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RISE, FALL AND RESURRECTION OF INFINITESIMALS

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RISE, FALL AND RESURRECTION OF INFINITESIMALS

af

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Abstract

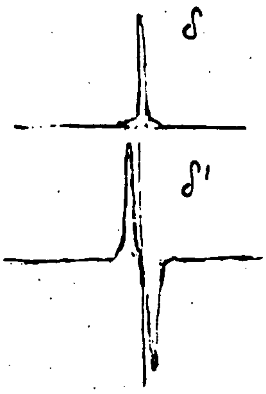
En historisk introduktion til ikke-standardanalyse.

Det vises, hvordan Eulers oprindelige analytiske beviser, som for en moderne matematiker virker uigennemskuelige og logisk tvivlsomme, både er elegante og logisk strigente, når de formuleres ved hjælp af uendelig store tal og infinitesimaler. Endvidere vises det, at distributionsteori og teorien for divergente rækker bliver simple og elegante, når de udvikles i ikke-standardanalyse.

D. Laugwitz

Rise, Fall and Resurrection of Infinitesimals[1]

Let me begin with a few personal reminiscences. When, almost three decades ago, I gave my first talks on the new theory of infinitesimals and infinitely large numbers invented by C. Schmieden, two remarks from the participants impressed me.
The first one came from physicists who stated that nothing at all was new to them: They claimed to have always calculated in that way. And, another one made by a mathematician: The (so called) installation of rigour in mathematics succeeded in sweeping infinitesimals under the table - and now you come and let these little insects creep up again through wormholes to the top of the table!
Actually, these worms always had, in the shape of differentials, lived as disguised outcasts of the community of mathematical subjects not only in physics but also in differential geometry. Having worked in differential geometry at the time when Schmieden's ideas came into my sight I could happily hail his infinitesimals. And, having had some training in 20th century physics, I even hailed his infinitely large numbers as a tool to give an intuitive meaning to such things like Dirac's delta function and dipoles. A delta function has infinitesimal values outside some infinitesimal region, and becomes infinitely large in some region inside, such that its integral has a certain finite value, say 1. If we accept the existence of some infinitely large number Ω and admit all rational operations for it, then



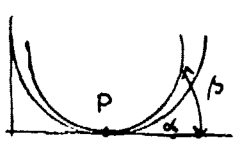
$$\delta(x) = \frac{1}{\pi}\,\frac{\Omega}{1+\Omega^2x^2}$$

might describe a delta function, and its derivative looks like a dipole.

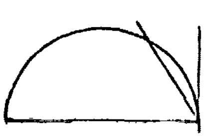
Other reminiscences go back to my childhood. I suppose I was a boy of about 12 when our mathematics teacher asked what a circle and its tangent have in common. My spontaneous answer was: Nothing. A point was "nothing" to me. If they had anything in common, this something must have some extension, very small, practically invisible of course. My teacher, whom I remember quite well, did not subdue this curious aberration of my mind. Moreover, at about the same time he even encouraged another glimpse of the infinitesimal which must have occured to everybody who does not simply believe in textbooks: Is $0.\bar{9}\ldots$ really equal to 1? Shouldn't it be less than 1? Actually, the usual "proofs" at that level are far from convincing, and objections like those ascribed to Zeno of Elea should come to the mind of everyone who is not infected by the 19th century dogma of the real numbers.

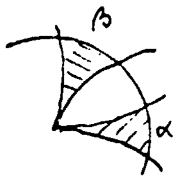
Indeed it is a historical fact that the first appearance of infinitesimals was in pre-Euclidean times, and I shall dwell for a few minutes on that mathematics of the "Stone Age" before entering the consideration of the well known rise of the infinitesimals at Leibniz' time and the summit of their success during the lifetime of Euler in the 18th century.

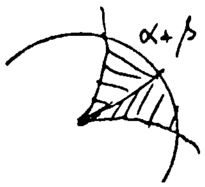
Infinitesimals in Greek mathematics entered, no one will be surprised, through geometry, namely in the shape of contact angles or hornlike angles. An angle is the space between two curves in a plane which meet at a point P, and in case these curves touch each other at P they enclose a contact angle. It is natural to call an angle α smaller than the angle β at P, if β embraces α in a sufficiently small neighbourhood of P. Then the angle between a circle and its tangent at P will be smaller than any straight angle with the tangent as one of its legs, and the angle of the half circle will be smaller than the right angle. (Euclid III,16)



Addition of angles is defined in the natural way, by adding the arcs which are cut out by the legs on circles of respective radii. It follows that $n\alpha < \beta$ for a finite straight angle β and a contact or horn angle α and any finite natural number n.







The property of Eudoxos-Archimedes is not valid for angles. There are not many traces of horn angles in Euclid, and he successfully eliminates these contact angles which have been of no use in geometry since then. The first half of Euclid III, 16 says that no straight line can be drawn in the space between a circular arc and its tangent. Contact angles are avoided. I cite the translation of Sir Thomas L. Heath[2] of Euclid Book III, Proposition 16: The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.

Nevertheless, the horn-like angles played some role in philosophical discussions on the foundations of mathematics through to the 17th century. Wallis emphasized that the contact angles are not magnitudes or quantities, whereas Leibniz stressed their character as examples of infinitesimal quantities in mathematics[3].

