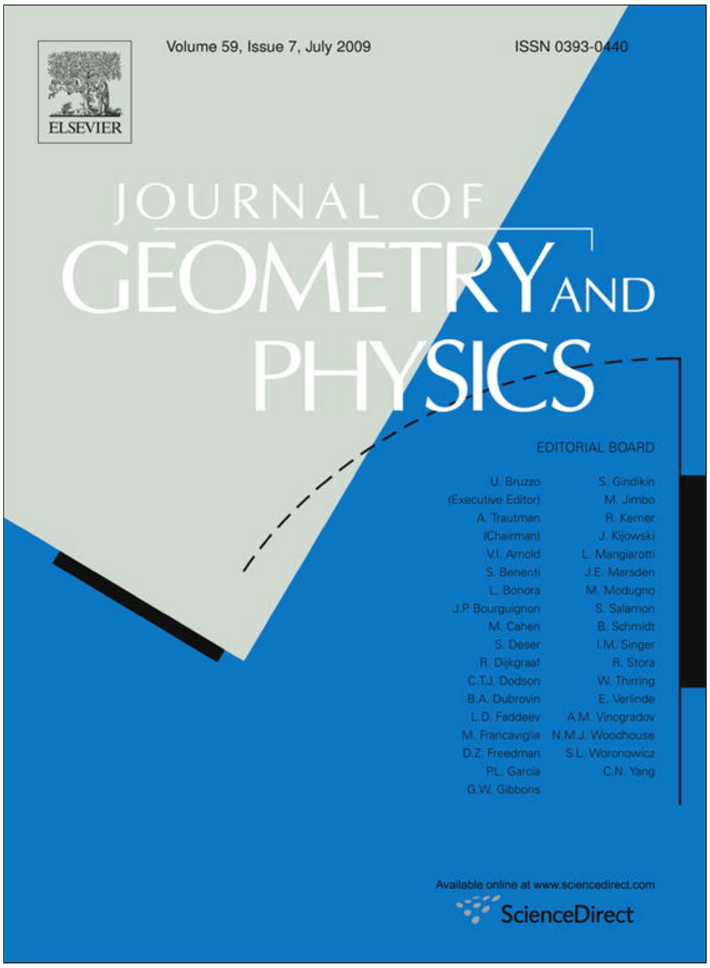
Volume 59, Issue 7, July 2009
ISSN 0393-0440
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Journal of Geometry and Physics 59 (2009) 784–826

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Journal of Geometry and Physics

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# The Calderón projection: New definition and applications

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ARTICLE INFO

*Article history:*
Received 16 December 2008
Received in revised form 19 March 2009
Accepted 22 March 2009
Available online 8 April 2009

*MSC:*
primary 58J32
secondary 35J67
58J50
57Q20

*Keywords:*
Calderón projection
Cauchy data spaces
Cobordism theorem
Continuous variation of operators and boundary conditions
Elliptic differential operator
Ellipticity with parameter
Lagrangian subspaces
Regular boundary value problem
Sectorial projection
Self-adjoint Fredholm extension
Sobolev spaces
Symplectic functional analysis

ABSTRACT

We consider an arbitrary linear elliptic first-order differential operator $A$ with smooth coefficients acting between sections of complex vector bundles $E, F$ over a compact smooth manifold $M$ with smooth boundary $\Sigma$. We describe the analytic and topological properties of $A$ in a collar neighborhood $U$ of $\Sigma$ and analyze various ways of writing $A\restriction U$ in product form. We discuss the sectorial projections of the corresponding tangential operator, construct various invertible doubles of $A$ by suitable local boundary conditions, obtain Poisson type operators with different mapping properties, and provide a canonical construction of the Calderón projection. We apply our construction to generalize the Cobordism Theorem and to determine sufficient conditions for continuous variation of the Calderón projection and of *well-posed* self-adjoint Fredholm extensions under continuous variation of the data.



## 1. Introduction

This paper is about basic analytical properties of elliptic operators on compact manifolds with smooth boundary. Our main achievements are (i) to develop the basic elliptic analysis in full generality, and not only for the generic case of operators of Dirac type in product metrics (i.e., we assume neither constant coefficients in normal direction nor symmetry of the tangential operator); (ii) to establish the cobordism invariance of the index in greatest generality; and (iii) to prove the continuity of the Calderón projection and of related families of global elliptic boundary value problems under parameter variation. We take our point of departure in the following observations.

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0393-0440/$ – see front matter 
doi:10.1016/j.geomphys.2009.03.012

### 1.1. Dirac operator folklore

Most analysis of geometrical and physical problems involving a Dirac operator $A$ on a compact manifold $M$ with smooth boundary $\Sigma$ acting on sections of a (complex) bundle $E$ seems to rely on quite a few basic facts which are part of the shared *folklore* of people working in this field of global analysis (e.g., see Booß–Bavnbek and Wojciechowski [1] for properties (WiUCP), (InvDoub) and (Cob), and Nicolaescu [2, Appendix] and Booß–Bavnbek, Lesch and Phillips [3] for property (Param)):

**WiUCP**: the weak inner unique continuation property (also called *weak UCP to the boundary*), i.e., there are no nontrivial elements in the null space $\ker A$ vanishing at the boundary $\Sigma$ of $M$;

**InvDoub**: the existence of a suitable elliptic invertible continuation $\widetilde{A}$ of $A$, acting on sections of a vector bundle over the closed double or another suitable closed manifold $\widetilde{M}$ which contains $M$ as submanifold; this yields a Poisson type operator $K_+$ which maps sections over the boundary into sections over $M$; and a precise Calderón projection $C_+$, i.e., an idempotent mapping of sections over the boundary onto the Cauchy data space which consists of the traces at the boundary of elements in the nullspace of $A$ (possibly in a scale of Sobolev spaces);

**Cob**: the existence of a self-adjoint regular Fredholm extension of any total (formally self-adjoint) Dirac operator $A$ in the underlying $L^2$-space with domain given by a pseudo-differential boundary condition; that implies the vanishing of the signature of the associated quadratic form, induced by the leading symbol in normal direction at the boundary; moreover, that actually is equivalent to the Cobordism Theorem asserting a canonical splitting of the induced tangential operator $B = B^+ \oplus B^-$ over $\Sigma$ with $\operatorname{ind} B^+ = 0$;

**Param**: the continuous dependence of a family of operators, their associated Calderón projections, and of any family of well-posed (elliptic) boundary value problems on continuous or smooth variation of the coefficients.

### 1.2. In search of generalization

With the renewed interest in geometrically defined elliptic operators of first order of general type, arising, e.g., from perturbations of Dirac operators, we ask to what extent the preceding list can be generalized to arbitrary linear elliptic differential operators with smooth coefficients. It is hoped that the results of this paper can serve as guidelines for similar constructions and results for hypo- and sub-elliptic operators where the symbolic calculus is not fully available.

There are immediate limits for generalization of some of the mentioned features by counter examples: UCP, even weak inner UCP may not hold for arbitrary elliptic systems of first order, see indications in that direction in Pliś [4, Corollary 1, p. 610] and the first-order Alinhac type counterexample to strong UCP [5, Example, p. 184]. Moreover, from just looking at the deficiency indices, we see that the formally self-adjoint operator $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ on the positive line does not admit a self-adjoint extension. This example is instructive because, quite opposite to the half-infinite domain, on a bounded one-dimensional interval *any* system of first-order differential equations satisfies property (Cob) by a deformation argument.

We go through the list.

*Property (**WiUCP**)*

It seems that the precise domain of validity is unknown. The local stability of weak inner UCP has been obtained by Booß and Zhu in [6, Lemma 3.2]. In spite of the *local* definition of UCP, the property (WiUCP) has a threefold *global* geometric meaning: (i) there are no ghost solutions, i.e., each section $u \in \Gamma^\infty(M; E)$ belonging to the null space $\ker A$ over the manifold $M$ has a non-trivial trace $u\restriction \Sigma$ at the boundary; (ii) equivalently, the maximal extension $A_{\max} = (A^t_{\min})^*$ is surjective in $L^2(M, E)$ for densely defined closed minimal $A_{\min} : \mathscr{D}(A_{\min}) = H^1_0(M, E) \to L^2(M, E)$; and (iii), as noted by Booß and Furutani in [7, Section 3.3] and in various follow-ups, it seems that assuming weak inner UCP of $A$ and $A^t$ is mandatory for obtaining the continuity of Cauchy data spaces and the continuous change of the Calderón projection under variation of the coefficients, i.e., property (Param).

*Property (**InvDoub**)*

Different approaches are available: one approach [1, Chapter 9] has been the gluing of $A$ and its formal adjoint $A^t$ to an invertible elliptic operator $\widetilde{A}$ over the closed double $\widetilde{M}$. This construction is explicit, if the metric structures underlying the Dirac operator's definition are product near the boundary. In the self-adjoint case, it yields at once the Lagrangian property of the Cauchy data space in the symplectic Hilbert space $L^2(\Sigma, E_\Sigma)$ of square integrable sections in $E_\Sigma := E\restriction \Sigma$. Then (Cob) follows.

Property (InvDoub) generalizes to the non-product case for operators of Dirac type and, as a matter of fact, for any elliptic operator satisfying weak inner UCP under the somewhat restrictive condition that the tangential operator has a self-adjoint leading symbol. Here the trick is that this condition permits the prolongation of the given operator to a slightly larger manifold $M'$ with boundary reaching constant coefficients in a normal direction close to the new boundary *and* maintaining UCP under the prolongation (as well as formal self-adjointness of the coefficients, if present at the old boundary). This is explained in the Appendix.

But what can be done for general elliptic operators? A very general and elegant construction of the Calderón projection was given by Hörmander in [8, Theorem 20.1.3] on the symbol level. Unfortunately, he obtains only an *almost* projection (up to smoothing operators) which limits its applicability in our context.

In this paper, we shall exploit another general definition of the Calderón projection which is due to Seeley [9, Theorem 1 and Appendix, Lemma]. Seeley's construction provides a precise projection, not only an approximate one, and does not require UCP. First he replaces $A$ by an invertible operator $A_1$ by adding the projection onto the finite-dimensional space of inner solutions. Then he extends the operator $D := \begin{pmatrix} 0 & A_1^t \\ A_1 & 0 \end{pmatrix}$ to the closed double $\widetilde{M}'$ of a slightly extended manifold $M'$ with boundary. In general, such a prolongation may, however, destroy weak inner UCP even when UCP was established on the original manifold. Seeley constructs on the symbol level (and by adding a suitable correction term) an elliptic extension $\widetilde{D}$ of $D$ over the whole of the closed manifold $\widetilde{M}'$ which is always invertible. He shows that $\widetilde{D}$ provides the wanted Poisson operator and a truly pseudo-differential Calderón projection $\mathscr{P}_+$ along the original boundary $\Sigma$. If the tangential operator is formally self-adjoint then $\mathscr{P}_+$ has a self-adjoint leading symbol and can be replaced by the orthogonal projection which is also pseudo-differential and has the same symbol (and may be denoted by the same letter $\mathscr{P}_+$). In this way, the choices in the *construction of the invertible double* are removed totally, as the operation of the resulting $\mathscr{P}_+$ is concerned. This makes $\mathscr{P}_+$ a good candidate for property (Cob).

However, Seeley's general construction, similarly the recent Grubb [10, Section 11.1], in difference to the simple gluing in the case of Dirac type operators of [1, Chapter 9], does not immediately lead to the Lagrangian property of im $\mathscr{P}_+$. Moreover and more seriously, when working with curves of elliptic problems Seeley's construction does not give a hint under what conditions the Calderón projections vary continuously when varying a parameter. There are too many choices involved in Seeley's construction.

### *1.3. Our present approach*

This motivates our present approach (inspired by Himpel, Kirk and Lesch [11]), namely the construction of the invertible double as a *canonically* given local boundary problem for the double $D$ exactly on the original manifold $M$, without any choices, prolongations etc. This leaves us with full control of the UCP situation; leads directly to the wanted Fredholm Lagrangian property (Cob); and, moreover and here most decisively, provides explicit formulas for treating the parameter dependence in the property (Param).

This program is opened in Section 2 by explaining our basic choice of product structures near the boundary for the sake of comprehensible analysis, even if the original geometric structures are non-product; moreover, for the convenience of the reader and for fixing our notation we summarize a few basic facts about regular boundary conditions.

To begin with, we do *not* assume a self-adjoint leading symbol of the tangential operator $B_0$ *nor* constant coefficients $B_x = B_0$ along an inward coordinate $x$. Most of our estimates depend on the single fact that $B_0 - \lambda$ is parameter dependent elliptic for $\lambda$ in a conic neighborhood of $\mathrm{i}\mathbb{R}$ in the sense of Shubin [12, Section II.9]. More precisely, we depend on the related concept of sectorial spectral projections introduced in 1970 by Burak [13] and recently further developed in Ponge [14, Section 3] as part of the current upsurge of interest in spectral properties of non-self-adjoint elliptic operators. Because of our interest in the continuous dependence of this kind of generalized positive spectral projections on the input data we found it necessary to develop the concept of sectorial projections once again from scratch. This is done in Section 3 where we develop an abstract Hilbert space framework for the concept of sectorial projections and apply it to tangential operators perceived as parametric elliptic operators.

In Section 4 we provide the construction and the relevant properties of the invertible double yielded by a local elliptic boundary value problem, induced by fixing an invertible bundle homomorphism $T$ over $\Sigma$.

In Section 5 we establish suitable Sobolev regularity of the inverse operator which leads to the definition and basic properties of Poisson operator and Calderón projection, both definitions made dependent of the choice of the above mentioned homomorphism $T$. We shall show that the range of the Calderón projection does not depend on the choice of $T$ and is, in fact, equal to the Cauchy data space. That yields the relation between the *canonical* Calderón projection defined as orthogonal projection onto the Cauchy data space, and our *relative* Calderón projections, which depend on $T$. However, it is not the canonical, but only the relative definition that establishes the Lagrangian property of the Cauchy data space and its continuous dependence of the coefficients for general elliptic differential operators of first order.

That is the subject of the two closing sections of this paper, which present the fruits of the analysis endeavour of Sections 1–5.

*Property (**Cob**)*

In Section 6, we give a first application of our construction of the Calderón projection: we give our reading of Ralston [15] and infer that the arguments of this 1970 paper establish the following findings for any formally self-adjoint differential operator $A$ over a compact manifold $M$ with smooth boundary $\Sigma$:

- the existence of a self-adjoint Fredholm extension $A_P$ given by a pseudo-differential boundary condition $P$;
- the vanishing of the signature of $\mathrm{i}\omega$ on the space $V(B_0)$ of eigensections to purely imaginary eigenvalues of the tangential operator $B_0$ of $A$ over the boundary (or on $\ker B_0$ in the case that $B_0$ is formally self-adjoint); here $\mathrm{i}\omega$ denotes the form induced by the Green form of $A$ on the symplectic von Neumann space $\beta(A) := \mathscr{D}(A_{\max})/\mathscr{D}(A_{\min})$, i.e., the leading symbol of $A$ over the boundary $\Sigma$ in normal direction;
- and, equivalently, but seemingly never recognized by people working in global analysis, the General Cobordism Theorem, stating that the index of *any* elliptic differential operator $B^+$ over a closed manifold $\Sigma$ must vanish, if $B^+$ can be written

as the left lower corner of a formally self-adjoint tangential operator over $\Sigma$ induced by an elliptic formally self-adjoint $A$ on a smooth compact manifold $M$ with $\partial M = \Sigma$.

*Property (**Param**)*

In Section 7, as a second application of our construction of the Calderón projection, we establish that property in great generality. Roughly speaking, we let an operator family $(A_z)$ and the family $(A_z^t)$ of formally adjoint operators vary continuously in the operator norm from $L_1^2(M)$ to $L^2(M)$ and assume that the leading symbol $(J_0(A_z))$ of $(A_z)$ over the boundary $\Sigma$ in normal direction also varies continuously in the $L_{1/2}^2(\Sigma)$ operator norm with $z$ running in a parameter space $Z$. We assume for all $A_z$ and $A_z^t$ property (WiUCP) or, almost equivalently, that the dimensions of the spaces of "ghost solutions" without trace at the boundary remain constant under the variation. Then in Theorems 7.2 and 7.9a we show that the inverse of the "invertible double" and, under slightly sharpened continuity, the Poisson operator in respective operator norms vary continuously; and so does the resolvent of a family $(A_{zP_z})$ of well-posed Fredholm extensions of now formally self-adjoint $(A_z)$ with orthogonal pseudo-differential projections $(P_z)$ varying in $L_{1/2}^2(\Sigma)$ operator norm.

Unfortunately, we can neither prove nor disprove the continuous variation of the Calderón projection in the same generality. However, if the leading symbol of the tangential operator is self-adjoint, we can prove the continuous variation of the sectorial projection (Proposition 7.15) and so (Corollary 7.4) of the Calderón projections by our correction formula (5.31) and Theorem 7.2b. Our Proposition 7.13 shows that the difficulties for proving continuous variation of the sectorial projection disappear also for a non-self-adjoint leading symbol, if the variation is of order $< 1$.

In the Appendix we discuss various special cases with emphasis on constant coefficients in normal direction in a collar around the boundary.

The main results of this paper have been announced in [16].

## 2. Elliptic differential operators of first order on manifolds with boundary

### 2.1. Product form and metric structures near the boundary

We shall begin with a basic observation: *Dirac operators* emerge from a Riemannian structure on the manifold and a Hermitian metric on the vector bundle (together with Clifford multiplication and a connection). Talking about a *general differential operator* it is in our view very misleading to pretend that the operator will depend on metrics and such. All we need is the operator and an $L^2$-structure on the sections. The latter basically only requires a density (take dvol in the Riemannian case) on the manifold and a metric on the bundle. In this paper, we prefer to *choose* metrics and such as simple as possible, and push all complications into the operator.

The message is this: we can always work in the product case and have to worry only about the operator. In detail:

Let $M$ be a compact manifold with boundary and $\pi : E \to M$ a vector bundle. Given a Hermitian metric $h$ on $E$ and a Riemannian metric $g$ on $M$ we can form the Hilbert space $L^2(M, E; g, h)$ which is the completion of $\Gamma_0^\infty(M \setminus \partial M; E)$ with regard to the scalar product

$$\langle u, v\rangle_{g,h} := \int_M h(u(x), v(x))\, \mathrm{dvol}_g(x). \tag{2.1}$$

The space of smooth sections of the vector bundle $E$ over $M$ is denoted by $\Gamma^\infty(M; E)$; the corresponding space of smooth compactly supported sections is denoted by $\Gamma_0^\infty(M; E)$. Given another Riemannian metric $g_1$ and another Hermitian metric $h_1$ on $E$ there is a smooth positive function $\varrho \in C^\infty(M)$ such that

$$\mathrm{dvol}_{g_1} = \varrho\, \mathrm{dvol}_g, \tag{2.2}$$

and there is a unique smooth section $\theta \in \Gamma^\infty(M; \mathrm{End}\, E)$ such that for $x \in M$, $\xi, \eta \in E_x$ we have

$$h_{1,x}(\xi, \eta) = h_x(\theta(x)\xi, \eta), \qquad h_x(\theta(x)\xi, \eta) = h_x(\xi, \theta(x)\eta). \tag{2.3}$$

With regard to $h$ the operator $\theta(x)$ is self-adjoint and positive definite, thus we may form $\sqrt{\theta}$ which is again a smooth self-adjoint and positive definite section of $(\mathrm{End}\, E, h)$. It is clear that (2.3) determines $\theta(x)$ uniquely and the claimed smoothness of $x \mapsto \theta(x)$ can be checked easily in local coordinates. In sum, we find for $u, v \in \Gamma^\infty(M; E)$

$$\begin{aligned}\langle u, v\rangle_{g_1,h_1} &= \int_M h_1(u(x), v(x))\, \mathrm{dvol}_{g_1}(x)\\ &= \int_M h(\theta(x)u(x), v(x))\varrho(x)\, \mathrm{dvol}_g(x)\\ &= \langle \sqrt{\varrho\theta}u, \sqrt{\varrho\theta}v\rangle_{g,h}.\end{aligned} \tag{2.4}$$

Thus we arrive at

**Lemma 2.1.** *The map*

$$\Psi : \Gamma^\infty(M;E) \longrightarrow \Gamma^\infty(M;E), \quad u \longmapsto \sqrt{\varrho\theta}^{-1} u$$

*extends to an isometry from* $L^2(M,E;g,h)$ *onto* $L^2(M,E;g_1,h_1)$.

Now assume that we are given a differential operator $A$ in $L^2(M,E;g,h)$ of first order. It may be a Dirac operator which is constructed from the metrics $g$ and $h$. $g$ and $h$ may be wildly non-product near the boundary. Suppose there are metrics $g_1$, $h_1$ which we like more, e.g., product near the boundary. Then consider the differential operator $\Psi A\Psi^{-1}$ in $L^2(M,E;g_1,h_1)$. $\Psi A\Psi^{-1}$ is still a differential operator and, since $\Psi$ is unitary, all spectral properties are preserved.

Let us be even more specific and choose a neighborhood $U$ of $\partial M =: \Sigma$ and a diffeomorphism $\phi : U \to [0,\varepsilon)\times\Sigma$ with $\phi\restriction\Sigma = \mathrm{id}_\Sigma$. Furthermore, we choose a metric $g_1$ on $M$ such that

$$\begin{aligned} &\phi_* g_1 = dx^2 \oplus g_\Sigma,\\ &g_\Sigma := g\restriction\Sigma, \end{aligned} \tag{2.5}$$

is a product metric which induces the same metric on the boundary as $g$. Here $x$ denotes the normal inward coordinate near the boundary in the metric $g_1$.

$\phi$ is covered by a bundle isomorphism $\mathscr{F} : E\restriction U \to [0,\varepsilon)\times E_\Sigma$, $E_\Sigma := E\restriction\Sigma$, i.e., we have the commutative diagram

$$\begin{array}{ccc} E\restriction U & \xrightarrow{\mathscr{F}} & [0,\varepsilon)\times E_\Sigma \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \mathrm{id}\times\pi} \\ U & \xrightarrow{\phi} & [0,\varepsilon)\times\Sigma. \end{array} \tag{2.6}$$

Likewise, we may now choose a metric $h_1$ on $E$ such that $h_1(x) := \mathscr{F}_* h_1\restriction\{x\}\times\Sigma = h_1\restriction\Sigma = h\restriction\Sigma =: h_\Sigma$ is independent of $x\in[0,\varepsilon)$. The mappings $\mathscr{F}$ and $\phi$ induce a map

$$\begin{aligned} \Psi_1 : \quad \Gamma^\infty(U;E) \quad &\longrightarrow \quad C^\infty([0,\varepsilon), \Gamma^\infty(E_\Sigma)) \\ f \quad &\longmapsto \quad \Big(x \mapsto \big(p \mapsto \mathscr{F}(f(\phi^{-1}(x,p)))\big)\Big), \end{aligned} \tag{2.7}$$

which extends to a unitary isomorphism $L^2(U,E;g_1,h_1) \to L^2([0,\varepsilon], L^2(\Sigma,E_\Sigma; g\restriction\Sigma, h\restriction\Sigma))$. On $\Sigma$ and $E_\Sigma$ we have the fixed metrics $g_\Sigma$ respectively $h_\Sigma$ and we will suppress the reference to them in the notation.

Together with the unitary isomorphism $\Psi$ of Lemma 2.1 we obtain the claimed isometry

$$\Phi := \Psi_1\circ\Psi : L^2(U,E;g,h) \longrightarrow L^2([0,\varepsilon], L^2(E_\Sigma)). \tag{2.8}$$

Now $\Phi A\Phi^{-1}$ is a first order differential operator in the product Hilbert space $L^2([0,\varepsilon))\otimes L^2(\Sigma,E_\Sigma;g_\Sigma,h_\Sigma)$ and hence it takes the form

$$D := \Phi A\Phi^{-1} =: J\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B\Big) \tag{2.9}$$

with a bundle endomorphism $J \in C^\infty([0,\varepsilon), \Gamma^\infty(\Sigma; \mathrm{End}\, E_\Sigma))$ and a smooth family of first order differential operators $B \in C^\infty([0,\varepsilon), \mathrm{Diff}^1(\Sigma;E_\Sigma))$; $\mathrm{Diff}^1(\Sigma;E_\Sigma)$ denoting the space of first order differential operators acting on sections of $E_\Sigma$. For the moment we consider here only the smooth case, but so far one can replace "smooth" by "continuous" or "Lipschitz" or whatever.

Let us repeat: now all metric structures are product near the boundary and we do not have to worry about them. If we start, e.g., with a Dirac operator $A$ on a Riemannian manifold with non-product metric, the 'non-product situation' is reflected in the varying coefficients of $D$. From now on we have to worry only about those varying coefficients and nothing else.

After these somewhat pedagogical remarks, we are ready to formulate the general set-up of the paper.

### *2.2. The general set-up*

We are going to fix some notation which will be used throughout the paper. Assume that the following data are given:

- a compact smooth Riemannian manifold $(M,g)$ with smooth boundary $\Sigma := \partial M$,
- Hermitian vector bundles $(E,h^E)$, $(F,h^F)$,
- a first order elliptic differential operator

$$A : \Gamma^\infty(M;E) \longrightarrow \Gamma^\infty(M;F). \tag{2.10}$$

- $A^t : \Gamma^\infty(M;F) \longrightarrow \Gamma^\infty(M;E)$ denotes the formal adjoint of $A$ with respect to the metrics $g$, $h^E$, $h^F$.

For further reference we record Green's formula for $A$.

**Lemma 2.2.** *Let $\nu \in \Gamma^\infty(\Sigma; TM\restriction\Sigma)$ be the outward normal vector field. Then we have with the notation of (2.1) and (2.5) for $u \in \Gamma^\infty(M; E)$, $v \in \Gamma^\infty(M; F)$*

$$\langle Au, v\rangle_{g,h} - \langle u, A^t v\rangle_{g,h} = \frac{1}{\mathrm{i}}\int_\Sigma h_\Sigma\big(\sigma_A^1(\nu^b)u\restriction\Sigma, v\restriction\Sigma\big)\mathrm{dvol}_{g_\Sigma} = -\langle J(0)u\restriction\Sigma, v\restriction\Sigma\rangle_{g_\Sigma,h_\Sigma}, \tag{2.11}$$

*where $\mathrm{i} := \sqrt{-1}$, $\nu^b$ denotes the cotangent vector field corresponding to $\nu$ in the metric $g$, and $\sigma_A^1$ denotes the leading symbol of $A$.*

Note that $\phi_*\nu = -\frac{\mathrm{d}}{\mathrm{d}x}$. Recall also from the previous Section 2.1 that, by construction, all transformations are trivial on the boundary, that is,

$$\begin{aligned}
&\phi\restriction\Sigma = \mathrm{id}, \qquad \mathscr{F}\restriction E_\Sigma = \mathrm{id},\\
&(\Psi_1 f)\restriction\Sigma = f\restriction\Sigma, \qquad (\Psi f)\restriction\Sigma = f\restriction\Sigma,\\
&(\Phi f)\restriction\Sigma = f\restriction\Sigma.
\end{aligned} \tag{2.12}$$

We consider $A$ as an unbounded operator between the Sobolev (and Hilbert) spaces

$$L_s^2(M, E; g, h^E), \quad L_s^2(M, F; g, h^F), \quad s \geq 0.^{1} \tag{2.13}$$

If $A$ acts as an unbounded operator between the Hilbert spaces $H_1, H_2$, we denote by $A^*$ its functional analytic adjoint. For 0th order operators and for elliptic operators on *closed* manifolds the distinction between formal adjoint and (true) adjoint does not really matter; so in this case we use both notations interchangeably.

The closure of $A\restriction\Gamma_0^\infty(M\setminus\Sigma; E)$ in $L^2$ is denoted by $A_{\min}$ and we put

$$\mathscr{D}(A_{\max}) := \left\{f \in L^2 \mid Af \in L^2\right\}, \tag{2.14}$$

the domain of an unbounded operator $T$ will always be denoted by $\mathscr{D}(T)$. As explained in Section 2.1 there exists a collar $U \approx [0, \varepsilon) \times \Sigma$ and linear isomorphisms

$$\Phi^G : \Gamma^\infty(U; G) \longrightarrow C^\infty([0, \varepsilon), \Gamma^\infty(G_\Sigma)), \quad G = E, F, \tag{2.15}$$

which extend to isometries

$$L^2(U, G; g, h^G) \longrightarrow L^2([0, \varepsilon], L^2(\Sigma, G_\Sigma; g_\Sigma, h^{G_\Sigma})), \quad G = E, F, \tag{2.16}$$

where $g_\Sigma = g\restriction\Sigma$, $G_\Sigma := G\restriction\Sigma$ and $h^{G_\Sigma} = h^G\restriction G_\Sigma$.

Now we consider

$$D := \Phi^F A(\Phi^E)^{-1} : C^\infty([0, \varepsilon), \Gamma^\infty(E_\Sigma)) \longrightarrow C^\infty([0, \varepsilon), \Gamma^\infty(F_\Sigma)). \tag{2.17}$$

Since $A$ is a first order elliptic differential operator we find

$$D = J_x\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_x\Big), \tag{2.18}$$

where $J_x \in \mathrm{Hom}(E_\Sigma, F_\Sigma)$, $0 \leq x \leq \varepsilon$, is a smooth family of bundle homomorphisms and $(B_x)_{0\leq x\leq\varepsilon}$ is a smooth family of first order elliptic differential operators between sections of $E_\Sigma$. Note that in view of (2.11) $J_0$ equals $i\sigma_A^1(\nu^b)$, where $\nu = -\frac{\mathrm{d}}{\mathrm{d}x}$ is the outward normal vector field.

