TEKST NR 52 1982


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# Desuspension of splitting elliptic symbols


TEKSTER fra

IMFUFA ROSKILDE UNIVERSITETSCENTER
INSTITUT FOR STUDIET AF MATEMATIK OG FYSIK SAMT DERES FUNKTIONER I UNDERVISNING, FORSKNING OG ANVENDELSER

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IMFUFA tekst nr. 52 (1982), RUC
103 s. ISSN 0106-6242

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This paper provides an algorithm for the conversion of the index of an elliptic first-order differential operator $A$ on the torus $Y \times S^1$ into the index of a canonically associated elliptic pseudo-differential operator $Q$ on $Y$. It is supposed that $Y$ is a closed smooth manifold and that $A$ "splits" into $\frac{\partial}{\partial t} + B_t$, where $\{B_t\}$ is a family of self-adjoint elliptic operators on $Y$ satisfying the periodicity condition $B_1 = g\, B_o\, g^{-1}$ for some unitary automorphism g. Then it will be shown that the operator $Q$ (the "desuspension" of $A$) can be written down explicitly in the form $Q = P_+ - g\, P_-$ where $P_\pm$ are projections onto the space of Cauchy data. Applications are given for the calculation of the index of the general linear conjugation problem ("cutting and pasting" of elliptic operators) and the intimate interrelations between the related procedures of algebraic topology, spectral theory and functional analysis are explained. Generalisations in various directions are indicated.

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## Introduction.

This paper is an effort to provide the necessary algorithmic machinery for the study of the behaviour of the index of elliptic operators under certain, not necessarily continuous deformations. Let $A$ be an elliptic first order differential operator acting on sections of the trivial bundle $\mathbb{C}^N \times X$ over a closed smooth Riemannian n-manifold $X$, i.e.

$$A: C^\infty(X;\mathbb{C}^N) \rightarrow C^\infty(X;\mathbb{C}^N)$$

where $A$ has in local coordinates the form

$$\sum_{i=1}^{n} A_i(x) \frac{\partial}{\partial x_i} + A_o(x)$$

with complex $N \times N$-matrices $A_o, A_1, \ldots, A_n$ such that the homogeneous first order polynomial (the "symbol")

$$\sigma(A)(x,\xi) := \Sigma A_i(x)\xi_i : \mathbb{C}^N \rightarrow \mathbb{C}^N$$

belongs to $GL(N,\mathbb{C})$ for each $\xi \in T^*X$, $\xi \neq 0$

It is well known that for such an $A$ the spaces

$$\ker A := \{ u \mid Au = 0 \}$$

and

$$\operatorname{coker} A := C^\infty(X;\mathbb{C}^N)/\mathrm{Image}\, A$$

have a finite dimension, cf. Treves [35, chapter II], and one defines

$$\operatorname{index} A := \dim \ker A - \dim \operatorname{coker} A.$$

The index depends only on the symbol and is a topological invariant, i.e. remains unchanged under continuous deformations of the operator.

Given a smooth closed submanifold $Y$ of codimension 1 which divides $X$ into two parts $X_-$, $X_+$

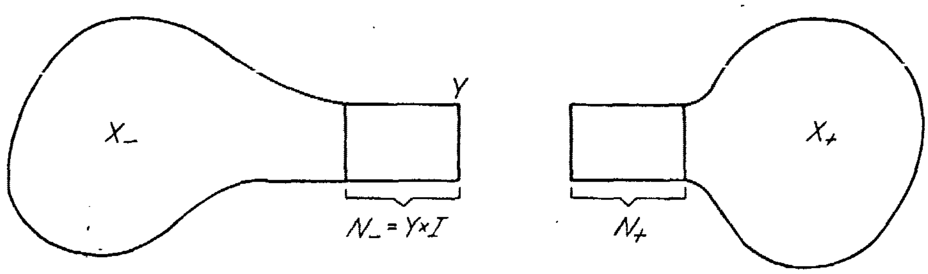


fig.1

and a map

$$g : Y \to U(N),$$

we consider a new operator (made up by "linear conjugation")

$$A^g : C^\infty(X;\mathbb{C}^N)_g \to C^\infty(X;\mathbb{C}^N)_g$$

on the space

$$C^\infty(X;\mathbb{C}^N)_g := \{ u \mid u_\pm := u|X_\pm \text{ is smooth and } u_+ = g\, u_- \text{ on } Y\}.$$

In [ 13 ], [ 14 ] it was noted that the "glued" operator $A^g$ is well defined on the symbol level if the symmetry condition

$$g(y)\ \sigma(A)(y,\xi)\ g^{-1}(y) = \sigma(A)(y,\xi) \qquad (*)$$

is satisfied for all $y \in Y$ and $\xi \in (T^*X)_y$, $\xi \neq 0$.

For simplicity we assume that $A$ splits near $Y$, i.e. (cf. the terminology introduced by Gilkey and Smith in [19]) $A$ takes the form

$$A(t,y) = \frac{\partial}{\partial t} + B(y)$$

in a neighbourhood $N$ of $Y$ in $X$ with $N_\pm := N \cap X_\pm$, where $t$ is the normal coordinate and $B$ an elliptic self-adjoint operator on $Y$. Condition (*) is then equivalent to

$$g\ \sigma(B)\ g^{-1} = \sigma(B).$$

Let $\{B_t\}_{t\in I}$ be a family of elliptic self-adjoint operators on $Y$ connecting

$$B_o := B \text{ and } B_1 := g\, Bg^{-1},$$

e.g.

$$B_t(y) := f(t)\ g(y)\ B(y)\ g^{-1}(y) + (1-f(t))\ B(y)$$

where $f$ is a smooth function equal o near o and equal 1 near 1. Then we can define the operator $A^g$ explicitly by

$$A^g := A \quad \text{on} \quad X \setminus N_-$$

and

$$A^g(t,y) := \frac{\partial}{\partial t} + B_t(y) \quad \text{on} \quad N_- .$$

Since

$$A^g(u_+)|Y = g(A^g u_-)|Y$$

for any $u \in C^\infty(X;\mathbb{C}^N)_g$, it follows that

$$A^g : C^\infty(X;\mathbb{C}^N)_g \to C^\infty(X;\mathbb{C}^N)_g$$

and one checks easily the ellipticity of $A^g$.

The aim of this paper is to develop an algorithm for the calculation of

$$\mu(g,A) := \text{index}\, A^g - \text{index}\, A$$

in terms of the geometry of $Y$ and $g$ and of the behaviour of $\sigma(A)$ near $Y$.

If $A$ is the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$, this is just the classical linear conjugation problem ("Riemann-Hilbert problem") solved by Hilbert and F.Noether, see [26]. The general linear conjugation problem was investigated in [13], [14] in a broader context in an effort to derive a recurrence formula for the index of elliptic operators on manifolds with a given decomposition in simpler parts e.g. via Morse theory. The machinery used there was based on the analytical theory of elliptic transmission problems and mixed boundary value problems. It turned out to be too delicate to lead to explicit formulas. Here however, we are able to derive explicit analytical and topological formulas for $\mu(g,A)$.

Our main result is the following

<u>0.1. THEOREM.</u> *Let $P_\pm : C^\infty(Y,\mathbb{C}^N) \to H_\pm$ be the projections onto the spaces of Cauchy data*

$$H_\pm := \{u_\pm|Y \mid u_\pm \in C^\infty(X_\pm;\mathbb{C}^N) \text{ and } A(u_\pm) = 0\}.$$

*Then $P_+ - g\,P_-$ is a Fredholm operator, in fact an elliptic pseudo-differential operator of order zero on $Y$, and:*

$$\mu(g,A) = \operatorname{index}\,(P_+ - g\,P_-). \tag{**}$$

To prove the theorem we first reduce the problem to the study of an elliptic operator over the torus $Y \times S^1$, second to the study of the associated family of operators over $Y$, parametrized by $S^1$. Finally we construct one single operator over $Y$ with the same index as the original operator over $Y \times S^1$. The main part of the paper is devoted to the development of the for that necessary method of "desuspension" of splitting elliptic symbols. In some sense our method is complementary to the approach given by Atiyah, Patodi and Singer in [ 4 ]. In order to provide information about the function $\eta(s)$, associated to a self-adjoint elliptic operator $B$ on $Y$, they had to find a manifold $X_+$ with $\partial X_+ = Y$ and an elliptic operator $A$ on $X$ extending $\frac{\partial}{\partial t} + B$. Whereas their approach can be thought of as "suspension", we are treating the opposite direction, namely how to go down from an elliptic operator on a n-manifold to an elliptic operator on a (n-1)-manifold.