The decline of infinitesimals took place well before the times of Eudoxos and Euclid. It was closely connected to the discoveries of irrationality and of difficulties with the concept of continuity. To cut a long story short and put it in modern language: The intermediate value theorem for continuous functions is violated over nonarchimedean fields. Consider the characteristic function of the infinitesimals, $f(x) = 1$ if x is infinitesimal or zero, and $f(x) = 0$ everywhere else. You may define continuity in any way you like, this function will be continuous at every x_0 of the nonarchimedean field. Take the ε-δ-property: To each $\varepsilon > 0$ there exists a $\delta > 0$, possibly infinitesimal, such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x-x_0| < \delta$. Or take Cauchy's definition of continuity: Whenever

$x-x_0$ is infinitesimal, then $f(x)-f(x_0)$ is again infinitesimal. Anyway, our function gives a counterexample to the intermediate value theorem. This is a very serious objection against any kind of analysis using infinitesimals, and this very objection has been repeated over and over again until recent times. An equivalent formulation is: There are bounded sets, e. g. the set of infinitesimals, which have no least upper bound or no greatest lower bound. The nonarchimedean fields lack a property which in the sense of Dedekind is a characteristic of the continuum. No wonder that Leibniz who initiated the most successful period of the use of infinitesimals felt that the continuum was a labyrinth!

The followers of Aristotle deny that the continuum is a set of points. In this sense no ordered field F, archimedean or nonarchimedean, can be a biunique image of the continuum, or, there are always more points in the continuum than any given set. The points whose coordinates are members of F do not exhaust the linear continuum. Now mathematicians of the 20th century seem to be unable to deal with anything but sets, like modern musicians are not able to play or even compose in the style of the old Greeks. Later in this talk I shall show you how to circumvent the difficulties of the continuum to arrive at safe grounds for infinitesimal mathematics.

The Greeks avoided the difficulties of the infinitesimal (and to some extent, the infinite) by the trick of Eudoxos. The existence of infinitesimals, e.g. in the shape of horn angles, could not be denied - such a denial was left to a prominent mathematician of the late 19th century who seriously announced that he could prove the impossibility of infinitesimals; his name was Georg Cantor.[4]

If infinitesimals were unnecessary for the first four books of Euclid, they should be eliminated from the conceptual basis of mathematics.
That was done by Eudoxos at the beginning of Book V.

Following Heath, the essential definitions in Euclid's Book V read:

Definition 4. Magnitudes are said to have a ratio to one another which are capable, when multiplied, to exceed one another.

Note that this definition does not say that entities which do not enjoy this property are excluded from the realm of magnitudes!

Definition 5. Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, of any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

In modern terminology, having the same ratio is an equivalence relation in the set of ordered pairs of magnitudes. Two pairs (a,b) and (a',b') are equivalent if both a-a' and b-b' are infinitesimal. The concept of an infinitesimal is successfully eliminated if only ratios of magnitudes and not the magnitudes *per se* are considered.

The ground for the rise of infinitesimals in the 17th century had been prepared by many mathematicians including Cavalieri and Pascal. In his first considerations Newton used infinitesimals which he rejected later in favour of limiting procedures. Let me concentrate our attention to the work and ideas of Leibniz. His calculus of infinitesimals in the shape of differentials is well known: All of the rational operations can be applied to differentials dx, dy, and in the final result of a calculation infinitesimal terms of a sum can be dropped when added to a finite magnitude. If $y = x^2$ then $dy = (x+dx)^2-x^2 = 2x \cdot dx + dx^2$ or $\frac{dy}{dx} = 2x + dx = 2x$. Here the equality sign is open to criticism which was certainly one of the roots of the decline of infinitesimal mathematics: How can dx which is a denominator on the left hand side of the equation be eventually be zero on the right hand side? The bishop Berkeley quite convincingly made this vanishing of something which was assumed to be different from zero an object of his mockery;

though directed against Newton's fluxions his criticism also applies to Leibniz and his followers, saying that "by virtue of a twofold mistake you arrive, though not at a science, yet at the truth", because errors were compensating for each other. "In every other science men prove their conclusions by their principles, and not their principles by their conclusions." This is a heavy attack against the more or less pragmatical use of infinitesimals to establish results the truth of which can be ascertained by correct methods like that of the Ancients, notably Archimedes. "He who can digest a second or third fluxion... need not, methinks, be squeamish about any point in Divinity." And, since the derivative regarded as the ratio of the evanescent increments dy and dx, what are these rates of change? They are nothing but "the ghosts of departed quantities" [5].

It is an easy task for 20th century mathematicians to find a way out of the dilemma: The equality sign is used in two different meanings. If we write $A \approx B$ when $A - B$ is infinitesimal, then $\frac{dy}{dx} = \frac{(x+dx)^2-x^2}{dx} = 2x + dx \approx 2x$. If A is a real number and B is an infinitesimal, then $A \pm B \approx A$, and we invent a new name. A is called the standard part of $A \pm B$. In our modern terminology the mapping $A \pm B \to A$ is a homomorphism of the ring of finite (plus infinitesimal) numbers onto the field of real numbers. The sign $\approx$ indicates an equivalence relation. The use of equivalence relations in mathematics became generally accepted only after the work of Frege [6] (1884). This concept, as used today, is definitely a child of set theory. The lack of this concept, or to put it in other words, the lack of different notations for equality (=) and equality up to an infinitesimal ($\approx$) was certainly one of the reasons for the decline of infinitesimal mathematics starting in the second half of the 18th century.

The Bernoulli brothers in Basle were the first to accept, develop, and propagate the new Calculus, the summit of which was certainly attained in the work of Leonhard Euler (1707-1783), a pupil of John Bernoulli.

It is not easy to find a proper foundation of the infinitesimal concept in Leibniz' writings. In many cases he gives popular explanations which are far from being lucid. It is only in letters to mathematicians when he expresses himself clearly as to our way of thinking. There is a famous letter of February 2, 1702 to Varignon in which Leibniz indicates how the Calculus of infinitesimals and infinities should be considered as a special case of his general Principle of Continuity: The laws of the very large finite should remain valid for the infinite, and the laws of the very small finite hold for the infinitely small.

In a modern interpretation we shall use this principle of Leibniz as a foundation of our version of Nonstandard Analysis.

Euler was very reluctant to explain his basic concepts. I claim that his actual use of infinitely small and infinitely large numbers obtains a correct sense if looked at in the light of Leibniz' principle.