To avoid an inflation of parentheses we will most often use the notation $B_x, J_x$ instead of $B(x), J(x)$ etc. Only to avoid double subscripts we will write $B(x), J(x)$ in subscripts.

Since $\Phi^E, \Phi^F$ are unitary (cf. (2.15) and (2.16)) we have $D^t = \Phi^E A^t(\Phi^F)^{-1}$ and hence

$$\begin{aligned}
-D^t &= J_x^t\frac{\mathrm{d}}{\mathrm{d}x} - B_x^t J_x^t + (J_x')^t\\
&= J_x^t\Big(\frac{\mathrm{d}}{\mathrm{d}x} - (J_x^t)^{-1}B_x^t J_x^t\Big) + (J_x')^t.
\end{aligned} \tag{2.19}$$

If $A$ is formally self-adjoint, we have the relations

$$J^t = -J, \qquad JB = J' - B^t J \tag{2.20}$$

($'$ denotes differentiation by $x$).

Alternatively, we may choose the following normal form in a collar of the boundary:

$$D = J_x\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_x\Big) + \frac{1}{2}J_x'. \tag{2.21}$$

[1] For simplicity we content ourselves with Sobolev spaces of nonnegative order. On a manifold with boundary, Sobolev spaces of negative order are a nuisance, although with some care they could be dealt with here, cf. [17].

In this normalization, $A = A^t$ implies the relations

$$J^t = -J, \qquad JB = -B^t J. \tag{2.22}$$

The normal form (2.21) determines $J$ and $B$ uniquely.

Returning to general (not necessarily formally self-adjoint) $A$ we remark that the ellipticity of $A$ and hence of $D$ imposes various restrictions. The obvious ones are that $J_x$ is invertible and that $B_x$ is elliptic for all $x$. What's more, ellipticity of $D$ means that for $\lambda \in \mathbb{R}$, $\xi \in T_p^*\Sigma$, $(\lambda, \xi) \neq (0, 0)$, the operator

$$\mathrm{i}\lambda + \sigma^1_{B(x)}(p, \xi) \tag{2.23}$$

is invertible for all $(x, p) \in [0, \varepsilon) \times \Sigma$. Here, $\sigma^1_{B(x)}$ denotes the leading symbol of $B_x$. In other words, for $\xi \in T_p^*\Sigma \setminus \{0\}$ the endomorphism $\sigma^1_{B(x)}(p, \xi) \in \mathrm{End}(E_p)$ has no eigenvalues on the imaginary axis $\mathrm{i}\mathbb{R}$.

### *2.3. Regular boundary conditions*

For the convenience of the reader, and to fix some notation, we briefly summarize a few basic facts about boundary conditions for $A$. Standard references are [1,8,17,9]. We will adopt the point of view of the elementary functional analytic presentation [18]. However, we try to be as self-contained as possible.

It is well-known that the trace map

$$\varrho : \Gamma_0^\infty(M; E) \longrightarrow \Gamma^\infty(\Sigma; E), \quad f \longmapsto f \restriction \Sigma \tag{2.24}$$

extends by continuity to a bounded linear map between Sobolev spaces

$$L^2_s(M, E) \longrightarrow L^2_{s-1/2}(\Sigma, E_\Sigma), \quad s > 1/2. \tag{2.25}$$

For the domain of $A_{\max}$ this can be pushed a bit further. Namely, for $s \geq 0$ the trace map extends by continuity to a bounded linear map

$$\mathscr{D}(A_{\max,s}) \longrightarrow L^2_{s-1/2}(\Sigma, E_\Sigma), \quad s \geq 0, \tag{2.26}$$

that is, there is a constant $C_s$, such that for $f \in L^2_s(M, E)$ with $Af \in L^2_s(M, E)$

$$\|\varrho f\|_{s-1/2} \leq C_s(\|f\|_s + \|Af\|_s) \quad (s \geq 0). \tag{2.27}$$

Here $\|f\|_s$ denotes the Sobolev norm of order s. Furthermore, norms of operators from $L^2_s$ to $L^2_{s'}$ will be denoted by $\|\cdot\|_{s,s'}$, and $\|\cdot\|_\infty$ denotes the sup-norm of a function.

The proof of (2.26) and (2.27) in [1, Theorem 13.8 and Corollary 13.9] simplifies [17] for operators of Dirac type but remains valid for any elliptic differential operator of first order, cf. also [18, Lemma 6.1].

**Definition 2.3.** (a) Let $\mathrm{CL}^0(\Sigma; E_\Sigma, G)$ denote the space of *classical pseudo-differential operators of order* 0, acting from sections of $E_\Sigma$ to sections of another smooth Hermitian vector bundle $G$ over $\Sigma$.
(b) Let $P \in \mathrm{CL}^0(\Sigma; E_\Sigma, G)$. We denote by $A_P$ the operator $A$ acting on the domain

$$\mathscr{D}(A_P) := \left\{ f \in L^2_1(M, E) \mid P(\varrho f) = 0 \right\}, \tag{2.28}$$

and by $A_{\max,P}$ the operator $A$ acting on the domain

$$\mathscr{D}(A_{\max,P}) := \left\{ f \in L^2(M, E) \mid Af \in L^2(M, F), P(\varrho f) = 0 \right\}. \tag{2.29}$$

(c) The boundary condition $P$ for $A$ is called *regular* if $A_{\max,P} = A_P$, i.e., if $f, Af \in L^2, P(\varrho f) = 0$ already implies that $f \in L^2_1(M, E)$.
(d) The boundary condition $P$ is called *strongly regular* if $f \in L^2, Af \in L^2_k, P(\varrho f) = 0$ already implies $f \in L^2_{k+1}(M, E)$, $k \geq 0$.

(2.27) shows that $\mathscr{D}(A_P)$ is in any case a closed subspace of $L^2_1(M, E)$.

**Proposition 2.4.** *Let $P$ be regular for $A$. Then $A_P$ is a closed semi-Fredholm operator with finite-dimensional kernel.*

**Proof.** Let $(f_n) \subset \mathscr{D}(A_P)$ be a sequence with $f_n \to f$ and $Af_n \to g \in L^2(M, F)$. Then $Af = g$ weakly, hence $f \in \mathscr{D}(A_{\max}) = \mathscr{D}(A_{\max,0})$ and in view of (2.26) and (2.27) we have $P(\varrho f) = 0$ and the regularity of $P$ implies $f \in L^2_1(M, E)$, thus $f \in \mathscr{D}(A_P)$.

Hence $A_P$ is closed and thus $\mathscr{D}(A_P)$ is complete in the graph norm. In view of the Closed Graph Theorem the previous argument shows that the inclusion $\iota : \mathscr{D}(A_P) \hookrightarrow L^2_1(M, E)$ is bounded. $\iota$ is thus an injective bounded linear map from the Hilbert space $\mathscr{D}(A_P)$ (equipped with the graph norm) onto a closed subspace of $L^2_1(M, E)$; the closedness is also a consequence of the argument at the beginning of this proof. Consequently, on $\mathscr{D}(A_P)$ the graph norm and the $L^2_1$-norm are equivalent, i.e., for $f \in \mathscr{D}(A_P)$ we have

$$\frac{1}{C}\|f\|_1 \leq \|f\|_0 + \|Af\|_0 \leq C\|f\|_1. \tag{2.30}$$

Since the inclusion $L^2_1(M,E) \hookrightarrow L^2(M,E)$ is compact, the inclusion $\mathscr{D}(A_P) \hookrightarrow L^2(M,E)$ is compact too. Consequently, $A_P$ is a semi-Fredholm operator with finite-dimensional kernel. □

**Remark 2.5.** (1) $P = \mathrm{Id}$ is strongly regular and its domain $\mathscr{D}(A_P) = L^2_{1,0}(M,E)$ equals the closure of $\Gamma_0^\infty(M \setminus \Sigma; E)$ in $L^2_1(M,E)$. This is seen by induction. Namely, if $f \in L^2$, $Af = g \in L^2$ and $\varrho f = 0$ we may extend[2] $f$ by 0 to obtain $\widetilde{f}, \widetilde{Af} \in L^2_{\mathrm{loc}}$ and hence $f \in L^2_1$.

For the induction we first note that similarly as in [18, Cor. 2.14] one shows that to the map

$$\varrho^{(k+1)} : L^2_{k+1}(M,E) \longrightarrow \bigoplus_{j=0}^{k} L^2_{k-j+1/2}(\Sigma, E_\Sigma), \qquad f \longmapsto (\varrho A^j f)_{j=0}^k \tag{2.31}$$

there exists a continuous linear right-inverse

$$\mathrm{e}^{(k+1)} : \bigoplus_{j=0}^{k} L^2_{k-j+1/2}(\Sigma, E_\Sigma) \longrightarrow L^2_{k+1}(M,E). \tag{2.32}$$

To complete the induction consider $f \in L^2$, $Af \in L^2_k$, $\varrho f = 0$. By induction we may assume $f \in L^2_k$. Put

$$f_1 := f - \mathrm{e}^{(k+1)}(0, \varrho Af, \ldots, \varrho A^k f). \tag{2.33}$$

Then $f - f_1 \in L^2_{k+1}$, $f_1 \in L^2_k$ and $\varrho A^j f_1 = 0$, $j = 0, \ldots, k$. Hence we may extend $f$ by 0 to obtain $\widetilde{f} \in L^2$ with $\widetilde{A^j f} \in L^2$, $j = 0, \ldots, k+1$. From local elliptic regularity we infer $\widetilde{f} \in L^2_{k+1,\mathrm{loc}}$ and thus $f \in L^2_{k+1}$.

$(\operatorname{im} A_{\mathrm{Id}})^\perp = \{ f \in L^2(M,F) \mid A^t f = 0 \}$ which is known to be (or see Proposition 5.12 and Theorem 6.1) infinite-dimensional if $\dim M > 1$. Hence we cannot expect regularity to imply that $A_P$ is Fredholm. However, if $P$ and the dual boundary condition for $A^t$ are regular then $A_P$ is Fredholm.

(2) Regular boundary conditions are closely related to the well-posed boundary conditions of Seeley [9,1]. One of the main results in [18] states that for symmetric Dirac operators and symmetric boundary conditions (given by operators $P$ with closed range), regularity and well-posedness are equivalent.

(3) It is well-known that if $P$ has closed range then $P$ may be replaced by an orthogonal projection with the same kernel. In this setting the dual boundary condition can easily be computed:

**Proposition 2.6.** *Let $P \in \mathrm{CL}^0(\Sigma; E_\Sigma)$, $P = P^2 = P^*$. Then*

$$(A_P)^* = A^t_{\max,(\mathrm{Id}-P)J_0^t}. \tag{2.34}$$

**Proof.** This follows easily from Green's formula Lemma 2.2. □

We recall from [8, Definition 20.1.1] (see also [1, Remark 18.2d]):

**Definition 2.7.** Let $P \in \mathrm{CL}^0(\Sigma; E_\Sigma, G)$. We say that $P$ defines a *local elliptic boundary condition* for our operator $A$ (or, equivalently, we say $P$ satisfies the Šapiro–Lopatinskiĭ condition for $A$), if and only if the leading symbol $\sigma^0_P$ of $P$ maps the space $M^+_{y,\zeta}$ isomorphically onto the fiber $G_y$ for each point $y \in \Sigma$ and each cotangent vector $\zeta \in T^*_y(\Sigma)$, $\zeta \neq 0$. Here $M^+_{y,\zeta}$ denotes the space of boundary values of bounded solutions $u$ on the positive real line of the linear system $\frac{\mathrm{d}}{\mathrm{d}t}u + \sigma^1_{B(0)}(y,\zeta)u = 0$ of ordinary differential equations.

**Remark 2.8.** Note that a solution of the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t}u + \sigma^1_{B(0)}(y,\zeta)u = 0$ is bounded if and only if the initial value $u_0$ belongs to the range of the positive spectral projection $P_+(\sigma^1_{B(0)}(y,\zeta))$ (cf. Section 3) of the matrix $\sigma^1_{B(0)}(y,\zeta)$. Hence $M^+_{y,\zeta} = \operatorname{im} P_+(\sigma^1_{B(0)}(y,\zeta))$ and local ellipticity means that $\sigma^0_P$ maps $\operatorname{im} P_+(\sigma^1_{B(0)}(y,\zeta))$ isomorphically onto $G_y$.

We obtain from [8, Theorem 20.1.2, Theorem 20.1.8] (differently also along the lines of [1, Theorem 19.6]):

**Proposition 2.9.** *Any $P$ satisfying the Šapiro–Lopatinskiĭ condition for $A$ is strongly regular and the corresponding $A_P$ is a Fredholm operator.*

## 3. Sectorial projections of an elliptic operator

### *3.1. Parameter dependent ellipticity*

Regarding properties of the tangential operator $B_0$ on $\Sigma$, it is natural to distinguish three situations of increasing generality:

(i) $B_0$ is formally self-adjoint,

[2] We think of $M$ as being a subset of an open manifold $\widetilde{M}$ to which $A$ can be extended as an elliptic operator.

(ii) $B_0 - B_0^t$ is an operator of order zero, and
(iii) $B_0$ is the tangential operator of an elliptic operator over the whole manifold $M$.

Whereas (i) implies that the spectrum $\operatorname{spec}(B_0)$ of $B_0$ is contained in the real axis and (ii) that for all $p \in \Sigma$ and $\xi \in T_p^*\Sigma$ the leading symbol $\sigma^1_{B(0)}(p,\xi) \in \operatorname{End}(E_p)$ is self-adjoint, the general case (iii) a priori only implies that $\sigma^1_{B(0)}(p,\xi)$ has no eigenvalues on the imaginary axis $\mathrm{i}\mathbb{R}$ for all $p \in \Sigma$ and $\xi \in T_p^*\Sigma \setminus \{0\}$, as explained above after (2.23).

One may ask, what consequences can be drawn from the general property (iii) for the spectrum of $B_0$? A first answer is Proposition 3.3. In fact, (iii) contains more information than just that the leading symbol $\sigma^1_{B(0)}(p,\xi)$ has no eigenvalues on $\mathrm{i}\mathbb{R}$.

For the convenience of the reader let us briefly recall the notion of a (pseudo)-differential operator with parameter, cf. Shubin [12, Section II.9].

Let $\Lambda \subset \mathbb{C}$ be an open conic subset, i.e., $z \in \Lambda, r > 0 \Rightarrow rz \in \Lambda$. For an open subset $U \subset \mathbb{R}^n$ let $S^m(U, \mathbb{R}^n \times \Lambda)$ denote the space of smooth functions

$$a : U \times \mathbb{R}^n \times \Lambda \longrightarrow \mathbb{C}, \qquad (x, \xi, \lambda) \longmapsto a(x, \xi, \lambda),$$

such that for multi-indices $\alpha, \beta \in \mathbb{Z}_+^n,\ \gamma \in \mathbb{Z}_+^2$ and each compact subset $K \subset U$ we have

$$\left| \partial_x^\alpha \partial_\xi^\beta \partial_\lambda^\gamma a(x,\xi,\lambda) \right| \le C_K \left(1 + |\xi| + |\lambda|\right)^{m-|\beta|-|\gamma|}.$$

We emphasize that $\partial_\lambda^\gamma$ denotes real partial derivatives — we do not require holomorphicity in $\lambda$.

In other words, $S^m(U, \mathbb{R}^n \times \Lambda)$ are the symbols of Hörmander type (1,0).

We shall call a symbol $a \in S^m(U, \mathbb{R}^n \times \Lambda)$ *classical* if it has an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_{m-j}, \tag{3.1}$$

where $a_{m-j} \in S^{m-j}(U, \mathbb{R}^n \times \Lambda)$ with homogeneity

$$a_{m-j}(r\xi, r\lambda) = r^{m-j} a_{m-j}(\xi, \lambda) \quad \text{for } r \ge 1,\ |\xi|^2 + |\lambda|^2 \ge 1.$$

We denote the classical symbols by $\mathrm{CS}^m(U, \mathbb{R}^n \times \Lambda) \subset S^m(U, \mathbb{R}^n \times \Lambda)$.

**Definition 3.1.** Let $E_\Sigma$ be a complex vector bundle of finite fiber dimension $N$ over a smooth closed manifold $\Sigma$ and let $\Lambda \subset \mathbb{C}$ be open and conic. A *classical pseudodifferential operator of order $m$ with parameter $\lambda \in \Lambda$* is a family $B(\lambda) \in \mathrm{CL}^m(\Sigma; E_\Sigma),\ \lambda \in \Lambda$, such that locally $B(\lambda)$ is given by

$$B(\lambda)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_U \mathrm{e}^{\mathrm{i}\langle x-y, \xi\rangle} b(x, \xi, \lambda) u(y) \mathrm{d}y \mathrm{d}\xi$$

with $b$ an $N \times N$ matrix of functions belonging to $\mathrm{CS}^m(U, \mathbb{R}^n \times \Lambda)$.

**Remark 3.2.** (1) A *pseudo-differential operator with parameter* is more than just a map from $\Lambda$ to the space of pseudo-differential operators.

(2) Our definition of a *pseudo-differential operator with parameter* is slightly different from that of Shubin [12, Section II.9]; however, the main results of [12] do also hold for this class of operators.

The leading symbol of a classical pseudo-differential operator $B$ of order $m$ with parameter is now a smooth function $\sigma_B^m(x, \xi, \lambda)$ on $T^*\Sigma \times \Lambda \setminus \{(x, 0, 0) \mid x \in \Sigma\}$ which is homogeneous in the following sense

$$\sigma_B^m(x, r\xi, r\lambda) = r^m \sigma_B^m(x, \xi, \lambda) \quad \text{for } (\xi, \lambda) \ne (0, 0),\ r > 0.$$

*Parameter dependent ellipticity* is defined as invertibility of this homogeneous leading symbol. The basic example of a pseudo-differential operator with a parameter is the resolvent of an elliptic differential operator.

**Proposition 3.3.** *Let $\Sigma$ be a closed manifold and let $B \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$ be a first order differential operator. Let $\Lambda \subset \mathbb{C}$ be an open conic subset such that $B - \lambda, \lambda \in \Lambda$, is parameter dependent elliptic, i.e., for each $(p, \xi) \in T^*\Sigma,\ \xi \ne 0$, and each $\lambda \in \Lambda$ the homomorphism*

$$\sigma_B^1(p, \xi) - \lambda : E_p \longrightarrow E_p$$

*is invertible. Then there exists $R > 0$ such that $B - \lambda$ is invertible for $\lambda \in \Lambda,\ |\lambda| \ge R$, and we have*

$$\|(B - \lambda)^{-1}\|_{s, s+\alpha} \le C_\alpha |\lambda|^{-1+\alpha} \tag{3.2}$$

*for such $\lambda$ and $0 \le \alpha \le 1$.*

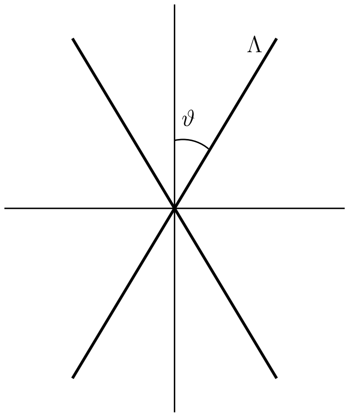


**Fig. 1.** Construction of a closed cone $\Lambda$ such that $B_0 - \lambda$ is elliptic with parameter $\lambda \in \Lambda$.

For the proof see [12, Theorem 9.3]. In our situation Proposition 3.3 has the following consequences:

**Proposition 3.4.** *Let $\Sigma$ be a closed manifold and let*

$$D = J_x\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_x\Big)$$

*be a first order elliptic differential operator on the collar $[0, \varepsilon) \times \Sigma$. Then*

(a) *$B_0 - \lambda$ is parameter dependent elliptic in an open conic neighborhood $\Lambda$ of the imaginary axis $\mathrm{i}\mathbb{R}$.*
(b) *$B_0$ is an operator with compact resolvent,* spec *$B_0$ consists of a discrete set of eigenvalues of finite multiplicity. At most finitely many eigenvalues lie on the imaginary axis $\mathrm{i}\mathbb{R}$.*

For an eigenvalue $\lambda$ even the *generalized eigenspace* $\bigcup_N \ker(B_0 - \lambda)^N$ is finite-dimensional; note that $B_0$ is not necessarily self-adjoint.

**Proof.** From the ellipticity of $D$ we infer that $\sigma^1_{B(0)}(p, \xi) - it$ is invertible for $(p, \xi, t) \in T^*\Sigma \times \mathbb{R},\ (\xi, t) \neq (0, 0)$. Since

$$\bigcup_{(p,\xi)\in T^*\Sigma,\ |\xi|=1} \operatorname{spec} \sigma^1_{B(0)}(p, \xi)$$

is bounded in $\mathbb{C}$ and in view of the homogeneity we find an angle $\vartheta > 0$ such that

$$\operatorname{spec} \sigma^1_{B(0)}(p, \xi) \cap \Lambda = \emptyset.$$

Here $\Lambda$ is as in Fig. 1.

(a) now follows from the previous proposition. Since $B_0$ is elliptic, its spectrum is either discrete or equals $\mathbb{C}$. The previous lemma implies that $B_0 - \lambda$ is invertible for $\lambda \in \Lambda$ large enough. Hence we conclude that spec $B_0$ is discrete and that (b) holds. □

### 3.2. Sectorial operators: Abstract Hilbert space framework

We shall now discuss the *positive* respectively *negative sectorial spectral projections* of an elliptic differential operator $B$ of first order on a closed manifold $\Sigma$. We start with a purely functional analytic discussion.

#### 3.2.1. Idempotents in a Hilbert space

Let us briefly summarize some facts about (not necessarily bounded) idempotents in a Hilbert space. A densely defined operator $P$ in the Hilbert space $H$ is an *idempotent* if $P \circ P = P$; as an identity between unbounded operators $P \circ P = P$ means $\operatorname{im} P \subset \mathscr{D}(P)$ and $P(Px) = Px$ for $x \in \mathscr{D}(P)$.

Given subspaces $U, V \subset H$ with

$$U \cap V = \{0\}, \tag{3.3}$$

$$U + V \text{ dense in } H, \tag{3.4}$$

the projection $P_{U,V}$ along $U$ onto $V$ is an (not necessarily bounded) idempotent and every idempotent $P$ in $H$ is of this form with $\mathscr{D}(P_{U,V}) = U + V$, $U = \ker P$ and $V = \operatorname{im} P$.

It is easy to see that $P^*_{U,V} = P_{V^\perp,U^\perp}$ is also an (not necessarily densely defined) idempotent. Thus $P_{U,V}$ is closable iff $U^\perp + V^\perp$ is dense or, equivalently $\overline{U} \cap \overline{V} = \{0\}$. In that case, the closure of $P_{U,V}$ is $\overline{P_{U,V}} = P_{\overline{U},\overline{V}}$. Consequently, $P_{U,V}$ is a closed operator if and only if $U, V$ are closed subspaces of $H$.

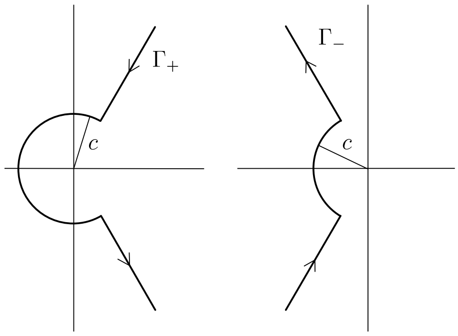


**Fig. 2.** The contours $\Gamma_\pm$ in the plane defining the semigroups $Q_\pm$.

**Lemma 3.5.** (a) *Let $P_{U,V}$ be an idempotent in the Hilbert space $H$, where $U, V$ are closed subspaces satisfying (3.3) and (3.4). Then $P_{U,V}$ is bounded if and only if $U + V = H$.*

(b) *Let $P = P_{U,V}$ be a bounded idempotent in the Hilbert space $H$. Then $P + \mathrm{Id} - P^*$ is an invertible operator.*

*Denote by $P_{\mathrm{ort}}$ the* orthogonalization *of $P$, i.e., $P_{\mathrm{ort}} = P_{V^\perp,V}$ is the orthogonal projection onto* im $P$. *Then we have*

$$P_{\mathrm{ort}} = P(P + \mathrm{Id} - P^*)^{-1}, \tag{3.5}$$

$$(P^*)_{\mathrm{ort}} = (P + \mathrm{Id} - P^*)^{-1}P. \tag{3.6}$$

**Proof.** (a) is a consequence of the Closed Graph Theorem.

(b) By (a) $U, V$ are closed subspaces of $H$ satisfying $U \cap V = \{0\}$, $U \oplus V = H$. Then $\mathrm{Id} - P^* = P_{U^\perp,V^\perp}$. Since bounded idempotents are bounded below $P{\restriction}U^\perp$ maps $U^\perp = \ker P^\perp$ bijectively onto $V$ and $\mathrm{Id} - P^*$ maps $U = \ker(\mathrm{Id} - P^*)^\perp$ bijectively onto $V^\perp$. Hence $P + \mathrm{Id} - P^*$ is invertible. Moreover, this description gives $(P + \mathrm{Id} - P^*)^{-1}$ explicitly: given $v \in V = \mathrm{im}\, P$ then $(P + \mathrm{Id} - P^*)^{-1}v$ is the unique element $\xi \in U^\perp$ with $P\xi = v$ and thus $P(P + \mathrm{Id} - P^*)^{-1}v = v$. Furthermore, if $v \in V^\perp$ then $(P + \mathrm{Id} - P^*)^{-1}v$ is the unique element $\eta \in U = \ker P$ with $(\mathrm{Id} - P^*)\eta = v$. This proves $P(P + \mathrm{Id} - P^*)^{-1} = P_{V^\perp,V} = P_{\mathrm{ort}}$. The equality (3.6) is proved similarly.

Alternatively, one may apply (3.5) to $P^*$ to find $(P^*)_{\mathrm{ort}} = P^*(P^* + \mathrm{Id} - P)^{-1}$. Then (3.6) follows from $(P + \mathrm{Id} - P^*)P^* = PP^* = P(P^* + \mathrm{Id} - P)$. □

Our construction of $P_{\mathrm{ort}}$ is a slight modification of the construction given by M. Birman and A. Solomyak and disseminated in [1, Lemma 12.8].

Lemma 3.5(a) shows that unbounded idempotents in a Hilbert space are abundant. See also Example 3.13.

### *3.2.2. The semigroups $Q_\pm(x)$ of a sectorial operator*

In this subsection let $H$ be a separable Hilbert space and $B$ a closed operator in $H$.

**Definition 3.6.** We call $B$ a *weakly sectorial* operator if

(1) $B$ has compact resolvent.
(2) There exists a closed conic neighborhood $\Lambda$ of $\mathrm{i}\mathbb{R}$ such that $\mathrm{spec}\, B \cap \Lambda$ is finite and

$$\|(B - \lambda)^{-1}\| = \mathcal{O}(|\lambda|^{-\alpha}), \quad |\lambda| \to \infty, \lambda \in \Lambda, \tag{3.7}$$

for some $0 < \alpha \leq 1$.

If $\alpha = 1$ then we call $B$ a *sectorial* operator.

We fix a weakly sectorial operator $B$ in the sense of Definition 3.6.

**Convention 3.7.** (a) $c > 0$ is chosen large enough such that

$$\mathrm{spec}\, B \cap \{z \in \mathbb{C} \mid |z| = c\} = \emptyset, \tag{3.8}$$

and such that $\{z \in \mathbb{C} \mid |z| = c\}$ contains all eigenvalues on the imaginary axis.

(b) We specify two complementary contours $\Gamma_\pm$ in the plane as sketched in Fig. 2 with $\Gamma_+$ encircling, up to finitely many exceptions, the eigenvalues of $B$ with real part $\geq 0$ and $\Gamma_-$ encircling the remaining eigenvalues. Of course, for this to be possible $c$ has to be large enough.

**Definition 3.8.**

$$Q_+(x) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+} \mathrm{e}^{-\lambda x} (\lambda - B)^{-1} \, \mathrm{d}\lambda, \quad x > 0, \tag{3.9}$$

$$= \mathrm{Id} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+} \mathrm{e}^{-\lambda x} \lambda^{-1} B (\lambda - B)^{-1} \, \mathrm{d}\lambda, \tag{3.10}$$

$$Q_-(x) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_-} \mathrm{e}^{-\lambda x} (\lambda - B)^{-1} \, \mathrm{d}\lambda, \quad x < 0, \tag{3.11}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_-} \mathrm{e}^{-\lambda x} \lambda^{-1} B (\lambda - B)^{-1} \, \mathrm{d}\lambda. \tag{3.12}$$

When the dependence on $B$ matters we will write $Q_\pm(x, B)$.

Formulas (3.10) and (3.12) are obtained by adding and subtracting $\lambda^{-1}$ inside the integral and taking into account that 0 lies inside $\Gamma_+$ but outside $\Gamma_-$.

$Q_\pm(x)$ are certainly bounded operators for $x > 0$ $(x < 0)$. Heuristically, $Q_\pm(0)$ should be the positive/negative sectorial spectral projection of $B$, obtained from holomorphic functional calculus. However, $Q_\pm(0)$ is not defined everywhere. To avoid ambiguities, we shall keep to the following two rigorous definitions instead of dealing directly with $Q_\pm(0)$.

