The Maslov index in symplectic Banach spaces

Bernhelm Booss-Bavnbek and Chaofeng Zhu

ABSTRACT. We consider a curve of Fredholm pairs of Lagrangian subspaces in a fixed Banach space with continuously varying *weak* symplectic structures. Assuming vanishing index, we obtain intrinsically a continuously varying splitting of the total Banach space into pairs of symplectic subspaces. Using such decompositions we define the curve's Maslov index by symplectic reduction to the classical finite-dimensional case. We prove the transitivity of repeated symplectic reductions and obtain the invariance of the Maslov index under symplectic reduction, while recovering all the standard properties of the Maslov index.

As an application, we consider curves of elliptic operators which have varying principal symbol, varying maximal domain and are not necessarily of Dirac type. For this class of operator curves, we derive a desuspension spectral flow formula for varying well-posed boundary conditions on manifolds with boundary and obtain the splitting of the spectral flow on partitioned manifolds.

CONTENTS

2010 *Mathematics Subject Classification.* Primary 53D12; Secondary 58J30.

Key words and phrases. Banach bundles, Calderón projection, Cauchy data spaces, elliptic operators, Fredholm pairs, general spectral flow formula, Lagrangian subspaces, Maslov index, symplectic reduction, unique continuation property, variational properties, weak symplectic structure, well-posed boundary conditions.

Corresponding author: CZ [zhucf@nankai.edu.cn].

List of Figures

Introduction

Purpose and message. The purpose of this paper is to establish a universal relationship between incidence geometries in finite and infinite dimensions. In finite dimensions, counting incidences is nicely represented by the *Maslov index*. It counts the dimensions of the intersections of a pair of curves of Lagrangian subspaces in a symplectic finite-dimensional vector space. The concept of the Maslov index is non-trivial: in finite dimensions, the Maslov index of a loop of pairs of Lagrangians does not necessarily vanish. In infinite dimensions, counting incidences is nicely represented by the *spectral flow*. It counts the number of intersections of the spectral lines of a curve of self-adjoint Fredholm operators with the zero line. In finite dimensions, the spectral flow is trivial: it vanishes for all loops of Hermitian matrices.

Over the last two decades there have been various, and in their way successful attempts to generalize the concept of the Maslov index to curves of Fredholm pairs of Lagrangian subspaces in strongly symplectic Hilbert space, to establish the correspondence between Lagrangian subspaces and self-adjoint extensions of closed symmetric operators, and to prove spectral flow formulae in special cases, namely for curves of Dirac type operators and other curves of closed symmetric operators with bounded symmetric perturbation and subjected to curves of self-adjoint Fredholm extensions (i.e., well-posed boundary conditions). While these approaches vary quite substantially, they all neglect the essentially finite-dimensional character of the Maslov index, and, consequently, break down when one deals with operator families of *varying* maximal domain. Quite simply, there is no directly calculable Maslov index when the symplectic structures are weak (i.e., the symplectic forms are not necessarily generated by anti-involutions J) and vary in an uncontrolled way.

In this paper we show a way out of this dilemma. We develop the classical method of symplectic reduction to yield an *intrinsic* reduction to finite dimension, induced by a given curve of Fredholm pairs of Lagrangians in a fixed Banach space with varying symplectic forms. From that reduction, we obtain an *intrinsic* definition of the Maslov index in symplectic Banach bundles over a closed interval. This Maslov index is calculable and yields a general spectral flow formula. In our application for elliptic systems, say of order one on a manifold M with boundary Σ, our fixed Banach space (actually a Hilbert space) is the Sobolev space $H^{1/2}(\Sigma;E|_\Sigma)$ of the traces at the boundary of the $H^1(M;E)$ sections of a Hermitian vector bundle E over the whole manifold. For $H^{1/2}(\Sigma;E|_\Sigma)$, we have a family of continuously varying weak symplectic structures induced by the principal symbol of the underlying curve of elliptic operators, taken over the boundary in normal direction. That yields a symplectic Banach bundle which is the main subject of our investigation.

Whence, the message of this paper is: The Maslov index belongs to finite dimensions. Its most elaborate and most general definitions can be reduced to the finite-dimensional case in a natural way. The key for that - and for its identification with the spectral flow - is the concept of Banach bundles with weak symplectic structures and intrinsic symplectic reduction. From a technical point of view, that is the main achievement of our work.

Upcoming and continuing interest in the Maslov index. Since the legendary work of V.P. Maslov [60] in the mid 1960s and the supplementary explanations by V. Arnol'd [3], there has been a continuing interest in the Maslov index for curves of Lagrangians in symplectic space. As explained by Maslov and Arnold, the interest arises from the study of dynamical systems in classical mechanics and related problems in Morse theory. This same index occurs as well in certain asymptotic formulae for solutions of the Schrödinger equations. For a systematic review of the basic vector analysis and geometry and for the physics background, we refer to Arnol'd [4] and M. de Gosson [36].

The Morse index of a geodesic is a special case of the Maslov index. Later, T. Yoshida [94] and L. Nicolaescu [70, 71] expanded the view by embracing also spectral problems for Dirac type operators on partitioned manifolds and thereby stimulating some quite new research in that direction. For a short review, we refer to our Section 4.1 below.

Data: $A(s)\colon C_0^\infty(M;E)\to C_0^\infty(M;E), s\in[0,1]$ curve of symmetric elliptic first order differential operators.
What fixed? $H^1(M;E)$ and $H^{1/2}(\Sigma;E|_\Sigma)\cong H^1(M;E)/H_0^1(M;E)$.
On $L^2(\Sigma;E|_\Sigma)$ strong $\omega(s)_{\rm Green}(x,y):=-\langle J(s)x,y\rangle_{L^2}$.
On $H^{1/2}(\Sigma;E|_\Sigma)$ induced weak $\omega(s)(x,y):=\omega(s)_{\rm Green}(x,y)$
$=-\langle J'(s)x,y\rangle_{H^{1/2}}$ with compact $J'(s)=(I+|B|)^{-1/2}J(s)$, B formally self-adjoint elliptic of first order on Σ.
Obstructions:

- $J'(s)^2\neq -I$, so $H^{1/2}\neq \ker(J'(s)-iI)\oplus\ker(J'(s)+iI)$;
- $\lambda^{\omega(s)\omega(s)}\neq\lambda$ for closed linear subspace λ; valid for ω-closed subspaces, where the topology is defined by the semi-norms $p_y(x):=|\omega(x,y)|$ (R. Schmid);
- $\operatorname{ind}(\lambda,\mu)\le 0$ for $(\lambda,\mu)\in\mathcal{FL}$; generally not equal to 0;
- $\mathcal{L}$ i.g. not contractible (Swanson); $\pi_1(\mathcal{FL}_0(X,\lambda))\overset{?}{=}\mathbb{Z}$ for $\lambda\in\mathcal{L}(X,\omega)$; valid for strong symplectic Hilbert space (X,ω).

FIGURE 1. Why going weak and what obstructions to circumvent?

Weak symplectic forms on Banach manifolds. Early in the 1970s, P. Chernoff, J. Marsden [32] and A. Weinstein [91] called attention to the practical and theoretical importance of symplectic forms on Banach manifolds. See R.C. Swanson [84, 85, 86] for an elaboration of the achievements of that period regarding linear symplectic structures on Banach spaces. It seems, however, that rigorous and operational definitions of the Maslov index of curves of Lagrangian subspaces in spaces of infinite dimension was not obtained until 25 years later. Our [22, Section 3.2] gives an account and compares the various definitions.

At the same place we emphasized a couple of rather serious obstructions (see Figure 1) to applying these concepts to arbitrary systems of elliptic differential equations of not-Dirac type: Firstly, some of the key section spaces for studying boundary value problems (the Sobolev space $H^{1/2}(\Sigma;E|_\Sigma)$ containing the traces over the boundary $\Sigma=\partial M$ of sections over the whole manifold M) are not carrying a strong symplectic structure, but are naturally equipped with a weak structure not admitting the rule $J^2=-I$. Secondly, in [22] our definition of the Maslov index in weak symplectic spaces requires a symplectic splitting which does not always exist, is not canonical, and therefore, in general, not obtainable in a continuous way for continuously varying symplectic structures. Thirdly, a priori, a symplectic reduction to finite dimensions is not obtainable for weak symplectic structures in the setting of [22].

An additional incitement to investigate weak symplectic structures comes from a stunning observation of E. Witten (explained by M.F. Atiyah in [5] in a heuristic way). He considered a weak presymplectic form on the loop space $\operatorname{Map}(S^1, M)$ of a finite-dimensional closed orientable Riemannian manifold M and noticed that a (future) thorough understanding of the infinite-dimensional symplectic geometry of that loop space "should lead rather directly to the index theorem for Dirac operators" (l.c., p. 43). Of course, restricting ourselves to the linear case, i.e., to the geometry of Lagrangian subspaces instead of Lagrangian manifolds, we can only marginally contribute to that program in this paper.

Symplectic reduction. In their influential paper [59, p. 121], J. Marsden and A. Weinstein describe the purpose of symplectic reduction in the following way:

> "... when we have a symplectic manifold on which a group acts symplectically, we can reduce this phase space to another symplectic manifold in which, roughly speaking, the symmetries are divided out."

and

> "When one has a Hamiltonian system on the phase space which is invariant under the group, there is a Hamiltonian system canonically induced on the reduced phase space."

The basic ideas go back to the work of G. Hamel [49, 50] and C. Carathéodory [30] in dynamical systems at the beginning of the last century, see also J.-M. Souriau [83]. For symplectic reduction in low-dimensional geometry see the monographs by S.K. Donaldson and P.B. Kronheimer, and by D. McDuff and D. Salamon [37, 62].

Our aim is less intricate, but not at all trivial: Following L. Nicolaescu [71] and K. Furutani [15] (joint work with the first author) we are interested in the finite-dimensional reduction of Fredholm pairs of Lagrangian *linear* subspaces in infinite-dimensional Banach space. The general procedure is well understood, see also P. Kirk and M. Lesch in [54, Section 6.3]: let $W \subset X$ be a closed co-isotropic subspace of a symplectic Banach space (X, ω). Then W/W^ω inherits a symplectic form from ω such that

$$R_W(\lambda) := \frac{(\lambda + W^\omega) \cap W}{W^\omega} \subset \frac{W}{W^\omega} \text{ isotropic for } \lambda \text{ isotropic.}$$

Here W^ω denotes the annihilator of W with respect to the symplectic form ω (see Definition 1.8c).

In general, however, the reduced space $R_W(\lambda)$ does not need to be Lagrangian in W/W^ω even for Lagrangian λ. In [**71**, **15**] a closer analysis of the reduction map R_W is given within the setting of strong symplectic structures; with emphasis on the topology of the space of Fredholm pairs of Lagrangians; and for fixed W. Now we drop the restriction to strong symplectic forms; our goal is to define the Maslov index for continuous curves $s \to (\lambda(s), \mu(s))$ of Fredholm pairs of Lagrangians with respect to continuously varying symplectic forms $\omega(s)$; and, at least locally (for $s \in (t-\varepsilon, t+\varepsilon)$ around $t \in [0,1]$), we let the pair $(\lambda(t), \mu(t))$ induce the reference space $W(t)$ for the symplectic reduction and the pair $(\lambda(s), \mu(s))$ induce the reduction map $R^{(s)}_{W(t)}$ in a natural way. The key to finding the reference spaces $W(t)$ and defining a suitable reduction map $R_{W(t)}$ is our Proposition 1.19. It is on decompositions of symplectic Banach spaces, naturally induced by a given Fredholm pair of Lagrangians of vanishing index. It might be, as well, of independent interest. The assumption of vanishing index is always satisfied for Fredholm pairs of Lagrangian subspaces in strong symplectic Hilbert spaces, and by additional global analysis arguments in our applications as well.

Thus for each path $\{(\lambda(s), \mu(s))\}_{s\in[0,1]}$ of Fredholm pairs of Lagrangian subspaces of vanishing index, we receive a finite-dimensional symplectic reduction *intrinsically*, i.e., without any other assumption. The reduction transforms the given path into a path of pairs of Lagrangians in finite-dimensional symplectic space. The main part of the paper is then to prove the invariance under symplectic reduction and the independence of choices made. That permits us a conservative view in this paper. Instead of defining the Maslov index in infinite dimensions via spectral theory of unitary generators of the Lagrangians as we did in [**22**], we elaborate the concept of the Maslov index in finite dimensions and reduce the infinite-dimensional case to the finite-dimensional case, i.e., we take the symplectic reduction as our beginning for re-defining the Maslov index instead of deploring its missing.

Structure of presentation. This paper is divided into four sections and one appendix. The first three sections present a rigorous definition of the Maslov index in Banach bundles by symplectic reduction. In Section 1, we fix the notation and establish our key technical device, namely the mentioned natural decomposition of a symplectic Banach

Levels of arguments:

1. Complex vector spaces
   - Pair (X, Y) with non-degenerate form $\Omega\colon X \times Y \to \mathbb{C}$
   - (X, ω) with ω symplectic, i.e., sesquilinear, skew-symmetric, non-degenerate
2. Banach spaces
   - $\mathcal{S}(X)$ closed linear subspaces of Banach space X
   - Gap topology $\widehat{\delta}\colon \mathcal{S}(X) \times \mathcal{S}(X) \to [0, 1]$
   - Weak symplectic Banach spaces
   - Banach bundles
3. Hilbert spaces
   - Strong symplectic
   - Weak symplectic
   - Hilbert bundles
4. Global analysis, elliptic problems
   - Compact manifold M with smooth boundary Σ
   - Partitioned manifold $M = M_- \cup_\Sigma M_+$

Levels of application:

- Weak symplectic Sobolev space $H^{1/2}(\Sigma; E|_\Sigma)$
- Unconstrained variation of elliptic problems

FIGURE 2. Natural levels of treatment

space into two symplectic spaces, induced by a pair of co-isotropic subspaces with finite codimension of their sum and finite dimension of the intersection of their annihilators. We introduce the symplectic reduction of arbitrary linear subspaces via a fixed co-isotropic subspace W and prove the transitivity of the symplectic reduction when replacing W by a larger co-isotropic subspace W'. For Fredholm pairs of Lagrangian subspaces of vanishing index, that yields an identification of the two naturally defined symplectic reductions. In Section 2, we recall and elaborate the Maslov index in strong symplectic Hilbert space, particularly in finite dimensions, to prove the invariance of our definition of the Maslov index under different symplectic reductions. In Section 3, we investigate the symplectic reduction to finite dimensions for a given path of Fredholm pairs of Lagrangian subspaces in fixed Banach space with varying symplectic structures and define the Maslov index in the general case via finite-dimensional symplectic reduction. In Section 3.3, we show that the Maslov index is invariant under symplectic reduction in the general case. For a first review of the entangled levels of treatment see Figure 2.

Section 4 is devoted to an application in global analysis and dynamical systems. We summarize the predecessor formulae, we prove a wide generalization of the Yoshida-Nicolaescu spectral flow formula, namely the identity Maslov index=spectral flow, both in general terms of Banach bundles and for elliptic differential operators of arbitrary positive order on smooth manifolds with boundary. That involves weak symplectic Hilbert spaces like the Sobolev space $H^{1/2}$ over the boundary. Applying substantially more advanced results we derive a corresponding spectral flow formula in all Sobolev spaces H^σ for $\sigma \geq 0$, so in particular in the familiar strong symplectic L^2.

In the Appendix A on closed subspaces in Banach spaces, we address the continuity of operations of linear subspaces. In gap topology, we prove some sharp estimates which might be of independent interest. E.g., they yield the following basic convergence result for sums and intersections of permutations of closed subspaces in Banach space in Proposition A.21 ([67, Lemma 1.5 (1), (2)]): Let $M' \to M$, $N' \to N$ and $M+N$ be closed. Then $M' \cap N' \to M \cap N$ iff $M'+N' \to M+N$. For each of the three technical main results of the Appendix, some applications are given to the global analysis of elliptic problems on manifolds with boundary.

Relation to our previous results. With this paper we conclude a series of our mutually related previous approaches to symplectic geometry, dynamical systems, and global analysis; in chronological order [**14**, **15**, **99**, **96**, **98**, **16**, **17**, **97**, **21**, **18**, **13**, **22**].

The model for our various approaches was developed in joint work with K. Furutani and N. Otsuki in [**14**, **15**, **16**]. Roughly speaking, there we deal with a *strong* symplectic Hilbert space $(X, \langle\cdot,\cdot\rangle, \omega)$, so that $\omega(x,y) = \langle Jx, y\rangle$ with $J^* = -J$ and $J^2 = -I$, possibly after continuous deformation of the inner product $\langle\cdot,\cdot\rangle$. Then the space $\mathcal{L}(X,\omega)$ of all Lagrangian subspaces is contractible and, for fixed $\lambda \in \mathcal{L}(X,\omega)$, the fundamental group of the Fredholm Lagrangian Grassmannian $\mathcal{FL}(X,\lambda)$ of all Fredholm pairs (λ,μ) with $\mu \in \mathcal{L}(X,\omega)$ is cyclic, see [**15**, Section 4] for an elementary proof. By the induced *symplectic splitting* $X = X^+ \oplus X^-$ with $X^\pm := \ker(J \mp iI)$ we obtain

(i) $\forall \lambda \in \mathcal{L}(X,\omega)\ \exists U\colon X^+ \to X^-$ unitary with $\lambda = \operatorname{graph}(U)$;
(ii) $(\lambda,\mu) \in \mathcal{FL}(X,\omega) \iff UV^{-1} - I_{X^-} \in \mathcal{F}(X)$; and
(iii) $\operatorname{Mas}(\lambda(s),\mu(s))_{s\in[0,1]} := \operatorname{sf}_{(0,\infty)}\big(U_sV_s^{-1}\big)_{s\in[0,1]}$ well defined.

Here $\mathcal{F}(X)$ denotes the space of bounded Fredholm operators on X and $\mathcal{FL}(X,\omega)$ the set of Fredholm pairs of Lagrangian subspaces of (X,ω) (see Definition 1.11).

This setting is suitable for the following application in operator theory: Let $\mathcal{H}$ be a complex separable Hilbert space and A a closed symmetric operator. We extend slightly the frame of the Birman-Kreĭn-Vishik theory of self-adjoint extensions of semi-bounded operators (see the review [**1**] by A. Alonso and B. Simon). Consider the space $\beta(A) :=$

$\operatorname{dom}(A^*)/\operatorname{dom}(A)$ of abstract boundary values. It becomes a strong symplectic Hilbert space with

$$\omega(\gamma(x),\gamma(y)) \;:=\; \langle A^*x,y\rangle - \langle x,A^*y\rangle,$$

and the projection $\gamma\colon \operatorname{dom}(A^*) \to \beta(A)$, $x \mapsto [x] := x + \operatorname{dom}(A)$. The inner product $\langle\gamma(x),\gamma(y)\rangle$ is induced by the graph inner product $\langle x,y\rangle_{\mathcal{G}} := \langle x,y\rangle + \langle A^*x,A^*y\rangle$ that makes $\operatorname{dom}(A^*)$ and, consequently, $\beta(A)$ to Hilbert spaces. Introduce the abstract *Cauchy data space* $\operatorname{CD}(A) := (\ker(A^*)+\operatorname{dom}(A))\,/\operatorname{dom}(A) = \{\gamma(x) \mid x \in \ker A^*\}$. From von Neumann's famous [68] we obtain the correspondence

$$A_D \text{ self-adjoint extension} \quad\Longleftrightarrow\quad [D] \subset \beta(A) \text{ Lagrangian},$$

for $\operatorname{dom}(A) \subset D \subset \operatorname{dom}(A^*)$. Now let A_D be a *self-adjoint Fredholm extension*, $\{C(s)\}_{s\in[0,1]}$ a C^0 curve in $\mathcal{B}^{\mathrm{sa}}(\mathcal{H})$, the space of bounded self-adjoint operators, and assume *weak inner Unique Continuation Property (UCP)*, i.e., $\ker(A^*+C(s)+\varepsilon)\cap\operatorname{dom}(A) = \{0\}$ for small positive ε. Then, [14] shows that

(i) $\{\operatorname{CD}(A+C(s)),\gamma(D)\}_{s\in[0,1]}$ is a continuous curve of Fredholm pairs of Lagrangians in the gap topology, and
(ii) $\operatorname{sf}\{(A+C(s))_D\}_{s\in[0,1]} = \operatorname{Mas}\{\operatorname{CD}(A+C(s)),\gamma(D)\}_{s\in[0,1]}$.

On one side, the approach of [14] has considerable strength: It is ideally suited both to Hamiltonian systems of ordinary differential equations of first order over an interval $[0,T]$ with varying lower order coefficients, and to curves of Dirac type operators on a Riemannian partitioned manifold or manifold M with boundary Σ with fixed Clifford multiplication and Clifford module (and so fixed principal symbol), but symmetric bounded perturbation due to varying affine connection (background field). Hence it explains Nicolaescu's Theorem (see below Section 4.1) in purely functional analysis terms and elucidates the decisive role of weak inner UCP. For such curves of Dirac type operators, the β-space remains fixed and can be described as a subspace of the distribution space $H^{-1/2}(\Sigma)$ with "half" component in $H^{1/2}(\Sigma)$. As shown in [15], the Maslov index constructed in this way is invariant under finite-dimensional symplectic reduction. Moreover, the approach admits varying boundary conditions and varying symplectic forms, as shown in [17, 21] and can be generalized to a spectral flow formula in the common $L^2(\Sigma)$ as shown in [16].

Unfortunately, that approach has severe limitations since it excludes varying maximal domain: there is no β-space when variation of the highest order coefficients is admitted for the curve of elliptic differential operators.

The natural alternative (here for first order operators) is to work with the Hilbert space

$$H^{1/2}(\Sigma; E|_\Sigma) \;\cong\; H^1(M;E)/H^1_0(M;E)$$

which remains fixed as long as we keep our underlying Hermitian vector bundle $E \to M$ fixed. So, let $A(s)\colon C^\infty_0(M;E) \to C^\infty_0(M;E), s \in [0,1]$ be a curve of symmetric elliptic first order differential operators. Green's form for $A(s)$ induces on $L^2(\Sigma; E|_\Sigma)$ a *strong* symplectic form $\omega(s)_{\mathrm{Green}}(x,y) := -\langle J(s)x, y\rangle_{L^2}$. On $H^{1/2}(\Sigma; E|_\Sigma)$ the induced symplectic form $\omega(s)(x,y) := \omega(s)_{\mathrm{Green}}(x,y) = -\langle J'(s)x,y\rangle_{H^{1/2}}$ is *weak*. To see that, we choose a formally self-adjoint elliptic operator B of first order on Σ to generate the metric on $H^{1/2}$ according to Gårding's Theorem. Then we find $J'(s) = (I+|B|)^{-1/2}J(s)$, which is a compact operator and so not invertible. This we emphasized already in our [20] where we raised the following questions:

- **Q1:** How to define $\mathrm{Mas}(\lambda(s),\mu(s))_{s\in[0,1]}$ for curves of Fredholm pairs of Lagrangian subspaces?
- **Q2:** How to calculate?
- **Q3:** What for?
- **Q4:** Dispensable? Non-trivial example?

Questions Q3 and Q4 are addressed below in Section 4 (see also our [20]). There we point to the necessity to work with the weak symplectic Hilbert space $H^{1/2}(\Sigma)$. Such work is indispensable when we are looking for spectral flow formulae for partitioned manifolds with curves of elliptic operators which are *not* of Dirac type.

To answer questions Q1 and Q2, we recall the following list of obstructions and open problems, partly from [20] (see also Figures 1, 3). For simplicity, we specify for Hilbert spaces instead of Banach spaces:

Let (X,ω) be a fixed complex Hilbert space with weak symplectic form $\omega(x,y) = \langle Jx,y\rangle$, and $(X(s),\omega(s)), s\in[0,1]$ a curve of weak symplectic Hilbert spaces, parametrized over the interval $[0,1]$ (other parameter spaces could be dealt with). Then in general we have in difference to strong symplectic forms:

(I) $J^2 \neq -I$;

(II) so, in general $X \neq X^- \oplus X^+$ with $X^\pm := \ker(J \mp iI)$; more generally, our Example 2.2 shows that there exist strong symplectic Banach spaces that do not admit any symplectic splitting;

(III) in general, for continuously varying $\omega(s)$ it does not hold that $X^\mp(s)$ is continuously varying;

Examples (blocking direct generalizations *strong* $\to$ *weak*)

1. No symplectic splitting: Let $(X,\omega) := \lambda \oplus \lambda^*$ and $\lambda := \ell^p$ $(p \in (1,+\infty) \setminus \{2\})$. Then X is a strong symplectic Banach space, but there is no splitting $X = X^+ \oplus X^-$ such that $\mp i\omega|_{X^\pm} > 0$, and $\omega(x,y) = 0$ for all $x \in X^+$ and $y \in X^-$. [See Section 2.1]
2. Double annihilator not always idempotent: Let (X,ω) be a weak symplectic Hilbert space and $\omega(x,y) = \langle Jx, y\rangle$. Let V be a proper closed linear subspace of X such that $V^\perp \cap JX = \{0\}$. Then $V^\omega = J^{-1}V^\perp = \{0\}$ and $V^{\omega\omega} = X \neq V$. [See Section 1.1]
3. Fredholm pair of Lagrangians with negative index: Let X be a complex Hilbert space and $X = X_1 \oplus X_2 \oplus X_3$ an orthogonal decomposition with $\dim X_1 = n \in \mathbb{N}$ and $X_2 \simeq X_3$. Then we can find a skew-self-adjoint injective, but not surjective J such that $\omega(x,y) = \langle Jx, y\rangle$ becomes a weak symplectic form on X and $\lambda_\pm = \{(\alpha, \pm\alpha); \alpha \in X_2\}$ becomes a pair of complementary Lagrangian subspaces of $X_2 \oplus X_3$ by identifying X_2 and X_3, and, in fact, a pair of Lagrangians of X with $\operatorname{ind}(\lambda_+, \lambda_-) = -n$. [See Section 1.2]

FIGURE 3. Three counterexamples

(IV) as shown in our Example 1.6, we have $\lambda^{\omega\omega} \supsetneqq \lambda$ for some closed linear subspaces λ; according to our Lemma 1.4, the double annihilator, however, is idempotent for ω-closed subspaces, where the topology is defined by the semi-norms $p_y(x) := |\omega(x,y)|$ (based on R. Schmid, [79]);

(V) by Corollary 1.13 we have $\operatorname{index}(\lambda,\mu) \leq 0$ for $(\lambda,\mu) \in \mathcal{FL}$; our Example 1.15 shows that there exist Fredholm pairs of Lagrangian subspaces with truly negative index; hence, in particular, the concept of the *Maslov cycle* $\mathcal{M}(X,\omega,\lambda_0) := \mathcal{FL}(\lambda_0,\cdot) \setminus \mathcal{FL}^0(\lambda_0,\cdot)$ of a fixed Lagrangian subspace λ_0 (comprising all Lagrangians that form a Fredholm pair with λ_0 but do not intersect λ_0 transversally) is invalidated: we can no longer conclude complementarity of μ and λ_0 from $\mu \cap \lambda_0 = \{0\}$;

(VI) in general, the space $\mathcal{L}(X,\omega)$ is not contractible and even not connected according to Swanson's arguments for counterexamples [86, Remarks after Theorem 3.6], based on A. Douady, [38];

(VII) $\pi_1(\mathcal{FL}_0(X,\lambda)) \stackrel{?}{=} \mathbb{Z}$ for $\lambda \in \mathcal{L}(X,\omega)$; valid for strong symplectic Hilbert space (X,ω).

Limited value of our previous pilot study. Anyway, our previous [22] deals with a continuous family of weak symplectic forms $\omega(s)$ on a curve of Banach spaces $X(s)$, $s \in [0,1]$. It gives a definition of the Maslov index for a path $(\lambda(s), \mu(s))_{s\in[0,1]}$ of Fredholm pairs of Lagrangian subspaces of index 0 under the assumption of a continuously varying symplectic splitting $X = X^+(s) \oplus X^-(s)$. The definition is inspired by the careful distinctions of planar intersections in [99, 96, 98, 97]. Then it is shown that all nice properties of the Maslov index are preserved for this general case. However, that approach has four serious drawbacks which render this definition incalculable:

1. In Section 2.1, our Example 2.2 provides a strong symplectic Banach space that does not admit a symplectic splitting.
2. Even when a single symplectic splitting is guaranteed, there is no way to establish such splitting for families in a continuous way (see also our obstruction III above).
3. The Maslov index, as defined in [22] becomes independent of the choice of the splitting only for strong symplectic forms.
4. That construction admits finite-dimensional symplectic reduction only for strong symplectic forms.

To us, our [22] is a highly valuable pilot study, but the preceding limitations explain why in this paper we begin again from scratch. For that purpose, an encouraging result was obtained in [18] combined with [13]: the continuous variation of the Calderón projection in $L^2(\Sigma)$ for a curve of elliptic differential operators of first order. We shall use this result in our Section 4.5.

Acknowledgements. We thank Prof. K. Furutani (Tokyo), Prof. M. Lesch (Bonn), and Prof. R. Nest (Copenhagen) for inspiring discussions about this subject. The second author was partially supported by NSFC (NO.11221091), LPMC of MOE of China, and Nankai University.

1. General theory of symplectic analysis in Banach spaces

We fix the notation and establish our key technical device in Proposition 1.19 and Corollary 1.20, namely a natural decomposition of a fixed symplectic Banach space into closed symplectic subspaces induced by a single Fredholm pair of Lagrangians of index 0. Reversing the order of the Fredholm pair, we obtain an alternative symplectic reduction. In Proposition 1.32, we show that the two symplectic reductions coincide by establishing a transitivity of symplectic reductions in Lemma 1.25 and Corollary 1.26. As we shall see later in Section 3, that yields the symplectic reduction to finite dimensions for a given path of Fredholm pairs of Lagrangian subspaces of index 0 in a fixed Banach space with varying symplectic structures and the invariance of the Maslov index under different symplectic reductions.

Our assumption of vanishing index is trivially satisfied in *strong* symplectic Hilbert space. More interestingly and inspired by and partly reformulating previous work by R. Schmid, and D. Bambusi [**79, 9**], we obtain in Lemma 1.4 a delicate condition for making the annihilator an involution, or differently put, the double annihilator idempotent. In Corollary 1.13 we show that the index of a Fredholm pair of Lagrangian subspaces can not be positive. In Corollary 1.16 we derive a necessary and sufficient condition for its vanishing for *weak* symplectic forms and in the concrete set-up of our global analysis applications in Section 4. In order to emphasize the intricacies of weak symplectic analysis, it seems worthwhile to clarify in Lemma 1.4 a potentially misleading formulation in [**79**, Lemma 7.1], and in Remark 1.5, to isolate an unrepairable error in [**9**, First claim of Lemma 3.2, pp.3387-3388], namely the wrong claim that the double annihilator is idempotent on all closed subspaces of reflexive weak symplectic Banach spaces.

To settle some of the ambiguities around weak symplectic forms once and for all, we provide two counterexamples in Examples 1.6 and 1.15. The first gives a closed subspace where the double annihilator is not idempotent. The second gives a Fredholm pair of Lagrangians with negative index.

1.1. Dual pairs and double annihilators. Our point of departure is recognizing the difficulties of dealing with both *varying* and *weak* symplectic structures, as explained in our [**22**]. As shown there, a direct way to define the Maslov index in that context requires a continuously varying symplectic splitting. As mentioned in the Introduction, neither the existence nor a continuous variation of such a splitting is guaranteed. Consequently, that definition is not very helpful for calculations in applications.

To establish an intrinsic alternative, we shall postpone the use of the symplectic structures to later sections and do as much as possible in the rather *neutral* category of linear algebra. A first taste of the use of purely algebraic arguments of linear algebra for settling open questions of symplectic geometry is the making of a kind of annihilator. For the true annihilator concept of symplectic geometry see below Definition 1.8.c.

Already here we can explain the need for technical innovations when dealing with weak symplectic structures instead of hard ones. To give a simple example, let us consider a complex symplectic Hilbert space $(X, \langle , \rangle, \omega)$ with $w(x,y) = \langle Jx, y\rangle$ for all $x, y \in X$ where $J \colon X \to X$ is a bounded, injective and skew-self-adjoint operator (for details see below Section 1.2). Then we get at once $\lambda^\omega = (J\lambda)^\perp$ and $\lambda^{\omega\omega} \supset \overline{\lambda}$ for all linear subspaces $\lambda \subset X$. We denote the orthogonal complement by the common orthogonality exponent $\perp$ and the symplectic annihilator by the exponent ω. Now, if we are in the *strong* symplectic case, we have J surjective and $J^2 = -I$, possibly after a slight deformation of the inner product. In that case, we have immediately

$$\lambda^{\omega\omega} = \left(J\left((J\lambda)^\perp\right)\right)^\perp = (\lambda^\perp)^\perp = \overline{\lambda}.$$

Hence the double annihilator is an idempotent on the set of closed subspaces in strong symplectic Hilbert space, like in the familiar case of finite-dimensional symplectic analysis. Moreover, from that it follows directly that the index of a Fredholm pair of Lagrangians (see Definition 1.11 and Corollaries 1.13 and 1.16) vanishes in strong symplectic Hilbert space.

The preceding chain of arguments breaks down for the double annihilator in *weak* symplectic analysis, and we are left with two basic technical problems:

(i) when do we have precisely $\lambda^{\omega\omega} = \overline{\lambda}$, and consequently,
(ii) when are we guaranteed the vanishing of the index of a Fredholm pair of Lagrangian subspaces?

As mentioned above, we are not the first who try to determine the precise conditions for the annihilator of an annihilator not to become larger than the closure of the original space. We are indebted to the previous work by R. Schmid [79, Arguments of the proof of Lemma 7.1] and D. Bambusi [9, Arguments around Lemmata 2.7 and 3.2]. They suggested to apply a wider setting and address the pair-annihilator concept of linear algebra. We shall follow - and modify - some of their arguments and claims.

Definition 1.1. Let X, Y be two complex vector spaces. Denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ the sets of real numbers, complex numbers and integers, respectively. Let $h\colon \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear isomorphism. Let $\Omega\colon X \times Y \to \mathbb{C}$ be a $\mathbb{R}$-linear map with $\Omega(ax, by) = ah(b)\Omega(ax, by)$ for all $a, b \in \mathbb{C}$ and $(x,y) \in X \times Y$.
a) For each of the subspaces $\lambda \subset X$ and $\mu \subset Y$, we define the right and left *annihilators* of λ and μ as real linear subspaces of X and Y by

$$\lambda^{\Omega,r} \;:=\; \{y \in Y; \Omega(x,y) = 0, \forall x \in \lambda\}, \tag{1.1}$$

$$\mu^{\Omega,l} \;:=\; \{x \in X; \Omega(x,y) = 0, \forall y \in \mu\}. \tag{1.2}$$

b) The form Ω is said to be *non-degenerate in* X (*in* Y) if $X^{\Omega,r} = \{0\}$ ($Y^{\Omega,l} = \{0\}$). The form Ω is said to be just *non-degenerate* if $X^{\Omega,r} = \{0\}$ and $Y^{\Omega,l} = \{0\}$. In that case one says that X, Y form an *algebraic* $\mathbb{R}$-*dual pair* (see also Pedersen [72, 2.3.8]).
c) We have the *reduced form*

$$\widetilde{\Omega}\colon X/Y^{\Omega,l} \times Y/X^{\Omega,r} \longrightarrow \mathbb{C}$$

defined by $\widetilde{\Omega}(x + Y^{\Omega,l}, y + X^{\Omega,r}) := \Omega(x,y)$ for each $(x,y) \in X \times Y$.
d) The *annihilator map* $\Omega^b\colon Y \to \operatorname{Hom}(X,\mathbb{C})$ is the $\mathbb{R}$-linear map defined by $\Omega^b(y)(x) := \Omega(x,y)$ for all $x \in X$.

NOTE. By definition, the reduced form $\widetilde{\Omega}$ is always non-degenerate, since

$$\begin{aligned} &(X/Y^{\Omega,l})^{\widetilde{\Omega},r} \\ &\quad = \{y + X^{\Omega,r}\,;\, \widetilde{\Omega}\left(x + Y^{\Omega,l}, y + X^{\Omega,r}\right) = \Omega(x,y) = 0\ \forall x \in X\} \\ &\quad = X^{\Omega,r} \ = \ \{0\} \quad \text{ in } Y/X^{\Omega,r}, \end{aligned}$$

making the form $\widetilde{\Omega}$ non-degenerate in $X/Y^{\Omega,l}$. Similarly, we obtain $\left(Y/X^{\Omega,r}\right)^{\widetilde{\Omega},l} = Y^{\Omega,l}$, making the form $\widetilde{\Omega}$ non-degenerate in $Y/X^{\Omega,r}$.

We list a few immediate consequences: First of all, we have $\ker_{\mathbb{R}} \Omega^b = X^{\Omega,r}$, as real vector spaces. Then we have $\lambda + Y^{\Omega,l} \subset (\lambda^{\Omega,r})^{\Omega,l}$, and $\lambda_1^{\Omega,r} \supset \lambda_2^{\Omega,r}$ if $\lambda_1 \subset \lambda_2 \subset X$. From that we get $\lambda^{\Omega,r} \supset ((\lambda^{\Omega,r})^{\Omega,l})^{\Omega,r} \supset \lambda^{\Omega,r}$, hence

$$\lambda^{\Omega,r} = ((\lambda^{\Omega,r})^{\Omega,l})^{\Omega,r}. \tag{1.3}$$

The following lemma generalizes our [22, Lemma 5, Corollary 1]. We shall use it below in the proof of Lemma 1.12 to establish the general result that the index of Fredholm pairs of Lagrangians in symplectic Banach space always is non-positive.

LEMMA 1.2. *(a) If* $\dim X < +\infty$ *and* $X^{\Omega,r} = \{0\}$, *we have*

$$\dim Y \ = \ \dim X/Y^{\Omega,l} \ \le \ \dim X.$$

The equality $\dim X = \dim Y$ *holds if and only if* $Y^{\Omega,l} = \{0\}$.
(b) Let $\lambda \subset X$ *be a linear subspace. If* $\dim X/(\lambda + Y^{\Omega,l}) < +\infty$, *we have*

$$\dim \lambda^{\Omega,r}/X^{\Omega,r} \ \le \ \dim X/(\lambda + Y^{\Omega,l}).$$

The equality holds if and only if $(\lambda^{\Omega,r})^{\Omega,l} = \lambda + Y^{\Omega,l}$.

(c) Let $\lambda \subset X$ be a linear subspace. If $\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} < +\infty$, we have

$$\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} \;=\; \dim Y/\lambda^{\Omega,r} \quad\text{and}\quad \lambda + Y^{\Omega,l} \;=\; (\lambda^{\Omega,r})^{\Omega,l}.$$

PROOF. (a): If $\dim X < +\infty$ and $X^{\Omega,r} = \{0\}$, Ω^b is injective. Then we have $2\dim Y = \dim_{\mathbb{R}} Y \le \dim_{\mathbb{R}} \operatorname{Hom}(X, \mathbb{C}) = 2\dim X$. So we have $\dim Y \le \dim X$.

If Ω is non-degenerate, we have $\dim X \le \dim Y$ and $\dim X = \dim Y$. Applying the argument for $\widetilde{\Omega}$, we have $\dim X/Y^{\Omega,l} = \dim Y$.

If $\dim X = \dim Y$, we have $\dim X = \dim X/Y^{\Omega,l}$ and $Y^{\Omega,l} = \{0\}$.
(b): Define the first λ-reduced form

$$f\colon X/(\lambda + Y^{\Omega,l}) \,\times\, \lambda^{\Omega,r}/X^{\Omega,r} \;\longrightarrow\; \mathbb{C}$$

by

$$f(x + \lambda + Y^{\Omega,l}, y + X^{\Omega,r}) \;:=\; \Omega(x, y), \quad \forall (x, y) \in X \times \lambda^{\Omega,r}.$$

Then we have

$$(X/(\lambda + Y^{\Omega,l}))^{f,r} \;=\; \{0\} \;\text{ and }\; (\lambda^{\Omega,r}/X^{\Omega,r})^{f,l} \;=\; (\lambda^{\Omega,r})^{\Omega,l}/(\lambda + Y^{\Omega,l}).$$

By (a), we get our results.
(c): Define the second λ-reduced form

$$g\colon (\lambda + Y^{\Omega,l})/Y^{\Omega,l} \,\times\, Y/\lambda^{\Omega,r} \;\longrightarrow\; \mathbb{C}$$

by

$$g(x + Y^{\Omega,l}, y + \lambda^{\Omega,r}) \;:=\; \Omega(x, y), \quad \forall (x, y) \in (\lambda + Y^{\Omega,l}) \times Y.$$

Then g is non-degenerate. By (a), we have $\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} = \dim Y/\lambda^{\Omega,r}$. By (b) we have $\dim Y/\lambda^{\Omega,r} \ge \dim(\lambda^{\Omega,r})^{\Omega,l}/Y^{\Omega,l}$. So we have $\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} = \dim(\lambda^{\Omega,r})^{\Omega,l}/Y^{\Omega,l}$. Since $\lambda + Y^{\Omega,l} \subset (\lambda^{\Omega,r})^{\Omega,l}$, we have $\lambda + Y^{\Omega,l} = (\lambda^{\Omega,r})^{\Omega,l}$. □

Assume that Ω is non-degenerate in Y. Then the family of semi-norms $\mathcal{F} := \{p_y(x) := |\Omega(x, y)|,\, x \in X\}_{y \in Y}$ is *separating*, i.e., for $x \ne x'$ in X, there is a $y \in Y$ such that $p_y(x - x') \ne 0$. We shall denote the topology on X induced by the family $\mathcal{F}$ by $\mathcal{T}_\Omega$ and call it the *weak topology induced by* Ω or shortly the Ω*-topology*. By [72, 1.5.3 and 3.4.2] $(X, \mathcal{T}_\Omega)$ becomes a Hausdorff separated, locally convex, topological vector space. The following two lemmata are proved implicitly by [79, Arguments of the proof of Lemma 7.1]. Clearly, we have

LEMMA 1.3. *Assume that Ω is non-degenerate in Y. Then the real linear map Ω^b maps Y onto $(X, \mathcal{T}_\Omega)^*$.*

Then the Hahn-Banach Theorem yields

LEMMA 1.4 (R. Schmid, 1987). *Assume that Ω is non-degenerate in Y and λ is a closed linear subspace of $(X, \mathcal{T}_\Omega)$. Then we have*

$$\lambda = (\lambda^{\Omega,r})^{\Omega,l}. \tag{1.4}$$

REMARK 1.5. a) Let (X, ω) be a complex weak symplectic Banach space. By definition (see below), the form $\omega\colon X \times X \to \mathbb{C}$ is non-degenerate. Then we have three topologies on X: the *norm-topology*, the canonical *weak topology* induced from the family X^* of continuous functionals on X, and the *ω-induced weak topology* $\mathcal{T}_\omega$. The weak topology is weaker than the norm topology; and the ω-induced topology is weaker than the weak topology. So, a closed subset $V \subset X$ is not necessarily weakly closed or closed in $(X, \mathcal{T}_\omega)$: the set V can have more accumulation points in the weak topology and even more in the ω-induced weak topology than in the norm topology. A standard example is the unit sphere that is not weakly closed in infinite dimensions (see, e.g., H. Brezis [**24**, Example 1, p. 59]. Fortunately, by [**24**, Theorem 3.7] every norm-closed *linear* subspace is weakly closed. Hence it is natural (but erroneous) to suppose that the difference between the three topologies does not necessarily confine severely the applicability of Schmid's Lemma, namely to linear subspaces.
b) It seems that D. Bambusi in [**9**, Lemmata 2.7,3.2] supposed erroneously that in reflexive Banach space all norm-closed subspaces are not only weakly closed but also ω-weakly closed. Rightly, in spaces where that is valid, Schmid's Lemma is applicable (or can be reproved independently).
c) Recall that a Banach space X is *reflexive* if the isometry

$$\iota\colon X \longrightarrow X^{**}, \text{ given by } \iota(x)(\varphi) := \varphi(x) \text{ for } x \in X,\ \varphi \in X^*$$

is surjective, i.e., its range is the whole bidual space X^{**}. Typical examples of reflexive spaces are all Hilbert spaces and the L^p-spaces for $1 < p < \infty$, but not L^1.
d) Unfortunately, in general the claim of [**9**, Lemma 3.2] (the validity of the idempotence of the double annihilator for closed linear subspaces in complex reflexive symplectic Banach space) is not correct. If it was correct, then, e.g., in (automatically reflexive) weak symplectic Hilbert space $(X, \langle\cdot,\cdot\rangle, \omega)$, the double annihilator $\lambda^{\omega\omega}$ of every closed subspace λ should coincide with λ. However, here is a counterexample: Let $(X, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $J\colon X \to X$ a bounded injective skew-self-adjoint operator. Then $\omega\colon X \times X \to \mathbb{C}$ defined by $\omega(x, y) := \langle Jx, y\rangle$ is a symplectic form on X. So $\operatorname{im} J$ is dense in X. For $V \subset X$ closed subspace, denote by $V^\perp$ the orthogonal complement of V with respect to the inner product on X, and by V^ω the symplectic

complement (i.e., the annihilator) of V. Then we have

$$V^\omega \;=\; (JV)^\perp \;=\; J^{-1}\left(V^\perp\right).$$

Now assume that $\operatorname{im} J \neq X$ (like in the weak symplectic Sobolev space $X := H^{1/2}(\Sigma; E|_\Sigma)$, as explained in the Introduction). Let $x \in X \setminus \operatorname{im} J$ and set $V := (\operatorname{span}\{x\})^\perp$. Then we have $J^{-1}\left(V^\perp\right) = \{0\}$, hence $V^\omega = \{0\}$ and $V^{\omega\omega} = X \neq V$. That falsifies the first part of Equation (13) in [9, Lemma 3.2].
e) The preceding example falsifies [9, Equation (11)], as well: For any closed subspace $V \subset X$ we have $J\left((JV)^\perp\right) \subset V^\perp$. Then Bambusi's Equation (11) is equivalent to

$$\overline{J\left((JV)^\perp\right)} \;=\; V^\perp.$$

For our concrete example $V := (\operatorname{span}\{x\})^\perp$, however, we obtain

$$J\left((JV)^\perp\right) \cap V^\perp = \{0\} \quad\text{and}\quad V^\perp = \operatorname{span}\{x\}.$$

Thus (11) is incorrect.
f) For any Lagrangian subspace λ in a complex symplectic Banach space (X, ω) we have $\lambda^{\omega\omega} = \lambda$ by definition. That follows also directly from the identity (1.3), and, alternatively, from Schmid's Lemma, since a Lagrangian subspace is always ω-closed.

