3 April, 2014.

[3] **Larson, Greger; et al.** Rethinking dog domestication by integrating genetics, archaeology, and biogeography. *Proc Natl Acad Sci U S A.* Jun 5, 2012; 109(23), 8878–8883.

[4] **Svartberg, Kenth; Forkman, Björn**. Personality traits in the domestic dog (Canis familiaris). *Applied Animal Behaviour Science* **79** (2002), 133–155; **McBride, Anne**. The human–dog relationship. In: Robinson, I. (Ed.), The Waltham Book of Human–Animal Interaction: Benefits and Responsibilities of Pet Ownership*. Pergamon, Oxford*, 1995. ISBN 978-1483234748, pp. 99–112; **Tami, Gabriela; Gallagher, Anne**. Description of the behaviour of domestic dog (Canis familiaris) by experienced and inexperienced people. *Applied Animal Behaviour Science* **120** (2009), 159–169.

When talking about learning and teaching mathematics, we also deal with the human mind: it is quite different from that of dogs, much more variable, namely much more able to adapt to new situations. But is there evolutionary evidence for deep changes in basic human behaviour?

One answer is the Cro-Magnon aesthetics in La grotte de Lascaux, in Le tombe di Tarquinia, and in the works of Tiziano Vecellio, Paula Modersohn-Becker, and Jackson Pollock. All of them give strong evidences of the apparently indestructible stability of *human curiosity* and *imagination*, of *creativity* and *concentration*. Exactly what we want from our students.

What then is the supposed ongoing secular media-generated change of consciousness about? Can we easily discard all the present talking on ongoing media-generated changes of our own and our students' life style and ways of thinking?

The short answer is yes and no!

- No: social influences cannot be discarded. Otherwise, showing our personal example and delivering our teaching would be dispensable. And we don't want to believe that.
- Yes: for doing math, disturbing social influences must be eliminated or at least confined; learning math requires time and full concentration, and a student will hardly make progress, if he or she is not able to let her be absorbed by mathematics for hours, days and years.

A. Luria (1902–1977). In 1974, the Soviet neuropsychologist and developmental psychologist Alexander Romanovich Luria published a comprehensive empirical study about cognitive changes induced by social conditions – under the extreme social changes of the first years after the Bolshevist revolution.[5]

He found indeed remarkable differences, e.g., that people raised in larger and more urban places were good at taxonomic classification, i.e., the use of *abstract categories* like *tools* while people raised in remote areas were better at classifications regarding *practical situations* like the work process of *sawing and chopping wood*.

For doing mathematical research and developing applications, for learning and teaching math, Luria's message is that all people can easily adapt to radical new ways of thinking in new environments. The only precondition is that the new environments are presented in a peaceful way, notwithstanding how dramatic and even painful the changes and the challenges may be.

Preliminary conclusion. There is no evidence of short term changes of the students' mind-set.

[5] **Luria, Alexander R**. The cognitive development: its cultural and social foundations. *Harvard University Press.* 1976 (Translation from the Russian original of 1974). ISBN 0-674-13731-0. Here pp. v and, in particular, pp. 64 and 164. Similarly, but from a religious (Roman Catholic) perspective the essay **Ong, Walter J**. Interfaces of the word: studies in the evolution of consciousness and culture. *Ithaca, N. Y. Cornell University Press,* 1977. - 352 s. ISBN 0-8014-1105-x.

PART II

5. Math is hard – How and why?

The vest trick. Some time ago, I was consultant/supervisor of a third semester project. In the semester opening I presented myself with the entertaining vest trick, to illustrate what a topologist (my speciality) is doing, namely to think about questions like *How is it possible that I can remove a vest under my jacket*? *Has it something to do with the uneven number of my heads or the even number of my arms etc.*? The students asked *Is this mathematics*? We discussed it. I had to admit that a similar problem in two dimensions is well understood. It is the question whether a closed curve in the punctured plane (a plane where one point, e.g., the origin is removed) is contractible. The question can be easily decided by calculating the winding number of the curve; and there are various and seemingly very different methods to do that (calculating a path integral, solving a differential equation, by combinatorics, etc.). The curve is contractible if and only if the winding number vanishes. But what can we do with the three-dimensional case?