**Definition 3.9.** We put $\mathscr{D}(P_{+,0}) := \{\xi \in H \mid \lim_{x\to 0+} Q_+(x)\xi \text{ exists}\}$ and $P_{+,0}\xi := \lim_{x\to 0+} Q_+(x)\xi$ for $\xi \in \mathscr{D}(P_{+,0})$. $P_{-,0}$ is defined analogously using $Q_-(x)$.

(3.10), (3.12) and the estimate (3.7) imply that $\mathscr{D}(B) \subset \mathscr{D}(P_{+,0})$ and for $\xi \in \mathscr{D}(B)$ we have

$$P_{+,0}\xi = \xi + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+} \lambda^{-1} (\lambda - B)^{-1} \, \mathrm{d}\lambda \; (B\xi) \tag{3.13}$$

$$P_{-,0}\xi = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_-} \lambda^{-1} (\lambda - B)^{-1} \, \mathrm{d}\lambda \; (B\xi), \tag{3.14}$$

thus $P_{\pm,0}$ is densely defined ($\mathscr{D}(B)$ is indeed a core for $P_{\pm,0}$). Note that $Q_\pm(x, B)^* = Q_\pm(x, B^*)$ (cf. Proposition 3.11), hence the densely defined operator $P_{\pm,0}(B^*)$ is contained in $P_{\pm,0}(B)^*$. Thus $P_{\pm,0}$ is closable:

**Definition 3.10.** The closure of $P_{\pm,0}$ will be called the *positive/negative sectorial spectral projection* $P_\pm$ of $B$.

**Proposition 3.11.** *For $x, y > 0$ we have*

(a) $Q_+(x, B)^* = Q_+(x, B^*)$, $Q_-(-x, B)^* = Q_-(-x, B^*)$.
(b) $Q_+(x)Q_+(y) = Q_+(x + y)$.
(c) $Q_+$ *is differentiable and* $\frac{\mathrm{d}Q_+}{\mathrm{d}x}(x) = -BQ_+(x)$.
(d) $Q_+(x)Q_-(-y) = Q_-(-y)Q_+(x) = 0$.
(e) $P_+Q_+(x) \subset Q_+(x)P_+$, $P_+Q_-(-x) = 0$.

**Proof.** The proof is straightforward and analogous to the proof in [14] of the fact that $P_+$ is an idempotent. □

**Corollary 3.12.** *$P_\pm$ are complementary, i.e., $P_+ = \mathrm{Id} - P_-$, (possibly unbounded) idempotents in $H$.*

**Proof.** Since $\mathscr{D}(B)$ is a core for $P_\pm$ it suffices to check that for $\xi \in \mathscr{D}(B)$ we have $P_\pm^2\xi = P_\pm\xi$ and $(P_+ + P_-)\xi = \xi$.

If $\xi \in \mathscr{D}(B)$ then using Proposition 3.11 we find

$$Q_+(x)P_+\xi = \lim_{y\to 0+} Q_+(x)Q_+(y)\xi = \lim_{y\to 0+} Q_+(x + y)\xi = Q_+(x)\xi, \tag{3.15}$$

hence $P_+\xi \in \mathscr{D}(P_{+,0}) \subset \mathscr{D}(P_+)$ and $P_+^2\xi = P_+\xi$.

Secondly, we take a $\xi \in \mathscr{D}(B)$ and find

$$(P_+ + P_-)\xi = \xi + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \lambda^{-1} (\lambda - B)^{-1} \, \mathrm{d}\lambda \; (B\xi),$$

where $\Gamma$ is chosen as in Fig. 3. Pushing the radius of the circle arches to $\infty$ shows $(P_+ + P_-)\xi = \xi$. □

The fact that the sectorial projections are a priori unbounded operators may seem strange. The following example shows that the phenomenon really occurs:

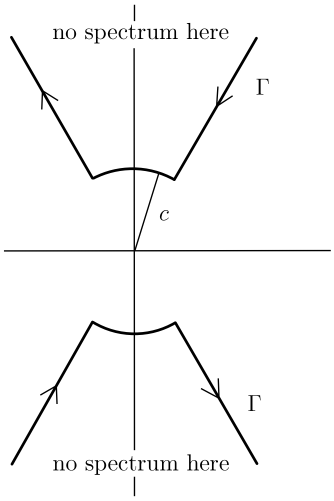


**Fig. 3.** A two-component contour $\Gamma$, separating an inner sector around the real axis where all eigenvalues of $B_0$ show up, from two outer sectors which totally belong to the resolvent set of $B_0$.

**Example 3.13.** Let $D$ be a discrete self-adjoint positive definite operator in $H$, i.e., there is an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $H$ such that $De_n = \lambda_n e_n$, where $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$.

Pick a parameter $0 \le \alpha \le 1$ and define the operator $B$ in $H \oplus H$ as follows:

$$\begin{aligned} \mathscr{D}(B) &:= \left\{ (u, v) \in H \oplus H \mid v \in \mathscr{D}(D), Du - D^{2-\alpha} v \in H \right\}, \\ B(u, v) &:= (Du - D^{2-\alpha} v, -Dv). \end{aligned} \tag{3.16}$$

One immediately checks that for $\lambda \notin \operatorname{spec} D \cup -\operatorname{spec} D$ the resolvent of $B$ is given by

$$(B - \lambda)^{-1}(\xi, \eta) = ((D - \lambda)^{-1}\xi - 2(D - \lambda)^{-1}D^{2-\alpha}(D + \lambda)^{-1}\eta, -(D + \lambda)^{-1}\eta). \tag{3.17}$$

Because of $0 \le \alpha \le 1$ the resolvent is indeed bounded. Furthermore, (3.17) shows that outside a conic neighborhood of the real axis, equivalently in a conic neighborhood of the $\mathrm{i}\mathbb{R}$, we have an estimate

$$\|(B - \lambda)^{-1}\| = \mathscr{O}(|\lambda|^{-\alpha}), \quad |\lambda| \to \infty. \tag{3.18}$$

Hence, if $0 < \alpha \le 1$ then $B$ is a weakly sectorial operator in the sense of Definition 3.6, $\operatorname{spec} B = \operatorname{spec} D \cup -\operatorname{spec} D$ and the positive/negative spectral subspaces of $B$ are given by

$$\begin{aligned} \operatorname{im} P_+(B) &= H \oplus 0, \\ \ker P_+(B) &= \left\{ (u, D^{\alpha-1}u) \mid u \in H \right\} = \operatorname{Graph}(D^{\alpha-1}). \end{aligned} \tag{3.19}$$

Consequently, if $0 < \alpha < 1$ then $\operatorname{im} P_+(B) \oplus \ker P_+(B)$ is not closed and hence the positive sectorial projection $P_+(B)$ is not bounded.

We leave it as an intriguing problem to find an example of a sectorial operator with decay rate $\alpha = 1$ in (3.7) such that $P_+$ is unbounded.

### *3.3. Sectorial operators: Parametric elliptic differential operators*

#### *3.3.1. The geometric situation*

We return to our geometric situation and consider the tangential operator $B$ (previously denoted by $B(0)$ or $B_0$, for convenience we omit $(0)$ as long as we do not need $B(x)$) of an elliptic differential operator $A$ on a compact manifold with boundary, cf. Section 2, in particular (2.18).

Then it is known that the positive sectorial projection is bounded:

**Theorem 3.14.** *Let $B$ be a first order elliptic differential operator on the closed manifold $\Sigma$. Furthermore, assume that $B - \lambda$ is parametric elliptic in an open conic neighborhood $\Lambda$ of $\mathrm{i}\mathbb{R}$. Then the positive/negative sectorial projections $P_\pm$ of $B$ are pseudo-differential operators of order 0. In particular $P_\pm$ acts as a bounded operator in each Sobolev space $L^2_s(\Sigma, E_\Sigma)$.*

The proof is an adaption of the classical complex power construction of Seeley [19]. See Burak [13], Wojciechowski [20], and recently Ponge [14].

We also note that it follows from Proposition 3.3 that $B$ is a sectorial operator in the sense of Definition 3.6. Also, recall from (3.2) the resolvent estimate:

For all $s \in \mathbb{R}$, $0 \le \alpha \le 1$, we have

$$\sup_{\lambda \in \Gamma_\pm} |\lambda|^{1-\alpha} \, \|(\lambda - B)^{-1}\|_{s,s+\alpha} \le C(s,\alpha), \tag{3.20}$$

where $\|\cdot\|_{s,s+\alpha}$ denotes the operator norm between the Sobolev spaces $L^2_s(\Sigma, E_\Sigma)$ and $L^2_{s+\alpha}(\Sigma, E_\Sigma)$, see also the following remark.

Here and in the following we shall denote the closed interval $[0,\infty)$ by $\mathbb{R}_+$. Similarly $\mathbb{Z}_+ := \{0,1,2,3,\ldots\}$.

**Remark 3.15.** (1) We recall that (cf. e.g. [18, Cor. 2.20])

$$L^2_s(\mathbb{R}_+ \times \Sigma, E_\Sigma) = L^2_s(\mathbb{R}_+, L^2(\Sigma, E_\Sigma)) \cap L^2(\mathbb{R}_+, L^2_s(\Sigma, E_\Sigma)), \quad s \ge 0. \tag{3.21}$$

In particular, if $s \in \mathbb{Z}_+$ then a Sobolev norm for $L^2_s(\mathbb{R}_+ \times \Sigma, E_\Sigma)$ is given by

$$\|f\|^2_{L^2_s(\mathbb{R}_+\times\Sigma,E_\Sigma)} = \int_0^\infty \|\partial_x^s f(x)\|^2_{L^2(\Sigma,E_\Sigma)} + \|(\mathrm{Id} + |B|)^s f(x)\|^2_{L^2(\Sigma,E_\Sigma)} \mathrm{d}x. \tag{3.22}$$

Since the spaces $L^2_s(\ldots)$ have the interpolation property [21, Sec. 4.2], [18, Sec. 2] for $s \ge 0$, it will be sufficient in most cases to deal with integer $s \in \mathbb{Z}_+$.

(2) Note that since $B$ is elliptic, the Sobolev norms on sections of $E_\Sigma$ can be defined using $B$, i.e.,

$$\|\xi\|^2_{L^2_s(\Sigma,E_\Sigma)} = \|(\mathrm{Id} + |B|)^s \xi\|^2_{L^2(\Sigma,E_\Sigma)}. \tag{3.23}$$

(3) Whenever it is clear from the context whether we are taking norms of sections over $\mathbb{R}_+ \times \Sigma$ or over $\Sigma$ we will, as before, denote Sobolev norms of order $s$ by a subscript $s$.

### 3.3.2. *Mapping properties of $Q_+$.*

The following proposition will be useful for the study of the mapping properties of the invertible double and of the remainder terms in the construction of the Poisson operator and the Calderón projection, see Sections 5.2 and 5.3. Proposition 3.16 establishes a weak convergence of $Q_+(x) \to P_+, x \to 0+$, in compensation for the generally not valid convergence in the operator norm.

**Proposition 3.16.** *Let $\varphi \in C_0^\infty(\mathbb{R}_+)$, $m \in \mathbb{Z}_+$.*

(a) *For $s \in \mathbb{R}$ the operator*

$$\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+ : \xi \longmapsto \big(x \mapsto x^m \varphi(x) Q_+(x)\xi\big) \tag{3.24}$$

*maps $L^2_s(\Sigma, E_\Sigma)$ continuously to $L^2_{\mathrm{comp}}(\mathbb{R}_+, L^2_{s+m+1/2}(\Sigma, E_\Sigma))$.*

(b) *For $s \ge -1/2$ it maps continuously to $L^2_{s+m+1/2,\mathrm{comp}}(\mathbb{R}_+ \times \Sigma, E_\Sigma)$.*

**Proof.** Let us first prove the claim (b). It is fairly easy to see that $\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+$ maps $L^2_s(\Sigma, E_\Sigma)$ continuously to $L^2_{\mathrm{comp}}(\mathbb{R}_+, L^2_{s'}(\Sigma, E_\Sigma))$ for some $s'$. Thus once we have proved that the range of $\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+$ is contained in the space $L^2_{\mathrm{comp}}(\mathbb{R}_+, L^2_{s+m+1/2}(\Sigma, E_\Sigma))$ the continuity will follow from the Closed Graph Theorem.

Furthermore, since $\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+$ commutes with $B$ it suffices to prove the claim for $s$ large enough: namely, we pick a $\lambda_0$ in the resolvent set of $B$. Then for arbitrary $s \ge -1/2$ we choose $k$ large enough such that the claim holds for $s+k$. The claim for $s$ now follows from the identity

$$\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+ | L^2_s = (\lambda_0 - B)^k \big(\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+ | L^2_{s+k}\big)\big((\lambda_0 - B)^{-k} | L^2_s\big). \tag{3.25}$$

Finally, by complex interpolation (cf. e.g. [21, Sec. 4.2]) it suffices therefore to consider $s = n + 1/2$, $n \in \mathbb{Z}_+$. Now pick $\xi \in L^2_{n+1/2}(\Sigma, E_\Sigma)$ and put $f(x) := x^m \varphi(x) Q_+(x)\xi$. It is straightforward to check that $f$ is smooth on $(0,\infty) \times \Sigma$. From

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + B\right)^j f(x) = \left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j x^m \varphi(x)\right) Q_+(x)\xi \tag{3.26}$$

we infer by the boundedness of $P_+(B)$ (according to Theorem 3.14) that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + B\right)^j f \bigg|_{x=0} = \begin{cases} 0, & j = 0,\ldots,m-1, \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \bigg|_{x=0} x^m \varphi(x) P_+(B)\xi \in L^2_{n+1/2}(\Sigma, E_\Sigma), & j \ge m, \end{cases}$$
$$\in L^2_{s+m+1/2-j-1/2}(\Sigma, E_\Sigma). \tag{3.27}$$

From (3.27) and (an obvious adaption of) [18, Cor. 2.17] (cf. also Remark 2.5) we infer that $f \in L^2_{s+m+1/2}(\mathbb{R}_+ \times \Sigma, E_\Sigma)$. Hence (b) is proved.

For $s \geq -1/2$ the claim (a) follows from (b). For arbitrary $s$ we again conjugate by $(\lambda_0 - B)^k$ as above and we reach the conclusion. □

**Remark 3.17.** The claim of the previous proposition also follows by applying more sophisticated pseudo-differential techniques (cf. Grubb [22, Thm. 2.5.7]). Our proof only uses the basic trace results for Sobolev spaces, the ellipticity of $B$, and the boundedness of the positive sectorial projection on Sobolev spaces. The previous proposition can therefore be generalized to situations where pseudo-differential techniques are not necessarily available. An abstract version is as follows (see also Section 5.1 where scales of Hilbert spaces are recalled to some extent):

**Proposition 3.18.** *Let $B$ be a sectorial operator in a Hilbert space. Let $H_s := \mathscr{D}((B^*B)^{s/2})$, $s \geq 0$, be the scale of Hilbert spaces of $B^*B$ and $\widetilde{H}_s := \mathscr{D}((BB^*)^{s/2})$ be the scale of Hilbert spaces of $BB^*$. For negative $s$ the spaces $H_s$ and $\widetilde{H}_s$ are defined by duality (cf. [18, Sec. 2.A]). Furthermore, put for $s \geq 0$*

$$\mathscr{H}_s(\mathbb{R}_+, H_\bullet) := \bigcap_{0 \leq t \leq s} L^2_s(\mathbb{R}_+, H_{s-t})$$

*(cf. [18, Sec. 2, Prop. 2.10] for other descriptions).*

*Assume that the positive sectorial projection $P_+$ of $B$ maps $H_s$ continuously to $\widetilde{H}_s$ for all $s$.*

*Let $\varphi \in C_0^\infty(\mathbb{R}_+)$, $m \in \mathbb{Z}_+$. Then*

(a) *For $s \in \mathbb{R}$ the operator*

$$\mathrm{id}^m_{\mathbb{R}_+} \varphi Q_+ : \xi \longmapsto \big(x \mapsto x^m \varphi(x) Q_+(x)\xi\big) \tag{3.28}$$

*maps $\mathscr{H}_s(\mathbb{R}_+, H_\bullet)$ continuously to $L^2_{\mathrm{comp}}(\mathbb{R}_+, \widetilde{H}_{s+m+1/2})$.*

(b) *For $s \geq -1/2$ it maps continuously to $\mathscr{H}_{s+m+1/2,\mathrm{comp}}(\mathbb{R}_+, \widetilde{H}_\bullet)$.*

For elliptic pseudo-differential operators the distinction between $H_s$ and $\widetilde{H}_s$ is, of course, unnecessary. For general unbounded operators, however, we cannot expect $H_s$ to be equal to $\widetilde{H}_s$.

## 4. The invertible double

We return to the set-up described at the beginning of Section 2.2 and give a construction of the invertible double of a general first order elliptic differential operator.

### 4.1. The construction of $\widetilde{A}_{P(T)}$

We introduce the operator

$$\widetilde{A} := A \oplus (-A^t) : \Gamma^\infty(M; E \oplus F) \longrightarrow \Gamma^\infty(M; F \oplus E). \tag{4.1}$$

We are going to consider a special class of boundary conditions for $\widetilde{A}$:

**Definition 4.1.** Let $T \in \mathrm{CL}^0(\Sigma; E_\Sigma, F_\Sigma)$ be a classical pseudo-differential operator of order 0, acting from sections of $E_\Sigma$ to sections of $F_\Sigma$. We put

$$P(T) = \begin{pmatrix} -T & \mathrm{Id} \end{pmatrix} \in \mathrm{CL}^0(\Sigma; E_\Sigma \oplus F_\Sigma, F_\Sigma). \tag{4.2}$$

Viewed as an operator in $L^2_s(\Sigma, E_\Sigma \oplus F_\Sigma)$ the operator $P(T)$ has closed range which equals

$$\operatorname{im} P(T) = L^2_s(\Sigma, F_\Sigma) \subset L^2_s(\Sigma, E_\Sigma \oplus F_\Sigma). \tag{4.3}$$

Since this is a closed subspace of $L^2_s(\Sigma, E_\Sigma \oplus F_\Sigma)$, the boundary condition for $\widetilde{A}$ given by $P(T)$ can be realized by a pseudo-differential orthogonal projection, as noted in Remark 2.5(3).

To be more specific we recall that the realization $\widetilde{A}_{P(T)}$ of $\widetilde{A}$ with respect to the boundary condition $P(T)$ has domain

$$\mathscr{D}(\widetilde{A}_{P(T)}) := \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2_1(M, E \oplus F) \mid \varrho f_- = T \varrho f_+ \right\}. \tag{4.4}$$

**Lemma 4.2.** *If $T$ is invertible, the dual of the boundary condition $P(T)$ for $\widetilde{A}$ is*

$$P(-J_0^{-1}(T^t)^{-1}J_0^t),$$

*i.e.,*

$$\big(\widetilde{A}_{P(T)}\big)^* = \widetilde{A}^t_{\max, P(-J_0^{-1}(T^t)^{-1}J_0^t)}. \tag{4.5}$$

**Proof.** Let $f = \binom{f_+}{f_-} \in \mathscr{D}(\widetilde{A}_{P(T)})$ and $g = \binom{g_+}{g_-} \in \mathscr{D}\big((\widetilde{A}_{P(T)})^*\big)$. Note that $g_+ \in L^2(M, F)$, $g_- \in L^2(M, E)$. Green's formula Lemma 2.2 yields

$$\begin{aligned} 0 &= \langle \widetilde{A}f, g\rangle - \langle f, \widetilde{A}^t g\rangle \\ &= \langle Af_+, g_+\rangle - \langle f_+, A^t g_+\rangle - \langle A^t f_-, g_-\rangle + \langle f_-, Ag_-\rangle \\ &= -\langle J_0\varrho f_+, \varrho g_+\rangle - \langle \varrho f_-, J_0\varrho g_-\rangle \\ &= -\langle \varrho f_+, J_0^t\varrho g_+ + T^t J_0 \varrho g_-\rangle. \end{aligned} \tag{4.6}$$

This holds for all $f \in \mathscr{D}(\widetilde{A}_{P(T)})$ if and only if $J_0^t\varrho g_+ + T^t J_0\varrho g_- = 0$ and we reach the conclusion. □

## 4.2. The local ellipticity of $P(T)$ for $\tilde{A}$

**Proposition 4.3.** *Let $T$ be an invertible bundle homomorphism from $E_\Sigma$ to $F_\Sigma$ with $J_0^t T > 0$. Then the boundary condition defined by $P(T)$ satisfies the Šapiro–Lopatinskiĭ condition for $\widetilde{A}$.*

**Remark 4.4.** (1) Obvious candidates for $T$ with $J_0^t T > 0$ are $J_0$ and $(J_0^t)^{-1}$. We can in addition choose $T$ to be unitary by putting $T := (J_0 J_0^t)^{-1/2} J_0$.

(2) Note that if $J_0^t T$ is positive definite then it is in particular self-adjoint and hence the dual condition for $\widetilde{A}^t$ (cf. Lemma 4.2) is given by

$$T^{\mathrm{dual}} = -J_0^{-1}(T^t)^{-1}J_0^t = -(T^t J_0)^{-1}J_0^t = -(J_0^t T)^{-1}J_0^t = -T^{-1}. \tag{4.7}$$

In particular we find that fulfilling the assumption $J_0^t T > 0$ of the preceding proposition implies that the boundary condition for $\widetilde{A}^t$ defined by $P(T^{\mathrm{dual}})$ also satisfies the Šapiro–Lopatinskiĭ condition. To see this we recall from (2.19) that the front bundle endomorphism $J_0(A^t)$ of $A^t$ is given by $J_0(A^t) = -J_0^t$, so

$$(-J_0^t)^t T^{\mathrm{dual}} = J_0 T^{-1} = J_0 (J_0^t T)^{-1} J_0^t > 0. \tag{4.8}$$

**Proof.** We refer to Remark 2.8 and use the language of idempotents. During this proof, for an endomorphism $b$ of a finite-dimensional vector space, $P_\pm(b)$ will denote the spectral projection corresponding to a closed contour encircling all eigenvalues $\lambda$ with $\operatorname{Re}\lambda \geq 0$ (respectively $< 0$), cf. Section 3.

From (2.18) and (2.19) we see that the tangential operator of $\widetilde{A}$ has leading symbol $b_0 \oplus -(J_0^t)^{-1} b_0^* J_0^t$, $b_0 := \sigma^1_{B(0)}$. Consequently the positive spectral projection of $b_0 \oplus -(J_0^t)^{-1}b_0^* J_0^t$ is given by $P_+(b_0) \oplus (J_0^t)^{-1}P_-(B^t)J_0^t$. In each $y \in \Sigma$ and $\zeta \in T_y^*(E_\Sigma)$, $\zeta \neq 0$, we consider the Šapiro–Lopatinskiĭ mapping from $\operatorname{im} P_+(b_0(y,\zeta)) \oplus (J_0^t)^{-1} \operatorname{im} P_-(b_0(y,\zeta)^*)$ to $F_y$, given by $\sigma^0_{P(T)} = (-T \;\; \mathrm{Id})$:

$$\begin{aligned} &\operatorname{im} P_+(b_0(y,\zeta)) \oplus (J_0^t)^{-1} \operatorname{im} P_-(b_0(y,\zeta)^*) \longrightarrow F_y \\ &\big(e_+, (J_0^t)^{-1}e_-\big) \longmapsto -Te_+ + (J_0^t)^{-1}e_-. \end{aligned}$$

Multiplying by $J_0^t$ we see that this map is bijective if and only if the map

$$\begin{aligned} &E_y = \operatorname{im} P_+(b_0(y,\zeta)) \oplus \operatorname{im} P_-(b_0(y,\zeta)^*) \longrightarrow E_y \\ &(e_+, e_-) \longmapsto -J_0^t T e_+ + e_- \end{aligned} \tag{4.9}$$

is bijective. To explain why $E_y = \operatorname{im} P_+(b_0(y,\zeta)) \oplus \operatorname{im} P_-(b_0(y,\zeta)^*)$ we note that $\operatorname{im} P_+(b_0(y,\zeta))^\perp = \ker P_+(b_0(y,\zeta))^* = \ker P_+(b_0(y,\zeta)^*) = \operatorname{im} P_-(b_0(y,\zeta)^*)$, so the sum on the left of (4.9) is indeed an orthogonal decomposition (cf. also Lemma 3.5).

Since the dimensions on the left and on the right side of (4.9) coincide, it suffices to show that the map in (4.9) is injective: so let $-J_0^t T e_+ + e_- = 0$, $e_+ \in \operatorname{im} P_+(b_0(y,\zeta))$, $e_- \in \operatorname{im} P_-(b_0(y,\zeta)^*) = \operatorname{im} P_+(b_0(y,\zeta))^\perp$. Taking scalar product with $e_+$ we find $0 = -\langle J_0^t T e_+, e_+\rangle$. This implies, since by assumption $J_0^t T > 0$, that $e_+ = 0$. But then $e_- = 0$ as well. □

## 4.3. The solution space $\ker \tilde{A}_{P(T)}$

Next we indicate why the boundary conditions of Definition 4.1 are significant. Before doing that we recall the various solution spaces and Cauchy data spaces associated to $A$.

**Definition 4.5.** (a) Put

$$Z^s(A) := \left\{ f \in L^2_s(M, E) \mid Af = 0 \right\}, \quad s \geq 0.$$

$Z^s(A^t) \subset L^2_s(M, F)$ is defined analogously. For brevity we often write

$$Z^s_+ := Z^s(A), \quad Z^s_- := Z^s(A^t). \tag{4.10}$$

It follows from (2.25) and (2.26) that the trace map sends $Z^s_\pm$ continuously to $L^2_{s-1/2}(\Sigma, E_\Sigma)$ respectively $L^2_{s-1/2}(\Sigma, F_\Sigma)$, $s \ge 0$.
(b) We define the *Cauchy data spaces* by

$$\begin{aligned} N^s_\pm &:= \varrho(Z^{s+1/2}_\pm), \quad s \ge -1/2, \\ N_\pm &:= N^0_\pm. \end{aligned} \tag{4.11}$$

(c) Finally let

$$Z_{+,0}(A) := \left\{ f \in L^2_1(M, E) \mid Af = 0, \varrho f = 0 \right\} \tag{4.12}$$

denote the space of all *inner solutions*. It is the finite-dimensional kernel of $A_{\mathrm{Id}}$ (cf. Proposition 2.4). $Z_{-,0}(A) := Z_{+,0}(A^t)$ denotes the corresponding kernel of $A^t_{\mathrm{Id}}$.
(d) We say that $A$ has the *weak inner unique continuation property* (UCP) if $Z_{+,0}(A) = \{0\}$.

**Proposition 4.6.** *Let $T$ be as in Definition 4.1. Then there is a canonical inclusion*

$$Z_{+,0}(A) \oplus Z_{-,0}(A) \subset \ker \widetilde{A}_{P(T)}.$$

*If, in addition, $J^t_0 T$ is positive definite, then the inclusion is an equality.*

**Proof.** If $f_\pm \in Z_{\pm,0}$ then $\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = f \in \ker \widetilde{A}_{P(T)}$ since $\varrho f_- = 0 = T\varrho f_+$, cf. (4.4).

Now assume that $J^t_0 T$ is positive definite and let $\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = f \in \ker \widetilde{A}_{P(T)}$. Then certainly $f_\pm \in Z^1_\pm$ and $\varrho f_- = T\varrho f_+$. Since $J^t_0 T$ is nonnegative and invertible, the operator $W := (J^t_0 T)^{1/2}$ exists and is invertible. Now Green's formula Lemma 2.2 yields

$$\begin{aligned} \|W\varrho f_+\|^2 &= \langle \varrho f_+, J^t_0 T \varrho f_+ \rangle \\ &= \langle J_0 \varrho f_+, \varrho f_- \rangle \\ &= -\langle A f_+, f_- \rangle + \langle f_+, A^t f_- \rangle \\ &= 0, \end{aligned} \tag{4.13}$$

and since $W$ is invertible we find $\varrho f_- = 0$. Thus $\varrho f_+ = T^{-1}\varrho f_- = 0$ and hence $f_\pm \in Z_{\pm,0}$. □

### *4.4. The main result*

Recall from Proposition 2.9 that by checking the Šapiro–Lopatinskiĭ condition we have not only proved the regularity of $P(T)$ for $\widetilde{A}$, but also that $\widetilde{A}_{P(T)}$ is a Fredholm operator. Let us summarize the results of Propositions 2.4, 2.9 and 4.3, Remark 4.4, and Proposition 4.6:

**Theorem 4.7.** *Let $M$ be a compact Riemannian manifold with boundary and*

$$A : \Gamma^\infty(M; E) \longrightarrow \Gamma^\infty(M; F)$$

*a first order elliptic differential operator. Write*

$$D = \Phi^F A (\Phi^E)^{-1} =: J_x \left( \frac{\mathrm{d}}{\mathrm{d}x} + B_x \right)$$

*as in (2.16) and (2.17) for suitable isometries $\Phi^E$, $\Phi^F$.*

*Let $T$ be an invertible bundle homomorphism from $E_\Sigma$ to $F_\Sigma$ and consider the boundary condition for $\widetilde{A} := A \oplus (-A^t)$ given by*

$$P(T) = \begin{pmatrix} -T & \mathrm{Id} \end{pmatrix}. \tag{4.14}$$

*Assume furthermore that*

$$J^t_0 T \text{ is positive definite, in particular self-adjoint.} \tag{4.15}$$

*Then*

(a) *$P(T)$ is strongly regular for $\widetilde{A} := A \oplus (-A^t)$.*
(b) *The dual condition is given by $P(T^{\mathrm{dual}}) = P(-T^{-1})$. It is strongly regular for $\widetilde{A}^t$.*

(c) *The operator* $\widetilde{A}_{P(T)}$ *is Fredholm with compact resolvent and*

$$\ker \widetilde{A}_{P(T)} = Z_{+,0}(A) \oplus Z_{-,0}(A),$$
$$\operatorname{coker} \widetilde{A}_{P(T)} \simeq Z_{-,0}(A) \oplus Z_{+,0}(A).$$

(d) *Finally, if* $A$ *and* $A^t$ *satisfy weak inner UCP then* $\widetilde{A}_{P(T)}$ *is invertible. Moreover, in this case, the inverse* $\left(\widetilde{A}_{P(T)}\right)^{-1}$ *maps* $L^2_s(M, F \oplus E)$ *continuously to* $L^2_{s+1}(M, E \oplus F)$ *for all* $s \geq 0$.