The counterexample of the preceding Remarks d and e can be generalized in the following form.

Example 1.6 (Closed subspaces with non-idempotent double annihilator). Let (X, ω) be a weak symplectic Hilbert space and $\omega(x, y) = \langle Jx, y\rangle$. Let V be a proper closed linear subspace of X such that $V^\perp \cap JX = \{0\}$. Then $V^\omega = J^{-1}V^\perp = \{0\}$ and $V^{\omega\omega} = X \supsetneqq V$.

For later use it is worth noting the following extension of Schmid's Lemma which is the weak and corrected version of [9, Lemma 3.2].

Lemma 1.7. *Assume that X, Y, Ω as above and Ω non-degenerate in Y and bounded in X. Assume that X is a reflexive Banach space. Then $\Omega^b(Y)$ is dense in X^* and we have*

$$\lambda = \left(\lambda^{\Omega,r}\right)^{\Omega,l} \quad \textit{for any linear and } \omega\textit{-closed subspace } \lambda \subset X. \tag{1.5}$$

1.2. Basic symplectic concepts. Before defining the Maslov index in symplectic Banach space by symplectic reduction to the finite-dimensional case, we recall the basic concepts and properties of symplectic functional analysis.

DEFINITION 1.8. Let X be a complex vector space.
(a) A mapping

$$\omega\colon X \times X \longrightarrow \mathbb{C}$$

is called a *symplectic form* on X, if it is sesquilinear, skew-symmetric, and non-degenerate, i.e.,

(i) $\omega(x,y)$ is linear in x and conjugate linear in y;
(ii) $\omega(y,x) \;=\; -\overline{\omega(x,y)}$;
(iii) $X^\omega \;:=\; \{x \in X \mid \omega(x,y) \;=\; 0 \text{ for all } y \in X\} \;=\; \{0\}$.

Then we call (X,ω) a *symplectic vector space.*
(b) Let X be a complex Banach space and (X,ω) a symplectic vector space. (X,ω) is called *(weak) symplectic Banach space*, if ω is bounded, i.e., $|\omega(x,y)| \le C\|x\|\|y\|$ for all $x,y \in X$.
(c) The *annihilator* of a subspace λ of X is defined by

$$\lambda^\omega \;:=\; \{y \in X \mid \omega(x,y) \;=\; 0 \quad \text{for all } x \in \lambda\}.$$

(d) A subspace λ is called *symplectic*, *isotropic*, *co-isotropic*, or *Lagrangian* if

$$\lambda \cap \lambda^\omega \;=\; \{0\}\,, \quad \lambda \subset \lambda^\omega\,, \quad \lambda \supset \lambda^\omega\,, \quad \lambda \;=\; \lambda^\omega\,,$$

respectively.
(e) The *Lagrangian Grassmannian* $\mathcal{L}(X,\omega)$ consists of all Lagrangian subspaces of (X,ω).

REMARK 1.9. In symplectic Banach spaces, the annihilator λ^ω is closed for any linear subspace λ, and we have the trivial inclusion

$$\lambda^{\omega\omega} \supset \overline{\lambda}. \tag{1.6}$$

In particular, all Lagrangian subspaces are closed, and trivially, as emphasized in Remark 1.5.f, we have an equality in the preceding (1.6).

If X is a complex Banach space, each symplectic form ω induces a uniquely defined injective mapping $J\colon X \to X^*$ such that

$$\omega(x,y) \;=\; (Jx,y) \quad \text{for all } x,y \in X, \tag{1.7}$$

where we set $(Jx,y) := (Jx)(y)$. The induced mapping J is a bounded, injective mapping $J\colon X \to X^*$ where X^* denotes the (topological) dual space.

DEFINITION 1.10. Let (X,ω) be a symplectic Banach space. If J is also surjective (hence with bounded inverse), the pair (X,ω) is called a *strong symplectic Banach space.*

We have taken the distinction between *weak* and *strong* symplectic structures from P. Chernoff and J. Marsden [32, Section 1.2, pp. 4-5]. If X is a Hilbert space with symplectic form ω, we identify X and X^*. Then the induced mapping J is a bounded, skew-self-adjoint operator (i.e., $J^* = -J$) on X with $\ker J = \{0\}$. As in the strong symplectic case, we then have that $\lambda \subset X$ is Lagrangian if and only if $\lambda^\perp = J\lambda$. As explained above, in Hilbert space, a main difference between weak and strong is that we can assume $J^2 = -I$ in the strong case (see [22, Lemma 1] for the required smooth deformation of the inner product), but not in the weak case. The importance of such an anti-involution is well-known from symplectic analysis in finite dimensions and exploited in strong symplectic Hilbert spaces, but, in general, it is lacking in weak symplectic analysis.

We recall the key concept to symplectic analysis in infinite dimensions:

Definition 1.11. The space of *Fredholm pairs* of Lagrangian subspaces of a symplectic vector space (X, ω) is defined by

$$\mathcal{FL}(X) := \{(\lambda, \mu) \in \mathcal{L}(X) \times \mathcal{L}(X) \mid \dim(\lambda \cap \mu) < +\infty \text{ and } \dim X/(\lambda + \mu) < +\infty\} \tag{1.8}$$

with

$$\operatorname{index}(\lambda, \mu) := \dim(\lambda \cap \mu) - \dim X/(\lambda + \mu). \tag{1.9}$$

For $k \in \mathbb{Z}$ we define

$$\mathcal{FL}_k(X) := \{(\lambda, \mu) \in \mathcal{FL}(X) \mid \operatorname{index}(\lambda, \mu) = k\}. \tag{1.10}$$

For $k \in \mathbb{Z}$ and $\mu \in \mathcal{L}(X)$ we define

$$\mathcal{FL}(X, \mu) := \{\lambda \in \mathcal{L}(X); (\lambda, \mu) \in \mathcal{FL}(X)\}, \tag{1.11}$$

$$\mathcal{FL}_k(X, \mu) := \{\lambda \in \mathcal{L}(X); (\lambda, \mu) \in \mathcal{FL}_k(X)\}, \tag{1.12}$$

$$\mathcal{FL}_0^k(X, \mu) := \{\lambda \in \mathcal{FL}_0(X, \mu); \dim(\lambda \cap \mu) = k\}. \tag{1.13}$$

It is well known that Fredholm pairs of Lagrangian subspaces in strong symplectic Hilbert spaces always have vanishing index. Here we give another proof for the well-known fact (proved before in our [22, Proposition 1]) that Fredholm pairs of Lagrangian subspaces in symplectic vector spaces never can have positive index. In Example 1.15 we give a Fredholm pair of Lagrangian subspaces in a weak symplectic Hilbert space with negative index. Hence, we can not take the vanishing of the index for granted for weak symplectic forms, neither in Hilbert spaces. In our applications, however, we shall deal only with Fredholm pairs of Lagrangians where the vanishing of the index is granted by arguments of global analysis.

LEMMA 1.12. *Let* (X,ω) *be a symplectic vector space and* $\lambda_1,\dots,\lambda_k$ *linear subspaces of* X. *Assume that* $\dim X/(\sum_{j=1}^k \lambda_j) < +\infty$. *Then the following holds.*
(a) We have

$$\dim(\bigcap_{j=1}^k \lambda_j^\omega) \le \dim X/(\sum_{j=1}^k \lambda_j). \tag{1.14}$$

The equality holds if and only if $\sum_{j=1}^k \lambda_j = (\sum_{j=1}^k \lambda_j)^{\omega\omega}$.
(b) If λ_j *is isotropic for each* j, *we have*

$$\dim(\bigcap_{j=1}^k \lambda_j) \le \dim X/(\sum_{j=1}^k \lambda_j). \tag{1.15}$$

The equality holds if and only if $\bigcap_{j=1}^k \lambda_j = \bigcap_{j=1}^k \lambda_j^\omega$ *and* $\sum_{j=1}^k \lambda_j = (\sum_{j=1}^k \lambda_j)^{\omega\omega}$.

PROOF. (a) Since $\bigcap_{j=1}^k \lambda_j^\omega = (\sum_{j=1}^k \lambda_j)^\omega$, our result follows from Lemma 1.2.b.
(b) By (a) and $\bigcap_{j=1}^k \lambda_j \subset \bigcap_{j=1}^k \lambda_j^\omega$. □

COROLLARY 1.13 (Non-positive Fredholm index). *Let* X *be a complex Banach space with symplectic form* ω. *Then each Fredholm pair* (λ,μ) *of Lagrangian subspaces of* (X,ω) *has negative index or is of index* 0.

REMARK 1.14. (a) The Corollary has a wider validity: Let (λ,μ) be a Fredholm pair of isotropic subspaces. Then we have by Lemma 1.12.b $\operatorname{index}(\lambda,\mu) \le 0$. If $\operatorname{index}(\lambda,\mu) = 0$, λ and μ are Lagrangians (see [22, Corollary 1 and Proposition 1]).
(b) By Lemma 1.12.b we obtain $\operatorname{index}(\lambda,\mu) = 0$, *if* we have

$$(\lambda+\mu)^{\omega\omega} = \lambda+\mu \quad\text{and}\quad \lambda = \lambda^\omega \quad\text{and}\quad \mu = \mu^\omega.$$

For Lagrangian subspaces the last two equations are satisfied by definition. By Lemma 1.4, the first equation is satisfied if the space $\lambda+\mu$ is ω-closed, i.e., closed in the weak topology $\mathcal{T}_\omega$ (see above). In a symplectic Banach space (X,ω) all Lagrangian subspaces are norm-closed, weakly closed and ω-weakly closed at the same time, as emphasized in Remark 1.5. Since λ, μ are norm-closed and $\dim X/(\lambda+\mu) < +\infty$, $\lambda+\mu$ is norm-closed by [15, Remark A.1] and [52, Problem 4.4.7]. However, that does not suffice to prove that $\lambda+\mu$ is ω-closed, see Remark 1.9.

Here is an example which shows that the index of a Fredholm pair of Lagrangian subspaces in weak symplectic Banach space need not vanish.

EXAMPLE 1.15 (Fredholm pairs of Lagrangians with negative index). Let X be a complex Hilbert space and $X = X_1 \oplus X_2 \oplus X_3$ an orthogonal decomposition with $\dim X_1 = n \in \mathbb{N}$ and $X_2 \simeq X_3$. Then we can find a bounded skew-self-adjoint injective, but not surjective $J\colon X \to X$ such that $\omega(x,y) = \langle Jx, y\rangle$ becomes a weak symplectic form on X. Let J be of the form

$$J = i\begin{pmatrix} A_{11} & A_{12} & \bar{k}A_{12} \\ A_{21} & A_{22} & 0 \\ kA_{21} & 0 & -A_{22} \end{pmatrix},$$

where $k \in \mathbb{C}, k \neq \pm 1$, $\operatorname{im} A_{21} \cap \operatorname{im} A_{22} = \{0\}$ and $\ker A_{21} = \ker A_{22} = \{0\}$.

Set $V = X_2 \oplus X_3$. We identify the vectors in X_2 and X_3. Then the pair (λ_+, λ_-) with $\lambda_\pm := \{(\alpha, \pm\alpha); \alpha \in X_2\}$ becomes a Fredholm pair of Lagrangian subspaces of $(V, \omega|_V)$ with $\lambda_+ \cap \lambda_- = \{0\}$ and

$$V \ = \ \lambda_+ \oplus \lambda_-.$$

We claim that $J^{-1}(X_1 \oplus \lambda_\pm) \subset V$. In fact, let $(x_1, x_2, x_3) \in J^{-1}(X_1 \oplus \lambda_\pm)$. Then there is an $\alpha \in X_2$ such that $A_{21}x_1 + A_{22}x_2 = \alpha$ and $kA_{21}x_1 - A_{22}x_3 = \pm\alpha$. So $(1 \mp k)A_{21}x_1 + A_{22}(x_2 \pm x_3) = 0$. Since $\operatorname{im} A_{21} \cap \operatorname{im} A_{22} = 0$ and $\ker A_{21} = 0$, we have $x_1 = 0$.

Note that $\lambda_\pm^\perp = X_1 \oplus \lambda_\mp$ and $\lambda_\pm^\omega \cap V = \lambda^\pm$. Then we have $\lambda_\pm^\omega = J^{-1}(X_1 \oplus \lambda_\mp) \subset V$ and $\lambda_\pm^\omega = \lambda_\pm^\omega \cap V = \lambda^\pm$. So $\lambda_\pm$ are Lagrangian subspaces of (X, ω). Then, by definition of J they form a Fredholm pair of Lagrangians of X with $\operatorname{index}(\lambda_+, \lambda_-) = -n$.

COROLLARY 1.16. *Let (X,ω) be a symplectic vector space and λ, μ two linear subspaces. Assume that*

$$\dim X/(\lambda+\mu) < +\infty \text{ and } \dim X/(\lambda^\omega + \mu^\omega) < +\infty.$$

Then the following holds.
(a) (λ,μ) and $(\lambda^\omega, \mu^\omega)$ are Fredholm pairs, and we have

$$\operatorname{index}(\lambda,\mu) \ + \ \operatorname{index}(\lambda^\omega, \mu^\omega) \ \le \ 0. \tag{1.16}$$

(b) The equality holds in (1.16) if and only if $\lambda + \mu = (\lambda+\mu)^{\omega\omega}$, $\lambda^\omega + \mu^\omega = (\lambda^\omega + \mu^\omega)^{\omega\omega}$, and $\lambda \cap \mu = \lambda^{\omega\omega} \cap \mu^{\omega\omega}$.

PROOF. (a) By Lemma 1.12, we have

$$\dim(\lambda^\omega \cap \mu^\omega) \le \dim X/(\lambda+\mu) < +\infty, \tag{1.17}$$

$$\dim(\lambda \cap \mu) \le \dim(\lambda^{\omega\omega} \cap \mu^{\omega\omega}) \le \dim X/(\lambda^\omega + \mu^\omega) < +\infty. \tag{1.18}$$

Then (λ,μ) and $(\lambda^\omega,\mu^\omega)$ are Fredholm pairs, and we have

$$\begin{aligned}\operatorname{index}(\lambda,\mu)+\operatorname{index}(\lambda^\omega,\mu^\omega) &= \dim(\lambda\cap\mu)-\dim X/(\lambda+\mu)\\ &\quad +\dim(\lambda^\omega\cap\mu^\omega)-\dim X/(\lambda^\omega+\mu^\omega)\\ &= \dim(\lambda\cap\mu)-\dim X/(\lambda^\omega+\mu^\omega)\\ &\quad +\dim(\lambda^\omega\cap\mu^\omega)-\dim X/(\lambda+\mu)\le 0.\end{aligned}$$

(b) By the proof of (a), the equality in (1.16) holds if and only if $\dim(\lambda^\omega\cap\mu^\omega)=\dim X/(\lambda+\mu)$ and $\dim(\lambda\cap\mu)=\dim(\lambda^{\omega\omega}\cap\mu^{\omega\omega})=\dim X/(\lambda^\omega+\mu^\omega)$. Since $\lambda\cap\mu\subset\lambda^{\omega\omega}\cap\mu^{\omega\omega}$, by Lemma 1.12, the equality in (1.16) holds if and only if $\lambda+\mu=(\lambda+\mu)^{\omega\omega}$, $\lambda^\omega+\mu^\omega=(\lambda^\omega+\mu^\omega)^{\omega\omega}$, and $\lambda\cap\mu=\lambda^{\omega\omega}\cap\mu^{\omega\omega}$. □

1.3. Natural decomposition of X induced by a Fredholm pair of Lagrangian subspaces with vanishing index. The following lemmata are the key to the definition of the Maslov index in symplectic Banach spaces by symplectic reduction to the finite-dimensional case. For technical reasons, in this section, Fredholm pairs of Lagrangians are always assumed to be of index 0.

We begin with some general facts.

LEMMA 1.17. *Let (X,ω) be a symplectic vector space and X_0,X_1 two linear subspaces with $X=X_0+X_1$. Assume that $X_0\subset X_1^\omega$. Then we have $X_0=X_1^\omega$, $X_1=X_0^\omega$, $X=X_0\oplus X_1$, and X_0,X_1 are symplectic.*

PROOF. Since $X_0\subset X_1^\omega$, we have $X_1\subset X_1^{\omega\omega}\subset X_0^\omega$. Since $X=X_0+X_1$, there holds

$$X_1\cap X_1^\omega\subset X_0^\omega\cap X_1^\omega=(X_0+X_1)^\omega=\{0\}.$$

So X_1 is symplectic, and we have $X_1^\omega=X_1^\omega\cap(X_0+X_1)=X_0+X_1^\omega\cap X_1=X_0$ and $X_1\cap X_0=X_1\cap X_1^\omega=\{0\}$. Hence we have $X=X_0\oplus X_1$. Since $X_1\subset X_0^\omega$ and $X=X_0+X_1$, we have $X_1=X_0^\omega$ and X_0 is symplectic. □

LEMMA 1.18. *Let (X,ω) be a symplectic vector space and λ,V two linear subspaces. Assume that $\dim V<+\infty$. Then we have*

$$\dim\lambda/(\lambda\cap V^\omega)\le\dim V. \tag{1.19}$$

The equality holds if and only if $\lambda+V^\omega=X$. In this case we have $\lambda^\omega\cap V=\{0\}$.

PROOF. By [22, Corollary 1], we have $\dim X/V^\omega=\dim V$. Hence we have

$$\dim\lambda/(\lambda\cap V^\omega)=\dim(\lambda+V^\omega)/V^\omega\le\dim X/V^\omega=\dim V.$$

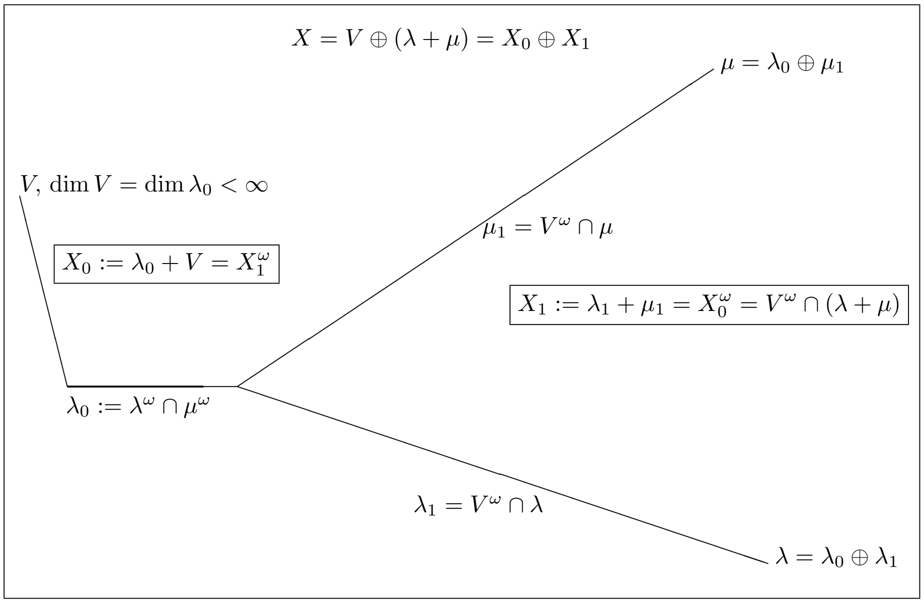

FIGURE 4. Natural decomposition of a symplectic vector space

The equality holds if and only if $\lambda + V^\omega = X$. In this case we have $\lambda^\omega \cap V = (\lambda + V^\omega)^\omega = \{0\}$. □

Now we turn to our key observation.

PROPOSITION 1.19. *Let (X, ω) be a symplectic vector space. Let (λ, μ) be a pair of co-isotropic subspaces with* $\dim \lambda_0 = \dim X/(\lambda+\mu) < +\infty$*, where $\lambda_0 = \lambda^\omega \cap \mu^\omega$. Let V be a linear subspace of X with $X = V \oplus (\lambda + \mu)$. Let $\lambda_1 = V^\omega \cap \lambda$ and $\mu_1 = V^\omega \cap \mu$. Let $X_0 = \lambda_0 + V$ and $X_1 = \lambda_1 + \mu_1$. Then the following holds.*
(a) $V^\omega + \lambda_0 = X$.
(b) $X_0 = \lambda_0 \oplus V$, $\lambda = \lambda_0 \oplus \lambda_1$ and $\mu = \lambda_0 \oplus \mu_1$. $X_1 = \lambda_1 \oplus \mu_1$ if λ and μ are Lagrangian subspaces of X.
(c) $\lambda_1 = \lambda \cap X_1$, $\mu_1 = \mu \cap X_1$ and $\lambda + \mu = \lambda_0 + X_1$.
(d) $X_1 = X_0^\omega = V^\omega \cap (\lambda + \mu)$, $X_0 = X_1^\omega$, $X = X_0 \oplus X_1$, and X_0 and X_1 are symplectic.
(e) The subspace λ_0 is a Lagrangian subspace of X_0. λ_1, μ_1 are Lagrangian subspaces of X_1 if λ and μ are Lagrangian subspaces of X.

For the notations of Proposition 1.19, see Fig. 4.

PROOF. (a) Since $X = V \oplus (\lambda + \mu)$, we have $V \cap \lambda_0 = \{0\}$ and $V^\omega \cap \lambda_0 = \{0\}$. By [22, Corollary 1], we have $\dim X/V^\omega = \dim V =$

$\dim \lambda_0$. So we have $X = V^\omega + \lambda_0$.
(b) Note that

$$\begin{aligned} \dim \lambda_0 &\le \dim \lambda/(V^\omega \cap \lambda) &= \dim(V^\omega + \lambda)/V^\omega \\ &\le \dim X/V^\omega \le \dim V &= \dim \lambda_0. \end{aligned}$$

We have $\lambda_1 \cap \mu_1 = V^\omega \cap \lambda \cap \mu = V^\omega \cap \lambda_0 = \{0\}$ if λ and μ are Lagrangian subspaces of X. So (b) holds.
(c) Since $X_1 = \lambda_1 + \mu_1 \subset V^\omega$, we have $\lambda \cap X_1 \lambda_1 \subset \lambda \cap X_1$. So $\lambda_1 = \lambda \cap X_1$ holds. Similarly we have $\mu_1 = \mu \cap X_1$. By (b) we have

$$\lambda + \mu = \lambda_0 + \lambda_1 + \lambda_0 + \mu_1 = \lambda_0 + \lambda_1 + \mu_1 = \lambda_0 + X_1.$$

(d) Since $X = X_0 + X_1$, our claim follows from Lemma 1.17 and the fact

$$X_0^\omega = V^\omega \cap \lambda_0^\omega \supset V^\omega \cap (\lambda + \mu) \supset X_1.$$

(e) By definition, λ_0 is isotropic. Moreover, $\dim \lambda_0 = \frac{1}{2} \dim X_0$. So λ_0 is Lagrangian in X_0.

Now assume that λ and μ are Lagrangian subspaces of X. Note that λ_1 and μ_1 are isotropic. Since $X_1 = \lambda_1 \oplus \mu_1$, by [22, Lemma 4], λ_1 and μ_1 are Lagrangian subspaces of X_1. □

Corollary 1.20. *Let (X, ω) be a symplectic vector space. Let (λ, μ) be a Fredholm pair of Lagrangian subspaces of index 0. Then there exists a Lagrangian subspace $\widetilde{\mu} \subset X$ such that $X = \lambda \oplus \widetilde{\mu}$ and $\dim \mu/(\mu \cap \widetilde{\mu}) = \dim \widetilde{\mu}/(\mu \cap \widetilde{\mu}) = \dim(\lambda \cap \mu)$.*

Proof. By Proposition 1.19, X_0 is symplectic and λ_0 is a Lagrangian subspace of X_0. Choose a Lagrangian $\widetilde{V}$ of X_0 with $X_0 = \lambda_0 \oplus \widetilde{V}$. Then set $\widetilde{\mu} := \widetilde{V} \oplus \lambda_1$. □

Lemma 1.21. *Let (X, ω) be a symplectic vector space and λ an isotropic subspace of X. Assume that $\dim \lambda = n < +\infty$. Then there exists a $2n$ dimensional symplectic subspace X_0 such that λ is a Lagrangian subspace of X_0, $X_0 = X_0^{\omega\omega}$ and $X = X_0 \oplus X_0^\omega$.*

Proof. Since $\dim \lambda = n < +\infty$, by [22, Corollary 1] we have $\lambda^{\omega\omega} = \lambda$ and $\dim X/\lambda^\omega = n$. Take an n dimensional linear subspace V of X such that $X = V \oplus \lambda^\omega$. Since $\lambda \subset \lambda^\omega$, we have

$$\lambda^\omega \cap (\lambda + V) = \lambda + \lambda^\omega \cap V = \lambda.$$

Since $\dim V = n < +\infty$, by [22, Corollary 1] we have $V^{\omega\omega} = V$ and $\dim X/V^\omega = n$. Set $X_0 := \lambda + V$. Then we have

$$X_0 \cap X_0^\omega = (\lambda + V) \cap \lambda^\omega \cap V^\omega = \lambda \cap V^\omega = (\lambda^\omega + V)^\omega = \{0\}.$$

By [22, Corollary 1], $\dim X/X_0^\omega = \dim X_0 = 2n$ and $X_0^{\omega\omega} = X_0$. So we have $X = X_0 \oplus X_0^\omega$. Since $\dim \lambda = n$ and λ is isotropic, λ is a Lagrangian subspace of X_0. □

Corollary 1.22. *Let ε be a positive number. Let $(X, \omega(s))$, $s \in (-\varepsilon, \varepsilon)$ be a family of symplectic Banach space with continuously varying $\omega(s)$. Let $X_0(s)$, $s \in (-\varepsilon, \varepsilon)$ be a continuous family of linear subspaces of dimension $2n < +\infty$ such that $(X_0(0), \omega(0)|_{X_0(0)})$ is symplectic. Let $\lambda(0)$ be a Lagrangian subspace of $(X_0(0), \omega(0)|_{X_0(0)})$. Then there exist a $\delta \in (0, \varepsilon)$ and a continuous family of linear subspaces $\lambda(s)$, $s \in (-\delta, \delta)$ such that $(X_0(s), \omega(s)|_{X_0(s)})$ is symplectic and $\lambda(s)$ is a Lagrangian subspace of $(X_0(s), \omega(s)|_{X_0(s)})$ for each $s \in (-\delta, \delta)$.*

Proof. Since $\dim X_0(s) = 2n < +\infty$ and $X_0(0)$ is symplectic, we have $X = X_0(0) \oplus X_0(0)^{\omega(0)}$. By Appendix A.3, there exists a $\delta_1 \in (0, \varepsilon)$ such that $X = X_0(s) \oplus X_0(0)^{\omega(s)}$ for each $s \in (-\delta_1, \delta_1)$.

By the proof of [52, Lemma III.1.40], there exists a closed subspace X_1 such that $X = X_0(0) \oplus X_1$. Then there exists a $\delta_2 \in (0, \delta_1)$ such that $X = X_0(s) \oplus X_1$ for each $s \in (-\delta_2, \delta_2)$. By [52, Lemma I.4.10], we can work on a finite-dimensional symplectic vector space $X_0(0)$ with continuously varying symplectic structure.

We give $X_0(0)$ an inner product $\langle \cdot, \cdot \rangle$. Let $J_0(s) \in \mathrm{GL}(X_0(0))$ be the operators that define symplectic structures on $X_0(0)$. Since $\lambda(0)$ is a Lagrangian subspace of $(X_0(0), \omega(0))$, $\operatorname{sign}(iJ_0(s)) = 0$. Then there exists a continuous family $T(s) \in \mathrm{GL}(X_0(0))$, $s \in (-\delta, \delta)$ with $\delta \in (0, \delta_2)$ such that $T(s)^* J_0(s) T(s) = J_{2n}$, where

$$J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Set $\lambda(s) := T(s)\lambda(0)$ and our result follows. □

1.4. Symplectic reduction of Fredholm pairs. We recall the general definition of symplectic reduction.

Definition 1.23. Let (X, ω) be a symplectic vector space and W a co-isotropic subspace.

(a) The space W/W^ω is a symplectic vector space with induced symplectic structure

$$\widetilde{\omega}(x + W^\omega, y + W^\omega) := \omega(x, y) \text{ for all } x, y \in W. \tag{1.20}$$

We call $(W/W^\omega, \widetilde{\omega})$ the *symplectic reduction* of X via W.

(b) Let λ be a linear subspace of X. The *symplectic reduction* of λ via W is defined by

$$R_W(\lambda) = R_W^\omega(\lambda) := \big((\lambda + W^\omega) \cap W\big)/W^\omega = \big(\lambda \cap W + W^\omega\big)/W^\omega. \tag{1.21}$$

Clearly, $R_W(\lambda)$ is isotropic if λ is isotropic. If $W^\omega \subset \lambda \subset W$ and λ is Lagrangian, $R_W(\lambda)$ is Lagrangian. We have the following lemma.

LEMMA 1.24. *Let (X,ω) be a symplectic vector space with isotropic subspace W_0. Let $\lambda \supset W_0$ be a linear subspace. Then λ is a Lagrangian subspace of X if and only if $W_0^{\omega\omega} \subset \lambda \subset W_0^\omega$ and $R_{W_0^\omega}(\lambda)$ is a Lagrangian subspace of $W_0^\omega/W_0^{\omega\omega}$.*

PROOF. By (1.3) we have $W_0^{\omega\omega\omega} = W_0^\omega$. Since $W_0 \subset W_0^\omega$, $W_0^{\omega\omega} \subset W_0^\omega$.

If $\lambda \in \mathcal{L}(X)$ and $\lambda \supset W_0$, we have $\lambda \subset W_0^\omega$ and $W_0^{\omega\omega} \subset \lambda$. Then we get $R_{W_0^\omega}(\lambda) = \lambda/W_0^{\omega\omega}$ and $(\lambda/W_0^{\omega\omega})^{\widetilde{\omega}} = (\lambda^\omega \cap W_0^\omega)/W_0^{\omega\omega} = \lambda/W_0^{\omega\omega}$, i.e., $R_{W_0^\omega}(\lambda) \in \mathcal{L}(W_0^\omega/W_0^{\omega\omega})$.

Assume that $W_0^{\omega\omega} \subset \lambda \subset W_0^\omega$, we have $W_0^{\omega\omega} \subset \lambda^\omega \subset W_0^\omega$. If $R_{W_0^\omega}(\lambda) \in \mathcal{L}(W_0^\omega/W_0^{\omega\omega})$, we have

$$\lambda/W_0^{\omega\omega} = (\lambda/W_0^{\omega\omega})^{\widetilde{\omega}} = (\lambda^\omega \cap W_0^\omega)/W_0^{\omega\omega} = \lambda^\omega/W_0^{\omega\omega}.$$

So we get $\lambda = \lambda^\omega$, i.e., $\lambda \in \mathcal{L}(X)$. □

LEMMA 1.25 (Transitivity of symplectic reduction). *Let (X,ω) be a symplectic vector space with two co-isotropic subspaces $W_1 \subset W_2$, hence clearly $W_1/W_2^\omega \subset W_2/W_2^\omega$ with $(W_1/W_2^\omega)^{\omega_2} = W_1^\omega/W_2^\omega$, where ω_2 denotes the symplectic form on W_2/W_2^ω induced by ω. Then the following holds.*
(a) Denote by $K_{W_1,W_2}\colon W_1/W_2^\omega \longrightarrow W_1/W_1^\omega$ the map induced by I_{W_1}, where I_W denotes the identity map on a space W. Then K_{W_1,W_2} induces a symplectic isomorphism

$$\widetilde{K}_{W_1,W_2}\colon (W_1/W_2^\omega)/(W_1^\omega/W_2^\omega) \longrightarrow W_1/W_1^\omega, \tag{1.22}$$

such that the following diagram becomes commutative:
(1.23)

$$\begin{tikzcd} W_1 \arrow[r, "{[\cdot+W_2^\omega]}"] \arrow[d, "I_{W_1}"] & W_1/W_2^\omega \arrow[r, "{[\cdot+W_1^\omega/W_2^\omega]}"] \arrow[d, "K_{W_1,W_2}"'] & (W_1/W_2^\omega)/(W_1^\omega/W_2^\omega) \arrow[dl, "\cong"', "\widetilde{K}_{W_1,W_2}"] \\ W_1 \arrow[r, "{[\cdot+W_1^\omega]}"'] & W_1/W_1^\omega & \end{tikzcd}$$

(b) For a linear subspace of λ of X, we have

$$R_{W_1/W_2^\omega}(R_{W_2}(\lambda)) = \widetilde{K}^{-1}_{W_1,W_2}(R_{W_1}(\lambda)). \tag{1.24}$$

Differently put, the following diagram is commutative:

$$
\begin{tikzcd}
\operatorname{Lin}(X) \arrow[r, "R_{W_1}"] \arrow[d, "R_{W_2}"'] & \operatorname{Lin}(W_1/W_1^\omega) \arrow[d, "(\widetilde{K}_{W_1,W_2})^{-1}"] \\
\operatorname{Lin}(W_2/W_2^\omega) \arrow[r, "R_{W_1/W_2^\omega}"'] & \operatorname{Lin}\big((W_1/W_2^\omega)/(W_1^\omega/W_2^\omega)\big)
\end{tikzcd}
\tag{1.25}
$$

Here $\operatorname{Lin}(X)$ *denotes the set of linear subspaces of the vector space* X*.*

Proof. (a) Since $W_1 \subset W_2$ and they are co-isotropic, we have $W_2^\omega \subset W_1^\omega \subset W_1 \subset W_2$. So K_{W_1,W_2} is well-defined. Since $\ker K_{W_1,W_2} = W_1^\omega/W_2^\omega$, $\widetilde{K}_{W_1,W_2}$ is a linear isomorphism. By Definition 1.23, $\widetilde{K}_{W_1,W_2}$ is a symplectic isomorphism.

(b) Note that

$$
\begin{aligned}
R_{W_2}(\lambda) \cap (W_1/W_2^\omega) + W_1^\omega/W_2^\omega &= \big((\lambda \cap W_2 + W_2^\omega) \cap W_1 + W_1^\omega\big)/W_2^\omega \\
&= (\lambda \cap W_1 + W_2^\omega + W_1^\omega)/W_2^\omega = (\lambda \cap W_1 + W_1^\omega)/W_2^\omega.
\end{aligned}
$$

So (1.24) holds. □

Corollary 1.26. *Let* (X,ω) *be a symplectic vector space with a co-isotropic subspace* W*, a Lagrangian subspace* μ *and two linear spaces* V,λ*. Assume that* $\dim W^\omega \cap \mu = \dim X/(W+\mu) = \dim V < +\infty$*,* $X = V \oplus (W+\mu)$ *and* $W^\omega \cap \mu \subset \lambda \subset W+\mu$*. Set* $X_0 := W^\omega \cap \mu + V$ *and* $X_1 := V^\omega \cap W + V^\omega \cap \mu$*. Denote by* $P_1 \colon X \to X_1$ *defined by* $X = X_0 \oplus X_1$ *(see Proposition 1.19). Then the following holds.*
(a) $W \cap X_1 = V^\omega \cap W$*,* $W^\omega \cap X_1 = W^\omega \cap V^\omega$*,* $\mu \cap X_1 = V^\omega \cap \mu$*,* $\lambda = W^\omega \cap \mu + \lambda \cap X_1$*, and* $(W \cap X_1)^\omega = W^\omega + V = X_0 + W^\omega \cap X_1$*.*
(b) P_1 *induces a symplectic isomorphism* $\widetilde{P}_1 \colon (W+\mu)/(W^\omega \cap \mu) \to X_1$ *and* $\widetilde{P}_1(R_{W+\mu}(\lambda)) = \lambda \cap X_1$*.*
(c) Denote by $R^{X_1}_{V^\omega \cap W}(\lambda \cap X_1)$ *the symplectic reduction of* $\lambda \cap X_1$ *in* X_1 *via* $V^\omega \cap W$*. Define* $\widetilde{L}_{W,W+\mu} \colon (W \cap X_1)/(W^\omega \cap X_1) \to W/W^\omega$ *by* $\widetilde{L}_{W,W+\mu}(x + W^\omega \cap X_1) = x + W^\omega$ *for all* $x \in W \cap X_1$*. Then the following diagram is commutative*

$$
\begin{tikzcd}
\operatorname{Lin}_{W,\mu}(X) \arrow[r, "R_W"] \arrow[d, "\cap X_1"'] & \operatorname{Lin}(W/W^\omega) \arrow[d, "\cong"', "(\widetilde{L}_{W,W+\mu})^{-1}"] \\
\operatorname{Lin}(X_1) \arrow[r, "R^{X_1}_{V^\omega \cap W}"'] & \operatorname{Lin}\big((W \cap X_1)/(W^\omega \cap X_1)\big)
\end{tikzcd}
\tag{1.26}
$$

and, in particular, we have

$$
R^{X_1}_{V^\omega \cap W}(\lambda \cap X_1) = \widetilde{L}^{-1}_{W,W+\mu}(R_W(\lambda)). \tag{1.27}
$$

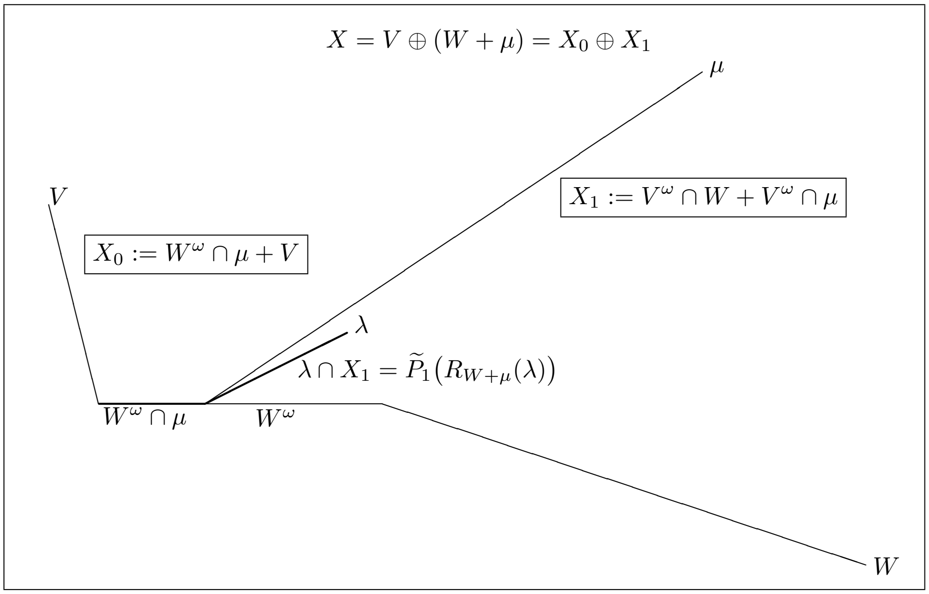

FIGURE 5. Data of the inner symplectic reduction

Here $\mathrm{Lin}_{W,\mu}(X) := \{\lambda \in \mathrm{Lin}(X) \mid W^\omega \cap \mu \subset \lambda \subset W + \mu\}$.
(d) W *is complemented (see Remark 1.27) in* X *if and only if* $W \cap X_1$ *is complemented in* X_1. *In the case of a Banach space we require all the appeared subspaces to be closed.*
(e) W^ω *is complemented in* W *if and only if* $W^\omega \cap X_1$ *is complemented in* $W \cap X_1$. *In the case of a Banach space we require all the appeared subspaces to be closed.*

For the notations of Corollary 1.26, see Fig. 5.