The paper is organized as follows. §1 describes the geometry of families of self-adjoint elliptic operators in terms of the spectral flow. That invariant was introduced by Atiyah, Patodi and Singer in [ 4 ] very briefly. Here the definition and the investigation of the spectral flow will be worked out in more detail. It turns out that, roughly speaking, the difference element of any elliptic first order differential operator on the torus $S^1 \times Y$, i.e. an element of the group $K(T(S^1 \times Y))$, can be obtained as the difference element of an associated family of self-adjoint elliptic differential operators of first order on $Y$ parametrized by $S^1$, i.e. an element of the group $K^{-1}(S^1 \times TY)$. Since $T(S^1 \times Y) \cong S^1 \times R \times Y$, both groups are isomorphic. As a corollary we ~~obtain~~ an independent proof of the spectral formula

$$\operatorname{index}(\{-\frac{\partial}{\partial t} + B_t\}) = \operatorname{sf}\{B_t\} \qquad (***)$$

which was already noted by Atiyah, Patodi and Singer [ 4 ] in the much more complex setting of the signature theorem and global elliptic boundary problems. Expressing the index of an operator over a n-manifold by the spectral flow of a family of operators over a (n-1)-manifold, (***) constitutes the first step in our program of desuspension. Unfortunately, a purely analytical proof of (***) is not yet known.

§2 establishes a symbolic calculus for the spectral projections $P_{\pm}(B)$ of an elliptic self-adjoint operator $B$ of non-negative order. Analytical conditions are given for the vanishing of the stable symbol class of $B$.

§3 generalizes a situation which is well known and quite essential for calculations with singular integrals, namely commutativity modulo compact operators. More precisely, let $S$ be the difference between two complementary orthogonal projections $P_+$ and $P_-$ in a complex Hilbert space, both with infinite dimensional range. Following a suggestion by Bojarski [ 9 ] we consider the space $GL_S$ of automorphisms of $H$ which commute with $S$ modulo compact operators and we show that $GL_S$ is homotopically equivalent to the space Fred(H) of Fredholm operators in $H$ and, accordingly, a minimal classifying space for K-theory.

§4 then establishes the second step of the desuspension: the spectral flow of a family $\{B_t\}$ of self-adjoint elliptic first order differential operators over $Y$, parametrized by $I$ with the periodicity condition $B_1 = g\, B_o\, g^{-1}$, will be represented as the index of an associated single elliptic pseudodifferential operator on $Y$:

$$sf\{B_t\} = \text{index } (P_+(B_o) - g\, P_-(B_o)). \quad (****)$$

That procedure has very much in common with the classical boundary integral method and other projection methods of the analysis of elliptic boundary value problems.

This is explained in §5. There we describe the both sided spaces $H_{\pm}(A)$ of the Cauchy data on $Y$ of the solutions of $Au = 0$ on $X_{\pm}$ where $A$ is an elliptic operator over $X$ which splits near $Y$ and $Y$ divides $X$ into $X_{\pm}$. Let $A$ take the form $\frac{\partial}{\partial t} + B$. In fact, then the spectral projections $P_{\pm}(B)$ are projections onto $H_{\pm}(A)$. However, they are not identical with the Calderon projections which are defined more generally without the assumption of self-adjointness. We have avoided the notion of Calderon projections since they require much harder analysis than the elementary functional analysis which is sufficient for the definition of our spectral projections. It should be noted that the range of these two different projections differs only by a finite dimensional space.

Our approach is based on the theory of Fredholm pairs of subspaces as introduced by Kato [24, IV.4]. Some relations with classical results about the unique continuation property of elliptic operators and their formal adjoints are investigated.

The main part of §5 is devoted to the study of classical transmission problems like the problem of linear conjugation and its generalization, the cutting and pasting of splitting elliptic symbols. Exploiting the local index formula by Seeley [31] we show that the integer $\mu(g,A)$ is a spectral invariant and in fact (by (***), the first step of desuspension) equal to $sf\{B_t\}$ where $\{B_t\}$ is a family of self-adjoint elliptic first order differential operators on $Y$ connecting

$$B_o := B \quad \text{and} \quad B_1 := g\,B\,g^{-1}$$

and $A$ splits into $\frac{\partial}{\partial t} + B$ near $Y$. Then the second step of desuspension completes the proof of Theorem 0.1.

§6 gives some indications of how our results can be extended to the more general situations of:

(A) splitting elliptic symbols where the part $B$ on $Y$ is not necessarily self-adjoint,

(B) arbitrary clutching functions $g: E|Y \to E|Y$ which need not induce the identity in the basis $Y$, but an arbitrary diffeomorphism $f:Y \to Y$, i.e. analytically speaking a theory of Carleman singular integral operators with a shift in the kernel, and topologically speaking a theory of elliptic symbols over the mapping torus $(Y\times S^1)^f$. The details of these generalizations will be worked out by the second author in a separate publication.

Several examples with explicit calculations are given at the end of each paragraph.

ACKNOWLEDGEMENTS. In preparing this report we are indebted to many people. People familiar with the subject will perceive the dominant influence of the thinking of Bogdan Bojarski in every chapter. It was his advice to analyze carefully the topological aspects of the most elementary analytical situations in the theory of singular integrals which lead us to the appropriate generalizations, to natural conjectures and transparent proofs. Our collaboration with Stephan Rempel and frequent conversations with Matthias Kreck, Tadeusz Mostowski and Bert-Wolfgang Schulze have been influential in large parts of this work. We also thank Johan Dupont and Isadore Singer for their constructive criticism of an earlier announcement of our results. Finally, we are indebted to Peter Gilkey for his continuing interest and encouragement, to our colleagues in Roskilde, Copenhagen, and Warsaw for their stimulating and supportive atmosphere, to Birthe Holm for rapid typing, and to Krystyna and Władysław Uscinowicz for providing us housing in Szczecin for our monthly meetings.

## 1. Periodic Families of Self-Adjoint Elliptic Operators.

Let $\{B_t\}_{t\in I}$ be a family of elliptic self-adjoint differential operators (of positive order r) over a closed Riemannian manifold Y acting on the smooth sections of a Hermitian vector bundle E over Y. We always assume that the coefficients of the operators are smooth and depend smoothly on the parameter t. Recall that elliptic self-adjoint operators of positive order have a discrete spectrum of finite multiplicity $\{\lambda_j\}_{j\in \mathbb{Z}}$. Moreover there exists no essential spectrum, and the eigenvectors span the whole $L^2E$ as shown in [27, chapter XI, theorem 14].

Now we assume that $B_0$ and $B_1$ have the same spectrum. Note that the eigenvalues change continuously when we vary the operator continuously, cf. Lemma 1.4. Roughly speaking, the spectral flow of the family $\{B_t\}_{t\in I}$ is the difference between the number of eigenvalues which change the sign from $-$ to $+$ on I and the number of eigenvalues which change the sign from $+$ to $-$. To put it more precisely: First, we deform the family $\{B_t\}_{t\in I}$ into a new family $\{\tilde{B}_t\}_{t\in I}$ such that 0 is an eigenvalue of $\tilde{B}_t$ only for a finite number of t. Then we define a function $\ell$ on I with the properties $\ell(0) = 0$, and $\ell(t)$ increases by 1 every time an eigenvalue $\lambda<0$ changes to one $\geq 0$ and decreases by 1 when the reverse happens. Then the spectral flow of the family is equal to $\ell(1)$. We denote it by $sf\{B_t\}_{t\in I}$. (See also definition 1.11).

In fact, it is not difficult to see that the definition is independant of the chosen deformation and that the spectral flow is a homotopy invariant of the family.[x)] We will put this in a slightly different form than [4], a form most useful for explicit calculations.

1.1. THEOREM. (a) *Let* $\hat{\mathrm{Ell}}(Y;E)$ *be the space of elliptic self-ajoint operators of positive order over the Riemannian manifold* $Y$ *in sections of the Hermitian bundle* $E$, *and let* $B : I \to \hat{\mathrm{Ell}}(Y;E)$ *be a family of operators where the coefficients depend smoothly on the parameter* $t \in I$. *Then the graph of the spectrum of* $B$ *can be parametrized near the 0-line through a finite set of continuous functions* $j_1 \leq j_2 \leq \ldots,\ j_k : I \to \mathbb{R}$. *Moreover, if* $B_0$ *and* $B_1$ *have the same spectrum, we get*

$$j_k(1) = j_{k+\alpha}(0)$$

*for some integer* $\alpha$ *which is the same for each* $k$. *That* $\alpha$ *will be called the spectral flow* $\mathrm{sf}\{B_t\}$ *of the family* $\{B_t\}$.