**Proof.** We only have to comment on the very last statement. As in the proof of Proposition 2.4 we infer from the strong regularity that on $\left\{f \in L^2_k \mid P(T)(\varrho f) = 0\right\}$ we have estimates

$$\frac{1}{C}\|f\|_k \leq \|f\|_{k-1} + \|\widetilde{A}f\|_{k-1} \leq C\|f\|_k. \tag{4.16}$$

Hence $\widetilde{A}^{-1}_{P(T)}$ maps $L^2_k$ continuously to $L^2_{k+1}$, $k \in \mathbb{Z}_+$. Now the claim follows from complex interpolation. □

**Remark 4.8.** We emphasize that the condition (4.15) holds for

$$T \in \left\{J_0, (J_0^t)^{-1}, (J_0 J_0^t)^{-1/2} J_0\right\}. \tag{4.17}$$

## 5. Calderón projection from the invertible double

### 5.1. Sobolev scale

Next we recall the purely functional analytic notion of the *Sobolev scale* of an operator (cf. [17,18]). For the moment let $D$ be a closed operator in the Hilbert space $H$. For $s \in \mathbb{R}$ let $H_s(D)$ be the completion of

$$H_\infty(D) := \bigcap_{s\geq 0} \mathscr{D}((D^*D)^{s/2}) \tag{5.1}$$

with respect to the scalar product

$$\langle x, y\rangle_s := \langle (\mathrm{Id} + D^*D)^s x, y\rangle. \tag{5.2}$$

Obviously $H_1(D) = \mathscr{D}(D)$ and the scalar product $\langle ., .\rangle_0$ extends to a perfect pairing between $H_s(D)$ and $H_{-s}(D)$. Furthermore, the spaces $H_s(D)$ have the interpolation property, that is for $s < t$ and $0 \leq \Theta \leq 1$ we have

$$H_{\Theta t+(1-\Theta)s}(D) = [H_s(D), H_t(D)]_\Theta \tag{5.3}$$

in the sense of complex interpolation theory (cf. e.g. [21, Sec. 4.2]).

Note that $D$ induces bounded linear maps $H_s(D) \to H_{s-1}(D^*)$. We will mostly use the case $|s| \leq 1$.

Since $H_{-1}(D)$ is canonically ($\mathbb{C}$-anti-)isomorphic to the dual of $H_1(D)$ it follows from (5.3) that $H_s(D)$, $|s| \leq 1$, depends only on the spaces $H_0(D)$ and $H_1(D)$; it does not depend on the particular operator $D$ generating the scale. This independence, of course, is not true for $|s| > 1$.

If the condition (4.15) ($J_0^t T > 0$) is fulfilled, then in view of (4.4) it is appropriate to put[3]

$$L^2_{s,-T^{-1}}(M, F \oplus E) := H_s((\widetilde{A}_{P(T)})^*), \quad -1 \leq s \leq 1. \tag{5.4}$$

Obviously, for $1/2 < s \leq 1$ we have by (4.7)

$$L^2_{s,-T^{-1}}(M, F \oplus E) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2_s(M, E \oplus F) \mid \varrho f_- = -T^{-1}\varrho f_+ \right\}. \tag{5.5}$$

The latter definition makes sense also for $s > 1$ but the equality (5.4) is limited to $|s| \leq 1$.

We have by construction $L^2_{s,T} \subset L^2_s$, hence $\varrho$ induces bounded linear maps

$$\begin{aligned} L^2_{s,-T^{-1}}(M, F \oplus E) &\longrightarrow L^2_{s-1/2}(\Sigma, F_\Sigma), \\ \begin{pmatrix} f \\ g \end{pmatrix} &\longmapsto f\restriction\Sigma, \end{aligned} \qquad 1/2 < s \leq 1. \tag{5.6}$$

Denote by $\varrho^*$ the $L^2$-dual of $\varrho$. I.e., $\varrho^*$ is a bounded linear map

$$\varrho^* : L^2_s(\Sigma, F_\Sigma) \longrightarrow L^2_{s-1/2,-T^{-1}}(M, F \oplus E), \quad -1/2 \leq s < 0, \tag{5.7}$$

such that for $\xi \in L^2_s(\Sigma, F_\Sigma)$ and $f \in L^2_{-s+1/2,-T^{-1}}(M, F \oplus E)$ we have $\langle \varrho^*\xi, f\rangle = \langle \xi, \varrho f\rangle$.

[3] Later on the following considerations will always be used for the dual boundary condition $T^{\mathrm{dual}} = -T^{-1}$, see (4.7). Therefore we present them already here for $-T^{-1}$ instead of $T$.

### 5.2. Induced Poisson type operators and inverses

We use the notation of the previous Section 4. Throughout the whole section we assume (4.15)

$$J_0^t T \text{ is positive definite, in particular self-adjoint} \tag{5.8}$$

and additionally

$$[J_0^t T, B_0^t] \text{ is of order } 0. \tag{5.9}$$

**Remark 5.1.** $[J_0^t T, B_0^t] = 0$ for the choice $T = (J_0^t)^{-1}$.
If $A = A^t$ and $B_0 - B_0^t$ is of order 0 then $[J_0^t T, B_0^t]$ is of order 0 for all three choices of $T$ in (4.17).

Recall from Remark 4.4 that condition (5.8) implies that the dual boundary condition for $\widetilde{A}^t$ is then given by $-T^{-1}$. According to Theorem 4.7 the boundary condition $P(T)$ is regular for $\widetilde{A}$ and

$$\begin{aligned}
\ker \widetilde{A}_{P(T)} &= Z_{+,0}(A) \oplus Z_{-,0}(A) = Z_{+,0}(\widetilde{A}),\\
\operatorname{coker} \widetilde{A}_{P(T)} &\simeq Z_{-,0}(A) \oplus Z_{+,0}(A) = Z_{-,0}(\widetilde{A}).
\end{aligned} \tag{5.10}$$

The orthogonal projections onto $Z_{+,0}(\widetilde{A}), Z_{-,0}(\widetilde{A})$ are denoted by $P_{Z_{+,0}(\widetilde{A})}, P_{Z_{-,0}(\widetilde{A})}$, respectively.

In a collar of the boundary we expand $A$ in the following form, omitting the explicit reference to $\Phi^E$, $\Phi^F$ etc. (cf. (2.17) and (2.18)),

$$A = J_0\Big(\frac{\mathrm{d}}{\mathrm{dx}} + B_0\Big) + C_1 x + C_0 \tag{5.11}$$

with a first order differential operator $C_1$ and $x$-independent bundle morphism $C_0 \in \Gamma^\infty(\Sigma; \operatorname{Hom}(E_\Sigma, F_\Sigma))$. Here and in the sequel, by slight abuse of notation, $x$ will also denote the operator of multiplication by the function $x \mapsto x$.

To see (5.11) we expand $J$ and $B$ near $x = 0$

$$\begin{aligned}
J_x &= J_0 + J_x^{(1)} x = J_0 + J_0' x + J_x^{(2)} x^2\\
B_x &= B_0 + B_x^{(1)} x.
\end{aligned} \tag{5.12}$$

Noting that $[\frac{\mathrm{d}}{\mathrm{dx}}, x] = 1$ we find

$$\begin{aligned}
J_x \frac{\mathrm{d}}{\mathrm{dx}} &= J_0 \frac{\mathrm{d}}{\mathrm{dx}} + J_x^{(1)} x \frac{\mathrm{d}}{\mathrm{dx}}\\
&= J_0 \frac{\mathrm{d}}{\mathrm{dx}} + \Big(J_x^{(1)} \frac{\mathrm{d}}{\mathrm{dx}}\Big) x - J_x^{(1)}\\
&= J_0 \frac{\mathrm{d}}{\mathrm{dx}} + \Big(J_x^{(1)} \frac{\mathrm{d}}{\mathrm{dx}} - J_x^{(2)}\Big) x - J_0'\\
J_x B_x &= J_0 B_0 + \big(J_0 B_x^{(1)} + J_x^{(1)} B_x\big) x,
\end{aligned} \tag{5.13}$$

thus

$$\begin{aligned}
C_1 &= J_x^{(1)} \frac{\mathrm{d}}{\mathrm{dx}} - J_x^{(2)} + J_0 B_x^{(1)} + J_x^{(1)} B_x,\\
C_0 &= -J_0'.
\end{aligned} \tag{5.14}$$

The formal adjoint can be written similarly as

$$A^t = \Big(-\frac{\mathrm{d}}{\mathrm{dx}} + B_0^t\Big) J_0^t + \widetilde{C}_1 x + \widetilde{C}_0, \tag{5.15}$$

where again $\widetilde{C}_1$ is a first order differential operator and $\widetilde{C}_0$ is an $x$-independent bundle morphism. More precisely,

$$\begin{aligned}
\widetilde{C}_1 &= C_1^t + \frac{1}{x}\big([x, C_1^t] - [x, C_1^t]_{|x=0}\big),\\
\widetilde{C}_0 &= C_0^t + [x, C_1^t]_{|x=0} = -(J_0')^t + (J_0')^t = 0,
\end{aligned} \tag{5.16}$$

thus $\widetilde{C}_0$ in fact vanishes. Note that in (5.11) and (5.15) $x$ is intentionally on the right of $C_1$. From this it also becomes clear that $\widetilde{C}_0 = 0$ because in the expansion of $A^t = \Big(-\frac{\mathrm{d}}{\mathrm{dx}} + B_x^t\Big) J_x^t$ near $x = 0$ the commutator $[x, \frac{\mathrm{d}}{\mathrm{dx}}]$ does not show up.

**Remark 5.2.** We note that if the tangential operator $B_0$ has a self-adjoint leading symbol we may replace $B_0$ by $\frac{1}{2}(B_0 + B_0^t)$ and still write $A, A^t$ in the form (5.11) and (5.15). This changes, of course, $C_0$ and $\widetilde{C}_0$.

We fix a real number $c > 0$ as in Section 3, Convention 3.7 and consider the corresponding family of operators $Q_\pm(x)$ of Definition 3.8. Then we define the following operators mapping (distributional) sections of $E \restriction \Sigma$ into (distributional) sections of $E \restriction \mathbb{R}_+ \times \Sigma$:

$$(R\xi)(x) := \begin{pmatrix} Q_+(x)\xi \\ Q_-(-x)^*\xi \end{pmatrix}, \qquad R_T\xi := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -T \end{pmatrix} R\xi. \tag{5.17}$$

$R_T$ will allow us to study the regularity properties of the (generalized) inverse of $A_{P(T)}$ (cf. (5.27)) and of the Poisson operator (cf. Definition 5.9). The Poisson operator is a map sending sections on the boundary into the kernel of $\widetilde{A}$ in the interior. $R$ and $R_T$ do almost have this property. For the constant coefficient operator $A = J_0\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_0\Big)$ one has indeed $\widetilde{A}Q_+ = 0$ by Proposition 3.11(a), (c). Even in the constant coefficient case $\widetilde{A}R$ is not necessarily 0 but, thanks to (5.9), small in a certain sense. This will become clear below. Note that $R_T$ does *not* map into the domain of $A_{P(T)}$. Its role will become transparent in formula (5.21).

For a cut-off function $\varphi \in C_0^\infty(\mathbb{R}_+)$ we consider $\varphi R_T$ as an operator from sections of $E_\Sigma = E\restriction\Sigma$ to sections of $E \oplus F$ over $M$; note that the range of $\varphi R_T$ consists of sections vanishing outside a collar of $\Sigma$. From Proposition 3.16 we infer that $\varphi R_T$ maps $L_s^2(\Sigma, E_\Sigma)$ continuously to $L^2_{s+1/2,\mathrm{comp}}(M, E \oplus F)$, $s \geq -1/2$.

To calculate $\widetilde{A}\varphi R_T$ we proceed by component:

$$A\varphi Q_+(x)\xi = \Big((C_1x + C_0)\varphi(x) + J_0\varphi'(x)\Big)Q_+(x)\xi \tag{5.18}$$

and

$$A^t\varphi T Q_-(-x)^*\xi = \Big((\widetilde{C}_1xT + \widetilde{C}_0T + [B_0^t, J_0^tT])\varphi(x) - J_0^t\varphi'(x)\Big)Q_-(-x)^*\xi. \tag{5.19}$$

The mapping properties of the right hand sides with respect to Sobolev spaces can be deduced from Proposition 3.16.

**Definition 5.3.** We write $S(A,T)\xi$ for the differential expression $\widetilde{A}$ applied to $\varphi R_T\xi$.

**Remark 5.4.** The "differential expression" is emphasized here since $\varphi R_T$ does *not* map into the domain of $\widetilde{A}_{P(T)}$. However, by duality (cf. Section 5.1) $\widetilde{A}_{P(T)}$ may also be viewed as a bounded operator $L^2(M, E\oplus F) \longrightarrow L^2_{-1,-T^{-1}}(M, F\oplus E)$. This should be viewed as applying $\widetilde{A}$ *in the distributional sense.*

The distinction between $S(A,T)$ and $\widetilde{A}_{P(T)}$ acting on $L^2(M, E \oplus F)$ is crucial. The difference between the two operators is (see (5.21)) $\varrho^*J_0(P_+ + P_-^*)$.

$S(A,T)$ will allow us to control the error (in terms of regularity, not in terms of size) between the approximate Poisson operator constructed from $R_T$ and the true Poisson operator.

$S(A,T)$ also depends on the choice of $c$ in Convention 3.7, but this will be suppressed in the notation. $S(A,T)$ is a $2 \times 1$ column consisting (up to sign) of the right hand sides of (5.18) and (5.19).

We single out an immediate but important consequence of Proposition 3.16:

**Proposition 5.5.** *$S(A,T)$ maps $L_s^2(\Sigma, E_\Sigma)$ continuously to $L^2_{s+1/2,\mathrm{comp}}(M, F \oplus E)$, $s \geq -1/2$.*

From the mapping properties of $\varphi R_T$ we conclude in particular that for $\xi \in L_s^2(\Sigma, E_\Sigma)$, $s \geq -1/2$, we have $\varphi R_T\xi \in L^2(M, E\oplus F) = H_0(\widetilde{A}_{P(T)})$. Hence we may apply $\widetilde{A}_{P(T)} \in \mathscr{B}(L^2(M, E \oplus F), L^2_{-1,-T^{-1}}(M, F \oplus E))$ (cf. Section 5.1) to $\varphi R_T$. Recall that this is defined by duality and the result will be different from the differential expression $\widetilde{A}$ applied to $\varphi R_T\xi$. Indeed for $f \in \mathscr{D}((\widetilde{A}_{P(T)})^*) = L^2_{1,-T^{-1}}(M, F \oplus E)$ we find using Green's formula, (5.8) and the boundary condition $\varrho_- f = -T^{-1}\varrho_+ f$; $\varrho_\pm f := \varrho(f_\pm)$:

$$\begin{aligned}\langle \varphi R_T\xi, (\widetilde{A}_{P(T)})^* f\rangle &= \langle J_0P_+\xi, \varrho_+ f\rangle - \langle J_0^tTP_-^*\xi, \varrho_- f\rangle + \langle S(A,T)\xi, f\rangle \\ &= \langle J_0(P_+ + P_-^*)\xi, \varrho f\rangle + \langle S(A,T)\xi, f\rangle \\ &= \langle \big(\varrho^*J_0(P_+ + P_-^*) + S(A,T)\big)\xi, f\rangle. \end{aligned} \tag{5.20}$$

Here $P_\pm$ denote the positive/negative sectorial spectral projections of $B_0$ in the sense of Definition 3.10. Recall $P_+(B_0^t) = P_+(B_0)^*$.

Thus as an identity in $H_{-1}((\widetilde{A}_{P(T)})^*) = L^2_{-1,-T^{-1}}(M, F \oplus E)$ we arrive at

$$\widetilde{A}_{P(T)}\varphi R_T\xi = \big(\varrho^*J_0(P_+ + P_-^*) + S(A,T)\big)\xi. \tag{5.21}$$

It is important to note that, even if $\xi$ has better regularity than $L^2_{-1/2}$, this is just an identity in $H_{-1}((\widetilde{A}_{P(T)})^*)$ since $\varphi R_T\xi$ does not fulfill the boundary condition for $A_{P(T)}$. The boundary condition plays no role as long as we view $\varphi R_T\xi$ as an element of $L^2$.

With some care we can now basically proceed as in [11, Sec. 3.2 and 3.3]: Firstly we note that by Lemma 3.5 $P_+ + P_-^*$ is invertible. Secondly we introduce the (Hilbert space) pseudo-inverse $\widetilde{G}$ of $\widetilde{A}_{P(T)}$, namely,

$$\widetilde{G}f := \begin{cases} \widetilde{A}_{P(T)}^{-1}f, & f \in \operatorname{im}\widetilde{A}_{P(T)}, \\ 0, & f \in \operatorname{im}\big(\widetilde{A}_{P(T)}\big)^{\perp}, \end{cases} \tag{5.22}$$

taking account of the possible absence of weak unique continuation. Here $\widetilde{A}_{P(T)}^{-1}f$ denotes the inverse image in $\big(\ker\widetilde{A}_{P(T)}\big)^{\perp}$ of $f$ under $\widetilde{A}_{P(T)}$.

Let

$$U : L^2(M, E \oplus F) \longrightarrow L^2(M, F \oplus E) \tag{5.23}$$

be the partial isometry which sends $Z_{+,0}(\widetilde{A})$ onto $Z_{-,0}(\widetilde{A})$ by interchanging the summands in (5.10) and which is zero on the orthogonal complement. Then

$$P_{Z_{+,0}(\widetilde{A})} = U^*U, \quad P_{Z_{-,0}(\widetilde{A})} = UU^*, \tag{5.24}$$

and

$$\widetilde{G} = (\mathrm{Id} - U^*U)(\widetilde{A}_{P(T)} + U)^{-1}. \tag{5.25}$$

We recall that $Z_{+,0}(\widetilde{A}), Z_{-,0}(\widetilde{A})$ are finite-dimensional and consist of sections which are smooth up to the boundary (cf. Remark 2.5(1)). Hence $U$ and $U^*$ are smoothing operators. Furthermore, from $\varrho U = 0$, $\varrho U^* = 0$ we immediately obtain

$$U^*\varrho^* = 0, \quad U\varrho^* = 0. \tag{5.26}$$

By construction $\ker\widetilde{A}_{P(T)} \subset H_\infty(\widetilde{A}_{P(T)})$. Hence $\widetilde{A}_{P(T)} + U$ induces invertible bounded linear maps from $H_s(\widetilde{A}_{P(T)})$ onto $H_{s-1}((\widetilde{A}_{P(T)})^*)$. Consequently, $\widetilde{G}$ induces bounded linear maps from $H_s((\widetilde{A}_{P(T)})^*)$ onto $H_{s+1}(\widetilde{A}_{P(T)})$. Together with the mapping properties of $\varrho^*$ we conclude that $\widetilde{G}\varrho^*$ maps $L^2_s(\Sigma, F_\Sigma)$ continuously to $L^2_{s+1/2}(M, E \oplus F)$ for $-1/2 \le s < 0$.

$G\varrho^*$ is the main building block of the Poisson operator (Definition 5.9) which should act at least on $L^2$. Therefore we have to improve the bound on $s$, which is now straightforward:

To apply the pseudo-inverse $\widetilde{G}$ to (5.21) it is enough that (5.21) is an identity in $H_{-1}$. Hence we find

$$\widetilde{G}\varrho^* = \Big((\mathrm{Id} - P_{Z_{+,0}(\widetilde{A})})\varphi R_T - \widetilde{G}S(A,T)\Big)(J_0(P_+ + P_-^*))^{-1}. \tag{5.27}$$

**Theorem 5.6.** *For $-1/2 \le s \le 1/2$ the operator $\widetilde{G}\varrho^*$ maps $L^2_s(\Sigma, F_\Sigma)$ continuously to $L^2_{s+1/2}(M, E \oplus F)$.*

**Proof.** This follows immediately from Proposition 3.16, (5.18), (5.19), (5.27), and Proposition 5.5. □

### 5.3. The Calderón projection

From the invertible double, the construction of the Calderón projection is straightforward. During the whole subsection we assume that the conditions (5.8) and (5.9) are fulfilled.

**Definition 5.7.** For a section $f = \binom{f_+}{f_-}$ of $E \oplus F$ we recall the notation $\varrho_\pm(f) := \varrho(f_\pm)$ and $r_\pm(f) := f_\pm$. Furthermore, we put for sections $f, g$ of $E, F$

$$e_+(f) := \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad e_-(g) := \begin{pmatrix} 0 \\ g \end{pmatrix}. \tag{5.28}$$

**Proposition 5.8.** *$\widetilde{G}\varrho^*$ maps $L^2_s(\Sigma, F_\Sigma)$ to $Z_+^{s+1/2} \oplus Z_-^{s+1/2}$, $-1/2 \le s \le 1/2$.*

**Proof.** In view of Theorem 5.6 it remains to be shown that $\widetilde{G}\varrho^*$ maps into the kernel of $\widetilde{A}$.

Let $f \in L^2_s(\Sigma, F_\Sigma)$, $-1/2 \le s \le 1/2$, and a test function $\varphi \in \Gamma_0^\infty(M \setminus \Sigma; F \oplus E)$ be given. In view of (5.7) we choose a real number $s'$ with $-1/2 \le s' < 0$, $s' \le s$. Then, by (5.7), we have $\varrho^*f \in L^2_{s'-1/2,-T^{-1}}(M, F \oplus E)$ and thus $\widetilde{G}\varrho^*f \in H_{s'+1/2}(\widetilde{A}_{P(T)})$; note $s' + 1/2 \ge 0$. Since $\varphi$ has compact support away from $\Sigma$ we certainly have $\varphi \in H_\infty\big((\widetilde{A}_{P(T)})^*i\big)$. Hence $\langle\widetilde{A}\widetilde{G}\varrho^*f, \varphi\rangle = \langle\widetilde{G}\varrho^*f, (\widetilde{A}_{P(T)})^*\varphi\rangle$. Viewing the rhs as the dual pairing between $H_{s'+1/2}(\widetilde{A}_{P(T)})$ and $H_{-s'-1/2}(\widetilde{A}_{P(T)})$ we may also move $\widetilde{G}$ to the right and find $\langle \widetilde{A}\widetilde{G}\varrho^* f, \varphi\rangle = \langle \varrho^* f, \widetilde{G}^*(\widetilde{A}_{P(T)})^*\varphi\rangle$, cf. Section 5.1. By construction of the generalized inverse we have, note again that $\varphi \in H_\infty\big((\widetilde{A}_{P(T)})^*\big) \subset L^2(M, F \oplus E)$,

$$\widetilde{G}^*(\widetilde{A}_{P(T)})^*\varphi = (I - U^*U)\varphi. \tag{5.29}$$

However, $U^*U\varphi \in H_\infty(\widetilde{A}_{P(T)})$ and $\varrho U^*U\varphi = 0$, cf. (5.26). Thus $\langle \varrho^* f, (\mathrm{Id} - U^*U)\varphi\rangle = \langle f, \varrho\varphi\rangle = 0$.

This calculation shows that if $\widetilde{A}$ is applied in the weak sense to $\widetilde{G}\varrho^* f$ one gets 0. But then $\widetilde{G}\varrho^* f \in Z_+^{s+1/2} \oplus Z_-^{s+1/2}$. □

**Definition 5.9.** (1) Define the *Poisson operator* by

$$K_\pm := \pm r_\pm \widetilde{G}\varrho^* J_0.$$

$K_\pm$ maps $L^2_s(\Sigma, E_\Sigma)$ continuously to $L^2_{s+1/2}(M, E)$ ($L^2_{s+1/2}(M, F)$) for $-1/2 \le s \le 1/2$.

(2)

$$C_+ := \varrho_+ K_+, \quad C_- := T^{-1}\varrho_- K_-.$$

$C_+$ is called the Calderón projection of $A$. Recall that $T$ is the operator defining the boundary condition for $\widetilde{A}$.

$K_\pm$ and $C_\pm$ depend on the pair $(A, T)$.

We summarize the result of the construction before Theorem 5.6, cf. in particular (5.27):

**Proposition 5.10.** *Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ and let $R_T$ be defined as in (5.17). Furthermore, let $P_+ := P_+(B_0)$, $P_- := P_-(B_0)$ be the positive respectively negative sectorial spectral projections of $B_0$ as introduced in Definition 3.10, cf. Theorem 3.14.*

*Then the Poisson operators are given by*

$$K_\pm = \pm r_\pm\Big((\mathrm{Id} - P_{Z_{+,0}(\widetilde{A})})\varphi R_T - \widetilde{G}S(A, T)\Big)(P_+ + P_-^*)^{-1}, \tag{5.30}$$

*and the Calderón projections are given by (see also (5.6) and (5.7))*

$$\begin{aligned} C_+ &= \Big(P_+ - \varrho_+\widetilde{G}S(A, T)\Big)(P_+ + P_-^*)^{-1} \\ C_- &= \Big(P_- + T^{-1}\widetilde{G}S(A, T)\Big)(P_+ + P_-^*)^{-1}. \end{aligned} \tag{5.31}$$

**Proof.** The theorem follows immediately from (5.27). □

**Remark 5.11.** Note that in the formula for $K_+$ in (5.30) $R_T$ can be replaced by $R$ (cf. (5.17)), hence the first summand for $K_+$ is independent of $T$.

Note that "our" Calderón projection differs from $P_{+,\mathrm{ort}} = P_+(P_+ + P_-^*)^{-1}$ (cf. Lemma 3.5, (3.5)) by an operator which regularizes by at least one Sobolev order. So our construction of the invertible double naturally yields a version of the *orthogonalized* Calderón projection; if $T = (J_0^t)^{-1}$ then $C_+$ is indeed an orthogonal projection, see the next proposition.

This is in contrast to the *classical* Calderón projection $\mathscr{P}_+$ of Seeley [9] which is a pseudo-differential operator of order 0 whose leading symbol coincides with that of $P_+$. Hence our $C_+$ differs from the orthogonalized Calderón projection $\mathscr{P}_{+,\mathrm{ort}}$ by an operator which regularizes by at least one Sobolev order. With some more work one can indeed show that $C_+$ is a pseudo-differential operator which differs from $\mathscr{P}_{+,\mathrm{ort}}$ by an operator of order $-1$.

If $B_0 = B_0^t$, or more generally if $B_0$ has a self-adjoint leading symbol, then Proposition 5.10 shows in particular the well-known fact (cf. [1, Corollary 14.3]) that $C_+ - P_+(B_0)$ is an operator of order $-1$.

Our approach also reproves a stronger result in the product situation: namely assume that $A = J_0(\frac{\mathrm{d}}{\mathrm{d}x} + B_0)$ in a collar of the boundary with $B_0 = B_0^t$. Using the following modified version[4] of $R$

$$(R'\xi)(x) := \begin{pmatrix} Q_+(x)\xi \\ (J_0^t T)^{-1} Q_-(-x)^*(J_0^t T)\xi \end{pmatrix}, \qquad R'_T\xi := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -T \end{pmatrix} R'\xi, \tag{5.32}$$

one then has $\widetilde{A}R_T\xi = 0$ and hence $S(A, T)\xi$ is supported in $\operatorname{supp}\varphi'$ away from the boundary. Then it is not difficult to see that $S(A, T)$ is smoothing (cf. [11, Prop 3.15 and Eq. (3.38)]) and thus $C_+ - P_+(B_0)$ is a smoothing operator.

This result was proved by Grubb [23, Prop. 4.1]. Before it was shown by S. Scott [24, Prop. 2.2] for self-adjoint Dirac operators on spin manifolds in the case $\ker B_0 = 0$.

In general, $C_+$ and $P_+(B_0)$ belong to different connected components of the Grassmannian of pseudo-differential projections with fixed leading symbol even for symmetric $B_0$, see [1, Remark 22.25]. For operators of Dirac type, however, $C_+$ can be continuously deformed into a finite range perturbation of $P_+(B_0)$ in the $L^2$ and the $L^2_{1/2}$ operator topology, see [2, Corollary C.3].

4 The whole discussion after (5.17) can be carried out with this modified $R'$, too.

**Proposition 5.12.** *$C_\pm$ are idempotents with $C_+ + C_- = \mathrm{Id}$ and*

$$\begin{aligned} C_+(L^2_s) &= N^s_+, \\ C_-(L^2_s) &= T^{-1}N^s_-, \end{aligned} \qquad -1/2 \le s \le 1/2.$$

*Furthermore if $T := (J_0^t)^{-1}$ then $C_\pm^* = C_\pm$, i.e., $C_\pm$ act as orthogonal projections on $L^2$. In that case $C_\pm \restriction L^2_0$ are $L^2$ extensions of pseudo-differential projections.*

**Remark 5.13.** In view of the previous proposition and Theorem 4.7 we can always construct $\widetilde{A}_{P(T)}$ in such a way that $C_\pm$ are orthogonal projections (as mentioned, by choosing $T := (J_0^t)^{-1}$). However, even if $A$ is symmetric it may happen that $\widetilde{A}_{P(T)}$ is not self-adjoint.

