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General spectral flow formula for fixed maximal domain *

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Abstract: We consider a continuous curve of linear elliptic formally self-adjoint differential operators of first order with smooth coefficients over a compact Riemannian manifold with boundary together with a continuous curve of global elliptic boundary value problems. We express the spectral flow of the resulting continuous family of (unbounded) self-adjoint Fredholm operators in terms of the Maslov index of two related curves of Lagrangian spaces. One curve is given by the varying domains, the other by the Cauchy data spaces. We provide rigorous definitions of the underlying concepts of spectral theory and symplectic analysis and give a full (and surprisingly short) proof of our General Spectral Flow Formula for the case of fixed maximal domain. As a side result, we establish local stability of weak inner unique continuation property (UCP) and explain its role for parameter dependent spectral theory.

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1 Statement of the problem and main result

1.1 Statement of the problem

Roughly speaking, the spectral flow counts the net number of eigenvalues changing from the negative real half axis to the non-negative one. The definition goes back to a famous paper by M. Atiyah, V. Patodi, and I. Singer [3], and was made rigorous by J. Phillips [23] for continuous paths of bounded self-adjoint Fredholm operators, by K.P. Wojciechowski [30] and C. Zhu and Y. Long [34] in various non-self-adjoint cases, and by B. Booss-Bavnbek, M. Lesch, and J. Phillips [7] in the unbounded self-adjoint case. We shall give a rigorous definition of spectral flow, most suitable for our purpose, below in Subsection 2.1 together with a review of its basic properties. For a definition of spectral flow admitting zero in the continuous spectrum, we refer to A. Carey and J. Phillips [13].

In various branches of mathematics one is interested in the calculation of the spectral flow of a continuous family of closed densely defined (not necessarily bounded) self-adjoint Fredholm operators in a fixed Hilbert space. We consider the following typical problem of this kind.

Assumption 1.1. *Let* $\{A_s : C^\infty(M;E) \to C^\infty(M;E)\}_{s\in[0,1]}$ *be a family of formally self-adjoint linear elliptic differential operators of first order with continuously varying smooth coefficients over a smooth compact Riemannian manifold* M *with boundary* Σ*, acting on sections of a Hermitian vector bundle* E *over* M*. Let* $\{P_s\}$ *be a continuous family of orthogonal pseudodifferential projections in* $L^2(\Sigma; E|_\Sigma)$*. Define* A_{s,P_s} *to be the unbounded operator in* $L^2(M;E)$ *with domain*

$$D_s := \{x \in H^1(M;E) \mid P_s(\gamma(x)) = 0\}, \tag{1}$$

where

$$\gamma : H^1(M;E) \to H^{\frac{1}{2}}(\Sigma; E|_\Sigma) \tag{2}$$

denotes the (continuous) trace map from the first Sobolev space over the whole manifold to the $\frac{1}{2}$ *Sobolev space over the boundary. (Note that in this paper the symbols* x *and* y *do not denote points of the underlying manifolds* M *or* Σ*, but points in Hilbert spaces, sections of vector bundles, etc., following the conventions of functional analysis and dynamical systems.) Assume that each* P_s *defines a self-adjoint elliptic boundary condition for* A_s*, i.e.,* A_{s,P_s} *is a self-adjoint Fredholm operator for each* $s \in [0,1]$*.*

Then the spectral flow $\mathrm{sf}\{A_{s,P_s}\ ;\ s \in [0,1]\}$ or, shortly, $\mathrm{sf}\{A_{s,P_s}\}$ is well defined. As a spectral invariant it is essentially a *quantum* variable which one may not always be able to determine directly by eigenvalue calculations. As an alternative, one is looking for a *classical* method of calculating the spectral flow. There are two different approaches. One setting expresses the spectral flow (of a loop of Dirac operators on a closed manifold) as an integral over a 1-form induced by the heat kernel (for a review see [13]). The other setting is reduction to the boundary, i.e., one expresses the spectral flow (of a path of self-adjoint boundary value problems on a compact manifold with boundary) in terms of the intersection geometry of the solution spaces of the homogeneous differential equations and the boundary conditions. That is the approach we shall follow in this paper.

Problem 1.2. Give a *classical* method of calculating the spectral flow of the family $\{A_{s,P_s}\}$ by reduction to the boundary, i.e., a method not involving the determination of the spectrum near 0 and yielding an expression on Σ.

The preceding spectral flow calculation problem is formulated for families by analogy with *Bojarski's Theorem* for single operators which expresses the index (which is the difference between the multiplicities of the 0-eigenvalue of the original and the formally adjoint problem and so a priori a quantum or spectral invariant) of an elliptic operator over a closed partitioned manifold $M = M_- \cup_\Sigma M_+$ by the index of the Fredholm pair of Cauchy data spaces from two sides along the hypersurface Σ (which are classical objects, see Bojarski [4] and Booss and Wojciechowski [10, Chapter 24]).

1.2 General functional analytic setting and announcement of the General Spectral Flow Formula

Now we translate our problem into a functional analytic setting. For any such family there are three geometrically defined relevant Hilbert spaces of global sections which remain fixed under variation of the coefficients of the operators and under variation of the boundary conditions:

$$L^2(M;E), \quad H^1_0(M;E), \quad \text{and} \quad H^1(M;E). \tag{3}$$

Here $H^1_0(M;E)$ denotes the closure of $C^\infty_0(M\setminus\Sigma;E)$ in the first Sobolev space $H^1(M;E)$, where $C^\infty_0(M\setminus\Sigma;E)$ denotes the smooth sections with support in the interior of $M\setminus\Sigma$. Since the trace map $\gamma : H^1(M;E) \to H^{\frac{1}{2}}(\Sigma;E|_\Sigma)$ is continuous, we have $H^1_0(M;E) = \ker\gamma$, i.e., the space $H^1_0(M;E)$ consists exactly of the elements of $H^1(M;E)$ which vanish on the boundary Σ.

For each $s \in [0,1]$, we shall denote the unbounded operator A_s acting in $L^2(M;E)$ with domain $H^1_0(M;E)$ also by A_s. Since the differential operator A_s is elliptic, the unbounded operator A_s is closed by *Gårding's inequality*

$$\|x\|_{H^1(M;E)} \le C\big(\|x\|_{L^2(M;E)} + \|A_s x\|_{L^2(M;E)}\big) \quad \text{for } x \in H^1_0(M;E)\,. \tag{4}$$

Denote by $\mathrm{dom}(A)$ the domain of an operator A, by A^* the adjoint operator of A, and

$$D_{\max}(A) := \mathrm{dom}(A^*). \tag{5}$$

Since A is closed and symmetric, it follows that $D_{\max}(A) = \{x \in L^2(M;E) \mid Ax \in L^2(M;E)\}$ with Ax taken in the distributional sense. For A_s formally self-adjoint, it follows immediately that $H^1(M;E) \subset D_{\max}(A_s)$ and that A_s (with domain $H^1_0(M;E)$) is symmetric.