(b) *The spectral flow is a* $C^\infty$*-homotopy invariant of periodic (i.e.* $B_1 = B_0$*) families of elliptic self-adjoint operators of positive order, i.e. it doesn't change under* $C^\infty$*-deformations of the coefficients of the operators involved.*

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x) For the definition of the spectral flow it is not necessary to suppose the periodicity $A_1 = A_0$ of the operators but only of the spectrum : $\mathrm{spec}\ A_1 = \mathrm{spec}\ A_0$.

The situation may be illustrated by the following picture

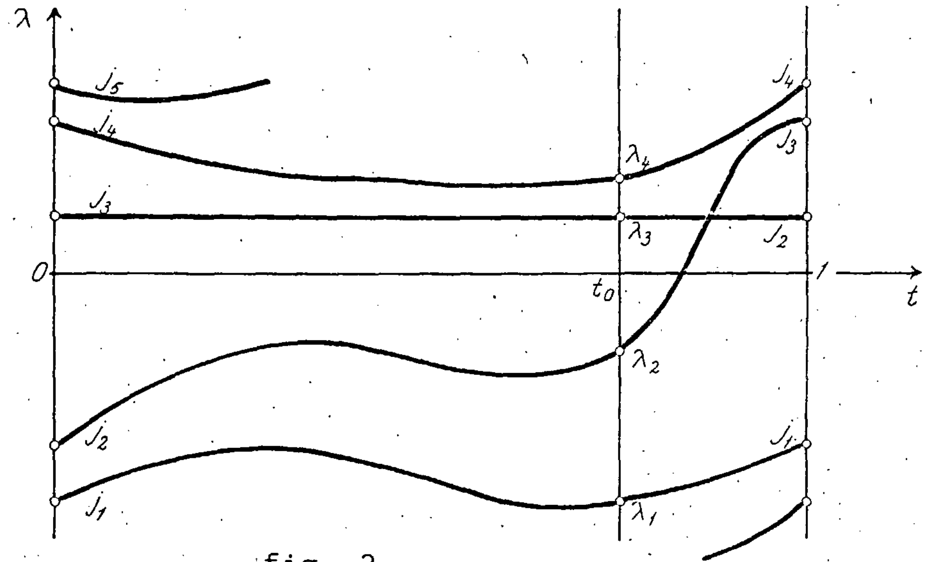


fig. 2

where e.g. $j_1(1) = j_2(0)$, $j_2(1) = j_3(0)$,

$j_3(1) = j_4(0)$ etc., hence $sf\{B_t\} = 1$.

Theorem 1.1 could be proved directly by exploiting the perturbation theory of the spectrum of closed, not necessarily bounded operators as presented in [24, chapter IV]. However, the topological meaning of the spectral flow will be more transparent if instead we work with self-adjoint (bounded) operators in Hilbert space first and carry over the arguments to operators of positive order later.

Following [ 4 ] and [ 5 ] we give a short introduction into the topology of the space of self-adjoint Fredholm operators in Hilbert space.

Let $H$ be a separable infinite-dimensional complex Hilbert space and let $F$ denote the space Fred(H) of all Fredholm operators on $H$, i.e. bounded linear operators with finite dimensional kernel and cokernel. Then any $A \in F$ has an index defined by

$$\text{index } A := \dim \ker A - \dim \operatorname{coker} A.$$

The space $F$ equipped with the operator norm, decomposes into $\mathbb{Z}$ connected components which are distinguished by the index. More generally for any continuous family $\{A_p\}_{p\in P}$ of Fredholm operators parametrized by a compact topological space $P$ we can define a homotopy invariant, the index bundle

$$\text{index } \{A_p\}_{p\in P} \in K(P)$$

where $K(P)$ is the Grothendieck group of complex vector bundles over $P$. The Atiyah-Jänich theorem states that the index bundle defines a bijection

$$[P,F] \to K(P)$$

where $[\ ,\ ]$ denotes the homotopy classes of mappings. Hence the homotopy type of the space $F$ is completely identified: it is a classifying space for the functor $K$. (For all that cf. [12]).

It turns out that one can develop an analogous theory for the space $\hat{F}$ of self-adjoint Fredholm operators, not based on the index which vanishes on $\hat{F}$ but on the spectral flow. We recall

1.2. PROPOSITION ([5, theorem B]). (a) *The space* $\hat{F}$ *has three components* $\hat{F}_+$, $\hat{F}_-$ *and* $F_*$ *characterized by*

$A \in \hat{F}_\pm \Leftrightarrow$ A *is essentially positive (negative), i.e.* A-k *is positive (negative) for some compact operator* k

$A \in \hat{F}_* \Leftrightarrow A \notin \hat{F}_\pm$ .

(b) *The components* $\hat{F}_+$ *and* $\hat{F}_-$ *are contractible.*

(c) *Define a map*

$$\alpha : \hat{F}_* \to \Omega F$$

*by assigning to each* $A \in \hat{F}_*$ *the path from* +id *to* -id *on* $F$ *given by*

$$\cos \pi t + iA \sin \pi t, 0 \leqq t \leqq 1.$$

*Then* $\alpha$ *is a homotopy equivalence, and so* $\hat{F}_*$ *is a classifying space for the functor* $K^{-1}$.

Note. - $K^{-1}(P)$ denotes the Grothendieck group of vector bundles over the suspension

SP: = (P × I)/identifying P×{1}
with a single point and P×{0}
with another single point.

Hence we have the following isomorphisms

$$K^{-1}(P) \cong [P,\hat{F}_*] \cong [P,\Omega F] \cong [SP,F] \cong K(SP) \cong [P,GL(\infty)] \cong [P,U(\infty)],$$

where $U(\infty) = \lim\limits_{n\to\infty} U(n)$ (cf. [23, I.3.14 and II.3.19]).

PROOF. The complete proof can be found in [ 5 ]. Here we restrict to the outlines of the proof. We present only those arguments in detail (and in a slightly different way) which are illuminating the methods of spectral deformation which are important for the calculation of the spectral flow and other spectral invariants.

Since

$$t \to A_t := tA + (1-t)\mathrm{id}\ , \quad t \in I$$

is a path in $\hat{\mathcal{F}}_+$ connecting any $A_1 = A \in \hat{\mathcal{F}}_+$ with $A_0 = \mathrm{id}$, it follows that $\hat{\mathcal{F}}_+$ (and similarily $\hat{\mathcal{F}}_-$) is contractible.

The determination of the homotopy type of the space $\hat{\mathcal{F}}_*$ will be carried out step by step through the following lemmata, which lead to a homotopy equivalence $\hat{\mathcal{F}}_* \to U(\infty)$. Since all classes of vector bundles over the suspension SP of S are generated through clutching over P of trivial bundles it follows

$$[P, \hat{\mathcal{F}}_*] \cong [P, U(\infty)] \cong K(SP) =: K^{-1}(P).$$

Applying the Atiyah-Jänich theorem we get

$$K(SP) \cong [SP, \mathcal{F}] \cong [P, \Omega\mathcal{F}]$$

where $\Omega\mathcal{F}$ is the space of loops in $\mathcal{F}$ which begin at the identity and end in $-\mathrm{id}$. □

Recall that a continuous map $f: X \to Y$ is called a weak homotopy equivalence if, for every point $x_o \in X$,

$$f_* : \pi_n(X, x_o) \to \pi_n(Y, f(x_o))\ , \quad n \geqq 0$$

is bijective. For spaces having the homotopy type of a CW-complex a weak homotopy equivalence is actually a homotopy equivalence. In [25] Milnor shows that a suitable convexity property ensures that a space has the homotopy type of a CW-complex. In particular this gives the following

1.3. LEMMA. *A weak homotopy equivalence between open subsets of a Banach space or deformation retracts of such sets is a homotopy equivalence.*

The next lemma shows (and gives a precise meaning to) the continuity of a finite system of eigenvalues.