Let me give a few examples from Euler's work. In his Introduction to the analysis of the infinite, published in 1748, he develops a theory of the elementary functions starting from his expression for the exponential function,

$$e(x) = \left(1 + \frac{x}{\Omega}\right)^{\Omega}$$

where Ω is a fixed infinitely large number (for which Euler at that time still writes i, which was later, even by himself, the generally adopted notation for $\sqrt{-1}$. Thus I prefer Ω.) As a consequence of the general principle we may write

$$(1)\qquad \begin{aligned}\left(1 + \frac{x}{\Omega}\right)^{\Omega} &= 1 + \Omega\cdot\frac{x}{\Omega} + \frac{\Omega(\Omega-1)}{2!}\cdot\frac{x^2}{\Omega^2} + \frac{\Omega(\Omega-1)(\Omega-2)}{3!}\cdot\frac{x^3}{\Omega^3} + \\ &\quad + \ldots + \frac{x^{\Omega}}{\Omega^{\Omega}} \\ &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{\Omega}}{\Omega!} \approx \sum_{k=o}^{\infty} \frac{x^k}{k!} \;.\end{aligned}$$

Are we really entitled to draw all of these conclusions to which Euler jumps, always writing = instead of our $\approx$, by simply remarking that

$$\frac{\Omega(\Omega-1)}{\Omega^2} = (1-\frac{1}{\Omega}) \approx 1, \quad \frac{\Omega(\Omega-1)(\Omega-2)}{\Omega^3} = (1-\frac{1}{\Omega})(1-\frac{2}{\Omega}) \approx 1,$$

etc?

The first equality is clear from the principle, and we can easily follow him with

$$\binom{\Omega}{k} \cdot \frac{1}{\Omega^k} = \frac{\Omega}{\Omega} \cdot \frac{\Omega-1}{\Omega} \cdot \frac{\Omega-2}{\Omega} \cdot \ldots \cdot \frac{\Omega-(k-1)}{\Omega} \cdot \frac{1}{k!}$$

(2)

$$= (1-\frac{1}{\Omega})(1-\frac{2}{\Omega}) \ldots (1-\frac{k-1}{\Omega}) \cdot \frac{1}{k!} \approx \frac{1}{k!}$$

as long as k is a finite natural number. For, an expression S is an infinitesimal if for each finite $\varepsilon > 0$ we have $|S| < \varepsilon$. Let S be a sum of finitely many terms, $S = S_0 + \ldots + S_N$, each of which is infinitesimal, i.e. $|S_j| < \frac{\varepsilon}{N+1}$ for each finite $\varepsilon > 0$, then $|S| < \varepsilon$. We may conclude that, for each finite N

$$(3) \quad \sum_{k=0}^{N} \binom{\Omega}{k} \frac{x^k}{\Omega^k} - \sum_{k=0}^{N} \left(\frac{x^k}{k!}\right) = \sum_{k=0}^{N} \left[\binom{\Omega}{k} \cdot \frac{1}{\Omega^k} - \frac{1}{k!}\right] x^k \approx 0.$$

But does this hold for the infinitely large Ω in place of N? Or, more generally, is it true that

$$\sum_{j=0}^{N} a_k \approx \sum_{j=0}^{N} b_k$$

for finite N implies that the same holds for all infinitely large N? Certainly not; let $a_k = \frac{1}{\Omega}$, $b_k = 0$ and $N = \Omega - 1$. Then the first sum is 1 (this is an expression for the definite integral $\int_0^1 dx$) and the second one is 0 and not infinitely close to 1. But in this particular case $\sum a_k$ is not convergent, and it can easily be seen that our implication is correct if both series converge. And this is true in Euler's deduction under consideration.

But have not writers, even in famous books on the history of mathematics, invariably told us that Euler knew little or nothing about convergence? Actually, this is a fairy tale which has nothing in common with truth even if repeated dozens of times. The concept of convergence was a triviality to Euler which he did not care to mention. As early as in 1734, in a paper on harmonic series,[7] he even gave a necessary and sufficient condition for convergence: $\sum a_k$ is convergent if and only if $\sum_{k\geq N} a_k$ is infinitesimal for all infinitely large N. Unfortunately Euler did not mention that in his famous textbooks, leaving his method of infinitesimals open to attacks. Here we see one reason for the beginning mistrust in and decline of infinitesimals.

Let me give you a few more examples of Euler's method.[8] If $y = (1 + \frac{x}{\Omega})^\Omega$ represents e^x, then x should represent log y. Now we obtain $x = \Omega(y^{1/\Omega}-1)$, and since the Ω-th root of y will have Ω values it follows that there are infinitely many values of the logarithm which can be calculated, thus ending a long standing controverse between Leibniz und John Bernoulli; everything will be clear if we consider, for the sake of briefness, $y = 1$, or $y^{1/\Omega} = \cos \frac{2\pi m}{\Omega} + i.\sin \frac{2\pi m}{\Omega}$:

$$\log 1 \approx x = \Omega[1- \frac{(2\pi m)^2}{\Omega^2 2!} + \frac{F}{\Omega^4} + i \frac{2\pi m}{\Omega} - \frac{G}{\Omega^3} - 1]$$

where F and G are finite. We conclude that for every finite m

$$\log 1 \approx x \approx 2\pi m\cdot i,$$

and by taking "standard parts" we obtain $\log 1 = 2m\pi\cdot i$.

This is a beautiful direct approach to the problem:
The logarithm is obtained by solving an algebraic equation of degree Ω. There are no tiresome detours through the complex plane! I should mention that Euler deduces his famous formula $e^{iz} = \cos z + i\cdot\sin z$ which was used here again in an algebraical way, as well as the series for cos and sin.