In local coordinates, we view each coefficient of $\{A_s\}$ as a continuous map which assigns to $s \in [0,1]$ a smooth section. Then the continuity of the curve $\{A_s\}_{s\in[0,1]}$ in the sense of *continuously varying coefficients* implies the continuity of the curve

$$[0,1] \ni s \mapsto A_s^*|_{H^1(M;E)} \in \mathcal{B}(H^1(M;E), L^2(M;E))\,, \tag{6}$$

as a curve of *bounded operators* from $H^1(M;E)$ to $L^2(M;E)$.

We denote by $Q_s : L^2(\Sigma; E|_\Sigma) \to L^2(\Sigma; E|_\Sigma)$ the *Calderón projection.* It is a projection onto the *Cauchy data space* of A_s^* which is defined as the L^2-closure of $\gamma(\ker(A_s^*|_{H^1(M;E)}))$. It can be described as a pseudodifferential operator, e.g., when continuing A_s to an elliptic operator on a closed manifold $\widetilde{M} \supset M$, see R.T. Seeley [29, Sections 4 and 8] and [10, Chapter 12]. For an alternative canonical construction based on a natural boundary value problem and avoiding the choices of closing the manifold and continuing the operator, see B. Himpel, P. Kirk, and M. Lesch [16, Section 3] and recent joint work of the authors with M. Lesch [8].

For each $s \in [0,1]$, there is a natural (strong) symplectic form ω_s on the quotient space $D_{\max}(A_s)/H_0^1(M;E)$ induced by *Green's form* of A_s as

$$\omega_s(\gamma(x), \gamma(y)) := \langle A_s^* x, y\rangle - \langle x, A_s^* y\rangle\,, \quad x, y \in D_{\max}(A_s). \tag{7}$$

Here γ denotes the natural projection

$$D_{\max}(A_s) \to D_{\max}(A_s)/H_0^1(M;E)\,.$$

Identifying the quotient space $D_{\max}(A_s)/H_0^1(M;E)$ with a subspace of the Sobolev (distribution) space $H^{-1/2}(\Sigma; E|_\Sigma)$, we obtain that this γ extends the Sobolev trace map of (2). A rigorous definition of *symplectic structures* and *Lagrangian subspaces* will be given below in Subsection 2.2.

For our formally self-adjoint differential operators of first order, we have an explicit description of the form in (7), restricted to $H^1(M;E)$, by Stokes' Theorem

$$\omega_s(\gamma(x), \gamma(y)) = -\int_\Sigma \langle \sigma_1(A_s)(\cdot, dt)\big(x|_\Sigma\big), y|_\Sigma\rangle d\mathrm{vol}_\Sigma\,, \tag{8}$$

where $\sigma_1(A_s)(\cdot, dt)$ denotes the principal symbol of A_s at the boundary, taken in inner (co-)normal direction dt. Notice that we do not require that the manifold M is orientable: for our application of Stokes' Theorem it suffices that any collar neighborhood of Σ in M is oriented by the normal structure. Then the form $\omega_s|_{H^1(M;E)}$ of (8) extends to a (strong) symplectic structure $\overline{\omega}_s$ on $L^2(\Sigma; E|_\Sigma)$. One can show that $\omega_s|_{H^1(M;E)/H_0^1(M;E)}$ is a *weak* (but not strong) symplectic form on the Hilbert space $H^1(M;E)/H_0^1(M;E) \cong H^{\frac12}(\Sigma; E|_\Sigma)$ (cf. Booss and Zhu [11, Remark 1.6b]).

We have $H^1(M;E) = D_{\max}(A_s)$ if and only if $\dim M = 1$. For higher dimensional case, the strict inclusion $H^1(M;E) \subset D_{\max}(A_s)$ and the weakness of $\omega_s|_{H^1(M;E)}$ causes technical difficulties.

However, we still have the following theorem (cf. Theorem 0.1 of [11]).

Theorem 1.3 (General Spectral Flow Formula). *Let* $\{A_s\}_{s\in[0,1]}$ *and* $\{P_s\}_{s\in[0,1]}$ *be operator families like in Assumption 1.1. We assume that* $\{\ker P_s\}_{s\in[0,1]}$ *is a continuous family of Lagrangian subspaces in* $(H,\overline{\omega}_s)$. *If* A_s *satisfies weak inner UCP, i.e.,* $\ker A_s = \{0\}$ *for each* $s \in [0,1]$, *we have:*
(a) *The family* $\{A_{s,P_s}\}_{s\in[0,1]}$ *of closed self-adjoint Fredholm operators on* X *is a continuous family* (*in the gap norm, or equivalently, in the projection norm*).
(b) *The Cauchy data spaces* $\operatorname{im} Q_s$ *are Lagrangian subspaces in the weak symplectic Hilbert space* $(H^{\frac{1}{2}}(\Sigma; E|_\Sigma),\overline{\omega}_s)$ *and form a continuous family in* $H^{\frac{1}{2}}(\Sigma; E|_\Sigma)$ *for* $s \in [0,1]$.
(c) *Finally, the following formula holds:*

$$\operatorname{sf}\{A_{s,P_s}\} = -\operatorname{Mas}\{\ker P_s, \operatorname{im} Q_s\}, \tag{9}$$

where the spectral flow sf *and the Maslov index* Mas *are defined by Definitions* 2.1 *and* 2.11 *below respectively.*

Remark 1.4. (a) The General Spectral Flow Formula contains and generalizes all previously known spectral flow formulae, as given by M. Morse [21], W. Ambrose [1], J.J. Duistermaat [14], A. Floer [15], P. Piccione and D.V. Tausk [24] and [25], and C. Zhu [32] and [33] for the 1-dimensional setting of the study of geodesics, and for the higher dimensional setting the formulae given by T. Yoshida [31], L. Nicolaescu [22], S.E. Cappell, R. Lee, and E.Y. Miller [12], B. Booss, K. Furutani, and N. Otsuki [5] and [6], and P. Kirk and M. Lesch [18].

(b) The main difference to [5] and [6] is that we admit varying maximal domain and varying Fredholm domain. The main difference to [18] is that we admit more general operators than Dirac type operators with constant coefficients in normal direction close to the boundary.

(c) The proof of the above theorem is rather technical and complicated. In this review article, we only prove the following fixed maximal domain case which completely covers all above cited one-dimensional cases (cf. Corollary 2.14 in [11]). Moreover, it contains [5] and [6] and generalizes it to varying Fredholm domains, and contains [18] for the case of fixed maximal domain and generalizes it under that restriction to more general operator families.

1.3 Statement of the result for fixed maximal domain

Let X be a Hilbert space, and $D_m \subset D_{\max}$ be two dense linear subspaces of X. Let $\{A_s\}_{s\in[0,1]}$ be a family of symmetric densely defined operators in $\mathcal{C}(X)$ with domain $\operatorname{dom}(A_s) = D_m$. Here we denote by $\mathcal{C}(X)$ all closed operators in X. Assume that $\operatorname{dom}(A_s^*) = D_{\max}$, i.e., the domain of the maximal symmetric extension A_s^* of A_s is independent of s.