Those were *not* the questions which the students were most interested in. *Neither* were they surprised over that I could not tell them a solution at once. What so triggered their curiosity and gave a full semester's work was their own question, namely can one tell from the outside whether a mathematical problem is hard or easy. More precisely: *How can it be that many problems in mathematics are easy to formulate but difficult to solve*? They investigated that question by looking carefully at various historical examples (long and fascinating stories). Anyway, in this way they gave me the idea for this Part II of my paper: What were the answers of the giants of mathematics?

J.-L. Lagrange (1736-1813). To me, Giuseppe Lodovico (Luigi) Lagrangia (Joseph-Louis Lagrange) is one of the most important figures in the history of mathematics. He was extremely successful in introducing radically new and often highly abstract concepts to make mathematical ideas clear and comprehensible also to a non-genius which otherwise would have remained the domain of the intuition of outstanding people. He replaced Euclid's polygons and Descartes's curves by homotopies; to me, his Second letter to Euler, of 12 August, 1755, is the birth certificate of deformation theory and differential topology; and he replaced the Eulerian mechanic that intended following the ever more confusing visible orbits of single pieces by his mechanic of the easier capable underlying invisible potentials.

When he died on the 10 April 1813, there were an official obituary (by Delambre) and a 'Supplement' by a *G.* The supplement dealt with his last days and the thoughts he expressed shortly before his death. Nobody knows who the G. was. There are some speculations which were investigated in a paper by the math historian Ivor Grattan-Guinness.[6] He argues that G.'s Supplement is believable.

The following quotes of Lagrange are from Grattan-Guinness' paper. Until his death, according to these documents, Lagrange war so sorry for his students, that they

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[6] **Grattan-Guinness, Ivor**. A Paris curiosity, 1814: Delambre's obituary of Lagrange, and its "supplement". *Mathemata,* 493--510, Boethius Texte Abh. Gesch. Exakt. Wissensch., XII, *Steiner, Wiesbaden,* 1985. MR0799763.

had to read his textbooks, Lagrange's masterpieces in Mechanics that are so much more elaborated, intricate, and harder than all previous mechanics treatises. Of course, Lagrange was right: His books were dispensable for the calculation of simple mechanical systems - but indispensable for making complex mechanical systems transparent for the human brain and understandable and calculable. He felt "sorry for the young geometers who have such thorns to swallow. If I had to start again, I would not study: These large in-4° would make me too scared". He proposed instead a one-volume reprint of original works of the calculus by Fermat, Leibniz, l'Hôpital and especially John Bernoulli's lectures on the integral calculus, together with another volume comprising items by Euler and d'Alembert.

Already Delambre quoted Lagrange for "If I had had a fortune, I would probably not have made my profession [état] in mathematics." G. supplemented by recalling an occasion when Lagrange had met 'a young man devoting himself to the exact sciences with much ardour', and upon asking him 'Do you have a fortune?' and receiving a negative answer had replied: 'so much the worse, sir. The lack of fortune and of the existence it can give in the world, is a constant stimulus which nothing can replace, and without which one cannot bring to hard tasks all the necessary progress [suite]'.

When Lagrange was teaching, his 'researching intelligence' (G.) could cause sudden lapses in conversation. G. described the effect on his lectures at the École Polytechnique:

> Who has not seen him suddenly interrupt himself thus in the lectures which he gave at the École Polytechnique, appearing sometimes embarrassed like a beginner, leaving the blackboard and coming to sit down opposite the audience, while teachers and students, confused on the seats [bans] expected in a respectful silence that he would have led his thought back from the spaces that it had gone to travel through.

To Lagrange, all mathematics was hard, also when it was seemingly easy for the student and would relieve its hardness only for the expert. So, the main goal of a mathematician's life was to think how to make math easier and more accessible, sometimes at the cost of introducing further abstract and more elaborated concepts.

In essence and in my reformulation: Math can be made easy and comprehensible only by accepting and enduring its hardness. Students are exposed to the cultural clash immanent in abstractions, formalism and symbol processing. Teachers must help them to experience that clash as a positive step like processes of adolescence or seeking work abroad, and not as a series of defeats. For sure, it doesn't help with well-intended lies or self-deception about easy access to mathematical abstractions as demanded by the new caste of administrators. Acquiring mathematical experience is nothing that falls from heaven or comes from playing on the ground. It requires work, concentration, exercises, and endurance: Ὁ μὴ δαρεὶς ἄνθρωπος οὐ παιδεύεται (The non-flayed human will not be educated, Menander, c. 341/42– c. 290 BCE, disseminated by J.W. Goethe as motto over his autobiography *Dichtung und Wahrheit*), or less draconic, Ohne Fleiss kein Preis (Without hard working no praise, after Hesiod, thought by scholars to have been active between 750 and 650 BCE).