If $A = A^t$ and if $T = J_0(-J_0^2)^{1/2}$ satisfies (5.9) we may construct a self-adjoint extension of $\widetilde{A}$ at the cost of a non-orthogonal Calderón projection.

Only if $J_0^2 = -\mathrm{Id}$ and $T = (J_0^t)^{-1}$ then $\widetilde{A}_{P(T)}$ is self-adjoint **and** $C_\pm$ are orthogonal projections.

**Proof of Proposition 5.12.** We already know from Proposition 5.8 that

$$\begin{aligned} &C_+(L^2_s) \subset N^s_+, \\ &C_-(L^2_s) \subset T^{-1}N^s_-. \end{aligned}$$

We show

(i) $N^s_+ \cap T^{-1}N^s_- = \{0\}$,
(ii) $C_+ + C_- = \mathrm{Id}$.

This easily implies the first claim.

(i) Let $\xi \in N^s_+ \cap T^{-1}N^s_-$. Then there are $f \in Z_+^{s+1/2}$, $g \in Z_-^{s+1/2}$ with $\varrho f = \xi = T^{-1}\varrho g$. Then

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \ker \widetilde{A}_{P(T)} = Z_{+,0}(\widetilde{A}). \tag{5.33}$$

Since elements of $Z_{+,0}(\widetilde{A})$ vanish on the boundary we infer $\xi = 0$.

(ii) Let $\xi \in L^2_s(\Sigma, E_\Sigma)$, $-1/2 \le s \le 1/2$, and $f \in \mathscr{D}\big((\widetilde{A}_{P(T)})^*\big)$. Then $\varrho_+ f = -T\varrho_- f$ (cf. Remark 4.4) and exploiting the self-adjointness of $J_0^t T$ we obtain

$$\begin{aligned} \langle (C_+ + C_-)\xi, J_0^t \varrho_+ f\rangle &= \langle \varrho_+ \widetilde{G}\varrho^* J_0 \xi - T^{-1}\varrho_- \widetilde{G}\varrho^* J_0\xi, J_0^t\varrho_+ f\rangle \\ &= \langle \varrho_+\widetilde{G}\varrho^* J_0\xi, J_0^t \varrho_+ f\rangle - \langle \varrho_-\widetilde{G}\varrho^* J_0 \xi, J_0 J_0^{-1}(T^{-1})^t J_0^t \varrho_+ f\rangle \\ &= \langle (\varrho_+ \oplus \varrho_-)\widetilde{G}\varrho^* J_0\xi, (J_0^t \oplus J_0)(\varrho_+ f \oplus \varrho_- f)\rangle \\ &= \langle \widetilde{G}\varrho^* J_0 \xi, (\widetilde{A}_{P(T)})^* f\rangle \\ &= \langle \varrho^* J_0\xi, f\rangle = \langle \xi, J_0^t \varrho f\rangle. \end{aligned} \tag{5.34}$$

This proves (ii).

Finally, let $T = (J_0^t)^{-1}$ and pick $\xi \in N^0_+$, $\eta \in T^{-1}N^0_-$. Choose $f \in Z_+^{1/2}$ with $\varrho_+ f = \xi$ and $g \in Z_-^{1/2}$ with $T^{-1}\varrho_+ g = \eta$. Then Green's formula Lemma 2.2 gives

$$\langle \xi, \eta\rangle = \langle J_0 \varrho_+ f, \varrho_+ g\rangle = -\langle Af, g\rangle + \langle f, A^t g\rangle = 0. \tag{5.35}$$

Hence $N^0_+ \perp T^{-1}N^0_-$ and we are done.

To prove the pseudo-differential property we recall from [9, Appendix] Seeley's construction of the Calderón projection which always yields a pseudo-differential projection $\mathscr{P}^s_+$ onto $N^s_+$. By orthogonalization, cf. Lemma 3.5, we obtain an orthogonal pseudo-differential projection onto $N^s_+$ which must coincide with $C_+$ for $s = 0$. □

## 6. The General Cobordism Theorem

In the preceding sections, we gave a new definition of the Calderón projection. We achieved a canonical construction, free of extensions and other choices, and in greatest generality. Our main goal with the new definition was a construction which admits to follow precisely a continuous variation of the coefficients of an elliptic differential operator up to the induced variation of the Calderón projection. We shall return to this application below in Section 7.

An added bonus of our construction of the Calderón projection is that it leads immediately, and somewhat surprisingly, to a simple proof and a wide generalization of the classical Cobordism Theorem. We shall now give five different formulations of the Cobordism Theorem and show that the first claim (I), expressed in the language of symplectic functional analysis, follows immediately from our construction of the Calderón projection, and that the four other definitely non-trivial claims (II)–(V) are easily derived from the first claim. Put differently, we shall show that the Cobordism Theorem and its various generalizations and reformulations are an almost immediate consequence of our construction of the Calderón projection.

In this section we shall assume that our first order elliptic operator $A$ over the smooth compact manifold $M$ with boundary $\Sigma$ is formally self-adjoint, i.e., $A = A^t$. For convenience we shall write $B_0 := B(0)$ and $J_0 := J(0)$ as in the previous sections.

### 6.1. The General Cobordism Theorem

The main result of this section is

**Theorem 6.1** (*The General Cobordism Theorem*). *Let $A : \Gamma^\infty(M; E) \to \Gamma^\infty(M; E)$ be a first order formally self-adjoint elliptic differential operator on a smooth compact manifold M with boundary acting between sections of the vector bundle E. We assume that (5.9) is satisfied*[5] *by $T = J_0(-J_0^2)^{-1/2}$*

*Then we have the following results:*

(I) *Let $C_\pm$ denote the Calderón projections introduced in Definition 5.9, constructed from the invertible double with $T \in \{(J_0^t)^{-1}, J_0, J_0(-{J_0}^2)^{-1/2}\}$. Then the range of $C_+$ is a Lagrangian subspace of the strongly symplectic Hilbert space $\left(L^2(\Sigma, E_\Sigma), -J_0\right)$. Note that $\operatorname{im} C_+$ is independent of T. Moreover, $\operatorname{im} C_-$ is also Lagrangian, if $T := J_0(-{J_0}^2)^{-1/2}$.*

(II) *We have $\operatorname{sign} iP_0J_0 \restriction W_0 = 0$. Here $W_0$ denotes the (finite-dimensional) sum of the generalized eigenspaces of $B_0$ corresponding to purely imaginary eigenvalues and $P_0$ denotes the orthogonal projection onto $W_0$; in general $J_0$ will not map $W_0$ to itself.*

*If $B_0 = {B_0}^t$, then $J_0$ anticommutes with $B_0$ and we have $\operatorname{sign} iJ_0{\restriction} \ker B_0 = 0$.*

(III) *Under the same additional assumption, i.e., for $B_0 = {B_0}^t$, the tangential operator $B_0$ is odd with respect to the grading given by the unitary operator $\alpha := iJ_0(-{J_0}^2)^{-1/2}$ and hence splits into matrix form $B_0 = \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix}$ with respect to the $\pm 1$-eigenspaces of $\alpha$. The index of $B^+ : \ker(\alpha - 1) \longrightarrow \ker(\alpha + 1)$ vanishes.*

(IV) *While we do not know whether $C_+$ is a pseudo-differential operator for $T = J_0(J_0^2)^{-1/2}$, we can prove the following:*

*There exists a pseudo-differential projection P over $\Sigma$ such that $\ker P$ is a Lagrangian subspace of the strongly symplectic Hilbert space $\left(L^2(\Sigma, E_\Sigma), -J_0\right)$, and $(\ker P, \operatorname{im} C_+)$ is a Fredholm pair of closed subspaces of $L^2(\Sigma, E_\Sigma)$.*

(V) *There exists a self-adjoint pseudo-differential Fredholm extension $A_P$.*

In the following, we shall first give our view of elements of symplectic functional analysis. Based on that, we shall prove the preceding theorem as indicated above, i.e., we shall prove (I) directly from our new construction of the Calderón projection, then the implications (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III), and then (II) $\Longrightarrow$ (IV), (V).

To us, our order of proving Theorem 6.1 is the most simple and natural, beginning with and footing on claim (I). However, at the end of this section we shall explain that one can reverse the order of the proof. In particular, we shall show that (V) was essentially proved by Ralston in 1970 in [15] and that (I) can be derived from (V) independently.

**Remark 6.2.** (1) An alternative reading of (III) is the following: the index of *any* elliptic first order differential operator $C$ with smooth coefficients over a smooth closed manifold $\Sigma$ must vanish, if the block operator $B := \begin{pmatrix} 0 & C^t \\ C & 0 \end{pmatrix}$ can be written as the tangential operator of an elliptic formally self-adjoint operator $A$ on a smooth compact manifold $M$ with $\partial M = \Sigma$. That is a new generalization of the illustrious *Cobordism Theorem* for Dirac operators on spin manifolds which played a decisive role for the first proof of the Atiyah-Singer Index Theorem (1963). Since then, it was slightly generalized and an impressive variety of different proofs were given. Our point here is to show that the Cobordism Theorem has nothing to do with the specific form of Dirac type operators but generalizes to all elliptic formally self-adjoint differential operators of first order.

(2) To some extent, our approach is motivated by [18, Section 1.C], and is similar to C. Frey's [25, Section 3.4]. The main difference between our approach and Frey's is that we reduce the arguments a bit more to purely algebraic reasoning. That permits us to get through, also in the general case, *not* assuming Dirac type, i.e., not assuming constant coefficients near the boundary in normal direction - as Frey does, as does all the other literature on cobordism invariance.

(3) From a geometric point of view, the assumption of constant coefficients may appear sufficiently general for many applications, justified for index problems by $K$-theory and homotopy invariance of the index, and sufficiently challenging for constantly attracting ever new and more simple and more ingenious proofs of the cobordism invariance of the index over the last 50 years. For a recent example and summary of the highlights, we refer to Braverman [26]. However, from an analysis point of view, it is not natural to assume constant coefficients near the boundary. Moreover, it seems more than timely that the pilot work by Ralston, admitting general coefficients and so showing a way for our General Cobordism Theorem, is taken into account in the analysis and topology literature.

### 6.2. Elements of symplectic functional analysis

Let us briefly summarize the basic set-up of symplectic functional analysis (see, e.g., [7,27] or [28]).

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[5] As noted in Remark 5.1 this is the case if $B_0 - B_0^t$ is of order 0.

*6.2.1. Basic definitions*

Let $H$ be a real or complex Hilbert space. A weakly symplectic form on $H$ is a non-degenerate Hermitian sesquilinear form $\omega$ on $H$. Sesquilinear means, following the tradition in mathematical physics, $\omega(\lambda x, y) = \overline{\lambda}\omega(x, y)$, $\omega(x, \lambda y) = \lambda\omega(x, y)$, and Hermitian means $\omega(y, x) = -\overline{\omega(x, y)}$ for $x, y \in H$. Finally, non-degeneracy means that the map $H \ni x \mapsto \omega(x, \cdot) \in H^*$ is an injective continuous linear map; it then follows that the range of this map is dense.

The pair $(H, \omega)$ is called a (complex) *weakly symplectic Hilbert space*. Since $\omega$ is continuous there is a unique skew-symmetric injective map $\gamma \in \mathscr{B}(H)$ such that

$$\omega(x, y) = \langle \gamma x, y\rangle, \quad x, y \in H. \tag{6.1}$$

Here, as before, $\mathscr{B}(H)$ denotes the space of bounded endomorphisms of $H$.

For a subspace $\lambda \subset H$ we have

$$\lambda^\omega := \{x \in H \mid \omega(x, y) = 0 \quad \text{for } y \in \lambda\} = \big(\gamma\lambda\big)^\perp, \tag{6.2}$$

hence $\lambda$ is Lagrangian (i.e. $\lambda^\omega = \lambda$) if and only if $\lambda = \big(\gamma\lambda\big)^\perp$.

The pair $(H, \omega)$ is called *strongly* symplectic, if the injective operator with dense range $H \ni x \mapsto \omega(x, \cdot) \in H^*$ is in fact surjective and hence has a bounded inverse. Equivalently, the skew-symmetric operator $\gamma$ which implements $\omega$ is invertible.

In a strongly symplectic Hilbert space we may choose an equivalent scalar product $\langle\cdot, \cdot\rangle_\gamma$ such that the operator which implements $\omega$ with respect to $\langle\cdot, \cdot\rangle_\gamma$ is the *unitary reflection* $\gamma(-\gamma^2)^{-1/2}$: namely

$$\omega(x, y) = \langle \gamma x, y\rangle = \langle(-\gamma^2)^{1/2}\gamma(-\gamma^2)^{-1/2}x, y\rangle = \langle\gamma(-\gamma^2)^{-1/2}x, y\rangle_\gamma \tag{6.3}$$

with $\langle\xi, \eta\rangle_\gamma := \langle(-\gamma^2)^{1/2}\xi, \eta\rangle$.

The scalar product $\langle\cdot, \cdot\rangle_\gamma$ is equivalent to $\langle\cdot, \cdot\rangle$ in the sense that there is a constant $C$ such that

$$C^{-1}\langle x, x\rangle \le \langle x, x\rangle_\gamma \le C\langle x, x\rangle, \quad x \in H. \tag{6.4}$$

In view of (6.3) the operator which implements $\omega$ with respect to $\langle\cdot, \cdot\rangle_\gamma$ is the unitary reflection $\gamma(-\gamma^2)^{-1/2}$.

Finally, we comment on isomorphisms: let $R : (H_1, \omega_1) \longrightarrow (H_2, \omega_2)$ be an invertible bounded linear map which is symplectic but *not* necessarily isometric. If $\omega_j(\cdot, \cdot) = \langle\gamma_j\cdot, \cdot\rangle$, $j = 1, 2$ with skew-symmetric $\gamma_j$ then $\gamma_1 = R^*\gamma_2 R$, in particular $(H_1, \omega_1)$ is strongly symplectic if and only if $(H_2, \omega_2)$ is.

**Remark 6.3.** For simplicity, we assume from now on that all vector spaces, Hilbert spaces, and symplectic spaces are complex. Note, however, that our definition of symplectic Hilbert space does not require the existence of a Lagrangian subspace (as opposed to e.g. [28, Sec. 6]). Indeed, it might be that there are no Lagrangian subspaces at all. Take, e.g., $H := \mathbb{C}$ and $\omega(x, y) := \mathrm{i}\overline{x}y$; see also [28, Rem. 6.13].

*6.2.2. Algebraic observations*

Dealing with elliptic problems on manifolds with boundary naturally leads to symplectic Hilbert spaces, see below Section 6.2.4. However, keeping the arguments on a purely algebraic level where possible may make proofs more transparent. Moreover, Sobolev chains of symplectic Hilbert spaces are equipped with a variety of non-equivalent norms but compatible symplectic forms. For such applications it is nice to establish results independent of topological choices.

So, let $(H, \omega)$ be a symplectic vector space; i.e., no boundedness of the symplectic form is assumed.

First we note that the quadratic form $x \mapsto \mathrm{i}\omega(x, x)$ has well-defined signature, if $H$ is finite-dimensional. In that case we have

$$\operatorname{sign} \mathrm{i}\omega = 0 \iff \exists\lambda \subset H \text{ Lagrangian subspace}, \tag{6.5}$$

cf. Remark 6.7(1).

Next, we recall a simple algebraic observation, taken from [27, Lemma 1.2]. Here, and in the proposition further below, the point is to establish the Lagrangian property for isotropic subspaces via a *purely algebraic Fredholm pair property*, i.e., finite-dimensional intersection and finite codimension of sum; no closedness is assumed. The algebraic Fredholm pair property can be considered as the natural generalization of the well-known definition of Lagrangian subspaces in finite dimension as isotropic subspaces of *maximal dimension*.

**Lemma 6.4.** *Let $(H, \omega)$ be a symplectic vector space with transversal subspaces $\lambda, \mu$, i.e., $\lambda + \mu = H, \lambda \cap \mu = 0$. If $\lambda, \mu$ are isotropic subspaces, then they are Lagrangian subspaces.*

**Proof.** From linear algebra we have

$$\lambda^\omega \cap \mu^\omega = (\lambda + \mu)^\omega = \{0\},$$

since $\lambda + \mu = H$. From

$$\lambda \subset \lambda^\omega, \qquad \mu \subset \mu^\omega \tag{6.6}$$

we get

$$H = \lambda^{\omega} \oplus \mu^{\omega}. \tag{6.7}$$

Since $H = \lambda \oplus \mu$ we conclude from (6.6) and (6.7) that $\lambda = \lambda^{\omega}$ and $\mu = \mu^{\omega}$. □

The preceding result can be generalized:

**Proposition 6.5.** *Let $(H, \omega)$ be a symplectic vector space with isotropic subspaces $\lambda, \mu$. If $(\lambda, \mu)$ forms an algebraic Fredholm pair with* $\operatorname{ind}(\lambda, \mu) \geq 0$, *then $\lambda$ and $\mu$ are Lagrangian subspaces of $H$ and we have*

$$\operatorname{ind}(\lambda, \mu) = 0, \qquad (\lambda + \mu)^{\omega} = \lambda \cap \mu, \quad \text{and} \quad (\lambda + \mu)^{\omega\omega} = \lambda + \mu.$$

For the proof we refer to [27, Proposition 1.13a].

*6.2.3. Symplectic reduction*

We recall a lemma on symplectic reduction from [28, Prop 6.12] (see also [29, Proposition 2.2] for a generalization to weakly symplectic Hilbert spaces):

**Lemma 6.6.** *Let $(H, \omega)$ be a strongly symplectic Hilbert space, $\lambda \subset H$ a Lagrangian subspace and $W \subset H$ a closed co-isotropic subspace. Assume that $\lambda + W^{\omega}$ is closed. Then the form*

$$\widetilde{\omega}(x + W^{\omega}, y + W^{\omega}) := \omega(x, y), \quad x, y \in W$$

*is a strongly symplectic form on $W/W^{\omega}$. Moreover, the symplectic reduction of $\lambda$ by $W$*

$$\operatorname{Red}_W(\lambda) := \big((\lambda + W^{\omega}) \cap W\big)/W^{\omega} \subset W/W^{\omega} \tag{6.8}$$

*is a Lagrangian subspace of $W/W^{\omega}$.*

**Remark 6.7.** (1) Without the assumption $\lambda + W^{\omega}$ closed $\widetilde{\omega}$ will still be a non-degenerate sesquilinear form on the quotient $W/W^{\omega}$. However, in that case it might be that there are no Lagrangian subspaces at all, as pointed out in Remark 6.3. If the form $\widetilde{\omega}$ is written as $\widetilde{\omega}(\xi, \eta) = \langle \widetilde{\gamma}\xi, \eta \rangle$ with a suitable scalar product and a skew-symmetric unitary operator $\widetilde{\gamma}$, then the existence of a Lagrangian subspace is equivalent to the fact that the $\pm i$ eigenspaces of $\widetilde{\gamma}$ have the same dimension, corresponding to (6.5).

(2) In [28, Prop. 6.12] the Lemma was formulated under seemingly more restrictive assumptions. Let us give an equivalent formulation of the Lemma which clarifies the link to [28] and which will be useful below:

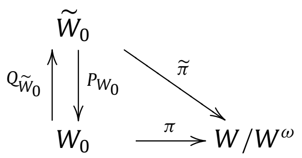
**Lemma 6.8.** *Let $(H, \omega)$, $\lambda \subset H$ and $W \subset H$ be as in Lemma 6.6. Let $\gamma$ be the invertible skew-symmetric operator in $H$ such that $\omega(\cdot, \cdot) = \langle \gamma \cdot, \cdot \rangle$.*

*Let $W_0 \subset W$ be a closed subspace such that $W = W_0 \oplus W^{\omega}$ (the sum is not necessarily orthogonal). Furthermore, let $P_0$ denote the orthogonal projection onto $W_0$ and let $Q_0$ denote the projection* along *$W^{\omega}$ onto $W_0$.*

*Then $\omega \restriction W_0 = \langle P_0 \gamma \cdot, \cdot \rangle$ is a* strongly *symplectic form on $W_0$, $Q_0(\lambda \cap W)$ is a Lagrangian subspace of $(W_0, \omega)$ and the quotient map $\pi : W_0 \longrightarrow W/W^{\omega}$ is a bounded invertible symplectic linear operator.*

**Proof.** Let us briefly sketch the proof of this and of the previous lemma. First as remarked at the beginning of this subsection we may choose a scalar product such that the corresponding $\gamma$ is unitary. Consider first $W_0 = (W^{\omega})^{\perp} = W \cap \gamma W$. [28, Prop. 6.12] and its proof show that $(W_0, \omega)$ is symplectic and that $Q_0 = P_0$ maps $\lambda \cap W$ onto a Lagrangian subspace of $W_0$. Furthermore, since $W_0 = (W^{\omega})^{\perp}$ in this case, the quotient map $\pi \restriction W_0 : (W_0, \omega) \to (W/W^{\omega}, \widetilde{\omega})$ is a unitary symplectic isomorphism. This proves that $(W/W^{\omega}, \widetilde{\omega})$ is indeed a strongly symplectic Hilbert space and that $\operatorname{Red}_W(\lambda)$ is a Lagrangian subspace. Hence Lemma 6.6 is proved.

To prove Lemma 6.8 for an arbitrary closed subspace $\widetilde{W}_0 \subset W$ with $W = \widetilde{W}_0 \oplus W^{\omega}$ we only have to note that we have the following commutative diagram




(6.9)

where $\pi, \widetilde{\pi}$ denote the quotient map $W \longrightarrow W/W^{\omega}$ restricted to $W_0, \widetilde{W}_0$ respectively, $P_{W_0}$ denotes the orthogonal projection onto $W_0$ and $Q_{\widetilde{W}_0}$ denotes the projection along $W^{\omega}$ onto $\widetilde{W}_0$. $\pi, \widetilde{\pi}$ are symplectic bounded invertible maps. From this, all remaining claims follow. □

*6.2.4. The von-Neumann quotient of all natural boundary values*

We recall the basic findings about self-adjoint extensions and the relation between Fredholm Lagrangian pairs in the von-Neumann quotient $\mathscr{D}(A_{\max})/\mathscr{D}(A_{\min}) =: \beta(A)$ and in $L^2(\Sigma, E_\Sigma)$:

Let $A_m$ be a closed symmetric operator with domain $\mathscr{D}_m$ in a Hilbert space $H$. Following von Neumann and the Russian tradition of M. Krein, Vishik, and Birman, the operator $A_m$ defines a (strongly) symplectic Hilbert space $\beta(A_m) := \mathscr{D}_{\max}/\mathscr{D}_m$ of *natural boundary values*. Here $\mathscr{D}_{\max}$ denotes the domain of $A_m^*$. The Hilbert space structure on $\beta(A_m)$ is given by the graph scalar product, and the symplectic form is given by Green's form

$$\omega(x+f, y+g) := \langle Ax, y\rangle - \langle x, Ay\rangle \quad \text{for } x, y \in \mathscr{D}_{\max}, \tag{6.10}$$

independent of the choice of $f, g \in \mathscr{D}_m$ (see our Eq. (2.11) and [7, Section 3]). It is well known that there is a one-to-one correspondence between

- domains $\mathscr{D}_m \subset \mathscr{D} \subset \mathscr{D}_{\max}$ which yield a self-adjoint operator $A_{\mathscr{D}} := A_m^* \restriction \mathscr{D}$
- and the Lagrangian subspaces $\lambda$ of $\beta(A_m)$

by

$$\mathscr{D} \mapsto \lambda := \mathscr{D}/\mathscr{D}_m \quad \text{and} \quad \lambda \mapsto \mathscr{D} := \{x \in H \mid x + \mathscr{D}_m \in \lambda\}.$$

In our situation, we set $H := L^2(M, E)$ and consider $A_m = A_{\min} = A \restriction \mathscr{D}(A_{\min})$ with $\mathscr{D}_m = \mathscr{D}(A_{\min}) = L^2_{1,0}(M, E)$, the closure of $\Gamma_0^\infty(M \setminus \Sigma; E)$ in $L^2_1(M, E)$ as in Remark 2.5. Then $\beta(A_m)$ naturally becomes a subspace of $L^2_{-1/2}(\Sigma, E_\Sigma)$. From now on we will use this identification and view $\beta(A_m)$ as a subspace of $L^2_{-1/2}(\Sigma, E_\Sigma)$.

To discuss self-adjoint extensions by boundary conditions given by pseudo-differential projections, it is helpful to consider two other symplectic spaces: the strongly symplectic Hilbert space $L^2(\Sigma, E_\Sigma)$ with symplectic form induced by $-J_0$ and the weakly symplectic Hilbert space $L^2_1(M, E)/L^2_{1,0}(M, E)$ with symplectic form induced by Green's form, or, equivalently, by $-\widetilde{J_0}$, as well. Here we identify the quotient with the subspace $L^2_{1/2}(\Sigma, E_\Sigma) \subset L^2(\Sigma, E_\Sigma)$ of sections over the boundary, but with different scalar product $\langle\cdot,\cdot\rangle_{L^2_{1/2}(\Sigma, E_\Sigma)}$, hence $\widetilde{J_0} = (\mathrm{Id} + |B|)^{-1} J_0$. Note that $\widetilde{J_0}$ is not invertible for $\dim M > 1$, as explained in [27, Remark 1.6b].

The relations between the Lagrangian subspaces of these three different symplectic spaces are somewhat delicate because neither $L^2(\Sigma, E_\Sigma) \subset \beta(A_m)$ nor $L^2(\Sigma, E_\Sigma) \supset \beta(A_m)$. We may, however, recall a very general result from [30, Theorem 1.2a]:

Let $\beta$ and $L$ be strongly symplectic Hilbert spaces with symplectic forms $\omega_\beta$ and $\omega_L$, respectively. Let

$$\beta = \beta_- \oplus \beta_+ \quad \text{and} \quad L = L_- \oplus L_+ \tag{6.11}$$

be direct sum decompositions by transversal (not necessarily orthogonal) pairs of Lagrangian subspaces. We assume that there exist continuous, injective mappings

$$i_- : \beta_- \longrightarrow L_- \quad \text{and} \quad i_+ : L_+ \longrightarrow \beta_+ \tag{6.12}$$

with dense images and which are compatible with the symplectic structures, i.e.,

$$\omega_L(i_-(x), a) = \omega_\beta(x, i_+(a)) \quad \text{for all } a \in L_+ \text{ and } x \in \beta_-. \tag{6.13}$$

Let $\lambda_0$ be a fixed Lagrangian subspace of $\beta$. We consider the *Fredholm Lagrangian Grassmannian* of $\lambda_0$

$$\mathscr{FL}_{\lambda_0}(\beta) := \{\mu \subset \beta \mid \mu \text{ Lagrangian subspace and } (\mu, \lambda_0) \text{ Fredholm pair}\}.$$

The topology of $\mathscr{FL}_{\lambda_0}(\beta)$ is defined by the operator norm of the orthogonal projections onto the Lagrangian subspaces.

**Theorem 6.9** (*Booß-Bavnbek, Furutani, Otsuki*). *Under the assumptions (6.11), (6.12), and (6.13), we have a natural continuous mapping*

$$\tau : \mathscr{FL}_{\beta_-}(\beta) \longrightarrow \mathscr{FL}_{L_-}(L), \qquad \mu \longmapsto \mu \cap L,$$

*where $\beta$ and $L$ are identified with subspaces of $\beta_+ \oplus L_-$.*

The following splitting lemmata are of independent interest. Note that we do not claim a direct sum decomposition of $\beta(A)$ into Lagrangian subspaces for now. Later, this will be a consequence of our Theorem 6.1. See also the recent [31, Section 1]. In the following lemma, we could use our $C_+$ instead of using Seeley's Calderón projection $\mathscr{P}_+$. All these direct sum decompositions of $\beta(A)$ are different, but equally valid, see Lemma 6.11 and Remark 6.12.