We recall from [5] (see also B. Lawruk, J. Śniatycki, and W.M. Tulczyjew [19] for early investigation of symplectic structures and boundary value problems) for each $s \in [0,1]$:

(1) The space $D_{\max}$ is a Hilbert space with the graph inner product

$$\langle x, y\rangle_{\mathfrak{G}_s} := \langle x, y\rangle_X + \langle A_s^* x, A_s^* y\rangle_X \quad \text{for } x, y \in D_{\max}\,. \tag{10}$$

(2) The space D_m is a closed subspace in the graph norm and the quotient space $D_{\max}/D_m$ is a strong symplectic Hilbert space with the (bounded) symplectic form induced by Green's form

$$\omega_s(x + D_m, y + D_m) := \langle A_s^* x, y\rangle_X - \langle x, A_s^* y\rangle_X \quad \text{for } x, y \in D_{\max}\,. \tag{11}$$

(3) If A_s admits a self-adjoint Fredholm extension $A_{s,D_s} := A_s^*|_{D_s}$ with domain D_s, then the *natural Cauchy data space* $(\ker A_s^* + D_m)/D_m$ is a Lagrangian subspace of $(D_{\max}/D_m, \omega_s)$.
(4) Moreover, self-adjoint Fredholm extensions are characterized by the property of the domain D_s that $(D_s + D_m)/D_m$ is a Lagrangian subspace of $(D_{\max}/D_m, \omega_s)$ and forms a Fredholm pair with $(\ker A_s^* + D_m)/D_m$.
(5) We denote the natural projection (which is independent of s) by

$$\gamma : D_{\max} \longrightarrow D_{\max}/D_m.$$

The main result of this paper is the following theorem which reproves parts of the preceding list.

Theorem 1.5 (General Spectral Flow Formula for fixed maximal domain). *We assume that on $D_{\max}$ the graph norms induced by A_s, $0 \le s \le 1$ are mutually equivalent. Then we fix a graph norm $\mathcal{G}$ on $D_{\max}$ induced by A_0. Assume that $\{A_s^* : D_{\max} \to X\}$ is a continuous family of bounded operators and each A_s is injective. Let $\{D_s/D_m\}$ be a continuous family of Lagrangian subspaces of $(D_{\max}/D_m, \omega_s)$, such that each A_{s,D_s} is a Fredholm operator. Then:*
(a) *Each $\big(D_s/D_m, \gamma(\ker(A_s^*)\big)$ is a Fredholm pair in $D_{\max}/D_m$.*
(b) *Each Cauchy data space $\gamma(\ker A_s^*)$ is a Lagrangian subspace of $(D_{\max}/D_m, \omega_s)$.*
(c) *The family $\{\gamma(\ker A_s^*)\}$ is a continuous family in $D_{\max}/D_m$.*
(d) *The family $\big\{A_{s,D_s}\big\}$ is a continuous family of self-adjoint Fredholm operators in $\mathcal{C}(X)$.*
(e) *Finally, we have*

$$\operatorname{sf}\{A_{s,D_s}\} = -\operatorname{Mas}\{\gamma(D_s), \gamma(\ker A_s^*)\}. \tag{12}$$

2 Definition of spectral flow and Maslov index

2.1 Spectral flow, revisited and generalized

Let X be a Hilbert space. For a self-adjoint Fredholm operator $A \in \mathcal{C}(X)$, there exists a unique orthogonal decomposition

$$X = X^+(A) \oplus X^0(A) \oplus X^-(A) \tag{13}$$

such that $X^+(A)$, $X^0(A)$ and $X^-(A)$ are invariant subspaces associated to A, and $A|_{X^+(A)}$, $A|_{X^0(A)}$ and $A|_{X^-(A)}$ are positive definite, zero and negative definite respectively. We introduce vanishing, natural, or infinite numbers

$$m^+(A) := \dim X^+(A),\ m^0(A) := \dim X^0(A),\ m^-(A) := \dim X^-(A),$$

and call them *Morse positive index*, *nullity* and *Morse index* of A respectively. For finite-dimensional X, the *signature* of A is defined by $\operatorname{sign}(A) = m^+(A) - m^-(A)$ which yields an integer. The *APS projection* Q_A (where APS stands for Atiyah-Patodi-Singer) is defined by

$$Q_A(x^+ + x^0 + x^-) := x^+ + x^0,$$

for all $x^+ \in X^+(A), x_0 \in X^0(A), x^- \in X^-(A)$.

Let $\{A_s\}$, $0 \le s \le 1$ be a continuous family of self-adjoint Fredholm operators. The spectral flow $\operatorname{sf}\{A_s\}$ of the family should be equal to $m^-(A_0) - m^-(A_1)$ if $\dim X < +\infty$. We will generalize this definition to general X.

For each $t \in [0,1]$, there exists a bounded open neighborhood N_t of 0 such that ∂N_t is of class C^1, $\sigma(A_t) \cap \partial N_t = \emptyset$, and $P(A_t, N_t)$ is a finite rank projection. Here we denote the spectrum of a closed operator A by $\sigma(A)$, and the spectral projection by

$$P(A,N) := -\frac{1}{2\pi\sqrt{-1}} \int_{\partial N} (A - zI)^{-1} dz$$

if N is a bounded open subset of $\mathbb{C}$ with C^1 boundary and $\partial N \cap \sigma(A) = \emptyset$. The orientation of ∂N is chosen to make N stay on the left side of ∂N. Since the family $\{A_s\}$, $0 \le s \le 1$ is continuous, there exists a $\delta(t) > 0$ for each $t \in [0,1]$ such that

$$\sigma(A_s) \cap \partial N_t = \emptyset,\ \text{for all } s \in (t - \delta(t), t + \delta(t)) \cap [0,1].$$

Then

$$\big\{P(A_s, N_t)\big\}_{s \in (t-\delta(t), t+\delta(t)) \cap [0,1]} \quad \text{for fixed } t \in [0,1],$$

is a continuous family of orthogonal projections. By Lemma I.4.10 in Kato [17], they have the same rank. We denote by $A(s,t)$ the operator A_s acting on the finite-dimensional space $\operatorname{im} P(A_s, N_t)$. Since $[0,1]$ is compact, there exists a partition $0 = s_0 < \ldots < s_n = 1$ and $t_k \in [s_k, s_{k+1}]$, $k = 0, \ldots, n-1$ such that $[s_k, s_{k+1}] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each $k = 0, \ldots, n-1$.

Definition 2.1. The *spectral flow* $\operatorname{sf}\{A_s\}$ of the family $\{A_s\}$, $0 \le s \le 1$ is defined by

$$\operatorname{sf}\{A_s\} := \sum_{k=0}^{n-1} \Big(m^-\big(A(s_k, t_k)\big) - m^-\big(A(s_{k+1}, t_k)\big) \Big). \tag{14}$$

After carefully examining the above definition, inspired by [23], we find that the necessary data for defining any spectral flow are the following:

- a co-oriented bounded real 1-dimensional regular C^1 submanifold ℓ of $\mathbb{C}$ without boundary (we call such an ℓ *admissible*, and denote by $\ell \in \mathcal{A}(\mathbb{C})$);
- a Banach space X;
- and a continuous family of admissible operators A_s, $0 \le s \le 1$ in $\mathcal{A}_\ell(X)$.