The mathematicians I admire most are very close to Lagrange's position in continuing a life-long interest in teaching math and insisting that the essence of math, triggering curiosity and creativity and its true place in applications is that it is hard, and that it becomes dispensable and replaceable by engineering arts and econometric analyses etc. when it becomes easy.

I.M. Singer (*1924). Rightly, he can be proud of his achievements, among others the Index Theorems, which brought him the Abel Prize in 2004 jointly with M.F. Atiyah. When afterwards he was asked what to do next, he did not hesitate: *Now I want to use more sophisticated mathematics not yet available to physics.*[7] Clearly, to Singer the role of mathematics is to deal with extremely hard problems.

Part of the story is that in all recent years this same man, now 90 years old, participates at MIT in the math teaching of beginners, and as he says with great intellectual satisfaction, nursing and watching the emerging math understanding of young students:

> A while back I decided to be a TA in the freshman calculus course. I think I was motivated to do so because I had been too far removed from undergraduates. Making contact with freshmen again was a wonderful experience… Teaching does integrate with my other work. I'm inclined to understand rather than solve. For me, doing research means understanding something nobody has understood, and then telling others about it. What makes me a good teacher is empathy. I can put myself in the position of a student and know what they don't understand. If I know them well enough, I can explain what they don't understand in terms they can comprehend.[8]

V.I. Arnold (1937-2010). Some attribute to him and his former students the most decisive advances in the mathematical understanding of dynamical systems since the seminal work of H. Poincaré more than 100 years ago. When he was asked about the situation of math in Russia after the fall of the Soviet Union he deplored in his sarcastic way: *Well, it's terrible. Now the professors are cleverer and know more than the students.*

How sad. Indeed, teaching and learning math is only interesting when the teacher in each meeting with the students, say of one hour, gets at least one new mathematical idea. Otherwise it doesn't work with our goal, namely to socialize a new generation of math students to the way of mathematical thinking. The hour would have been lost – or could have been left to an electronic instruction device - with the same default result.

In an article[9] tracing the history of his own research, Arnold showed how apparently unrelated subjects are linked by a kind of mycelium from which theorems pop up like mushrooms. Continuing his life-long battle against formalism and Bourbakism, he distinguishes the easiness of communicating formal theorems from the hardship of explaining the underlying ideas in the following parable:

> When you are collecting mushrooms, you only see the mushroom itself. But if you are a mycologist, you know that the real mushroom is in the earth. There's an enormous thing down there, and you just see the fruit, the body that you eat.

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[7] **Singer, I.M.** Transcript of May 12, 2010 MIT150 interview, http://mit150.mit.edu/infinite-history/isadore-singer, accessed May 21, 2015.
[8] L.c.
[9] **Arnold, Vladimir I.** From Hilbert's superposition problem to dynamical systems. *Mathematical events of the twentieth century,* 19--47, *Springer, Berlin,* 2006. MR2182777.

> In mathematics, the upper part of the mushroom corresponds to theorems that you see, but you don't see the things which are below, that is: *problems, conjectures, mistakes, ideas,* and so on.
>
> You might have several unrelated mushrooms being unable to see what their relation is unless you know what is behind. And that's what I am now trying to describe. This is difficult, because to study the visible part of the mathematical mushroom you use the left half of the brain, the logic, while for the other part the left brain has no role at all, since this part is highly illogical. It is hence difficult to communicate it to others.

F. Hirzebruch (1927-2012). Since the 1950s he was the outstanding figure of mathematics in Western Germany. He was the natural candidate as director of a Max-Planck Institute (MPI) in mathematics, and he became the director of the first MPI in mathematics in Bonn in the 1980s. But for decades there was no MPI for mathematics. Shortly after the end of World War II, many MPIs were founded and lavishly financed to bring the sciences in the Federal Republic of Germany rapidly back to international top level after the decline and demolition during Nazi time.

Once I asked Hirzebruch why the Bonn MPI for mathematics came so late? He frankly told me that at least one reason was a controversy between him and the Board of the Max-Planck Gesellschaft (MPG).