1.4. LEMMA. *Let* $A$ *be a self-adjoint bounded linear operator in a Hilbert space* $H$ *and let* $a$ *be a positive real number such that the intersection*

$$\operatorname{spec} A \cap ]-a, a[$$

*consists only of a finite system of eigenvalues*

$$-a < \lambda_k \leq \lambda_{k+1} \leq \dots < 0 \leq \lambda_0 \leq \dots \leq \lambda_m < a$$

*(all eigenvalues repeated according to their multiplicity). Then for all operators* $A'$ *sufficiently close to* $A$ *the intersection*

$$\operatorname{spec} A' \cap ]-a, a[$$

*consists also of the same number of eigenvalues*

$$-a < \lambda'_k \leq \lambda'_{k+1} \leq \dots \leq \lambda'_m < a$$

*and one has*

$$| \lambda_j - \lambda'_j | \leq \| A - A' \|$$

*for each* $j \in \{k, .., m\}$.

PROOF. We first decompose $A$ into the difference $A_+ - A_-$ of two non-negative operators (see [21, §32]). $A_+$ has the eigenvalues $\lambda_0, \ldots, \lambda_m$ and $A_-$ has the eigenvalues $-\lambda_k, \ldots, -\lambda_{-1}$. Since $\|A_\pm - A'_\pm\| \leq \|A - A'\|$ it suffices to consider the case when $A$ is non-negative.

We recall briefly that then the jth eigenvalue $\lambda_j$ can be characterized by the minimum-maximum principle (cf. [17, §27]) through

$$\lambda_j = \sup\{U_A(v_1, \ldots, v_j) \mid v_1, \ldots, v_j \in H\}$$

where

$$U_A(v_1, \ldots, v_j) := \inf\{\langle Aw, w\rangle \mid \|w\| = 1 \text{ and } w \in \operatorname{span}(v_1, \ldots, v_j)^\perp\}.$$

Now, for each choice of $v_1, \ldots, v_j$ and for each $\varepsilon > 0$ we choose a vector $w$ with $\|w\| = 1$, orthogonal to $v_1, \ldots, v_j$ such that

$$\langle Aw, w\rangle - U_A(v_1, \ldots, v_j) < \varepsilon.$$

It follows that

$$\begin{aligned} U_{A'}(v_1, \ldots, v_j) &\leq \langle A'w, w\rangle \\ &= \langle Aw, w\rangle + \langle (A'-A)w, w\rangle \\ &\leqq U_A(v_1, \ldots, v_j) + \|A-A'\| + \varepsilon, \end{aligned}$$

hence

$$U_{A'}(v_1, \ldots, v_j) \leqq U_A(v_1, \ldots, v_j) + \|A-A'\|.$$

In the same way we get

$$U_A(v_1, \ldots, v_j) \leq U_{A'}(v_1, \ldots, v_j) + \|A-A'\|,$$

which proves the lemma. □

NOTE. The minimum-maximum properties of eigenvalues were derived and employed to study the effect which a change of an operator has on its eigenvalues first by Weyl (1912) and more widely later by Courant for integral equations of the second kind ("essentially positive operators" in our terminology) and several vibration problems of mathematical physics (see [16, ch. III and VI].

The continuity of a finite system of eigenvalues can be obtained for all closed not necessarily bounded nor self-adjoint operators as shown in [24, IV.3.5]. In the case of self-adjoint elliptic operators of positive order over a closed manifold we get the continuity of the eigenvalues immediately from lemma 1.4. As usual operators of order $r$ can be reduced to operators of order $0$ by the substitution

$$B \mapsto (1 + B^2)^{-1/2} \circ B =: \hat{B}.$$

More precisely, let $\{\lambda_j, e_j\}_{j \in \mathbb{Z}}$ be a spectral decomposition of the Hilbert space of square Lebesgue-integrable sections in a hermitian vector bundle $E$ generated by $B$, i.e. the $\{\lambda_j\}$ are the eigenvalues of

$$B : H^r E \to L^2 E,$$

where $H^r E$ is the rth Sobolev space, with the corresponding eigenfunctions $\{e_j\}$ spanning the whole $L^2 E$. Then

$$\{\lambda_j/(1 + \lambda_j^2)^{1/2} \; ; \; e_j\}_{j \in \mathbb{Z}}$$

is a spectral decomposition of $L^2 E$ generated by $\hat{B}$ with all eigenvalues bounded by $\pm 1$. Hence the continuity of the eigenvalues of $\hat{B}$ carries over to the eigenvalues of $B$.

The following lemma is of independent interest:

1.5. LEMMA. *The topological non-trivial component $\hat{F}_*$ of the space $\hat{F}$ of bounded self-adjoint Fredholm operators can be deformed into the space*

$$\hat{F}(\infty) := \{A \in \hat{F} \mid \mathrm{spec}(A) \text{ is a finite subset of } [-1,1] \text{ and the essential spectrum } \sigma_{ess}(A) \text{ is equal to } \{-1,1\}\}.$$

*where*

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda\, \mathrm{id} \notin F\}.$$

PROOF. Let $A \in \hat{F}_*$. We deform first $A$ into an operator $\tilde{A} \in \hat{F}_*$ with

$$|\inf \sigma_{ess}(\tilde{A})| = 1$$

through the linear homotopy

$$A_t := ((1-t) + t|\inf \sigma_{ess}(A)|)^{-1} A.$$

Then we choose a symmetric deformation retraction $r_t$ of the real axis onto the closed interval $[-1,1]$. Then the map $A \to r_1(\tilde{A})$ yields a deformation retraction of $\hat{F}_*$ onto

$$\hat{F}_* := \{A \in \hat{F}_* \mid \|A\| = 1 \wedge \sigma_{ess}(A) = \{1,-1\}\}.$$

Next we introduce the spaces

$$\hat{F}(m) := \{A \in \hat{F}_* \mid \mathrm{spec}(A) \cap\, ]-1,1[\, = \{\lambda_1,..,\lambda_k\},\ k \leq m\}$$

for arbitrary $m \geq 1$.

Let $B \in \hat{F}(m)$, $\mathrm{spec}(B) \cap\, ]-1,1[\, = \{\lambda_1,..,\lambda_m\}$.

We choose an $\varepsilon > o$ such that

$$-1 + \varepsilon \lesseqgtr \lambda_1 \lesseqgtr \ \ldots \ \lesseqgtr \lambda_m \lesseqgtr 1 - \varepsilon .$$

Let $h_t$ be a deformation of the interval $[-1,1]$ which shrinks $[-1, -1 + \varepsilon/2]$ onto $\{-1\}$ and $[1 - \varepsilon/2, 1]$ onto $\{1\}$ and which is the identity on $[-1+\varepsilon, 1-\varepsilon]$. Then $h_t$ induces a deformation $\hat{F}_* \to \hat{F}_*$ given by $A \mapsto h_1(A)$ with the following properties:

(i) $h_t(B) = B$ for all $t \in [0,1]$.

(ii) For any compact $X \subset \hat{F}_*$ there exists an integer $n$ such that $h_1(X) \subset \hat{F}(n)$. In fact, if $\|A_1 - A_2\| < \varepsilon/2$ then $h_1(A_1)$, $h_1(A_2)$ lie in the same space $\hat{F}(n)$ because their eigenvalues differ less than $\varepsilon/2$ by lemma 1.4.

This proves that the inclusion maps $\hat{F}(m) \subset \hat{F}_*$ induce bijections

$$\pi_k(\hat{F}(m), B) \approx \pi_k(\hat{F}_*, B)$$

for all $k \geq 0$, for any choice of the "base point" $B \in \hat{F}(m)$, and for $m$ sufficiently large.

Since $\hat{F}(\infty) = \lim_{m\to\infty} \hat{F}(m)$ lemma 1.5 follows by lemma 1.3. □

For $A \in \hat{F}(\infty)$ we have an orthonormal basis $\{e_j\}_{j \in \mathbb{Z}}$ of $H$ such that

$$A = \sum_{j=m+1}^{\infty} p_j - \sum_{-\infty}^{k-1} p_j + \sum_{j=k}^{m} \lambda_j p_j$$

where

$$p_j : H \to \operatorname{span}\{e_j\}$$

are the orthogonal projections.

Let $V := \operatorname{span}\{e_k, e_{k+1}, \ldots, e_m\}$, then we have

$$-\exp(i\pi A) = \begin{cases} \text{id} & \text{on } V^{\perp} \\ \sum_{j=k}^{m} (e^{-i\pi\lambda_j}) p_j & \text{on } V . \end{cases}$$

This operator is an element of the topological non-trivial subgroup

$$\tilde{U}(\infty) := \{U \mid U \text{ unitary and id}-U \text{ has finite rank}\}$$

of the contractible group $\mathcal{U}(H)$ of all unitary operators, and we get:

**1.6. LEMMA.** *The map*

$$-\exp i\pi : \hat{F}(\infty) \to \tilde{U}(\infty)$$

*given by*

$$A \to -\exp i\pi A$$

*is a homotopy equivalence.*

The proof is not very difficult but laborious. All details can be found in [ 5 ].