PROOF. (a) By Proposition 1.19, we have $W \cap X_1 = V^\omega \cap W$, $\mu \cap X_1 = V^\omega \cap \mu$, $W + \mu = W^\omega \cap \mu + X_1$, and $X = V^\omega + W = V^\omega + \mu$. Since $W^\omega \cap \mu \subset \lambda \subset W + \mu$, we have

$$\lambda = \lambda \cap (W + \mu) = \lambda \cap (W^\omega \cap \mu + X_1) = W^\omega \cap \mu + \lambda \cap X_1.$$

Note that $W = W^\omega \cap \mu + W^\omega \cap X_1$. By Lemma 1.18 we have $\dim(W^\omega \cap X_1)^w / W^\omega \le \dim(W^\omega \cap \mu) = \dim V$. Moreover, we have

$$(W \cap X_1)^\omega \supset W^\omega + X_1^\omega = W^\omega + X_0 = W^\omega + V.$$

Since $W^\omega \cap V = (W + V^\omega)^\omega = \{0\}$, we have

$$(W \cap X_1)^\omega = W^\omega + V = X_0 + W^\omega \cap X_1.$$

(b) Since $W^\omega \cap \mu$ is isotropic and $W + \mu = W^\omega \cap \mu + X_1$, P_1 induces a symplectic isomorphism $\widetilde{P}_1\colon (W+\mu)/(W^\omega \cap \mu) \to X_1$. Since $W^\omega \cap \mu \subset \lambda \subset W + \mu$, we have $R_{W+\mu}(\lambda) = \lambda/(W^\omega \cap \mu)$. So it holds that $\widetilde{P}_1(R_{W+\mu}(\lambda)) = \lambda \cap X_1$.
(c) Let $\widetilde{K}_{W,W+\mu}$ denote the symplectic isomorphism defined by (1.22). Note that $\widetilde{K}_{W,W+\mu} = \widetilde{L}_{W,W+\mu}$ under the symplectic isomorphism $\widetilde{P}_1$. So (1.27) follows from (b) and Lemma 1.25.
(d) If $W \cap X_1$ is complemented in X_1, there exists a linear subspace M_1 such that $X_1 = W \cap X_1 \oplus M_1$. Since $\dim X_0 < +\infty$, there exists a linear subspace M_0 such that $X_0 = W^\omega \cap \mu \oplus M_0$. Take $M = M_0 \oplus M_1$ and we have $X = W \oplus M$.

Conversely, if W is complemented in X, there exists a linear subspace of M such that $X = W \oplus M$. By (a), we have $W = W \cap X_1 \oplus W^\omega \cap \mu$. So we have

$$X_1 = X_1 \cap (W \cap X_1 + W^\omega \cap \mu + M) = W \cap X_1 \oplus X_1 \cap (W^\omega \cap \mu + M).$$

(e) If $W^\omega \cap X_1$ is complemented in $W \cap X_1$, there exists a linear subspace N_1 such that $W \cap X_1 = W^\omega \cap X_1 \oplus N_1$. Then we have $W = W^\omega \cap \mu \oplus W^\omega \cap X_1 \oplus N_1 = W^\omega \oplus N_1$.

Conversely, if W^ω is complemented in W, there exists a linear subspace of N such that $W = W^\omega \oplus N$. By (a), we have

$$\begin{aligned} W \cap X_1 &= (W \cap X_1 \oplus W^\omega \cap \mu) \cap X_1 \\ &= (W^\omega \cap X_1 \oplus N \oplus W^\omega \cap \mu) \cap X_1 \\ &= (W^\omega \cap X_1) \oplus (N \oplus W^\omega \cap \mu) \cap X_1. \end{aligned}$$

□

Remark 1.27. A linear subspace M of a vector space X is called *complemented* in X if there exists another linear subspace N of X such that $X = M \oplus N$. In Banach space we require M, N to be closed. Note that any linear subspace in a vector space is complemented by Zorn's lemma. Our Corollary 1.26 (d), (e) is not trivial if either X is a Banach space or one does not want to use Zorn's lemma.

To ensure that symplectic reduction does not lead us out of our class of pairs of Fredholm Lagrangian subspaces of index 0, we prove the following Proposition 1.30.

Lemma 1.28. *Let X be a vector space and $W_1 \subset W_2$, λ, μ four linear subspaces of X. For each linear subspace V, set $R(V) := \big(V \cap W_2 + W_1\big)/W_1$. Assume that $W_1 \subset \lambda \subset W_2$. Then (λ, μ) is a Fredholm pair of subspaces of X if and only if $(R(\lambda), R(\mu))$ is a Fredholm pair of subspaces of W_2/W_1, $\dim(\mu \cap W_1) < +\infty$ and $\dim X/(W_2 + \mu) < +\infty$.*

In this case it holds that

$$\begin{aligned}\dim(R(\lambda)\cap R(\mu)) &= \dim(\lambda\cap\mu)-\dim(\mu\cap W_1),\\ \dim(W_2/W_1)/(R(\lambda)+R(\mu)) &= \dim X/(\lambda+\mu)-\dim X/(W_2+\mu),\\ \operatorname{index}(R(\lambda),R(\mu)) &= \operatorname{index}(\lambda,\mu)\\ &\quad -\dim(\mu\cap W_1)+\dim X/(W_2+\mu).\end{aligned}$$

PROOF. Since $W_1\subset\lambda\subset W_2$, we have

$$\begin{aligned}R(\lambda)\cap R(\mu) &= (\lambda/W_1)\cap\big(((\mu+W_1)\cap W_2)/W_1\big)=(\lambda\cap\mu+W_1)/W_1\\ &\cong (\lambda\cap\mu)/(\lambda\cap\mu\cap W_1),\end{aligned}$$

and

$$\begin{aligned}(W_2/W_1)/(R(\lambda)+R(\mu)) &\cong\ W_2/(\lambda+\mu\cap W_2)=W_2/((\lambda+\mu)\cap W_2)\\ &= (W_2+\lambda+\mu)/(\lambda+\mu)=(W_2+\mu)/(\lambda+\mu)\\ &\cong (X/(\lambda+\mu))/(X/(W_2+\mu)).\end{aligned}$$

So our lemma follows. $\square$

Now we can prove the basic calculation rule of symplectic reduction:

PROPOSITION 1.29 (Symplectic quotient rule). *Let (X,ω) be a symplectic vector space and λ, μ, W subspaces. Assume that $\lambda\subset W$, $\mu=\mu^\omega$ and*

$$\operatorname{index}(\lambda,\mu)+\operatorname{index}(\lambda^\omega,\mu)=0. \tag{1.28}$$

Then we have $\dim(W^\omega\cap\mu)=\dim X/(W+\mu)<+\infty$ *and we have* $W+\mu=W^{\omega\omega}+\mu$.

PROOF. Since $\lambda\subset W$, we have $W^\omega\subset\lambda^\omega$. Since $\mu=\mu^\omega$, we have $(W+\mu)^\omega=W^\omega\cap\mu\subset W+\mu$. Denote by $\widetilde{\omega}$ the symplectic structure on $(W+\mu)/(W^\omega\cap\mu)$. Then we have

$$\begin{aligned}&\lambda\cap(W+\mu)=\lambda,\qquad \lambda^\omega+W^\omega\cap\mu=\lambda^\omega,\\ &(\lambda+W^\omega\cap\mu)^\omega\cap(W+\mu)=\lambda^\omega\cap(W+\mu),\\ &R_{W+\mu}(\lambda)=(\lambda+W^\omega\cap\mu)/(W^\omega\cap\mu),\\ &R_{W+\mu}(\lambda^\omega)=(\lambda^\omega\cap(W+\mu))/(W^\omega\cap\mu)=(R_{W+\mu}(\lambda))^{\widetilde{\omega}},\\ &R_{W+\mu}(\mu)=\mu/(W^\omega\cap\mu)=(R_{W+\mu}(\mu))^{\widetilde{\omega}}.\end{aligned}$$

By Lemma 1.28 and (1.28) we have

$$\begin{aligned}\operatorname{index}(\lambda,\mu) &= \operatorname{index}(R_{W+\mu}(\lambda), R_{W+\mu}(\mu)) \\ &\quad + \dim(\lambda\cap W^\omega\cap\mu) - \dim X/(W+\mu), \\ \operatorname{index}(\lambda^\omega,\mu) &= \operatorname{index}(R_{W+\mu}(\lambda^\omega), R_{W+\mu}(\mu)) \\ &\quad + \dim(W^\omega\cap\mu) - \dim X/(\lambda^\omega+W+\mu).\end{aligned}$$

Note that $(\lambda^\omega+W+\mu)^\omega = \lambda^{\omega\omega}\cap W^\omega\cap\mu \supset \lambda\cap W^\omega\cap\mu$. By Lemma 1.12 and Corollary 1.16 we have

$$\begin{aligned}&\operatorname{index}(R_{W+\mu}(\lambda), R_{W+\mu}(\mu)) + \operatorname{index}(R_{W+\mu}(\lambda^\omega), R_{W+\mu}(\mu)) \le 0, \\ &\dim(\lambda\cap W^\omega\cap\mu) \le \dim X/(\lambda^\omega+W+\mu), \\ &\dim(W^\omega\cap\mu) \le \dim X/(W+\mu).\end{aligned}$$

By (1.28), the above three inequalities take equalities.

By (1.3), we have $W^{\omega\omega\omega} = W^\omega$. Apply the above result to $W^{\omega\omega}$, we have $\dim(W^\omega\cap\mu) = \dim X/(W^{\omega\omega}+\mu)$. Since $W\subset W^{\omega\omega}$, we have $W+\mu = W^{\omega\omega}+\mu$. □

The following proposition is inspired by [14, Proposition 3.5].

Proposition 1.30. *Let (X,ω) be a symplectic vector space with a co-isotropic subspace W. Let (λ,μ) be a Fredholm pair of Lagrangian subspaces of X with index 0. Assume that $W^\omega\subset\lambda\subset W$. Then we have $\dim(W^\omega\cap\mu) = \dim X/(W+\mu) < +\infty$, $W+\mu = W^{\omega\omega}+\mu$, and $(R_W(\lambda), R_W(\mu))$ is a Fredholm pair of Lagrangian subspaces of W/W^ω with index 0.*

Proof. By Proposition 1.29 we have $\dim(W^\omega\cap\mu) = \dim X/(W+\mu) < +\infty$ and $W+\mu = W^{\omega\omega}+\mu$.

By Lemma 1.28, $(R_W(\lambda), R_W(\mu))$ is a Fredholm pair of subspaces of W/W^ω, $\dim(W^\omega\cap\mu) < +\infty$, and $\dim X/(W+\mu) < +\infty$. Since λ and μ are Lagrangian subspaces of X, $R_W(\lambda)$ and $R_W(\mu)$ are isotropic subspaces of W/W^ω. By Lemma 1.12, we have $\dim(W^\omega\cap\mu) \le \dim X/(W+\mu)$ and $\operatorname{index}(R_W(\lambda), R_W(\mu)) \le 0$. By Lemma 1.28, we have $\dim(W^\omega\cap\mu) = \dim X/(W+\mu)$ and $\operatorname{index}(R_W(\lambda), R_W(\mu)) = 0$. By [22, Proposition 1], $R_W(\lambda)$ and $R_W(\mu)$ are Lagrangian subspaces of W/W^ω. □

Corollary 1.31. *Let (X,ω) be a symplectic vector space with a finite-dimensional linear subspace V. Let (λ,μ) be a Fredholm pair of Lagrangian subspaces of X with index 0. Assume that $V+\lambda+\mu = X$ and $V\cap\lambda = \{0\}$. Then we have $V^\omega+\lambda = X$.*

Proof. Set $W := V+\lambda$. Then $W^\omega = V^\omega\cap\lambda$. By Lemma 1.18, we have $\dim\lambda/W^\omega \le \dim V$. Since $V\cap\lambda = \{0\}$, we have $\dim W/W^\omega = \dim V + \dim\lambda/W^\omega$.

By Proposition 1.30, $R_W(\lambda) = \lambda/W^\omega$ is a Lagrangian subspace of W/W^ω. Then we have

$$\dim \lambda/W^\omega = \frac{1}{2}\dim W/W^\omega = \dim V.$$

By Lemma 1.18 we have $V^\omega + \lambda = X$. □

The following proposition gives us a new understanding of the symplectic reduction.

Proposition 1.32. *Let (X,ω) be a symplectic vector space and λ_0, V linear subspaces. Let λ and μ be Lagrangian subspaces. Set $\lambda_1 := V^\omega \cap \lambda$, $\mu_1 := V^\omega \cap \mu$, $X_0 := \lambda_0 + V$ and $X_1 := \lambda_1 + \mu_1$. Assume that*

$$X = \lambda_0 \oplus V \oplus \lambda_1 \oplus \mu_1 = \lambda \oplus (V + \mu_1) = \mu \oplus (V + \lambda_1). \tag{1.29}$$

Denote by $P_0\colon X \to X_0$ the projection defined by $X = X_0 \oplus X_1$. Then the following holds.
(a) There exist $A_1 \in \operatorname{Hom}(\lambda_0, V), A_2 \in \operatorname{Hom}(\lambda_0, \mu_1), B_1 \in \operatorname{Hom}(\lambda_0, V)$ and $B_2 \in \operatorname{Hom}(\lambda_0, \lambda_1)$ such that

$$\lambda = \{x_0 + A_1x_0 + x_1 + A_2x_0; x_0 \in \lambda_0, x_1 \in \lambda_1\}, \tag{1.30}$$

$$\mu = \{y_0 + B_1y_0 + B_2y_0 + y_1; y_0 \in \lambda_0, y_1 \in \mu_1\}, \tag{1.31}$$

where $\operatorname{Hom}(X,Y)$ denotes the linear maps from X to Y.
(b) The linear maps $P_0|_{(V+\lambda)}$ and $P_0|_{(V+\mu)}$ induce linear isomorphisms $T_l\colon (V+\lambda)/\lambda_1 \to X_0$ and $T_r\colon (V+\mu)/\mu_1 \to X_0$ respectively, and

$$\dim(\lambda \cap \mu) = \dim(P_0(\lambda) \cap \dim P_0(\mu)). \tag{1.32}$$

(c) We have

$$T_l(R_{V+\lambda}(\lambda)) = T_r(R_{V+\mu}(\lambda)) = P_0(\lambda), \tag{1.33}$$

$$T_l(R_{V+\lambda}(\mu)) = T_r(R_{V+\mu}(\mu)) = P_0(\mu). \tag{1.34}$$

(d) Denote by ω_l the symplectic structure of X_0 induced by T_l from $(V+\lambda)/\lambda_1$ and ω_r the symplectic structure of X_0 induced by T_r from $(V+\mu)/\mu_1$. Then we have

$$\omega_l(x_0+v, x_0'+v') = \omega(x_0+v, x_0'+v') - \omega(x_0+A_1x_0, x_0'+A_1X_0')$$
$$= \omega_r(x_0+v, x_0'+v') = \omega(x_0+v, x_0'+v') - \omega(x_0+B_1x_0, x_0'+B_1X_0')$$

for all $x_0, x_0' \in \lambda_0$ and $v, v' \in V$. If either $\lambda_0 \subset \lambda_1^\omega$ or $\lambda_0 \subset \mu_1^\omega$, we have $\omega_l = \omega_r = \omega|_{X_0}$.
(e) Assume that V is isotropic.

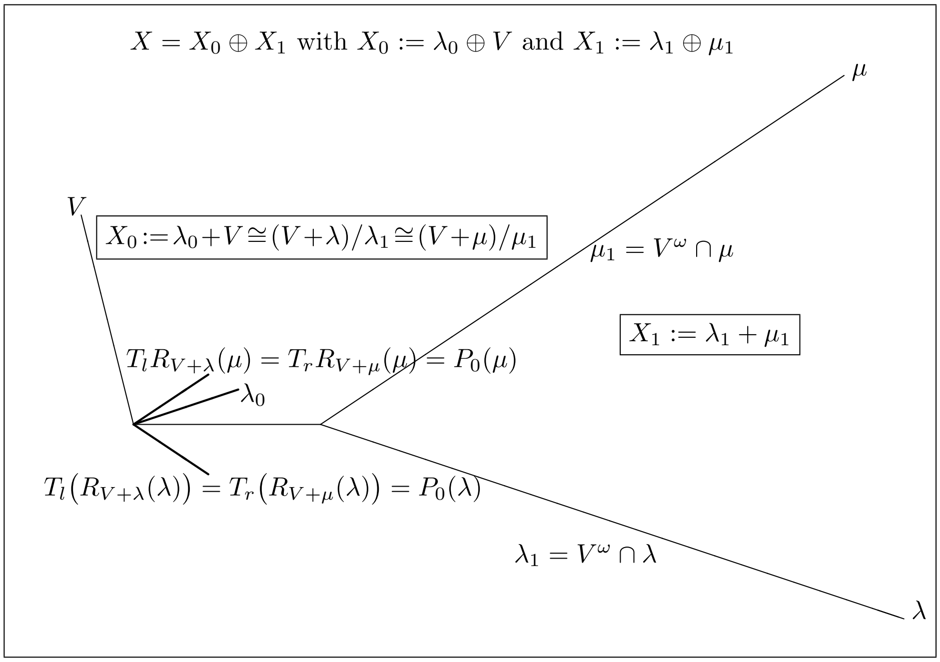

FIGURE 6. Invariance of the two natural symplectic reductions of a symplectic vector space

(i) *The sesquilinear form* $Q(x_0, x_0') := \omega(x_0, (A_1 - B_1)x_0')$ *on* λ_0 *is a quadratic form. We call the form* Q *the* intersection from of (λ, μ) on λ_0 at V. *If* $\lambda_0 = \mu$ *and* V *is a Lagrangian subspace* W *of* X, *we set* $Q(\mu, W; \lambda) := Q$ *(see* [39, (2.4)]*).*

(ii) *Assume that there is another pair* $(\widetilde{\lambda}, \widetilde{V})$ *that satisfies the conditions for the pair* (λ_0, V), $\lambda_0 = \lambda \cap \mu \subset \widetilde{\lambda}$ *and* $V \subset \widetilde{V}$. *Then we have*

$$\omega(x_0, A_1 x_0') = \overline{\omega(x_0', A_1 x_0)} = \omega(x_0, \widetilde{A}_1 x_0'), \tag{1.35}$$

$$\omega(x_0, B_1 x_0') = \overline{\omega(x_0', B_1 x_0)} = \omega(x_0, \widetilde{B}_1 x_0') \tag{1.36}$$

for all $x_0, x_0' \in \lambda_0$.

(f) We have $V + \lambda + \mu = X$.

For the notations of Proposition 1.32, see Fig. 6.

PROOF. (a) Note that $\lambda_1 = V^\omega \cap \lambda \subset \lambda$ and $\mu_1 = V^\omega \cap \mu \subset \mu$. Our claim follows from the assumptions.

(b) By (a) we have

$$V + \lambda = \{x_0 + v + x_1 + A_2 x_0; x_0 \in \lambda_0, v \in V, x_1 \in \lambda_1\}.$$

So $P_0|_{(V+\lambda)}$ induces a linear map $T_l\colon (V+\lambda)/\lambda_l \to X_0$. Clearly, $\ker T_l = \{0\}$. By Corollary A.2, T_l is surjective. Thus T_l is a linear isomorphism. Similarly we get that the map $P_0|_{(V+\mu)}$ induces a linear isomorphism $T_r\colon (V+\mu)/\mu_1 \to X_0$. the equation (1.32) follows from Lemma 1.28.
(c) By (a) and (b) we have $T_l(R_{V+\lambda}(\lambda)) = P_0(\lambda)$. Note that

$$\mu\cap(V+\lambda) = \{x_0 + B_1x_0 + B_2x_0 + A_2x_0; x_0 \in \lambda_0\}.$$

By (a) and (b) we have $T_l(R_{V+\lambda}(\mu)) = P_0(\mu)$. Similarly we get the result for T_r.
(d) Since $\lambda_1 = (V+\lambda)^\omega$ and $\mu_1 = (V+\mu)^\omega$, $(V+\lambda)/\lambda_1$ and $(V+\mu)/\mu_1$ are symplectic vector spaces. Let $x_0, x_0' \in \lambda_0$ and $v, v' \in V$ be vectors in X. By (a) and (b), we have

$$\begin{aligned}\omega_l(x_0+v, x_0'+v') =&\omega(x_0+v+A_2x_0, x_0'+v'+A_2x_0')\\ =&\omega(x_0+v, x_0'+v') + \omega(x_0+v, A_2x_0')\\ &+\omega(A_2x_0, x_0'+v')\\ =&\omega(x_0+v,x_0'+v') + \omega(x_0, A_2x_0') + \omega(A_2x_0, x_0'). \end{aligned}\tag{1.37}$$

So we have $\omega_l = \omega|_{X_0}$ if $X_0 = X_1^\omega$. Note that $A_1x_0, A_1x_0' \in V$. Then we have

$$\begin{aligned}0 &= \omega(x_0 + A_1x_0 + A_2x_0, x_0' + A_1x_0' + A_2x_0')\\ &= \omega(x_0+A_1x_0, x_0'+A_1x_0') + \omega(x_0, A_2x_0') + \omega(A_2x_0, x_0').\end{aligned}$$

Thus it holds

$$\omega_l(x_0+v, x_0'+v') = \omega(x_0+v, x_0'+v') - \omega(x_0+A_1x_0, x_0'+A_1X_0').$$

Similarly we get the expression for ω_r. Since $P_0(\mu) = T_l(R_{V+\lambda}(\mu))$ is isotropic in (X_0, ω_l), we have

$$\omega(x_0+B_1x_0, x_0'+B_1X_0') = \omega(x_0+A_1x_0, x_0'+A_1X_0')$$

for all $x_0, x_0' \in \lambda_0$ and $v, v' \in V$. So we have $\omega_l = \omega_r$.

If $\lambda_0 \subset \mu_1^\omega$, by (1.37) we have $\omega_l = \omega_r = \omega|_{X_0}$. Similarly, we have $\omega_l = \omega_r = \omega|_{X_0}$ if $\lambda_0 \subset \lambda_1^\omega$.
(e) (i) By (d).
(ii) We have

$$\begin{aligned}0 =&\omega(x_0 + A_1x_0 + A_2x_0, x_0' + \widetilde{A}_1x_0' + \widetilde{A}_2x_0')\\ =&\omega(A_1x_0, x_0' + \widetilde{A}_2x_0') + \omega(x_0+A_2x_0, \widetilde{A}_1x_0')\\ =&\omega(A_1x_0, x_0') + \omega(x_0, \widetilde{A}_1x_0')\end{aligned}$$

for all $x_0, x_0' \in \lambda_0$. By taking $\widetilde{\lambda}_0 = \lambda_0$ and $\widetilde{V} = V$, we have

$$0 = \omega(A_1x_0, x_0') + \omega(x_0, A_1x_0') = -\overline{\omega(x_0', A_1x_0)} + \omega(x_0, A_1x_0').$$

Then we obtain (1.35). Similarly we have (1.36).
(f) Since $V+\lambda+\mu\supset X_1$ and $X_0\supset P_0(V+\lambda)\supset P_0(V+\lambda)=X_0$, by Corollary A.2 we have $V+\lambda+\mu=X$. □

2. The Maslov index in strong symplectic Hilbert space

2.1. The Maslov index via unitary generators. Let $p\colon E\to[0,1]$ be a Hilbert bundle with fibers $X(s):=p^{-1}(s)$ for each $s\in[0,1]$. Let $(X(s),\omega(s))$, $s\in[0,1]$ be a family of strong symplectic Hilbert spaces with continuously varying Hilbert inner product $\langle\cdot,\cdot\rangle_s$ and continuously varying symplectic form $\omega(s)$. As usual, we assume that we can write $\omega(s)(x,y)=\langle J(s)x,y\rangle_s$ with invertible $J(s)\colon X\to X$ and $J(s)^*=-J(s)$. The fiber bundle E is always trivial. So we can actually assume that $X(s)\equiv X$. By [52, Lemma I.4.10] and Lemma A.29, the set of closed subspaces is a Hilbert manifold and can be identified locally with bounded invertible linear maps of X. Let $N\subset M\subset X$ be closed linear subspaces, so that $M/N\cong N^{\perp_M}=N^\perp\cap M$.

Denote by $X^\mp(s)$ the positive (negative) subspace of $iJ(s)$ with respect to the spectral decomposition. Then the quadratic form $-i\omega(s)$ is negative definite, respectively, positive definite on the subspaces $X^\mp(s)$ and we have a symplectic splitting $X=X^-(s)\oplus X^+(s)$.

Definition 2.1. Let $(\lambda(s),\mu(s))$, $s\in[0,1]$ be a path of Fredholm pairs of Lagrangian subspaces of (X,ω_s). Let $U(s),V(s)\colon X_{-,s}\to X_{+,s}$ be generators for $(\lambda(s),\mu(s))$, i.e., $\lambda(s)=\operatorname{graph}(U(s))$ and $\mu(s)=\operatorname{graph}(V(s))$ (see [22, Proposition 2]). Then $U(s)V(s)^{-1}$ is a continuous family of unitary operators on continuous families of Hilbert spaces $X^+(s)$ with Hilbert structure $-i\omega(s)|_{X^+(s)}$ and $U(s)V(s)^{-1}-I_{X^+(s)}$ is a family of Fredholm operators with index 0. Denote by $\ell_\pm:=(1-\varepsilon,1+\varepsilon)$ with real $\varepsilon\in(0,1)$ and with upward (downward) co-orientation. The Maslov index $\operatorname{Mas}_\pm\{\lambda_s,\mu_s\}$ of the path $(\lambda(s),\mu(s))$, $s\in[0,1]$ is defined by

$$\operatorname{Mas}\{\lambda_s,\mu_s\}=\operatorname{Mas}_+\{\lambda_s,\mu_s\}=-\operatorname{sf}_{\ell_-}\{U(s)V(s)^{-1}\},\tag{2.1}$$

$$\operatorname{Mas}_-\{\lambda_s,\mu_s\}=\operatorname{sf}_{\ell_+}\{U(s)V(s)^{-1}\}.\tag{2.2}$$

Here we refer to [98, Definition 2.1] and [22, Definition 13] for the definition of the spectral flow sf.

The following simple example shows that the preceding definition of the Maslov index can not be generalized literally to symplectic Banach spaces or weak symplectic Hilbert spaces. It shows that there exist strong symplectic Banach spaces that do not admit a symplectic splitting in the preceding sense. That may seem to contradict Zorn's Lemma. However, in a symplectic Banach space (X,ω) Zorn's Lemma can only provide the existence of a maximal subspace X^+ where the form $-i\omega$ is positive definite. Then $-i\omega$ is negative definite on $X^- := (X^+)^\omega$ and vanishing on $X^+\times X^-$. However, one can not show that $X = X^+\oplus X^-$. Denote by $V := X^+\oplus X^-$, then $V^\omega = \{0\}$. We see from it that $\overline{V}^{\mathcal{T}} = X$, where $\mathcal{T}$ denotes the locally convex topology defined by ω.

EXAMPLE 2.2 (Symplectic splittings do not always exist). Let $(X,\omega) := \lambda\oplus\lambda^*$ and $\lambda := \ell^p$ with $p\in(1,+\infty)$ and $p\neq 2$. Then X is a strong symplectic Banach space, but there is no splitting $X = X^+\oplus X^-$ such that $\mp i\omega|_{X^\pm} > 0$, and $\omega(x,y) = 0$ for all $x\in X^+$ and $y\in X^-$. Otherwise we could establish an inner product on X that makes X a Hilbert space.

Moreover, even when a symplectic splitting exists, there is no way to establish such splitting for families of symplectic Banach spaces in a continuous way, as emphasized in the Introduction.

Consider the special case $\dim X = 2n < +\infty$. Note that the eigenvalues of $U(s)V(s)^{-1})$ are on the unit circle S^1. Recall that each map in $C([0,1],S^1)$ can be lifted to a map $C([0,1],\mathbb{R})$. By [52, Theorem II.5.2], there are n continuous functions $\theta_1,\dots,\theta_n\in C([0,1],\mathbb{R})$ such that the eigenvalues of the operator $U(s)V(s)^{-1}$ for each $s\in[0,1]$ (counting algebraic multiplicities) have the form

$$e^{i\theta_j(s)},\ j=1,\dots,n.$$

Denote by $[a]$ the integer part of $a\in\mathbb{R}$. Define

$$E(a) := \begin{cases} a, & a\in\mathbb{Z} \\ [a]+1, & a\notin\mathbb{Z}. \end{cases} \tag{2.3}$$

In this case, we have

$$\operatorname{Mas}_+\{\lambda(s),\mu(s); s\in[0,1]\} = \sum_{j=1}^{n}\left(E\big(\frac{\theta_j(1)}{2\pi}\big) - E\big(\frac{\theta_j(0)}{2\pi}\big)\right), \tag{2.4}$$

$$\operatorname{Mas}_-\{\lambda(s),\mu(s); s\in[0,1]\} = \sum_{j=1}^{n}\left([\frac{\theta_j(1)}{2\pi}] - [\frac{\theta_j(0)}{2\pi}]\right). \tag{2.5}$$

By definition, $\operatorname{Mas}_\pm\{\lambda(s),\mu(s); s\in[0,1]\}$ is an integer that does not depend on the choices of the arguments $\theta_j(s)$. By [22, Proposition

6], it does not depend on the particular choice of the paths of the symplectic splittings.

2.2. Properties of the Maslov index in Hilbert space. From the properties of the spectral flow, we get all the basic properties of the Maslov index for strong symplectic Hilbert spaces (see S. E. Cappell, R. Lee, and E. Y. Miller [28, Section 1] for a more comprehensive list). The proof of Proposition 2.3.d is less trivial (see [97, Corollary 4.1]).

The properties of the following list will first be used for establishing a rigorous and calculable concept of the Maslov index in weak symplectic Banach space. For the Maslov index defined in that way by symplectic finite-dimensional reduction, we shall later recover the full list of valid properties in Theorem 3.3 for the general case.

PROPOSITION 2.3. *(a) The Maslov index is* invariant under homotopies *of curves of Fredholm pairs of Lagrangian subspaces of index* 0 *with fixed endpoints. In particular, the Maslov index is invariant under* re-parametrization *of paths.*
(b) The Maslov index is additive under catenation*, i.e.,*

$$\operatorname{Mas}_{\pm}\{\lambda,\mu\} = \operatorname{Mas}_{\pm}\{\lambda|_{[0,a]},\mu|_{[0,a]}\} + \operatorname{Mas}_{\pm}\{\lambda|_{[a,1]},\mu|_{[a,1]}\}\,,$$

for any $a\in[0,1]$.
(c) The Maslov index is additive under direct sum*, i.e.,*

$$\operatorname{Mas}_{\pm}\{\lambda_1\oplus\lambda_2,\mu_1\oplus\mu_2\} = \operatorname{Mas}_{\pm}\{\lambda_1,\mu_1\} + \operatorname{Mas}_{\pm}\{\lambda_2,\mu_2\}\,,$$

where $\{\lambda_j(s)\},\{\mu_j(s)\}$ *are paths of Lagrangian subspaces in* $(X_j,\omega_j(s))$, $j=1,2$ *and* $\lambda_1\oplus\lambda_2$ *is a path of subspaces in* $(X_1\oplus X_2,\omega_1(s)\oplus\omega_2(s))$.
(d) The Maslov index is natural *under symplectic action: given a second Banach bundle* $\{X'(s)\}$*, a path of symplectic structures* $\omega'(s)$ *on* $X'(s)$*, and a bundle isomorphism* $\{L(s)\in\mathcal{B}(X(s),X'(s))\}$ *such that* $L(s)^*(\omega'(s))=\omega(s)$*, then we have*

$$\operatorname{Mas}_{\pm}\{\lambda(s),\mu(s);\omega(s)\} = \operatorname{Mas}_{\pm}\{L(s)\lambda(s),L(s)\mu(s);\omega'(s)\}.$$

(e) The Maslov index vanishes*, if* $\dim(\lambda(s)\cap\mu(s))$ *constant for all* $s\in[0,1]$.
(f) Flipping. *We have*

$$\begin{aligned}\operatorname{Mas}_{+}\{\lambda(s),\mu(s)\} &+ \operatorname{Mas}_{+}\{\mu(s),\lambda(s)\}\\ &= \operatorname{Mas}_{+}\{\lambda(s),\mu(s)\} - \operatorname{Mas}_{-}\{\lambda(s),\mu(s)\}\\ &= \dim(\lambda(0)\cap\mu(0)) - \dim(\lambda(1)\cap\mu(1)),\end{aligned}$$

and $\operatorname{Mas}_{\pm}\{\lambda_s,\mu_s\} = \operatorname{Mas}_{\pm}\{\mu_s,\lambda_s;-\omega(s)\}$.

(g) Local range. *Given a Fredholm pair of Lagrangian subspaces* (λ, μ) *of* (X, ω) *of index* 0*, there exists an* $\varepsilon > 0$ *such that*

$$0 \le \operatorname{Mas}_+\{\lambda(s), \mu(s); \omega(s)\} \le \dim(\lambda \cap \mu) - \dim(\lambda(1)) \cap \mu(1)$$

if $\lambda(0) = \lambda$*,* $\mu(0) = \mu$*,* $\omega(0) = w(0)$*, and*

$$\hat{\delta}(\lambda(s), \lambda), \hat{\delta}(\mu(s), \mu), \|\omega(s) - \omega\| < \varepsilon.$$

We have the following lemma (see [77, Theorem 2.3, Localization] for constant symplectic structure case).

LEMMA 2.4. *Let* $(X, \omega(s))$ *be a continuous family of* $2n$ *dimensional symplectic vector spaces with Lagrangian subspaces* λ_0*,* μ_0 *such that* $X = \lambda_0 \oplus \mu_0$*. Let* $A(s) \in \operatorname{Hom}(\lambda_0, \mu_0)$*,* $s \in [0,1]$ *be a path of linear maps such that* $\lambda(s) = \operatorname{graph}(A_s)$ *is a Lagrangian subspace of* $(\mathbb{C}^{2n}, \omega(s))$ *for each* $s \in [0,1]$*. Define* $Q(s)(x, y) = \omega(s)(x, A(s)y)$ *for all* $s \in [0,1]$*,* $x \in \lambda_0$ *and* $y \in \mu_0$*. Then* $Q(s)$ *is a quadratic form on* λ_0 *and we have*

$$\operatorname{Mas}_+\{\lambda(s), \lambda_0; s \in [0,1]\} = m^+(Q(1)) - m^+(Q(0)), \tag{2.6}$$

$$\operatorname{Mas}_-\{\lambda(s), \lambda_0; s \in [0,1]\} = m^-(Q(0)) - m^-(Q(1)), \tag{2.7}$$

where $m^{\pm}(Q)$*,* $m^0(Q)$ *denote the positive (negative) Morse index and nullity of* Q *respectively for a quadratic form* Q*.*

PROOF. Clearly, $\lambda(s)$ is Lagrangian if and only if $Q(s)$ is quadratic. By choosing a frame, we can assume that $X = \mathbb{C}^{2n}$, $\lambda_0 = \mathbb{C}^n \times \{0\}$ and $\mu_0 = \{0\} \times \mathbb{C}^n$. Let $J(s)$ be defined by $\omega(s)(x, y) = \langle J(s)x, y\rangle$ for each $s \in [0,1]$. Then we have $J(s) = \begin{pmatrix} 0 & -K(s)^* \\ K(s) & 0 \end{pmatrix}$ for some $K(s) \in \mathrm{GL}(n, \mathbb{C})$. Set $T(s) := \operatorname{diag}(K(s)^{-1}, I_n)$. Then we have $T(s) * J(s)T(s) = J_{2n}$. By Proposition 2.3.d, we can assume that $J(s) = J_{2n}$. Then we have $X^{\pm} = \{(x, \mp ix); x \in \mathbb{C}^n\}$. The generator of $\lambda(s)$ is the map $(x, ix) \mapsto (U(s)x, -iU(s)x)$, $x \in \mathbb{C}^n$. So $U(s) = (I_n + iA(s))(I_n - iA(s))^{-1}$. We have $U(s) = 0$ if $A(s) = 0$. Note that $A(s)$ is a continuous family of self-adjoint operators. By the definition of the spectral flow we have

$$\begin{aligned} \operatorname{Mas}_+\{\lambda(s), \lambda_0\} &= \operatorname{sf}_{\ell_-}\{U(s)\} = -\operatorname{sf}\{-A(s)\} \\ &= m^+(A(1)) - m^+(A(0)) = m^+(Q(1)) - m^+(Q(0)). \end{aligned}$$

Similarly we have (2.7). □

PROPOSITION 2.5. *Let* $(X, \omega(s))$*,* $s \in (-\varepsilon, \varepsilon)$ *be a family of strong symplectic Hilbert spaces with continuously varying symplectic form* $\omega(s)$*, where* $\varepsilon > 0$*. Let* $(\lambda(s), \mu(s))$*,* $s \in (-\varepsilon, \varepsilon)$ *be a path of Fredholm pairs of Lagrangian subspaces of* (X, ω_s)*. Let* $V(s)$ *be a path of* *finite-dimensional subspaces of X with $X = V(0) \oplus (\lambda(0)+\mu(0))$. Then there exists a $\delta \in (0,\varepsilon)$ such that*

$$X = V(0) + \lambda(s) + \mu(s) = V(s)^{\omega(s)} + \lambda(s) = V(s)^{\omega(s)} + \mu(s)$$

for all $s \in (-\delta, \delta)$, and

$$\begin{aligned} &\mathrm{Mas}_{\pm}\{\lambda(s), \mu(s); s \in [s_1, s_2]\} \\ &= \mathrm{Mas}_{\pm}\big\{R^{\omega(s)}_{V(s)+\lambda(s)}(\lambda(s)), R^{\omega(s)}_{V(s)+\lambda(s)}(\mu(s)); s \in [s_1, s_2]\big\} \qquad (2.8) \\ &= \mathrm{Mas}_{\pm}\big\{R^{\omega(s)}_{V(s)+\mu(s)}(\lambda(s)), R^{\omega(s)}_{V(s)+\mu(s)}(\mu(s)); s \in [s_1, s_2]\big\} \qquad (2.9) \end{aligned}$$

for all $[s_1, s_2] \subset (-\delta, \delta)$.

Proof. Set $\lambda_0(0) := \lambda(0) \cap \mu(0)$, $\lambda_1(s) := V(s)^{\omega(s)} \cap \lambda(s)$, $\mu_1(s) := V(s)^{\omega(s)} \cap \mu(s)$, $X_1(s) := \lambda_1(s) + \mu_1(s)$, and $X_0(s) := X_1(s)^{\omega(s)}$. By Proposition 1.19 we have

$$X = \lambda_0(0) \oplus V(0) \oplus \lambda_1(0) \oplus \mu_1(0),$$

$X_0(0) = \lambda_0(0) + V(0)$, and $X_1(0) = X_0(0)^{\omega}$.

By Appendix A.3 and Corollary 1.31, there exists a $\delta_1 \in (0, \varepsilon)$ such that

$$\begin{aligned} X &= V(s) + \lambda(s) + \mu(s) = V(s)^{\omega(s)} + \lambda(s) = V(s)^{\omega(s)} + \mu(s) \\ &= \lambda(s) \oplus (V(s) + \mu_1(s)) = \mu \oplus (V(s) + \lambda_1(s)), \end{aligned}$$

$X_1(s) = \lambda_1(s) \oplus \mu_1(s)$, and $X = X_0(s) \oplus X_1(s)$ for all $s \in (-\delta, \delta)$. Set $X_0(s) := X_1(s)^{\omega(s)}$. Then we have $V(s) \subset X_0(s)$. Since $X_0(s)$ is a Hilbert space, there exists a path $\lambda_0(s)$, $s \in (-\delta, \delta)$ such that $X_0(s) = \lambda_0(s) \oplus V(s)$.

Denote by $P_0(s)\colon X \to X_0(s)$ the projection defined by $X = X_0(s) \oplus X_1(s)$. By Proposition 1.19.c,d and Proposition 2.3.c,d,e, we have

$$\begin{aligned} &\mathrm{Mas}_{\pm}\{\lambda(s), \mu(s); s \in [s_1, s_2]\} \\ &\qquad = \mathrm{Mas}_{\pm}\{P_0(s)(\lambda(s)), P_0(s)(\mu(s)); s \in [s_1, s_2]\} \\ &\qquad\quad + \mathrm{Mas}_{\pm}\{\lambda_1(s)), \mu_1(s)); s \in [s_1, s_2]\} \\ &\qquad = \mathrm{Mas}_{\pm}\big\{T_l(R^{\omega(s)}_{V(s)+\lambda(s)}(\lambda(s))), T_l(R_{V(s)+\lambda(s)}(\mu(s))); s \in [s_1, s_2]\big\} \\ &\qquad = \mathrm{Mas}_{\pm}\big\{(R^{\omega(s)}_{V(s)+\lambda(s)}(\lambda(s))), (R^{\omega(s)}_{V(s)+\lambda(s)}(\mu(s))); s \in [s_1, s_2]\big\}. \end{aligned}$$

Note that by Proposition 1.19 and Appendix A.3, the Maslov indices in the above calculations are well-defined. The equality (2.9) follows similarly. $\square$

3. The Maslov index in Banach bundles over a closed interval

3.1. The Maslov index by symplectic reduction to a finite-dimensional subspace.

In this section, we make the following assumption

ASSUMPTION 3.1. Let $p\colon E \to [0,1]$ be a Banach bundle. Denote by $X(s) := p^{-1}(s)$ the fiber of p at $s \in [0,1]$. Let $\omega(s)$ be a continuous family of symplectic structure on $X(s)$, $s \in [0,1]$. Let $(\lambda(s), \mu(s))$ be a path of Fredholm pairs of Lagrangian subspaces of $(X(s), \omega(s))$ of index 0.

Here for a fiber bundle $p\colon E \to [0,1]$, a path $c(s)$, $s \in [0,1]$ of E is a continuous map $c\colon [0,1] \to E$ such that $c(s) \in p^{-1}(s)$ for each $s \in [0,1]$. We refer to [95] for the concept of Banach bundles. The fiber bundle E is always trivial. So we can actually assume that $X(s) \equiv X$. By [52, Lemma I.4.10] and Lemma A.29, the set of complemented closed subspaces is a Banach manifold and can be identified locally with bounded invertible linear maps of X.

As shown in Corollary 1.13, the assumption of vanishing index is always satisfied, if $F(s)$ is a Hilbert space or, more generally, a reflexive Banach space.

To define the Maslov index via finite-dimensional symplectic reduction, we begin with a purely formal definition.

We make Assumption 3.1. By definition of Fredholm pairs, for each $t \in [0,1]$, there exists $V(t) \subset X(t)$ such that $V(t) \oplus (\lambda(t) + \mu(t)) = X(t)$. Set $\lambda_0(t) := \lambda(t) \cap \mu(t)$ and $X_0(t) := \lambda_0(t) \oplus V(t)$. Then there exists for each t an $\delta(t) > 0$ such that
(i) there exists a local frame $L(t,s)\colon X(t) \to X(s)$, $s \in (t - \delta(t), t + \delta(t)) \cap [0,1]$ of the bundle E,
(ii) $X(s) = L(t,s)V(t) + \lambda(s) + \mu(s) = (L(t,s)V(t))^{\omega(s)} + \lambda(s)$ for all $s \in (t - \delta(t), t + \delta(t)) \cap [0,1]$, and
(iii) we have

$$
\begin{aligned}
X(s) &= L(t,s)X_0(t) \oplus \lambda_1(t,s) \oplus \mu_1(t,s) \\
&= \lambda(s) \oplus (L(t,s)V(t) + \mu_1(t,s)) \\
&= \mu(s) \oplus (L(t,s)V(t) + \lambda_1(t,s))
\end{aligned}
\tag{3.1}
$$

for all $s \in (t - \delta(t), t + \delta(t)) \cap [0,1]$, where $\lambda_1(t,s) := (L(t,s)V(t))^{\omega(s)} \cap \lambda(s)$ and $\mu_1(t,s) := (L(t,s)V(t))^{\omega(s)} \cap \mu(s)$.

Denote by $X_1(t,s) := \lambda_1(t,s)+\mu_1(t,s)$. Denote by $P_0(t,s)\colon X(s) \to X_0(s)$ the projection defined by $X(s) = L(t,s)X_0(t) \oplus X_1(t)$. Denote by $\omega_l(t,s) = \omega_r(t,s)$ the symplectic structure defined by Proposition 1.32.d. We have the finite-dimensional symplectic vector space $(X_0(t), L(t,s)^*(\omega_l(t,s))$ with continuous varying symplectic structure for fixed $t \in [0,1]$.