Contrary to the MPI tradition of teaching-free research, Hirzebruch had insisted that research in mathematics without teaching is meaningless, that, as a rule, new mathematical results are too hard to be digested at the distance; that they will falter rapidly when they are not forwarded instantaneously to new generations in interpersonal communication; that most young students need the contact and the role model of a successful researcher to overcome the hardships of acquiring math. Consequently, there should only be very few permanent positions for the directors and support staff, while the main human resources should consist of university teachers on leave as guest researchers for midterm stays. It took him several decades to reach the MPG's acceptance for this deviating status of math research, that it is meaningless without the umbilical cord to teaching and that all members of the MPI for math had to have an association with teaching.

Y.I. Manin (*1937). Like Hirzebruch, Manin is a magician who can create a world of deep interrelated concepts and results to his audience within 60 minutes, and so that most people in his audience have a strong feeling of having understood a lot, of being almost able to walk on water. Of course, when you go home and begin to work your way through your notes, your feeling will change and you will feel stupid and discouraged: too many things you can't understand in detail, and that means in math that you don't understand.

Manin himself commented that in his textbook on Mathematical Logic[10], namely that mathematical abstractions are hard to grasp; that thinking in symbols, while extremely effective in many contexts and indispensable in some, is deeply against the human nature. He explains that very carefully in his book and partially with references to

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[10] **Manin, Yuri I.** A course in mathematical logic for mathematicians. Second edition. Chapters I–VIII translated from the Russian by Neal Koblitz. With new chapters by Boris Zilber and the author. Graduate Texts in Mathematics, 53. *Springer, New York,* 2010. xviii+384 pp. ISBN: 978-1-4419-0614-4 MR2562767

facts based on some observations made by the psychologist A. Luria on patients with brain injuries. Some of the patients had preserved a sound judgement of the situation in hospital, e.g., of differences between various doctors and nurses in their competences and engagement, but lost the ability to think in relations: is an elephant bigger than a fly or a fly bigger than an elephant? His claim: *Abstraction is in essence more difficult for the human nature than doing judgements on personal relationships etc.*

Note that fully, and consciously and explicitly, Manin's insight or claim is directed against the traditional claims of all logicists and many adepts of mathematization who consider the process of abstraction and formalization as a process of simplification and clarification.

C.S. Peirce (1839-1914). He had an anthropological message that our concepts, also our scientific concepts have evolved in human praxis of more than 100.000 years in experience with the various contexts humans have had over time.

The good side of the message is, that in most situations common sense and scientific, mathematics based arguments need not contradict each other. The bad side of the message is, with view to the emerging quantum mechanics at Peirce's time, that we have a problem when dealing with phenomena in an artificial environment that our mind has not been accustomed to for thousands of years. Then we must transgress common sense because it will for the most part be systematically misleading.

C.F. Gauss (1777-1855) and **L. Hörmander** (1931-2012). They were masters in standard formulations when they reviewed the work of other mathematicians: *Incomprehensible --- wrong --- I did it a long time ago.*

To me, such typical referee reports prove that reading math papers is always hard, even for the greatest math geniuses. Correspondingly, we have in mathematics two very different exclamations of agreement, *it's trivial* and *it's clear*. The first is pejorative: don't waste my time with your boring stuff; the second is highest acclamation: aha, now I see! This is really hard stuff you are telling me!

H. Cramér (1893 – 1985). In his monumental monograph *Mathematical Methods of Statistics* of 1945, Cramér proved that the chi-square test statistic, i.e., the sum of relative errors between observed and expected magnitudes with f degrees of freedom, is distributed like the corresponding chi-square distribution with f degrees of freedom. For $f = 1$, it is the classical result by Karl Pearson of 1900, the proof of it is reproduced in most textbooks of mathematical statistics. For applications in material sciences, biology and medicine, Cramér's theorem is applied. Perhaps it is the most applied mathematical theorem of the 20th century. But to my best knowledge its proof has never (!) been reproduced. You can only find it in Cramér's textbook[11]. It is lengthy and not very inspiring. It is laborious – and boring. The main idea is much clearer for $f = 1$ than in the general case.

[11] **Cramér, Harald**. Mathematical Methods of Statistics. Princeton Mathematical Series, vol. 9. *Princeton University Press, Princeton, N. J.,* 1946. xvi+575 pp. MR0016588, here chapter 29.

Such is mathematics that it has theorems that are easy to apply but hard to understand and, in practice, perhaps understood only by the author of the theorem and a handful readers of the original publication.