Since $\tilde{U}(\infty)$ is homotopy equivalent to the space $U(\infty) := \operatorname{inj\,lim} U(n)$ (see [28]), lemma 1.6 completes the proof of proposition 1.2.

**1.7. DEFINITION.** Let $A : P \to \hat{F}_*$ be a family of self-adjoint Fredholm operators parametrized over a compact topological space $P$. The *analytical index* of the family is the homotopy class of this map - interpreted by proposition 1.2 as an element of the group $K^{-1}(P)$:

$$a\text{-index } A := [\alpha \circ A] \in K^{-1}(P)$$

NOTE. We can describe a-index A explicitly in terms of vector bundles. Let $A$ be deformed through the above explained homotopies into a continuous mapping

$$g : P \to U(n).$$

Then we have

$$\text{a-index}(A) = [P\times\mathbb{C}^n, g] := [E^g] - [S^1 \times P \times \mathbb{C}^n],$$

where

$$E^g := I \times P \times \mathbb{C}^n/\sim$$

is a vector bundle over $S^1 \times P$ which is defined by the identification

$$(1, p, e) \sim (0, p, ge) ; \quad p \in P, \quad e \in \mathbb{C}^n.$$

Thus we defined an element a-index $A \in K(S^1 \times P)$ whose restriction to $\{0\} \times P$ is trivial and so can be regarded as an element of the group

$$K^{-1}(P) = K((S^1 \times P)/P).$$

Recall the ring homomorphism ("Chern character")

$$ch : K(P) \to H^{even}(P;\mathbb{Q}) = \bigoplus_{i=o}^{\infty} H^{2i}(P;\mathbb{Q})$$

defined by

$$ch([E]-[F]) = ch(E) - ch(F)$$

where

$$ch(E) = \sum_{i=1}^{n} e^{x_i} = n + c_1 + \frac{1}{2}(c_1^2 - 2c_2)+\dots$$

is expressed as a polynomial in the Chern classes of the vector bundle $E$ of fibre dimension n, cf. [23, V.3]. As explained there we may extend ch by a homomorphism, also called Chern character

$$ch : K^{-1}(P) \to H^{odd}(P;\mathbb{Q}).$$

Let $\alpha \in K^{-1}(P)$ be represented by

$$\alpha = [P \times \mathbb{C}^n, g]$$

with a continuous map

$$g : P \to U(n)$$

or more generally

$$\alpha = [E,h]$$

where $E$ is a vector bundle over $P$ and $h$ an automorphism of $E$. Then

$$ch(\alpha) = ch(E^h) - ch(S^1 \times E)$$

defines an element in $H^*(S^1 \times P)$ which actually belongs to

$$H^{even}(I \times P, \{0,1\} \times P) \cong H^{odd}(P)$$

(by Thom isomorphism).

If $P = S^1$ all bundles are trivial, i.e. every $\alpha \in K^{-1}(S^1)$ has the form

$$\alpha = [S^1 \times \mathbb{C} , e^{ikx}] , \quad k \in \mathbb{Z}.$$

Hence we get

$$ch\ \alpha = k.$$

This proves the following

<u>1.8. LEMMA.</u> *The Chern character for* $P = S^1$ *is an isomorphism*

$$ch : K^{-1}(S^1) \overset{\cong}{\to} H^1(S^1) \cong \mathbb{Z}$$

*with*

$$\deg(g) = (ch\ [g])[S^1]$$

*for all continuous maps*

$$g : S^1 \to U(n)$$

where $\deg(g)$ is the winding number of $g$, which is defined by first deforming

$$g \sim \begin{pmatrix} h & o \\ o & id_{n-1} \end{pmatrix}$$

with $h : S^1 \to U(1) = S^1$ and then taking

$$\deg(g) := \text{mapping degree } (h),$$

$[S^1] \in H_1(S^1)$ is the fundamental class and

$$[g] := [S^1 \times \mathbb{C}^n, g] = [E^g] - [S^1 \times S^1 \times \mathbb{C}^n].$$

1.9. COROLLARY. We have a commutative diagram of isomorphisms

$$\begin{array}{ccc} \pi_1(\hat{F}_*) & \underset{-\exp i\pi}{\cong} & \pi_1(S^1) \\ \text{a-index} \wr\| & & \deg \wr\| \\ K^{-1}(S^1) & \underset{ch}{\cong} & \mathbb{Z} \end{array}$$

More precisely,

let $A_t$ be a family of self-adjoint Fredholm operators parametrized by $t \in S^1$. Then we have

$$\deg(-\exp i\pi A_t)_{t\in S^1} = ch \text{ a-index } \{A_t\}_{t\in S^1} [S^1].$$

Without proof we give the following generalization of lemma 1.8 to arbitrary smooth manifolds as parameter space (cf.[ 37]):

1.10. LEMMA. *Let* $P$ *be a smooth manifold,* $E$ *a smooth vector bundle over* $P$ *and* $h : E \to E$ *an automorphism. Then:*

$$\mathrm{ch}[E,h] = \sum_{k=1} \mathrm{ch}_k(h) \in \bigoplus_{k=1} H^{2k-1}(P;\mathbb{R})$$

*with*

$$\mathrm{ch}_k(h) = \frac{1}{(2\pi i)^k} \frac{1}{k!} \left( \sum_{m=0}^{k-1} \frac{(-1)^m}{m+1} \binom{k-1}{m} \right) \mathrm{tr}\, \omega^{2k-1}$$

*where*

$$\omega = h^{-1}\, dh.$$

Corollary 1.9 gives two different characterizations of the integer valued homotopy invariant of a continuous family

$$A : S^1 \to \hat{F}_* :$$

a *topological* definition by the winding number of a continuous map

$$g : S^1 \to U(n)$$

obtained from $A$ by deformations, and a *differential* definition by the Chern character of the analytical index. Rather surprisingly, it turns out that there exists also a *spectral* definition of this homotopy invariant:

1.11. DEFINITION. *Let*

$$A : S^1 \to \hat{F}(\infty)$$

*be a continuous map, then the graph of the spectrum of* $A$ *is given by a finite monotone sequence of continuous functions*

$$-1 \leq j_0 \leq j_1 \leq \ldots \leq j_m \leq 1 \ , \quad j_i : I \to [-1,1],$$

i.e.

$$\text{spec } A_t = \{j_0(t), j_1(t), \ldots, j_m(t)\}, \quad t \in I.$$

Here we have parametrized the circle $S^1$ by $t \in I$.

Let $\ell$ be the integer such that for all $k$

$$j_k(1) = j_{k+\ell}(0),$$

where we define

$$j_{-\ell} = j_{-\ell+1} = \ldots = j_0 = -1$$

and

$$j_{m+\ell} = j_{m+\ell-1} = \ldots = j_m = 1.$$

The number $\ell$ is called the spectral flow of the family A.

1.12. PROPOSITION. The spectral flow defines an isomorphism

$$sf : \pi_1(\hat{F}_*) \to \mathbb{Z}.$$

PROOF. First we show that the spectral flow is a homotopy invariant of the families. Let $A = \{A_t\}$ and $B = \{B_t\}$ be two periodic families parametrized by $t \in I$ which can be deformed into each other, i.e. there exists a two-parameter family $\{F_{t,u}\}_{t,u \in I}$ such that

$$F_{t,0} = A_t \ , \ F_{t,1} = B_t \ , \ \text{and} \ F_{0,u} = F_{1,u} \ \text{for each } u.$$

Let $C = \{C_s\}_{s \in I}$ be the boundary of $\{F_{t,u}\}$, e.g.

$$C_s = \begin{cases} A_{4s} & 0 \leq s \leq 1/4 \\ F_{1,4s-1} & 1/4 \leq s \leq 1/2 \\ B_{3-4s} & 1/2 \leq s \leq 3/4 \\ F_{0,4-4s} & 3/4 \leq s \leq 1 \end{cases}$$

Then

$$\text{sf } C = \text{sf } A - \text{sf } B$$

and $C$ can be deformed into a constant family.