As another application of his method of polynomials of infinite degree I shall repeat Euler's deduction of the logarithmic series. Let $y = 1 + h$ and take the binomial theorem to obtain

$$\log(1+h) \approx \Omega[(1+h)^{1/\Omega} - 1] = \Omega \cdot \sum_{k=1}^{\Omega} \binom{1/\Omega}{k} h^k$$

$$= h + \Omega \frac{1/\Omega(1/\Omega-1)}{2!} h^2 + \Omega \frac{(1/\Omega-1)(1/\Omega-2)}{3!} h^3 + \ldots$$

$$= h - \frac{1-\omega}{2} h^2 + \frac{(1-\omega)(1-\omega/2)}{3} h^3 - \frac{(1-\omega)(1-\omega/2)(1-\omega/3)}{4} h^4 + - \ldots$$

where $\omega = \frac{1}{\Omega}$.

Now this series is convergent for $|h| < 1$, having the majorizing series $\sum |h|^i$, and the same applies to the series obtained by dropping the infinitesimal ω's, $h - \frac{h^2}{2} + \frac{h^3}{3} - + \ldots$. Since finite partial sums of both series are infinitely close to each other we may apply our earlier reasoning to obtain the well known series development of the logarithmic function.

These examples may suffice to show you that Euler's use of infinitesimals goes far beyond the technique of Leibniz who needed only dx and its powers dx^2, dx^3, etc. With Euler, any explicit expression containing infinitely large or infinitesimal numbers makes sense if it does for ordinary numbers in place of them.

Moreover, the order properties of numbers follow from Leibniz' principle though Euler does not mention it.

People have tried to translate Euler's proofs into the language of limits. This is always possible, but almost any proof will lose its flavour and elegance. In most cases you are easily mislead and find a wrong translation. But do not blame Euler for your own clumsiness! As an example let us consider the exponential series (1). Replace Ω by the variable n for a natural number, aiming at $n \to \infty$. Equation (2) and (3) will be translated into

$$\lim_{n \to \infty} \left[\sum_{k=0}^{N} \binom{n}{k} \frac{x^k}{n^k} - \sum_{k=0}^{N} \frac{x^k}{k!} \right] = 0$$

and, when we let $N \to \infty$ we finally have the desired result

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad .$$

Of course, I made a mistake typical for beginners, by letting first $n \to \infty$ and then $N \to \infty$. But Euler is not responsible for my mistakes.

I suppose that many readers interpreted Euler's proof in the sense of limits, and when applying similar ways of reasoning in other cases and getting false results, denounced the infinitesimal method as unreliable. Actually the method was not foolproof. It was a powerful tool in the hands of the master but useless and even dangerous when taken up by minor or unexperienced novices.

Thus, when more and more people entered a study of higher mathematics, and the average level of their I. Q. presumably went down, the trust in the methodus inveniendi of infinitesimal mathematics was bound to decline. There were, up to the 1850's, some more or less isolated ingenious uses of the method, notably by Cauchy, and the language of infinitesimals was still alive, e. g. with Riemann, and generally in physics and differential geometry. But it was felt to lack rigour and was dislodged by the apparently rigourous method of limits. Still, it remains an interesting fact that Cauchy actually felt free to use infinitesimal reasoning, besides being one of the promoters of the ε-δ-techniques. At least in the work of such prominent mathematicians there was not an instantaneous fall of infinitesimals but rather a decline.

In the second half of the 19th century we conceive a decisive change in the attitude to the fundamental concepts of analysis culminating in the dogma that everything in analysis had to be founded on set theory. A function, which was an "analytical expression" in Euler's mathematics, became a subset of the Cartesian product of two sets. Euler's concept was intentional and open, the form of the "analytical expression" could vary with the concrete problems which were under consideration. The new concept is extensional and closed, there "is" a fixed and invariable set of functions once the ranges of definition and of values are given.

There were some attempts to revive infinitesimals around 1900, in connection with nonarchimedean <u>geometry</u> considered by Veronese and others. The discussions in the volumes of Jahresbericht der DMV of that time are most instructive to read, and they explain why a resurrection of infinitesimal analysis was bound to fail. I feel that necessary conditions for such a revival would have been:

a) The times of intentional definitions are over. Use sets if you want to be listened to by your contemporaries!

b) Find a suitable concept to replace Euler's "analytical expression", and give a precise meaning to Leibniz' Principle in terms of 20th century mathematics!

c) Prove important lemmas like the intermediate value theorem and the l.u.b. and g.l.b. properties for classes of functions and sets which actually occur in analysis! That is, find a good restriction of the concepts of set and function which cover the classical ground and are wide enough to include newer applications like distributions.

We know by now several approaches to Nonstandard Analysis, a word which was coined by A. Robinson in his first paper on the subject matter in 1961[9] (Incidentally, I do not like this name; our intention is to be closer to the Leibniz-Euler Calculus from which point of view the analysis of the late 19th century looks rather non-standard.) I shall now sketch the approach which was initiated by Schmieden and myself in a paper of 1958.[10]

Guided by Euler's methods and Leibniz' principle the approach starts by extending the usual number field. We adjoin a new "number" Ω, and everything that is valid in the "usual" theory for all sufficiently large natural numbers n is postulated to hold in the extended theory for Ω in place of n. Let us look at a few examples:

Since for all sufficiently large n, $n > 10^{10}$, we obtain $\Omega > 10^{10}$, and, in the same way, $\Omega > n_o$ for each fixed natural number n_o: Ω is an infinitely large number. The following are proved immediately:

$$0 < \frac{1}{\Omega^\Omega} < \frac{1}{2\Omega} < \frac{1}{\Omega} < 1 < \sqrt[\Omega]{\Omega} < 2 < (1+\frac{1}{\Omega})^\Omega < \sum_{k=0}^{\Omega} \frac{1}{k!} < 3 < \sqrt{\Omega} < \Omega < 2^\Omega,$$

$$(1+h)^\Omega = \sum_{k=0}^{\Omega} \binom{\Omega}{k} h^k.$$

A precise formulation of Leibniz' Principle will be obvious to you: Take any sentence form $A(n)$, that is a formula constituted by finitely many of the symbols of the "usual" theory, and n a symbol for a natural number variable. Then $A(\Omega)$ is a true theorem in the new theory if, for all sufficiently large natural numbers n, the sentences $A(n)$ are true theorems in the usual theory. In a precise sense, the formulas $A(\cdot)$ can be considered as the equivalent in modern terms of Euler's "analytical expressions". In some sense, we have now realized condition (b).