Definition 3.2 (Maslov index by symplectic reduction). With the notations above, let $0 = a_0 < a_1 < \ldots < a_n = 1$ be a partition with $[a_k, a_{k+1}] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for some $t_k \in [0,1]$, $k = 0, \ldots n-1$. Define

(3.2)

$$\mathrm{Mas}_\pm\{\lambda(s), \mu(s); s \in [0,1]\} := \sum_{k=0}^{n-1}$$

$$\mathrm{Mas}_\pm^{\omega(s)}\{L(t,s)^{-1}P_0(t,s)(\lambda(s)), L(t,s)^{-1}P_0(t,s)(\mu(s)); s \in [a_k, a_{k+1}]\}.$$

We call $\mathrm{Mas}_\pm$ the *positive (negative) Maslov index*. We call the positive Maslov index $\mathrm{Mas} := \mathrm{Mas}_+$ the *Maslov index*.

To lift the formal concepts of Definition 3.2 to a useful definition of the Maslov index in Banach spaces, we prove the following theorem:

Theorem 3.3 (Main Theorem). *Under Assumption 3.1, the mappings* $\mathrm{Mas}_\pm$ *are well-defined and the common properties of the Maslov index (listed in Proposition 2.3) are preserved.*

Remark 3.4. By the definition of the spectral flow, our definition coincides with that in Definition 2.1, and more generally, [22, Definition 7] in their special cases. Our definition of the Maslov index generalizes the ideas in [15, 84].

We firstly show that Theorem 3.3 is true in the local case.

Lemma 3.5. *We make Assumption 3.1 with $X(s) = X$. Let V be a finite-dimensional subspace of X. Assume that*

$$X = V + \lambda(s) + \mu(s) = V^{\omega(s)} + \lambda(s) = V^{\omega(s)} + \mu(s) \tag{3.3}$$

$$= X_0 \oplus \lambda_1(s) \oplus \mu_1(s) \tag{3.4}$$

$$= \lambda(s) \oplus (V + \mu_1(s)) = \mu(s) \oplus (V + \lambda_1(s)) \tag{3.5}$$

for all $s \in [0,1]$, where $\lambda_1(s) := V^{\omega(s)} \cap \lambda(s)$ and $\mu_1(s) := V^{\omega(s)} \cap \mu(s)$.

Denote by $X_1(s) := \lambda_1(s) + \mu_1(s)$. Denote by $P_0(s)\colon X \to X_0$ the projection defined by $X = X_0 \oplus X_1(s)$. Denote by $\omega_l(s) = \omega_r(s)$ the symplectic structure defined by Proposition 1.32.d. By definition of Fredholm pairs, for each $t \in [0,1]$, there exists $V(t)$ such that $V(t) \oplus$

$(\lambda(t)+\mu(t)) = X$. *Then there exists a* $\delta(t) > 0$ *for each* $t \in [0,1]$ *such that, for all* $[s_1, s_2] \subset (t-\delta(t), t+\delta(t)) \cap [0,1]$*, (ii) and (iii) in Definition 3.2 is satisfied with* $L(t,s) = I$*, and*

$$\begin{aligned} &\operatorname{Mas}_{\pm}\{P_0(s)(\lambda(s)), P_0(s)(\mu(s)); \omega_l(s); s \in [s_1, s_2]\} \\ &\quad = \operatorname{Mas}_{\pm}\{P_0(t,s)(\lambda(s)), P_0(t,s)(\mu(s)); \omega_l(t,s); s \in [s_1,s_2]\}, \end{aligned} \tag{3.6}$$

where $P_0(t,s)$ *and* $\omega_l(t,s)$ *is defined by Definition 3.2.*

Proof. Our assumption (3.3) implies that there exists a linear subspace $V(t)'$ of V such that $V(t)' \oplus (\lambda(t)+\mu(t)) = X$ for each $t \in [0,1]$. By Proposition 1.32, Lemma 1.25 and Proposition 2.5, there exists a $\delta_1(t) > 0$ for each t such that 3.6) holds for $V(t)$ replaced by $V(t)'$ and $[s_1, s_2] \subset (t-\delta_1(t), t+\delta_1(t)) \cap [0,1]$.

By Lemma A.29, there exists a path $f(t,\cdot)\colon [0,1] \to G(X, \lambda(t) + \mu(t))$ with $f(t,0) = V(t)$ and $f(t,1) = V(t)'$ for each $t \in [0,1]$. Set $\lambda_0(t) := \lambda(t) \cap \mu(t)$, $\lambda_1(t,a,s) = f(t,a,s)^{\omega(s)} \cap \lambda(s)$, and $\mu_1(t,a,s) = f(t,a,s)^{\omega(s)} \cap \mu(s)$. By Proposition 1.19 and Appendix A.3, there exists $\delta(t) \in (0, \delta_1(t))$ for each t

$$\begin{aligned} X &= \lambda_0(t) \oplus f(t,a) \oplus \lambda_1(t,a,s) \oplus \mu_1(t,a,s) \\ &= \lambda(s) \oplus (f(t,a) + \mu_1(t,a,s) = \mu(s) \oplus (f(t,a) + \lambda_1(t,a,s)). \end{aligned}$$

Note that in our case the symplectic reduction do not change the dimension of the intersection of Lagrangian subspaces. By Proposition 2.3, the Maslov indices in our lemma is unchanged if we replace $V(t)$ by $V(t)'$. Then (3.6) holds. $\square$

Proof of Theorem 3.3. By taking a common refinement of the partitions, the first part of the Theorem follows from Lemma 3.5. The second part of the Theorem is a repetition of the list of properties given in Proposition 2.3 for the case of a strong symplectic Hilbert space. The validity in the general case follows from the proposition and our definition of the Maslov index. $\square$

We have the following lemma from ([22, Lemma 8]:

Lemma 3.6. *Let* (X,ω) *be a symplectic vector space. Let* Δ *denote the diagonal (i.e., the canonical Lagrangian) in the product symplectic space* $(X \oplus X, (-\omega) \oplus \omega)$*, and* λ, μ *are linear subspaces of* (X,ω)*. Then*

$$(\lambda,\mu) \in \mathcal{FL}^2(X) \iff (\lambda \oplus \mu, \Delta) \in \mathcal{FL}^2(X \oplus X)$$

and

$$\operatorname{index}(\lambda,\mu) = \operatorname{index}(\lambda \oplus \mu, \Delta),$$

where $\lambda \oplus \mu := \{(x,y); x \in \lambda, y \in \mu\}$.

The following proposition generalizes [22, Proposition 4 (b)].

PROPOSITION 3.7. *Denote by* $\Delta(s)$ *the diagonal of* $X(s) \times X(s)$. *Under Assumption 3.1, we have*

$$\operatorname{Mas}\{\lambda(s) \oplus \mu(s), \Delta(s); \omega(s) \oplus (-\omega(s))\} = \operatorname{Mas}\{\lambda(s), \mu(s); \omega(s)\} \tag{3.7}$$

$$= \operatorname{Mas}\{\mu(s), \lambda_s; -\omega(s)\} \tag{3.8}$$

$$= \operatorname{Mas}\{\Delta(s), \lambda(s) \oplus \mu_s; (-\omega(s)) \oplus \omega(s)\}. \tag{3.9}$$

PROOF. By [22, Proposition 4 (b)], our results hold in the finite-dimensional case. The general case follows from the definition of the Maslov index. □

3.2. Calculation of the Maslov index. We start with the general case.

THEOREM 3.8. *Let* $\varepsilon > 0$ *be a positive number. Let* X *be a (complex) Banach space with continuously varying symplectic structure* $\omega(s)$, $s \in (-\varepsilon, \varepsilon)$. *Let* $(\lambda(s), \mu(s))$ *be a path of Fredholm pairs of Lagrangian subspaces of* $(X, \omega(s))$ *of index* 0. *Let* $V(s)$ *and* $\lambda_0(s)$ *be a path of finite-dimensional subspaces of* X *with* $\lambda_0(0) = \lambda(0) \cap \mu(0)$ *and* $V(0) \oplus (\lambda(0)+\mu(0)) = X$. *Set* $\lambda_1(s) := V(s)^{\omega(s)} \cap \lambda(s)$, $\mu_1(s) := V(s)^{\omega(s)} \cap \mu(s)$, $X_0(s) = \lambda_0(s) + V(s)$, $X_1(s) = \lambda_1(s) + \mu_1(s)$. *Then there exists a* $\delta > 0$ *such that for each* $s \in (-\delta, \delta)$ *and* $[s_1, s_2] \subset (-\delta, \delta)$ *we have*
(a) $X = V(s) + \lambda(s) + \mu(s) = V(s)^{\omega(s)} + \lambda(s) = V(s)^{\omega(s)} + \mu(s)$,
(b) $X = \lambda_0(s) \oplus V(s) \oplus \lambda_1(s) \oplus \mu_1(s) = V(s)^{\omega(s)} \oplus \lambda_0(s)$,
(c) $\lambda(s)$ *and* $\mu(s)$ *are expressed by*

$$\lambda(s) = \operatorname{graph}(\begin{pmatrix} A_1(s) & 0 \\ A_2(s) & 0 \end{pmatrix} : \lambda_0(s) \oplus \lambda_1(s) \to V(s) \oplus \mu_1(s)), \tag{3.10}$$

$$\mu(s) = \operatorname{graph}(\begin{pmatrix} B_1(s) & 0 \\ B_2(s) & 0 \end{pmatrix} : \lambda_0(s) \oplus \mu_1(s) \to V(s) \oplus \lambda_1(s)) \tag{3.11}$$

and $A_1(0) = A_2(0) = B_1(0) = B_2(0) = 0$,
(d) $\omega_l(s) = w_r(s)$ *and* $P_0(s)(\mu(s))$ *are Lagrangian subspaces of symplectic vector space* $(X_0(s), \omega_l(s))$, *where* $P_0(s)\colon X \to X_0(s)$ *is the projection defined by* $X = X_0(s) \oplus X_1(s)$,

$$\omega_l(s) := \omega(s)|_{X_0(s)} - \begin{pmatrix} I & 0 \\ A_1(s) & 0 \end{pmatrix}^* (\omega(s)|_{X_0(s)}), \text{ and} \tag{3.12}$$

$$\omega_l(s) := \omega(s)|_{X_0(s)} - \begin{pmatrix} I & 0 \\ A_1(s) & 0 \end{pmatrix}^* (\omega(s)|_{X_0(s)}). \tag{3.13}$$

(e) The following equalities hold

$$\mathrm{Mas}_\pm\{\lambda(s),\mu(s); s\in[s_1,s_2]\} \\ = \mathrm{Mas}_\pm\{P_0(s)(\lambda(s)),P_0(s)(\mu(s));\omega_l(s); s\in[s_1,s_2]\}, \tag{3.14}$$

and

$$\dim(\lambda(s)\cap\mu(s)) \;=\; \dim\big(P_0(s)(\lambda(s))\cap P_0(s)(\mu(s))\big). \tag{3.15}$$

Proof. (a) By Appendix A.3.

(b), (c), (d) By (a), Proposition 1.32 and Appendix A.3.

(e) If $V(s)\equiv V(0)$, our result follows from the definition of the Maslov index and Proposition 1.32.c.

In the general case, by [52, Lemma I.4.10] there exists a path of bounded invertible map $L(s)\in\mathcal{B}(X)$ such that $L(s)X_0=X_0(s)$ with $L(0)=I$. By Proposition 2.3.d we have

$$\mathrm{Mas}_\pm\{P_0(s)(\lambda(s)),P_0(s)(\mu(s));\omega_l(s); s\in[s_1,s_2]\} \\ = \mathrm{Mas}_\pm\{L(s)^{-1}P_0(s)(\lambda(s)),L(s)^{-1}P_0(s)(\mu(s)); L(s)^*\omega_l(s); s\in[s_1,s_2]\}. \tag{3.16}$$

Note that $L(s)$, $P_0(s)$ and $\omega_l(s)$ depend continuously on $\lambda_0(s)$, $V(s)$, $\lambda(s)$, $\mu(s)$ and $\omega(s)$. Replacing $\lambda_0(s)$ by $\lambda_0(ts)$ and $V(s)$ by $V(ts)$ for $t\in[0,1]$, we get a homotopy of the right hand side of (3.16). Note that in our case $\dim(P_0(s)(\lambda(s))\cap P_0(s)(\mu(s)))=\dim(\lambda(s)\cap\mu(s))$. Then our result follows from the special case and Proposition 2.3. ☐

By Corollary 1.20 and [67, Lemma 0.2], we have a path $\lambda_0(s)\subset\mu(s)$ with $\lambda_0(0)=\lambda(0)\cap\mu(0)$. By Corollary 1.22 and Proposition 1.19.d, we have a Lagrangian path $V(s)\in X_0(s)$ of $X_0(s)$. We have the following corollary.

Corollary 3.9. *Assume that $\lambda_0(s)\subset\mu(s)$ as in Theorem 3.8 and let $\delta>0$ be found correspondingly. Then for each $s\in(-\delta,\delta)$ and $[s_1,s_2]\subset(-\delta,\delta)$, we have $\omega_l(s)=\omega(s)|_{X_0(s)}$, and*

$$\mathrm{Mas}_\pm\{\lambda(s),\mu(s); s\in[s_1,s_2]\} \\ = \mathrm{Mas}_\pm\{P_0(s)(\lambda(s)),\lambda_0(s);\omega(s)|_{X_0(s)}; s\in[s_1,s_2]\}. \tag{3.17}$$

Proof. In this case we have $B_1(s)=B_2(s)=0$, $P_0(s)(\mu(s))=\lambda_0(s)$ and $\omega_l(s)=\omega(s)|_{X_0(s)}$. By Theorem 3.8, our results follow. ☐

Proposition 3.10. *Assume that $V(s)$ is isotropic in Theorem 3.8. Let δ be as given there. Then for each $s\in(-\delta,\delta)$ and $[s_1,s_2]\subset(-\delta,\delta)$*

we have a quadratic form $Q(s)$ such that

$$\operatorname{Mas}_+\{\lambda(s),\mu(s);s\in[s_1,s_2]\}=m^+(Q(s_2))-m^+(Q(s_1)),\tag{3.18}$$

$$\operatorname{Mas}_-\{\lambda(s),\mu(s);s\in[s_1,s_2]\}=m^-(Q(s_1))-m^-(Q(s_2)),\tag{3.19}$$

$$\dim(\lambda(s)\cap\mu(s))=m^0(Q(s)),\tag{3.20}$$

where $Q(s)(x,y):=\omega(s)(x,(A_1(s)-B_1(s))y)$ for all $x,y\in\lambda_0(s)$ and $s\in(-\delta,\delta)$.

Proof. Since $V(s)\subset X_0(s)$ is isotropic, $P_0(\mu(s))$ and $V(s)$ is Lagrangian in $(X_0(s),\omega_l(s))$. We have $X_0(s)=P_0(\mu(s))\oplus V(s)$ and $Q(s)$ is a quadratic form. For each $x\in\lambda_0(s)$, we have

$$\begin{aligned}&x+A_1(s)x=x+B_1(s)x+(A_1(s)-B_1(s))x,\ \text{and}\\ &\omega_l(s)(x+A_1(s)x,(A_1(s)-B_1(s))x)\\ &\qquad=\omega(s)(x+A_1(s)x,(A_1(s)-B_1(s))x)=Q(s)(x,x).\end{aligned}$$

By Theorem 3.8, Lemma 2.4 and Proposition 2.3.b, our results follow. □

We now calculate $Q(s)$.

Lemma 3.11. *Let (X,ω) be a symplectic vector space with Lagrangian subspaces λ, μ, isotropic spaces α_0, V and a linear subspace λ_0. Assume that $\dim\alpha_0=\dim\lambda_0=\dim V<+\infty$. Set $\lambda_1:=V^\omega\oplus\lambda$ and $\mu_1:=\mu\oplus\mu$. Let $\alpha_1,\beta_1\subset V^\omega$ be isotropic subspaces. Assume that*

$$X=\alpha_0\oplus V\oplus\alpha_1\oplus\beta_1=\lambda_0\oplus V\oplus\lambda_1\oplus\mu_1.$$

Assume that $\lambda=\operatorname{graph}(A)=\operatorname{graph}\widetilde{A}$ and $\mu=\operatorname{graph}(B)$, where

$$A=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}:\alpha_0\oplus\alpha_1\to V\oplus\beta_1,\tag{3.21}$$

$$\widetilde{A}=\begin{pmatrix}A_1&0\\A_2&0\end{pmatrix}:\lambda_0\oplus\lambda_1\to V\oplus\mu_1,\tag{3.22}$$

$$B=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}:\alpha_0\oplus\beta_1\to V\oplus\alpha_1.\tag{3.23}$$

Then the following holds.
(a) $V^\omega=V\oplus\alpha_1\oplus\beta_1$ and $\lambda_1=\operatorname{im}(A_{12}+I_{\alpha_1}+A_{22})$.
(b) If $\mu=\alpha_0\oplus\beta_1$ and $\lambda_0=\alpha_0$ hold, we have $\mu_1=\beta_1$, $A_1=A_{11}$ and $A_2=A_{22}$.
(c) Set

$$f:=I_{\alpha_0}+B_{11}+B_{21}\colon\alpha_0\to X,\quad g\colon=B_{12}+B_{22}+I_{\beta_1}\colon\beta_1\to X.$$

Assume that $\lambda_0 = f(\alpha_0)$ *and* $I_{\beta_1} - A_{22}B_{22}$ *is invertible. Then we have*

$$\begin{aligned} A_1 f =& A_{11} - B_{11} + A_{12}B_{21} - (B_{12} - A_{12}B_{22}) \\ & (I_{\beta_1} - A_{22}B_{22})^{-1}(A_{21} + A_{22}B_{21}), \end{aligned} \tag{3.24}$$

$$A_2 f = g(I_{\beta_1} - A_{22}B_{22})^{-1}(A_{21} + A_{22}B_{21}). \tag{3.25}$$

PROOF. (a), (b) By definition.
(c) Let $x \in \alpha_0$. Set $\widetilde{x} := f(x)$ and $w := g^{-1}(A_2\widetilde{x})$. Then we have $\widetilde{x} + A_1\widetilde{x} + A_2\widetilde{x} \in \lambda$. Since $\lambda = \operatorname{graph}(A)$, we have

$$\begin{pmatrix} B_{11}x + A_1\widetilde{x} + B_{12}w \\ w \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ B_{21}x + B_{22}w \end{pmatrix}.$$

By direct calculations we get (3.24) and (3.25). □

We generalize the notion of crossing forms of [77] to the case of C^1 varying symplectic structures.

Let (X, ω) be a symplectic Banach space and let $\lambda = \{\lambda(s)\}_{s\in[0,1]}$ be a C^1 curve of Lagrangian subspaces. Assume that $\lambda(t)$ is complemented. Let W be a fixed Lagrangian complement of $\lambda(t)$. The form

$$Q(\lambda, t) := Q(\lambda, W, t) = \frac{d}{ds}|_{s=t} Q(\lambda(t), W; \lambda(s)) \tag{3.26}$$

on $\lambda(t)$ is independent of the choice of W, where $Q(\alpha, \beta; \gamma)$ is defined by Proposition 1.32.e (see [39, (2.3)]).

If $X = \alpha \oplus W = \lambda(s) \oplus W$ for two Lagrangian subspaces α and W and $|s - t| << 1$, then, by Lemma A.29, there exists a path $A(s) \in \mathcal{B}(\alpha, W)$ with $\lambda(s) = \{x + A(s)x; x \in \alpha\}$. By definition we have

$$Q(\lambda, W, t)(x + A(t)x, y + A(t)y) = \frac{d}{ds}|_{s=t} Q(\alpha, W; \lambda(s))(x, y). \tag{3.27}$$

LEMMA 3.12. *Let* $(X, \omega(s))$ $s \in (-\varepsilon, \varepsilon)$, *be a* C^1 *path of symplectic Banach spaces with two* C^1 *families of Lagrangian subspaces* $\alpha(s)$, $\beta(s)$. *Assume that* $X = \alpha(s) \oplus \beta(s)$. *Let* $x(s), y(s) \in \alpha(s)$ *be two* C^1 *paths. Let* $A(s), B(s), C(s) \in \mathcal{B}(\alpha(s), \beta(s))$ *and* $D(s) \in \mathcal{B}(\beta(s), \alpha(s))$ *are* C^1 *families of bounded linear maps with* $A(0) = B(0) = C(0) = 0$. *Set* $\lambda(s) := \operatorname{graph}(A(s))$, $\mu(s) := \operatorname{graph}(B(s))$, $\widetilde{\alpha}(s) := \operatorname{graph}(C(s))$ *and* $\widetilde{\beta}(s) := \operatorname{graph}(D(s))$. *Then the following holds.*
(a) There exists a $\delta \in (0, \varepsilon)$ *such that for all* $s \in (-\delta, \delta)$, *we have*

$$X = \widetilde{\alpha}(s) \oplus \widetilde{\beta}(s) = \lambda(s) \oplus \widetilde{\beta}(s) = \mu(s) \oplus \widetilde{\beta}(s).$$

(a) For $s \in (-\delta, \delta)$, *let* $u(s), v(s) \in \beta(s)$ *be such that*

$$y(s) + C(s)y(s) + w_1(s) \in \lambda(s), y(s) + C(s)y(s) + w_2(s) \in \lambda(s),$$

where $w_1(s) := u(s) + D(s)u(s)$, $w_2(s) := v(s) + D(s)u(s)$. *Then we have*

$$\Gamma(\lambda, \mu, 0)(x(0), y(0)) :$$

$$= \frac{d}{ds}|_{s=0}\omega(s)(x(s) + C(s)x(s), w_1(s) - w_2(s)) \tag{3.28}$$

$$= \frac{d}{ds}|_{s=0}\omega(0)(x(0), (A(s) - B(s))y(s)). \tag{3.29}$$

(b) The form $\Gamma(\lambda, \mu, 0)$ *is a quadratic form if* $\lambda(s)$ *and* $\mu(s)$ *are Lagrangian subspaces of* $(X, \omega(s))$.

Proof. (a) By the continuity of the given families.
(b) By the definitions we have $y(s)+D(s)u(s)+u(s)+C(s)y(s) \in \lambda(s)$. Then we have $u(s) + C(s)y(s) = A(s)(y(s) + D(s)u(s))$, and

$$u(s) = (I_{\beta(s)} - A(s)D(s))^{-1}(A(s) - C(s))y(s).$$

Since $\alpha(s)$ and $\beta(s)$ are Lagrangian subspaces, we have

$$\frac{d}{ds}|_{s=0}\omega(s)(x(s) + C(s)x(s), w_1(s))$$

$$= \frac{d}{ds}|_{s=0}\omega(s)(x(s), u(s))$$

$$= \frac{d}{ds}|_{s=0}\omega(s)(x(s), (A(s) - C(s))y(s))$$

$$= \frac{d}{ds}|_{s=0}\omega(0)(x(0), (A(s) - C(s))y(s)). \tag{3.30}$$

Applying (3.30) for $\mu(s)$, we obtain (3.29).
(c) If $\lambda(s)$ and $\mu(s)$ are Lagrangian subspaces of $(X, \omega(s))$, the forms $\omega(s)(x(s), A(s)y(s))$ and $\omega(s)(x(s), A(s)y(s))$ are quadratic. Then the form $\Gamma(\lambda, \mu, 0)$ is quadratic. If $\omega(s) \equiv \omega$ is fixed, we can take $\alpha(s) \equiv \alpha(0)$ and $\beta(s) \equiv \beta(0)$. By (3.27) and the definitions, the form $\Gamma(\lambda, \mu, 0)$ defined here coincides with that defined by (3.34). ☐

Lemma 3.13. *We use the notations of Theorem 3.8. Assume that* $\omega(s)$, $V(s)$, $\lambda(s)$, $\mu(s)$ *with* $s \in (-\delta, \delta)$ *are* C^1 *families and* $\lambda_0(s) = \lambda_0 = \lambda(0) \cap \mu(0)$. *Then the following holds.*
(a) The families $V(s)^{\omega(s)}$, $\lambda_1(s)$, $\mu_1(s)$ *are* C^1 *families.*
(b) The form

$$\Gamma(\lambda, \mu, \lambda_0, V, 0)(x(0), y(0)) := \frac{d}{ds}|_{s=0}\omega(s)(x(s), (A_1(s) - B_1(s))y(s)) \tag{3.31}$$

on $\lambda_0(0)$ is quadratic, where $x(s), y(s) \in \lambda_0(s)$ are two C^1 paths.
(c) The form $\Gamma(\lambda,\mu,0) := \Gamma(\lambda,\mu,\lambda_0,V,0)$ does not depend on the choices of paths λ_0 and V. In particular,

(i) *if $V(s)$ is a C^1 isotropic path and $Q(s)$ is defined as in Proposition 3.10, we have*

$$\Gamma(\lambda,\mu,0)(x(0),y(0)) = \frac{d}{ds}|_{s=0} Q(s)(x(s),y(s)); \tag{3.32}$$

(ii) *for $A_{11}(s)$ and $B_{11}(s)$ as defined in Lemma 3.11, we have*

$$\Gamma(\lambda,\mu,0)(x(0),y(0)) = \frac{d}{ds}|_{s=0}\omega(s)(x(s),(A_{11}(s)-B_{11}(s))y(s)); \tag{3.33}$$

(iii) *if $\omega(s)\equiv\omega$ is fixed, we have*

$$\Gamma(\lambda,\mu,0) = (Q(\lambda,0)-Q(\mu,0))|_{\lambda(0)\cap\mu(0)}. \tag{3.34}$$

Proof. (a) By Lemma A.33.
(b) By (a), the families $A_1(s), A_2(s), B_1(s), B_2(s)$ are of class C^k. Note that $A_1(0)=A_2(0)=B_1(0)=B_2(0)=0$. Then we have

$$\begin{aligned} 0 =& \frac{d}{ds}|_{s=0}\omega_l(s)(x(s)+B_1(s)x(s), y(s)+B_1(s)y(s))) \\ =& \frac{d}{ds}|_{s=0}\omega(s)((B_1(s)-A_1(s))x(s),y(s)) \\ & + \frac{d}{ds}|_{s=0}\omega(s)(x(s),(B_1(s)-A_1(s))y(s)). \end{aligned}$$

Since $\omega(s)$ is symplectic, we get our result.
(c) Fix a C^1 path $\widetilde{\lambda}_0(s)\subset\mu(s)$ with $\widetilde{\lambda}_0(0)=\lambda_0(0)$. Consider the C^1 symplectic space $(V(s)+\mu(s))/\mu_1(s))$. By Lemma 3.12, we have

$$\Gamma(\lambda,\mu,\lambda_0,V,0) = \Gamma(\lambda,\mu,\widetilde{\lambda}_0,V,0). \tag{3.35}$$

Take a C^1 path of Lagrangian subspaces $\widetilde{V}(s)$ of symplectic subspaces $\widetilde{X}_0(s) := \widetilde{\lambda}_0(s)+V(s)$. Then we have $\omega_l(s) = \omega(s)|_{\widetilde{X}_0(s)}$. By Lemma 3.12, we have

$$\Gamma(\lambda,\mu,\widetilde{\lambda}_0,V,0) = \Gamma(\lambda,\mu,\widetilde{\lambda}_0,\widetilde{V},0). \tag{3.36}$$

Fix a C^1 isotropic path $\overline{V}(s)$ with $X(0)=\overline{V}(0)\oplus(\lambda(0)+\mu(0))$. Fix C^1 paths $\widetilde{x}(s),\widetilde{y}(s)\in\widetilde{\lambda}_0(s)$. Let $\widetilde{A}_1(s),\widetilde{A}_2(s)$ and $\overline{A}_1(s),\overline{A}_2(s)$ be C^1 paths defined by Theorem 3.8 for $(\lambda,\mu,\widetilde{\lambda}_0,\widetilde{V})$ and $(\lambda,\mu,\widetilde{\lambda}_0,\overline{V})$. Since $\widetilde{x}(s)+\widetilde{A}_1(s)\widetilde{x}(s)+\widetilde{A}_2(s)\widetilde{x}(s), \widetilde{y}(s)+\overline{A}_1(s)\widetilde{y}(s)+\overline{A}_2(s)\widetilde{y}(s)\in\lambda(s)$, we have

$$\begin{aligned}
0 =&\frac{d}{ds}|_{s=0}\omega(s) \\
&(\widetilde{x}(s)+\widetilde{A}_1(s)\widetilde{x}(s)+\widetilde{A}_2(s)\widetilde{x}(s),\widetilde{y}(s)+\overline{A}_1(s)\widetilde{y}(s)+\overline{A}_2(s)\widetilde{y}(s)) \\
=&\frac{d}{ds}|_{s=0}\omega(s)(\widetilde{x}(s),\widetilde{y}(s)+\overline{A}_1(s)\widetilde{y}(s)+\overline{A}_2(s)\widetilde{y}(s)) \\
&+\frac{d}{ds}|_{s=0}\omega(s)(\widetilde{x}(s)+\widetilde{A}_1(s)\widetilde{x}(s)+\widetilde{A}_2(s)\widetilde{x}(s),\widetilde{y}(s)) \\
=&\frac{d}{ds}|_{s=0}\Big(\omega(s)(\widetilde{x}(s),\overline{A}_1(s)\widetilde{y}(s))+\omega(s)(\widetilde{A}_1(s)\widetilde{x}(s),\widetilde{y}(s))\Big) \\
=&\Gamma(\lambda,\mu,\widetilde{\lambda}_0,\overline{V})(x(0),y(0))-\Gamma(\lambda,\mu,\widetilde{\lambda}_0,\widetilde{V})(x(0),y(0)).
\end{aligned}$$

By (3.35) and (3.36), we obtain

$$\Gamma(\lambda,\mu,\lambda_0,V,0)=\Gamma(\lambda,\mu,\widetilde{\lambda}_0,\overline{V},0). \tag{3.37}$$

For the special cases, (i) is clear, (ii) by taking $\lambda_0(s)=\alpha_0(s)$, and (iii) by taking $\lambda_0(s)\equiv\lambda_0(0)$ and $V(s)\equiv V(0)$ to be an isotropic subspace. □

Let $p\colon E\to[0,1]$ be a C^1 Banach bundle with $p^{-1}(s)=X(s)$. Let $\{(\lambda(s),\mu(s))\}$, $0\le s\le 1$ be a curve of Fredholm pairs of Lagrangian subspaces of C^1 family $(X(s),\omega(s))$ of index 0. By Corollary 1.20, $\lambda(s)$ and $\mu(s)$ are complemented. For $t\in[0,1]$, the *crossing form* $\Gamma(\lambda,\mu,t)$ is a quadratic form on $\lambda(t)\cap\mu(t)$ defined by Lemma 3.13.

A *crossing* is a time $t\in[0,1]$ such that $\lambda(t)\cap\mu(t)\neq\{0\}$. A crossing is called *regular* if $\Gamma(\lambda,\mu,t)$ is non-degenerate. It is called *simple* if it is regular and $\lambda(t)\cap\mu(t)$ is one-dimensional. As before, we shall denote by m^+,m^-,m^0 the dimensions of the subspaces where the form is positive-definite, negative-definite, or vanishing, respectively.

Now we give a method for using the crossing form to calculate Maslov indices (see [77] for the fixed finite-dimensional symplectic vector space case, [14, Theorem 2.1] and [97, Proposition 4.1] for the fixed strong symplectic Hilbert space case).

PROPOSITION 3.14. *Let $(X(s),\omega(s))$ be a C^1 family of symplectic Banach space and $\{(\lambda_s,\mu_s)\}$, $0\le s\le 1$ be a C^1 curve of Fredholm pairs of Lagrangian subspaces of X of index 0 with only regular crossings. Then we have*

$$\operatorname{Mas}\{\lambda,\mu\}=m^+(\Gamma(\lambda,\mu,0))-m^-(\Gamma(\lambda,\mu,1))+\sum_{0<t<1}\operatorname{sign}(\Gamma(\lambda,\mu,t)). \tag{3.38}$$

PROOF. For each crossing $t \in [0,1]$, we consider the path $(\lambda(s+t), \mu(s+t))$ for $|s-t| << 1$. By Proposition 1.19.c, we can take an isotropic V with $X = V \oplus (\lambda(t) + \mu(t))$. Then the assumptions of Lemma 3.11 can be satisfied. Let $Q(s)$ be defined in Proposition 3.10. By Lemma 3.13.c we have

$$\frac{d}{ds}|_{s=0} Q(s) \;=\; \Gamma(\lambda, \mu, t).$$

Since the crossing t is regular, for $0 < |s| << 1$, by (3.32) $Q(s)$ and $s\Gamma(\lambda, \mu, t)$ are non-degenerate and they have the same positive (negative) Morse index. Thus the set of crossings is discrete (and then finite, for $[0,1]$ is compact). By Proposition 3.10 and Proposition 2.3.b, our results hold. □

We recall (see [23, Definition 3.1] for the finite-dimensional case).

DEFINITION 3.15. Let (X, ω) be a symplectic Banach space and let $\{\lambda(s)\}_{0\in[0,1]}$, be a C^1 curve of complemented Lagrangian subspaces. We call the curve $\{\lambda(s)\}$ *(semi-)positive* at $t \in [0,1]$, if $Q(\lambda, t)$ is positive definite, respectively semi-positive definite. The curve $\{\lambda(s)\}$ is called *(semi-)positive* if it is (semi-)positive at all $t \in [0,1]$, respectively.

LEMMA 3.16. *Let (X, ω) be a symplectic Banach space and $\{\lambda(s)\}_{0\in[0,1]}$ a C^1 curve with a Lagrangian complement W. Then $\{\lambda(s)\}$ is (semi-) positive if and only if the path of quadratic forms $Q(\lambda(0)), W; \lambda(s))$ is strictly increasing (respectively, increasing).*

PROOF. By (3.27). □

LEMMA 3.17. *Let X be a finite-dimensional Hilbert space and*

$$A\colon (-\varepsilon, \varepsilon) \longrightarrow \mathcal{B}(X)$$

a family of self-adjoint operators. Assume that $A(s_1) \le A(s_2)$ for all $-\varepsilon < s_1 \le s_2 < \varepsilon$. Then the following holds.
(a) There exists a $\delta \in (-\varepsilon, \varepsilon)$ such that the functions $m^{\pm,0}(A(s))$ are constant for $s \in (-\delta, 0)$ or $s \in (0, \delta)$.
(b) Assume that $A(s)$ is continuous at s. Then we have

$$m^+(A(s)) = m^+(A_0), m^-(A_s) - m^-(A_0) = m^0(A_0) - m^0(A_s) \tag{3.39}$$

for $s \in (-\delta, 0)$, and

$$m^-(A(s)) = m^+(A_0), m^+(A_s) - m^+(A_0) = m^0(A_0) - m^0(A_s) \tag{3.40}$$

for $s \in (0, \delta)$.

Proof. (a) For all linear subspace V of X, and $-\varepsilon < s_1 \le s_2 < \varepsilon$, we have that $A(s_1)|_V > 0$ implies $A(s_2) > 0$. So $m^+(A(s))$ is an increasing function. Similarly, $m^-(A(s))$ is a decreasing function. Since the two functions are bounded integer valued, they have finitely many discontinuous points. Since $m^0(A(s)) = \dim X - m^+(A(s)) - m^-(A(s))$, the same result holds for $m^0(A(s))$. So we obtain (a).
(b) Since $A(s)$ is continuous at s, we have $m^\pm(0) \le m^\pm(s)$. Note that $m^+(A(s))$ is an increasing function and $m^-(A(s))$ is a decreasing function. Then the first equalities of (3.39) and (3.40) follow. The second equalities of (3.39) and (3.40) follow from the first ones. □

Proposition 3.18. *Let (X,ω) be a symplectic Banach space and let $\{(\lambda(s),\mu)\}$, $0 \le s \le 1$ be a C^1 curve of Fredholm pairs of Lagrangian subspaces of X of index 0 with a semi-positive path λ and constant path μ. Then $\dim(\lambda(s)\cap\mu)$ is locally constant except for finitely many points $s \in [0,1]$, and we have*

$$\operatorname{Mas}\{\lambda,\mu\} = \sum_{0<t\le 1} \left(\dim(\lambda(t)\cap\mu) - \lim_{s\to t^-} \dim(\lambda(s)\cap\mu)\right). \tag{3.41}$$

Proof. Let $t \in [0,1]$ and consider the path $(\lambda(s+t),\mu)$ for $|s| << 1$. By Proposition 1.19, there exists an isotropic V such that $X = V \oplus (\lambda(s)+\mu)$. Set $\lambda_1(t+s) = V^\omega \cap \lambda(t+s)$, $\mu_1 := V^\omega \cap \mu$ and $W := V + \lambda_1$. Then W is a Lagrangian subspace and $X = \lambda(t+s) \oplus W$. Let $Q(s)$ be defined by Proposition 3.10 in our case. Then we have

$$Q(s) = Q(\lambda(t), W; \lambda(t+s))|_{\lambda(t)\cap\mu}.$$

By Lemma 3.16, the family of forms $Q(s)$ is increasing. By Proposition 3.10 and Lemma 3.17, we obtain our results. □

Theorem 3.19. *Let $p\colon E \to [0,1]$ and $\widetilde{p}\colon \widetilde{E} \to [0,1]$ be two Banach bundles with $X(s) := p^{-1}(s)$, $\widetilde{X}(s) := \widetilde{p}^{-1}(s)$ for each $s \in [0,1]$. Let $\{\omega_s\}, \{\widetilde{\omega}_s\}$ be paths of symplectic forms for $X(s)$, respectively, $\widetilde{X}(s)$, $0 \le s \le 1$. For $0 \le a \le \delta$, $\delta > 0$, we are given continuous two-parameter families*

$$\begin{aligned} &\{(\lambda(s,a),\mu(s)) \in \mathcal{L}^2(X(s),\omega(s))\} \text{ and} \\ &\{(\widetilde{\lambda}(s,a),\widetilde{\mu}(s)) \in \mathcal{L}^2(\widetilde{X}(s),\widetilde{\omega}(s))\}. \end{aligned} \tag{3.42}$$

We assume that

$$(3.43)\quad (\lambda(s,0),\mu(s))\in\mathcal{FL}_0(X(s)) \text{ and } (\widetilde{\lambda}(s,0),\widetilde{\mu}(s))\in\mathcal{FL}_0(\widetilde{X}(s)),$$

$$(3.44)\quad \{\lambda(s,a)\} \text{ differentiable in } a \text{ and semi-positive for fixed } s,$$

$$(3.45)\quad \{\widetilde{\lambda}(s,a)\} \text{ differentiable in } a \text{ and positive for fixed } s,$$

$$(3.46)\quad \dim(\lambda(s,a)\cap\mu(s))-\dim(\widetilde{\lambda}(s,a)\cap\widetilde{\mu}(s))=c(s).$$

Then we have

$$(3.47)\qquad \operatorname{Mas}\{\lambda(s,0),\mu(s);\omega(s)\}=\operatorname{Mas}\{\widetilde{\lambda}(s,0),\widetilde{\mu}(s);\widetilde{\omega}(s)\}.$$

Proof. Since $[0,1]$ is compact, after making δ smaller, we may assume that the given two families (3.42) are families of Fredholm pairs of index 0.

Fix $t\in[0,1]$. Since $\widetilde{\lambda}(t,a)$ is differentiable in a and positive, by Proposition 3.14 there exists a $\delta(t)\in(0,\delta)$ such that $\widetilde{\lambda}(t,a)\cap\widetilde{\mu}(t)=\{0\}$ for $a\in(0,\delta(t)]$. From the continuity of our family $(\widetilde{\lambda}(s,a),\widetilde{\mu}(s))$, there exists an $\varepsilon(t)>0$ such that

$$(3.48)\qquad \widetilde{\lambda}(s,\delta(t))\cap\widetilde{\mu}(s)=\{0\} \text{ for } s\in(t-\varepsilon(t),t+\varepsilon(t))\cap[0,1].$$

By compactness of $[0,1]$, there exists a partition $0=s_0<s_1<\ldots<s_n=1$ of $[0,1]$ and $t_1,\ldots t_n\in[0,1]$ with $s_{k-1},s_k\in(t_k-\varepsilon(t_k),t_k+\varepsilon(t_k))$ for $k=1,\ldots,n$.

We now prove the formula (3.47) for a small interval $[s_{k-1},s_k]$. We consider the two-parameter families (3.42) for $s\in[s_{k-1},s_k]$ and $a\in[0,\delta(t_k)]$. Because of the homotopy invariance of Maslov index, both integers $\operatorname{Mas}\{\lambda(s,a),\mu(s)\}$ and $\operatorname{Mas}\{\widetilde{\lambda}(s,a),\widetilde{\mu}(s)\}$ must vanish for the boundary loop going counter clockwise around the rectangular domain from the corner point $(s_{k-1},0)$ via the corner points $(s_k,0)$, $(s_k,\epsilon(t_k))$, and $(s_{k-1},\epsilon(t_k))$ back to $(s_{k-1},0)$.

Moreover, by (3.46) and (3.48), for all $s\in[s_{k-1},s_k]$ we have

$$\dim(\lambda(s,\delta(t_k))\cap\mu(s))=c_{\delta(t_k)} \text{ and } \widetilde{\lambda}(s,\delta(t_k))\cap\widetilde{\mu}(s)=\{0\}.$$

Hence, our two Maslov indices must vanish on the top segment of our box.

Finally, by Proposition 3.18 and (3.46) we have

$$\begin{aligned}
&\operatorname{Mas}\{\lambda(s_{k-1},a),\mu(s_{k-1});a\in[0,\delta(t_k)]\}\\
&\quad-\operatorname{Mas}\{\widetilde{\lambda}(s_{k-1},a),\widetilde{\mu}(s_{k-1});a\in[0,\delta(t_k)]\}\\
&=\sum_{0<a\le\delta(t_k)}(\dim(\lambda(s_{k-1},a)\cap\mu(s_{k-1})-\lim_{b\to a^-}\dim(\lambda(s_{k-1},b)\cap\mu(s_{k-1}))\\
&\quad-\sum_{0<a\le\delta(t_k)}\dim(\widetilde{\lambda}(s_{k-1},a)\cap\widetilde{\mu}(s_{k-1}))\\
&=\sum_{0<a\le\delta(t_k)}(c_a-\lim_{b\to a^-}c_b)\\
&=\sum_{0<a\le\delta(t_k)}(\dim(\lambda(s_k,a)\cap\mu(s_k))-\lim_{b\to a^-}\dim(\lambda(s_k,b)\cap\mu(s_k)))\\
&\quad-\sum_{0<a\le\delta(t_k)}\dim(\widetilde{\lambda}(s_k,a)\cap\widetilde{\mu}(s_k))\\
&=\operatorname{Mas}\{\lambda(s_k,a),\mu(s_k);a\in[0,\delta(t_k)]\}\\
&\quad-\operatorname{Mas}\{\widetilde{\lambda}(s_k,a),\widetilde{\mu}(s_k);a\in[0,\delta(t_k)]\}.
\end{aligned}$$

By additivity under catenation, the formula (3.47) holds for the small interval $[s_{k-1},s_k]$. Again by additivity under catenation, the formula (3.47) holds for the whole interval $[0,1]$. □

3.3. Invariance of the Maslov index under symplectic operations. In this section we show that the Maslov index is invariant under symplectic embedding and symplectic reduction under natural conditions.

Our first theorem generalizes Definition 3.2. We begin with a lemma.