Here we define $A \in \mathcal{C}(X)$ to be *admissible* with respect to ℓ, if there exists a bounded open neighborhood N of ℓ in $\mathbb{C}$ with C^1 boundary ∂N such that (i) $\partial N \cap \sigma(A) = \emptyset$; (ii) $N \cap \sigma(A) \subset \ell$ is a finite set; and (iii) $P^0_\ell(A) := P(A,N)$ is a finite rank projection.

Note that $P^0_\ell(A)$ does not depend on the specific choice of N. We call $\nu_{h,\ell}(A) := \dim \operatorname{im} P^0_\ell(A)$ the *hyperbolic nullity* of A with respect to ℓ. We denote by $\mathcal{A}_\ell(X)$ the set of closed admissible operators with respect to ℓ. It is an open subset of $\mathcal{C}(X)$.

Similarly as before, we can define the spectral flow $\mathrm{sf}_\ell\{A_s\}$. It counts the number of spectral lines of A_s coming from the negative side of ℓ to the non-negative side of ℓ. For the details, see [34].

Example 2.2. a) In the above self-adjoint case, $\ell = \sqrt{-1}(-\epsilon,\epsilon)$ ($\epsilon > 0$) with co-orientation from left to right. Then a self-adjoint operator A is admissible with respect to ℓ if and only if A is Fredholm.
b) Another important case is that $\ell = (1-\epsilon, 1+\epsilon)$ ($\epsilon \in (0,1)$) with co-orientation from downward to upward, and all A_s unitary. A unitary operator A is admissible with respect to ℓ if and only if $A - I$ is Fredholm.

The spectral flow has the following properties (cf. [23] and Lemma 2.6 and Proposition 2.2 in [34]).

Proposition 2.3. *Let $\ell \in \mathcal{A}(\mathbb{C})$ be admissible and let $\{A_s\}$, $0 \le s \le 1$ be a curve in $\mathcal{A}_\ell(X)$. Then the spectral flow $\mathrm{sf}_\ell\{A_s\}$ is well defined, and the following holds:*

(1) **Catenation.** *Assume $t \in [0,1]$. Then we have*

$$\mathrm{sf}_\ell\{A_s; 0 \le s \le t\} + \mathrm{sf}_\ell\{A_s; t \le s \le 1\} = \mathrm{sf}_\ell\{A_s; 0 \le s \le 1\}. \tag{15}$$

(2) **Homotopy invariance.** *Let $A(s,t)$, $(s,t) \in [0,1]\times[0,1]$ be a continuous family in $\mathcal{A}_\ell(X)$. Then we have*

$$\mathrm{sf}_\ell\{A(s,t); (s,t) \in \partial([0,1]\times[0,1])\} = 0. \tag{16}$$

(3) **Endpoint dependence for Riesz continuity.** *Let $\mathcal{B}^{\mathrm{sa}}(X)$, respectively $\mathcal{C}^{\mathrm{sa}}(X)$ denote the spaces of bounded, respectively closed self-adjoint operators in X. Let*

$$\begin{aligned} R : \mathcal{C}^{\mathrm{sa}} &\to \quad \mathcal{B}^{\mathrm{sa}}(X) \\ A &\mapsto A(A^2+I)^{-\frac{1}{2}} \end{aligned}$$

denote the Riesz transformation. *Let $A_s \in \mathcal{C}^{\mathrm{sa}}(X)$ for $s \in [0,1]$. Assume that $\{R(A_s)\}$, $0 \le s \le 1$ is a continuous family. If $m^-(A_0) < +\infty$, then $m^-(A_1) < +\infty$ and we have*

$$\mathrm{sf}\{A_s\} = m^-(A_0) - m^-(A_1). \tag{17}$$

(4) **Product.** *Let* $\{P_s\}$ *be a curve of projections on* X *such that* $P_sA_s \subset A_sP_s$ *for all* $s \in [0,1]$. *Set* $Q_s = I - P_s$. *Then we have* $P_sA_sP_s \in \mathcal{A}_\ell(\operatorname{im} P_s) \subset \mathcal{C}(\operatorname{im} P_s)$, $Q_sA_sQ_s \in \mathcal{A}_\ell(\operatorname{im} Q_s) \subset \mathcal{C}(\operatorname{im} Q_s)$, *and*

$$\mathrm{sf}_\ell\{A_s\} = \mathrm{sf}_\ell\{P_sA_sP_s\} + \mathrm{sf}_\ell\{Q_sA_sQ_s\}. \tag{18}$$

(5) **Bound.** *For* $A \in \mathcal{A}_\ell(X)$, *there exists a neighborhood* $\mathcal{N}$ *of* A *in* $\mathcal{C}(X)$ *such that* $\mathcal{N} \subset \mathcal{A}_\ell(X)$, *and for curves* $\{A_s\}$ *in* $\mathcal{N}$ *with endpoints* $A_0 =: A$ *and* $A_1 =: B$, *the* relative Morse index $I_\ell(A,B) := -\mathrm{sf}_\ell\{A_s, 0\,; \le s \le 1\}$ *is well defined and satisfies*

$$0 \le I_\ell(A,B) \le \nu_{h,\ell}(A) - \nu_{h,\ell}(B). \tag{19}$$

(6) **Reverse orientation.** *Let* $\hat{\ell}$ *denote the curve* ℓ *with opposite co-orientation. Then we have*

$$\mathrm{sf}_\ell\{A_s\} + \mathrm{sf}_{\hat{\ell}}\{A_s\} = \nu_{h,\ell}(A_1) - \nu_{h,\ell}(A_0). \tag{20}$$

(7) **Zero.** *Suppose that* $\nu_{h,\ell}(A_s)$ *is constant for* $s \in [0,1]$. *Then* $\mathrm{sf}_\ell\{A_s\} = 0$.

(8) **Invariance.** *Let* $\{T_s\}_{s\in[0,1]}$ *be a curve of bounded invertible operators. Then we have*

$$\mathrm{sf}_\ell\{T_s^{-1}A_sT_s\} = \mathrm{sf}_\ell\{A_s\}. \tag{21}$$

Now we give a method of calculating the spectral flow of differentiable curves, inspired among others by J.J. Duistermaat [14] and J. Robbin and D. Salamon [28].

Definition 2.4. Let $\ell \in \mathcal{A}(\mathbb{C})$ be admissible and $\{A_s\}_{s\in[0,1]}$ be a curve in $\mathcal{A}_\ell(X)$.

(1) A *crossing* for A_s is a number $t \in [0,1]$ such that $\nu_{h,\ell}(A_t) \neq 0$.

(2) Set $P_s = P_\ell^0 A_s$. A crossing t is called *regular* if $\mathrm{dom}(A_s) = D$ fixed for s near t, A_sx is differentiable at $s = t$ for all $x \in D$, and $P_t\dot{A}_tP_t$ is *hyperbolic*, i.e. $\nu_{h,\ell}(P_t\dot{A}_tP_t) = 0$, where $\dot{A}_s$ is the unbounded operator with domain D defined by

$$\dot{A}_sx = \frac{d}{ds}A_sx$$

for all $x \in D$.

(3) A crossing t is called *simple* if it is regular and $\nu_{h,\ell}(A_t) = 1$.