Let $a(n)$, $b(n)$, ... be sequences of ordinary numbers. Then, if $A(n) : a(n) = b(n)$, for all sufficiently large natural n, then $A(\Omega) : a(\Omega) = b(\Omega)$ is true in the new theory. The new objects obtained by this equivalence relation in the set of sequences of ordinary numbers will be called omega numbers. If the ordinary numbers come from an ordered field like $\mathbb{Q}$ or $\mathbb{R}$ then the omega numbers have the defining properties of an ordered field. E.g.,

$$c(\Omega) = d(\Omega) \quad \vee \quad c(\Omega) < d(\Omega) \quad \vee \quad c(\Omega) > d(\Omega)$$

and

$$c(\Omega) = 0 \quad \vee \quad \bigvee_{d} c(\Omega)\cdot d(\Omega) = 1 .$$

These theorems state that the omega numbers are linearly ordered, and that every number which is different from 0 can serve as a denominator. You will observe that the logical symbols are included in our alphabet which lists the symbols of the basic "usual" theory. Though trained mathematicians usually may formulate and understand theorems in a more or less informal manner a "fool-proof" statement should, at least in principle, be given in this formalized way. One may speculate and suppose that Leibniz had no objections against such a formalization.

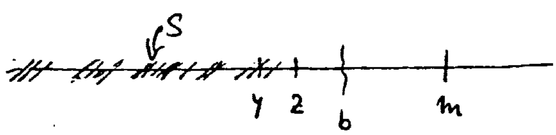
What about our postulate (c)? The set of infinitesimal omega numbers is certainly bounded, without having a least upper bound. But this set is defined by a word which cannot be formulated in the alphabet of the basic theory. It is quite natural to restrict sets and functions of the omega theory to those which can be defined in an internal way, using only this alphabet. For instance, the interval $[0,\Omega]$ is internal: If for all sufficiently large n, $0 \leqq x(n) \leqq n$, then $0 \leqq x(\Omega) \leqq \Omega$. More generally, if a sequence of sets $S(n)$ of ordinary numbers is given, then the internal set $S(\Omega)$ will have $x(\Omega)$ as an element if for all sufficiently large n, $x(n) \in S(n)$. It is elementary to show that this definition is independent of the particular choice of the representing sequences $x(n)$, $S(n)$. The method of proving theorems on internal sets is straightforward, fool-proof, and consequently tiresome. For instance, each non-empty internal set $S(\Omega)$ of real omega numbers which is bounded from above has a least upper bound $b(\Omega)$:

$$[S(\Omega) = \phi] \vee \left[\bigvee_{m(\Omega)} \bigwedge_{x(\Omega)} x(\Omega) \in S(\Omega) \Rightarrow x(\Omega) \leqq m(\Omega)\right] \Rightarrow$$

$$\Rightarrow \left[\bigvee_{b(\Omega)} \left\{\bigwedge_{x(\Omega)} x(\Omega) \in S(\Omega) \Rightarrow x(\Omega) \leqq b(\Omega)\right\} \wedge \left\{y(\Omega) < b(\Omega) \Rightarrow\right.\right.$$

$$\left.\left.\Rightarrow \bigvee_{z(\Omega)} z(\Omega) \in S(\Omega) \wedge y(\Omega) < z(\Omega)\right\}\right]$$

Actually, if you replace Ω by n, then the sentence is true for ordinary real numbers and every natural n. You should not feel shocked by the lengthy and tiresome formal statement of the l.u.b. property. We are used to avoid it by stating it in plain English: A set is empty, or, if it has an upper bound m, then there even exists an upper bound b, such that for each y smaller than b there exists some member z of the set which exceeds y. In teaching mathematics we draw some picture like that



and we do this appealing to a certain correspondence between real numbers and the supposed geometry of the linear continuum.

Actually, as soon as you have got some experience with proving theorems in the extended theory, you may easily proceed in a similar way.

Of course, non-internal sets, or external sets, make sense, like the set of all infinitesimals $\{x(\Omega) \mid x(\Omega) \approx 0\}$. Only it is impossible to define them in the "usual" alphabet; $\approx$ is a new, external symbol. Internal sets have, in some sense, the same properties as standard sets; external sets enjoy different properties, and it is from them that we can expect results which extend the given theory.

The situation is quite similar with functions. The example of a delta function which I gave at the beginning of this talk is internal, it belongs to a sequence $f_n(x) = \frac{n}{\pi(1+x^2n^2)}$ in the basic theory. The intermediate value theorem follows easily if the f_n are continuous in the basic theory. Instead of proving this I prefer to show you how this theorem for ordinary (!) continuous real functions follows in our infinitesimal framework.

Let f be some real function defined on the real interval [0,1]. There is a canonical extension to all omega numbers x_Ω of this interval: If $y_n = f(x_n)$ for almost all natural n, then $y_\Omega = f(x_\Omega)$. As soon as you have said yes to Leibniz' Principle you must accept this. Now consider an infinitesimal subdivision $N.\omega$, $\omega = 1/\Omega$, of the interval, the natural omega integers N running from 0 to Ω. Now suppose that $f(0) < 0 < f(1)$. Then there will exist a smallest element M of the set of all natural numbers N such that $f(N\omega) > 0$. This set is internal and not empty, since it contains Ω. From $M > 0$ and the defining property we have that $f((M-1)\omega) \leq 0 < f(M\omega)$. If f is continuous then $f(x) \approx f(x')$ whenever $x \approx x'$. Let x_0 be the standard part of $M\omega$, that is the uniquely determined real number infinitely close to $M\omega$, and incidentally to $(M-1)\omega$. Now $f(x_0)$ has to be 0 since it is a real number infinitely close to $f(M\omega) > 0$ and $f((M-1)\omega) \leq 0$. This proves the intermediate value theorem.