LEMMA 3.20. *Let (X,ω) be a symplectic Banach space and $W\subset X$ a co-isotropic subspace. Assume that W is a Banach space (not induced by the norm on X) such that the injection $j\colon W\to X$ is bounded. Then the symplectic reduction $(W/W^\omega,\widetilde{\omega})$ is a symplectic Banach space.*

PROOF. Note that W^ω is closed in X. By Proposition A.35.a, W^ω is closed in W, so the quotient W/W^ω with the norm induced by W is a Banach space. Since j is bounded, $\widetilde{W}$ is bounded on W/W^ω. Then $(W/W^\omega)^{\widetilde{\omega}}=W^\omega/W^\omega=\{0\}$. So $\widetilde{\omega}$ is non-degenerate. Since W is a Banach space, $(W/W^\omega,\widetilde{\omega})$ is a symplectic Banach space. □

ASSUMPTION 3.21. We make the following assumptions.
(i) We are given Banach bundles $q_0\colon F_0\to[0,1]$, $q\colon F\to[0,1]$, $\widetilde{q}\colon\widetilde{F}\to$

$[0,1]$, and $p\colon E\to[0,1]$ with fibers $q_0^{-1}(s):=W_0(s)$, $q^{-1}(s):=W(s)$, $\widetilde{q}^{-1}(s):=\widetilde{W}(s)$ and $p^{-1}(s):=X(s)$ for each $s\in[0,1]$, respectively. Assume that we have Banach subbundle maps $F_0\to F$, $F\to\widetilde{F}$, $\widetilde{F}\to E$.
(ii) We are given a path of symplectic structures $\omega(s)$ on $X(s)$.
(iii) We have a path $(\lambda(s),\mu(s))$ of Fredholm pairs of Lagrangian subspaces of $(X(s),\omega(s))$ of index 0.
(iv) Assume that $W_0(s)=W(s)^{\omega(s)}$, $\dim(W(s)^{\omega(s)}\cap\mu(s))=k$ for each $s\in[0,1]$, and $W(s)+\mu(s)$, $s\in[0,1]$ is a path of closed subspaces of $X(s)$ (it holds automatically if $W(s)$ is closed in $X(s)$).
$W(s)$ is closed in $\widetilde{W}(s)$, and $\widetilde{W}(s)+\mu(s)=X(s)$ (there are three special cases: $\widetilde{W}(s)$ is dense in $X(s)$, $k=0$, and $W(s)$ is closed in $X(s)$ which yields the existence of such $\widetilde{W}(s)$) for each $s\in[0,1]$.

THEOREM 3.22 (Invariance under symplectic reduction). *Under Assumption 3.21, we have the following.*
(a) For each $s\in[0,1]$, we have $\dim X(s)/(W(s)+\mu(s))=k$ and $W(s)+\mu(s)=W(s)^{\omega(s)\omega(s)}+\mu(s)$
(b) The family

$$\{\big(R_W(s)^{\omega(s)}(\lambda(s)),R_W(s)^{\omega(s)}(\mu(s))\big);s\in[0,1]\}$$

is a path of Fredholm pairs of Lagrangian subspaces of

$$(W(s)/W(s)^{\omega(s)},\widetilde{\omega}(s))$$

of index 0.
(c) We have

$$\operatorname{Mas}_\pm\{\lambda(s),\mu(s)\}=\operatorname{Mas}_\pm\{R^{\omega(s)}_{W(s)}(\lambda(s)),R^{\omega(s)}_{W(s)}(\mu(s))\}. \tag{3.49}$$

PROOF. We divide the proof into three steps.

Step 1. By Proposition 1.30, (a) and (b) hold.

By Lemma A.34, $\lambda(s)$, $\mu(s)\cap W(s)$, $W(s)^{\omega(s)}$ is closed in $W(s)$ for each $s\in[0,1]$. Since $R_W(s)^{\omega(s)}(\mu(s))$ is closed in $W(s)/W(s)^{\omega(s)}$, $\mu(s)\cap W(s)+W(s)^{\omega(s)}$ is closed in $W(s)$ for each $s\in[0,1]$. By Corollary A.36, $\lambda(s)\in\mathcal{S}(W(s))$ is a path. By Lemma A.9, $R_W(s)^{\omega(s)}(\lambda(s))\in\mathcal{S}(W(s)/W(s)^{\omega(s)})$ is a path.

Since $W(s)+\mu(s)$ and $\widetilde{W}(s)+\mu(s)$ are finite-dimensional extensions of the closed subspace $\lambda(s)+\mu(s)\in X(s)$, by Proposition A.4 they are closed in $X(s)$. Note that we have $\widetilde{W}(s)+\mu(s)=X(s)$ (if $\widetilde{W}(s)$ is dense in $X(s)$ or $k=0$).

Since $W(s)\subset\widetilde{W}(s)$, we have

$$W(s)+\mu(s)\cap\widetilde{W}(s)=(W(s)+\mu(s))\cap\widetilde{W}(s).$$

By Lemma A.34, $\mu(s) \cap \widetilde{W}(s)$ and $(W(s) + \mu(s)) \cap \widetilde{W}(s)$ are closed in $\widetilde{W}(s)$. By Corollary A.22, $\mu(s) \cap W(s)$, $\mu(s) \cap W(s)^{\omega(s)}$, $\mu(s) \cap W(s) + W(s)^{\omega}(s)$ are paths of closed subspaces of $W(s)$. By Lemma A.9, $R_W(s)^{\omega(s)}(\mu(s)) \in \mathcal{S}(W(s)/W(s)^{\omega(s)})$ is a path. Therefore the Maslov index on the right hand side of (3.49) is well-defined.

Step 2. Reduce to the case of $W(s) + \mu(s) = X(s)$.

Since $W(s) + \mu(s)$, $s \in [0,1]$ is a path of closed subspaces of $X(s)$ of finite codimension, we have $W(s) + \mu(s) \in \mathcal{S}^c(X)$. By Lemma A.32 (see also [67, Lemma 0.2]), $\bigcup_{s\in[0,1]}(W(s) + \mu(s))$ is a Banach bundle over $[0,1]$, and there exists a finite-dimensional Banach subbundle $\bigcup_{s\in[0,1]} V(s)$ of E such that $V(s) \oplus (W(s) + \mu(s)) = X(s)$. Note that we can take $\widetilde{W}(s) := W(s) + V(s)$ if $W(s)$ is closed in $X(s)$.

We use the notations from Corollary 1.26. Set $E_j := \bigcup_{s\in[0,1]} X_j(s)$, $j = 0, 1$. By Step 1, $W(s)^{\omega(s)}$, $s \in [0,1]$ is a path of $\mathcal{S}^c(\widetilde{W}(s))$. By Corollary A.36, it is a path of $\mathcal{S}^c(X(s))$. So $X_0(s) := W(s)^{\omega(s)} + V(s)$, $s \in [0,1]$ is a path of $\mathcal{S}^c(X(s))$, and E_0 is a Banach subbundle of E. By Proposition 1.19, we have

$$X_1(s) := V(s)^{\omega(s)} \cap W(s) + V(s)^{\omega(s)} \cap \mu(s) = V(s)^{\omega(s)} \cap (W(s) + \mu(s)).$$

Note that $X(s) = X_0(s) \oplus X_1(s)$. Then $X_1(s)$ is a path of $\mathcal{S}^c(X_s)$. By Lemma A.32 (see also [67, Lemma 0.2]), E_1 is a Banach subbundle of E. Set $W_{01}(s) := W(s)^{\omega(s)} \cap V(s)^{\omega(s)}$, $W_1(s) := V(s)^{\omega(s)} \cap W(s)$, and $F_j(s) := \bigcup_{s\in[0,1]} W_j(s)$ for $j = 01, 1$. By Lemma A.33, F_{01} is a Banach subbundle of F_0, F_1 is a Banach subbundle of F, and F_{01} is a Banach subbundle of F_1. Then we can replace $X_1(s)$ for $X(s)$.

Set $\lambda_0(s) = \mu_0(s) := W(s)^{\omega(s)} \cap \mu(s)$, $\lambda_1(s) := \lambda(s) \cap X_1(s)$ and $\mu_1(s) := \mu \cap X_1(s)$. By Proposition 2.3, for a local path $s \in [s_1, s_2] \subset (t - \delta(t), t + \delta(t))$ we have

$$\begin{aligned}\mathrm{Mas}_\pm\{\lambda(s), \mu(s)\} &= \mathrm{Mas}_\pm\{\lambda_0(s), \mu_0(s)\} + \mathrm{Mas}_\pm\{\lambda_1(s), \mu_1(s)\}\\ &= \mathrm{Mas}_\pm\{\lambda_1(s), \mu_1(s)\}.\end{aligned}$$

Then our result follows from the compactness of $[0,1]$ and Definition 3.2.

Step 3. The case of $W(s) + \mu(s) = X(s)$.

Fix $t \in [0,1]$. Let $V_1(t) \subset W(t)$ be a linear subspace such that $X(s) = V_1(t) \oplus (\lambda(t) + \mu(t))$. Let $L(t,s)\colon W(t) \to W(s)$ be the local frame of the bundle F. Set $V_1(t,s) := L(t,s)V_1(t) \subset W(s)$. By Lemma 1.25 and Theorem 3.8, for a local path $s \in [s_1, s_2] \subset (t - \delta(t), t + \delta(t))$

we have

$$\begin{aligned}\mathrm{Mas}_{\pm}\{\lambda(s),\mu(s)\} &= \mathrm{Mas}_{\pm}\{R_{V_1(t,s)+\lambda(s)}(\lambda(s)), R_{V_1(t,s)+\lambda(s)}(\mu(s))\}\\ &= \mathrm{Mas}_{\pm}\{R_{W(s)}(\lambda(s)), R_{W(s)}(\mu(s))\}.\end{aligned}$$

Then our result follows from the compactness of $[0,1]$ and Definition 3.2. $\square$

COROLLARY 3.23. *The equation (3.49) holds if $W(s)\in\mathcal{S}^c(X(s))$ is a path of closed subspaces and $W(s)^{\omega(s)}\in\mathcal{S}^c(W(s))$ is a path of closed subspaces.*

PROOF. Set $F := \bigcup_{s\in[0,1]} W(s)$ and $F_0 := \bigcup_{s\in[0,1]} W(s)^{\omega(s)}$. By Lemma A.32 (see also [67, Lemma 0.2]), F is a Banach subbundle of E, and F_0 is a Banach subbundle of F. By Theorem 3.22, our result follows. $\square$

COROLLARY 3.24. *The equation (3.49) holds for $W(s)=V(s)\oplus\lambda(s)$ if $V(s)$ is a path of finite-dimensional linear subspaces of $X(s)$ and it holds that*

$$X(s)=V(s)+\lambda(s)+\mu(s). \tag{3.50}$$

PROOF. Since $V(s)\cap\lambda(s)=\{0\}$ and since Equation (3.50) holds, by Corollary 1.31 we have $V(s)^{\omega(s)}+\lambda = X(s)$. Note that $X(s) = W(s)+\mu(s)$, $V(s)\cap\lambda(s)=\{0\}$. By Lemma 1.18, we have

$$\dim W(s)/\lambda(s) \;=\; \dim\lambda(s)/W(s)^{\omega(s)} \;=\; \dim V(s).$$

By Proposition A.4, $W(s)\in\mathcal{S}^c(X)$. Clearly, $W(s)^{\omega(s)}\in\mathcal{S}^c(W(s))$ since it is closed and of finite codimension. Since $V(s)^{\omega(s)}$ and $\lambda(s)$ are paths and $V(s)^{\omega(s)}+\lambda(s)=X$, by Proposition A.21, $V(s)^{\omega(s)}\cap\lambda(s)$ is also a path. By Theorem 3.22, our result follows. $\square$

Our second theorem generalizes [22, Lemma 12]. We make some preparations for it.

LEMMA 3.25. *Let (X,ω) be a reflexive symplectic Banach space and X_0, X_1 two symplectic subspaces with $X=X_0\oplus X_1$ and $X_0=X_1^{\omega}$. Let $\lambda\subset X$ be a Lagrangian subspace of X. Assume that $\lambda\cap X_0$ is a Lagrangian subspace of X_0. Then $\lambda\cap X_1$ is a Lagrangian subspace of X_1, and we have*

$$\lambda=\lambda\cap X_0\oplus\lambda\cap X_1. \tag{3.51}$$

PROOF. By Lemma 1.17, we have $X_1=X_0^{\omega}$ and X_0, X_1 are closed. Since $X=X_0\oplus X_1$, $\lambda\cap X_0+X_1$ is closed. By [9, Lemma 3.2] we have $(\lambda\cap X_0+X_1)^{\omega\omega}=\lambda\cap X_0+X_1$. Since $\lambda\cap X_0$ is a Lagrangian subspace of X_0, we have $(\lambda \cap X_0)^\omega \cap X_0 = \lambda \cap X_0 \subset \lambda$. Then we have $\lambda \cap X_0 + X_1 \supset \lambda$. Thus there holds

$$\lambda = \lambda \cap (\lambda \cap X_0 + X_1) = \lambda \cap X_0 + \lambda \cap X_1, \text{ and}$$
$$\lambda^\omega = (\lambda \cap X_0)^\omega \cap X_0 + (\lambda \cap X_1)^\omega \cap X_1.$$

Consequently, $(\lambda \cap X_1)^\omega \cap X_1 = \lambda \cap X_1$ and $\lambda \cap X_1$ is a Lagrangian subspace of X_1. $\square$

PROPOSITION 3.26. *Let $p\colon E \to [0,1]$ be a Banach bundle such that $X(s) := p^{-1}(s)$ is a reflexive Banach space for each $s \in [0,1]$. Let $\omega(s)$ be a path of symplectic structures on $X(s)$. Let $(\lambda(s), \mu(s))$ be a path of Fredholm pairs of Lagrangian subspaces of $(X, \omega(s))$ of index 0. Let $p_j\colon E_j \to [0,1]$ be two Banach subbundles of $p\colon E \to [0,1]$ with $X_j(s) := p_j^{-1}(s)$, $s \in [0,1]$, $j = 1,2$. We assume that*

- *$\omega(s)|_{X_j(s)}$ are continuously varying for $j = 0,1$,*
- *$X(s) = X_0(s) \oplus X_1(s)$ and $X_0(s) = X_1(s)^{\omega(s)}$, and*
- *$(\lambda(s) \cap X_0(s), \mu(s) \cap X_0(s))$ is a family of pairs of Lagrangian subspaces in $(X_0(s), \omega(s)|_{X_0(s)})$.*

Then $(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s))$ is a path of Fredholm pairs of Lagrangian subspaces in $(Y(s), \omega(s)|_{X_j(s)})$ of index 0, $j = 0,1$, and

$$\begin{aligned}\operatorname{Mas}_\pm\{\lambda(s), \mu(s)\} = &\operatorname{Mas}_\pm\{\lambda(s) \cap X_0(s), \mu(s) \cap X_0(s)\}\\ &+ \operatorname{Mas}_\pm\{\lambda(s) \cap X_1(s), \mu(s) \cap X_1(s)\}.\end{aligned} \tag{3.52}$$

PROOF. By (ii), we have an injective continuous map $f\colon \mathcal{S}(X_0(s)) \times \mathcal{S}(X_1(s)) \to \mathcal{S}(X(s)$ by $f(M,N) = M + N$, and f is a homeomorphism onto its image. By Lemma 3.25, $(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s))$ is a path of Fredholm pairs of Lagrangian subspaces in $(Y(s), \omega(s)|_{X_j(s)})$, $j = 0,1$, and

$$\begin{aligned}\operatorname{index}(\lambda(s), \mu(s)) = &\operatorname{index}(\lambda(s) \cap X_0(s), \mu(s) \cap X_0(s))\\ &+ \operatorname{index}(\lambda(s) \cap X_1(s), \mu(s) \cap X_1(s)).\end{aligned}$$

By Lemma 1.12, we have $\operatorname{index}(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s)) \le 0$, $j = 0,1$. So we have $\operatorname{index}(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s)) = 0$, $j = 0,1$. Then the Equation (3.52) follows from Proposition 2.3.c. $\square$

THEOREM 3.27 (Invariance under symplectic embedding). *Let $p\colon E \to [0,1]$ be a Banach bundle. Denote by $X(s) := p^{-1}(s)$ the fiber of p at $s \in [0,1]$. Let $\omega(s)$ be a path of symplectic structures on $X(s)$. Let $(\lambda(s), \mu(s))$ be a path of Fredholm pairs of Lagrangian subspaces of $(X, \omega(s))$ of index 0. Let $p_1\colon F \to [0,1]$ be a second Banach bundle which is a linear subbundle of $p\colon E \to [0,1]$ (in general the inclusion*

$Y(s) \hookrightarrow X(s)$ is neither continuous nor dense), where $Y(s) := p_1^{-1}(s)$. We assume that

- *$\omega(s)|_{Y(s)}$ is continuously varying,*
- *$(\lambda(s) \cap Y(s), \mu(s) \cap Y(s))$ is a path of Fredholm pairs of Lagrangian subspaces in $(Y(s), \omega(s)|_{Y(s)})$ of index 0, and*
- *$\dim(\lambda(s) \cap \mu(s)) - \dim(\lambda(s) \cap \mu(s) \cap Y(s))$ is a constant k.*

Then we have

$$\mathrm{Mas}_\pm\{\lambda(s), \mu(s)\} = \mathrm{Mas}_\pm\{\lambda(s) \cap Y(s), \mu(s) \cap Y(s)\}. \tag{3.53}$$

PROOF. Since $[0,1]$ is compact, by Definition 3.2 we need only consider the local case. In this case the bundles E and F are both trivial, i.e., we can assume that $X(s) = X$ and $Y(s) = Y$.

Fix $t \in [0,1]$. Set $\lambda_Y(s) := \lambda(s) \cap Y$, $\mu_Y(s) := \mu(s) \cap Y$, and $\omega_Y(s) := \omega(s)|_Y$ for all $s \in [0,1]$. By the Fredholm properties, there exist finite-dimensional linear subspaces $V_1 \subset Y$ and $V_2 \subset X$ such that

$$Y = V_1 \oplus (\lambda(t) \cap Y + \mu(t) \cap Y), \quad X = V_2 \oplus (Y + \lambda(t) + \mu(t)).$$

Set $V := V_1 \oplus V_2$. Then we have

$$V + \lambda(t) + \mu(t) = V_1 + V_2 + \lambda(t) + \mu(t) = V_2 + Y + \lambda(t) + \mu(t) = X.$$

Note that

$$\begin{aligned} V \cap \lambda(t) &= (V_1 + V_2) \cap (V_1 + \lambda(t)) \cap \lambda(t) \\ &= (V_1 + V_2 \cap (V_1 + \lambda(t)) \cap \lambda(t) \\ &= V_1 \cap \lambda(t) = V_1 \cap Y \cap \lambda(t) = \{0\}. \end{aligned}$$

By Appendix A.3, there exists a $\delta > 0$ such that for $s \in (-\delta, \delta) \cap [0,1]$, we have $V + \lambda(s) + \mu(s) = X$, $V_1 + \lambda_Y(s) \cap Y + \mu(s) \cap Y = Y$, and $V \cap \lambda(s) = V_1 \cap \lambda_Y(s) = \{0\}$. By Corollary 3.24, for all paths $[s_1, s_2] \subset (-\delta, \delta) \cap [0,1]$ we have

$$\begin{aligned} \mathrm{Mas}_\pm\{\lambda(s), \mu(s)\} &= \mathrm{Mas}_\pm\{R^{\omega(s)}_{V+\lambda(s)}(\lambda(s)), R^{\omega(s)}_{V+\lambda(s)}(\mu(s))\}, \\ \mathrm{Mas}_\pm\{\lambda_Y(s), \mu_Y(s)\} &= \mathrm{Mas}_\pm\{R^{\omega_Y(s)}_{V_1+\lambda_Y(s)}(\lambda(s)), R^{\omega_Y(s)}_{V+\lambda_Y(s)}(\mu(s))\}. \end{aligned}$$

Consider the symplectic linear maps

$$\frac{V_1+\lambda_Y(s)}{V_1^{\omega_Y(s)} \cap \lambda(s)} \xrightarrow{f(s)} \frac{V_1+\lambda(s)}{V_1^{\omega(s)} \cap \lambda(s)} \xrightarrow{g(s)} \frac{V+\lambda(s)}{V^{\omega(s)} \cap \lambda(s)}, \tag{3.54}$$

where f, g are induced by natural embeddings. Note that a symplectic linear map is an injection. By comparing dimensions, f is an isomorphism and g is an injection. For any linear subspace M of X, we have

$$f(s)(R^{\omega_Y(s)}_{V_1+\lambda_Y(s)}(M \cap Y)) \subset R^{\omega(s)}_{V_1+\lambda(s)}(M). \tag{3.55}$$

If $M=\lambda(s)$ or $M=\mu(s)$, then

- $R^{\omega_Y(s)}_{V_1+\lambda_Y(s)}(M\cap Y)$ is a Lagrangian subspace in the reduced space $(V_1+\lambda_Y(s))/(V_1^{\omega_Y(s)}\cap\lambda(s))$, and
- $R^{\omega(s)}_{V_1+\lambda(s)}(M)$ is a Lagrangian subspace in $(V_1+\lambda(s))/(V_1^{\omega(s)}\cap\lambda(s))$.

So (3.55) is an equality. Then we can apply Lemma 1.25, Lemma 1.28 and Proposition 1.30. Our problem is then reduced to the case of symplectic embeddings $g(s)f(s)$, which replace the linear embedding of the bundles.

Now we are in the finite-dimensional case, i.e., we can assume that $\dim X<+\infty$. In this case, the embedding is always continuous, and $X=Y(s)\oplus Y(s)^{\omega(s)}$. By Proposition 3.26 and Proposition 2.3.e we have

$$\begin{aligned}\mathrm{Mas}_\pm\{\lambda(s),\mu(s)\}&=\mathrm{Mas}_\pm\{\lambda(s)\cap Y(s),\mu(s)\cap Y(s)\}\\&\quad+\mathrm{Mas}_\pm\{\lambda(s)\cap Y(s)^{\omega(s)},\mu(s)\cap Y(s)^{\omega(s)}\}\\&=\mathrm{Mas}_\pm\{\lambda(s)\cap Y(s),\mu(s)\cap Y(s)\}.\end{aligned}$$

Our result is then proved. □

3.4. The Hörmander index. In this section we fix the symplectic Banach space (X,ω). Firstly we give some preparations. Recall from Definition 1.11: for $k,m\in\mathbb{Z}$ and $\mu\in\mathcal{L}(X)$, we define

$$\begin{aligned}\mathcal{FL}_k(X)&:=\{(\lambda,\mu)\in\mathcal{FL}(X)\mid \mathrm{index}(\lambda,\mu)=k\},\\ \mathcal{FL}_k(X,\mu)&:=\{\lambda\in\mathcal{L}(X);(\lambda,\mu)\in\mathcal{FL}_k(X)\},\\ \mathcal{FL}_0^m(X,\mu)&:=\{\lambda\in\mathcal{FL}_0(X,\mu);\dim(\lambda\cap\mu)=m\}.\end{aligned}$$

LEMMA 3.28. *Let $\mu\in\mathcal{L}(X)$. Then we have that*
(a) $\mathcal{FL}_0^0(\mu)$ is an affine space (hence contractible),
(b) $\mathcal{FL}_0^0(\mu)$ is dense in $\mathcal{FL}_0(\mu)$ and $\mathcal{FL}_0(\mu)$ is path connected.

PROOF. (a) Let $\lambda\in\mathcal{FL}_0^0(\mu)$. By Lemma A.29, we have

$$\begin{aligned}\mathcal{FL}_0^0(X,\mu)&=\{\mathrm{graph}(A)\in\mathcal{L}(X);A\in\mathcal{B}(\lambda,\mu)\}\\&=\{\mathrm{graph}(A);A\in\mathcal{B}(\lambda,\mu),\omega(x,Ay)+\omega(y,Ax)=0,\forall x,y\in\lambda\}.\end{aligned}$$

So (a) is proved.
(b) Let $\lambda\in\mathcal{FL}_0(X,\mu)$. By Proposition 1.19, we have $X=X_0\oplus X_1$, where $X_0:=V\oplus\lambda_0$, $X_1:=\lambda_1\oplus\mu_1$, $\lambda_0:=\lambda\cap\mu$, $\lambda_1=V^\omega\cap\lambda$, $\mu_1=\mu^\omega\cap\mu$, and V is chosen to be isotropic. We have $X_0=X_1^\omega$ is of finite dimension, and X_0, X_1 are symplectic. Note that $V,\lambda_0\in\mathcal{L}(X_0)$ and $\lambda_1,\mu_1\in\mathcal{L}(X_1)$. Let $A\colon\lambda_0\to V$ be a linear isomorphism with

$\omega(x, Ay) + \omega(y, Ax) = 0, \forall x, y \in \lambda_0$. Set $c_1(s) := \operatorname{graph}(sA)$, $s \in [0,1]$ and $c(s) = c_1(s) \oplus \lambda_1$. The $c(0) = \lambda$ and $c(s) \in \mathcal{FL}_0^0(X, \mu)$. By (a), we get (b). □

LEMMA 3.29. *Let $\lambda, \mu \in \mathcal{L}(X)$ and $X = \lambda \oplus \mu$. Then we have*
(a) $\mathcal{FL}_0^0(X, \mu) \cap \mathcal{CP}_0(\lambda)$ is an affine space (hence contractible),
(b) $\mathcal{FL}_0^0(X, \mu) \cap \mathcal{CP}_0(\lambda)$ is dense in $\mathcal{FL}_0(X, \mu) \cap \mathcal{CP}_0(\lambda)$ and $\mathcal{FL}_0(X, \mu) \cap \mathcal{CP}_0(\lambda)$ is path connected.

PROOF. The proof of Lemma 3.29 is similar to that of Corollary A.45 and we omit it. □

COROLLARY 3.30. *Let $(\lambda_j(s), \mu_1) \in \mathcal{FL}_0^2(X)$, $0 \le s \le 1$ for $j = 1, 2$ be two paths with the same endpoints. Let $\mu_2 \in \mathcal{L}(X)$ such that $\mu_1 \sim^c \mu_2$ and $[\mu_1 - \mu_2] = 0$. Then we have $(\lambda_j(s), \mu_1) \in \mathcal{FL}_0^2(X)$ and*

$$\operatorname{Mas}\{\lambda_1, \mu_2\} - \operatorname{Mas}\{\lambda_1, \mu_1\} = \operatorname{Mas}\{\lambda_1, \mu_1\} - \operatorname{Mas}\{\lambda_2, \mu_1\}. \tag{3.56}$$

PROOF. By Lemma 3.29, there is a path $\mu(s)$ with $\mu(0) = \mu_1$ and $\mu(1) = \mu_2$, $\mu(s) \sim^c \mu_2$ and $[\mu_1 - \mu(s)] = 0$. By Proposition A.44.g, we have $\mathcal{F}_{0,\mu_1}(X) = \mathcal{F}_{0,\mu_2}(X)$ and $\mathcal{FL}_0(X, \mu_1) = \mathcal{FL}_0(X, \mu(s))$. Then we have $(\lambda_j(s), \mu(s)) \in \mathcal{FL}_0^2(X)$. Then we have two homotopies $(\lambda_j(s), \mu(t)) \in \mathcal{FL}_0^2(X)$, $(s,t) \in [0,1]$ for $j = 1, 2$. By Proposition 2.3 we have

$$\begin{aligned}
\operatorname{Mas}\{\lambda_1, \mu_2\} - \operatorname{Mas}\{\lambda_1, \mu_1\} &= \operatorname{Mas}\{\lambda_1(1), \mu\} - \operatorname{Mas}\{\lambda_1(0), \mu\} \\
&= \operatorname{Mas}\{\lambda_2(1), \mu\} - \operatorname{Mas}\{\lambda_2(0), \mu\} \\
&= \operatorname{Mas}\{\lambda_2, \mu_2\} - \operatorname{Mas}\{\lambda_2, \mu_1\}.
\end{aligned}$$ □

Now we are in the position of defining the Hörmander index.

DEFINITION 3.31. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathcal{L}(X)$ be Lagrangian subspaces of X. Assume that $\lambda_1, \lambda_2 \in \mathcal{FL}_0(X, \mu_1)$, $\mu_1 \sim^c \mu_2$ and $[\mu_1 - \mu_2] = 0$. By Lemma 3.28, there is a path $\lambda\colon [0,1] \to \mathcal{FL}_0(X, \mu_1)$ with $\lambda(0) = \lambda_1$, $\lambda(1) = \lambda_2$. By Lemma 3.30, we can define the *Hörmander index* $\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2)$ by

$$\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2) = \operatorname{Mas}\{\lambda, \mu_2\} - \operatorname{Mas}\{\lambda, \mu_1\}. \tag{3.57}$$

We note the condition that $[\mu_1 - \mu_2] = 0$ is automatically satisfied if X is reflexive.

LEMMA 3.32. *Let (X, ω) be a reflexive symplectic Banach space. Let $(\lambda, \mu_1) \in \mathcal{FL}^2(X)$ and $\mu_1 \sim^c \mu_2$. Then we have $[\mu_1 - \mu_2] = 0$.*

PROOF. By Proposition A.44.g, we have $(\lambda, \mu_2) \in \mathcal{FL}^2(X)$. By Corollary 1.13, we have $\operatorname{index}(\lambda, \mu_1) = \operatorname{index}(\lambda, \mu_2) = 0$. By Proposition A.44.g, we have $[\mu_1 - \mu_2] = 0$. □

4. The desuspension spectral flow formula

In this section, we study self-adjoint Fredholm extensions of symmetric operators, and prove a general spectral flow formula under the assumption of a certain weak inner unique continuation property (wiUCP).

4.1. Short account of predecessor formulae. To begin with, we describe the topological and analytic background of our applications.

4.1.1. *The spectral flow.* In various branches of mathematics one is interested in the calculation of the spectral flow of a continuous family of closed densely defined (not necessarily bounded) self-adjoint Fredholm operators in a fixed Hilbert space. Roughly speaking, the spectral flow is an intersection number between the spectrum and the real line and counts the net number of eigenvalues changing from the negative real half axis to the nonnegative one.

The spectral flow for a one parameter family of linear self-adjoint Fredholm operators was introduced by M. Atiyah, V. Patodi, and I. Singer [7] in their study of index theory on manifolds with boundary. Since then, other significant applications have been found; many of them were inspired by C. Vafa and E. Witten's use of the spectral flow to estimate uniform bounds for the spectral gap of Dirac operators in [87]. The spectral flow was implicit already in Atiyah and Singer [8] as the isomorphism from the fundamental group of the non-trivial connected component of bounded self-adjoint Fredholm operators in complex Hilbert space onto the integers. Later this notion was made rigorous for not necessarily closed curves of bounded self-adjoint Fredholm operators in J. Phillips [73] and for gap-continuous curves of self-adjoint (generally unbounded) Fredholm operators in Hilbert spaces in [17] by the Cayley transform. The notion was generalized to the higher dimensional case in X. Dai and W. Zhang [34] for Riesz-continuous families, and to more general operators by K.P. Wojciechowski and C. Zhu and Y. Long in [92, 98, 99].

4.1.2. *Switch between symmetric and symplectic category.* In this section we derive spectral flow formulae in the following sense. We are given a continuous curve of self-adjoint Fredholm operators, or more generally, a continuous curve of self-adjoint Fredholm relations. Such curves arise typically from a family of elliptic operators over a compact manifold with boundary with smoothly varying coefficients and smoothly varying regular boundary conditions. Then we consider two mutually related invariants: within the *symmetric* category, we have the number of negative eigenvalues or, more generally, the spectral flow;

that is our first invariant. Basically, it is an intersection number of the spectral lines with the real axis. It is well defined, but, being a spectral invariant, difficult to determine in general. To define the second invariant, we switch from the symmetric category to the *symplectic* category. We notice that self-adjoint extensions are characterized by Lagrangian subspaces in corresponding symplectic Hilbert spaces coming from the domains, i.e., from the boundary values. That consideration yields another intersection number, the Maslov index. The Maslov index does not arise from the spectrum, but can be calculated directly from the boundary values of the solutions. Speaking roughly, the Maslov index counts the changes of the intersection dimensions of two curves of Lagrangians. In our case, the one curve is made of the continuously varying Lagrangians coming from the Fredholm domains. The other curve is made of the Cauchy data spaces, which also form Lagrangians and vary continuously under suitable assumptions. Then the type of spectral flow formulae we are interested in are formulae where the spectral flow of a given curve of self-adjoint Fredholm relations or operators is expressed by a related Maslov index. Here the point is that the calculation of the Maslov index is different from the calculation of the spectral flow, and, in general, easier.

4.1.3. *Origin and applications in Morse theory.* The first spectral flow formula was the classical Morse index theorem (cf. M. Morse [**63**]) for geodesics on Riemannian manifolds. It was extended by W. Ambrose [**2**] in 1961 to more general boundary conditions, which allowed the two end points of the geodesics to vary in two submanifolds of the manifold. In 1976, J.J. Duistermaat [**39**] completely solved the problem of calculating the Morse index for the one-dimensional variational problems, where the positivity of the second order terms was required. In 2000-2002, P. Piccione and D.V. Tausk [**74, 75**] were able to prove the Morse index theorem for semi-Riemannian manifolds for the same boundary conditions as in [**2**], and certain non-degeneracy conditions were needed. In 2001, the second author [**98**] was able to solve the general problem for the calculation of the Morse index of index forms for regular Lagrangian systems. See also the work of M. Musso, J. Pejsachowicz, and A. Portaluri on a Morse index theorem for perturbed geodesics on semi-Riemannian manifolds in [**65**] which has in particular lead N. Waterstraat to a K-theoretic proof of the Morse Index Theorem in [**90**].

4.1.4. *From ordinary to partial differential equations.* In 1988, A. Floer [**42**] emphasized that the notion of a Morse index of a function on a finite-dimensional manifold cannot be generalized directly to the symplectic action function α on the loop space of a manifold. He defined for any pair of critical points of α a relative Morse index, which corresponds to the difference of the two Morse indices in finite dimensions. It is based on the spectral flow of the Hessian of α. That paper opened another line of studying spectral flow formulae, namely for partial differential operators:

Let $\{A_s : C^\infty(M;E) \to C^\infty(M;E)\}_{s\in[0,1]}$ be a family of continuously varying formally self-adjoint linear elliptic differential operators of first order over a smooth compact Riemannian manifold M with boundary Σ, acting on sections of a Hermitian vector bundle E over M. Fixing a unitary bundle isomorphism between the original bundle and a product bundle in a collar neighborhood N of the boundary, the operators A_s can be written in the form

$$A_s|_N = J_{s,t}(\frac{\partial}{\partial t} + B_{s,t}) \tag{4.1}$$

with invertible skew-self-adjoint bundle isomorphisms $J_{s,t}$ and first order elliptic differential operators $B_{s,t}$ on Σ. Here t denotes the inward normal coordinate in N. For details see B. Booß-Bavnbek, M. Lesch and C. Zhu [**18**, Section 1].

Following another seminal paper by A. Floer [**41**], in 1991 T. Yoshida [**94**] elaborated on Floer homology of 3-manifolds by studying a curve $\{A_s\}$ of Dirac operators with invertible ends, such that $J_{s,t} = J_s$ are unitary operators and $B_{s,t} = B_s$ symmetric in the preceding notation. In 1995, L. Nicolaescu [**70**] generalized Yoshida's results to arbitrary $\dim M$. One year later, S.E. Cappell, R. Lee, and E.Y. Miller [**29**, Theorem G] found a somewhat intricate spectral flow formula for curves of arbitrary elliptic operators of first order under the conditions of constant coefficients close to the boundary in normal direction and symmetric induced tangential operators. In 2000, M. Daniel [**35**] removed the nondegenerate conditions in [**70**]. In 1998-2001, the first author, jointly with K. Furutani and N. Otsuki [**15, 16**] proved the case that the A_s differ by 0th order operators, and the boundary condition is fixed. In 2001, P. Kirk and M. Lesch [**54**, Theorem 7.5] proved the case that A_s is of Dirac type, $J_{s,t}$ is fixed unitary, and $B_{s,t} = B_s$ symmetric. Later in this section we shall only assume that each A_s satisfies weak inner unique continuation property (wiUCP), i.e., $\ker A_s|_{H^1_0(M;E)} = \{0\}$.

The formulae are of varying generality: Some deal with a fixed (elliptic) differential operator with varying self-adjoint extensions (i.e., varying boundary conditions); others keep the boundary condition fixed and let the operator vary. An example for a path of operators with fixed principal symbol is a curve of Dirac operators on a manifold with fixed

Riemannian metric and Clifford multiplication but varying defining connection (varying background field which is a zero-order perturbation and as such does not inflict the principal symbol). See also the results by the present authors in [21] for varying operator and varying boundary conditions but fixed maximal domain. Recently, M. Prokhorova [76] considered a path of Dirac operators on a two-dimensional disk with a finite number of holes subjected to local elliptic boundary conditions of chiral bag type. She obtained a beautiful explicit formula for the spectral flow (respectively, the Maslov index) which recently was re-proved and generalized by M. Katsnelson, V. Nazaikinskii, A. Gorokhovsky, and M. Lesch in [53, 46].

4.1.5. *Our contribution in this paper.* In this paper we have substantially expanded and settled the validity range of the predecessor formulae. Roughly speaking, we have achieved the following results:

(i) In the language of Banach bundles we present the list of assumptions on operator families that yield an abstract general spectral flow formula. The list can be found in Assumption 4.11 and the obtained formulae in Equations (4.13) and (4.14) of Theorem 4.12. This result expands substantially the validity of the functional analytic spectral flow formula of [14]. The novelty of the approach is the replacement of a fixed strong symplectic Hilbert space (the β-space of the quotients of maximal and minimal domain of a closed symmetric operator) by the quotient of a fixable intermediate domain with the minimal domain, equipped with varying weak symplectic forms.

(ii) In Section 4.5 we turn to the geometric setting. We consider a smooth family of formally self-adjoint elliptic differential operators of fixed order acting on sections of varying vector bundles over varying manifolds with boundary and impose varying well-posed self-adjoint boundary conditions. Under a technical condition that generalizes weak inner UCP, we obtain an array of general spectral flow formulae in Equations (4.23) and (4.24) of Theorem 4.15 in all Sobolev spaces over the boundaries of non-negative order. This result removes the restriction of previous formulae to curves of Dirac type operators or curves with only lower order variation.

(iii) In Theorem 4.17 we give the conditions for the validity of two formulae for the spectral flow of a curve of formally self-adjoint elliptic differential operators over a curve of closed partitioned manifolds $M(s) = M(s)^+ \cup_{\Sigma(s)} M(s)^-$ with separating hypersurfaces $\Sigma(s)$, $s \in [0,1]$. The first Formula (4.27) expresses the spectral flow over the whole manifold(s) in terms of a spectral flow of a canonically associated curve of well posed boundary problems over one part. The second Formula (4.28) expresses the spectral flow over the whole manifold(s) by the Maslov index of the corresponding Cauchy data spaces from both sides along the separating hypersurface(s). This result generalizes the splitting formulae of T. Yoshida [94] and L. Nicolaescu [70] and determines the limits of their validity.

4.1.6. *Spectral flow formulae also for higher order operators.* Usually one is only interested in the spectral flow of elliptic differential operators of first order. Typically, elliptic differential operators of second order like the various Laplacians are essentially positive (or essentially negative). For such operators the spectral flow of loops must vanish and the spectral flow of curves is just the difference between the number of negative (respectively positive) eigenvalues at the endpoints, hence trivial. However, the formula $\mathrm{sf} = \mathrm{Mas}$ is not trivial and can give radically new insight also in the case of second order operators. Therefore, we have not restricted our treatment to operators of order one.

For first order operators, in most applications the formula $\mathrm{sf} = \mathrm{Mas}$ will be read as a *desuspension*-type formula, namely expressing the spectral flow (a kind of quantum variable arising from the spectrum) over a manifold by the Maslov index (a kind of classical variable arising from solution spaces) over a submanifold of codimension 1. Then for essentially positive second order elliptic differential operators, in the applications we have in mind (e.g., a higher order Morse index theorem) the formula $\mathrm{sf} = \mathrm{Mas}$ should be read as a *suspension*-type formula, namely expressing the a-priori unknown Maslov index by the in that case trivial spectral flow via the introduction of an additional parameter.

4.1.7. *Partitioned manifolds in topology, geometry, and analysis.* In topology, the interest in partitioned manifolds is connected to the name of P. Heegaard who in his dissertation [51] introduced ways of splitting 3-manifolds and gaining corresponding graphs for algebraic investigation. In that way he could point to essential differences between homology and homotopy theory that had been missed by H. Poincaré (see, e.g., the elementary presentation by M. Scharlemann in [78]). Later his ideas were lavishly generalized in the concepts of cobordism, surgery, and cutting and pasting of the 1950-60s, see C.T.C. Wall [89]. In spite of the great expectations, the concept of partitioned manifolds has not proved valuable for proving Poincaré's Conjecture. Years before G. Perelman's final proof of the Conjecture, A. Floer expressed in [41] his expectation that the approach via Heegaard splittings or more general decompositions most probably would not solve the Poincaré Conjecture but would support the complementary topological program, namely to determine all groups that can show up as fundamental groups of 3-manifolds.

In geometry, the interest in partitioned manifolds is connected both to the concept of coarse geometry and to the geometry of singular spaces. In the first case one separates arduous, but topologically uninteresting parts out of complete (non-compact) manifolds, e.g., in the relative index theorems of M. Gromov and H.B. Lawson [47]. In the second case one focuses on the geometry around singularities by separating them out.

In analysis, the interest in partitioned manifolds is connected with the Riemann-Hilbert Problem of complex analysis. Classically, one looks for pairs of functions where one is holomorphic inside, and the other outside the disc and that are linearly conjugated by a transmission condition along the circle, see, e.g., N.I. Muskhelishvili [64]. In [12] B. Bojarski conjectured the general validity of a Riemann-Hilbert type index formula for elliptic operators on even-dimensional closed partitioned manifolds in terms of the index of the Fredholm pair of Cauchy data spaces along the separating hypersurface. That Bojarski Conjecture was proved by K.P. Wojciechowski and the first author in [19, Chapter 24]. We missed the odd-dimensional case which was then treated by L. Nicolaescu in [70]. While his result is restricted to Dirac type operators it served as the model for the present treatise.

There is a remarkable difference between the topological and the analytic approach to invariants of partitioned manifolds. The topological approach is characterized by the ease of achieving additivity formulae for topological invariants like the Euler characteristic or the signature solely by means of singular homology. Deriving the same results by analytic means, e.g., via the Atiyah-Singer Index Theorem is much more demanding. For finer topological invariants and spectral or differential invariants, homology theory may not suffice and harder means are demanded either from homotopy theory or, after all, from analysis. On the analysis level, there is clearly no recognizable splitting of the spectrum of a Dirac or Laplace operator on a partitioned manifold in its components from the parts. For the index (the chiral multiplicity of the zero-eigenvalues) we have both topological and analytical splitting formulae ([19, Chapters 23-25]). For the analytic torsion we have a topological splitting formula by W. Lück in [58]. For the η-invariant we have an analytic splitting formula by K.P. Wojciechowski in [93].

Similarly, our Theorem 4.17 should be considered an analytic splitting formula for the spectral flow.

4.1.8. *Wider perspectives.* In this section, we focus solely on the intertwining of the symmetric category (here the spectral flow) and the anti-symmetric category (here symplectic analysis). Clearly, each side deserves independent investigations and poses puzzles of their own. *Symplectic error terms in global analysis of singular manifolds* One such puzzle is to find the correct place of symplectic invariants (like the Maslov index and the Hörmander index) in the hierarchy of invariants in global analysis, compared with the index, the η-invariant, and the ζ-regularized determinant. We meet the *index* of Fredholm operators as the index of elliptic problems on closed manifolds, on manifolds with boundary, and on manifolds with singularities. From the viewpoint of global analysis, however, the index of elliptic problems on *closed* manifolds is distinguished because there the index can be expressed by an integral over an integrand that is locally expressed by the coefficients of the operator. The *η-invariant* arises in *boundary value problems.* It is not given by an integral, not by a local formula. It depends, however, only on finitely many terms of the symbol of the resolvent and will not change when one changes or removes a finite number of eigenvalues. Its derivative is local.