Let $\{G_{t,u}\}_{t,u\in I}$ be such a two-parameter family with $G_{t,o} = C_t$, $G_{t,1} = \tilde{C}$, $G_{0,u} = G_{1,u}$, let $m$ be the spectral flow of the family $\{G_{o,u}\}_{u\in I}$, and let $j_k(t,u)$ be the sequence of functions parametrizing the spectrum. Then

$$j_k(0,0) = j_{k-m}(0,1) = j_{k-m}(1,1) = 1_j(1,0),$$

hence $\text{sf } C = 0$.

Now it is clear that the spectral flow defines a homomorphism

$$\text{sf} : \pi_1(\hat{F}_*) \to \mathbb{Z}.$$

We show that it is in fact an isomorphism. We construct a periodic family with spectral flow equal to $1$: Let $\{e_k\}_{k\in\mathbb{Z}}$ be an orthonormal basis of $H$. Take

$$A_t := \sum_{k=1}^{\infty} (p_k - p_{-k}) + (2t-1)p_o$$

where $p_k$ is the orthogonal projection onto $\text{span}\{e_k\}$ and hence

$$\text{spec } A_t = \{-1, 2t-1, 1\}, \quad t \in I. \quad \square$$

1.13. COROLLARY. *Let* $A : S^1 \to \hat{F}_*$ *be a continuous family. Then:*

$$\mathrm{sf}\ A \;=\; \deg(-\exp i\pi A) \;=\; \mathrm{ch}(\text{a-index } A)\ [S^1].$$

PROOF. The last equation was obtained in corollary 1.9. By proposition 1.12 it follows that sf(A) and deg(-exp $i\pi A$) coincide up to sign. Consider again the "test" family

$$A_t = \sum_{k=1}^{\infty} (p_k - p_{-k}) \;+\; (2t-1)p_o \;, \quad t \in I$$

from the proof above. Then we get

$$-\exp i\pi\, A_t(e_k) = \begin{cases} e_k & \\ & \text{for} \\ -e^{i\pi(2t-1)}\, e_k & \end{cases} \qquad \begin{matrix} k \neq o \\ \\ k = o \end{matrix}$$

which is homotopically equivalent to the map

$$g : S^1 \to U(1)$$

with

$$g(t) \;=\; -e^{i\pi(2t-1)} .$$

Since

$$\deg g = \deg e^{i2\pi t} \;=\; 1$$

the sign is clarified. □

Without proof we give the following generalization, see [37]:

1.14. PROPOSITION. *Let* $\{A_p\}_{p\in P}$ *be a family of self-adjoint Fredholm operators parametrized over* $p$ *in a compact space* $P$. *Denote by* $\mathrm{sf}\,A$ *the singular 1-cochain given on any singular 1-simplex*

$$f : S^1 \to P$$

*by the formula*

$$\mathrm{sf}\,A\ [f] := \mathrm{sf}\{A_{f(t)}\}_{t\in S^1}.$$

*Then we have in* $H^1(P;\mathbb{R})$

$$\mathrm{sf}\,A = ch_1(\text{a-index } A).$$

NOTE. Proposition 1.12. proves theorem 1.1 by the reduction argument given in the note after lemma 1.4. It turns out that the spectral flow of a family of self-adjoint elliptic operators doesn't change when we change the Hermitian or Riemannian structure. This is due to the fact that the corresponding $L^2$-spaces are equivalent or, more precisely, that all finite dimensional eigenspaces are subspaces of $C^\infty$-sections.

Before we proceed further with the spectral analysis of periodic families of self-adjoint elliptic differential operators it might be appropriate to comment upon the situation.

In principle, there are two main directions in spectral analysis. One direction of research is restricted to positive self-adjoint operators like $\sqrt{\Delta}$ of Riemannian geometry, see [20]+). This direction of research exploits the extreme asymmetry of the spectrum. The second direction focuses on the asymmetry itself of the spectrum of self-adjoint not necessarily positive operators. From the point of view of geometry the second approach is more flexible since natural operators such as the signature operator take the form of self-adjoint operators near any submanifold of codimension 1 with the spectrum on both sides of the real line, c.f. [3], [4]. In fact, there exists a whole theory on measuring the asymmetry of the spectrum of one single self-adjoint operator, the so-called "$\eta$-function" and "$\eta$-invariant", see [19], [36].

The spectral flow measures only the change of the spectral asymmetry of a family of self-adjoint operators and is thus less intricate than the $\eta$-function. It is related to the $\eta$-function by several formulas (cf. [4, p.90 and 93]).

The significance of the spectral flow is indicated by proposition 1.12, namely that the vanishing of the spectral flow is a necessary and sufficient criterion for the possible deformation of a periodic family of self-adjoint Fredholm operators (or elliptic operators) into the constant family.

+) See also Duistermaat, J. & Guillemin, V., The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. 29 (1975), 39-79.

Now we want to proceed with the topological study of self-adjoint elliptic operators. Recall that the homotopy class of an elliptic operator $B$ over a closed manifold $Y$ depends only on the homotopy class of its leading symbol $\sigma_B$. For any such symbol the usual theory of elliptic symbols constructs a "symbol class" $[\sigma_B]$ which is an element of $K(TY)$ and which leads to the computation of the index of $B$ by the "topological index" homomorphism $K(TY) \to \mathbb{Z}$ (see e.g. [12]).

If $B$ is a self-adjoint elliptic operator over $Y$ its homotopy class depends again only on the homotopy class of its leading symbol, i.e. $B$ can be deformed into another self-adjoint elliptic operator $B'$ through a smooth 1-parameter family $\{B_t\}$ of self-adjoint elliptic operators if and only if $\sigma_B$ can be deformed into $\sigma_{B'}$ in the class of self-adjoint and (for each non-zero cotangent vector) invertible symbols. This is a standard exercise. However, the usual symbol class in $K(TY)$ is vanishing. In [4, §3] it was found out that the characteristic element of a self-adjoint elliptic operator lies in the group $K^{-1}(TY)$.

<u>1.15. DEFINITION.</u> (a) *Let $B$ be a self-adjoint elliptic operator acting on the $C^\infty$-sections of a Hermitian vector bundle $E$ over a closed $C^\infty$-manifold $Y$ and let*

$$\sigma_B(y,\eta) : E_y \overset{\cong}{\to} E_y , \quad y \in Y, \quad \eta \in TY_y \setminus 0$$

be the principal symbol of B. We consider the family

$$\tilde{\sigma}_t(y,\eta) = \begin{cases} \cos t \, id + i \sin t \, \sigma_A(y,\eta) & 0 \leq t \leq \pi \\ & \text{for} \\ e^{it} \, id & \pi < t < 2\pi \end{cases}$$

of elliptic symbols over $Y$ parametrized by a point $t$ on the circle $S^1$. This gives an automorphism of the bundle $\pi^*E$ where

$$\pi : S^1 \times SY \to Y$$

is the projection ($SY \subset TY$ is the sphere bundle of cotangent vectors) and hence defines an element $[\tilde{\sigma}_B] \in K(S^1 \times TY)$. Its restriction to $\{0\} \times TY$ is trivial and so $[\tilde{\sigma}_B]$ can be regarded as an element of $K^{-1}(TY)$. It will be called the stable symbol class of B. (Compare §2 and [4, p.80] for an alternative characterization of $[\tilde{\sigma}]$ involving a standard "stability" argument).

(b) For a family $B = \{B_p\}_{p \in P}$ with a compact parameter space $P$ we obtain similarily a 2-parameter family of elliptic symbols $\tilde{\sigma}_{p,t}$ over $Y$ parametrized by $p \in P$ and $t \in S^1$ and hence an element in $K^{-1}(P \times TY)$, the stable symbol class $[\tilde{\sigma}_B]$ of the family B.