What about Leibniz' original use of infinitesimals in the shape of differentials dx? A dx may be an infinitesimal, and the differential dy of $y = f(x)$ will be $dy = f(x+dx) - f(x)$. If for a given x all differential quotients $\frac{dy}{dx}$ happen to have the same standard part, then this real number will be called the derivative $f'(x)$. Recall that the notion of a derivative entered the scene about a century after the inventions of Leibniz, but let us jump to this concept immediately for the sake of convenience. Please do not write $dy = f'(x)dx$, a bad habit which led to many confusions in the past and eventually discredited infinitesimals. Actually, if $f'(x)$ exists, then a true and useful formula will be $dy = f'(x)dx + o \cdot dx$, where o is an infinitesimal, which may depend on x and dx, $o = o(x,dx)$. The essential formulas of differential calculus are proved in a straightforward manner.

A definite integral $\int_a^b f(x)dx$ is the standard part of a sum $\sum_{m=1}^{M} f(x_m)dx_m$, where $a = \hat{x}_0 < \hat{x}_1 < \ldots < \hat{x}_m < \hat{x}_{m+1} < \ldots < \hat{x}_M = b$, and $dx_m = \hat{x}_m - \hat{x}_{m-1}$ is infinitesimal for all m, and $\hat{x}_{m-1} \leqq x_m \leqq \hat{x}_m$.

Of course, we have to assume for integrability that this real number depends only on f, a, b but not on the particular infinitesimal subdivision produced by the $\hat{x}_m$'s. The proof that any continuous f is integrable is simpler here than in the conventional setting since the concept of equicontinuity is avoided.

Incidentally, since the notations introduced by Leibniz have been used through the centuries, even though his fundamental concepts were not always accepted, the introduction of infinitesimals does not really change anything in Calculus. But now the notation regains its original sense and is not a mere symbolism.

Finally, let me mention two types of methods whose formulation and application would be not elegant in the analysis of limits;

the first one is the method of divergent series, the second one the representation of "generalized functions" or distributions by internal functions.

The method of divergent series was widely used by Euler. For the sake of brevity, I give only very easy examples which are in the style of Euler.

Let be our fixed infinitely large number, and let $E = \sum_{k=1}^{\Omega} \frac{1}{k}$ which is certainly infinitely large. (Euler knew that this is infinitely close to $\log \Omega + C$ where $C = .577\ldots$, but we shall not need this.) Let M be any infinitely large number, and N a natural number, finite or infinite. Then

$$\sum_{k=M+1}^{M+N} \frac{1}{k} = \sum_{j=1}^{N} \frac{1}{1+\frac{j}{M}} \cdot \frac{1}{M} \approx \int_1^b \frac{dx}{x} = \log b$$

where $b = 1 + \frac{N}{M}$.

In particular, if $M = \Omega$ and $M + N = K$ then

$$\sum_{k=1}^{K} \frac{1}{k} = \sum_{k=1}^{\Omega} \frac{1}{k} + \sum_{k=\Omega+1}^{K} \frac{1}{k} \approx E + \log \frac{K}{\Omega} = (E - \log \Omega) + \log K$$

$$= C + \log K,$$

now using C as an abbreviation for $E - \log \Omega$.

We conclude that, for any infinitely large M,

$$\sum_{k=1}^{2M} \frac{(-1)^{k+1}}{k} = \sum_{k=1}^{2M} \frac{1}{k} - 2\sum_{k=1}^{M} \frac{1}{2k} \approx (\log 2M + C) - (\log M + C)$$

$$= \log 2$$

Consider the series which is usually called a rearrangement of this series of Leibniz,

$$S = (1 + \frac{1}{3} - \frac{1}{2}) + (\frac{1}{5} + \frac{1}{7} - \frac{1}{4}) + \ldots + (\frac{1}{4M-3} + \frac{1}{4M-1} - \frac{1}{2M})$$

$$= \sum_{k=1}^{4M-1} \frac{1}{k} - 2\sum_{k=1}^{M} \frac{1}{2k} - \sum_{k=M+1}^{2M-1} \frac{1}{2k}$$

$$\approx C + \log(4M-1) - (C + \log M) - \frac{1}{2} \log \frac{2M-1}{M}$$

$$\approx \log \frac{4M-1}{M} - \frac{1}{2} \log 2 \approx \frac{3}{2} \log 2 .$$

Actually, when taking a definite last term of an infinite series into account, S is not a rearrangement of the series for log 2, but contains twice as much positive terms as negative ones, this increasing the value to $\frac{3}{2}$ log 2.

We come back to the internal function $\delta(x) = \frac{1}{\pi} \frac{\Omega}{1+x^2\Omega^2}$, with $\delta(x) \approx 0$ for all $|x| \geqq \beta$ and some $\beta \approx 0$. e.g. $\beta = \frac{1}{\sqrt[4]{\Omega}}$, and $\delta(x)$ infinitely large for x very close to 0. It is an easy exercise to show that $\int_{\infty}^{+\infty} \delta(x) f(x) dx \approx f(0)$ whenever f is a bounded real function which is continuous at 0. Of course, there are many internal functions sharing this property which is essential for the delta "distribution". Let A be any internal function such that $A(x) \approx \frac{|x|}{2}$ for all finite and infinitesimal x. Then the second derivative A is a delta function. (Derivatives of an internal function $f_\Omega(x)$ are defined in a canonical manner, via the sequence $f_n'(x)$.)