P.J. Davis (*1923). In a public talk in Roskilde, Davis gave a similar example when he confessed that he never had completed his checks of the proof of the *principal axis theorem* on block-diagonalization of normal matrices in linear algebra in spite of the fact that that theorem was a central tool in many of his works on effective numerical methods.

In a recent paper, Davis expanded on his view that we must live with some imperfections also in mathematics, that some basic tasks in numerical analysis are too hard to admit a rigorous approach.[12] Among his examples he refers to the concept of *numerical stability* in iterations, when, contrary to the toy examples of elementary classes in numerical analysis, no estimates are available about the achieved precision of an approximate result. Nevertheless, we have to stop the iterations at some point. For solving systems of differential equations, a common stop rule is when the results become unchanged under further iteration or refinement of the underlying discretization. Then a result seems to become stable and reliable, while we have examples where numerical stability can be achieved far from the true result. To comfort our mathematical fears and incertitude Davis usually cites Richard Hamming (1915-1998) for having said *I would never fly with a plane where the construction depends on the difference between Riemann and Lebesgue integral.*[13]

Davis points to another symptom of the difficulty of doing math, namely our almost unlimited *freedom* to add or to remove assumptions that is though sharply *restrained* by logical demands regarding the formulation and consistency and even more sharply restrained by respectful regards to the history of a topic and which examples or expansions might be considered meaningful and which not.

Warning 1. From a technological point of view, *hard and presently unsolved problems* are wonderful and highly applicable, like the present lack of efficient algorithms to factorize a given product of two large prime numbers into its two components, or other presently unsolved problems regarding elliptic curves that keep the common public keys in cryptology relatively safe!

Warning 2. For numerical algorithms in the analysis of dynamical systems and of combinatorial tasks, e.g., in graph theory, mathematicians try to give *asymptotic estimates* about the complexity (i.e., the expected time necessary for a solution) of a problem. By

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[12] **Davis, Philip J.** The relevance of the irrelevant beginning, *ScienceOpen Research*, 2014, 5 pp, DOI: 10.14293/A2199-1006.01.SOR-MATH.6G464.v1.

[13] **Hamming, Richard W.** Mathematics on a distant planet. *Amer. Math. Monthly* **105** (1998), no. 7, 640--650. MR1633089, The full quote is "for more than 40 years I have claimed that if whether an airplane would fly or not depended on whether some function that arose in its design was Lebesgue but not Riemann integrable, then I would not fly in it. Would you? Does Nature recognize the difference? I doubt it!" [p. 644]. Certainly, Hamming's insistence on robustness in applications is a relief. However, it is a fact that certain highly applicable concepts like the Hilbert space L^2 of equivalence classes of measurable, square-integrable functions can only be established by embracing all Lebesgue integrable functions to obtain the indispensable completeness.

definition, the problems that are hardest to solve are the so-called NP-complete problems like the travelling salesman problem. The for practical purposes perfect organization of just-in-time delivery for retail chains shows that one never should become blocked in search for practical solutions by asymptotic seemingly insurmountable estimates.

Piet Hein (1905-1996).

Problems worthy
of attack
prove their worth
by hitting back.

6. Math is easy – Really?

In the previous Section 5, I tried to explain why learning and doing math is hard, by necessity. But what about the many people, pupils, students, teachers, researchers, who love to spend many hours thinking about a mathematical problem; some early in the morning when one is fresh, some late in the night when one is not disturbed, some on their desk and some while jogging or walking their dogs? And what about the rich treasures of investigations, suggestions and predictions how doing math can be made easier and more accessible?

Let me comment upon the most outspoken positions.

A. Schopenhauer (1818). In his treatise *Die Welt als Wille und Vorstellung (The World as Will and Representation*), the philosopher – or rather a philosopher-poet like the many other German philosopher-poets Hegel, Nietzsche, Heidegger with their love for extensive formulations – Arthur Schopenhauer released the following torrent of words[14] against the mathematicians' arrogance and stupidity making math, according to Schopenhauer unnecessarily hard and non-intelligible, and that Euclid's classical arguments were monstrous and dispensable:

> … mathematical knowledge *that* something is the case is the same thing as knowledge of *why* it is the case, even though the Euclidean method separates these two completely, letting us know only the former, not the latter. But, in Aristotle's splendid words from the *Posterior Analytics*, I, 27: 'A science is more exact and more excellent if it tells us simultaneously *what* something is and *why* it is, not *what* it is and *why* it is separately.' In physics we are satisfied only when our recognition *that* something is the case is united with our recognition of *why* it is, so the fact that the mercury in a Torricelli tube is 28 inches high is a poor kind of knowledge if we do not add that it is held at this height to counterbalance the atmosphere. So why should we be satisfied in mathematics with the following occult quality of the circle: the fact that the segments of any two intersecting chords always contain equal rectangles? Euclid certainly demonstrates it in the 35th proposition of the third book, but why it is so remains in doubt. Similarly, Pythagoras' theorem tells us about an occult quality of