**Lemma 6.10.** *Let $A$ be an elliptic formally self-adjoint first order differential operator on a compact smooth manifold $M$ with smooth boundary $\Sigma$ and let $\mathscr{P}_+$ denote Seeley's corresponding (pseudo-differential) Calderón projection. Then the space $\beta(A) := \mathscr{D}(A_{\max})/\mathscr{D}(A_{\min})$ can be described explicitly as the direct sum*

$$\beta(A) = \mathrm{im}(\mathscr{P}_{+,-1/2}) \oplus \mathrm{im}(\mathrm{Id} - \mathscr{P}_{+,1/2}), \tag{6.14}$$

*where $\mathscr{P}_{+,s}$ denotes the extension/restriction of the pseudo-differential $\mathscr{P}$ to $L^2_s(\Sigma, E_\Sigma)$.*

**Proof.** (1) First we show the inclusion $\operatorname{im}(\mathscr{P}_{+,-1/2}) \subset \beta(A)$. Let $f$ belong to $\operatorname{im}(\mathscr{P}_{+,-1/2})$. Then there exists $f_1 \in L^2_{-1/2}(\Sigma, E_\Sigma)$ with $f = r_+\widetilde{G}\rho^* J_0 f_1$, where $\widetilde{G}$ denotes Seeley's 'inverse on the double' (which in contrast to our $\widetilde{G}$ is not canonically defined) and $\widetilde{\rho}^*$ Seeley's dual of the trace (which, once again, in contrast to our $\rho^*$ is neither canonically defined). We observe that $g := \widetilde{G}\rho^* J_0 f_1 \in \mathscr{D}(\widetilde{A}_{\max})$ with $\widetilde{A}g = 0$. Note that this is Seeley's $\widetilde{A}$ which is not canonical and therefore not suitable for discussing the parameter dependence, but has the advantage of delivering a pseudo-differential Calderón projection. So $f = \rho r_+ g$ with $r_+ g \in \mathscr{D}(A_{\max})$. Hence $f \in \beta(A)$.
(2) Next we observe that $L^2_{1/2}(\Sigma, E_\Sigma) \subset \beta(A)$ since $L^2_1(M, E) \subset \mathscr{D}(A_{\max})$.
(3) Together with argument (1) this implies

$$\operatorname{im}(\mathscr{P}_{+,-1/2}) \oplus \operatorname{im}(\operatorname{Id} - \mathscr{P}_{+,1/2}) \subset \beta(A).$$

(4) To show the equality, we notice $\beta(A) \subset L^2_{-1/2}(\Sigma, E_\Sigma)$. Applying Seeley's result

$$\mathscr{P}_{+,s} + \mathscr{P}_{-,s} = \operatorname{Id} \tag{6.15}$$

for $s = -1/2$ we can write each $f \in \beta(A)$ in the form

$$f = f_1 + f_2, \text{ where } f_1 \in \operatorname{im}(\mathscr{P}_{+,-1/2}) \text{ and } f_2 \in \operatorname{im}(\mathscr{P}_{-,-1/2}).$$

By (1), $f_1 \in \beta(A)$, so $f_2 = f - f_1 \in \beta(A)$, i.e., there exists a $g \in \mathscr{D}(A_{\max})$ such that $f_2 = \rho g$. Note $\mathscr{P}_+ f_2 = 0$ by the splitting (6.15). Applying one version of Gårding's inequality (see, e.g., [1, Chapter 18])

$$\|g\|_{L^2_1(M,E)} \le C\big(\|g\|_{L^2(M,E)} + \|Ag\|_{L^2(M,E)} + \|\mathscr{P}_+\rho g\|_{L^2_{1/2}(\Sigma,E_\Sigma)}\big), \tag{6.16}$$

we obtain $g \in L^2_1(M, E)$ and so $f_2 = \rho g \in L^2_{1/2}(\Sigma, E_\Sigma)$, i.e., $f_2 \in \operatorname{im}(\mathscr{P}_{-,1/2})$. □

**Lemma 6.11.** *Let $P, Q$ be two pseudo-differential projections with the same leading symbol. Then*

$$\operatorname{im}(P_{-1/2}) \oplus \operatorname{im}(\operatorname{Id} - P_{1/2}) = \operatorname{im}(Q_{-1/2}) \oplus \operatorname{im}(\operatorname{Id} - Q_{1/2}). \tag{6.17}$$

**Proof.** So, let $f \in \operatorname{im}(P_{-1/2}) \oplus \operatorname{im}(\operatorname{Id} - P_{1/2})$. Then there are $\varphi \in L^2_{-1/2}(M, E)$, $\psi \in L^2_{1/2}(M, E)$ such that

$$f = P\varphi + (I - P)\psi = Q\varphi + (I - Q)\psi + \underbrace{(P - Q)(\varphi - \psi)}_{=:h}.$$

By assumption $P - Q$ is a pseudo-differential operator of order $\le -1$, hence $h \in L^2_{1/2}(M, E)$ and thus $f = Q(\varphi + h) + (I - Q)(\psi + h)$ with $\varphi + h \in L^2_{-1/2}(M, E)$ and $\psi + h \in L^2_{-1/2}(M, E)$, proving the claim. □

**Remark 6.12.** By combining the two preceding lemmata we obtain the useful formula

$$\beta(A) = \operatorname{im}(P_{-1/2}) \oplus \operatorname{im}\big((\operatorname{Id} - P)_{1/2}\big) \tag{6.18}$$

for all pseudo-differential projections $P$ with $\sigma_0(P) = \sigma_0(\mathscr{P}_+)$ where $\mathscr{P}_+$ denotes Seeley's (pseudo-differential) Calderón projection.

Eq. (6.18) generalizes a previous result in [32, Proposition 7.15] obtained for the spectral Atiyah-Patodi-Singer projection $P := P_{\ge}(B_0)$. There, the von Neumann space $\beta(A)$ was expressed as the direct sum of

- the $L^2_{1/2}(\Sigma, E_\Sigma)$-closure of the linear span of the negative eigenspaces of the tangential operator $B_0$ with
- the $L^2_{-1/2}(\Sigma, E_\Sigma)$-closure of the linear span of the nonnegative eigenspaces of $B_0$

in the special case that the operator $A$ is of Dirac type in product metric near the boundary (in particular, that $A$ has a formally self-adjoint tangential operator $B_0$ and constant coefficients in normal direction near the boundary).

### 6.3. *Proof of* Theorem 6.1

As announced above, we prove the five claims successively.

#### 6.3.1. *Application of the new construction of the Calderón projection*

**Proof of (I).** We recall that $(u, v) \mapsto \omega(u, v) := \langle -J_0 u, v\rangle$ is a symplectic form for the Hilbert space $\big(L^2(\Sigma, E_\Sigma), \langle\cdot,\cdot\rangle\big)$. It is strong since $J_0$ is a bundle isomorphism. Then the range $\operatorname{im}(C_+) = N^0_+$ is an isotropic subspace because of Green's formula (A.13), applied to the kernel of the formally self-adjoint operator $A$. Here we use $T := J_0\big(-J_0{}^2\big)^{-1/2}$ to construct $C_\pm$, see Remark 5.13. Then also $\operatorname{im}(C_-) = T^{-1}(N^0_+)$ is an isotropic subspace since the chosen $T$ is clearly symplectic. By Proposition 5.12, we have $C_+ + C_- = \operatorname{Id}$, so $N^0_+$ and $T^{-1}(N^0_+)$ make a pair of transversal isotropic subspaces of $L^2(\Sigma, E_\Sigma)$. Then (I) follows by applying Lemma 6.4. □

**Remark 6.13.** We notice that the splitting $\mathscr{P}_+ + \mathscr{P}_- = \mathrm{Id}$ in Seeley, [33, Lemma 5] does not provide two transversal Lagrangian subspaces but only an isotropic range of $\mathscr{P}_+$ with the preceding argument in the case of symmetric $A$. The problem is that even for symmetric $A$ Seeley's continuation into a collar and further over the double does not preserve symmetry in general.

*6.3.2. Stability arguments*

Before deriving **(II)** we shall address stability aspects of the issue.

We see at once that any formally self-adjoint operator of the form

$$A := -j_t \frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} j_t\right) - b_t, \quad j_t \text{ invertible}$$

on the unit interval $t \in [0, 1]$ admits self-adjoint boundary conditions. The symplectic form on the space of boundary values is given by $J := j_0 \oplus (-j_1)$ with respect to the reversed orientation at the ends of the interval. By continuity, we have $\operatorname{sign} j_0 = \operatorname{sign} j_1$. So, $\operatorname{sign} iJ = 0$. (Note that there is no tangential operator in the 1-dimensional case).

In higher dimension, a similar continuity argument does not work in general.

The following lemma yields the stability of the signature of the almost complex form $J_t$ on the nullspace $\ker B_t$ under variation of the parameter $t$. It can be considered as an index stability statement (and certainly can be proved also that way instead of the proof given below).

**Lemma 6.14.** *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $(S_t)$ and $(J_t)$ be two continuous families of bounded invertible operators on $H$, $t \in [0, 1]$. Assume that all $S_t$ are positive definite. Let $(B_t)$ be a continuous family of closed Fredholm operators. We assume that all $\mathrm{i}J_t$ and $B_t$ are self-adjoint with respect to the new metric $\langle x, y\rangle_t := \langle S_t x, y\rangle$. Moreover, we assume that all $J_t$ have bounded inverses, and $J_t B_t = -B_t J_t$ for all $t$. Then we have*

$$\operatorname{sign}(\mathrm{i}J_t{\restriction}\ker B_t) = \textit{constant}. \tag{6.19}$$

**Proof.** We divide the proof into three steps.
*Step* 1. We can assume that $S_t = \mathrm{Id}$. Indeed, denote by $A^{*_t}$ the adjoint operator of $A$ with respect to the scalar product induced by $S_t$. Then we have

$$\langle S_t x, A^{*_t} y\rangle = \langle S_t A x, y\rangle.$$

So $A^{*_t} = S_t^{-1} A^* S_t$. Set

$$J_t' := S_t^{\frac{1}{2}} J_t S_t^{-\frac{1}{2}}, \quad B_t' := S_t^{\frac{1}{2}} B_t S_t^{-\frac{1}{2}}.$$

Then $\mathrm{i}J_t'$ and $B_t'$ are self-adjoint, $J_t' B_t' = -B_t' J_t'$, and

$$\operatorname{sign}(\mathrm{i}J_t{\restriction}\ker B_t) = \operatorname{sign}(\mathrm{i}J_t'{\restriction}\ker B_t').$$

So we can assume $S_t = \mathrm{Id}$.
*Step* 2. We reduce to the finite-dimensional case: For each $t \in [0, 1]$, there is a small $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \cap \sigma(B_t) \subset \{0\}$. Then for $s$ close to $t$, $\pm\varepsilon \notin \sigma(B_s)$. Let $D_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$. Define

$$P_s := -\frac{1}{2\pi \mathrm{i}} \int_{\partial D_\varepsilon} (B_s - z\,\mathrm{Id})^{-1} \mathrm{d}z.$$

Then $P_s$ is a continuous family of orthogonal projections of finite rank, and $P_s J_s P_s$, $P_s B_s P_s : \operatorname{im} P_s \to \operatorname{im} P_s$ satisfy our assumptions, and $P_t B_t P_t = 0$.
*Step* 3. Since in the finite-dimensional case $\operatorname{sign}(\mathrm{i}J_t)$ is constant, it suffices to prove the following
*Claim.* Let $H$ be finite-dimensional. Then we have $\operatorname{sign}(\mathrm{i}J_t) = \operatorname{sign}(\mathrm{i}J_t{\restriction}\ker B_t)$.

In fact, let $V_t$ denote the orthogonal complement of $\ker B_t$. Then both $\ker B_t$ and $V_t$ are invariant subspaces of $J_t$. Since on $V_t$, $B_t$ is invertible, and since we have $-J_t = B_t^{-1} J_t B_t$, we have $\operatorname{sign}(\mathrm{i}J_t{\restriction}V_t) = 0$. Hence our claim follows. □

*6.3.3. Proof of (II), (III)*

We now proceed to show the General Cobordism Theorem (II).

We exploit the formal self-adjointness of $A$ and choose, in a collar of the boundary, the normal form

$$A = J_x\left(\frac{\mathrm{d}}{\mathrm{d}x} + B_x\right) + \frac{1}{2} J_x'$$

of Eq. (2.21) with the relations $J^* = -J$, $JB = -B^t J$. The relation $JB = -B^t J$ has consequences for the positive/negative sectorial spectral subspaces with regard to the natural symplectic structure on $L^2(\Sigma, E_\Sigma)$, see the proof of Lemma 6.15.

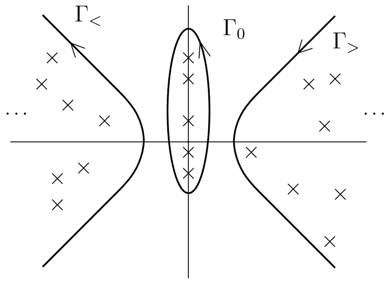



**Fig. 4.** Three contours encircling all eigenvalues in the right half plane, on the imaginary axis, and all eigenvalues in the left half plane, respectively.

Similarly to Fig. 1 we now choose contours $\Gamma_<$, $\Gamma_>$ and $\Gamma_0$ as follows (see Fig. 4): $\Gamma_<$ encircles all eigenvalues in the left half plane, $\Gamma_>$ encircles all eigenvalues in the right half plane, and $\Gamma_0$ encircles all eigenvalues on the imaginary axis $\mathrm{i}\mathbb{R}$. The corresponding spectral projections are denoted by $P_<(B_0)$, $P_>(B_0)$ and $P_0(B_0)$. In view of Theorem 3.14, these are pseudo-differential operators of order 0, and hence bounded, and as closed idempotents they do have closed range (see Section 3.2.1). $P_0(B_0)$ is of finite-rank and hence a smoothing operator with range being the sum of the generalized eigenspaces of $B_0$ to imaginary eigenvalues. We therefore have a direct sum decomposition

$$L^2(\Sigma, E_\Sigma) = \operatorname{im} P_<(B_0) \oplus \operatorname{im} P_0(B_0) \oplus \operatorname{im} P_>(B_0) =: W_< \oplus W_0 \oplus W_>. \tag{6.20}$$

In particular, $(W_>, W_<)$ is a Fredholm pair of closed subspaces of $L^2(\Sigma, E_\Sigma)$. Recall that the closedness of $W_<$, $W_>$ and the finite codimension of $W_< + W_>$ in $L^2(\Sigma, E_\Sigma)$ imply that $W_< + W_>$ is closed, see [32, Remark A.1].

**Lemma 6.15.** *$W_>$, $W_<$ are isotropic subspaces with $W_>^\omega = W_0 \oplus W_>$ and $W_<^\omega = W_< \oplus W_0$.*

**Proof.** This is a consequence of the relation $J_0 B_0 = -B_0^t J_0$ which implies for $\xi \in L_1^2(\Sigma, E_\Sigma)$ (cf. Section 3.2)

$$\begin{aligned} J_0 P_>(B_0)\xi &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_>} \lambda^{-1} J_0 (\lambda - B_0)^{-1} \mathrm{d}\lambda \; B_0 \xi \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_>} \lambda^{-1} (\lambda + B_0^t)^{-1} \mathrm{d}\lambda \; B_0^t J_0 \xi \\ &= P_<(B_0^t) J_0 \xi. \end{aligned} \tag{6.21}$$

Then

$$\begin{aligned} W_>^\omega &= (J_0 W_>)^\perp = (J_0 \operatorname{im} P_>(B_0))^\perp = \operatorname{im} P_<(B_0^t)^\perp \\ &= \ker P_<(B_0^t)^* = \ker P_<(B_0) = W_0 \oplus W_>. \end{aligned}$$

The other claim is proved analogously. □

Lemma 6.15 is now the key to the proofs of the remaining implications in this paragraph. Before proceeding, we note:

**Lemma 6.16.** *Let $P$ and $Q$ be bounded idempotents in a Banach space $H$, and $P - Q$ compact. Then the pair $(\ker P, \operatorname{im} Q)$ is Fredholm.*

**Proof.** Consider the operator

$$\begin{aligned} R &:= QP + (\mathrm{Id} - P)(\mathrm{Id} - Q) = Q(Q + P - Q) + (\mathrm{Id} - Q + Q - P)(\mathrm{Id} - Q) \\ &= Q + Q(P - Q) + \mathrm{Id} - Q + (Q - P)(\mathrm{Id} - Q) \\ &= \mathrm{Id} + Q(P - Q) + (Q - P)(\mathrm{Id} - Q). \end{aligned}$$

Since $P - Q$ is compact, $R$ is Fredholm. Since $(\ker P \cap \operatorname{im} Q) \subset \ker R$ and $(\ker P + \operatorname{im} Q) \supset \operatorname{im} R$, we have $\dim(\ker P \cap \operatorname{im} Q) < +\infty$ and $\dim(H/(\ker P + \operatorname{im} Q)) < +\infty$. Thus the pair $(\ker P, \operatorname{im} Q)$ is algebraically Fredholm, and hence Fredholm, since the spaces $\ker P$, $\operatorname{im} Q$ are closed. □

**Proof of (I) $\Longrightarrow$ (II).** Let $\mathscr{P}_+$ denote Seeley's corresponding (pseudo-differential) Calderón projection. Then we have $\operatorname{im} C_+ = \operatorname{im} \mathscr{P}_+$. Since the difference between $\mathscr{P}_+$ and $P_>(B_0)$ is of order -1, we have that $(\operatorname{im} \mathscr{P}_+, W_<)$ is a Fredholm pair by the preceding argument (and finite-dimensional perturbation). Hence $\operatorname{im} \mathscr{P}_+ \oplus W_<$ is a closed subspace.

By (I), $\operatorname{im} \mathscr{P}_+$ is a Lagrangian subspace. Applying Lemma 6.8 to the co-isotropic subspace $W_<^\omega = W_0 \oplus W_<$ we obtain that

$$\mathrm{Red}_{W_<^\omega}(\operatorname{im} \mathscr{P}_+) \simeq \pi(\mathrm{Red}_{W_<^\omega}(\operatorname{im} \mathscr{P}_+)) \subset W_0$$

is a Lagrangian subspace of

$$W^{\omega}_{<}/W^{\omega\omega}_{<} = (W_< + W_0)/W_< \simeq W_0.$$

So, the finite-dimensional symplectic Hilbert space $\big(W_0, \langle \mathrm{i}P_0J_0\cdot, \cdot\rangle\big)$ has a Lagrangian subspace. Therefore

$$\operatorname{sign} \mathrm{i}P_0J_0\restriction W_0 = 0. \quad \square$$

We assume $B_0 = {B_0}^t$ and put $\alpha = -iJ_0(-{J_0}^2)^{-1/2}$. Then the special case of claim (II) of Theorem 6.1 follows. Moreover, we have under that assumption:

**Proof of (II) ⇔ (III).** We follow the conventional lines and refer, e.g., to [1, Theorem 21.5] with the immediate modifications: Note that $\alpha$ is a grading which thanks to $B_0 = {B_0}^t$ and $B_0J_0 = -{B_0}^tJ_0$ anticommutes with $B_0$, making $B_0$ an odd operator. More precisely, let $(E_\Sigma)^\pm$ denote the positive (negative) eigenspace of $\alpha$. Under the direct sum decomposition $E_\Sigma = (E_\Sigma)^+ \oplus (E_\Sigma)^-$, the operator $B_0$ takes the form $B_0 = \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix}$, where $B^- := (B^+)^t$ and $B^+ : \ker(\alpha - 1) \to \ker(\alpha + 1)$. Then we have $\ker B_0 = \ker B^+ \oplus \ker B^-$, and the positive (negative) eigenspace of $\mathrm{i}J_0|_{\ker B_0}$ is $\ker B^\pm$. That proves

$$\operatorname{sign}(\mathrm{i}J_0|_{\ker B_0}) = \operatorname{ind} B^+. \quad \square$$

### 6.3.4. *Proof of (II) ⟹ (IV), (V)*

To show that (IV) and (V) are trivial consequences of (I), one is tempted to set $P := C_+$. However, in Proposition 5.12 we have established that $C_+$ is the $L^2$ extension of a pseudo-differential projection only for the boundary operator $T := (J_0^t)^{-1}$, contrary to our assumption $T := J_0(-J_0^2)^{-1/2}$ in (I) for achieving that $\ker C_+ = \operatorname{im} C_-$ becomes a Lagrangian subspace of $L^2(\Sigma, E_\Sigma)$. To prove that $C_+$ is also pseudo-differential for the last choice of $T$ would require applying more advanced elliptic boundary value theory. Instead of that, we give a simple construction of the wanted $P$ as a perturbation of the positive spectral projection $P_>(B_0)$ by a suitable finite-rank operator, and let simple symplectic analysis do the remainder of the work:

**Proof of (II) ⟹ (IV), (V).** The vanishing of the signature $\operatorname{sign} iP_0J_0 \restriction W_0$ on the finite-dimensional space $W_0$ implies the existence of a transversal pair of Lagrangian subspaces $(\lambda, \mu) \subset W_0$. The pair $\big(W_> + \lambda, W_< + \mu\big)$ is a transversal pair of Lagrangian subspaces of $L^2(\Sigma, E_\Sigma)$. Denote by $P$ the projection of $L^2(\Sigma, E_\Sigma)$ onto $W_> + \lambda$. Then $P$ is a zeroth order pseudo-differential operator, and $P - \mathscr{P}_+$ is of $-1$ order. Then $(\ker P, \operatorname{im} \mathscr{P}_+)$ is a Fredholm pair, and $\ker P = W_< + \mu$ is a Lagrangian subspace of $L^2(\Sigma, E_\Sigma)$. Since $\operatorname{im} C_+ = \operatorname{im} \mathscr{P}_+$, (IV) follows.

Now we consider (V). By Remark 6.12 we have $\beta(A) = \operatorname{im} P_{-1/2} \oplus \operatorname{im}(\mathrm{Id} - P_{1/2})$. Clearly $\operatorname{im} P_{-1/2}$ and $\operatorname{im}(\mathrm{Id} - P_{1/2})$ are isotropic subspaces of $\beta(A)$. By Lemma 6.4, they are Lagrangian subspaces. Then the extension $A_P$ is a self-adjoint extension. Fredholmness follows from leading symbol consideration. $\square$

## 6.4. *Alternative routes to the General Cobordism Theorem*

We describe alternative routes to prove Theorem 6.1. In the present context (I) is an immediate consequence of our Calderón construction. Therefore, we began the proof of the General Cobordism Theorem with a proof of (I).

One true alternative is to begin with a proof of (V): That claim was proved in [15, Theorem I] for bounded regions $M$ in Euclidean space. However, Ralston's arguments fully generalize and can be summarized in the following way (in our present notation):

**Outlines of a proof of (V).** (1) First we notice the pointwise vanishing of the signature of the form $\mathrm{i}J_0\restriction p$ on the fiber $E_p$ for each $p \in \Sigma$. This can be obtained by deforming $J_0\restriction p$ into a strict anti-involution $\widetilde{J_0}\restriction p$ with $\big(\widetilde{J_0}\restriction p\big)^2 = -\mathrm{Id}$ and exploiting the anti-commutative relation (2.20) with the elliptic symbol. Consequently, the fiber dimension of the bundle $E$ must be even. That permits the first trick, namely to split $E = E_+ \oplus E_-$ and to show that there exists a well-posed symmetric Fredholm extension given by the graph of a pseudo-differential elliptic operator $P : \Gamma^\infty(\Sigma; E_+\restriction \Sigma) \to \Gamma^\infty(\Sigma; E_-\restriction \Sigma)$.
(2) Next we show that the deficiency indices $\kappa_1, \kappa_2$ of $A$ are finite and that their difference is equal to the index $\operatorname{ind} P$.
(3) Then we show that $-\operatorname{ind} P = \operatorname{sign} -i\omega\restriction \mathrm{vN}(P)$ where $\omega$ denotes the Green form and $\mathrm{vN}(P)$ denotes a suitably defined subspace of the von Neumann quotient $\mathscr{D}(A_{\max})/\mathscr{D}(A_{\min}) =: \beta(A)$.
(4) Finally we show that $\operatorname{ind} P$ vanishes, and hence that $A_P$ can be extended to a self-adjoint $A_{\widetilde{P}}$ with domain given by a pseudo-differential projection and preserving the Fredholm property. $\square$

Note that we can deduce (I) from (V) in the following way, and independently of the delicacies of our Calderón construction. We firstly show that we have (V) ⟹ (II).

**Proof of (V) ⟹ (II).** Since $A_P$ is a self-adjoint Fredholm operator, by [7, Proposition 3.5], $\operatorname{im} \mathscr{P}_{+,-1/2}$ is a Lagrangian subspace of $\beta(A)$.

By Remark 6.12 and Eq. (6.20), we have

$$\beta(A) = \operatorname{im} P_{<,1/2}(B_0) \oplus \operatorname{im} P_0(B_0) \oplus \operatorname{im} P_{>,-1/2}(B_0). \tag{6.22}$$

Using the same method as in the proof of (I) $\Longrightarrow$ (II) and applying Lemma 6.8 to the co-isotropic subspace $\operatorname{im}\big(P_{<,1/2}(B_0)\big)^{\omega} = \operatorname{im} P_0(B_0) \oplus \operatorname{im} P_{<,1/2}(B_0)$, we obtain that

$$\operatorname{Red}_{(P_{<,1/2}(B_0))^{\omega}}(\operatorname{im}\mathscr{P}_{+,-1/2}) \simeq \pi(\operatorname{Red}_{(P_{<,1/2}(B_0))^{\omega}}(\operatorname{im}\mathscr{P}_{+,-1/2})) \subset W_0$$

is a Lagrangian subspace of $W_0$. So, the finite-dimensional symplectic Hilbert space $\big(W_0, \langle \mathrm{i}P_0 J_0\cdot,\cdot\rangle\big)$ has a Lagrangian subspace. Therefore

$$\operatorname{sign} \mathrm{i}P_0 J_0\restriction W_0 = 0. \quad \square$$

**Proof of (V) $\Longrightarrow$ (I).** By the above proofs we have $L^2(\Sigma, E_\Sigma) = \operatorname{im} P \oplus \operatorname{im}(\mathrm{Id} - P)$ and $\beta(A) = \operatorname{im} P_{-1/2} \oplus \operatorname{im}(\mathrm{Id} - P_{1/2})$. By Theorem 6.9, $\operatorname{im}\mathscr{P}_{+,-1/2} \cap L^2(\Sigma, E_\Sigma) = \operatorname{im}\mathscr{P}_+ = \operatorname{im} C_+$ is a Lagrangian subspace of $L^2(\Sigma, E_\Sigma)$. $\square$

## 7. Parameter dependence

In this section we discuss the continuous dependence of the Calderón projection and the Poisson operator on the input data. That is, given a first order elliptic differential operator $A \in \operatorname{Diff}^1(M; E, F)$ and $T \in \operatorname{Diff}^0(\Sigma; E_\Sigma, F_\Sigma)$, $\Sigma = \partial M$ (cf. Definition 4.1), we want to have criteria to ensure that $(A, T) \mapsto K_+ = K_+(A, T)$ respectively $(A, T) \mapsto C_+ = C_+(A, T)$ is continuous in an appropriate sense.

Therefore we first introduce various metrics on the spaces of pairs $(A, T)$. Referring to (5.11) and (5.15) we consider $J_0, B_0, \overset{(\sim)}{C_0}, \overset{(\sim)}{C_1}$ as functions of $A$.

**Definition 7.1.** (a) Let $\mathscr{V}(M; E, F)$ denote the subspace of $\operatorname{Diff}^1(M; E, F) \times \operatorname{Diff}^0(\Sigma; E_\Sigma, F_\Sigma)$ consisting of those $(A, T)$ for which $[B_0^t, J_0^t T]$ is of order 0 (cf. (5.9)). By $\mathscr{E}(M; E, F)$ we denote the subspace of $\mathscr{V}(M; E, F)$ consisting of those pairs $(A, T)$ where

(1) $A$ is elliptic
(2) $T$ is invertible and satisfies (5.8).

Finally, we denote by $\mathscr{E}_{\mathrm{UCP}}(M; E, F)$ the subspace of $\mathscr{E}(M; E, F)$ consisting of pairs $(A, T)$ where $A$ and $A^t$ satisfy weak inner UCP.
(b) On the linear space $\mathscr{V}(M; E, F)$ we introduce two norms:

$$N_0(A, T) := \|A\|_{1,0} + \|A^t\|_{1,0} + \|T\|_{1/2,1/2}, \tag{7.1}$$

and

$$N_1(A, T) := \|B_0\|_{1,0} + \|B_0^t\|_{1,0} + \|[B_0^t, J_0^t T]\|_0 + \|T\|_0 + \|J_0\|_0 + \|C_1\|_{1,0} + \|C_0\|_0 + \|\widetilde{C}_1\|_{1,0} + \|\widetilde{C}_0\|_0. \tag{7.2}$$

Except for $C_1, \widetilde{C}_1$ the norms in (7.2) are the mapping norms between Sobolev spaces over $\Sigma$ while the norms for $C_1, \widetilde{C}_1$ are mapping norms between Sobolev spaces over the collar $[0, \varepsilon) \times \Sigma$.
(c) We equip the space $\mathscr{E}(M; E, F)$ with the metric $d_0$ induced by the metric $N_0$ of (7.1). I.e.

$$d_0((A, T), (A', T')) := N_0(A - A', T - T'). \tag{7.3}$$

The norms $N_0, N_1$ induce metrics on subspaces of $\mathscr{V}(M; E, F)$, in particular on $\mathscr{E}(M; E, F)$.

To study the dependence of $K_+$ and $C_+$ on $(A, T)$ the formulas (5.30) and (5.31) in Proposition 5.10 are crucial.

To illustrate this let us consider a map $Z \ni z \mapsto (A(z), T(z)) \in \mathscr{E}(M; E, F)$ from a metric space $Z$ to $\mathscr{E}(M; E, F)$. To conclude that the corresponding map $z \mapsto K_+(A(z), T(z)) \in \mathscr{B}(L^2(\Sigma, E_\Sigma), L^2_s(M, E))$ is continuous for some fixed $0 \le s \le 1/2$ it suffices to show the continuity of

(1) $z \mapsto \varphi R_{T(z)}(A(z)) \in \mathscr{B}(L^2(\Sigma, E_\Sigma), L^2_s(M, E \oplus F))$,
(2) $z \mapsto \underset{\sim}{S}(A(z), T(z)) \in \mathscr{B}(L^2(\Sigma, E_\Sigma), L^2(M, F \oplus E))$,
(3) $z \mapsto \widetilde{G}(A(z), T(z)) \in \mathscr{B}(L^2(M, F \oplus E), L^2_1(M, E \oplus F))$.