Proposition 2.5 (cf. Theorem 4.1 of [34]). *Let* X *be a Banach space and* $\ell = \sqrt{-1}(-\epsilon,\epsilon)$ $(\epsilon > 0)$ *with co-orientation from left to right. Let* A_s, $-\epsilon \le s \le \epsilon$ $(\epsilon > 0)$, *be a curve in* $\mathcal{A}_\ell(X)$. *Suppose that* 0 *is a regular crossing of* A_s. *Set* $P = P_\ell^0(A_0)$, $A = A_0$ *and* $B = \dot{A}_s|_{s=0}$. *Assume that*

$$P(AB - BA)P = 0. \tag{22}$$

Then there is a $\delta \in (0,\epsilon)$ *such that* $\nu_{h,\ell}(A_s) = 0$ *for all* $s \in [-\delta,0) \cup (0,\delta]$ *and*

$$\mathrm{sf}_\ell\{A_s; 0 \le s \le \delta\} = -m^-(PBP), \tag{23}$$
$$\mathrm{sf}_\ell\{A_s; -\delta \le s \le 0\} = m^+(PBP). \tag{24}$$

Here we denote by $m^+(PBP)$ $(m^-(PBP))$ *the total algebraic multiplicity of eigenvalues of* PBP *with positive (negative) imaginary part respectively.*

2.2 Symplectic functional analysis and Maslov index

A main feature of symplectic analysis is the study of the *Maslov index*. It is an intersection index between a path of Lagrangian subspaces with the *Maslov cycle*, or, more generally, with another path of Lagrangian subspaces. The Maslov index assigns an integer to each continuous path of Fredholm pairs of Lagrangian subspaces of a fixed Hilbert space with continuously varying symplectic structures.

Firstly we define symplectic Hilbert spaces and Lagrangian subspaces.

Definition 2.6. Let H be a complex vector space. A mapping

$$\omega : H \times H \longrightarrow \mathbb{C}$$

is called a (weak) *symplectic form* on H, if it is sesquilinear, skew-hermitian, and non-degenerate, i.e.,
(i) $\omega(x, y)$ is linear in x and conjugate linear in y;
(ii) $\omega(y, x) = -\overline{\omega(y, x)}$;
(iii) $H^\omega := \{x \in H \mid \omega(x, y) = 0 \text{ for all } y \in H\} = \{0\}$.
Then we call (H, ω) a *complex symplectic vector space.*

Definition 2.7. Let (H, ω) be a complex symplectic vector space.
(a) The *annihilator* of a subspace λ of H is defined by

$$\lambda^\omega := \{y \in H \mid \omega(x, y) = 0 \text{ for all } x \in \lambda\}.$$

(b) A subspace λ is called *isotropic*, *co-isotropic*, or *Lagrangian* if

$$\lambda \subset \lambda^\omega, \quad \lambda \supset \lambda^\omega, \quad \lambda = \lambda^\omega,$$

respectively.
(c) The *Lagrangian Grassmannian* $\mathcal{L}(H, \omega)$ consists of all Lagrangian subspaces of (H, ω).

Definition 2.8. Let H be a complex Hilbert space. A mapping $\omega : H \times H \to \mathbb{C}$ is called a (strong) *symplectic form* on H, if $\omega(x, y) = \langle Jx, y\rangle_H$ for some bounded invertible skew-adjoint operator J. (H, ω) is called a (strong) *symplectic Hilbert space.*

Before giving a rigorous definition of the Maslov index, we fix the terminology and give a simple lemma.

We recall:

Definition 2.9. (a) The space of (algebraic) *Fredholm pairs* of linear subspaces of a vector space H is defined by

$$\mathcal{F}^2_{\text{alg}}(H) := \{(\lambda, \mu) \mid \dim(\lambda \cap \mu) < +\infty \text{ and } \dim\big(H/(\lambda + \mu)\big) < +\infty\} \tag{25}$$

with

$$\operatorname{index}(\lambda, \mu) := \dim(\lambda \cap \mu) - \dim(H/(\lambda + \mu)). \tag{26}$$

(b) In a Banach space H, the space of (topological) *Fredholm pairs* is defined by

$$\mathcal{F}^2(H) := \{(\lambda, \mu) \in \mathcal{F}^2_{\rm alg}(H) \mid \lambda, \mu, \text{ and } \lambda + \mu \subset H \text{ closed}\}. \tag{27}$$

We need the following well-known lemma (see, e.g., [11, Lemma 1.7]).

Lemma 2.10. *Let (H, ω) be a (strong) symplectic Hilbert space. Then*
(1) there is a 1-1 *correspondence between the space*

$$\mathcal{U}^J = \{U \in \mathcal{B}\big(H^+, H^-\big) \mid U^* J|_{H^-} U = -J|_{H^+}\}$$

and $\mathcal{L}(H, \omega)$ under the mapping $U \to L := \mathfrak{G}(U)$ (= graph of U), where $H^\pm = H^\mp(\sqrt{-1}J)$ in the sense of the decomposition (13)*;*
(2) if $U, V \in \mathcal{U}^J$ and $\lambda := \mathfrak{G}(U)$, $\mu := \mathfrak{G}(V)$, then (λ, μ) is a Fredholm pair if and only if $U - V$, or, equivalently, $UV^{-1} - I$ is Fredholm. Moreover, we have a natural isomorphism

$$\ker(UV^{-1} - I) \simeq \lambda \cap \mu\,. \tag{28}$$

Definition 2.11. Let $(H, \langle\cdot,\cdot\rangle_s)$, $s \in [0,1]$ be a continuous family of Hilbert spaces, and $\omega_s(x, y) = \langle J_s x, y\rangle_s$ be a continuous family of symplectic forms on H, i.e., $\{A_{s,0}\}$ and $\{J_s\}$ are two continuous families of bounded invertible operators, where $A_{s,0}$ is defined by

$$\langle x, y\rangle_s = \langle A_{s,0} x, y\rangle_0 \quad \text{for all } x, y \in H.$$

Let $\{(\lambda_s, \mu_s)\}$ be a continuous family of Fredholm pairs of Lagrangian subspaces of $(H, \langle\cdot,\cdot\rangle_s, \omega_s)$. Then there is a continuous splitting

$$H = H_s^-(\sqrt{-1}J_s) \oplus H_s^+(\sqrt{-1}J_s) \tag{29}$$

associated to the self-adjoint operator $\sqrt{-1}J_s \in \mathcal{B}(H, \langle\cdot,\cdot\rangle_s)$ for each $s \in [0,1]$. By Lemma 2.10, $\lambda_s = \mathfrak{G}_s(U_s)$ and $\mu_s = \mathfrak{G}_s(V_s)$ with U_s, $V_s \in \mathcal{U}^{J_s}$, where $\mathfrak{G}_s$ denotes the graph associated to the splitting (29). We define the *Maslov index* $\operatorname{Mas}\{\lambda_s, \mu_s\}$ by

$$\operatorname{Mas}\{\lambda_s, \mu_s\} = -\operatorname{sf}_\ell\{U_s V_s^{-1}\}, \tag{30}$$

where $\ell := (1-\epsilon, 1+\epsilon)$ with, $\epsilon \in (0,1)$ and with upward co-orientation.