[14] **Schopenhauer, Arthur**. Die Welt als Wille und Vorstellung. *Werke in 5 Bänden, hrsg. von L. Lütkehaus. Haffmans, Zürich* 1991, vol. 1, §15, p. 119. English translation in: The world as will and representation; translated and edited by Judith Norman, Alistair Welchman, Christopher Janaway; with an introduction by Christopher Janaway. *The Cambridge Edition of the Works of Schopenhauer. Cambridge University Press, Cambridge and New York*, 2010, p. 98.

the right-angled triangle: Euclid's stilted (stelzbeinig), indeed underhand (hinterlistig), proof leaves us without an explanation of why, while the following simple and well-known figure yields more insight into the matter in one glance than that proof, and also gives us a strong inner conviction of the necessity of this property and of its dependence on the right angle:

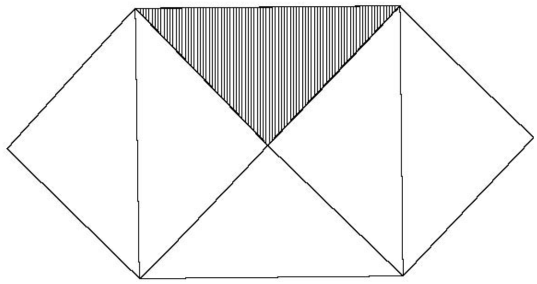

Figure 2. Schopenhauer's fantasied "simplification"

As often when outsiders comment on math it strikes me how little they understand of the crux of a mathematical achievement. So also Schopenhauer: The crux of Pythagoras' Theorem is its validity for *all* rectangular triangles in the plane, i.e., even when the sides at the right angle are unequal. By the way, that's until today the most typical application of the theorem in construction: To check whether the walls in a room or house are rectangular, a carpenter would mark a 3-meter (or yards) point upward in a corner, a 4-meter (or yards) point along a wall on the floor, and then check whether the straight line between the two marks is exactly $5 = \sqrt{(3^2 + 4^2)}$ meters (or yards).

One would expect an error term; but no, Pythagoras claims and Euclid proves that the error term vanishes even when we deform the rectangular triangle, within the class of rectangular triangles. Later generations proved that Pythagoras' theorem remains basically valid even for non-rectangular triangles, incorporating an error term coming from the cosine of the included angle, and for rectangular triangles on a sphere, incorporating a curvature error term coming from the sphere's radius.
So, for a mathematician the Pythagoras' Theorem is an approximation theorem, that you can change something with controlled effects, sometimes with zero effect, sometimes with nonvanishing, but calculable effects.

Of course, Schopenhauer is right: math can be much easier when we remove the key points and reduce it to trivialities. Actually, we can answer Schopenhauer, that math would become even easier, when we reduce it to the empty set. So far Schopenhauer only shows his lack of understanding.

However, rightly he points to the difference between checking a proof, line by line, as opposed to grasping the reason for the validity of a claim. Every mathematician has experienced it: that we still don't understand a given proof after we have checked it step-by-step. Hence, in modern textbooks and for papers in learned journals, authors are praised when they explain the underlying idea of a proof before the reproduction of the proof in its details.

C.F. Gauss (1777-1855). His reply to Schopenhauer was: *On the contrary! Mathematics is so difficult that we never should tell the reader how we got the idea. In most cases it* *will be either impossible or distracting to make the idea explicit. To make results accessible we shall hide all complications we had to meet and overcome, and keep silent about the wrong tracks we went when searching and finding the proof. What counts in mathematics is only a presentation of the purified final form*.[15] For 200 years, Gauss perception of simplicity through hiding the birth bangs and presenting only the sleek version has dominated the publication culture of mathematics. Fortunately it has been on the retreat along with the retreat of Bourbakism.