This gives us a hint for the definition of "distributions": Let F be any real function which is defined and continuous for all real x, and f, g internal functions which are n-times differentiable and $f(x) \approx F(x) \approx g(x)$ for all finite and infinitesimal x. Then $f^{(m)}$ and $g^{(m)}$ are called equivalent in the distributional sense, it can be shown that the equivalence classes are the distributions in the sense of L. Schwartz. Thus, these distributions can be represented by genuine functions, meeting the more intuitive expectations of physicists and others.

Of particular interest, and quite easily to deal with, are distributions of period 2π. If for some finite m_o, for real a_k, b_k and for all k

$$|a_k| \leqq \frac{m_o}{k^2}, \quad |b_k| \leqq \frac{m_o}{k^2}$$

then

$$g(x) = \frac{a_o}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

is a real continuous function of period 2π. For any infinite M and all omega-numbers x,

$$F_M(x) = \frac{a_o}{2} + \sum_{k=1}^{M} a_k \cos kx + b_k \sin kx \approx g(x) .$$

The series obtained by differentiation of $F_M(x)$ represent distributions. An example is

$$(4) \quad \delta_M(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{M} \cos kx = \frac{1}{2\pi} \sum_{k=-M}^{+M} e^{ikx} ,$$

a delta function of period 2π . Actually, if f is any continuous real function then

$$\int_{-\pi}^{+\pi} \delta_M(x) f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx + \sum_{k=1}^{M} \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx \approx f(o)$$

provided that the Fourier series of f converges to $f(o)$ at $x = 0$.

Note that the series $\sum_k \cos kx$ diverges in the conventional sense for every x. Nevertheless, the internal function δ_M provides a most useful tool in analysis.

In the distributional sense one may say that for all real $x \neq 2k\pi$ the "value" of $\delta_M(x)$ is 0. Indeed Euler stated that

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \dots = 0$$

for $x \neq 2k\pi$, a statement which was not accepted by his contemporaries. In a precise way we can now say that this series behaves analytically like a delta function which has values vanishing outside some infinitesimal regions around $x = 2k\pi$. Geometrically, the graph of δ_M behaves in a somewhat unexpected manner. By using the sum of the geometrical series $\sum_{K=-M}^{+M} e^{ikx}$ one obtains Dirichlet's kernel function

$$(5) \qquad \delta_M(x) = \frac{1}{2\pi} \frac{\sin(M+\frac{1}{2})x}{\sin \frac{x}{2}}$$

whose graph looks like an infinitesimally fine saw enveloped by the curves $y = \pm \frac{1}{2\pi \sin\frac{x}{2}}$. By integrating (4) for $|x|<\pi, x \in \mathbb{R}$,

$$(6) \quad \frac{\text{sign } x}{2} \approx \int_0^x \delta_M(t)\,dt = \sigma_M(x) = \frac{x}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{M} \frac{\sin kx}{k} .$$

Perhaps you may find it not so satisfactory that the graphs of internal functions representing distributions may show a peculiar behaviour. But at least they are functions and not objects of some complicated mathematical structure. The tools of topological vector spaces are eliminated from the theory of distributions which now can be treated in an elementary way. Moreover, there are equivalent representations which display the expected types of graphs.

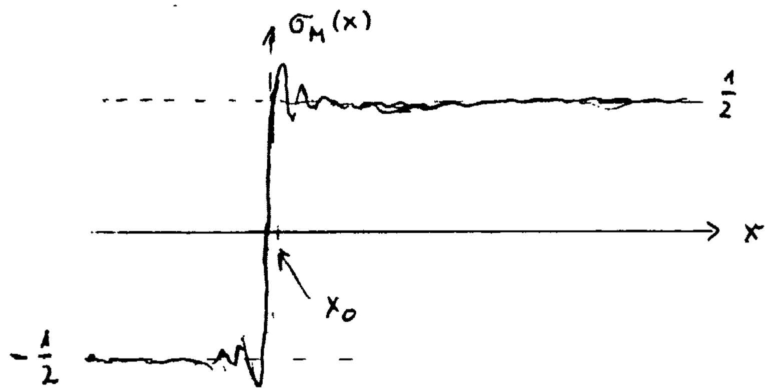
Let us dwell for one more minute on the function σ_M! What is the maximal value attained by it? We should expect it to be approximately $\frac{1}{2}$. Since $\sigma'_M = \delta_M$, we obtain $(M + \frac{1}{2})x_o = \pi$ for the first $x_o > 0$ with a maximum of σ_M. Letting $(M + \frac{1}{2})t = s$, we obtain

$$\sigma_M(x_o) = \int_0^{x_o} \delta_M(t)dt = \frac{1}{\pi}\int_0^{\pi} \frac{\sin s}{\sin \frac{s}{2M+1}} \cdot \frac{ds}{2M+1} \, .$$

Since $\frac{\sin \lambda}{\lambda} \approx 1$ for $\lambda = \frac{s}{2M+1} \approx 0$,

$$\sigma_M(x_o) \approx \frac{1}{\pi}\int_0^{\pi} \frac{\sin s}{s} ds = .59 \, ,$$

which exceeds the expected value .5 at the rate of about 18%. This fact is known as the Gibbs phenomenon (1898), and had been discovered as early as in 1848 by Wilbraham who used infinitesimal analysis in a manner similar to ours [11].



$$\sigma_M(x) = \int_0^x \delta_M(t)\,dt$$

These few examples will suffice to draw some more general conclusions. First of all, we gave little contributions to the Leibnizian use of infinitesimals. Actually it turns out that the Leibnizian notation was designed in a manner which needs no improvement. But we have seen that it is not only a convenient notation: Quotients of differentials are not mere symbols but numbers which are infinitely close to the value of the derivative, and the chain rule $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ appears as an almost trivial equation. An integral is represented by a sum, up to an infinitesimal error, and the rule for substitutions of variables, $\int y\, dx = \int y \frac{dx}{dt}\, dt$, is one more equation, much easier to invent, to prove, and to remember than its equivalent in derivatives.