Remark 2.12. For finite-dimensional H, constant $\mu_s = \mu_0$, and a loop $\{\lambda_s\}$, i.e., for $\lambda_0 = \lambda_1$, we notice that $\operatorname{Mas}\{\lambda_s, \mu_s\}$ is the winding number of the closed curve $\{\det(U_s^{-1} V_0)\}_{s\in[0,1]}$. This is the original definition of the Maslov index as explained in Arnol'd, [2].

Lemma 2.13. *The Maslov index is independent of the choice of the complete inner product of H.*

Proof. Let $\langle\cdot,\cdot\rangle_{s,k}$, $s\in[0,1]$ with $k=0,1$ be two continuous families of complete inner products of H. We define

$$\langle\cdot,\cdot\rangle_{s,t}=(1-t)\langle\cdot,\cdot\rangle_{s,0}+t\langle\cdot,\cdot\rangle_{s,1}$$

for each $(s,t)\in[0,1]\times[0,1]$. Let (λ_s,μ_s) be a continuous family of Fredholm pairs of Lagrangian subspaces of (H,ω_s). For each inner product $\langle\cdot,\cdot\rangle_{s,t}$, we denote by $U_{s,t}$ and $V_{s,t}$ the associated generated "unitary" operators of λ_s and μ_s respectively. We also denote by Mas_t the Maslov index defined with $\langle\cdot,\cdot\rangle_{s,t}$ for each $t\in[0,1]$. By Proposition 2.3 we have

$$\begin{aligned}\operatorname{Mas}_0\{\lambda_s,\mu_s\}\,&-\,\operatorname{Mas}_1\{\lambda_s,\mu_s\}\\&=-\operatorname{sf}_\ell\{U_{s,0}V_{s,0}^{-1}\}+\operatorname{sf}_\ell\{U_{s,1}V_{s,1}^{-1}\}\\&=-\operatorname{sf}_\ell\{U_{s,t}V_{s,t}^{-1};(s,t)\in\partial\big([0,1]\times[0,1]\big)\}\\&=0.\end{aligned}$$

□

Now we give a method of using the crossing form to calculate Maslov indices (cf. [14], [28], [5, Theorem 2.1]; for a full proof of the following Proposition see [33, Corollary 3.1]).

Let $\lambda=\{\lambda_s\}_{s\in[0,1]}$ be a C^1 curve of Lagrangian subspaces of H. Let W be a fixed Lagrangian complement of λ_t. For $v\in\lambda_t$ and $|s-t|$ small, define $w(s)\in W$ by $v+w(s)\in\lambda_s$. The form

$$Q(\lambda,t):=Q(\lambda,W,t)(u,v)=\frac{d}{ds}|_{s=t}\omega(u,w(s)),\quad\forall u,v\in\lambda_t \tag{31}$$

is independent of the choice of W. Let $\{(\lambda_s,\mu_s)\}$, $0\le s\le1$ be a curve of Fredholm pairs of Lagrangian subspaces of H. For $t\in[0,1]$, the *crossing form* $\Gamma(\lambda,\mu,t)$ is a quadratic form on $\lambda_t\cap\mu_t$ defined by

$$\Gamma(\lambda,\mu,t)(u,v)=Q(\lambda,t)(u,v)-Q(\mu,t)(u,v),\quad\forall u,v\in\lambda_t\cap\mu_t. \tag{32}$$

A *crossing* is a time $t\in[0,1]$ such that $\lambda_t\cap\mu_t\neq\{0\}$. A crossing is called *regular* if $\Gamma(\lambda,\mu,t)$ is nondegenerate. It is called *simple* if it is regular and $\lambda_t\cap\mu_t$ is one-dimensional.

Proposition 2.14. *Let (H,ω) be a symplectic Hilbert space and $\{(\lambda_s,\mu_s)\}$, $0\le s\le1$ be a C^1 curve of Fredholm pairs of Lagrangian subspaces of H with only regular crossings. Then we have*

$$\operatorname{Mas}\{\lambda,\mu\}=m^+(\Gamma(\lambda,\mu,0))-m^-(\Gamma(\lambda,\mu,1))+\sum_{0<t<1}\operatorname{sign}(\Gamma(\lambda,\mu,t)). \tag{33}$$

3 Symplectic analysis of symmetric operators

3.1 Local stability of weak inner UCP

Let X be a complex Hilbert space and $A\in\mathcal{C}(X)$ a linear, closed, densely defined operator in X. We assume that A is symmetric, i.e., $A^*\supset A$ where A^* denotes the adjoint operator.

We denote the domains of A by D_m (the *minimal* domain) and of A^* by $D_{\max}$ (the *maximal* domain).

Definition 3.1. Let X be a Hilbert space and $A \in \mathcal{C}(X)$ with $\operatorname{dom} A = D_m$ and $A^* \supset A$. We shall say that the operator A satisfies the *weak inner Unique Continuation Property (UCP)* if $\ker A = \{0\}$.

It is well known that weak UCP and weak inner UCP can be established for a large class of Dirac type operators, see the first author with Wojciechowski [10, Chapter 8], and the first author with M. Marcolli and B.-L. Wang [9]. However, it is not valid for all linear elliptic differential operators of first order as shown by one of the Pliś counterexamples [26]. Moreover, one has various quite elementary examples of linear and non-linear perturbations which *invalidate* weak inner UCP for Dirac operators. Two such examples are listed in [9]. In the same paper, however, it was shown that weak UCP is *preserved* under certain 'small' perturbations of Dirac type operators. Here we show an elementary result, namely the local stability of weak inner UCP.

Lemma 3.2. *Let X be a Hilbert space. Let $A_s \in \mathcal{C}(X)$, $0 \le s \le 1$ be a family of symmetric operators with* $\operatorname{dom} A_s = D_m$ *and* $\operatorname{dom} A_s^* = D_{\max}$ *independent of s. Assume that $\{A_s^* : D_{\max} \to X\}$ is a continuous curve of bounded operators, where the norm on $D_{\max}$ is the graph norm induced by A_0^*. If A_0 satisfies weak inner UCP and there exists a self-adjoint Fredholm extension $A_0^*|_D$ of A_0, then for all $s \ll 1$ the operators A_s^* are surjective and the operators A_s satisfy weak inner UCP.*

Proof. By our assumptions, $\operatorname{im} A_0^*|_D$ is closed and is of finite codimension. Since $\operatorname{im} A_0^*|_D \subset \operatorname{im} A_0^* \subset X$, the full range $\operatorname{im} A_0^*$ is closed. Since A_0 satisfies weak inner UCP, $\operatorname{im} A_0^* = X$. Then A_0^* is semi-Fredholm. By Theorem IV.5.17 of Kato [17] we have $\operatorname{im} A_s^* = X$ for $s \ll 1$. Since A_s are symmetric, A_s satisfy weak inner UCP for $s \ll 1$. □

3.2 Continuity of the family $\{A_{s,D_s}\}$

Let X be a complex Hilbert space, and $M, N \subset X$ be two closed linear subspaces. Let P_M, P_N be the orthogonal projections onto M, N respectively. Then the distance $d(M,N)$ is defined by $d(M,N) = \|P_M - P_N\|$ and called the *gap* between M and N. For any two closed operators A, B on X, we define $d(A,B)$ as the distance between their graphs.