For the continuity of $z \mapsto C_+(A(z), T(z)) \in \mathscr{B}(L^2(\Sigma, E_\Sigma))$ (1) has to be replaced by the continuity of the map

(1′) $z \mapsto P_+(B_0(A(z))) \in \mathscr{B}(L^2(\Sigma, E_\Sigma))$.

The continuity of these maps is by no means necessary for ensuring the continuous dependence of $K_+, C_+$. In order to keep the presentation reasonable in size we estimate generously — we are not striving for optimality here.

Let us now state the main result of this section.

We define the *strong metric* on the space $\mathscr{E}(M; E, F)$ by

$$d_{\mathrm{str}}((A, T), (A', T')) := N_0(A - A', T - T') + N_1(A - A', T - T'). \tag{7.4}$$

Note that by complex interpolation $\|T - T'\|_{s,s} \le d_{\mathrm{str}}((A, T), (A', T'))$ for all $0 \le s \le 1/2$.

**Theorem 7.2.** (a) *The map*

$$(\mathscr{E}_{UCP}(M; E, F), d_{\mathrm{str}}) \longrightarrow \mathscr{B}(L^2(\Sigma, E_\Sigma), L^2_s(M, E))$$

*sending* $(A, T)$ *to the Poisson operator* $K_+(A, T)$ *is continuous for* $0 \le s < 1/2$.

(b) *Let* $(A(z), T(z))_{z\in Z}$ *be a continuous family in* $(\mathscr{E}_{UCP}(M; E, F), d_{\mathrm{str}})$ *parametrized by a metric space* $Z$. *Assume that the corresponding family*

$$z \mapsto P_+(B_0(A(z))) \in \mathscr{B}(L^2_s(\Sigma, E_\Sigma))$$

*of positive spectral projections of the tangential operator is continuous for some fixed* $s \in [-1/2, 1/2]$. *Then the map*

$$Z \longrightarrow \mathscr{B}(L^2_s(\Sigma, E_\Sigma))$$

*sending* $z$ *to the Calderón projection* $C_+(A(z), T(z))$ *is continuous.*

**Remark 7.3.** (1) Of course, analogous statements hold for $K_-, C_-$. We leave it as an intriguing problem whether the statement about $K_+$ still holds for $s = 1/2$. This would be more natural since $K_+$ maps $L^2$ to $L^2_{1/2}$.

(2) The fact that the continuous dependence of $P_+$ in (b) has to be assumed is not very satisfactory. The point here is *not* that the construction of $P_+$ requires a spectral cut. Suppose a spectral cut for $B_0 = B_0(A(z_0))$ is chosen. Then $P_+$ should vary continuously as long as no eigenvalues approach the contours $\Gamma_\pm$ (cf. Fig. 2). Unfortunately we cannot prove the continuity of $B \mapsto P_+(B) \in \mathscr{B}(L^2(\Sigma, E_\Sigma))$ if we equip $\mathrm{Diff}^1(\Sigma; E_\Sigma)$ say with the norm $\|\cdot\|_{1,0}$; we cannot prove it for any other norm either. We will come back to this problem below in Section 7.5, where we will give a criterion for the continuity of $P_+$ in special cases.

**Proof.** The Theorem follows from Propositions 5.10, 7.8, 7.12, and Theorem 7.9. □

The discussion in Section 7.5 will give at least the following result:

**Corollary 7.4.** *Denote by* $\mathscr{E}^{\mathrm{sa}}_{\mathrm{UCP}}(M; E, F)$ *the subspace of* $\mathscr{E}_{UCP}(M; E, F)$ *consisting of pairs* $(A, T)$ *where the corresponding tangential operator* $B_0(A)$ *has a self-adjoint leading symbol. Then for* $s \in [-1/2, 1/2]$ *the map*

$$(\mathscr{E}^{\mathrm{sa}}_{\mathrm{UCP}}(M; E, F), d_{\mathrm{str}}) \longrightarrow \mathscr{B}(L^2_s(\Sigma, E_\Sigma))$$

*sending* $(A, T)$ *to the Calderón projection* $C_+(A, T)$ *is continuous.*

**Proof.** We may adopt the language of a family $(A(z), T(z))_{z\in Z}$ (with $Z = \mathscr{E}^{\mathrm{sa}}_{\mathrm{UCP}}(M; E, F)$!) of the previous Theorem 7.2.

The point is that locally (cf. Convention 3.7) one can choose a continuous family $z \mapsto P_+(B_0(z))$. Then the result follows from Theorem 7.2b.

To see this we recall from Remark 5.2 that if the tangential operator has a self-adjoint leading symbol, the $B_0 = B_0(A(z))$ in (5.11) can be chosen to be self-adjoint. Hence let $B_0(z)$ now denote this self-adjoint operator in the representation (5.11). To show continuity at $z_0$ pick a spectral cut $c$ for $B_0(z_0)$. Then by Proposition 7.15 the family $P_+(B_0(z)) := 1_{[c,\infty)}(B_0(z)) \in \mathscr{B}(L^2_s(\Sigma, E_\Sigma))$ depends continuously on $z$ in a neighborhood of $z_0$. Hence Theorem 7.2b yields the claim. □

**Remark 7.5.** In [7, Section 3.3] Booß-Bavnbek and Furutani give a purely functional analytic proof of the continuous variation of the Cauchy data spaces as subspaces of the von-Neumann quotient $\beta(A)$ of all natural boundary values, as defined above in Section 6.2.4, in great generality: only the symmetry of $A$, weak inner UCP and the existence of a self-adjoint Fredholm extension are assumed. In particular, no product form near the boundary or symmetry of a tangential operator is assumed. However, [7] is restricted to continuous variation by bounded perturbations, i.e., perturbations of lower order in the operator norm, whereas the preceding corollary admits arbitrary continuous variations, though in the strong metric.

Below in Proposition 7.13 we shall explain why lower order perturbations of a fixed operator lead to continuous variation of the sectorial projection in our setting, and hence to continuous variation of the Calderón projection according to the preceding theorem.

We now proceed to give criteria for the continuity of the maps (1)–(2), (1′). We start with some basic estimates.

## 7.1. Some estimates

We fix a first order elliptic differential operator $B \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$ such that $B - \lambda$ is parameter dependent elliptic in a conic neighborhood of $i\mathbb{R}$ (cf. Section 3.1). Choose contours $\Gamma_\pm$ accordingly as in Fig. 2.

Before we address the continuous dependence of the sectorial projections on the data, i.e., on $B$, we shall give some useful estimates.

We will frequently use that for $V \in \mathrm{CL}^1(\Sigma; E_\Sigma)$ we have by duality $\|V\|_{0,-1} = \|V^t\|_{1,0}$.

Our first result is the following perturbation lemma.

**Lemma 7.6.** *Let* $V \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$. *If* $\|V\|_{1,0} + \|V^t\|_{1,0}$ *is sufficiently small, then* $B + V - \lambda$ *is a parameter dependent elliptic in a conic neighborhood of* $i\mathbb{R}$ *containing* $\Gamma_+, \Gamma_-$.

*Furthermore, for* $|s|, |s'|, |s - s'| \le 1$ *we have for* $\lambda \in \Gamma_- \cup \Gamma_+$

$$\|(\lambda - (B + V))^{-1} - (\lambda - B)^{-1}\|_{s,s'} \le C(s, s', B)\left(\|V\|_{1,0} + \|V^t\|_{1,0}\right)|\lambda|^{-1+s'-s}.$$

**Proof.** The first claim is clear. The second follows from a straightforward application of the Neumann series for the resolvent of $B+V$ and complex interpolation. For the convenience of the reader we present some details of the estimate.

For $s \in \{0,1\}$ Eq. (3.20) yields for $\lambda \in \Gamma_+ \cup \Gamma_-$

$$\begin{aligned}\|(B-\lambda)^{-1}V\|_{s,s} &\le \|(B-\lambda)^{-1}\|_{s-1,s}\,(\|V\|_{1,0}+\|V^t\|_{1,0})\\ &\le C(s)\,(\|V\|_{1,0}+\|V^t\|_{1,0}),\end{aligned} \tag{7.5}$$

and similarly

$$\|V(B-\lambda)^{-1}\|_{s,s} \le C(s)\,(\|V\|_{1,0}+\|V^t\|_{1,0}). \tag{7.6}$$

Furthermore, complex interpolation (or Hadamard's three line theorem) gives that there is a constant $C$ such that

$$\sup_{0\le s\le 1,\,\lambda\in\Gamma_+\cup\Gamma_-}\|V(B-\lambda)^{-1}\|_{s,s} + \sup_{0\le s\le 1,\,\lambda\in\Gamma_+\cup\Gamma_-}\|(B-\lambda)^{-1}V\|_{s,s} \le C\,(\|V\|_{1,0}+\|V^t\|_{1,0}). \tag{7.7}$$

Choose $V$ such that $C\,(\|V\|_{1,0}+\|V^t\|_{1,0}) < 1/2$. Consequently, $B+V-\lambda$ is invertible for all $\lambda\in\Gamma_+\cup\Gamma_-$ and as operator $L^2_s \to L^2_{s-1}$, $0\le s\le 1$, its inverse is given by the Neumann series

$$\begin{aligned}(B+V-\lambda)^{-1} &= (\mathrm{Id}+(B-\lambda)^{-1}V)^{-1}(B-\lambda)^{-1}\\ &= \sum_{n\ge 0}(-1)^n\big((B-\lambda)^{-1}V\big)^n(B-\lambda)^{-1}.\end{aligned}$$

Hence

$$\|(B+V-\lambda)^{-1}-(B-\lambda)^{-1}\|_{s,s'} \le \sum_{n\ge 1}\left\|\big((B-\lambda)^{-1}V\big)^n(B-\lambda)^{-1}\right\|_{s,s'}. \tag{7.8}$$

Now one has to check case by case.

1. Let $s'\ge 0,\ s'-s\ge 0$. Then

$$\begin{aligned}\left\|\big((B-\lambda)^{-1}V\big)^n(B-\lambda)^{-1}\right\|_{s,s'} &\le \|(B-\lambda)^{-1}\|_{s,s'}\,\|(B-\lambda)^{-1}V\|^n_{s',s'}\\ &\overset{(3.20)}{\le} C'\,|\lambda|^{-1+s'-s}\big(C\,(\|V\|_{1,0}+\|V^t\|_{1,0})\big)^n\\ &\le \widetilde{C}|\lambda|^{-1+s'-s}(1/2)^{n-1}\,(\|V\|_{1,0}+\|V^t\|_{1,0}).\end{aligned} \tag{7.9}$$

Summing up gives the claim in this case.

2. Let $-1\le s\le s'\le 0$. Then

$$\begin{aligned}\left\|\big((B-\lambda)^{-1}V\big)^n(B-\lambda)^{-1}\right\|_{s,s'} &\le \|(B-\lambda)^{-1}V\|_{0,s'}\,\left\|\big((B-\lambda)^{-1}V\big)^{n-1}\right\|_{0,0}\,\|(B-\lambda)^{-1}\|_{s,0}\\ &\le C\,|\lambda|^{-1+s'+1}\,(\|V\|_{1,0}+\|V^t\|_{1,0})\,(1/2)^{n-1}\,|\lambda|^{-1-s}.\end{aligned}$$

Again, summing up gives the claim also in this case.

For estimating $\left\|\big((B-\lambda)^{-1}V\big)^n(B-\lambda)^{-1}\right\|_{s,s'}$, the roles of $s$ and $s'$ are symmetric. Hence the cases $s'\le s$ follow analogously. □

We shall investigate the stability of the sectorial projections under perturbation of the input data $B$ by $V$ and show that the operator norm of $\varphi(Q_\pm(B+V)-Q_\pm(B))$ from $L^2_s(\Sigma,E_\Sigma)$ to $L^2_{s',\mathrm{comp}}(\mathbb{R}_+\times\Sigma,E_\Sigma)$ is bounded by a constant depending on $s$, $s'$, $B$, $\varphi$ times $(\|V\|_{1,0}+\|V^t\|_{1,0})$.

**Proposition 7.7.** *Let $\varphi\in C_0^\infty(\mathbb{R}_+)$ and $V$ as in Lemma 7.6. $Q_\pm(B)$ and $Q_\pm(B+V)$ are the operator families of $B$ respectively $B+V$ introduced in Definition 3.8.*

(a) *For $-1/2 < s\le s' < s+1/2$, $s'\le 1$ we have for $\xi\in L^2_s(\Sigma,E_\Sigma)$*

$$\left\|\varphi\big(Q_\pm(B+V)-Q_\pm(B)\big)\xi\right\|_{s'} \le C(s,s',B,\varphi)\,(\|V\|_{1,0}+\|V^t\|_{1,0})\,\|\xi\|_s. \tag{7.10}$$

(b) *For $-1\le s\le 0$ we have for $\xi\in L^2_s(\Sigma,E_\Sigma)$*

$$\left\|\mathrm{id}_{\mathbb{R}_+}\varphi\big(Q_\pm(B+V)-Q_\pm(B)\big)\xi\right\|_{s+1} \le C(s,B,\varphi)\,(\|V\|_{1,0}+\|V^t\|_{1,0})\,\|\xi\|_s. \tag{7.11}$$

**Proof.** We use Lemma 7.6 and estimate for $-1 \le s \le s' \le s+1$, $s' \le 1$, $\xi \in L^2_s(\Sigma, E_\Sigma)$ and $m \in \{0, 1\}$

$$\begin{aligned}
&\left\| x^m \varphi(x) \big(Q_\pm(B+V) - Q_\pm(B)\big)(x)\,\xi \right\|_{s'} \\
&\le \frac{x^m}{2\pi}\varphi(x) \int_{\Gamma_\pm} |e^{-x\lambda}|\, \left\| \big((\lambda-(B+V))^{-1} - (\lambda - B)^{-1}\big)\xi \right\|_{s'} |d\lambda| \\
&\le C(s,s',B)\frac{x^m\varphi(x)}{2\pi}\int_{\Gamma_\pm} |e^{-x\lambda}|\,|\lambda|^{-1+s'-s}\,|d\lambda|\,(\|V\|_{1,0} + \|V^t\|_{1,0})\,\|\xi\|_s \\
&\le C(s,s',B)(\|V\|_{1,0} + \|V^t\|_{1,0})\,\|\xi\|_s\, x^{m-s'+s}|\log x|\varphi(x).
\end{aligned} \tag{7.12}$$

Analogously and using the previous estimate (7.12) we find for $0 \le s \le s' = 1$

$$\begin{aligned}
\left\| \partial_x \Big( x^m\varphi(x)\big(Q_\pm(B+V) - Q_\pm(B)\big)\xi(x)\Big)\right\|_{L^2(\Sigma,E_\Sigma)} &\le C(s,B,\varphi)(\|V\|_{1,0} + \|V^t\|_{1,0})\,\|\xi\|_s \\
&\quad \times \Big( x^s\partial_x(x^m\varphi(x)) + \varphi(x)x^{m-1+s}\Big)|\log x|.
\end{aligned} \tag{7.13}$$

The $\log x$-terms on the right of (7.12) and (7.13) are necessary only in the case $s = s'$. They are obsolete if $s < s'$.

From these two estimates, both claims will follow in a straightforward manner:

(a) It suffices to prove the claim for $-1/2 < s < s' = 0$ and for $1/2 < s \le s' = 1$. It is clear that it then holds for $-1/2 < s \le s' \le 0$ and $1/2 < s \le s' \le 1$. The general case then follows from the complex interpolation method.

So let us start with the case $-1/2 < s \le s' = 0$. Since $s > -1/2$ we may integrate the square of (7.12) and reach the conclusion.

If $1/2 < s \le s' = 1$ then apply (7.12) and (7.13) with $m = 0$. Squaring and integrating the inequality gives the claim in view of (3.21).

(b) By interpolation theory it is enough to deal with the cases $s = -1$ and $s = 0$. If $s = -1$ apply (7.12) with $s' = s + 1 = 0$ and if $s = 0$ apply (7.12) with $s' = s + 1 = 1$ and (7.13). Again referring to (3.21) we are done. □

### 7.2. Continuous dependence of $\varphi R_T(B_0)$

Since $R_T$ is the multiplication of $R$ by a simple matrix containing Id and $T$ it suffices to study the dependence of $R = R(B_0)$ on $B_0$:

**Proposition 7.8.** *Let $(A, T) \in \mathscr{E}(M; E, F)$. Fix a real number $c > 0$ such that $\operatorname{spec} B_0 \cap \{z \in \mathbb{C} \mid |z| = c\} = \emptyset$, cf. Convention 3.7. Let $V \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$ be a first order differential operator. According to Lemma 7.6 assume that $(\|V\|_{1,0} + \|V^t\|_{1,0})$ is small enough so that $B_0 + V - \lambda$ is a parameter dependent elliptic in a conic neighborhood of $i\mathbb{R}$ containing $\Gamma_+ \cup \Gamma_-$.*

*Then for $-1/2 < s \le s' < s + 1/2$, $s' \le 1$ we have*

$$\left\| \varphi\big(R(B_0 + V) - R(B_0)\big)\right\|_{s,s'} \le C(s, s', B_0, \varphi)(\|V\|_{1,0} + \|V^t\|_{1,0}). \tag{7.14}$$

**Proof.** This follows immediately from Proposition 7.7 and Eqs. (5.18) and (5.19). □

### 7.3. Continuous dependence of the invertible double

Dealing with the generalized inverse would make the discussion of the parameter dependence of the invertible double rather tedious. Therefore we assume in this Section 7.3 that UCP holds. Although with some care the results of this section carry over to families of operators where the dimensions of the spaces of "ghost solutions" $Z_{+,0}(A), Z_{-,0}(A)$, see (4.12), remain fixed (cf. [11, Thm. 3.16]).

Recall the definition of spaces and norms of Definition 7.1. The goal of this subsection is to prove:

**Theorem 7.9.** *The map $(\mathscr{E}_{UCP}, d_0) \longrightarrow \mathscr{B}(L^2(M, F \oplus E), L^2_1(M, E \oplus F))$, $(A, T) \mapsto \widetilde{A}^{-1}_{P(T)}$ is continuous.*

Note that this is much more than just graph continuity of $(A, T) \mapsto \widetilde{A}^{-1}_{P(T)}$. Namely, by construction of the metric $d_0$ this means that also $(A, T) \mapsto \big((\widetilde{A}_{P(T)})^*\big)^{-1} = (\widetilde{A}^t_{P(-T^{-1})})^{-1}$ (cf. (4.7)) is continuous as a map to $\mathscr{B}(L^2(M, F \oplus E), L^2_1(M, E \oplus F))$. Graph continuity of $(A, T) \mapsto \widetilde{A}^{-1}_{P(T)}$ means that $(A, T) \mapsto A^{-1}_{P(T)} \in \mathscr{B}(L^2, L^2)$ and $(A, T) \mapsto \big((A_{P(T)})^*\big)^{-1} \in \mathscr{B}(L^2, L^2)$ are continuous.

The result should not come as a surprise. We should take the invertible double as a guideline. Under more restrictive assumptions on $A$ one can construct from $A$ an invertible operator $\widetilde{A}$ on the double $\widetilde{M}$ of $M$. If $A$ varies continuously in the metric $d_0$ then the geometric invertible double is a continuously varying family in $\mathscr{B}(L^2_1(\widetilde{M}, \widetilde{E} \oplus \widetilde{F}), L^2(\widetilde{M}, \widetilde{F} \oplus \widetilde{E}))$ and hence[6] its inverse varies continuously in $\mathscr{B}(L^2(\widetilde{M}, \widetilde{F} \oplus \widetilde{E}), L^2_1(\widetilde{M}, \widetilde{E} \oplus \widetilde{F}))$. In the case of a geometric invertible double the nice thing is that the domains of all first order elliptic differential operators coincide with $L^2_1$.

The difficulty we are facing here is that the domains $L^2_{1,T}$ vary with $T$. So our first task will be to transform, at least locally, the whole situation to families of operators with constant domain.

**Definition 7.10.** Let $e : L^2_s(\Sigma, F_\Sigma) \to L^2_{s+1/2}(M, F)$, $s > 0$, be a linear right-inverse to the trace map (cf. e.g. Remark 2.5.1 and [1, Definition 11.7e]). For $T, T' \in \mathrm{Diff}^0(\Sigma; E_\Sigma, F_\Sigma)$ we put

$$\Phi_{T,T'} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := \begin{pmatrix} f_+ \\ f_- + e(T' - T)\varrho f_+ \end{pmatrix}. \tag{7.15}$$

We record some properties of $\Phi_{T,T'}$ which are straightforward to verify.

**Lemma 7.11.** *$\Phi_{T,T'} \in \mathscr{B}(L^2_1(M, E \oplus F))$ and we have $\Phi_{T',T''} \circ \Phi_{T,T'} = \Phi_{T,T''}$. In particular $\Phi_{T,T'}$ is invertible with inverse $\Phi_{T',T}$. Furthermore, $\Phi_{T,T'}$ maps $L^2_{1,T}(M, E \oplus F)$ bijectively onto $L^2_{1,T'}(M, E \oplus F)$.*

*Finally, we have*

$$\|\Phi_{T,T'} - \mathrm{Id}\,\|_{1,1} \le C(e)\|T - T'\|_{1/2,1/2}. \tag{7.16}$$

After these preparations, the proof of Theorem 7.9 is straightforward:

**Proof of Theorem 7.9.** The map $(A, T) \mapsto \widetilde{A}^{-1}_{P(T)}$ can be factorized as follows: fix a $T_0 \in \mathrm{Diff}^0(\Sigma; E_\Sigma, F_\Sigma)$. Then the following maps are continuous:

$$\begin{aligned} \mathscr{E} &\longrightarrow G\mathscr{B}(L^2_{1,T_0}, L^2) \overset{\text{inversion}}{\longrightarrow} \mathscr{B}(L^2, L^2_{1,T_0}) \\ (A, T) &\longmapsto \widetilde{A}_T \circ \Phi_{T_0,T}. \end{aligned} \tag{7.17}$$

Here, $G\mathscr{B}(L^2_{1,T_0}, L^2)$ denotes the *invertible* bounded linear maps between $L^2_{1,T_0}$ and $L^2$.

The continuity is seen as follows:

$$\begin{aligned} \|\widetilde{A}_{P(T)} \circ \Phi_{T_0,T} - \widetilde{A}'_{P(T')} \circ \Phi_{T_0,T'}\|_{L^2_{1,T_0} \to L^2} &\le \|\widetilde{A}_{P(T)} \circ \Phi_{T_0,T} - \widetilde{A}'_{P(T')} \circ \Phi_{T_0,T'}\|_{1,0} \\ &\le \|\widetilde{A}\|_{1,0}\|\Phi_{T_0,T} - \Phi_{T_0,T'}\|_{1,1} + \|\widetilde{A} - \widetilde{A}'\|_{1,0}\|\Phi_{T_0,T'}\|_{1,1}. \end{aligned} \tag{7.18}$$

Furthermore, the map

$$\begin{aligned} \mathrm{Diff}^0(\Sigma; E, F) \times \mathscr{B}(L^2, L^2_{1,T_0}) &\longrightarrow \mathscr{B}(L^2, L^2_1) \\ (T, T) &\longmapsto \Phi_{T_0,T} \circ T \end{aligned} \tag{7.19}$$

is continuous in view of Lemma 7.11. Hence

$$(A, T) \mapsto \widetilde{A}^{-1}_{P(T)} = \Phi_{T_0,T} \circ (\widetilde{A} \circ \Phi_{T_0,T} \restriction L^2_{1,T_0})^{-1} \tag{7.20}$$

is continuous as claimed. □

### 7.4. Continuous dependence of $S(A, T)$

Next we give a simple criterion for the continuous dependence of $S(A, T)$ on the input data.

**Proposition 7.12.** *The map $(A, T) \mapsto S(A, T) \in \mathscr{B}(L^2(\Sigma, E_\Sigma), L^2_{\mathrm{comp}}(M, E \oplus F))$ is continuous with respect to the norm $N_1$ introduced in Definition 7.1.*

**Proof.** This follows immediately from (5.18) and (5.19). □

### 7.5. Continuous dependence of $P_\pm$ on input data

Finally we study the dependence of $P_\pm$ on $B$, where $B \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$ satisfies the usual assumptions (cf. Section 7.1).

The definition of $P_+ = P_+(B)$ for $B \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$ requires a choice of a *spectral cut*, i.e., a $c > 0$ such that

$$\mathrm{spec}\, B \cap \{z \in \mathbb{C} \mid |z| = c\} = \emptyset. \tag{7.21}$$

Obviously, for a choice of $c$ the map $B \mapsto P_+(B)$ has a discontinuity at $B$'s where (7.21) is violated.

6 Note that for any pair of Banach spaces $X, Y$ the inversion map $\mathscr{B}(X, Y) \longrightarrow \mathscr{B}(Y, X)$, $S \mapsto S^{-1}$ is continuous.

Apart from that $P_+(B)$ should depend continuously on $B$ with respect to the norm $V \mapsto \|V\|_{1,0} + \|V^t\|_{1,0}$. Unfortunately, we cannot prove or disprove this conjecture. Instead we mention two simple continuity criteria. The first deals with lower order perturbations of a fixed operator and the second deals with self-adjoint operators where the Spectral Theorem yields continuity in a rather simple fashion.

**Proposition 7.13.** *Let $B \in \mathrm{Diff}^1(\Sigma; E_\Sigma)$ be a first order elliptic differential operator such that $B - \lambda$ is parameter dependent elliptic for $\lambda$ in a conic neighborhood of $\mathrm{i}\mathbb{R}$. Furthermore, let $V \in \mathrm{CL}^\alpha(\Sigma; E_\Sigma)$, $\alpha < 1$, with $\|V\|_{\alpha,0}$ sufficiently small. Then $B+V-\lambda$ is also parameter dependent elliptic in a conic neighborhood of $\mathrm{i}\mathbb{R}$ and for $c > 0$ such that $\mathrm{spec}(B + tV) \cap \{z \in \mathbb{C} \mid |z| = c\} = \emptyset, \quad 0 \le t \le 1$, we have the estimate*

$$\|P_+(B + V) - P_+(B)\|_{0,0} \le C(B)\|V\|_{\alpha,0}.$$

**Proof.** By a Neumann series argument it is clear that such a $c$ exists. Let $\Gamma_+$ (see Fig. 2) be the usual contour. Analogously to Lemma 7.6 one shows the estimate

$$\|(\lambda - (B + V))^{-1} - (\lambda - B)^{-1}\|_0 \le C(B)\|V\|_{\alpha,0}|\lambda|^{-2+\alpha}, \quad \lambda \in \Gamma_+, \tag{7.22}$$

from which the claim, thanks to $-2 + \alpha < -1$, follows by invoking the contour integral. □

We now turn to (formally) self-adjoint $B$. We first give a slight improvement of [34, Prop. 2.2] on the continuity of the *Riesz*-map (cf. [34] Eq. (2.2)).

**Proposition 7.14.** *Let $\mathrm{Ell}^{1,sa}(\Sigma; E_\Sigma) \subset \mathrm{Diff}^1(\Sigma; E_\Sigma)$ denote the space of* selfadjoint *first order elliptic differential operators. Then the* Riesz*-map*

$$B \mapsto B(\mathrm{Id} + B^2)^{-1/2}$$

*is continuous $(\mathrm{Ell}^{1,sa}, \|\cdot\|_{1,0}) \to \mathscr{B}(L^2_s(\Sigma, E_\Sigma))$ for all $s \in [-1/2, 1/2]$.*

**Proof.** For $s = 0$ this was proved in [34, Prop. 2.2]. The proof in [34], however shows the claimed stronger statement (cf. [34, Equ. (2.19)]). For the convenience of the reader let us present the argument.

It suffices to prove the continuity of $F$ for $s = 1/2$. Since $F(B)$ is self-adjoint $\|F(B) - F(\widetilde{B})\|_{1/2,1/2} = \|F(B) - F(\widetilde{B})\|_{-1/2,-1/2}$ and hence by complex interpolation (cf. [34, Appendix]) $\|F(B) - F(\widetilde{B})\|_{s,s} \le \|F(B) - F(\widetilde{B})\|_{1/2,1/2}$ for $|s| \le 1/2$.