Secondly, and much more in the spirit of Euler than Leibniz, the use of divergent series has now been vindicated. Though writers of 'rigourous' textbooks rarely hesitate to write down $\sum \frac{1}{k}$ they will insist elsewhere that divergent series are meaningless. Many a teacher will confirm my impression that critical students have difficulties in following the dogma of convergence as a necessary requirement for mathematical existence.

Thirdly, the supply of functions which are accessible to ordinary calculus and which are expressed as $y = f(x)$ in finitely many terms is considerably enlarged. I feel that this is a field of fruitful exploitation.

Let me mention a fourth aspect: Fundamental concepts can be formulated and handled easier. A function f is continuous at a (standard) point x iff $f(x') \approx f(x)$ for all $x' \approx x$, and it is uniformly continuous on a set iff $f(x') \approx f(x'')$ for all (standard or nonstandard) x', x'' of the set, $x' \approx x''$.
It is an easy exercise to prove the uniform continuity of a function which is continuous at each standard x of a closed real inverval. - The example which we considered for the phenomenon of Gibbs can be exploited in a different way to investigate concepts related to sequences of functions. Let σ_M be as in (6), M a natural number, finite or infinitely large. We shall say that $\sigma(x)$ is the limit function of the series if $\sigma_M(x) \approx \sigma(x)$ for every infinitely large M. Obviously, $\sigma(x) = \frac{1}{2}$ sign x is the limit of our series for all <u>standard</u> x, $-\pi < x < \pi$.

The Gibbs phenomenon shows that there is no convergence for some infinitesimal x! This throws a new light on a theorem of Cauchy which was denounced as an error ever since N.H.Abel: If a series $\sigma_M(x)$ of functions converges <u>everywhere</u> (toujours) on an interval [a,b], each of the σ_M being continuous on [a,b] for finite M, then the limit function is again continuous. This is a true theorem if 'everywhere' means <u>all</u> (standard and nonstandard) numbers of the interval. This convergence 'everywhere' is a nice and intuitive equivalent for uniform convergence, and much easier to formulate and to apply [12].

I see some justification of the recent rivival of infinitesimals in this fact, that it appears to serve as a better background than the analysis of limits to concepts and methods of Leibniz, Euler, Cauchy, and others. Please mind: A background in terms of our century, which does not mean that it could be identified with the framework of any earlier mathematics.

Many people view the relation of "old" and "new" infinitesimals vice versa. Let me quote from the Presidential Address of the International Congress at Vancouver 1974. In remembrance of his friend Abraham Robinson, H.S.M.Coxeter said: "When I was a boy, I was introduced to calculus the 'easy' way, using infinitesimals. At college I was told to put away childish things and become rigorous. How wonderful it is that the name 'infinitesimal calculus' has been restored to respectability!"

Notes

1 Extended version of lectures of September 10 and 13, 1984, at Dansk Matematisk Forening, Copenhagen, and at IMFUFA.

2 The Thirteen Books of Euclid's Elements, translated by Sir Thomas L. Heath. Dover Publ.

3 For the history of hornlike angles see Heath l.c.[2] Vol.I p.176 sq. and Vol. II p. 39 sq.
Sir Thomas Heath, A History of Greek Mathematics I, p. 178 sq., Oxford 1921; M. Cantor, Geschichte der Mathematik Vol. II (see Contingenzwinkel in the index); F. Beckmann: Neue Gesichtspunkte zum 5. Buch Euklids. Arch. Hist. Ex. Sci. 4, p. 1-144, 1967/68.
For a treatment from the contemporary point of view of ordered structures see: D. Laugwitz, Die Messung von Kontingenzwinkeln, J. f. d. r. u. angew.Math. 245, p.133-142, 1970.

4 Joseph W. Dauben: Georg Cantor, Harvard U.P. 1979; p.129-132.

5 Georg Berkeley, The Analyst (1734). Quotations from Morris Kline, Mathematical Thought from Ancient to Modern Times, O.U.P. 1972, p.427-428

6 Gottlob Frege, Die Grundlagen der Arithmetik. Reprint Darmstadt 1961

7 L. Euler, De progressionibus harmonicis observationes (1734/35). Op. omn. I 14, p. 87-100

8 L. Euler, De la controverse entre Mrs. Leibnitz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires. Op. omn. I 17, p. 195-232

9 Abraham Robinson, Non-standard analysis. Proc. Roy. Acad. Amsterdam Ser.A 64, 432-440 (1961) -
Non-Standard Analysis. North Holland, Amsterdam 1966

10 C. Schmieden, D. Laugwitz, Eine Erweiterung der Infinitesimalrechnung. Math. Z. 69, 1-39 (1958). - D. Laugwitz, Ω-calculus as a generalization of field extension. In: Nonstandard Analysis-Recent Developments. Ed. by A. Hurd. Lecture Notes in Mathematics, Vol. 983, p. 120-133 (1983).

11 Henry Wilbraham, On a certain periodic function. The Cambridge and Dublin Math. J. 3, 198-201 (1848); Josiah Willard Gibbs, Fourier's series, Nature 59, p. 606, (1898/99). For an extensive discussion see Detlef D. Spalt, Vom Mythos der mathematischen Vernunft. Darmstadt, Wiss. Buchges.,1981

12 Actually, if $x \approx 0$ (and $x > 0$) then $\sigma_M(x) \approx \sigma(x)$ merely for all sufficiently large infinite M. One might call that an infinitely slow convergence, in contrast to convergence in the sense of Euler. For a historical account of infinitely slow convergence as related to non-uniform convergence see Spalt,l.c.[11]

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Bernhelm Booss & Mogens Niss (eds.).

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