Let $A \in \mathcal{C}(X)$ be a linear, closed, densely defined operator in X. By Footnote 1 (page 198), Theorems IV.1.1 and IV.2.14 in [17], it is easy to verify the following

Lemma 3.3. *Let $B \in \mathcal{B}(\operatorname{dom}(A), X)$ be a bounded operator, where the norm on* $\operatorname{dom}(A)$ *is the graph norm $\mathcal{G}_A$ induced by A. Let $d := \|B - A\|_{\mathcal{G}_A} < \frac{1}{2}$. Then we have*
(1) $B \in \mathcal{C}(X)$, and it holds that

$$(1-2d)\langle x, x\rangle_{\mathcal{G}_A} \le \langle x, x\rangle_{\mathcal{G}_B} \le (1+d)^2 \langle x, x\rangle_{\mathcal{G}_A} \quad \text{for } x \in D.$$

(2) $d(B, A) \leq \frac{\sqrt{2}d}{(1-d)^{-1}}$.

Lemma 3.4. *Let X be a Hilbert space, and Y be a closed linear subspace of H. Then there exists a bijection between the space of closed linear subspaces of X containing Y and that of closed linear subspaces of X/Y which preserves the metric.*

Proof. We view X/Y as $Y^\perp$. Let $M, N \subset Y^\perp$ be two closed subspaces and P_M, P_N be the orthogonal projections onto M, N respectively. Then we have

$$d(M + Y, N + Y) = \|P_{M+Y} - P_{N+Y}\| = \|P_M - P_N\| = d(M, N).$$

□

From the definition of the gap norm and by some computations we have

Lemma 3.5. *Let $D_m \subset D_{\max} \subset X$ be three Hilbert spaces such that D_m is a closed subspace of $D_{\max}$ and a dense subspace of X. Let $\{A_s \in \mathcal{C}(X)\}_{s\in[0,1]}$ be a family of densely defined symmetric operators with domain D_m, and $\{D_s\}_{s\in[0,1]}$ be a family of closed subspaces of $D_{\max}$ containing D_m. We assume that* $\operatorname{dom}(A_s^*) = D_{\max}$*, each graph norm $\mathcal{G}_s$ of $D_{\max}$ induced by A_s^* is equivalent to the original norm $\mathcal{G}$ of $D_{\max}$, and $\{A_s^* \in \mathcal{B}(D_{\max}, X)\}$, $\{D_s/D_m \subset D_{\max}/D_m\}$ are two continuous families. Then $\{A_{s,D_s} \in \mathcal{C}(X)\}_{s\in[0,1]}$ is a continuous family of closed operators.*

3.3 Continuity of natural Cauchy data spaces

In this subsection we generalize the proof of the continuity of Cauchy data spaces given in [5, Section 3.3]. We need the following

Proposition 3.6 (Proposition 3.5 of [5]). *Let X be a Hilbert space, and $A \in \mathcal{C}(X)$ be a symmetric operator. Set $D_m =$* $\operatorname{dom}(A)$ *and $D_{\max} =$* $\operatorname{dom}(A^*)$*. If A admits a self-adjoint Fredholm extension with domain D, then the quotient space D/D_m and the natural Cauchy data space* $(\ker A^* + D_m)/D_m$ *form a Fredholm pair of Lagrangian subspaces of the* (*strong*) *symplectic Hilbert space $D_{\max}/D_m$* (*introduced above in Subsection* 1.3*, Item* (*ii*)).

Remark 3.7. From the arguments in Ralston [27] one can deduce (see [8]) that all linear formally self-adjoint elliptic differential operators over a compact smooth Riemannian manifold with smooth boundary admit a self-adjoint Fredholm extension.

Now we can prove

Proposition 3.8. *Let X be a Hilbert space, and $D_m \subset D_{\max}$ be two dense linear subspaces of X. Let $\{A_s : D_m \to X\}_{s\in[0,1]}$ be a family of closed symmetric densely defined operators in X. We assume that*

(1) each A_s admits a self-adjoint Fredholm extension with domain D_s ;
(2) $\mathrm{dom}(A_s^*) = D_{\max}$ *is independent of s and that all graph norms $\mathcal{G}_s$ of $D_{\max}$ induced by A_s^* are mutually equivalent;*
(3) each A_s satisfies weak inner UCP relative to D_m ; and
(4) $\{A_s^* : D_{\max} \to X\}$ forms a continuous family of bounded operators, where the norm on $D_{\max}$ is the graph norm $\mathcal{G}$ induced by A_0.

Then the natural Cauchy data spaces $\big(D_m + \ker A_s^*\big)/D_m$ are continuously varying in $D_{\max}/D_m$.

Proof. We denote the projection of $D_{\max}$ onto $D_{\max}/D_m$ by γ . Note that $\ker A_s^*$ is closed in $D_{\max}$.

To prove the continuity, we need only to consider the local situation at $s = 0$. First we show that $\{\ker A_s^*\}_{s\in[0,1]}$ is a continuous family of subspaces of $D_{\max}$; then we show that $\gamma(\ker A_s^*)$ is a continuous family in $D_{\max}/D_m$.

We consider the bounded operator

$$\begin{array}{rcl} F_s : D_{\max} & \longrightarrow & X \oplus \ker A_0^* \\ x & \mapsto & (A_s^* x, P_0 x) \end{array} ,$$

where $P_0 : D_{\max} \to \ker A_0^*$ denotes the orthogonal projection of the Hilbert space $D_{\max}$ onto the closed subspace $\ker A_0^*$. By definition, the family $\{F_s\}$ is a continuous family of bounded operators.

Clearly, F_0 is injective. Since $\mathrm{im}\, A_0^*|_{D_0} \subset \mathrm{im}\, A_0^* \subset X$ and $A_0^*|_{D_0}$ is Fredholm, $\mathrm{im}\, A_0^*$ is closed. From weak inner UCP we get $\mathrm{im}\, A_0^* = X$. So the operator F_0 is also surjective. This proves that F_0 is invertible with bounded inverse. Then all operators F_s are invertible for small $s \geq 0$, since F_s is a continuous family of operators.

Note that

$$F_s(\ker A_s^*) \subset \{0\} \oplus \ker A_0^*, \quad (F_s)^{-1}(\{0\} \oplus \ker A_0^*) \subset \ker A_s^*.$$

Since F_s are invertible for small $s \geq 0$, we have

$$F_s(\ker A_s^*) = \{0\} \oplus \ker A_0^*. \tag{34}$$

We define

$$\varphi_s := F_s^{-1} \circ F_0 : D_{\max} \ \cong \ D_{\max} \text{ and } \varphi_s^{-1} = F_0^{-1} \circ F_s : D_{\max} \ \cong \ D_{\max}$$

for s small. Since F_s are invertible for small $s \geq 0$, from (34) we obtain that

$$\varphi_s(\ker A_0^*) = \ker A_s^* . \tag{35}$$

From (35) we get that

$$\{P_s := \varphi_s P_0 \varphi_s^{-1} : D_{\max} \longrightarrow \ker A_s^*\}$$

is a continuous family of projections onto the solution spaces $\ker A_s^*$. The projections are not necessarily orthogonal, but can be orthogonalized and remain continuous in s like in [10, Lemma 12.8]. This proves the continuity of the family $\{\ker A_s^*\}$ in $D_{\max}$.