M.F. Atiyah (*1929). While he personally has contributed to the creation of many new mathematical theories, concepts, and methods, he insists that math is getting easier and more transparent by the emergence of any new mathematical achievement. He compares math with a ware house*. Looking for a box of nails in a small country shop and finding the right ones can be harder than looking around in a big specialized department store like Bauhaus.* Clearly, it is easier to find your way around in a big, well-organized modern department store than an old fashioned pop store. Making math more complex opens many new crossroads and makes search and communication easier. Such is the argument.[16]

Atiyah's optimistic claim is based on his view of ever clearer emerging unity of mathematics. In a recent paper, that unity belief was refuted by Davis and me as a myth.[17]

J.-L. Lagrange (1736-1813). This is the same Lagrange who appeared before as witness of the inevitable hardness of math. Now I call him as witness for the ease and simplicity of mathematical physics. Regarding the celestial bodies of our Solar system, he noticed that the planets are moving on almost circular orbits and most comets on very eccentric elliptic orbits. He concluded that *Nature favours planetary approximations by grouping heavenly bodies according to very small and enormous eccentricity*.[18] Indeed, for each of the two extreme cases we have specific and very powerful expansions, which would fail in the middle range. Modern astrophysics teaches us, however, that this is a very special

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[15] A typical example is provided by Gauss' first proof of the Fundamental Theorem of Algebra of 1799, **Gauss, Carl Friedrich.** Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*. Helmstedt: C. G. Fleckeisen*. 1799 (tr. New proof of the theorem that every integral rational algebraic function of one variable can be resolved into real factors of the first or second degree). German translation in: **Netto, Eugen** (ed.): Die vier Gauss'schen Beweise für die Zerlegung ganzer algebraischer Funktionen in reelle Factoren ersten oder zweiten Grades (1799–1849), *Ostwald's Klassiker der Exakten Wissenschaften Nr. 14, Wilhelm Engelmann, Leipzig* 1890 , pp. 3-36, 83 (figures). Accessible at University of Toronto, https://archive.org/details/dieviergausssche00gausuoft. Contrary to d'Alembert's predecessor proof of 1746, Gauss keeps this proof deliberately in purely real terms. However, one can easily trace the underlying complex constructions in the real presentation.

[16] **Atiyah, Michael F.** Trends in pure mathematics. In: *Proc. of the 3rd Internat. Congress on Mathematical Education (Karlsruhe 1976).* 1979, pp. 61-74. Reprinted in Collected works vol. 1, pp. 261—276. MR0951896.

[17] **Booß-Bavnbek, Bernhelm; Davis, Philip J.** Unity and Disunity in Mathematics. *Newsletter ofthe European Mathematical Sociey* No. 87 (March 2013), 28-31.

[18] Here is the full quote of Lagrange given in his above cited obituary by the anonymous G., offering a Lagrange type witticism: "It seems that nature had disposed these orbits [of the heavenly bodies] specially so that one may calculate them. Thus the [sic] eccentricity of the planets is very small, and that of the comets is enormous. Without this disparity [,] so favourable to approximations, and if these constants [of the orbits] were of an average magnitude*, goodbye geometers*; one could do nothing."

property of our solar system due to the dominance of the two gas planets Jupiter and Saturn which, at the whole, make our system so surprisingly stabile. Other solar systems in the Milky Way seem to provide for more challenging mathematical problems.

We may expand Lagrange's argument for *nature provided simplicity* to large parts of mathematical physics where we, e.g., not have to deal with very general differential equations with arbitrarily varying coefficients but with geometrically defined operators with strong inherent symmetries like the Laplace or the Dirac operator that are, moreover, often controlled by potentials and other background fields. Therefore large parts of mathematical physics that are based on first principles and geometry are mathematically easier and more accessible than some parts of biology that are less mathematized, based on ad-hoc assumptions and so tangled up in non-controllable generalizations.

More, and more trustworthy arguments. Until now, in this Section, I discarded common, partly ingenious suggestions and beliefs why and how doing math can become easier. I shall now turn to considerations that are also controversial, but definitely not to be discarded by me. It seems to me that they have the potential to explain why and how learning and doing math can appear personally satisfactory, natural and, from time to time even easy for some people in lucky moments and periods of their life.

I have written about the following quite different approaches separately and extensively[19] and shall be brief in this paper.

N. Chomsky (*1928). His message, or at least the message disseminated by his student Pinker[20], is that *Math is easy. Every child has solved the greatest math exercise of her or his life at the age of two, when it forms the generative grammar of the child's mother tongue and assembles the patterns and basic structures out of single words.* I'd better add that some of Chomsky's claims are controversial, in particular his biologistic assumption of special genetic grammar traits of the human race that are not confirmed by molecular geneticists.