Fix a $B \in \mathrm{Ell}^{1,sa}(\Sigma, E_\Sigma)$ and we have to prove the continuity of $F$ at $B$. Let $0 < q < \frac{1}{2}$ and consider $\widetilde{B} \in \mathrm{Ell}^{1,sa}(\Sigma, E_\Sigma)$ with

$$\|(B - \widetilde{B})(B \pm i)^{-1}\|_{0,0} + \|(B \pm i)^{-1}(B - \widetilde{B})\|_{0,0} \le q. \tag{7.23}$$

Note that by ellipticity the graph norm of $B$ is equivalent to the Sobolev norm $\|\cdot\|_1$, hence the left hand side of (7.23) induces a metric on $\mathrm{Ell}^{1,sa}(\Sigma, E_\Sigma)$ which is equivalent to the metric induced by $\|\cdot\|_{1,0}$. The Neumann series then immediately implies

$$\|(\widetilde{B} + i)^{-1}(B + i)\|_{0,0} \le \frac{1}{1 - q}. \tag{7.24}$$

Thus, for $f \in L^2(\Sigma, E_\Sigma)$ we have

$$\|(\widetilde{B} + i)^{-1} f\|_0 \le \frac{1}{1 - q}\|(B + i)^{-1} f\|_0 \tag{7.25}$$

and

$$\|(B + i)^{-1} f\|_0 \le \|(B + i)^{-1}(\widetilde{B} + i)\|_{0,0}\, \|(\widetilde{B} + i)^{-1} f\|_0 \le (1 + q)\|(\widetilde{B} + i)^{-1} f\|_0. \tag{7.26}$$

This implies the operator inequalities

$$\frac{1}{(1 + q)^2}|B + i|^{-2} \le |\widetilde{B} + i|^{-2} \le \frac{1}{(1 - q)^2}|B + i|^{-2}. \tag{7.27}$$

Since the square root is an operator-monotonic increasing function [35, Prop. 4.2.8] we may take the square root of these inequalities and after subtracting $|B + i|^{-1}$ we arrive at

$$-\frac{q}{1 + q}|B + i|^{-1} \le |\widetilde{B} + i|^{-1} - |B + i|^{-1} \le \frac{q}{1 - q}|B + i|^{-1}. \tag{7.28}$$

This gives

$$\||B + i|^{1/2}|\widetilde{B} + i|^{-1}|B + i|^{1/2} - \mathrm{Id}\|_{0,0} \le \frac{q}{1 - q}. \tag{7.29}$$

After these preparations we find

$$\begin{aligned}
\|F(B)-F(\widetilde{B})\|_{1/2,1/2} &= \big\||i+B|^{1/2}(F(B)-F(\widetilde{B}))|i+B|^{-1/2}\big\|_{0,0}\\
&= \big\||i+B|^{-1/2}(F(B)-F(\widetilde{B}))|i+B|^{1/2}\big\|_{0,0}\\
&\le \big\||i+B|^{-1/2}(B-\widetilde{B})|i+B|^{-1/2}\big\|_{0,0} + \big\| |B+i|^{-1/2}\big(\widetilde{B}(|i+B|^{-1}-|i+\widetilde{B}|^{-1})\big)|i+B|^{1/2}\big\|_{0,0}\\
&\le \||i+B|^{-1}(B-\widetilde{B})\|_{0,0} + \||i+B|^{-1/2}\widetilde{B}|i+B|^{-1/2}\| \,\|\mathrm{Id}-|i+B|^{1/2}|i+\widetilde{B}|^{-1}|i+B|^{1/2}\|_{0,0}\\
&\le q + \||i+B|^{-1}\widetilde{B}\|_{0,0}\frac{q}{1-q}\\
&\le q\left(1+\frac{1+q}{1-q}\right).
\end{aligned} \tag{7.30}$$

Here we have used that for a first order operator $V$ one has

$$\||i+B|^{-1/2}V|i+B|^{-1/2}\|_{0,0} \le \||i+B|^{-1}V\|_{0,0}. \tag{7.31}$$

This inequality also follows from complex interpolation (see [34, Appendix]). This shows that if $\|B_n - B\|_{1,0} \to 0$ then $F(B_n) \to F(B)$ in $L^2_{1/2}(\Sigma, E_\Sigma)$ and we are done. □

**Proposition 7.15.** *Let* $\mathrm{Ell}^{1,sa}_c(\Sigma; E_\Sigma) \subset \mathrm{Ell}^{1,sa}_c(\Sigma; E_\Sigma) \subset \mathrm{Diff}^1(\Sigma; E_\Sigma)$ *denote the space of* selfadjoint *first order elliptic differential operators B with* $\pm c \notin \operatorname{spec} B$. *Then for* $|s| \le 1/2$ *the map*

$$\begin{aligned}
\left(\mathrm{Ell}^1_c, \|\cdot\|_{1,0}\right) &\longrightarrow \mathscr{B}(L^2_s(\Sigma, E_\Sigma))\\
B &\mapsto 1_{[c,\infty)}(B)
\end{aligned} \tag{7.32}$$

*is continuous.*

**Proof.** We first note that for $B \in \mathrm{Ell}^{1,\mathrm{sa}}_c$ we have by the Spectral Theorem

$$1_{[c,\infty)}(B) = 1_{[F(c),\infty)}\big(F(B)\big).$$

Since, independently of $s \in [-1/2, 1/2]$, we have

$$\operatorname{spec}\big(F(B)\big) \subset [-1, 1], \tag{7.33}$$

$1_{[c,\infty)}(B)$ is given by the contour integral

$$\frac{1}{2\pi \mathrm{i}}\oint_{|z-(F(\lambda)+2)|=2} \Big(z - F(B)\Big)^{-1} \mathrm{d}z. \tag{7.34}$$

In view of the previous Proposition 7.14 this proves the claim. cf. also [34, Lemma 3.3]. □

*7.6. Continuity of families of well-posed self-adjoint Fredholm extensions*

In Theorems 3.8 and 3.9 in [3] the continuous dependence of the invertible double, the Calderón and Poisson operators and the graph continuity of realizations of well-posed boundary value problems are discussed in a special case. More precisely it was assumed that $J^2 = -\mathrm{Id}$ and that the tangential operator had a self-adjoint leading symbol.

Unfortunately the proof of Theorem 3.9 in [3] was incomplete. Now we can present correct statements with complete proofs. Our result is more general than [3] in the sense that we neither have to assume that $J^2 = -\mathrm{Id}$, nor do we have to assume a self-adjoint leading symbol of the tangential operator in all cases. On the negative side, we must admit that the topology we have to impose on the space of differential operators is stronger than we hoped at the time of writing of [3]. The correct replacement for Theorem 3.9a in [3] is Theorem 7.2a and the correct replacement for Theorem 3.9b in [3] are Theorem 7.2b and Corollary 7.4.

Theorem 7.9 generalizes Theorem 3.8 in [3] and [2, Proposition B.1]

Next we deal with families of realizations of well-posed boundary conditions (cf. Theorem 3.9d in [2]).

**Theorem 7.16.** *Consider the space of pairs* $(A, P)$ *where* $A \in \mathrm{Diff}^1(M; E)$ *is elliptic and formally self-adjoint and* $P \in \mathrm{CL}^0(\Sigma; E_\Sigma)$ *is an orthogonal projection which is well-posed with respect to A and such that* $A_P$ *is self-adjoint. Equip this space with the metric* $d_0$, *i.e.,*

$$d_0((A, P), (A', P')) = N_0(A - A', P - P'). \tag{7.35}$$

*Then the map* $(A, P) \mapsto (A_P + \mathrm{i})^{-1} \in \mathscr{B}(L^2(M, E), L^2_1(M, E))$ *is continuous with respect to the* $d_0$ *metric on the space of such pairs* $(A, P)$. *In particular* $(A, P) \mapsto (A_P + \mathrm{i})^{-1}$ *is continuous or, equivalently,* $(A, P) \mapsto A_P$ *is graph continuous.*

**Proof.** The proof is basically the same as the proof of Theorem 7.9 once the analogue of the maps $\Phi_{T,T'}$ is established. Note that if $P, Q \in \mathrm{CL}^0(\Sigma; E_\Sigma)$ are orthogonal projections with $\|P - Q\|_{1/2,1/2} < 1$ they form an *invertible pair*, i.e., $P : \operatorname{im} Q \longrightarrow \operatorname{im} P$ is invertible. For such $P, Q$ we put analogously to Definition 7.10

$$\Psi_{P,Q}(f) := f - e(P - Q)\varrho f. \tag{7.36}$$

Then as in Lemma 7.11 we have

$$\|\Psi_{P,Q} - \mathrm{Id}\|_{1,1} \le C(e)\|P - Q\|_{1/2,1/2}, \tag{7.37}$$

and thus $\Psi_{P,Q}$ is invertible for $\|P - Q\|_{1/2,1/2}$ small enough. Furthermore, $\Psi_{P,Q}$ maps $\mathscr{D}(A_Q) = \{f \in L^2_1(M, E) \mid Q\varrho f = 0\}$ bijectively onto $\mathscr{D}(A_P) = \{f \in L^2_1(M, E) \mid Q\varrho f = 0\}$.

Now one mimics the proof of Theorem 7.9 with $\Psi_{P,Q}$ instead of $\Phi_{T,T'}$ and $A_P + i$ instead of $\widetilde{A}_{P(T)}$. $\square$

Finally we state a more precise version of [3], Theorem 3.9c. Note that the following version applies to a much wider class of operators than [3].

**Theorem 7.17.** *Let $\mathrm{Ell}^{1,sa}_{UCP}(M; E) \subset \mathrm{Diff}^1(M; E)$ denote the space of formally self-adjoint elliptic differential operators acting on sections of the Hermitian vector bundle $E$ which satisfy UCP and whose tangential operator $B_0$ has a self-adjoint leading symbol. We equip this space with the strong metric induced by the embedding $\mathrm{Ell}^{1,sa}_{UCP}(M; E) \to \mathscr{E}_{UCP}(M; E)$, $A \mapsto (A, (J_0^t(A))^{-1})$*

*Then the map*

$$\mathrm{Ell}^{1,sa}(M; E) \longrightarrow \mathscr{B}(L^2_1(M, E), L^2(M, E)), \qquad A \longmapsto A_{C_+}$$

*sending $A$ to the self-adjoint well-posed realization associated to the Calderón projection is continuous. Here $C_+$ denotes the version of the Calderón projection constructed from $(J_0^t(A))^{-1}$, cf. Proposition 5.12.*

**Proof.** Note that $A_{C_+}$ is self-adjoint by Theorem 6.1(II), Proposition 5.12. $A_{C_+}$ is indeed a self-adjoint realization of a well-posed boundary value problem.

By Corollary 7.4 $A \mapsto (A, C_+)$ is now a continuous map from $\mathrm{Ell}^{1,sa}_{UCP}(M, E)$ to the space of pairs described in Theorem 7.16 and hence Theorem 7.16 yields the claim. $\square$

## Acknowledgments

This work was supported by the network "Mathematical Physics and Partial Differential Equations" of the Danish Agency for Science, Technology and Innovation. The second named author was partially supported by Sonderforschungsbereich/Transregio "Symmetries and Universality in Mesoscopic Systems" (Bochum–Duisburg/Essen–Köln–Warszawa) and the Hausdorff Center for Mathematics (Bonn). The third named author was partially supported by FANEDD 200215, 973 Program of MOST, Fok Ying Tung Edu. Funds 91002, LPMC of MOE of China, and Nankai University.

## Appendix. Smooth symmetric elliptic continuations with constant coefficients in normal direction

In this Appendix, we restrict ourselves to formally self-adjoint operators. To begin with, we write $D = J(\frac{d}{dx} + B)$ as in (2.9) with the relations $J^* = -J$ and $JB = J' - B^tJ$ of (2.20) without loss of generality.

Sometimes one is interested in operators satisfying additional relations. We shall consider the following cases:

(I) $D$ arbitrary symmetric elliptic, e.g. no additional relations.
(II) $J^2(x) = -\mathrm{Id}$. We will see below that then, after a suitable coordinate transformation, we can even obtain that $J_x = J$ is constant. This is the Dirac operator case.
(III) $B_0 - B_0^t$ of order 0. In view of (2.20) this implies that $J_0B_0 + B_0J_0$ is of order 0, too.
(IV) $B - B^t$ of order 0. Analogously then $JB + BJ$ is of order 0, too.
(V) $D = J\left(\frac{d}{dx} + B\right) + \frac{1}{2}J' + C$ with $J^* = -J$, $B = B^t$, $JB + BJ = 0$, $J' = \frac{dJ}{dx}$, and $C$ of order 0. Then automatically $C = C^*$.

One can think of even more cases. But the preceding cases suffice for the moment. From now on we shall write

$$D = J\left(\frac{d}{dx} + B\right) + \frac{1}{2}J' + C \tag{A.1}$$

in all five cases although the notation is redundant in cases (I)–(IV). Recall, that here $C$ is of order 0 and $B$ is of order 1.

**Proposition A.1.** *Consider the case* (II). *Then there is a smooth unitary gauge transformation $U \in C^\infty([0, \varepsilon), \Gamma^\infty(\Sigma; \mathscr{U}(E_\Sigma)))$ such that*

$$J_x = U(x)J_0U(x)^*.$$

*With the unitary transformation*

$$\begin{array}{cccccc} \Phi_1 & : & L^2([0,\varepsilon)\times\Sigma, E_\Sigma) & \longrightarrow & L^2([0,\varepsilon)\times\Sigma, E_\Sigma), \\ & & f & \longmapsto & Uf \end{array}$$

*we find*

$$\Phi_1^{-1} D \Phi_1 = J_0\Big(\frac{\mathrm{d}}{\mathrm{d}x} + U^* B U\Big) + \widetilde{C}. \tag{A.2}$$

Here and in the following we use the abbreviation $E_\Sigma := E\restriction\Sigma$ introduced in Section 2.1 and, by slight abuse of notation, $E_\Sigma := E\restriction[0,\varepsilon)\times\Sigma$ as well. Note that self-adjointness of $B$ or $C$ and the relation $JB + BJ = 0$ (of lower order) are preserved under $\Phi_1$.

**Proof.** We give a brief sketch; it is basically a standard fact often used in $K$-theory [36, Prop. 4.3.3], see also [18, Section 3].

We only show a bit less, namely that the claim is true after making $\varepsilon$ a bit smaller; but this is not really a loss of generality. After possibly making $\varepsilon$ smaller we may assume that

$$\|\mathrm{Id} + J_x J_0\| = \|\mathrm{Id} - J_x^{-1} J_0\| < 2, \quad \text{for } 0 \le x \le \varepsilon. \tag{A.3}$$

Then the operator $Z_x := \mathrm{Id} - \frac{1}{2}(J_x J_0 + \mathrm{Id}) = \frac{1}{2}(\mathrm{Id} - J_x J_0)$ is invertible. Moreover, since $J_x^2 = -\mathrm{Id}$ we find $J_x Z_x = \frac{1}{2}(J_x + J_0) = Z_x J_0$ and thus $J_x = Z_x J_0 Z_x^{-1}$.

One now checks by direct calculation that $Z_x$ is normal and that $Z_x Z_x^*$ commutes with $J_x$ and $J_0$ [18, (3.7)]. Hence we may put $U(x) := Z_x^{-1}\sqrt{Z_x Z_x^*}$ to reach the conclusion.

The remaining assertions are now clear. □

**Remark A.2.** (1) One may ask under what conditions we can obtain unitary $J_0$, i.e., $J_0^2 = -\mathrm{Id}$? It is not clear whether this question has a definite answer, e.g., it seems impossible to find a coordinate transformation preserving the symmetry of the fixed given differential operator $D$ and providing a unitary leading symbol $\widetilde{J}$ in the new normal direction, unless all eigenvalues of the original $J_0$ are of the form $\pm i$.

(2) For the symplectic Hilbert space $\Big(L^2(\Sigma, E_\Sigma), \langle\cdot,\cdot\rangle, \omega(\cdot,\cdot)\Big)$, however, there is a simple answer. Here we set

$$\langle f, g\rangle := \int_\Sigma \langle f(p), g(p)\rangle_{E_p}\, \mathrm{dvol} \quad \text{and} \quad \omega(f,g) := \langle J_0 f, g\rangle.$$

As always in symplectic Hilbert spaces (see, e.g., [27, Lemma 1.5]), we can preserve the symplectic form $\omega$ while deforming the inner product $\langle\cdot,\cdot\rangle$ of $L^2(\Sigma, E_\Sigma)$ over $\Sigma$ smoothly into

$$\langle\cdot,\cdot\rangle^\sim := \langle Sf, g\rangle \quad \text{with self-adjoint } S := \sqrt{J_0^* J_0}$$

in such a way that $\omega(f,g) = \langle\widetilde{J} f, g\rangle^\sim$ with $\widetilde{J}^* = -\widetilde{J}$ and $\widetilde{J}^2 = -\mathrm{Id}$.

The notation (A.1) has another advantage in case (IV). Namely, replacing $B$ by $\frac{1}{2}(B+B^t)$ and $C$ by $C + \frac{1}{2}J(B - B^t)$ if necessary we see that we may assume that $B = B^t$. Note that this does not work so easily in case (III).

Summing up, the cases (I)–(V) may now be described as follows (when $D$ is written as in (A.1)):

(I) $D$ arbitrary symmetric elliptic, e.g. no additional relations.
(II) $J^2 = -\mathrm{Id}$, $J_x = J_0$ constant. Again, this is called the Dirac operator case.
(III) $B_0 - B_0^t$ and $J_0 B_0 + B_0 J_0$ of order 0.
(IV) $B = B^t$, $JB + BJ$ of order 0.
(V) $B = B^t$, $JB + BJ = 0$, $C = C^*$.

We consider $D$ as before. The goal of this Appendix is to prove the following theorem.

**Theorem A.3.** *Let $D$ be as in (A.1) with families $J_x$, $B_x$, $C_x$ smoothly depending on $x$. Then there is a $\delta > 0$ and a symmetric elliptic first order differential operator $\widetilde{D}$ (i.e., with smooth coefficients) on $\Gamma^\infty([-\delta,\varepsilon)\times\Sigma; E_\Sigma)$ with the following properties:*

(1) *$\widetilde{D}\restriction[0,\varepsilon)\times\Sigma = D$, i.e., $\widetilde{D}$ extends $D$.*
(2) *$\widetilde{D}\restriction([-\delta,-2/3\delta)\times\Sigma) = J_0\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_0\Big) + C_0$. In particular, $\widetilde{D}$ has constant coefficients near the new boundary $\{-\delta\}\times\Sigma$, and the constant coefficients are given just by smoothly "rewinding" to the coefficients of $D$ at 0.*

Note that in the formula for $\widetilde{D}$ near the boundary $\frac{1}{2}J'$ is left out deliberately to make the constant coefficient operator symmetric.

Note also that due to the concrete formula for $\widetilde{D}$ near the boundary the additional relations in the cases (II)–(V) still hold for the extended operator. It is not claimed (and it is open in some cases) that the relations in (IV), (V) are preserved on the whole interval $[-\delta, 0]$.

**Proof.** 1. Let

$$D_0 := J_0\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_0\Big) + C_0. \tag{A.4}$$

Since $D_0$ has constant coefficients we may think of $D_0$ as acting on $\Gamma_0^\infty(\mathbb{R}\times\Sigma; E_\Sigma)$. Note that since $D$ is formally self-adjoint, so is $D_0$. To see that we recall that $D = D^t$ is (thanks to our singling-out of $\frac{1}{2}J'$ in Eq. (A.1)) equivalent to the relations

$$J^* = -J, \quad -B^tJ + C^* = JB + C. \tag{A.5}$$

Then

$$D_0^t = J_0\frac{d}{dx} - B_0^tJ_0 + C_0^* = J_0\frac{d}{dx} + J_0B_0 + C_0 = D_0. \tag{A.6}$$

2. Now we apply the definition of smoothness of maps defined on a manifold with boundary to conclude that $J, B, C$ have smooth extensions to the whole negative half-line. More precisely there exist

$$\widetilde{J} \in \Gamma^\infty((-\infty,\varepsilon)\times\Sigma; E_\Sigma), \qquad \widetilde{C} \in \Gamma^\infty((-\infty,\varepsilon)\times\Sigma; E_\Sigma) \quad \text{and} \quad \widetilde{B} \in C^\infty((-\infty,\varepsilon), \mathrm{Diff}^1(\Sigma; E_\Sigma))$$

such that

$$\widetilde{J}\restriction[0,\varepsilon)\times\Sigma = J, \qquad \widetilde{C}\restriction[0,\varepsilon)\times\Sigma = C, \quad \text{and} \quad \widetilde{B}\restriction[0,\varepsilon)\times\Sigma = B.$$

Replacing $\widetilde{J}$ by $\frac{1}{2}(\widetilde{J} - \widetilde{J}^*)$ if necessary, we can, additionally, obtain that $\widetilde{J}^* = -\widetilde{J}$.

The extensions of $J$ and $C$ are immediate. However, for $B$ one might feel a bit uneasy because of the target space $\mathrm{Diff}^1(\Sigma; E_\Sigma)$. Well, first extend the leading symbol of $B$, which works like for $J$ and $C$. Then choose a right inverse to the symbol map on $\Sigma$ to obtain an operator map $\widetilde{B}_1$. On $[0,\varepsilon)\times\Sigma$, $\widetilde{B}_1$ coincides with $B$ up to order 0. The difference $\widetilde{B}_1 - B\restriction[0,\varepsilon)\times\Sigma$ is again smooth. Extend it and subtract it to make up for the 0 order defect. This yields the wanted $\widetilde{B}$.

So, we can form the differential operator

$$\widetilde{D}_1 := \widetilde{J}\Big(\frac{\mathrm{d}}{\mathrm{d}x} + \widetilde{B}\Big) + \frac{1}{2}\widetilde{J}' + \widetilde{C} \tag{A.7}$$

which is now defined on $(-\infty,\varepsilon)\times\Sigma$, has smooth coefficients and $\widetilde{D}_1\restriction[0,\varepsilon)\times\Sigma = D$.

So far $\widetilde{D}_1$ is not necessarily formally self-adjoint. Put

$$\widetilde{D}_2 := \frac{1}{2}(\widetilde{D}_1 + \widetilde{D}_1^t) =: \widetilde{J}\Big(\frac{\mathrm{d}}{\mathrm{d}x} + \widetilde{B}_2\Big) + \frac{1}{2}\widetilde{J}' + \widetilde{C}_2. \tag{A.8}$$

Next, consider a cut-off function $\varphi \in C^\infty(\mathbb{R})$ with

$$\varphi(x) = \begin{cases} 1, & x \le -\frac{2}{3}\delta, \\ 0, & x \ge -\frac{1}{3}\delta. \end{cases} \tag{A.9}$$

Then we consider the operator

$$\widetilde{D} := \varphi D_0 + (1-\varphi)\widetilde{D}_2 + \frac{1}{2}\varphi'(J_0 - \widetilde{J}). \tag{A.10}$$

The last summand was left out in [3, (3.14)]; the additional term is, however, necessary to make $\widetilde{D}$ formally self-adjoint.

$\widetilde{D}$ has the following properties:

(1) $\widetilde{D}^t = \widetilde{D}$. That follows immediately from the formal self-adjointness of $D_0, \widetilde{D}_2$ and the relations $[D_0,\varphi] = J_0\varphi'$, $[\widetilde{D}_2,\varphi] = \widetilde{J}\varphi'$.
(2) $\widetilde{D}\restriction[0,\varepsilon)\times\Sigma = D$.
(3) $\widetilde{D}\restriction[-\delta,-2/3\delta)\times\Sigma = J_0\Big(\frac{\mathrm{d}}{\mathrm{d}x} + B_0\Big) + C_0$.

It remains to discuss the ellipticity of $\widetilde{D}$. So let

$$x \in [-\delta, 0], \qquad p \in \Sigma, \qquad \xi \in T_p^*(\Sigma) \quad \text{and} \quad \lambda\mathrm{d}x + \xi \in S_{x,p}^*(\mathbb{R}\times\Sigma),$$

$S^*$ denoting the cosphere bundle. Then $|\lambda|^2 + |\xi|^2 = 1$ and we find for the leading symbol of $\widetilde{D}$

$$\begin{aligned} \sigma_{\widetilde{D}}(x,p)[\lambda dx + \xi] &= \Big(\varphi(x)\sigma_{D_0}(x,p) + (1-\varphi(x))\sigma_{\widetilde{D}_2}(x,p)\Big)[\lambda dx + \xi] \\ &= \varphi(x) J_0\big(i\lambda + \sigma_{B(0)}(\xi)\big) + (1-\varphi(x)) J_x\big(i\lambda + \sigma_{B(x)}(\xi)\big) \\ &= J_0\big(i\lambda + \sigma_{B(0)}(\xi)\big) + (1-\varphi(x))\big\{J_x(i\lambda + \sigma_{B(x)}(\xi)) - J_0(i\lambda + \sigma_{B(0)}(\xi))\big\}. \end{aligned} \tag{A.11}$$

Hence, by the compactness of $\Sigma$ and by the continuity of $J_x$ and $\sigma_{B(x)}$ we may choose $\delta$ so small that $\widetilde{D}$ is elliptic.

The theorem is proved. □

**Remark A.4.** We briefly discuss for the various cases (I)–(V) whether the construction of $\widetilde{D}$ can be modified such that the relations continue to hold.

(I) Since there are no relations, there is nothing to worry about.
(II) Since $J$ is constant, we may extend it constantly.
(III) At the new boundary $\{-\delta\} \times \Sigma$ we have by construction $B(-\delta) = B(0)$ hence (III) also works fine.
(IV), (V) By construction of $D_0$ it is clear that $\widetilde{D}$ also satisfies (IV) or (V) in the collar $[-\delta, -2/3\delta] \times \Sigma$ of the new boundary. However, we do not know so far whether $\widetilde{D}$ can be constructed in such a way that the relations hold on the whole interval $[-\delta, 0]$.

We leave it to the reader to prove that the latter is indeed possible in the case (II) + (IV) or (II) + (V).

In Section 2.1 we first applied a unitary transformation to our operator. It remains to clarify what happens if we transform the whole construction back to the original situation. The result reads as follows:

**Theorem A.5.** *Let $(M, g_1)$ be a compact Riemannian manifold with boundary, $(E, h_1)$ a Hermitian vector bundle over $M$ and $A : \Gamma_0^\infty(M; E) \to \Gamma_0^\infty(M; E)$ a first order elliptic differential operator which is formally self-adjoint in the Hilbert space $L^2(M, E; g_1, h_1)$.*

*Choose metrics $g$ on $M$, $h$ on $E$ which are product near the boundary $\Sigma = \partial M$ and such that $g{\restriction}\partial M = g_1{\restriction}\partial M$, $h{\restriction}\partial M = h_1{\restriction}\partial M$. Let $\Phi$ be the isometry of (2.8) and assume that we have*

$$\Phi A \Phi^{-1} = J\Big(\frac{d}{dx} + B\Big) + \frac{1}{2}J' + C. \tag{A.12}$$

*For $\delta > 0$ form $M_\delta := \big([-\delta, 0] \times \Sigma\big) \cup_\Sigma M$.*

*Then for $\delta$ sufficiently small there are a Hermitian vector bundle $(E_\delta, h_{1,\delta})$ over $M_\delta$, a Riemannian metric $g_{1,\delta}$ on $M_\delta$ and a first order symmetric elliptic differential operator $A_\delta$ on $M_\delta$ such that:*

(1) *$E_\delta, g_{1,\delta}, h_{1,\delta}, A_\delta$ are extensions of $E, g_1, h_1, A$ respectively.*
(2) *$g_{1,\delta}$ and $h_{1,\delta}$ are product metrics near $\partial M_\delta$. More precisely, we have on $[-\delta, -2/3\delta] \times \Sigma$*

$$\begin{aligned} g_{1,\delta} &= dx^2 \oplus g(0), \\ h_{1,\delta}(x) &= h(0). \end{aligned} \tag{A.13}$$

(3) *With the natural extension of $\Phi$ to $M_\delta$ we have on $[-\delta, -2/3\delta] \times \Sigma$:*

$$\Phi A \Phi^{-1} = J_0\Big(\frac{d}{dx} + B_0\Big) + C_0. \tag{A.14}$$

**Proof.** We apply the previous Theorem to $\Phi A \Phi^{-1}$ in $L^2(M, E; g, h)$. Since the spaces of metrics on $M$ and on $E$ are positive cones, we may extend the metrics $g_1, h_1$ to smooth metrics $\widetilde{g}_\delta, h_{1,\delta}$ on $M_\delta$ in such a way that on the collar $[-\delta, -2/3\delta] \times \Sigma$ they are of the form (A.13). Now consider the unitary transformation $\Psi : L^2(M_\delta, E; g_{1,\delta}, h_{1,\delta}) \to L^2(M_\delta, E; g, h)$ as in Lemma 2.1 ($g, h$ extend trivially to $M_\delta$ since they are already product metrics). By the construction explained before Lemma 2.1 we see that $\Psi{\restriction}[-\delta, -2/3\delta] \times \Sigma = \mathrm{Id}$ and therefore we reach the conclusion from the previous theorem. □

**Remark A.6.** (1) The preceding theorem shows that it is always possible to extend a given symmetric elliptic differential operator (of first order) $A$ to a symmetric elliptic differential operator $\widetilde{A}$ by attaching a collar with finite cylindrical end with new boundary $\Sigma'$ such that the Riemannian and Hermitian structures become product close to $\Sigma'$ and the operators $J_x$, $B_x$ and $C_x$ become independent of the normal variable $x$ close to $\Sigma'$ ; i.e., we can always bring the operator in product form near a new boundary by suitable prolongation under preservation of the symmetry property.

(2) The previous discussion shows that $J_0^2 = -\mathrm{Id}$ is enough to obtain, by coordinate transformation, $J_x = J_0$ and hence $J_x^2 = -\mathrm{Id}$ in a whole neighborhood of the new boundary. See also Remark A.2 for conditions for $J_0^2 = -\mathrm{Id}$.

(3) The previous discussion can be made parameter dependent with the right notion of parameter dependency, cf. our Section 7.

(4) The question remains whether weak inner UCP can be preserved under the symmetric prolongation. The short answer is

- *yes* in the cases (III)–(V), i.e., when the leading symbol of the tangential operator is symmetric;
- *yes* in a very restricted sense, namely that the UCP-defect dimension

$$d(x) := \dim \{u \mid Au = 0 \text{ and } u\restriction \Sigma(x) = 0\} \tag{A.15}$$

is constant on the last part of the collar, i.e., for sufficiently negative tangential coordinate $x$ when constant coefficients in normal direction are obtained. Here $\Sigma(x) := \{x\} \times \partial M$ denotes the parallel surface in the collar.

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