Now we must show that $\{\gamma(\ker A_s^*)\}$ is a continuous family in the quotient space $D_{\max}/D_m$. This is not proved by the formula $\gamma(\ker A_s^*) = \gamma(\varphi_s(\ker A_0^*))$ alone. We must modify the endomorphism φ_s of $D_{\max}$ in such a way that it keeps the subspace D_m invariant.

By Proposition (3.6), the Cauchy data space $\gamma(D_m + \ker A_0^*)$ is closed in $D_{\max}/D_m$. So $D_m + \ker A_0^*$ is closed in $D_{\max}$. We define a continuous family of mappings by

$$\begin{array}{rcccl} \psi_s : D_{\max} = D_m + \ker A_0^* & + & (D_m + \ker A_0^*)^\perp & \longrightarrow & D_{\max} \\ x + y & + & z & \mapsto & x + \varphi_s(y) + z \end{array}$$

with $\psi_0 = \mathrm{id}$. Hence all ψ_s are invertible for $s \ll 1$, and $\psi_s(D_m) = D_m$ for such small s. Hence we obtain a continuous family of mappings $\{\widetilde{\psi}_s : D_{\max}/D_m \to D_{\max}/D_m\}$ with $\widetilde{\psi}_s(\gamma(\ker A_0^*)) = \gamma(\ker A_s^*)$. From that we obtain a continuous family of projections as above. □

Remark 3.9. From the preceding arguments it also follows that the Cauchy data spaces form a differentiable family, if $\{A_s^*\}$ is a differentiable family.

3.4 Proof of the spectral flow formula

We begin with a simple case.

Lemma 3.10. *Let X be a Hilbert space, and $A \in \mathcal{C}(X)$ be a symmetric operator with $\mathrm{dom}(A) = D_m$ and $\mathrm{dom}(A^*) = D_{\max}$. Let $A_D := A^*|_D$ be a self-adjoint Fredholm extension of A. We assume that A satisfies weak inner UCP. Then there exists an $\epsilon > 0$ such that $A_D + aI$ is Fredholm and satisfies weak inner UCP for each $a \in [0,\epsilon]$. Let $\gamma : D_{\max} \to D_{\max}/D_m$ denote the natural projection. Then we have*

$$\mathrm{sf}\{A_D + aI; a \in [0,\epsilon]\} = -\operatorname{Mas}\{\gamma(D), \gamma(\ker(A^* + aI)); a \in [0,\epsilon]\}.$$

Proof. By the definition of the spectral flow we have

$$\mathrm{sf}\{A_D + aI; a \in [0,\epsilon]\} = \sum_{a \in (0,\epsilon]} \dim\ker(A_D + aI). \qquad (36)$$

Let ω denote the Green form on $D_{\max}$ induced by A^*. Let $W \in \mathcal{L}(D_{\max}/D_m)$ be a Lagrangian complement of $\gamma(\ker(A^* + a_0 I))$. By Proposition 3.8, $\gamma(\ker(A^* + aI))$ and $\ker(A^* + aI)$ are two differentiable families. For each $y(a_0) \in \ker(A_D + a_0 I)$, there exists a continuous family $w(a) \in W + D_m$, $|a - a_0|$ small, such that $w(a_0) = 0$ and $y(a) := y(a_0) + w(a) \in \ker(A^* + aI)$. Since $A^*(y(a)) = -ay(a)$ and the family $\{y(a)\}$ is continuous in $D_{\max}$, the family $\{y(a)\}$ is also continuous in X. For all $x(a_0) \in \ker(A_D + a_0 I)$, we have

$$\begin{aligned}\omega(\gamma(x(a_0)), \gamma(w(a)) &= \langle A^*(x(a_0)), y(a) - y(a_0)\rangle - \langle x(a_0), A^*(w(a))\rangle \\ &= \langle -a_0 x(a_0), y(a) - y(a_0)\rangle - \langle x(a_0), A^*(y(a)) - A^*(y(a_0))\rangle \\ &= \langle -a_0 x(a_0), y(a) - y(a_0)\rangle - \langle x_(a_0), -ay(a) + a_0 y(a_0)\rangle \\ &= (a - a_0)\langle x(a_0), y(a)\rangle\end{aligned}$$

Let the crossing forms Q and Γ be defined by (31) and (32) respectively. Then we have $Q(\gamma(\ker(A^* + aI)), a_0)(\gamma(x(a_0)), \gamma(y(a_0)) = \langle x(a_0), y(a_0)\rangle$ and

$$\Gamma(\gamma(D), \gamma(\ker(A^* + aI)), a_0)(\gamma(x(a_0)), \gamma(y(a_0)) = -\langle x(a_0), y(a_0)\rangle.$$

By Proposition 2.14 we have

$$\mathrm{Mas}\{\gamma(D), \gamma(\ker(A^* + aI)); a \in [0, \epsilon]\} = -\sum_{a\in(0,\epsilon]} \dim\ker\big(A_D + aI\big). \tag{37}$$

Combine equations (36), (37), and our lemma follows. □

Now our main result follows at once.

Proof of Theorem 1.5. By Lemma 3.2, for each s_0 there exists an $\epsilon(s_0) > 0$ such that the operators $A_s + aI$ satisfy weak inner UCP for all s, a with $|s - s_0|, |a| < \epsilon(s_0)$. Here we use the continuity of the family $\{A_s^*\}$ as bounded operators from $D_{\max}$ to X. Since $[0, 1]$ is compact and A_{s,D_s} are Fredholm operators for all $s \in [0, 1]$, there exists an $\epsilon > 0$ such that the operators $A_s + aI$ satisfy weak inner UCP and $A_{s,D_s} + aI$ are Fredholm operators for all $s \in [0, 1]$ and $|a| < \epsilon$.

We only need to prove the formula (12) in a small interval $[s_0, s_1]$. We consider the two-parameter families

$$\{A_{s,D_s} + aI\} \text{ and } \{\gamma(D_s), \gamma(\ker A_s^* + aI)\}$$

for $s \in [s_0, s_1]$ and $a \in [0, \epsilon]$. Because of the homotopy invariance of spectral flow and Maslov index, both integers must vanish for the boundary loop going counter clockwise around the rectangular domain from the corner point $(s_0, 0)$ via the corner points $(s_1, 0)$, (s_1, ϵ), and (s_0, ϵ) back to $(s_0, 0)$.

Moreover, for s_1 sufficiently close to s_0 we can choose ϵ sufficiently small so that $\ker(A_{s,D_s} + \epsilon I) = \{0\}$ for all $s \in [s_0, s_1]$. Hence, spectral flow and Maslov index must vanish on the top segment of our box.

Finally, by the preceding lemma, the left and the right side segments of our curves yield vanishing sum of spectral flow and Maslov index. So, by additivity under catenation, our assertion follows. □

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