C.S. Peirce (1839-1914)[21], **A. Gramsci** (1891-1937)[22], **P. Freire** (1921-1997)[23], **P. Naur** (*1928)[24]. To me, these four names stand both for

- deep insight into the complexity of human thinking and communication, and for
- demystification of feeling, learning, and doing by relating it to human *habits* and *forms of life*.

Their teaching for the topic of this talk can be roughly summarized in the following short formula:

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[19] **Booß-Bavnbek, Bernhelm**. On the difficulties of acquiring mathematical experience, *EM TEIA – Revista de Educação Matemática e Tecnológica Iberoamericana* **5** - número 1 (2014), 1-24. Also: http://milne.ruc.dk/~Booss/Math_Easy-and-hard_Presentation/2014_BBB_EMTEIA.pdf

[20] **Pinker, Steven.** The language instinct. *New York: William Morrow***,** 1994.

[21] l.c.

[22] **Gramsci, Antonio**. Selections from the Prison Notebooks. *New York: International,* 1971.

[23] **Freire, Paolo.** Pedagogy of the oppressed. *New York: Herder and Herder*, 1972.

[24] **Naur, Peter.** Computing: a human activity. *New York: ACM/Addison-Wesley*, 1992.

1. Trace the *habits of nature*;
2. relate our feeling, thinking, and doing to our *form(s) of life*, take the risks and jumps of adolescence and accept the related clash of cultures;
3. for mathematics, exploit the translational power (and handle the 2 contradictions below) by *coding math experiences* and make them *transferrable for adaption* in new contexts.

Two contradictions. All math learning and teaching has to live with and to handle the two following contradictions:

A. Result v. process. We need to teach *results*, not only *processes*, not only ways of thinking; one needs results in sciences and mathematics.
B. Context v. abstraction. Students learn best in *context*, when they can see meaning and embedding in context; however, the power of mathematics is that it can be separated from the context; that is the true power of *abstractions* that we can transport experiences from one context to another one.

We cannot discard or bridge these two contradictions firmly. We must tell the math education administrators, that doing, learning, teaching math is difficult and requires time for the body and peace for the mind. We cannot deliver what they want, an easier, faster and more accessible teaching in the sense they want. Our only hope is:

S. Kierkegaard (1813 -1855). In *Enten-eller*, he explained the two most difficult situations in life for him, the love for another human and the love for God[25]. I don't agree fully with Kierkegaard, neither with the first situation where I have some personal experience, nor with the second, where I'm blank. Anyway, Kierkegaard emphasizes that both of these two situations require deep feelings: Let yourself be *seduced* and develop the *passion*!

Afterword. This is what math doing, learning, and teaching is good for when it is successful. On some occasions you'd better lie and follow the love advice of Elias Canetti (1905- 1994): *Don't tell me who you are. I want to adore you.* So, you need not tell the students the full truth[26] every day, e.g., about the

- destructive sides of math supported technology; about
- math induced inhuman formatting of social organisation; and
- the deformations of the mind by naïve belief in logic and modelling.

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[25] **Kierkegaard, Søren**. Either/Or. Volume I. *Princeton: Princeton University*, 1959. See *The Immediate Stages of the Erotic or the Musical Erotic*, pp. 43-134, in particular pp. 62, 93, 114.

[26] In **Hardy, G. H.** A mathematician's apology. With a foreword by C. P. Snow. Reprint of the 1967 edition. Canto. *Cambridge University Press, Cambridge,* 1992. 153 pp. ISBN: 0-521-42706-1 MR1148590, p. 33, n .16, Hardy ponders about his 1915 quote: "a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life." See also **Arnold, Vladimir I.** Polymathematics, l.c., p. 403, paraphrasing Hardy: "All mathematics is divided into three parts: cryptography (paid by CIA, KGB and the like), hydrodynamics (supported by manufacturers of atomic submarines), and celestial mechanics (financed by military and other institutions dealing with missiles, such as NASA)." and the anthology Mathematics and war. Papers from the International Meeting held in Karlskrona, August 29–31, 2002. Edited by Bernhelm Booß-Bavnbek and Jens Høyrup. *Birkhäuser Verlag, Basel,* 2003. viii+416 S. ISBN: 3-7643-1634-9 MR2033623, free download at http://www.springer.com/gp/book/9783764316341#otherversion=9783034880930.