Keeping this difference in mind, we meet a question repeatedly put forward by I.M. Gelfand: "what comes next?" To this, I.M. Singer remarked in personal communication [82]: "Just as η arises in boundary value problems for smooth boundaries, I think the next level will come from corner contributions when the boundary has corners." Indeed, when the boundary has corners, a third term, the *Hörmander index* of symplectic analysis appears, see C.T.C. Wall [88]. He noticed the non-additivity of the signature for the Hopf bundle with fibre D^2 over S^2 having signature ± 1 depending on the choices of sign: this is the union of the induced bundles over the upper and lower hemispheres of S^2, each of which (being contractible) has signature zero. Wall's observation was in striking contradiction to the common wisdom in topology, first observed by S.P. Novikov: If two manifolds are glued by an orientation-preserving diffeomorphism of their boundaries, then the signature of their union is the sum of their signatures. So, Wall found that this additivity property does not hold for the more general situation where one glues two $4k$-manifolds $Y_\pm$ along a common submanifold X_0 of the boundaries, which itself has a boundary Z. That yields an abstract Zaremba problem (see also B.-W. Schulze, C.-C. Chang, and N. Habal [31]). Wall determined the non-additivity term as the Hörmander index of three associated Lagrangian subspaces of an induced finite-dimensional symplectic vector space. His result was extended to the gluing of η-invariants by U. Bunke in [**26**, **27**].

That supports the claim of a *hierarchy of asymmetry invariants*, placing the index of elliptic problems on closed manifolds at the bottom; placing the *eta*-invariant a little higher, namely as an error term for smooth boundary value problems; and placing symplectic invariants even higher, namely as error terms for boundary value problems with corners. In this paper, we shall not follow that line of thoughts any further and content with placing the Maslov index on the level of smoothly partitioned manifolds for now.

4.1.9. *Other approaches to the spectral flow.* It may be worth mentioning that there is a multitude of *other* formulae involving spectral flow, e.g., as error term under cutting and pasting of the index (see the first author with K.P. Wojciechowski [**19**, Chapter 25]) or under pasting of the eta-invariant as in [**54**]. Whereas these formulae typically relate the spectral flow of a family on a closed manifold of dimension $n-1$ to the index or the eta-invariant of a single operator on a manifold of dimension n, this paper addresses the opposite direction, namely how to express the spectral flow of a family over a manifold of dimension n by objects (here by the Maslov index) defined on a hypersurface of dimension $n-1$. See also our Subsection 4.1.6 above.

4.2. Spectral flow for self-adjoint relations. In this section, we show that one can easily obtain a formula expressing the spectral flow for curves of Fredholm self-adjoint linear relations in Hilbert spaces by the Maslov index on a very basic and abstract level. This leads us to the general definition of the spectral flow for curves of Fredholm self-adjoint operators of index 0 in Banach spaces (see Definition 4.1 below).

Recall that a *linear relation* A between two linear spaces X and Y is a linear subspace of $X\times Y$. We use the notions of linear relations and spectral flow in [**22**, Appendix A.2, A.3]. A is (the graph of) an operator if and only if $A(0)=\{0\}$. Here we identify A and the graph of A. By [**22**, Lemma 16 (a)], A is Fredholm if and only if $(A, X\times\{0\})$ is Fredholm. In this case, we have

$$\operatorname{index} A = \operatorname{index}(A, X\times\{0\}). \tag{4.2}$$

Inspired by C. Bennewitz [**10**] and I. Ekeland [**40**], we define

Definition 4.1. Let X, Y be two complex vector spaces and $\Omega\colon X\times Y\to\mathbb{C}$ be a sesquilinear non-degenerate map. Set $Z:=X\times Y$ and

$$\omega((x_1,y_1),(x_2,y_2)) := \Omega(x_1,y_2) - \overline{\Omega(x_2,y_1)}. \tag{4.3}$$

Then (Z,ω) is a symplectic vector space with two Lagrangian subspaces $X\times\{0\}$ and $\{0\}\times Y$. We call ω on Z the *symplectic structure induced by* Ω. The *adjoint* of a linear relation $A\in\mathcal{S}(Z)$ is defined to be A^ω. A linear relation $A\in\mathcal{S}(Z)$ is called *symmetric*, respectively *self-adjoint*, if $A\subset(Z,\omega)$ is isotropic, respectively Lagrangian. If A is symmetric, we define the form $Q(A)$ on $\operatorname{dom}(A)$ by $Q(A)(x,y):=\Omega(x,z)$ for all $x,y\in\operatorname{dom}(A)$ and $z\in A(y)$. The form $Q(A)$ is a well-defined quadratic form. We call the form $Q(A)$ the *quadratic form associated to* A.

Note that Ω corresponds to a conjugate linear injection $\tau\colon Y\to X^*$ such that $\bigcap_{y\in Y}\ker\tau(y)=\{0\}$ by

$$\Omega(x,y)=(\tau(y))(x),\ \text{for all } x\in X, y\in Y. \tag{4.4}$$

We consider the case when τ is an $\mathbb{R}$-linear isomorphism. In the real case we set $Y=X^*$ and $\tau=I_Y$. If X is a Hilbert space, we set $Y=X$ and $\tau(y)(x)=\langle x,y\rangle$. The space Y becomes a Banach space with the norm $\|y\|_Y:=\|\tau(y)\|_{X^*}$. Then $(X\times Y,\omega)$ is a symplectic Banach space for

$$\omega((x_1,y_1),(x_2,y_2))=(\tau(y_2))(x_1)-\overline{(\tau(y_1))(x_2)}, \tag{4.5}$$

$X\times\{0\}$ and $\{0\}\times Y$ are two natural Lagrangian subspaces of $X\times Y$. The symplectic structure ω is strong if and only if X is reflexive (see [**86**]). In this case we call $(X\times Y,\omega)$ *Darboux* (See [**91**]).

Let X be a Hilbert space. Clearly, an operator $A\colon X\supset\operatorname{dom}(A)\to X$ is symmetric, respectively self-adjoint, if and only if $\operatorname{graph}(A)$ is a symmetric, respectively self-adjoint linear relation.

LEMMA 4.2. *Let X be a Hilbert space and $A\in\mathcal{S}(X\times X)$ a symmetric relation. Then we have*
(a) the Cayley transform $\kappa(A)$ is a well-defined partial isometry on X, and $\sigma(\kappa(A))=\kappa(\sigma(A))$, where $\kappa\colon z\mapsto\frac{z-i}{z+i}$,
(b) $\kappa(A)\in\mathcal{B}(X)$ is unitary if A is self-adjoint,
(c) $\sigma(A)\subset\mathbb{R}$ if A is self-adjoint, and
(d) 1 is discrete in $\sigma(A)\cup\{1\}$ if A is self-adjoint Fredholm.

PROOF. To (a), (b): We have a symplectic decomposition $X\times X=X^-\oplus X^+$ with

$$X^\mp:=\{(x,\pm ix);x\in X\}.$$

Define the relation $U:=\{(y,z)\in X\times X;(y,iy)+(z,-iz)\in A\}$. By [**22**, Lemma 3], $\widetilde{U}$ is a partial isometry. If A is self-adjoint, $\operatorname{dom}(U)=\operatorname{im}U=X$ and $U\in\mathcal{B}(X)$ is unitary.

Given $(x,x')\in A$, $x,x'\in X$, we decompose

$$(x,x')=(y,iy)+(z,-iz)\text{ with suitable } y,z\in X.$$

We obtain at once $y = \frac{x-ix'}{2}$ and $z = \frac{x+ix'}{2}$, or, equivalently in the language of linear relations,

$$y \in \frac{I-iA}{2} \cdot x \text{ and } z \in \frac{I+iA}{2} \cdot x\,. \tag{4.6}$$

Inverting the y–formula in (4.6) yields

$$z \in (I-iA)^{-1}(I+iA) \cdot y. \tag{4.7}$$

Conversely, if (4.7) holds, we get $(x,x') \in A$. Then we have $U = (I-iA)^{-1}(I+iA) = -\kappa(A)$. By functional calculus we have $\sigma(\kappa(A)) = \kappa(\sigma(A))$.
To (c), (d): By (a), (b). □

We have the following basic coincidence of spectral flow and Maslov index.

PROPOSITION 4.3. *Let $E \to [0,1]$ be a Hilbert bundle with $X(s) := p^{-1}(s)$, continuous varying inner product $\langle\cdot,\cdot\rangle_s$ on X, and $\{s \mapsto A_s \in \mathcal{S}^c(X \times X)\}_{s\in[0,1]}$ a continuous curve of self-adjoint Fredholm relations of X_s. Then we have*

$$\operatorname{sf}\{A(s)\} = \operatorname{Mas}_-\{A(s), X(s) \times \{0\}\}.$$

PROOF. Note that $X(s) \times \{0\} = \operatorname{graph}(0)$. By Lemma 4.2 and Definition 2.1 we have

$$\begin{aligned}\operatorname{sf}\{A(s)\} &= \operatorname{sf}_{\ell_+}(\kappa(A(s))\} = \operatorname{sf}_{\ell_+}\left(-\kappa(A(s))(-\kappa(0))^{-1}\right)\}\\ &= \operatorname{Mas}_-\{A(s), X(s) \times \{0\}\}.\end{aligned}$$ □

The proposition leads to the following definition.

DEFINITION 4.4. Let $p\colon E \to [0,1]$, $\widetilde{p}\colon \widetilde{E} \to [0,1]$ be Banach bundles with fibers $X(s) := p^{-1}(s)$, $Y(s) := (\widetilde{p})^{-1}(s)$ for each $s \in [0,1]$ respectively. Let $\Omega(s)\colon X(s) \times Y(s) \to \mathbb{C}$ be a path of bounded non-degenerate sesquilinear form, and $\omega(s)$ is the weak symplectic structure on $Z(s) := X(s) \times Y(s)$ induced by $\Omega(s)$. Let $A(s)$, $s \in [0,1]$ be a path of self-adjoint Fredholm linear relations of index 0. By [22, Lemma 16], we have $\operatorname{index}(A_s, X(s) \times \{0\}) = 0$. The *spectral flow* of $A(s)$ is defined by

$$\operatorname{sf}\{A(s)\} := \operatorname{Mas}_-\{A(s), X(s) \times \{0\}\}. \tag{4.8}$$

4.3. Symplectic analysis of operators and relations. Let X, Y be two vector spaces and let $\Omega\colon X \times Y \to \mathbb{C}$ be a sesquilinear non-degenerate map. Set $Z := X \times Y$. Let ω be defined by (4.3) Then (Z,ω) is a symplectic Banach space with two Lagrangian subspaces $X \times \{0\}$ and $\{0\} \times Y$.

PROPOSITION 4.5. *Let $A \subset W \subset Z$ be two linear relations. Assume that* $\operatorname{index}(A) + \operatorname{index}(A^\omega) = 0$. *Then we have* $\dim\ker W^\omega = \dim Y/\operatorname{im} W$ *and* $\operatorname{im} W = \operatorname{im} W^{\omega\omega}$.

PROOF. We apply Proposition 1.29, taking our Z as the underlying symplectic vector space and setting $\lambda := A$ and $\mu := X \times \{0\}$. Then we have $\dim(W^\omega \cap X \times \{0\}) = \dim(Z/(W + X \times \{0\}))$ and $W + X \times \{0\} = W^{\omega\omega} + X \times \{0\}$. By [22, Lemma 16] and its proof, our results follow. □

Let $A_m \in \mathcal{C}(X, Y)$ be a closed linear operator with $(\operatorname{dom}(A_m))^{\Omega,r} = \{0\}$. Then the adjoint relation A_m^ω is (the graph of) a closed operator. We assume that A_m is symmetric, i.e., $A_m^\omega \supset A_m$. We denote the domains of A_m by D_m (the *minimal* domain) and of A_m^ω by $D_{\max}$ (the *maximal* domain). We have (see [14] for the symmetric operator on Hilbert space):

(i) The space $D_{\max}$ is a Banach space with the graph norm

$$\|x\|_{\mathfrak{G}} := \|x\|_X + \|A_m^\omega x\|_Y \quad \text{for } x \in D_{\max}\,. \tag{4.9}$$

(ii) The space D_m is a closed subspace in the graph norm and the quotient space $D_{\max}/D_m$ is a Banach space with the minus Green's form

$$\widetilde{\omega}(x + D_m, y + D_m) := \Omega(x, A_m^\omega y) - \overline{\Omega(y, A_m^\omega x)} \text{ for } x, y \in D_{\max}\,. \tag{4.10}$$

The form is symplectic if and only if $A_m^{\omega\omega} = A$.

(iii) Let $B \supset A_m$ be an extension of A_m. By Lemma 1.24, the operator B is self-adjoint if and only if $A_m^{\omega\omega} \subset B \subset A_m^\omega$, and

$$R_{\operatorname{graph}(A_m^\omega)}(\operatorname{graph}(B)) \in \mathcal{L}\left(\operatorname{graph}(A_m^\omega)/\operatorname{graph}(A_m^{\omega\omega})\right).$$

(iv) We denote by γ the natural projection

$$\gamma\colon D_{\max} \longrightarrow D_{\max}/D_m.$$

For any linear subspace $D \subset X$, set

$$\gamma(D) := (D \cap D_{\max} + D_m)/D_m.$$

In our applications, we consider families of self-adjoint Fredholm operators with varying domain and varying maximal domain. To us, there is no natural way to identify the different symplectic spaces and to define continuity of Lagrangian subspaces and continuity of symplectic forms in these varying symplectic spaces. Fortunately, in most applications the minimal domain is fixed and also an intermediate (reduced) Hilbert space D_M, typically the Sobolev space H^d for elliptic differential operators of order d. We shall show that meaningful modifications of the preceding statements can be obtained when we replace $D_{\max}$ by this reduced (intermediate) space D_M under the following assumptions.

ASSUMPTIONS 4.6. (i) As at the beginning of this section, we let X, Y be two Banach spaces and $\Omega\colon X\times Y\to\mathbb{C}$ a bounded sesquilinear non-degenerate map. We set $Z:=X\times Y$ and let ω be defined by (4.3).
(ii) Our data are now four Banach spaces with continuous inclusions

$$D_m\hookrightarrow D_M\hookrightarrow D_{\max}\hookrightarrow X,$$

where the Banach space structure is given on $D_{\max}$ and D_m by the graph inner product of a fixed closed symmetric operator $A_m\in\mathcal{C}(X,Y)$ with $\operatorname{dom}(A_m)=D_m$. Assume that $(D_m)^{\Omega,r}=\{0\}$.
(iii) We assume that the $(A_M)^\omega=A_m$, where $A_M:=A_m^\omega|_{D_M}$.
(iv) Finally, we assume that there exists a self-adjoint Fredholm extension A_D of A_m of index 0 with domain $D_m\subset D\subset D_M$.

Assumption 4.6 (ii) implies

$$\|x\|_{\mathfrak{G}}=\|x\|_X+\|A_Mx\|_X\le C_1\|x\|_{D_M}\text{ for all }x\in D_M\,. \tag{4.11}$$

In particular, it follows that $A_M\colon D_M\to X$ is bounded.

By Proposition A.35.a, D_m is closed in D_M, and on D_m the graph norm and the norm induced by the Banach space D_M are equivalent. Then we have the opposite estimate to (4.11), namely

$$\|x\|_{D_M}\le C_2\big(\|x\|_X+\|A_Mx\|_X\big)=C_2\|x\|_{\mathfrak{G}}\text{ for all }x\in D_m\,. \tag{4.12}$$

By (1.3), Assumption 4.6 (iii) implies $A_m^{\omega\omega}=A_m$.

LEMMA 4.7. *Under Assumptions 4.6 (a), (b), (c) the quotient space D_M/D_m is a weak symplectic Banach space with the symplectic form induced by the minus Green's form on $D_{\max}$.*

PROOF. By Lemma 3.20. □

The lemma shows that any intermediate space D_M satisfying Assumptions 4.6 (i), (ii), (iii) is big enough to permit a meaningful symplectic analysis on the reduced quotient space D_M/D_m. The point of this construction is that the norm in D_M/D_m does not come from the graph norm in $D_{\max}$ but from the norm of D_M. Therefore, it can be kept fixed even when our operator varies. The *symplectic structure* of D_M/D_m, however, is induced by the minus Green's form and therefore will change with varying operators.

In [14, Proposition 3.5], in the spirit of the classical von-Neumann program, self-adjoint Fredholm extensions were characterized by the property that their domains, projected down into the strong symplectic space $\boldsymbol{\beta}(A_m):=D_{\max}/D_m$ of abstract boundary values, make Fredholm pairs of Lagrangian subspaces with the abstract, reduced Cauchy data space $(\ker A_m^*+D_m)/D_m$. Immediately, this does not help for operator families with varying maximal domain. Surprisingly, however, the arguments generalize to the weak symplectic space D_M/D_m intrinsically, i.e., without additional topological conditions.

LEMMA 4.8. *Denote by* $P_X\colon Z = X\times Y \to X$ *the projection onto the first component. Set* $W := \operatorname{graph}(A_M)$, $\lambda := \operatorname{graph}(A_D)$ *and* $\mu := X\times\{0\}$. *Then* P_X *induces a symplectic Banach isomorphism* $\widetilde{P}_X\colon W/W^\omega \to D_M/D_m$, *and we have*

$$\widetilde{P}_X(R_W(\lambda)) \;=\; \gamma(D), \qquad \widetilde{P}_X(R_W(\mu)) \;=\; \gamma(\ker A_M).$$

PROOF. By definition and direct calculation. □

PROPOSITION 4.9. *Under the Assumptions 4.6, the quotient space* D/D_m *and the* reduced Cauchy data space $(\ker A_M + D_m)/D_m$ *form a Fredholm pair of Lagrangian subspaces of the (weak) symplectic Banach space* D_M/D_m *with index* 0, *and* $\dim\ker A_m = \dim Y/(\operatorname{im} A_M)$. *Moreover, it follows that* $\operatorname{im} A_M = \operatorname{im} A_m^\omega$.

PROOF. Take Z as the symplectic vector space, $W := \operatorname{graph}(A_M)$, $\lambda := \operatorname{graph}(A_D)$ and $\mu := X\times\{0\}$. By Lemma 4.8 and Proposition 1.30, our results follow. □

REMARK 4.10. If we remove the topological requirements in Assumptions 4.6, the algebraic results of Lemma 4.7, Lemma 4.8 and Proposition 4.9 still hold.

4.4. Proof of the abstract spectral flow formula. In this section we prove an abstract spectral flow formula by Theorem 3.22. We shall make the following new assumptions. They are all natural in our applications, as we shall see later in Section 4.5.

ASSUMPTION 4.11. (i) Let $r_0\colon G_0\to[0,1]$, $r\colon G\to[0,1]$, $p\colon E\to[0,1]$, and $\widetilde{p}\colon \widetilde{E}\to[0,1]$ be Banach bundles with fibers $r_0^{-1}(s) := D_m(s)$, $r^{-1}(s) := D_M(s)$, $p^{-1}(s) := X(s)$ and $\widetilde{p}^{-1}(s) := Y(s)$ for each $s\in[0,1]$ respectively. Assume that we have Banach subbundle maps $G_0\to G$, $G\to E$.
(ii) Let $\Omega(s)\colon X(s)\times Y(s)\to\mathbb{C}$ be a path of bounded non-degenerate sesquilinear forms, and denote by $\omega(s)$ the weak symplectic structure on $Z(s) := X(s)\times Y(s)$ induced by $\Omega(s)$. Assume that $(D_m(s))^{\Omega,r} = \{0\}$, which implies that the adjoint of an operator with domain $D_m(s)$ is an operator.
(iii) Let $A_m(s)\colon X(s)\supset D_m(s)\to Y(s)$ be a family of closed symmetric operators such that the norm on $D_m(s)$ is equivalent to the graph norm of $D_m(s)$ defined by $A_m(s)$. Assume that there exists a constant integer k such that $\dim\ker A(s) = k$. Set $A_M(s) := A_m(s)^{\omega(s)}|_{D_M(s)}$. Assume

that $(A_M(s))^{\omega(s)} = A_m(s)$, and $A_M(s) \in \mathcal{B}(D_M(s), Y(s))$ is a path of bounded operators.
(iv) Let $D(s) \in \mathcal{S}(D_M(s))$ be a path of closed subspaces with $D_m(s) \subset D(s) \subset D_M(s)$. Assume that $A(s, D(s)) := A_M(s)|_{D(s)}$, $s \in [0,1]$ is a family of self-adjoint operators of index 0.

Theorem 4.12 (Abstract spectral flow formula). *Under Assumption 4.11, we have the following.*
(a) We have $\operatorname{im} A_M(s) = \operatorname{im}(A_m(s))^{\omega(s)}$ *and* $\dim Y(s)/(\operatorname{im} A_M(s)) = k$ *holds for each* $s \in [0,1]$.
(b) The family

$$\big(\gamma(D(s)), \gamma(\ker A_M(s))\big)$$

is a path of Fredholm pairs of Lagrangian subspaces of the symplectic Banach space $D_M(s)/D_m(s)$ *of index* 0.
(c) We have

$$\operatorname{sf}\{A(s, D(s))\} = -\operatorname{Mas}\{\gamma(D(s)), \gamma(\ker A_M(s)); -\widetilde{\omega}(s)\} \tag{4.13}$$

$$= -\operatorname{Mas}\{\gamma(\ker A_M(s), \gamma(D(s))); \widetilde{\omega}(s)\}. \tag{4.14}$$

Proof. Let $s \in [0,1]$. By [22, Lemma 16], the pair

$$(\operatorname{graph}(A(s, D(s)), X(s) \times \{0\})$$

is a Fredholm pair of Lagrangian subspaces of the symplectic Banach space $(Z(s), \omega(s))$. By Lemma 4.7, the quotient space $D_M(s)/D_m(s)$ is a weak symplectic Banach space with the symplectic form induced by the minus Green's form on $D_{\max}(s)$. By Proposition 4.9, (a) holds and the pair

$$\big(\gamma(D(s)), \gamma(\ker A_M(s))\big)$$

is a Fredholm pair of Lagrangian subspaces of the symplectic Banach space

$$(D_M(s)/D_m(s), \widetilde{\omega}(s)) \text{ of index 0 for each } s \in [0,1].$$

By Proposition A.35, the norm on $D_m(s)$ is uniformly equivalent to the graph norm of $D_m(s)$ defined by $A_m(s)$.

By Lemma A.9, $\gamma(D(s)) \subset \mathcal{S}(D_M(s)/D_m(s))$, $s \in [0,1]$ is a continuous path. By Corollary A.37, $\{A(s, D(s)) \in \mathcal{C}(X(s), Y(s))\}$, $s \in [0,1]$ is a continuous family.

We shall use the following notations:

$$W_0(s) := \operatorname{graph}(A_m(s)), W(s) := \operatorname{graph}(A_M(s)), W_1(s) := D_m(s) \times Y(s)$$
$$\widetilde{W}(s) := D_W(s) \times Y(s), \mu(s) := X(s) \times \{0\},$$
$$F_0 := \bigcup_{s\in[0,1]} W_0(s), F := \bigcup_{s\in[0,1]} W(s), F_1 := \bigcup_{s\in[0,1]} W_1(s),$$
$$\widetilde{F} := \bigcup_{s\in[0,1]} \widetilde{W}(s), \mathcal{E} := \bigcup_{s\in[0,1]} Z(s).$$

Then we have $W_0(s) \in \mathcal{S}^c(W_1(s))$, $W(s) \in \mathcal{S}^c(\widetilde{W}(s))$. By Lemma A.32.b (see also [67, Lemma 0.2]), F_0 is a subbundle of F_1, and F is a subbundle of $\widetilde{F}$. The bundle $\widetilde{F}$ is a subbundle of $\mathcal{E}$, and we have $\widetilde{W}(s) + \mu(s) = Z(s)$. By Definition 4.4, Theorem 3.22, Lemma 4.8 and Proposition 2.3.d,f, we have

$$\begin{aligned}
\operatorname{sf}\{A(s, D(s))\} &= \operatorname{Mas}_-\{\operatorname{graph}(A(s, D(s)), X(s) \times \{0\})\} \\
&= \operatorname{Mas}_-\{R_W(s)^{\omega(s)}(\operatorname{graph}(A(s, D(s))), R_W(s)^{\omega(s)}(X(s) \times \{0\})\} \\
&= \operatorname{Mas}_-\{\gamma(D(s)), \gamma(\ker A_M(s)); \widetilde{\omega}(s)\} \\
&= -\operatorname{Mas}\{\gamma(D(s)), \gamma(\ker A_M(s)); -\widetilde{\omega}(s)\} \\
&= -\operatorname{Mas}\{\gamma(\ker A_M(s)), \gamma(D(s)); \widetilde{\omega}(s)\}. \qquad \square
\end{aligned}$$

4.5. An application: A general desuspension formula for the spectral flow of families of elliptic boundary value problems. Having expanded weak symplectic linear algebra and analysis to some length and detail in the two preceding sections, we shall turn to the geometric setting and the geometric applications.

Consider a (big) Hermitian vector bundle $\mathbb{E}$ over a (big) compact Hausdorff space $\mathbb{M}$. We assume that $\mathbb{M}$ itself is a fiber bundle over the interval $[0,1]$ such that $\mathbb{M}$ is a continuous family of compact smooth Riemannian manifolds $j(s)\colon M(s) \hookrightarrow \mathbb{M}$ with boundary Σ_s. We require that the vector bundle structure is compatible with the boundary part. More precisely, we shall have a trivialization

$$\varphi\colon M(0) \times [0,1] \simeq \mathbb{M}$$

such that $\pi \circ \varphi^{-1} \circ j(s)\colon (M(s), \Sigma(s)) \to (M(0), \Sigma(0))$ is a diffeomorphism. Here π denotes the natural projection $\pi\colon M(0) \times [0,1] \to M_0$. We do not assume that M_s or $\Sigma(s)$ are connected. Note that the trivialization define smooth structures on $\operatorname{im} \varphi|_{(M(0)\setminus\Sigma(0))\times[0,1]}$ and so on $\mathbb{M}$ and $\mathbb{E}$.

Let $E(s) \to M(s)$ be the induced bundle, i.e., the pull back $j(s)^*(\mathbb{E})$. Denote by $C_0^\infty(M(s); E(s))$ the space of smooth sections with support in the interior $M(s)^0 := M(s) \setminus \Sigma(s)$ of $M(s)$. Assume that $d > 0$ is a positive integer and $\sigma \geq 0$ a non-positive real (on manifolds with boundary, Sobolev spaces of negative order are a nuisance and shall be avoided here). We define the Hilbert space

$$H_0^\sigma(M(s); E(s)) := \overline{C_0^\infty(M(s); E(s))}^{H^\sigma(M(s);E(s))}.$$

Here $H^\sigma(M(s); E(s))$ denotes the Sobolev space of order σ defined in [19, Chapter 11] as the restrictions to $M(s)$ of sections belonging to $H^\sigma(\widetilde{M(s)}; \widetilde{E(s)})$, where $\widetilde{E(s)} \to \widetilde{M(s)}$ is a smooth extension of the given vector bundle $E(s) \to M(s)$ over a smooth closed extension of $M(s)$, e.g., the closed double. C. Frey [43, p. 14] has shown that these definitions of $H^\sigma(M(s); E(s)), H_0^\sigma(M(s); E(s))$ coincide with the definitions given in J.-L. Lions and E. Magenes [57, Chapter 9]. The inner product is given by the Sobolev inner product. Set

(4.15)
$$D_m(s;\sigma) \ := \ H_0^{\sigma+d}(M(s); E(s)), \qquad D_M(s;\sigma) \ := \ H^{\sigma+d}(M(s); E(s)),$$
(4.16)
$$X(s) = Y(s) := L^2(M(s); E(s)), \quad S(s;\sigma) := \sum_{j=1}^{d} H^{\sigma+d-j}(\Sigma(s); (E(s))),$$
(4.17)
$$G_0 \ := \ \bigcup_{s\in[0,1]} D_m(0,s), \qquad G \ := \ \bigcup_{s\in[0,1]} D_M(0,s), \qquad \widetilde{E} \ := \ \bigcup_{s\in[0,1]} X(s).$$

Then G_0, G, $\widetilde{E}$ have Banach bundle structures over $[0,1]$, and the natural inclusions $G_0 \to G$, $G \to \widetilde{E}$ are Banach subbundle maps. So Assumption 4.11 (i) holds.

By the trace theorem, for $\sigma > -1/2$ we have

$$S(s;\sigma+1/2) \ \cong \ (D_M(s;\sigma))/(D_m(s;\sigma)). \tag{4.18}$$

Let $\Omega(s)\colon X(s) \times X(s) \to \mathbb{C}$ denote the L^2 inner product, and let $\omega(s)$ be the strong symplectic structure on $Z(s) := X(s) \times X(s)$ induced by $\Omega(s)$. Since $D_m(0,s)$ is dense in $X(s)$, we have $(D_m(0,s))^{\Omega,r} = (D_m(0,s))^\perp = \{0\}$. So Assumption 4.11 (ii) holds. For any closed operator $T \in \mathcal{C}(X(s))$, we have $T^* = T^{\omega(s)}$.

We consider a smooth linear differential operator $\mathbb{A}\colon C^\infty(\mathbb{M};\mathbb{E}) \to C^\infty(\mathbb{M};\mathbb{E})$ which induces a smooth family of elliptic differential operators $A(s)$ of order $d > 0$

$$A(s)\colon C_0^\infty(M(s);E(s)) \longrightarrow C^\infty(M(s);E(s)). \tag{4.19}$$

For each $s \in [0,1]$ and $\sigma \ge 0$, the operator $A(s)$ extends to a bounded operator

$$A_m(s;\sigma)\colon D_m(s;\sigma) \to H^\sigma(M(s);E(s)), \tag{4.20}$$

$$A_M(s;\sigma)\colon D_m(s;\sigma) \to H^\sigma(M(s);E(s)). \tag{4.21}$$

For each $s \in [0,1]$, by the interior elliptic estimate for $A(s)$, the operator

$$A_m(0,s)\colon X(s) \supset D_m(0,s) \ \longrightarrow \ X(s)$$

is a closed operator, and the graph norm on $D_m(0,s)$ defined by $A_m(0,s)$ is equivalent to the Sobolev norm (see, e.g., [**19**, Proposition 20.7] for the first order case and [**43**, Proposition 1.1.1] in the higher order case). The family $\{A_M(0,s) \in \mathcal{B}(D_M(0,s),X(s))\}$, $s \in [0,1]$ is a path of bounded operators. The family of the minus Green's form $\widetilde{\omega}(s)$ defined on $S(s,\Sigma)$ is a continuous family of bounded non-degenerate sesquilinear form for $\sigma \ge 0$. The form is invertible for $\sigma = 0$, and it is not well-defined for $\sigma < 0$. We have $(A_M(s;0))^* = A_m^t(s;0)$, where $A^t(s)$ denotes the formal adjoint of $A(s)$.

Let $Q(s;\sigma)\colon S(s;\sigma) \to S(s;\sigma)$ denote the projection defined by the orthogonal pseudo-differential Calderón projection $Q(s)$ belonging to the operator A_s. For the construction of the Calderón projection (first depending on choices and then orthogonalized), we refer to R.T. Seeley [**80**, Section 4], [**81**, Theorem 1]. There it is shown that it is a pseudo-differential idempotent with the Cauchy data space as its range. Recall that the Cauchy data space of a differential operator A of order d consists of the closure in $S(\sigma)$ of the array of derivatives up to order $d-1$ in normal direction along the boundary. For operators of Dirac type, Seeley's definition was worked out and made canonical in [**19**, Chapter 12], see also [**43**, Section 2.3] for elliptic differential operators of arbitrary order.

Proposition 4.13 (Continuity of Calderón projection). *Assume that*

$$\dim\ker A_m(s;0) = k \ \text{and} \ \dim\ker A_m^t(s;0) = l$$

are independent of s and $\sigma \ge -1/2$. Then the family $\{\operatorname{im} Q(s;\sigma)\}$ is continuous.

REMARK 4.14. If $d = 1$, A_s are of Dirac type and $M(s)$ has product structure near $\Sigma(s)$ compatible with the fiber structure for each s, then, by the construction, the projectors $Q(s)$ form a continuous family of pseudo-differential projectors and the family $\{\operatorname{im} Q(s;\sigma)\}$ is continuous for all real σ.

PROOF. By (4.18), for $\sigma > 0$ we have

$$\operatorname{im} Q(\sigma, s) = \gamma(\ker A_M(s; \sigma - 1/2)). \tag{4.22}$$

Let $\sigma \geq 1/2$. Since $A(s)$ is elliptic, $\ker A_m(s;\sigma-1/2) = \dim\ker A_m(s;0)$ and $\ker A_m^t(s;\sigma-1/2) = \ker A_m^t(s;0)$ consist of smooth sections. Since $\dim\ker(A^t(s))_m^0 = l$ is constant, by Corollary A.23, $\{\ker A_M(s;\sigma - 1/2)\}$, $s \in [0,1]$ is a continuous family . Since $\operatorname{im} Q(\sigma, s)$ is closed, the subspace $\ker A_M(s;\sigma-1/2)+D_m(s;\sigma-1/2)$ is closed in $D_M(s;\sigma-1/2)$. by Lemma A.9 and Corollary A.22, the family $\{\operatorname{im} Q(s;\sigma)\}$ is continuous.

By our [13, Theorem 5.3], based on our [18, Theorem 7.2b], the family $\{\operatorname{im} Q(s;\sigma)\}$, $s \in [0,1]$ is continuous for $\sigma \in [-1/2, 1/2]$. Then $\{\operatorname{im} Q(s;\sigma)\}$, $s \in [0,1]$ is a continuous family for all $\sigma \geq -1/2$. Note that the continuity is only proved there for the case when $d = 1$ and $k = 0$. The proof can be made much simpler by G. Grubb [48], and the proof can be easily transferred to the general case. □

We assume that all $A(s)$ are formally self-adjoint, i.e., $A_m(0,s) \subset (A_m(0,s))^*$. Note that we make no assumptions about product structures near the boundary $\Sigma(s)$. Then $(A_M(0,s))^* = A_m(0,s)$. Assume that there exists a constant integer k such that $\dim\ker A_m(0,s) = k$. Then Assumption 4.11 (iii) holds.

For each s we choose a well-posed self-adjoint boundary condition $P(s) \in \operatorname{Grass}_{\mathrm{sa}}(A(s))$ in the sense of [43, Definition 1.2.5] (see Brüning and Lesch [25] for the first order case). It is a projection $P(s;\sigma)\colon S^\sigma(s) \to S^\sigma(s)$ defined by pseudo-differential projection $P(s)$ which defines a self-adjoint Fredholm extension $A(s, P(s))$ in $X(s)$ with

$$\operatorname{dom} A(s, P(s)) \;=\; D(s) \;:=\; \{x \in D_M(0,s); P(s;1/2)(\gamma(x)) = 0\}.$$

Fix $\sigma \geq 0$. We assume $\{P(s;1/2)\}$ and $\{P(s;\sigma)\}$, $s \in [0,1]$ are continuous families. By Lemma A.9, $\{D(s)\}$, $s \in [0,1]$ is a continuous family. Then Assumption 4.11 (iv) holds.

We then have the following spectral flow formula.

THEOREM 4.15 (Desuspension spectral flow formula). *Under the above assumptions, we have the following.*
(a) We have

$$\operatorname{im} A_M(s;0) = \operatorname{im}(A_m(s;0))^* \;\text{ and }\; \dim X(s)/(\operatorname{im} A_M(s;0)) = k$$

holds for each $s \in [0,1]$.
(b) The family

$$\big(\ker P(s;\sigma), \operatorname{im} Q(s;\sigma)\big)$$

is a path of Fredholm pairs of Lagrangian subspaces of the symplectic Banach space $S(s;\sigma)$ *of index* 0.
(c) We have

$$\operatorname{sf}\{A(s, D(s))\} = -\operatorname{Mas}\{\ker P(s;\sigma), \operatorname{im} Q(s;\sigma); -\widetilde{\omega}(s)\} \tag{4.23}$$

$$= -\operatorname{Mas}\{\operatorname{im} Q(s;\sigma), \ker P(s;\sigma); \widetilde{\omega}(s)\}. \tag{4.24}$$

Remark 4.16. To us, the preceding Formulae (4.23) and (4.24) are most natural for $\sigma = 1/2$, i.e., when we evaluate the Maslov index on the right side of the formulae in the weak symplectic quotient spaces $H^{1/2}\big(\Sigma(s); E(s)|_{\Sigma(s)}\big) = H^1\big(M(s); E(s)\big) \,/H^1_0\big(M(s); E(s)\big)$. In that case the arguments are most easily derived from the abstract spectral flow formula in the preceding section. Note, however, that the two formulae remain valid for all $\sigma \geq 0$, so, in particular also for $\sigma = 0$, i.e., for calculating the Maslov index in the continuous family of the common strong symplectic Hilbert spaces $L^2\big(\Sigma(s); E(s)|_{\Sigma(s)}\big)$. The arguments are getting more involved, though, as indicated by the double continuity requirement for the boundary projections $P(s)$, namely requiring continuity both in $H^{1/2}$ and H^σ.

Proof. Since $X(s)$ is a Hilbert space and $A(s, P(s))$ is a self-adjoint Fredholm operator, $\operatorname{index} A(s, P(s)) = 0$ and $P(s)$ is a well-posed boundary value condition for $A(s)$ in the sense of [43, Definition 1.2.5]. By Theorem 4.12, (b), (c) hold for $\sigma = 1/2$ and (a) holds.

By [43, Theorem 2.1.4] and the regularity theory for elliptic operators, we have

$$\begin{aligned}&\dim(\ker P(s;\sigma) \cap \operatorname{im} Q(s;\sigma))\\ &\qquad = \dim(\ker P(s;1/2) \cap \operatorname{im} Q(s;1/2)),\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\dim S(s;\sigma)/(\ker P(s;\sigma) \cap \operatorname{im} Q(s;\sigma))\\ &\qquad = \dim S(s;1/2)/(\ker P(s;1/2) \cap \operatorname{im} Q(s;1/2)).\end{aligned} \tag{4.26}$$

Then the pair $\big(\ker P(s;\sigma), \operatorname{im} Q(s;\sigma)\big)$ is a Fredholm pair of isotropic subspaces of the symplectic Banach space $S(s;\sigma)$ of index 0, so it is a Lagrangian pair by [22, Proposition 1]. Then (b) holds.

Note that we have $S(s;\sigma) \subset S(s;1/2)$ for $\sigma \geq 1/2$ and $S(s;\sigma) \supset S(s;1/2)$ for $\sigma \in [0,1/2]$. By (4.25) and (4.25), we can apply Theorem 3.27 and obtain (c). $\square$

Now we assume that the manifold $M(s) = M(s)^+ \cup_{\Sigma(s)} M(s)^-$ is a partitioned closed manifold with a hypersurface $\Sigma(s)$. Let $\sigma \geq 0$. We denote the restrictions of $A(s)$ to the parts by $A(s)^\pm$. Note that we now have a pair of Calderón projections $(Q(s)^+, Q(s)^-)$ for each $s \in [0,1]$ with $\operatorname{im} Q(s;\sigma)^\pm$ Lagrangian subspaces in $S(s;\sigma)$ with symplectic form again defined by the minus Green's form $-\widetilde{\omega}(s)$.

THEOREM 4.17. *For the partitioned case we assume that $\sigma \geq 0$ and*

$$\dim\ker A_m^\pm(s;0) = k^\pm.$$

Then we have

$$\operatorname{sf}\{A(s)\} = \operatorname{sf}\{A^-(s, I - Q^+(s))\} \tag{4.27}$$

$$= -\operatorname{Mas}\{\operatorname{im} Q^-(s,\sigma), \operatorname{im} Q^+(s,\sigma); -\widetilde{\omega}(s)\}. \tag{4.28}$$

PROOF. Let $M^\sharp(s)$ denote the compact manifold

$$M^+(s) \sqcup M^-(s) = \big(M(s) \setminus \Sigma(s)\big) \,\cup\, \big((\Sigma(s) \sqcup (-\Sigma(s))\big)$$

with boundary

$$\partial M^\sharp(s) = \partial M^+(s) \sqcup \partial M^-(s) = \Sigma(s) \sqcup (-\Sigma(s)) =: \Sigma^\sharp(s)$$

and $E^\sharp(s) \to M^\sharp(s)$ the corresponding Hermitian bundle. For $M(s)$ and $M^a(s), \Sigma(s)$ with $a = \pm, \sharp$ we have notations $X(s)$, $X^a(s)$, $D_M(s;\sigma)$, $D_M^a(s;\sigma)$, $D_m(s;\sigma)$, $D_m^a(s;\sigma)$, $S(s,\sigma)$, and $S^\sharp(s,\sigma)$. Fixing $\Sigma(s)$ induces a decomposition

$$X(s) \cong X^+(s) \oplus X^-(s) = X^\sharp(s),$$

and for the Sobolev space

$$D_M^+(s;\sigma) \oplus D_M^-(s;\sigma) = D_M^\sharp(s,\sigma),\ D_m^+(s;\sigma) \oplus D_m^-(s;\sigma) = D_m^\sharp(s,\sigma).$$

We have the symplectic decomposition

$$(S^\sharp(s,\sigma), \widetilde{\omega}^\sharp(s)) = (S(s,\sigma) \times S(s,\sigma), \widetilde{\omega}(s) \oplus (-\widetilde{\omega}(s))).$$

Correspondingly, we obtain an operator $A(s)^\sharp$ for each $s \in [0,1]$ which is a formally self-adjoint elliptic differential operator of order d according to the assumptions made for Theorem 4.17.

For the Calderón projection of $A^\sharp$ we have

$$\operatorname{im} Q^\sharp(s) = \operatorname{im} Q^+(s) \oplus \operatorname{im} Q^-(s). \tag{4.29}$$

Let $\Delta(s;\sigma)$ denote the diagonal in $S(s;\sigma) \times S(s;\sigma)$. By Lemma 3.6, for each $s \in [0,1]$, the diagonal $\Delta(s;\sigma)$ is a Lagrangian subspace of $S^\sharp(s,\sigma)$ with respect to $\widetilde{\omega}^\sharp(s)$ and makes a Fredholm pair with each $\operatorname{im} Q^\sharp(s)$. By [43, Theorem 2.1.4], the projection of $S^\sharp(s)$ onto $\Delta(s)$ is well-posed for $A^\sharp(s)$ in the sense of [43, Definition 1.2.5] (even if it is not a pseudo-differential operator over the manifold $\Sigma^\sharp(s)$, as noticed in [54, Section 5] in the $d=1$ case).

Consequently, we have on the manifold $M^\sharp(s)$ a natural self-adjoint elliptic boundary condition (in the sense of our Theorem 4.15) defined for $A^\sharp(s)$ by the *pasting* domain

$$D^\sharp(s) := \{(x,y)\in D_M^\sharp(s;0); (\gamma^+(s))(x) = (\gamma^-(s))(y)\} \tag{4.30}$$

$$= \{(x,y)\in D_M^\sharp(s;0) \mid (\gamma^\sharp(s))(x,y)\in \Delta(s;1/2)\}, \tag{4.31}$$

where $\gamma^a(s)\colon D_M^\sharp(s;0)\to S^\sharp(s,1/2)$ denotes the trace maps for $s\in[0,1]$ and $a=\pm,\sharp$. Let $A^\sharp(s,D^\sharp(s))$ denote the operator which acts like $A^\sharp(s)$ and has domain $D^\sharp(s)$.