1.16. LEMMA. Let $B = \{B_t\}_{t\in S^1}$ be a family of elliptic self-adjoint first order operators acting on sections of a Hermitian bundle $E$ over a closed manifold $Y$ and let

$$\tilde{B} := \{-\frac{\partial}{\partial t} + B_t\}_{t\in S^1}$$

be the associated first order elliptic operator over the "torus" $S^1 \times Y$ acting on sections $u(t,y)$ of the vector bundle $\rho^*E$ ($E$ lifted to $S^1 \times Y$ be the canonical projection $\rho$). Then we have

$$K^{-1}(S^1 \times TY) \ni [\tilde{\sigma}_B] = [\sigma_{\tilde{B}}] \in K(T(S^1 \times Y)).$$

PROOF. Clearly $\tilde{B}$ is elliptic but, in general, not self-adjoint since its formal adjoint is

$$\tilde{B}^* = \{\frac{\partial}{\partial t} + B_t\}_{t\in S^1} .$$

Since the tension space of $S^1 \times Y$ is diffeomorphic to the product $S^1 \times \mathbb{R} \times TY$ we get a natural isomorphism

$$K(T(S^1 \times Y)) \cong K(S^1 \times \mathbb{R} \times TY) \cong K^{-1}(S^1 \times TY).$$

Now we consider

$$\sigma_{\tilde{B}}(t,y;\tau,\eta) = -i\tau + \sigma_{B_t}(y,\eta),$$

where $t \in S^1$, $y \in Y$, $(\tau,\eta) \in T(S^1 \times Y)$ with

$$|\tau|^2 + |\eta|^2 = 1.$$

Since $B_t$ is of first order and hence $\sigma_{B_t}$ homogeneous in $\eta$ we can rewrite

$$\sigma_{\tilde{B}}(t,y;\tau,\eta) = -i(\tau + i|\eta|\sigma_{Bt}(y,\frac{\eta}{|\eta|}))$$

$$= -i(\cos r + i\sin r\,\sigma_{B_t}(y,\frac{\eta}{|\eta|}))$$

by substituting $\tau = \cos r$ for $o \leq r \leq \pi$.
As usual we join $-i$ with the identity and so we get a homotopy between the clutching isomorphism

$$-i\tau + \sigma_{B_t}(y,\eta) \quad ; \quad \tau \in S^1,\ t \in S^1,\ (y,\eta) \in SY$$

which defines the symbol class $[\sigma_{\tilde{B}}]$ of the operator $\tilde{B}$ and the clutching isomorphism

$$\left.\begin{array}{l} \cos r + i \sin r \ \sigma_{B_t}(y,\eta) \\ e^{ir} \end{array}\right\} \text{for} \quad \begin{array}{l} 0 \leq r \leq \pi \\ \pi \leq r \leq 2\pi \end{array}$$

where $r \in S^1$, $t \in S^1$, $(y,\eta) \in SY$, which defines the stable symbol class $[\tilde{\sigma}_B]$ of the family $B$. Since glueing by homotopic isomorphisms gives the same classes in K-theory the lemmma is proved. □

As usual in index theory we want to relate the analytical index which is obtained globally, namely from the space of "solutions", with the symbol class which is obtained locally, namely from the "coefficients" of the differential equations involved. In the case of one single elliptic operator $B$ over a closed manifold $Y$ this relation is given by the Atiyah-Singer index theorem (c.f. [12])

$$\text{a-index } B = \text{t-index } [\sigma_B]$$

where

$$\text{a-index } B := [\ker B] - [\operatorname{coker} B] \in K(\text{point})$$

and

$$\text{t-index} \quad : \quad K(TY) \to K(\text{point}) \cong \mathbb{Z}$$

the topological index given by an embedding $j : Y \to \mathbb{R}^m$, the excision of a tubular neighborhood $N$ of $j(Y)$ and the Bott periodicity $K(T\mathbb{R}^m) \simeq K(\mathbb{R}^{2m}) \simeq K(\text{point})$.

In the language of characteristic classes this becomes

$$\text{t-index}[\sigma_B] = (-1)^{n-1}(\varphi\, ch[\sigma_B] \cup T(Y))[Y]$$

where it is supposed that $Y$ is oriented, $[Y] \in H_n(Y)$ the fundamental cycle of the orientation, $n$ the dimension of $Y$, $T(Y) \in H^*(Y)$ the Todd class of $Y$, ch the Chern character $K(TY) \to H^{even}_{comp}(TY;\mathbb{Q})$ and $\varphi : H^{even}_{comp}(TY;\mathbb{Q}) \to H^*(Y)$ the Thom isomorphism.

More generally, a family $B = \{B_p\}_{p\in P}$ of elliptic operators parametrized by a compact space $P$ has a "topological index" in $K(P)$ which is computed from the symbol class $[\sigma_B]$ by a homomorphism

$$\text{t-index} : K(P \times TY) \to K(P).$$

This is explained in [ 6 ].

Of course one has to make the appropriate changes in the definition in order to get a meaningful topological index for families of self-adjoint elliptic operators.

<u>1.17. DEFINITION.</u> Let $B = \{B_p\}_{p\in P}$ be a family of self-adjoint elliptic operators over a closed manifold $Y$, parametrized by a compact space $P$ with stable symbol class.

$$[\tilde{\sigma}_B] \in K^{-1}(P \times TY) \subset K(S^1 \times P \times TY).$$

Applying the topological index we get an element in $K(S^1 \times P)$ which lies in fact in $K(S^1 \times P/P) \simeq K^{-1}(P)$. Hence we have a (stable) topological index

$$\text{t-index} : K^{-1}(P \times TY) \to K^{-1}(P) .$$

1.18. PROPOSITION. *Let* $B = \{B_p\}_{p\in P}$ *be a family of self-adjoint elliptic operators over a closed manifold* $Y$ *parametrized by a compact space* $P$ *and let* a-index $B \in K^{-1}(P)$ *be its analytical index (which is defined as in 1.7). Then we have*

$$\text{a-index } B = \text{t-index } [\tilde{\sigma}_B].$$

PROOF. The proposition follows from the corresponding index theorem for families of elliptic operators (cf. [6,th.3.1]) by factorization.

1.19. THEOREM. *Let* $\{B_t\}_{t\in S^1}$ *be a family of self-adjoint elliptic operators of first order over a closed Riemannian manifold* $Y$ *parametrized by* $t \in S^1$. *Then*

$$\text{sf } B = \text{index } \tilde{B}$$

*where* $\tilde{B} = \{-\frac{\partial}{\partial t} + B_t\}$ *is the elliptic operator on* $Y \times S^1$ *naturally associated to the family.*

PROOF. By corollary 1.13 we have

$$\text{sf } B = (\text{ch a-index } B)\,[S^1],$$

where $[S^1]$ is the fundamental cycle of $S^1$ in standard orientation and a-index $B \in K^{-1}(S^1 \times Y)$ the analytical index of $B$ which equals its topological index by proposition 1.18, hence

$$\text{sf } B = (\text{ch t-index } [\tilde{\sigma}_B])[S^1].$$

Recall now the "double" character of $[\tilde{\sigma}_B] = [\sigma_{\tilde{B}}]$ by lemma 1.16 and consider the corresponding diagram

$$\begin{tikzcd} K^{-1}(S^1 \times TY) \arrow[r, "t\text{-index}"] \arrow[d, "\cong"'] & K^{-1}(S^1) \arrow[dr, "ch"] & \\ K(S^1 \times \mathbb{R} \times TY) \arrow[r, phantom, "\cong"] & K(T(S^1 \times Y)) \arrow[r, "t\text{-index}"'] & \mathbb{Z}. \end{tikzcd}$$

The diagram is commutative (check the commutativity for the family $\{-i\frac{d}{dt} + a\}_{a \in S^1}$,

hence

$$(ch\ t\text{-index}\ [\tilde{\sigma}_B])\ [S^1] = t\text{-index}\ [\sigma_{\tilde{B}}]$$

which proves the theorem by the index theorem applied to the single operator $\tilde{B}$. □

1.20. COROLLARY. *Under the assumptions of the proceeding theorem we get the following topological formula for the spectral flow:*

$$sf\ B = (-1)^n \int_{SY} ch[\sigma_{\tilde{B}}]\ \tau(Y),$$

*where* $\dim Y = n-1$.

We close this chapter by a series of examples. Some of them were already mentioned earlier in some proofs.

1.21. EXAMPLES. (a) Let $H$ be a complex separable Hilbert space and let $P_+, P_-$ be complementary projections, i.e. $P_+ + P_- = id$ with both having infinite dimensional range. Then for any real $a$ and $e \in$ image $P_-$ the operator

$$P_+ - P_- + a\, p_e$$

belongs to $\hat{F}_*$. Here $P_e$ denotes the projection of $H$ onto span{e}. The spectrum of $P_+ - P_- + ap_e$ consists of the essential spectrum $\{-1, 1\}$ and the eigenvalue $\lambda = -1 + a$ of multiplicity 1.