By these definitions and applying Proposition 3.7.b and Theorem 4.15 to the operator family $\{A^\sharp(s,D^\sharp(s))\}$ we obtain

$$\begin{aligned}
&\operatorname{sf}\{A(s)\} = \operatorname{sf}\{A^\sharp(s,D^\sharp(s))\}\\
&\overset{Th.4.15}{=} -\operatorname{Mas}\{\Delta(s;\sigma), \operatorname{im} Q^+(s;\sigma)\oplus \operatorname{im} Q^-(s,\sigma); (-\widetilde{\omega}(s))\oplus\widetilde{\omega}(s)\}\\
&\overset{(3.9)}{=} -\operatorname{Mas}\{\operatorname{im} Q^-(s;\sigma), \operatorname{im} Q^+(s;\sigma); -\widetilde{\omega}(s)\}\\
&\overset{(3.8)}{=} \operatorname{Mas}\{\operatorname{im} Q^+(s;\sigma), \operatorname{im} Q^-(s;\sigma); \widetilde{\omega}(s)\}\\
&\overset{Th.4.15}{=} \operatorname{sf}\{A^-(s, I-Q^+(s))\}. \qquad \square
\end{aligned}$$

REMARK 4.18. If one is only interested in the equality (4.27), one needs not argue with the Maslov index, as we do, but can find a direct proof in [54, Corollary 5.6] based solely on the homotopy invariance of the spectral flow of a related two-parameter family.

Appendix A. Perturbation of closed subspaces in Banach spaces

This appendix serves as an introduction to the topology of closed linear subspaces in Banach spaces with applications to families of closed operators with nested domains and perturbations of Fredholm pairs. Denote by $\mathcal{S}(X)$ ($\mathcal{S}^c(X)$) the set of all (complemented) closed linear subspaces of a Banach space X. Denote by $\mathcal{B}(X,Y)$ ($\mathcal{C}(X,Y)$) the set of all bounded operators (closed, not necessarily bounded operators) between Banach spaces X and Y. Then, we shall solve the following problems:

(I) Under what conditions do the elementary linear operations (intersection, sum and making quotients) become continuous for pairs of closed subspaces?

(II) Under what conditions do we obtain a continuous mapping $(A, D) \mapsto A_D$, where the operator A varies continuously in $\mathcal{B}(X, Y)$, the domain D varies continuously in $\mathcal{S}(X)$, and A_D denotes the restriction of A to the domain D and varies in $\mathcal{C}(X, Y)$?

(III) How can we control changes of a space of Fredholm pairs under finite or compact perturbation of one factor?

Question (I) will be answered in Propositions A.13 and A.21. Question (II) will be answered in Corollary A.37. Question (III) will be answered in Proposition A.44.

These results will be formulated and proved in general terms. We shall emphasize, however, the various applications to solving variational problems of the global analysis of elliptic operators on manifolds with boundary. Problem (I) has two immediate applications: The first application is the local stability of weak inner UCP, see Corollary A.17. The second application is the continuous variation of the Cauchy data spaces under variation of the operator under the assumption of weak inner UCP (or fixed dimension of the inner solution spaces), see Corollary A.23.

Problem (II) settles the intricate delicacies of independent variation of operator and boundary condition, yielding continuous variation of the induced Fredholm extension.

Problem (III) addresses the changes, roughly speaking, when we replace one boundary condition by another one under *small* perturbation. Here *small* means by finite or compact change of the domain, to be defined rigorously below. To give an idea of what kind of changes we are dealing with, we refer to the Grassmannian of pseudo-differential projections with the same principal symbols, that define large classes of well-posed and mutually intimately related boundary problems, as in [19].

This program requires rather detailed investigations of the topology of graphs and domains of closed operators. Our topological approach is based on the *gap* $\widehat{\delta}\colon \mathcal{S}(X) \times \mathcal{S}(X) \to \mathbb{R}_+$ and the *angular distance* $\widehat{\gamma}\colon \mathcal{S}(X) \times \mathcal{S}(X) \to [0, 1]$ (also called *minimal gap*), see Definition A.3 below. According to E. Berkson in [11], the concept of *opening* (as the *gap* was called in the 1940s and 1950s) was first introduced in Hilbert space in 1947 by M. G. Krein and M.A. Krasnosel'ski in [55]. The definition was one year later extended to arbitrary Banach spaces in [56] by M.G. Krein, M.A. Krasnosel'ski, and D.P. Mil'man. Ten years later, it was supplemented by the definition of the *minimal gap/angular distance* $\widehat{\gamma}$ in [45] by I. Gohberg and A.S. Markus.

We shall use T. Kato's [52, Chapter IV] as our general reference. We apply considerable diligence to the estimates to guarantee the sharpest versions of our invariance results. Some of the results, often in different and weaker form, can be found in the quoted original papers and the classical treatises [33, 44, 61, 66, 67, 69] by H.O. Cordes and J.-P. Labrousse, I. Gohberg and M.G. Krein, J.L. Massera and J.J. Schäffer, G. Neubauer, and J.D. Newburgh.

A.1. Some linear algebra facts. We have the following elementary fact of linear algebra.

LEMMA A.1. *Let X be a vector space and V_1, V_2, V_3 three linear subspaces. If $V_1 \subset V_3$, we have*

$$(V_1 + V_2) \cap V_3 = V_1 + V_2 \cap V_3. \tag{A.1}$$

COROLLARY A.2. *Let X be a vector space and V, X_0, X_1 three linear subspaces with $X = X_0 \oplus X_1$. Denote by $P_0\colon X \to X_0$ the projection defined by the decomposition $X = X_0 \oplus X_1$. Assume that $V \supset X_1$. Then we have $V = P_0V + X_1$. In particular, we have $V = X$ if $P_0V = X_0$.*

PROOF. Since $V \supset X_1$, by Lemma A.1 we have $V = V \cap (X_0 + X_1) = V \cap X_0 + X_1$. So we have $P_0V = V \cap X_0$, and $V = P_0V + X_1$. If $P_0V = X_0$, we have $V = X$. □

A.2. The gap topology.

DEFINITION A.3. Let X be a Banach space. Denote by $S(M)$ the unit sphere of M for any closed linear subspace M of X.
(a) For any two closed linear subspaces M, N of X, we set

$$\begin{aligned}\delta(M,N) &:= \begin{cases}\sup_{u\in S(M)} \operatorname{dist}(u,N), & \text{if } M \neq \{0\},\\ 0, & \text{if } M = \{0\},\end{cases}\\ \hat{\delta}(M,N) &:= \max\{\delta(M,N), \delta(N,M)\}.\end{aligned}$$

$\hat{\delta}(M,N)$ is called the *gap* between M and N.
(b) For any two closed linear subspaces M, N of X, we set

$$\begin{aligned}\gamma(M,N) &:= \begin{cases}\inf_{u\in M\setminus N} \frac{\operatorname{dist}(u,N)}{\operatorname{dist}(u,M\cap N)}\ (\le 1), & \text{if } M \not\subset N,\\ 1, & \text{if } M \subset N,\end{cases}\\ \hat{\gamma}(M,N) &:= \min\{\gamma(M,N), \gamma(N,M)\}.\end{aligned}$$

$\hat{\gamma}(M,N)$ is called the *minimal gap* between M and N. If $M \cap N = \{0\}$, we have

$$\gamma(M,N) = \inf_{u\in S(M)} \operatorname{dist}(u,N).$$

We have the following [24, Proposition 11.4] ([52, Lemma III.1.9] is Proposition A.4(a)).

Proposition A.4. *Let X be a Banach space and M be a closed subspace of X. Let $M' \supset M$ be a linear subspace of X with* $\dim M'/M < +\infty$. *Then we have*
(a) M' is closed, and
(b) $M \in \mathcal{S}^c(X)$ if and only if $M \in \mathcal{S}^c(X)$.

Definition A.5. (a) The space of (algebraic) *Fredholm pairs* of linear subspaces of a vector space X is defined by

$$\mathcal{F}^2_{\mathrm{alg}}(X) := \{(M,N) \mid \dim(M \cap N) < +\infty \ \text{ and} \dim X/(M+N) < +\infty\} \tag{A.2}$$

with

$$\operatorname{index}(M,N) := \dim(M \cap N) - \dim X/(M+N). \tag{A.3}$$

(b) In a Banach space X, the space of (topological) *Fredholm pairs* is defined by

$$\mathcal{F}^2(X) := \{(M,N) \in \mathcal{F}^2_{\mathrm{alg}}(X) \mid M, N, \text{ and } M+N \subset X \text{ closed}\}. \tag{A.4}$$

A pair (M,N) of closed subspaces is called *semi-Fredholm* if $M+N$ is closed, and at least one of $\dim(M \cap N)$ and $X/(M+N)$ is finite.
(c) Let X be a Banach space, $M \in \mathcal{S}(X)$ and $k \in \mathbb{Z}$. We define

$$\mathcal{F}_M(X) : = \{N \in \mathcal{S}(X); (M,N) \in \mathcal{F}^2(X)\}, \tag{A.5}$$

$$\mathcal{F}_{k,M}(X) :=\{N \in \mathcal{S}(X); (M,N) \in \mathcal{F}^2(X), \operatorname{index}(M,N) = k\}. \tag{A.6}$$

Remark A.6. Actually, in Banach space the closedness of $\lambda + \mu$ follows from its finite codimension in X in combination with the closedness of λ, μ (see [15, Remark A.1] and [52, Problem 4.4.7]).

The following lemma is from [52, Problem IV.4.6].

Lemma A.7. *Let X be a vector space and M', M, N be linear subspaces. Assume that $M' \supset M$ and $\dim M'/M = n < +\infty$. Then we have* $\operatorname{index}(M', N) = \operatorname{index}(M,N) + n$.

We give the following elementary fact.

Lemma A.8. *Let X be a Banach space and $(M,N) \in \mathcal{F}^2(X)$. Then we have $M, N \in \mathcal{S}^c(X)$.*

Proof. Since $(M,N) \in \mathcal{F}^2(X)$, there exist closed linear subspaces $M_1 \subset M$, $N_1 \subset N$ and a finite-dimensional linear subspace $V \subset X$ such that

$$M = M \cap N \oplus M_1, N = M \cap N \oplus N_1, X = V \oplus (M+N).$$

Then we have $N_1 \cap M = N_1 \cap M \cap N = \{0\}$, and

$$X = M \cap N \oplus M_1 \oplus N_1 \oplus V. \tag{A.7}$$

So $M, N \in \mathcal{S}^c(X)$ holds. $\square$

A.3. Continuity of operations of linear subspaces. We study the continuity of M/L, $M \cap N$ and $M + N$ for varying closed subspaces M and N and fixed closed subspace of a Banach space X.

For the quotient space, we have the following lemma.

Lemma A.9. *Let X be a Banach space with closed subspaces $M, N, L \in \mathcal{S}(X)$ such that $M, N \supset L$. Denote by p the natural map $p \colon X \to X/L$. Then we have*
(a) $d(p(u), p(M)) = d(u, M)$ for $u \in X$,
(b) $\gamma(p(M), p(N)) = \gamma(M, N)$,
(c) $d(u, N) \le d(u, L)\delta(M, N)$ for $u \in M$, and
(d) $\delta(M, N) = \delta(p(M), p(N))$.

Proof. (a), (b) By the last paragraph of the proof of [52, Theorem IV.4.2].
(c) Let $\varepsilon \in (0, 1)$. By [52, Lemma III.1.12], for any $u \in M$, there exists a $v \in L$ such that $d(u, L) \ge (1 - \varepsilon)\|u - v\|$. Since $L \subset N$, we have

$$d(u, N) = d(u - v, N) \le \|u - v\|\delta(M, N) \le (1 - \varepsilon)^{-1} d(u, L)\delta(M, N).$$

Let $\varepsilon \to 0$, and we have $d(u, N) \le d(u, L)\delta(M, N)$.
(d) If $M = L$, we have $\delta(M, N) = \delta(p(M), p(N)) = 0$. Assume that $M \ne L$. By definition and the first equality we have

$$\begin{aligned}\delta(M, N) &= \max\{d(u, N); u \in S(M)\} \\ &\le \max\{d(u, N); u \in M, d(u, L) \le 1\} = \delta(p(M), p(N)).\end{aligned}$$

By (a) and (c) we have

$$\delta(p(M), p(N)) = \max\{d(u, N); u \in M, d(u, L) = 1\} \le \delta(M, N).$$

Thus we obtain (d). $\square$

Firstly, we consider the case of $\dim(M \cap N) < +\infty$. We need the following uniform estimate of the given Banach norm by the coefficients with regard to a basis for finite-dimensional subspaces.

Lemma A.10. *Let X be a complex Banach space and $u_1, \ldots, u_n \in S(X)$. Set*

$$V_k := \begin{cases} \{0\}, & \text{for } k = 0, \\ \operatorname{span}\{u_1, \ldots, u_k\}, & \text{for } k = 1 \ldots, n. \end{cases}$$

Assume that $\operatorname{dist}(u_k, V_{k-1}) \geq \delta$ *for* $k = 1, \ldots, n-1$ *and* $\delta > 0$. *Then we have* $\delta \leq 1$, $\dim V_k = k$, *and*

$$\frac{1}{n}\left(\frac{\delta}{1+\delta}\right)^{n-1} \sum_{k=1}^{n} |a_k| \;\leq\; \|\sum_{k=1}^{n} a_k u_k\| \;\leq\; \sum_{k=1}^{n} |a_k|$$

for all $a_1, \ldots, a_k \in \mathbb{C}$.

Proof. Only the left inequality needs a proof. Clearly we have $1 = \|u_1\| = \operatorname{dist}(u_1, V_0) \geq \delta$. Since $\delta > 0$, we have $u_k \notin V_{k-1}$, and by induction we have $\dim V_k = k$.
Also by induction: $\|a_1 u_1\| = |a_1|$, and so

$$\begin{aligned}
\|a_1u_1 + a_2u_2\| \;&\geq\; \max\{\delta\,|a_2|, |a_1| - |a_2|\} \\
&\geq\; \max\left\{\frac{\delta}{1+\delta}|a_1|, \delta|a_2|\right\}, \ldots, \\
\|a_1u_1 + \ldots + a_nu_n\| \;&\geq \max\left\{\left(\frac{\delta}{1+\delta}\right)^{n-1}|a_1|, \left(\frac{\delta}{1+\delta}\right)^{n-k}\delta|a_k|;\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. k = 2, \ldots, n\right\} \\
&\geq\; \frac{1}{n}\left(\frac{\delta}{1+\delta}\right)^{n-1} \sum_{k=1}^{n} |a_k|.
\end{aligned}$$

Since $u_1, \ldots, u_n \in S(X)$, we have $\|\sum_{k=1}^n a_k u_k\| \;\leq\; \sum_{k=1}^n |a_k|$. □

In general, the distances $\delta(M, N)$ and $\delta(N, M)$ can be very different and, even worse, behave very differently under small perturbations. However, for finite-dimensional subspaces of the same dimension in a Hilbert space we can estimate $\delta(M, N)$ by $\delta(N, M)$ in a uniform way. We can give the following generalization of [22, Lemma 14], which is different from [67, Lemma 1.7]:

Lemma A.11. *Let* X *be a Banach space and* M, N *be two linear subspaces with* $\dim M = \dim N = n$. *Then we have*

$$\delta(M, N) \;\leq\; \frac{2^{n-1} n \delta(N, M)}{(1 - \delta(N, M))^n}\,,$$

if $1 - \delta(N, M) > 0$.

Proof. Take $\varepsilon \in (0, 1 - \delta(N, M))$. By induction and [52, Lemma IV.2.3], there exist $v_1, \ldots, v_n \in S(N)$ and $u_1, \ldots u_n \in M$ such that

$$\operatorname{dist}(v_k, V_{k-1}) \;=\; 1 \text{ and } \|u_k - v_k\| \;\leq\; \delta(N, M) + \varepsilon$$

for

$$V_k := \begin{cases} \{0\}, & \text{for } k = 0, \\ \operatorname{span}\{u_1, \ldots, u_k\}, & \text{for } k = 1 \ldots, n. \end{cases}$$

Then $1 - \delta(N, M) - \varepsilon \le \|u_k\| \le 1 + \delta(N, M) + \varepsilon$ and $\operatorname{dist}(u_k, V_{k-1}) \ge 1 - \delta(N, M) - \varepsilon$. By Lemma A.10, $V_n = M$. For any $u \in S(M)$, there exist $a_1, \ldots, a_n \in \mathbb{C}$ with $u = \sum_{k=1}^n a_k u_k$. By Lemma A.10, we also have

$$\begin{aligned} 1 \;&=\; \|\sum_{k=1}^n a_k u_k\| \;\ge\; \frac{1}{n}\Big(\frac{1 - \delta(N, M) - \varepsilon}{2}\Big)^{n-1} \sum_{k=1}^n |a_k|\|u_k\| \\ &\ge\; \frac{(1 - \delta(N, M) - \varepsilon)^n}{2^{n-1} n} \sum_{k=1}^n |a_k|. \end{aligned}$$

Set $v := \sum_{k=1}^n a_k v_k$. Then we have:

$$\begin{aligned} \|u - v\| \;&=\; \|\sum_{k=1}^n a_k(u_k - v_k)\| \;\le\; \sum_{k=1}^n |a_k|\delta(N, M) \\ &\le\; \frac{2^{n-1} n \delta(N, M)}{(1 - \delta(N, M) - \varepsilon)^n}. \end{aligned}$$

So $\delta(M, N) \le \frac{2^{n-1} n\delta(N,M)}{(1-\delta(N,M)-\varepsilon)^n}$. Let $\varepsilon \to 0$, then we have $\delta(M, N) \le \frac{2^{n-1} n\delta(N,M)}{(1-\delta(N,M))^n}$. □

The diligence with the preceding estimates pays back with the following Proposition A.13 that confines possible changes of the dimensions of intersections and the co-dimensions of sums of pairs of closed linear subspaces under variation. For that, we shall use the concepts of approximate nullity (approximate deficiency) defined by [52, §IV.4]:

Definition A.12. Let M, N be closed linear manifolds (i.e., closed subspaces) of a Banach space Z.
a) We define the *approximate nullity* of the pair M, N, denoted by $\operatorname{nul}'(M, N)$, as the least upper bound of the set of integers m ($m = +\infty$ being permitted) with the property that, for any $\varepsilon > 0$, there is an m-dimensional closed linear subspace $M_\varepsilon \subset M$ with $\delta(M_\varepsilon, N) < \varepsilon$.
b) We define the *approximate deficiency* of the pair M, N, denoted by $\operatorname{def}'(M, N)$, by $\operatorname{def}'(M, N) := \operatorname{nul}'(M^\perp, N^\perp)$.

Note. While $\operatorname{nul}(M, N) := \dim M \cap N$ and $\operatorname{def}(M, N) := \dim Z/(M + N)$ are defined in a purely algebraic fashion, the definition of $\operatorname{nul}'(M, N)$ and $\operatorname{def}'(M, N)$ depends on the topology of the underlying space Z.

Moreover, it is easy to show (see l.c., Theorems 4.18 and 4.19) that

$$\mathrm{nul}'(M,N) = \begin{cases} \mathrm{nul}(M,N), & \text{for } M+N \text{ closed},\\ +\infty, & \text{else}, \end{cases} \quad \text{and}$$

$$\mathrm{def}'(M,N) = \begin{cases} \mathrm{def}(M,N), & \text{for } M+N \text{ closed},\\ +\infty, & \text{else}. \end{cases}$$

We are now ready for the first main result of this appendix:

Proposition A.13. *Let Z be a Banach space and M, N, M', N' be closed linear subspaces. Assume that $M+N$ is closed. Then $\gamma(M,N) > 0$ by* [52, Theorem IV.4.2]*, and we have*

(a) $\delta(M' \cap N', M \cap N) \leq \frac{2}{\gamma(M,N)}(\delta(M',M) + \delta(N',N))$,
(b) $\dim(M' \cap N') \leq \mathrm{nul}'(M',N') \leq \dim(M \cap N)$ *if* $\delta(M',M)(1+\gamma(M,N)) + \delta(N',N) < \gamma(M,N)$,
(c) $\dim Z/(M' + N') \leq \mathrm{def}'(M',N') \leq \dim Z/(M + N)$ *if* $\delta(M,M') + \delta(N,N')(1+\gamma(M,N)) < \gamma(M,N)$, *and*
(d) $M' \cap N' \to M \cap N$ *if* $\dim(M' \cap N') = \dim(M \cap N) < +\infty$ *and* $\delta(M',M) + \delta(N',N) \to 0$.

Proof. (a): If $M' \cap N' = \{0\}$, we have

$$\delta(M' \cap N', M \cap N) = 0 \leq \frac{2}{\gamma(M,N)}(\delta(M',M) + \delta(N',N)).$$

If $M' \cap N' \neq \{0\}$, (a) follows from [52, Lemma IV.4.4].
(b) and (c): Similar to the proof of [52, Theorem IV.4.24].
(d): By Lemma A.11. □

We give a first application of the preceding proposition.

Assumption A.14. Assume that the following data are given:

- a compact smooth Riemannian manifold (M,g) with smooth boundary $\Sigma := \partial M$,
- Hermitian vector bundles (E,h^E) and (F,h^F) over M,
- an order $d > 0$ elliptic differential operator

$$A\colon C^\infty(M;E) \longrightarrow C^\infty(M;F), \tag{A.8}$$

- A^t denotes the formal adjoint of A with respect to the metrices g, h^E, h^F.
- Let $\sigma \geq 0$. Then $A_{m,\sigma}$ denotes the operator $A\colon H_0^{d+\sigma}(M;E) \to H^\sigma(M;E)$, and $A_{M,\sigma}$ denotes the operator $A\colon H^{d+\sigma}(M;E) \to H^\sigma(M;E)$.

The following lemma is standard in elliptic operator theory.

LEMMA A.15. *Let A satisfy Assumption A.14. Then $A_{m,\sigma}$ and $A_{M,\sigma}$ are semi-Fredholm operators,* $\ker A_{m,\sigma} = \ker A_{m,0}$ *consists of smooth sections, and we have* $\dim(H^\sigma(M;E))/(\operatorname{im} A_{M,\sigma}) = \dim\ker A^t_{m,0}$.

PROOF. By [43, Proposition A.1.4] and Gårding's inequality, $A_{m,\sigma}$ is left-Fredholm, i.e., $\dim\ker A_{m,\sigma} < +\infty$ and $\operatorname{im} A_{m,\sigma}$ is closed in $H^\sigma(M;E)$. By the regularity, $\ker A_{m,\sigma}$ consists of smooth sections and hence $\ker A_{m,\sigma} = \ker A_{m,0}$.

Denote by $C_+(A)$ the Calderón projection of A. Denote by γ the trace map. Set

$$D_\sigma := \{u \in H^{d+\sigma}(M;E); C_+(A)(\gamma(u)) = 0\}.$$

Denote by A_{D_σ} the operator $A\colon D_\sigma \to H^\sigma(M;E)$. Then A_{D_σ} is a Fredholm operator. Since $\operatorname{im} A_{M,\sigma} \supset \operatorname{im} A_{D_\sigma}$, the space $\operatorname{im} A_{D_\sigma}$ is closed and we have $\dim(H^\sigma(M;E))/(\operatorname{im} A_{D_\sigma}) < +\infty$. Then we have

$$\dim(H^\sigma(M;E))/(\operatorname{im} A_{M,\sigma}) = \dim\ker A^t_{m,\sigma} = \dim\ker A^t_{m,0}. \qquad \square$$

DEFINITION A.16. Let A satisfy Assumption A.14. The elliptic operator A is said to have *weak inner unique continuation property (UCP)* if $\ker A_{m,0} = \{0\}$.

COROLLARY A.17 (Local stability of weak inner UCP). *Let X, Y be Banach spaces and $A \in \mathcal{B}(X,Y)$ a bounded operator. Assume that* $\ker A = \{0\}$ *and* $\operatorname{im} A$ *is closed in Y. Then there exists a $\delta > 0$ such that for all $A' \in \mathcal{B}(X,Y)$ and $\|A' - A\| < \delta$, we have* $\ker A = \{0\}$.

PROOF. Set $Z := X \times Y$, $M := \operatorname{graph}(A)$, $M' := \operatorname{graph}(A')$ and $N = N' := X \times \{0\}$. By Proposition A.13.b and the proof of [22, Lemma 16], our result follows. $\square$

Now we refine our estimates to investigate the deformation behavior a bit further.

LEMMA A.18. *Let X be a Banach space and M, N be closed subspaces of X. Assume that $M \not\subseteq N$. Then for any $\varepsilon \in (0,1)$ and $u \in M\setminus N$, there exists a $u_0 \in M\setminus N$ such that* $\operatorname{dist}(u_0, N) = \operatorname{dist}(u, N)$ *and* $\operatorname{dist}(u_0, M\cap N) = \operatorname{dist}(u, M\cap N) \geq (1-\varepsilon)\|u_0\|$.

PROOF. There exists $v \in M\cap N$ such that $\operatorname{dist}(u, M\cap N) \leq (1-\varepsilon)\|u - v\|$. Set $u_0 := u - v$. $\square$

We have the following estimate. See [67, (1.4.2)] for a different estimate.

LEMMA A.19. *Let X be a Banach space and M, N, M', N' be closed linear subspaces. Assume that $\frac{1-\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} > \delta(M, M')$. Then we have*

$$\gamma(M', N') \le \frac{(1+\delta(N,N'))\gamma(M,N) + \delta(M,M') + \delta(N,N')}{\frac{1-\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} - \delta(M,M')}. \tag{A.9}$$

PROOF. **1.** If $M \subset N$, we have $\gamma(M', N') \le 1 = \gamma(M, N)$. So (A.9) holds.
2. Assume that $M \not\subseteq N$. Then for any $\varepsilon > 0$ and $u \in M \setminus N$, there exists $u' \in M'$ such that $\|u - u'\| \le \|u\|(\delta(M, M') + \varepsilon)$. By [52, Lemma IV.2.2] we have

$$\begin{aligned}\operatorname{dist}(u', M'\cap N') &\ge \operatorname{dist}(u, M'\cap N') - \|u - u'\| \\ &\ge \frac{\operatorname{dist}(u, M\cap N) - \|u\|\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} \\ &\qquad - \|u\|(\delta(M,M') + \varepsilon).\end{aligned}$$

If the right side of the inequality is larger than 0, we have $u' \in M' \setminus N'$. By [52, Lemma IV.2.2] we also have

$$\begin{aligned}\gamma(M', N') &\le \frac{\operatorname{dist}(u', N')}{\operatorname{dist}(u', M'\cap N')} \\ &\le \frac{(1+\delta(N,N'))\operatorname{dist}(u,N) + \|u\|\delta(N,N') + \|u\|(\delta(M,M')+\varepsilon)}{\frac{\operatorname{dist}(u, M\cap N) - \|u\|\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} - \|u\|(\delta(M,M')+\varepsilon)}.\end{aligned}$$

Let $\varepsilon \to 0$, and we have

$$\gamma(M', N') \le \frac{(1+\delta(N,N'))\operatorname{dist}(u,N) + \|u\|(\delta(M,M') + \delta(N,N'))}{\frac{\operatorname{dist}(u, M\cap N) - \|u\|\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} - \|u\|\delta(M,M')}.$$

By Lemma A.18, for any

$$\varepsilon_1 \in (0, 1 - \delta(M'\cap N', M\cap N) - \delta(M,M')(1 + \delta(M'\cap N', M\cap N))),$$

there exists $v \in M$ such that

$$\operatorname{dist}(v, N) \le (\gamma(M,N) + \varepsilon_1)\operatorname{dist}(v, M\cap N) \le \|v\|(\gamma(M,N) + \varepsilon_1)$$

and $\operatorname{dist}(v, M\cap N) \ge (1-\varepsilon_1)\|v\|$. Then we have

$$\begin{aligned}&\gamma(M', N') \\ &\quad\le \frac{(1+\delta(N,N'))\|v\|(\gamma(M,N)+\varepsilon_1) + \|v\|(\delta(M,M')+\delta(N,N'))}{\frac{\|v\|(1-\varepsilon_1) - \|v\|\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} - \|v\|\delta(M,M')} \\ &\quad= \frac{(1+\delta(N,N'))(\gamma(M,N)+\varepsilon_1) + \delta(M,M') + \delta(N,N')}{\frac{1-\varepsilon_1-\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)} - \delta(M,M')}.\end{aligned}$$

Let $\varepsilon_1 \to 0$, and we have

$$\gamma(M', N') \leq \frac{(1+\delta(N,N'))\gamma(M,N)+\delta(M,M')+\delta(N,N')}{\frac{1-\delta(M'\cap N', M\cap N)}{1+\delta(M'\cap N', M\cap N)}-\delta(M,M')}. \qquad \square$$

By [52, Theorem IV.4.2 and Lemma IV.4.4], we have

Corollary A.20. *Let X be a Banach space and M, N be closed linear subspaces of X. Then we have*
(a) $\limsup_{M'\to M, N'\to N}\gamma(M',N') \leq \gamma(M,N)$ *if $M+N$ is closed, and*
(b) $\lim_{M'\to M, N'\to N, M'\cap N'\to M\cap N}\gamma(M',N') = \gamma(M,N)$.

Now we are ready to investigate the deformation behavior, following some lines of [67, Lemma 1.5 (1), (2)]:

Proposition A.21. *Let $M' \to M$, $N' \to N$ and let $M+N$ be closed. Then $M'\cap N' \to M\cap N$ if and only if $M'+N' \to M+N$.*

Proof. By Theorem [52, IV.4.8], we have $\gamma(N^\perp, M^\perp) = \gamma(M,N)$. Here $N^\perp, M^\perp \subset X^*$ denote the annihilators in the dual space X^*. By [52, Theorem IV.4.8], $M^\perp + N^\perp$ is closed. So the proposition follows from the above lemma. $\square$

Corollary A.22. *Let $M' \to M$, $N' \to N$ and let $M+N$ be closed. Assume that $\dim(M'\cap N') = \dim(M\cap N) < +\infty$ or $\dim X/(M'+N') = \dim X/(M+N) < +\infty$. Then we have $M'\cap N' \to M\cap N$ and $M'+N' \to M+N$.*

Proof. If $\dim(M'\cap N') = \dim(M\cap N) < +\infty$, by Proposition A.13.d we have $M'\cap N' \to M\cap N$. By Proposition A.21 we have $M'+N' \to M+N$. If $\dim X/(M'+N') = \dim X/(M+N) < +\infty$, by [52, Section IV.4.11] we have $\dim((M')^\perp\cap(N')^\perp) = \dim(M^\perp\cap N^\perp) < +\infty$. By [52, Theorem 4.2.9, Theorem 4.4.8] and the above arguments we have $(M')^\perp\cap(N')^\perp \to M^\perp\cap N^\perp$ and $(M')^\perp+(N')^\perp \to M^\perp+N^\perp$. By [52, Theorem 4.2.9, Theorem 4.4.8] we have $M'\cap N' \to M\cap N$ and $M'+N' \to M+N$. $\square$

The following corollary generalizes [14, Theorem 3.8].

Corollary A.23 (Continuity of the family of the inner solution spaces and the Cauchy data spaces). *Let X, Y be Banach spaces and $A', A \in \mathcal{S}(X\times Y)$ be closed linear relations with $A' \to A$ and $\operatorname{im} A$ closed. If $\dim\ker A' = \dim\ker A < +\infty$ or $\dim Y/\operatorname{im} A' = \dim Y/\operatorname{im} A < +\infty$, we have $\ker A' \to \ker A$ and $\operatorname{im} A \to \operatorname{im} A'$.*

Proof. We have

$$\ker A \times \{0\} = A \cap (X\times\{0\}), \quad X\times \operatorname{im} A = A + X\times\{0\}.$$

By Corollary A.22, our results follows. □

Similar to the proof in [52, Section IV.4.5], we have (see [52, Remark IV.4.31] for discussions):

PROPOSITION A.24. *Let X be a Banach space and let (M, N) be a (semi-)Fredholm pair. Then there is a $\delta > 0$ such that $\hat{\delta}(M', M) + \hat{\delta}(N', N) < \delta$ implies that (M', N') is a (semi-)Fredholm pair and*

$$\operatorname{index}(M', N') \;=\; \operatorname{index}(M, N).$$

A.4. Smooth family of closed subspaces in Banach spaces. We begin with the definition.

DEFINITION A.25. Let X be a Banach space and B a C^k manifold, k is a nonnegative integer or $+\infty$ or ω. A map $f\colon B \to \mathcal{S}(X)$ is called C^k *at* $b_0 \in B$ if there exist a neighborhood U of b_0 and a C^k map $L\colon U \to \mathcal{B}(X)$ such that $L(b)$ is invertible and $L(b)f(b_0) = f(b)$ for each $b \in U$. f is called a C^k *map* if and only if f is C^k at each point $b \in B$. For the C^0 case we need B to be a topological space only.

By the definition we have

LEMMA A.26. *Let X be a Banach space and B a topological space. Let $f\colon B \to \mathcal{S}(X)$ be a map. If f is C^0 at $b_0 \in B$, f is continuous at b_0.*

The converse is not true in general (see [67, Lemma 0.2]).

Recall that $\mathcal{S}^c(X)$ denotes the set of complemented subspaces of a Banach space X. We omit the proof of the following standard facts.

LEMMA A.27. *Let X be a Banach space and $M \in \mathcal{S}(X)$. We have*
(a) $M \in \mathcal{S}^c(X)$ if and only if there exists a $P \in \mathcal{B}(X)$ such that $P^2 = P$ and $\operatorname{im} P = M$, and
(b) $M \in \mathcal{S}^c(X)$ if either $\dim M < +\infty$ or $\dim X/M < +\infty$.

LEMMA A.28. *Let X be a Banach space. Let $n \geq 0$ be an integer. Set $G(n, X) := \{V \in \mathcal{S}(X); \dim V = n\}$. Then the set $G(n, X)$ is open and path connected in $\mathcal{S}(X)$.*

PROOF. By [52, Corollary IV.2.6], the set $G(n, X)$ is open. Let V_1 and V_2 be in $G(n, X)$. Since $G(n, V_1 + V_2)$ is path connected, V_1 and V_2 can be joined by a path in $G(n, V_1 + V_2)$. So our result follows. □

LEMMA A.29. *Let X be a Banach space with a closed linear subspace X_1. Set $G(X, X_1) := \{M \in \mathcal{S}(X); X = M \oplus X_1\}$ (can be an empty set). Then the set $G(X, X_1)$ is an open affine subspace of $\mathcal{S}(X)$.*

PROOF. If $G(X, X_1) = \emptyset$, our results hold. Now we assume that $H(X, X_1) \neq \emptyset$. By [52, Lemma IV.4.29], the set $G(X, X_1)$ is open. Let $X_0 \in G(X, X_1)$. Denote by $\operatorname{graph}(A) := \{x + Ax; x \in X_0\}$ for all bounded operators $A \in \mathcal{B}(X_0, X_1)$. By the closed graph theorem, we have

$$G(X, X_1) = \{\operatorname{graph}(A); A \in \mathcal{B}(X_0, X_1)\}. \tag{A.10}$$

Since the topology of $G(X, X_1)$ coincides with that of $\mathcal{B}(X_0, X_1)$, our results follow. □

COROLLARY A.30. *The set $\mathcal{S}^c(X)$ is a Banach manifold. The local chart at $X_0 \in \mathcal{S}^c(X)$ is defined by the equation (A.10).*

COROLLARY A.31. *Let X be a Banach space with a closed linear subspace X_1. Then the set $G(X, X_1)$ is dense in $\mathcal{F}_{0,X_1}(X)$, and the set $\mathcal{F}_{0,X_1}(X)$ is path connected.*

PROOF. Let $M \in \mathcal{F}_{0,X_1}(X)$. Then there exist closed subspaces M_1, X_2 and a finite-dimensional subspace V such that $M = M \cap X_1 \oplus M_1$, $X_1 = M \cap X_1 \oplus X_2$, and $X = V \oplus (M + X_1)$. By (A.7) we have

$$X = M \cap X_1 \oplus M_1 \oplus X_2 \oplus V.$$

Since $\operatorname{index}(M, X_1) = 0$, we have $\dim M \cap X_1 = \dim V$. Let $A\colon M \cap X_1 \to V$ be a linear isomorphism. Set $c_1(s) := \operatorname{graph}(sA)$ for $s \in [0, 1]$. Then the path $c_1\colon [0, 1] \to \mathcal{S}(M \cap X_1 \oplus V)$ satisfies that $c_1(0) = M \cap X_1$ and $M \cap X_1 \oplus V = M \cap X_1 \oplus c_1(s)$ for each $s \in (0, 1]$. Set $c(s) := c_1(s) \oplus M_1$. Then we have $c(0) = M$ and $c(s) \in G(X, X_1)$ for $s \in (0, 1]$. So the set $G(X, X_1)$ is dense in $\mathcal{F}_{0,X_1}(X)$. By Lemma A.29, the set $\mathcal{F}_{0,X_1}(X)$ is path connected. □

The "if" part of the following Lemma A.32.b is [67, Lemma 0.2].

LEMMA A.32. *Let X be a Banach space and B a C^k manifold. For the C^0 case we need B to be a topological space only. Let $f\colon B \to \mathcal{S}(X)$ be a map. Let $b_0 \in B$ be a point. Assume that $f(b_0)$ is complemented in X. Then*
(a) f is C^k at b_0 if and only if there exist a neighborhood U of b_0 and a C^k map $P\colon U \to \mathcal{B}(X)$ such that $P(b)^2 = P(b)$ and $\operatorname{im} P(b) = f(b)$ for each $b \in U$, and
(b) f is C^0 at b_0 if and only if f is continuous at b_0.

PROOF. (a) Since $f(b_0)$ is complemented in X, there exists a projection $P_0 \in \mathcal{B}(X)$ such that $f(b_0) = \operatorname{im} P_0$. If f is C^k at b_0, there exist a neighborhood U of b_0 and a C^k map $L\colon U \to \mathcal{B}(X)$ such that $L(b)$ is invertible and $L(b)f(b_0) = f(b)$ for each $b \in U$. Define $P(b) := L(b)P_0L(b)^{-1}$. Then $P\colon U \to \mathcal{B}(X)$ is of class C^k and $\operatorname{im} P(b) = L(b) \operatorname{im} P_0 = f(b)$ for each $b \in U$.

Conversely, if there exists a neighborhood U of b_0 and a C^k map $P\colon U \to \mathcal{B}(X)$ such that $P(b)^2 = P(b)$ and $\operatorname{im} P(b) = f(b)$ for each $b \in U$, there exists a neighborhood $U_1 \subset U$ of b_0 such that $\|P(b) - P(b_0)\| < 1$. By [52, Lemma I.4.10], there exist a C^k map $L\colon U_1 \to \mathcal{B}(X)$ such that $L(b)$ is invertible, $L(b_0) = I$, and $L(b)P(b_0) = P(b)L(b)$ for each $b \in U_1$. So for $b \in U_1$, we have $L(b)f(b_0) = f(b)$.
(b) By Lemma A.26 and [67, Lemma 0.2]. $\square$

Lemma A.33. *Let X be a Banach space and B a C^k manifold. For the C^0 case we need B to be a topological space only. Let $f\colon B \to \mathcal{S}(X^*)$ be a C^k map. Assume that $\dim f(b) = n < +\infty$ for each $b \in B$. Then the map $b \mapsto f(b)^\perp$ is of class C^k.*

Proof. Fix $b_0 \in B$. Let $x_1^*, \ldots x_n^*$ be a base of $f(b_0)$. Since f is C^k, there exist a neighborhood U of b_0 and a C^k map $L\colon U \to \mathcal{B}(X)$ such that $L(b)$ is invertible and $L(b)f(b_0) = f(b)$. Set $x_k^*(b) := L(b)x_k^*$ for $k = 1, \ldots, n$ and $b \in U$. Then $x_k^*\colon U \to X^*$ is a C^k map for each $k = 1, \ldots, n$.

Since $\dim f(b_0) = n$, there exist $x_1, \ldots, x_n \in X$ such that the matrix $M(b_0)$ is invertible, where $M(b) := ((x_j^*(b))(x_k))_{j,k=1,\ldots,n}$. The map $M\colon U \to \mathrm{gl}(n, \mathbb{C})$ is of class C^k. Then there exists a neighborhood $U_1 \subset U$ of b_0 such that $\det M(b) \neq 0$. Set $N(b, x) := ((x_k^*(b))(x)x_j)_{j,k=1,\ldots,n}$. Define $P(b) \in \mathcal{B}(X)$ by $P(b)x = x - M(b)^{-1}N(b, x)$. Then $P\colon U_1 \to \mathcal{B}(X)$ is C^k, $P(b)^2 = P(b)$, and $f(b)^\perp = \operatorname{im} P(b)$ for each $b \in U_1$. By Lemma A.32, the map $b \mapsto f(b)^\perp$ is of class C^k. $\square$

A.5. Embedding Banach spaces. Let $j\colon W \to X$ be a Banach space embedding. In this subsection we study the continuous family of closed subspaces in W which is also closed in X.

Lemma A.34. *Let W, X be topological spaces and $j\colon W \to X$ a continuous injective map. Let $A \in X$ be a closed subset. Then $j^{-1}(A)$ is closed in W.*

Proof. Since $j\colon W \to X$ is an injective map, we have $j^{-1}(X \setminus A) = W \setminus j^{-1}(A)$. Since $A \in X$ is closed and j is continuous, we have $X \setminus A$ is open, $W \setminus j^{-1}(A)$ is open and $j^{-1}(A)$ is closed. $\square$

Proposition A.35. *Let W, X be Banach spaces. Let $j \in \mathcal{B}(W, X)$ be an injective bounded linear map. Let $M \subset W$ be such that $j(M) \in \mathcal{S}(X)$. Then the following hold.*
(a) M is closed in W, and the linear map $(j|_M)^{-1}\colon j(M) \to M$ is

bounded.
(b) Denote by $C(M) = \|(j|_M)^{-1}\|$. *Let* $N \subset \mathcal{S}(W)$ *be a closed linear subspace. Assume that* $M \neq \{0\}$ *and* $\delta(N, M) < (1 + \|j\|C(M))^{-1}$. *Then we have* $j(N) \in \mathcal{S}(X)$ *and*

$$C(N) \le C(M)(1 - (1 + \|j\|C(M))\delta(N, M))^{-1}. \tag{A.11}$$

(c) Under the assumptions of (b), we have

$$\hat{\delta}(j(M), j(N)) \le C(M)\|j\|(1 - (1 + \|j\|C(M))\delta(N, M))^{-1}\hat{\delta}(M, N). \tag{A.12}$$

Proof. (a) By Lemma A.34, $M = j^{-1}j(M)$ is closed. By the closed graph theorem, the linear map $(j|_M)^{-1} \colon j(M) \to M$ is bounded.
(b) Since $M \neq \{0\}$, we have $C(M) > 0$. Let $x \in N$ and $\varepsilon \in (0, 1 - (1 + \|j\|C(M))\delta(N, M))$. Then there exists a $y \in M$ such that

$$(1 - \varepsilon)(\|x\|_W - \|y\|_W) \le (1 - \varepsilon)\|x - y\|_W \le d(x, M) \le \delta(N, M)\|x\|_W.$$

So we have

$$\|x\|_W \le \frac{(1 - \varepsilon)\|y\|_W}{1 - \varepsilon - \delta(N, M)}. \tag{A.13}$$

Note that

$$\begin{aligned} C(M)^{-1}\|y\|_W - \|j(x)\|_X &\le \|j(y)\|_X - \|j(x)\|_X \\ &\le \|j(x) - j(y)\|_X \le \|j\|\|x - y\|_W. \end{aligned}$$

By (A.13) we have

$$\begin{aligned} \|j(x)\|_X &\ge C(M)^{-1}\|y\|_W - \|j\|\|x - y\|_W \\ &\ge \frac{C(M)^{-1}(1 - \varepsilon - \delta(N, M))\|x\|_W}{1 - \varepsilon} - \frac{\|j\|\delta(N, M)\|x\|_W}{1 - \varepsilon} \\ &= C(M)^{-1}\|x\|_W \left(1 - \frac{(1 + \|j\|C(M))\delta(N, M)}{1 - \varepsilon}\right). \end{aligned}$$

Let $\varepsilon \to 0$, and we have

$$\|j(x)\|_X \ge C(M)^{-1}\|x\|_W \left(1 - (1 + \|j\|C(M))\delta(N, M)\right).$$

Since N is closed in Banach space W, we have $j(N) \in \mathcal{S}(X)$ and the equation (A.11) holds.
(c) By the definition of the gap we have

$$\delta(j(M), j(N)) \le C(M)\|j\|\delta(M, N), \tag{A.14}$$

$$\delta(j(N), j(M)) \le C(N)\|j\|\delta(N, M). \tag{A.15}$$

Then we have

$$\hat{\delta}(j(M), j(N)) \le \max\{C(M), C(N)\}\|j\|\hat{\delta}(M, N). \tag{A.16}$$

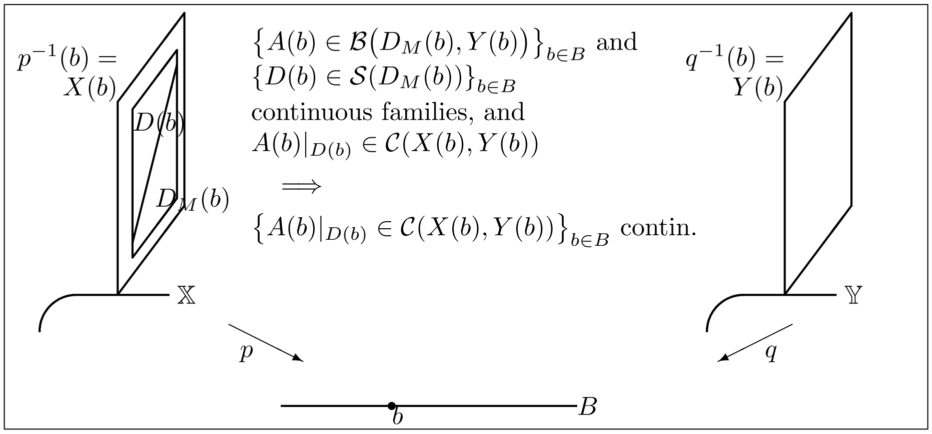

FIGURE 7. From the continuity of domains to the continuity of the operator family

By (b), our result follows. □

COROLLARY A.36. *Let B be a topological space. Let $q\colon F \to B$, $\widetilde{q}\colon \widetilde{F} \to B$, and $p\colon E \to B$ be Banach bundles with fibers $q^{-1}(b) := W(b)$, $\widetilde{q}^{-1}(b) := W(b)$ and $p^{-1}(b) := X(b)$ for each $b \in B$ respectively. Assume that we have Banach subbundle maps $F \to \widetilde{F}$, $\widetilde{F} \to E$, and there is a family $M(b) \in \mathcal{S}(X(b))$, $b \in B$ such that $M(b) \subset W(b)$ for each $b \in B$, and the family $M(b) \in \mathcal{S}(W(b))$, $b \in B$ is continuous. Then the family $M(b) \in \mathcal{S}(X(b))$, $b \in B$ is continuous.*

PROOF. By Proposition A.35.a, we have $M(b) \in \mathcal{S}(W(b))$ and $M(b) \in \mathcal{S}(\widetilde{W}(b))$ for each $b \in B$. By Proposition A.35.c, the families $M(b) \in \mathcal{S}(\widetilde{W}(b))$ and $M(b) \in X(b)$, $b \in B$ are continuous. □

The following corollary is the second main result of this appendix. It generalizes [71, Proposition B.1] and [18, Theorem 7.16].

COROLLARY A.37 (Continuity of the operator family). *Let B be a topological space and $p\colon \mathbb{X} \to B$, $p_1\colon \mathbb{X}_1 \subset \mathbb{X} \to B$, $q\colon \mathbb{Y} \to B$ three Banach bundles with fibers $p^{-1}(b) = X(b)$, $p_1^{-1}(b) = D_M(b)$ and $q^{-1}(b) = Y(b)$ for each $b \in B$ respectively. Assume that $\mathbb{X}_1$ is a subbundle of $\mathbb{X}$, and we have two continuous families $A(b) \in \mathcal{B}(D_M(b), Y(b)))$ and $D(b) \in \mathcal{S}(D_M(b))$, $b \in B$ such that $A(b)|_{D(b)}\colon X(b) \supset D(b) \to Y(b)$ is a closed operator for each $b \in B$. Then the family of operators $\{A(b)|_{D(b)} \in \mathcal{C}(X(b), Y(b))\}_{b\in B}$ is continuous.*

For the notations of Corollary A.37, see Fig. 7.

Proof. Set $W(b) := \mathrm{graph}(A(b)|D_M(b))$, $\widetilde{W}(b) := D_M(b) \times Y(b)$, $Z(b) := X(b) \times Y(b)$, $F := \bigcup_{b\in B} W(b)$, $\widetilde{F} := \bigcup_{b\in B} \widetilde{W}(b)$, and $E_2 := \bigcup_{b\in B} Z(b)$. Then we have a subbundle map $\widetilde{F} \to E_2$. Since the family $A(b) \in \mathcal{B}(D_M(b), Y(b)))$, $b \in B$ is continuous and $W(b)$ is complemented in $\widetilde{W}(b)$, we have subbundles $F \to \widetilde{F}$. Then our result follows from Corollary A.36. □

Remark A.38. (a) By Lemma A.9, our condition means $P_n \to P$ in $H^\sigma(\Sigma) \to H^\sigma(\Sigma)$ (in [71] Nicolaescu uses the notation L^2_σ for the Sobolev space $H^\sigma(\Sigma)$) for $\sigma = 1/2$. We do not require the condition for $\sigma = 0$.
(b) In [17, Theorem 3.9 (d)], it is assumed that $P_n \to P$ in $H^\sigma(\Sigma) \to H^\sigma(\Sigma)$ for $\sigma = 0$. The proof is incomplete there. For the correct proof, see [18, Theorem 7.16].

A.6. Compact perturbations of closed subspaces. Let X be a Banach space and M a closed subspace of X. In this subsection we study compact perturbations of M.

We recall the notion of relative index between projections.

Definition A.39. Let $P, Q \in \mathcal{B}(X)$ be projections and $QP\colon \operatorname{im} P \to \operatorname{im} Q$ is Fredholm. The *relative index* $[P - Q]$ is defined by

$$[P - Q] := \operatorname{index}(QP\colon \operatorname{im} P \to \operatorname{im} Q). \tag{A.17}$$

The relative index have the following properties.

Lemma A.40. *Let X be a Banach space and $P, Q, R, P_1, Q_1 \in \mathcal{B}(X)$ projections.*
(a) we have $[P - Q] = \operatorname{index}(\operatorname{im} P, \ker Q) = [(I - Q) - (I - P)]$.
(b) If $P - Q$ is compact, $QP\colon \operatorname{im} P \to \operatorname{im} Q$ is Fredholm.
(c) If $P - Q$ or $Q - R$ is compact, we have $[P - Q] + [Q - R] = [P - R]$. In particular, we have $[P - Q] = -[Q - P]$ if $P - Q$ is compact.
(d) If $PP_1 = P_1P = 0$, $QQ_1 = Q_1Q = 0$, we have $[(P + P_1) - (Q + Q_1)] = [P - Q] + [P_1 - Q_1]$.
(e) If $T \in \mathcal{B}(X, Y)$ is invertible, we have $[TPT^{-1} - TQT^{-1}] = [Q - P]$.

Proof. (a) We have $\ker(QP\colon \operatorname{im} P \to \operatorname{im} Q) = \operatorname{im} P \cap \ker Q$. Note that $\operatorname{im} P + \ker Q = Q(\operatorname{im} P) + \ker Q = \operatorname{im}(QP) + \ker P$. Then we have

$$X/(\operatorname{im} P + \ker Q) = (\operatorname{im} Q + \ker Q)/(\operatorname{im} P + \ker Q) \simeq \operatorname{im} Q/\operatorname{im}(QP).$$

So we have

$$\begin{aligned}[P - Q] &= \operatorname{index}(\operatorname{im} P, \ker Q) = \operatorname{index}(\operatorname{im}(I - Q), \ker(I - P)) \\ &= [(I - Q) - (I - P)].\end{aligned}$$

(b)-(e) See [99, Lemma 2.2,2.3]. Note that (c) follows from the proof of [99, Lemma 2.3], and (d) follows from the definition. $\square$

DEFINITION A.41. Let X be a Banach space and M, N be closed subspaces of X.
(a) We define $M \sim^f N$ if $\dim M/(M\cap N), \dim N/(M\cap N) < +\infty$, and call N a *finite change* of M (see [67, p. 273]).
(b) We define $M \sim^c N$ if there exist closed subspaces $M_1 \subset M$, $N_1 \subset N$ and a compact operator $K \in \mathcal{B}(X)$ such that $I + K$ is invertible, $N_1 = (I + K)M_1$ and $\dim M/M_1, \dim N/N_1 < +\infty$, and call N a *compact perturbation* of M. In this case we define the *relative index* $[M - N] := \dim M/M_1 - \dim N/N_1$.

LEMMA A.42. *Let X be a vector space and M, N, W three linear subspaces. If $N + W \subset M$, we have $\dim M/(N \cap W) \le \dim M/N + \dim M/W$.*

PROOF. We have

$$\begin{aligned}\dim M/(N\cap W) &= \dim M/N + \dim N/(N\cap W)\\ &= \dim M/N + \dim(N+W)/W\\ &\le \dim M/N + \dim M/W.\end{aligned}$$

$\square$

LEMMA A.43. *Let X be a Banach space and M, M_1, M_2 be closed linear subspaces. Assume that M_1, M_2 are subspaces of M with finite codimension in M, and there exists a compact operator $K \in \mathcal{B}(M_1, X)$ such that $(I_{M_1} + K)M_1 = M_2$ and $I + K \in \mathcal{B}(M_1, M_2)$ is invertible. Then there exists an invertible operator $L \in \mathcal{B}(M)$ such that $M - I_M$ is compact and $L_{M_1} = I_{M_1} + K$. In particular, we have $\dim M/M_1 = \dim M/M_2$.*

PROOF. Let V_1, V_2 be finite-dimensional subspaces of M such that $M = M_1 \oplus V_1 = M_2 \oplus V_2$. Let $A \in \mathrm{gl}(V_1, V_2)$ be a linear map. Set $L := (I_{M_1} + K) \oplus A$. Then $L \in \mathcal{B}(M)$ is Fredholm and

$$\operatorname{index} L = \operatorname{index} A = \dim V_1 - \dim V_2.$$

For any bounded set B of M, the sets $\{x \in M_1; x+v \in B \text{ for some } v \in V_1\}$ and $\{v \in V_1; x+v \in B \text{ for some } x \in M_1\}$ are bounded. Since K is compact, the set $L(B) = \{Kx+(A-I_M)v; x \in M_1, v \in V_1, x+v \in B\}$ is a sequentially compact set. Thus $L - I_M$ is compact and $\operatorname{index} M = 0$. So we have $\dim V_1 - \dim V_2 = \dim M/M_1 - \dim M/M_2 = 0$. Then we can choose A such that A is invertible. In this case L is invertible. $\square$

Now we are ready for the third main result of this appendix:

PROPOSITION A.44. *(a) The relations* $\sim^f$ *and* $\sim^c$ *are equivalence relations.*
(b) If $M \sim^c N$ *holds, the relative index* $[M-N]$ *is well-defined. In the case of* $[M-N]=0$*, there exists a compact operator* $K \in \mathcal{B}(M,X)$ *such that* $(I_M+K)M=N$ *and* $I_M+K \in \mathcal{B}(M,N)$ *is invertible.*
(c) If $M \sim^c N$ *and* $\dim M_1, \dim N_1 < +\infty$ *hold, we have* $[M-N] = -[N-M]$ *and* $[M_1-N_1] = \dim M_1 - \dim N_1$*.*
(d) If $M \sim^c N \sim^c W$ *holds, we have* $[M-N]+[N-W]=[M-W]$*.*
(e) If $M\cap M_1 = N\cap N_1 = \{0\}$*,* $\dim M_1, \dim N_1 < +\infty$*,* $M \sim^c N$ *holds if and only if* $M+M_1 \sim^c N+N_1$*. In this case we have* $[(M+M_1)-(N-N_1)] = [M-N]+[M_1-N_1]$*.*
(f) Assume that $M \in \mathcal{S}^c(X)$*. Then* $M \sim^c N$ *holds if and only if* $N \in \mathcal{S}^c(X)$*, and there exist projections* $P,Q\in\mathcal{B}(X)$ *such that* $P-Q$ *is compact,* $\operatorname{im} P = M$ *and* $\operatorname{im} Q = N$*. In this case we have* $[M-N] = [P-Q]$*. In the case of* $[P-Q]=0$*, there exists a compact operator* $K\in\mathcal{B}(X)$ *such that* $I+K$ *is invertible and* $(I+K)M=N$*.*
(g) If $M \sim^c N$ *and* $M \in \mathcal{S}^c(X)$*, we have* $\mathcal{F}_{k+[M-N],M}(X) = \mathcal{F}_{k,N}(X)$*.*

PROOF. (a) (i) If $M \sim^f N$, we have $N \sim^f M$. If $M \sim^f N \sim^f W$, we have

$$\begin{aligned}\dim(M\cap N)/(M\cap N\cap W) &= \dim(M\cap N + N\cap W)/(N\cap W)\\ &\le \dim N/(N\cap W) < +\infty.\end{aligned}$$

Then we have $\dim M/(M\cap N\cap W), \dim N/(M\cap N\cap W) < +\infty$. Similarly, we have $\dim W/(M\cap N\cap W) < +\infty$ and $M \sim^f W$.

(ii) If $M \sim^c N$, there exist closed subspaces $M_1 \subset M$, $N_1\subset N$ and a compact operator $K\in\mathcal{B}(X)$ such that $I+K$ is invertible, $N_1 = (I+K)M_1$ and $\dim M/M_1, \dim N/N_1 < +\infty$. Then $(I+K)^{-1}-I$ is compact and $M_1 = (I+K)^{-1}N_1$. So we have $M \sim^c N$.

(iii) If $M \sim^c N \sim^c W$, there exist closed subspaces M_1, N_1, N_2, W_2 and compact operators $K, L \in \mathcal{B}(X)$ such that $I+K$, $I+L$ is invertible, $N_1 = (I+K)M_1$, $W_2 = (I+L)N_2$ and

$$\dim M/M_1, \dim N/N_1, \dim N/N_2, \dim W/W_2 < +\infty.$$

By Lemma A.42 we have $\dim N/(N_1\cap N_2) < +\infty$. Set

$$\begin{aligned}M_3 &:= (I+K)^{-1}(N_1\cap N_2) \subset M_1 \subset M,\\ W_3 &:= (I+L)(N_1\cap N_2) \subset W_2 \subset W.\end{aligned}$$

Then we have that $(I+L)(I+K)-I$ is compact, $W_3 = (I+L)(I+K)M_3$, $\dim M_1/M_3 = \dim N_1/(N_1\cap N_2) < +\infty$, and $\dim W_1/W_3 = \dim N_2/(N_1\cap N_2) < +\infty$. Thus we have $\dim M/M_3, \dim W/W_3 < +\infty$

and $M \sim^c W$.
(b) Assume that we are given closed subspaces $M_j \subset M$, $N_j \subset N$ and compact operators $K_j \in \mathcal{B}(X)$, $j = 1, 2$ such that $I + K_j$ is invertible, $N_j = (I + K_j)M_j$ and $\dim M/M_j, \dim N/N_j < +\infty$ for $j = 1, 2$. Since $(I + K_2)(I + K_1)^{-1} - I$ is compact and $(I + K_j)(M_1 \cap M_2) \subset N$, by Lemma A.43 we have

$$\dim N/((I + K_1)(M_1 \cap M_2)) = \dim N/((I + K_2)(M_1 \cap M_2)).$$

So we have

$$\begin{aligned}
\dim M/M_1 - \dim M/M_2 &= \dim M_2/(M_1 \cap M_2) - \dim M_1/(M_1 \cap M_2) \\
&= \dim((I + K_2)M_2)/((I + K_2)(M_1 \cap M_2)) \\
&\quad - \dim((I + K_1)M_1)/((I + K_1)(M_1 \cap M_2)) \\
&= \dim N/((I + K_2)M_1) - \dim N/((I + K_1)M_2) \\
&= \dim N/N_1 - \dim N/N_2,
\end{aligned}$$

and therefore $\dim M/M_1 - \dim N/N_1 = \dim M/M_2 - \dim N/N_2$. Then $[M - N]$ is well-defined when $M \sim^c N$. If $[M - N] = 0$, by the proof of Lemma A.43 we get the desired K.
(c) By definition.
(d) We use the notations of the proof of (a) (iii). Then we have

$$\begin{aligned}
[M - N] - [N - K] &= (\dim M/M_3 - \dim N/(N_1 \cap N_2)) \\
&\quad + (\dim N/(N_1 \cap N_2) - \dim W/W_3) \\
&= \dim M/M_3 - \dim W/W_3 = [M - W].
\end{aligned}$$

(e) Since $M \sim^c M + M_1$ and $N \sim^c N + N_1$, by (b), $M \sim^c N$ holds if and only if $M + M_1 \sim^c N + N_1$. By (d), in this case we have

$$\begin{aligned}
[(M + M_1) - (N + N_1)] &= [(M + M_1) - M] + [M - N] \\
&\quad + [N - (N + N_1)] \\
&= \dim M_1 + [M - N] - \dim N_1 \\
&= [M - N] + [M_1 - N_1].
\end{aligned}$$

(f) (i) If $M \sim^c N$, there exist closed subspaces $M_1 \subset M$, $N_1 \subset N$ and a compact operator $K \in \mathcal{B}(X)$ such that $I + K$ is invertible, $N_1 = (I + K)M_1$ and $\dim M/M_1, \dim N/N_1 < +\infty$. Then we have $\dim((I + K)M)/N_1, \dim N/N_1 < +\infty$. Since $M \in \mathcal{S}^c(X)$, $(I + K)M \in \mathcal{S}^c(X)$. By Proposition A.4, $N_1 \in \mathcal{S}^c(X)$ and $N \in \mathcal{S}^c(X)$.

Let V_1, V_2 be finite-dimensional subspaces such that $(I + K)M = N_1 \oplus V_1$ and $N = V_2 \oplus N_1$. Set $W_1 := (I + K)\ker P$, $W_2 := \ker Q$ and

$Y := (N_1+W_1)\cap(N_1+W_2)$. By Lemma A.42 we have $\dim X/Y < +\infty$. Set $W_3 := W_1 \cap (N_1 + W_2)$ and $W_4 := W_2 \cap (N_1 + W_1)$. Then W_3 and W_4 are closed, and we have

$$Y = N_1 \oplus W_3 = N_1 \oplus W_4.$$

So $W_1/W_3 \simeq (N_1 + W_1)/Y$ and $W_2/W_4 \simeq (N_1 + W_2)/Y$ are finite-dimensional. Let $V_3 \subset W_1$, $V_4 \subset W_2$ be finite-dimensional subspaces such that

$$W_3 \oplus V_3 = W_1 \text{ and } W_4 \oplus V_4 = W_2.$$

So we have

$$\begin{aligned} X &= ((I+K)M) \oplus W_1 = (N_1 \oplus V_1) \oplus (W_3 \oplus V_3) &\text{(A.18)}\\ &= N \oplus W_2 = (N_1 \oplus V_2) \oplus (W_4 \oplus V_4) \\ &= (N_1 \oplus V_2) \oplus (W_3 \oplus V_4). &\text{(A.19)}\end{aligned}$$

The projection of X on $(I + K)M$ defined by (A.18) is $\widetilde{P} := (I + K)P(I + K)^{-1}$. Denote by $\widetilde{Q}$ the projection of X on N defined by (A.18). Then $\operatorname{im}(\widetilde{P} - \widetilde{Q}) = \widetilde{P}(V_2 \oplus V_4)$ is finite-dimensional. Since $P - \widetilde{P}$ is compact, $P - \widetilde{Q}$ is compact.

(ii) If there exist projections $P, Q \in \mathcal{B}(X)$ such that $\operatorname{im} P = M$ and $\operatorname{im} Q = N$, we set $R := QP + (I - Q)(I - P)$. Assume that R is Fredholm with index 0. Set

$$\begin{aligned} V_5 &:= \ker(QP\colon \operatorname{im} P \to \operatorname{im} Q),\\ W_6 &:= \operatorname{im}(QP\colon \operatorname{im} P \to \operatorname{im} Q),\\ V_7 &:= \ker((I-Q)(I-P)\colon \ker P \to \ker Q),\\ W_8 &:= \operatorname{im}((I-Q)(I-P)\colon \ker P \to \ker Q).\end{aligned}$$

Let W_5, W_7 be closed subspaces and V_6, V_8 finite-dimensional subspaces such that

$$\operatorname{im} P = V_5 \oplus W_5, \operatorname{im} Q = V_6 \oplus W_6, \ker P = V_7 \oplus W_7, \ker Q = V_8 \oplus W_8.$$

Then we have a bounded invertible linear map

$$\widetilde{R} := R|_{W_5+W_7}\colon W_5 + W_7 \to W_6 + W_8$$

with $\widetilde{R}(W_5) = W_6$, $\widetilde{R}(W_7) = W_8$. Since $\operatorname{index} R = 0$, we have $\dim V_5 + \dim V_7 = \dim V_6 + \dim V_8$. Let $A \in \operatorname{GL}(V_5 + V_7, V_6 + V_8)$ be invertible. Set $L := \widetilde{R} \oplus A$. Then $L \in \mathcal{B}(X)$ is invertible, $L(W_5) = W_6$ $L(W_7) = W_8$. Note that $\dim M/W_5, N/W_6 < +\infty$.

In this case, we have

$$[M - N] = \dim M/W_5 - \dim N/W_6 = \dim V_5 - \dim V_6 = [P - Q].$$

If $[P-Q]=0$, we can require $A(V_5)=V_6$, $A(V_7)=V_8$ and then $L(M)=N$.

Now assume that $P-Q$ is compact. Then $R-I=(Q-P)(2P-I)$ is compact and $\operatorname{index} R=0$. So we can apply the above argument. In this case $L-I$ is compact and our result is obtained.

(g) By Lemma A.8, we have $\mathcal{F}_{k+[M-N],M}(X), \mathcal{F}_{k,N}(X) \subset \mathcal{S}^c(X)$. Let $W \in \mathcal{S}^c(X)$. By (f), there exist projections $P, Q, R \in \mathcal{B}(X)$ such that $\operatorname{im} P = M$, $\operatorname{im} Q = N$, $\operatorname{im} R = W$, and $P-Q$ is compact. By (f) and Lemma A.40.a,c we have

$$\begin{aligned}\operatorname{index}(M,W) &= [P-(I-R)] = [P-Q]+[Q-(I-R)]\\ &= [M-N]+\operatorname{index}(N,W),\end{aligned}$$

and one side of each equality is well-defined if and only if the other side is. Thus we have $\mathcal{F}_{k+[M-N],M}(X)=\mathcal{F}_{k,N}(X)$. □

Corollary A.45. *Let X be a Banach space with a complemented closed linear subspace M. Let $P \in \mathcal{B}(X)$ be a projection onto M with $N := \ker P$. Set*

$$\mathcal{CP}_0(X,M) := \{(W \in \mathcal{S}^c(X); W \sim M, [W-M]=0\}, \tag{A.20}$$

$$\mathcal{CP}_0(P) := \{W \in \mathcal{CP}_0(X,M); X = W \oplus N\}. \tag{A.21}$$

Then
(a) the set $\mathcal{CP}_0(P)$ is an affine space (hence contractible), and
(b) the set $\mathcal{CP}_0(P)$ is dense in $\mathcal{CP}_0(X,M)$, and the set $\mathcal{CP}_0(X,M)$ is path connected.

Proof. (a) Let $W \in \mathcal{CP}_0(P)$. By Lemma A.29 we have $W = \operatorname{graph}(A)$ for some $A \in \mathcal{B}(M,N)$. Denote by P_W the projection of X onto W along N, and we have $P_W(x+y)=x+Ax$ for $x \in M$, $y \in N$. By Proposition A.44.f, A is compact. Conversely, for a given compact operator $A \in \mathcal{B}(M,N)$, the space $W := \operatorname{graph}(A) \in \mathcal{CP}_0(P)$. So we have

$$\mathcal{CP}_0(P) = \{\operatorname{graph}(A); A \in \mathcal{B}(M,N) \text{ is compact}\} \tag{A.22}$$

and the set $\mathcal{CP}_0(P)$ is an affine space (hence contractible).

Let $W \in \mathcal{CP}_0(X,M)$. By the proof of Corollary A.31, there exists a path $c\colon [0,1] \to \mathcal{S}^c(M)$ such that $c(0)=W$, $c(s) \sim^c W$ $c(s) \in G(X,N)$ and $[W-c(s)]=0$ for $s \in (0,1]$. Since $W \sim^c M$ and $[W-M]=0$, by Proposition A.44.a,d we have $c(s) \sim^c M$ and $[c(s)-M]=0$. So we have $c(s) \in \mathcal{CP}_0(X,M) \cap G(X,N) = \mathcal{CP}_0(P)$ for $s \in (0,1]$. Our results then follow from (a). □

References

[1] A. Alonso and B. Simon, 'The Birman-Kreĭn-Vishik theory of selfadjoint extensions of semibounded operators'. *J. Operator Theory* **4**/2 (1980), 251–270, addenda in *J. Operator Theory* **6** (1981), 407.

[2] W. Ambrose, 'The index theorem in Riemannian geometry'. *Ann. of Math. (2)* **73** (1961), 49–86.

[3] V. Arnol'd, 'Characteristic class entering in quantization conditions.'. *Funkts. Anal. Prilozh., Funct. Anal. Appl. (English transl.)* **1**/1 (1967), 1–14 (Russian).

[4] — *Mathematical methods of classical mechanics. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein. Corrected reprint of the second edition*, Graduate Texts in Mathematics, vol. 60. Springer-Verlag, New York, 1989, Available at http://www.math.boun.edu.tr/instructors/ozturk/eskiders/guz10m455/ClMech.pdf.

[5] M. F. Atiyah, 'Circular symmetry and stationary-phase approximation'. *Astérisque* (1985), 43–59, Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983), reprinted in [6, Vol. 5, pp.667–685].

[6] — *Collected works. Vol. 1-5*, Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1988.

[7] M. F. Atiyah, V. K. Patodi and I. M. Singer, 'Spectral asymmetry and Riemannian geometry. III'. *Math. Proc. Cambridge Philos. Soc.* **79**/1 (1976), 71–99, reprinted in [6, Vol. 4, pp.141–169].

[8] M. F. Atiyah and I. M. Singer, 'Index theory for skew-adjoint Fredholm operators'. *Inst. Hautes Études Sci. Publ. Math.* **37**/1 (1969), 5–26, reprinted in [6, Vol. 3, pp.349–372].

[9] D. Bambusi, 'On the Darboux theorem for weak symplectic manifolds'. *Proc. Amer. Math. Soc.* **127**/11 (1999), 3383–3391.

[10] C. Bennewitz, 'Symmetric relations on a Hilbert space'. In: *Conference on the Theory of Ordinary and Partial Differential Equations (Univ. Dundee, Dundee, 1972).* Springer, Berlin, 1972, pp. 212–218. Lecture Notes in Math., Vol. 280.

[11] E. Berkson, 'Some metrics on the subspaces of a Banach space'. *Pacific J. Math.* **13** (1963), 7–22.

[12] B. Bojarski, 'The abstract linear conjugation problem and Fredholm pairs of subspaces'. In: *In Memoriam I.N. Vekua, Tbilisi Univ.*, Tbilisi, 1979, pp. 45–60, Russian.

[13] B. Booss-Bavnbek, G. Chen, M. Lesch and C. Zhu, 'Perturbation of sectorial projections of elliptic pseudo-differential operators'. *J. Pseudo-Differ. Oper. Appl.* **3** (2012), 49–79, 10.1007/s11868-011-0042-5. arXiv:1101.0067v4[math.SP].

[14] B. Booss-Bavnbek and K. Furutani, 'The Maslov index: a functional analytical definition and the spectral flow formula'. *Tokyo J. Math.* **21**/1 (1998), 1–34.

[15] — 'Symplectic functional analysis and spectral invariants'. In: *Geometric aspects of partial differential equations (Roskilde, 1998)*, Contemp. Math., vol. 242. Amer. Math. Soc., Providence, RI, 1999, pp. 53–83.

[16] B. Booss-Bavnbek, K. Furutani and N. Otsuki, 'Criss–cross reduction of the Maslov index and a proof of the Yoshida–Nicolaescu Theorem'. *Tokyo J. Math.* **24** (2001), 113–128.

[17] B. Booss-Bavnbek, M. Lesch and J. Phillips, 'Unbounded Fredholm operators and spectral flow'. *Canad. J. Math.* **57**/2 (2005), 225–250. arXiv:math/0108014v3[math.FA].

[18] B. Booss-Bavnbek, M. Lesch and C. Zhu, 'The Calderón projection: new definition and applications'. *J. Geom. Phys.* **59**/7 (2009), 784–826. arXiv:0803.4160v1[math.DG].

[19] B. Booss-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1993.

[20] B. Booss-Bavnbek and C. Zhu, *Weak symplectic functional analysis and general spectral flow formula*, 2004. arXiv:math.DG/0406139.

[21] — 'General spectral flow formula for fixed maximal domain'. *Cent. Eur. J. Math.* **3**/3 (2005), 558–577 (electronic). arXiv:math/0504125v2[math.DG].

[22] — 'The Maslov index in weak symplectic functional analysis'. *Ann. Global. Anal. Geom.* **44** (2013), 283–318. arXiv:1301.7248[math.DG].

[23] R. Bott, 'On the iteration of closed geodesics and the Sturm intersection theory'. *Comm. Pure Appl. Math.* **9** (1956), 171–206.

[24] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext. Springer, New York, 2011.

[25] J. Brüning and M. Lesch, 'On boundary value problems for Dirac type operators. I. Regularity and self-adjointness'. *J. Funct. Anal.* **185**/1 (2001), 1–62.

[26] U. Bunke, 'On the gluing problem for the η-invariant'. *J. Differential Geom.* **41**/2 (1995), 397–448.

[27] — 'Index theory, eta forms, and Deligne cohomology'. *Mem. Amer. Math. Soc.* **198**/928 (2009), vi+120.

[28] S. E. Cappell, R. Lee and E. Y. Miller, 'On the Maslov index'. *Comm. Pure Appl. Math.* **47**/2 (1994), 121–186.

[29] — 'Self-adjoint elliptic operators and manifold decompositions. II. Spectral flow and Maslov index'. *Comm. Pure Appl. Math.* **49**/9 (1996), 869–909.

[30] C. Carathéodory, *Calculus of variations and partial differential equations of the first order. Transl. from the German original of 1935 by Robert B. Dean, ed. by Julius J. Brandstatter. 2nd ed.*. Chelsea Publishing Company, New York, 1982 (English).

[31] D.-C. Chang, N. Habal and B.-W. Schulze, 'The edge algebra structure of the Zaremba problem'. *J. Pseudo-Differ. Oper. Appl.* **5**/1 (2014), 69–155.

[32] P. R. Chernoff and J. E. Marsden, *Properties of infinite dimensional Hamiltonian systems*, Lecture Notes in Mathematics, Vol. 425. Springer-Verlag, Berlin, 1974.

[33] H. O. Cordes and J.-P. Labrousse, 'The invariance of the index in the metric space of closed operators'. *J. Math. Mech.* **12** (1963), 693–719.

[34] X. Dai and W. Zhang, 'Higher spectral flow'. *J. Funct. Anal.* **157**/2 (1998), 432–469.

[35] M. Daniel, 'An extension of a theorem of Nicolaescu on spectral flow and the Maslov index'. *Proc. Amer. Math. Soc.* **128**/2 (2000), 611–619.

[36] M. de Gosson, *The principles of Newtonian and quantum mechanics. The need for Planck's constant, h.*. London: Imperial College Press, 2001 (English).

[37] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990, Oxford Science Publications.

[38] A. Douady, 'Un espace de Banach dont le groupe linéaire n'est pas connexe'. *Nederl. Akad. Wetensch. Proc. Ser. A 68 = Indag. Math.* **27** (1965), 787–789.

[39] J. J. Duistermaat, 'On the Morse index in variational calculus'. *Advances in Math.* **21**/2 (1976), 173–195.

[40] I. Ekeland, *Convexity methods in Hamiltonian mechanics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 19. Springer-Verlag, Berlin, 1990.

[41] A. Floer, 'An instanton-invariant for 3-manifolds'. *Comm. Math. Phys.* **118**/2 (1988), 215–240.

[42] — 'A relative Morse index for the symplectic action'. *Comm. Pure Appl. Math.* **41**/4 (1988), 393–407.

[43] C. Frey, *On Non-local Boundary Value Problems for Elliptic Operators*, http://d-nb.info/1037490215/34, 2005, Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln.

[44] I. Z. Gohberg and M. G. Krein, 'The basic propositions on defect numbers, root numbers and indices of linear operators'. *Amer. Math. Soc. Transl. (2)* **13** (1960), 185–264.

[45] I. Z. Gohberg and A. S. Markus, 'Two theorems on the gap between subspaces of a Banach space'. *Uspehi Mat. Nauk* **14**/5 (89) (1959), 135–140.

[46] A. Gorokhovsky and M. Lesch, *On the spectral flow for Dirac operators with local boundary conditions*, 2013. arXiv:1310.0210[math.AP].

[47] M. Gromov and H. B. Lawson, Jr., 'Positive scalar curvature and the Dirac operator on complete Riemannian manifolds'. *Inst. Hautes Études Sci. Publ. Math.* (1983), 83–196 (1984).

[48] G. Grubb, 'The sectorial projection defined from logarithms'. *Math. Scand.* **111**/1 (2012), 118–126.

[49] G. Hamel, 'Die Lagrange-Eulerschen Gleichungen der Mechanik.'. *Schlömilch Z.* **50** (1904), 1–57 (German).

[50] — *Theoretische Mechanik. Eine einheitliche Einführung in die gesamte Mechanik. Ber. Reprint der Erstaufl.*, Grundlehren der mathematischen Wissenschaften - A Series of Comprehensive Studies in Mathematics, vol. 57. Springer-Verlag, Berlin-Heidelberg-New York, 1978 (German).

[51] P. Heegaard, *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang (Preliminary studies for a topological theory of the connectivity of algebraic surfaces)*, 1898 (Danish), Kjöbenhavn. 104 S. 8°, Dissertation. French translation: 'Sur l'Anlalysis Situs', *Bull. Soc. Math. France* **44** (1916) 161–242. A translation into English of the latter half of the dissertation is available at http://www.imada.sdu.dk/~hjm/agata.ps.

[52] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 19661995, Reprint of the 1980 edition.

[53] M. Katsnelson and V. Nazaikinskii, 'The Aharonov-Bohm effect for massless Dirac fermions and the spectral flow of Dirac type operators with

classical boundary conditions'. *Theor.Math.Phys.* **172** (2012), 1263–1277. arXiv:1204.2276[math.AP].

[54] P. Kirk and M. Lesch, 'The η-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary'. *Forum Math.* **16**/4 (2004), 553–629.

[55] M. G. Krein and M. A. Krasnosel'skii, 'Fundamental theorems on the extension of Hermitian operators and certain of their applications to the theory of orthogonal polynomials and the problem of moments'. *Uspehi Matem. Nauk (N. S.)* **2**/3(19) (1947), 60–106 (Russian).

[56] M. G. Krein, M. A. Krasnosel'skii and D. P. Mil'man, 'Concerning the deficiency numbers of linear operators in Banach space and some geometric questions'. *Sbornik Trudov Instit. Mat. Akad. Nauk. Ukr. S.S.R.* **11** (1948), 97–112 (Russian).

[57] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1*, Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.

[58] W. Lück, 'Analytic and topological torsion for manifolds with boundary and symmetry'. *J. Differential Geom.* **37**/2 (1993), 263–322.

[59] J. Marsden and A. Weinstein, 'Reduction of symplectic manifolds with symmetry'. *Rep. Mathematical Phys.* **5**/1 (1974), 121–130, Also on http://www.cds.caltech.edu/~marsden/bib/1974/01-MaWe1974/MaWe1974.pdf.

[60] V. Maslov, *Théorie des perturbations et méthodes asymptotiques. Suivi de deux notes complementaires de V. I. Arnol'd et V. C. Bouslaev, Traduit par J. Lascoux et R. Seneor*, Études mathématiques. Dunod Gauthier-Villars, Paris, 1972 (French), Russian original Izdat. Moskov. Univ., Moscow, 1965; updated Moskva: Nauka. 312 pp. 1988.

[61] J. L. Massera and J. J. Schäffer, 'Linear differential equations and functional analysis. I'. *Ann. of Math. (2)* **67** (1958), 517–573.

[62] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1995, Oxford Science Publications.

[63] M. Morse, *The calculus of variations in the large*, American Mathematical Society Colloquium Publications, vol. 18. American Mathematical Society, Providence, RI, 1996, Reprint of the 1932 original.

[64] N. I. Muskhelishvili, *Singular integral equations.* Noordhoff International Publishing, Leyden, 1977, Boundary problems of function theory and their application to mathematical physics, Revised translation from the Russian, edited by J. R. M. Radok, Reprinted of the 1958 edition.

[65] M. Musso, J. Pejsachowicz and A. Portaluri, 'A Morse index theorem for perturbed geodesics on semi-Riemannian manifolds'. *Topol. Methods Nonlinear Anal.* **25**/1 (2005), 69–99.

[66] G. Neubauer, 'Über den Index abgeschlossener Operatoren in Banachrämen'. *Math. Ann.* **160** (1965), 93–130 (German).

[67] — 'Homotopy properties of semi-Fredholm operators in Banach spaces'. *Math. Ann.* **176** (1968), 273–301.

[68] J. v. Neumann, 'Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren'. *Math. Ann.* **102**/1 (1930), 49–131.

[69] J. D. Newburgh, 'A topology for closed operators'. *Ann. of Math. (2)* **53** (1951), 250–255.

[70] L. I. Nicolaescu, 'The Maslov index, the spectral flow, and decomposition of manifolds'. *Duke Math. J.* **80** (1995), 485–533.

[71] — 'Generalized symplectic geometries and the index of families of elliptic problems'. *Mem. Amer. Math. Soc.* **128**/609 (1997), 1–80.

[72] G. K. Pedersen, *Analysis now*, Graduate Texts in Mathematics, vol. 118. Springer-Verlag, New York, 1989.

[73] J. Phillips, 'Self-adjoint Fredholm operators and spectral flow'. *Canad. Math. Bull.* **39**/4 (1996), 460–467.

[74] P. Piccione and D. V. Tausk, 'The Maslov index and a generalized Morse index theorem for non-positive definite metrics'. *C. R. Acad. Sci. Paris Sér. I Math.* **331**/5 (2000), 385–389.

[75] — 'The Morse index theorem in semi-Riemannian geometry'. *Topology* **41**/6 (2002), 1123–1159. arXiv:math.DG/0011090.

[76] M. Prokhorova, 'The Spectral Flow for Dirac Operators on Compact Planar Domains with Local Boundary Conditions'. *Comm. Math. Phys.* **322**/2 (2013), 385–414. arXiv:1108.0806[math-ph].

[77] J. Robbin and D. Salamon, 'The Maslov index for paths'. *Topology* **32**/4 (1993), 827–844.

[78] M. Scharlemann, 'Heegaard splittings of 3-manifolds'. In: *Low dimensional topology*, New Stud. Adv. Math., vol. 3. Int. Press, Somerville, MA, 2003, pp. 25–39.

[79] R. Schmid, *Infinite-dimensional Hamiltonian systems*, Monographs and Textbooks in Physical Science. Lecture Notes, vol. 3. Bibliopolis, Naples, 1987.

[80] R. T. Seeley, 'Singular integrals and boundary value problems.'. *Am. J. Math.* **88** (1966), 781–809.

[81] — 'Topics in pseudo-differential operators'. In: *Pseudo-Diff. Operators (C.I.M.E., Stresa, 1968)*. Edizioni Cremonese, Rome, 1969, pp. 167–305.

[82] I. M. Singer, *Personal communication*, 1999, letter, unpublished.

[83] J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de mathématiques. Dunod, Paris, 1970.

[84] R. C. Swanson, 'Fredholm intersection theory and elliptic boundary deformation problems. I'. *J. Differential Equations* **28**/2 (1978), 189–201.

[85] — 'Fredholm intersection theory and elliptic boundary deformation problems. II'. *J. Differential Equations* **28**/2 (1978), 202–219.

[86] — 'Linear symplectic structures on Banach spaces'. *Rocky Mountain J. Math.* **10**/2 (1980), 305–317.

[87] C. Vafa and E. Witten, 'Eigenvalue inequalities for fermions in gauge theories'. *Comm. Math. Phys.* **95**/3 (1984), 257–276.

[88] C. T. C. Wall, 'Non-additivity of the signature'. *Invent. Math.* **7** (1969), 269–274.

[89] — *Surgery on compact manifolds*, second ed., Mathematical Surveys and Monographs, vol. 69. American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by A. A. Ranicki.

[90] N. Waterstraat, 'A K-theoretic proof of the Morse index theorem in semi-Riemannian geometry'. *Proc. Amer. Math. Soc.* **140**/1 (2012), 337–349.

[91] A. Weinstein, 'Symplectic manifolds and their Lagrangian submanifolds'. *Advances in Math.* **6** (1971), 329–346.

[92] K. P. Wojciechowski, 'Spectral flow and the general linear conjugation problem'. *Simon Stevin* **59**/1 (1985), 59–91.

[93] — 'The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian'. *Comm. Math. Phys.* **201**/2 (1999), 423–444.

[94] T. Yoshida, 'Floer homology and splittings of manifolds'. *Ann. of Math. (2)* **134**/2 (1991), 277–323.

[95] M. G. Zaĭdenberg, S. G. Kreĭn, P. A. Kučment and A. A. Pankov, 'Banach bundles and linear operators'. *Uspehi Mat. Nauk* **30**/5(185) (1975), 101–157, English cover-to-cover translation in Russian Math. Surveys.

[96] C. Zhu, *Maslov-type index theory and closed characteristics on compact convex hypersurfaces in* $\mathbb{R}^{2n}$, PhD Thesis, Nankai Institute, Tianjin, 2000, in Chinese.

[97] — 'A generalized Morse index theorem'. In: *Analysis, geometry and topology of elliptic operators*. World Sci. Publ., Hackensack, NJ, 2006, pp. 493–540.

[98] — *The Morse index theorem for regular Lagrangian systems*, Leipzig MPI Preprint. 2003. No. 55 (modified version), 2001, (first version). arXiv:math.DG/0109117.

[99] C. Zhu and Y. Long, 'Maslov-type index theory for symplectic paths and spectral flow. I'. *Chinese Ann. Math. Ser. B* **20**/4 (1999), 413–424.

Department of Sciences, Systems and Models/IMFUFA, Roskilde University, Postboks 260, DK-4000 Roskilde, Denmark

E-mail address: booss@ruc.dk

Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: zhucf@nankai.edu.cn