(b) The spectrum of the family

$$B_e = \{P_+ - P_- + 2tp_e\}_{t\in I}$$

is given by the graph

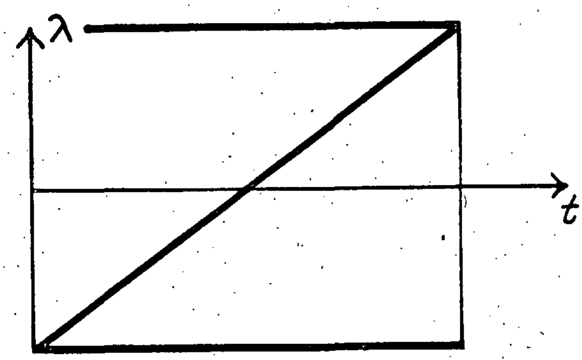


fig.3

and hence $\mathrm{sf}\, B_e = 1$. This family was already considered in the proof of proposition 1.12. Taking $p_U$ instead of $p_e$, where $U$ is a subspace of image $(P_-)$ of dimension $N$ we get the same graph and a spectral flow equal $N$ by multiplicity argument.

(c) Now consider the family $B = \{B_t\}_{t\in S^1}$

$$B_t := -i\frac{d}{dx} + 2\pi t$$

of ordinary differential operators over the circle $S^1$, parametrized by $t \in S^1 = I/\{0,1\}$. We have a spectral decomposition of $H := L^2(S^1)$ by the system $\{e^{ik2\pi x}\}_{k\in\mathbb{Z}}$ of eigenfunctions with corresponding eigenvalues $\{k+t\}_{k\in Z}$, i.e. the spectrum of $B$ is given by the following graph

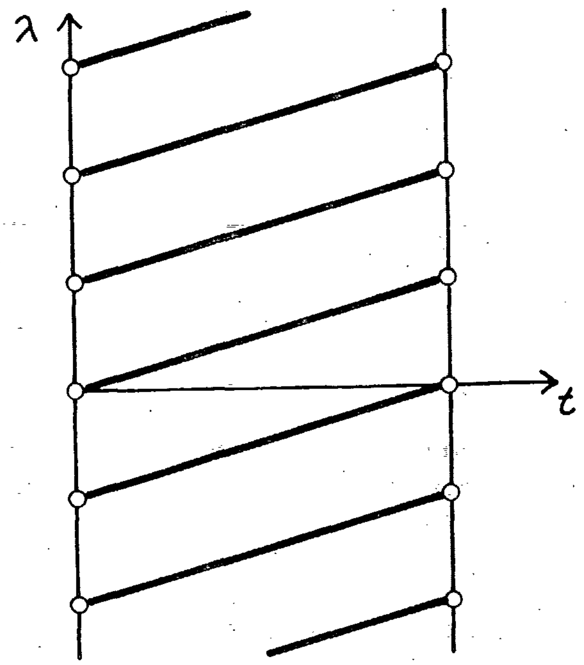


fig. 4

and hence $\mathrm{sf}\, B = 1$.

Note that the reduction

$$B \mapsto B' := (1+B^2)^{-\frac{1}{2}} \circ B$$

leads to an operator $B' \in \hat{F}_*$, which is in fact an element of $\hat{F}_*$ and which can be deformed further into the family $b_e$ of (a) where $P_+, P_-$ are the projections onto $\mathrm{span}\{e^{i2\pi kx}\}$ with $k > 0$ and $k \leq 0$ and $e$ is the constant function equal 1 $(k=0)$. A further analysis of this example is given in 2.4 (a).

(d) Let $\tilde{E}$ be the bundle over the torus $S^1 \times S^1$ which is obtained from the bundle $I \times S^1 \times \mathbb{C}$ by identifying

$$(1, x, z) \sim (0, x, e^{-ix}z) \quad , \quad x \in S^1, \quad z \in \mathbb{C}.$$

Consider the operator

$$A = \frac{\partial}{\partial t} - i\frac{\partial}{\partial x} + t$$

acting on the space

$$C^\infty \tilde{E} \simeq \{f \in C^\infty(I \times S^1) \mid f(1,x) = e^{-i2\pi x} f(0,x) \text{ for all } x \in S^1 = I/\{0,1\}\}$$

Then we have

$$[\sigma_A] = [\tilde{\sigma}_B]$$

where $B$ is the family of ordinary differential operators of (c) and hence by theorem 1.19 (see also example 5.2(b))

$$\text{index } A = \text{sf } B = 1.$$

## 2. Elliptic self-adjoint symbols and spectral projections.

Let $B$ be an elliptic self-adjoint operator of non-negative order acting on the smooth sections of a Hermitian bundle $E$ over a smooth Riemannian manifold $B$. We are going to define an involution $S$ (i.e. $S^2 = \mathrm{id}$) on the Hilbert space $L^2(Y;E)$ which takes the form $P_+ - P_-$ where $P_+, P_-$ are complementary orthogonal (pseudodifferential) projections (i.e. $P_\pm = P_\pm^* = P_\pm^2$, $P_+ + P_- = \mathrm{id}$, $P_+P_- = 0 = P_-P_+$). It turns out that $S$ is a self-adjoint elliptic operator of order zero and its symbol gives the same class in $K^{-1}(TY)$ as the symbol of $B$.

2.1. DEFINITION. First we define the bounded operator (of order zero)

$$\hat{B} := (\mathrm{id} + B^2)^{-\frac{1}{2}} \circ B.$$

As noted in the remark after lemma 1.4 it has a discrete spectrum of the form

$$\{\lambda_j / \sqrt{1 + \lambda_j^2}\}_{j\in\mathbb{Z}}$$

where $\{\lambda_j\}_{j\in\mathbb{Z}}$ are the eigenvalues of $B$. This set is contained in the interval $[-1,1]$, and only $\{\pm 1\}$ may be points of the essential spectrum. Then we can find and orthonormal basis $\{e_j\}_{j\in Z}$ of $L^2(Y;E)$ consisting of eigenfunctions of $B$ (which are also the eigenfunctions of $\hat{B}$)

$$Be_j = \lambda_j e_j, \quad j \in \mathbb{Z}.$$

We denote with $P_{\pm}$ the projections of $L^2(Y;E)$ onto the span$\{e_j\}_{j \gtrless 0}$ respectively. It is well known from spectral theory (cf. e.g. [24, III. 6.5]) that $P_{\pm}$ have the integral representation

$$P_{\pm} = \frac{1}{2\pi i} \int_{\Gamma_{+}} (\hat{B} - \lambda)^{-1} \, d\lambda$$

where (see fig. 5 )

$$\Gamma_{+} := \{(1+\varepsilon)e^{it} - 1\} \;, \quad \Gamma_{-} := \{(1-\varepsilon)e^{it} - 1\}$$

and $\varepsilon$ chosen so small that

$$[\varepsilon, 0[ \; \cap \; \operatorname{spec}(\hat{B}) = \emptyset$$

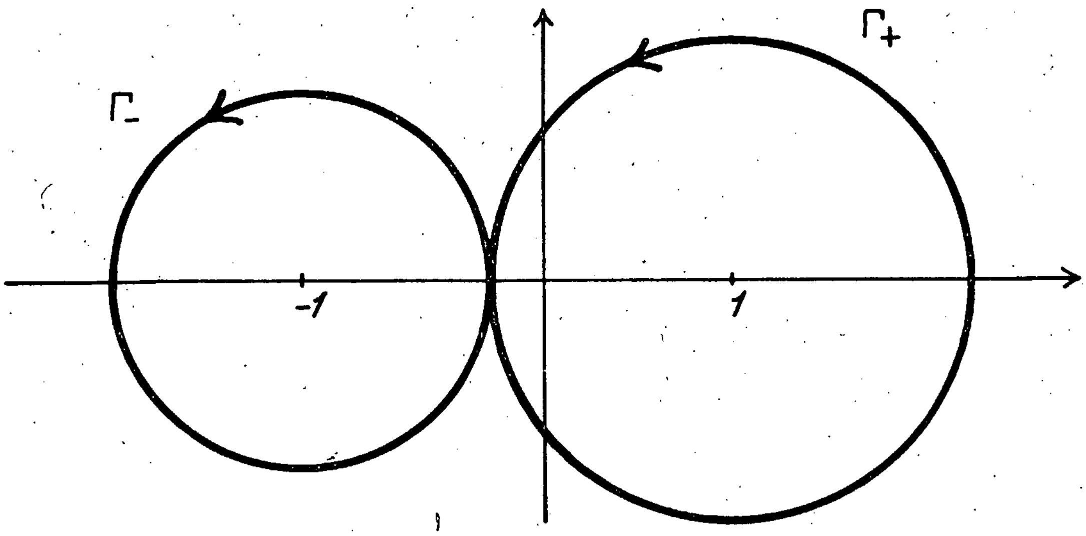


fig. 5

$P_{\pm}$ are called the *spectral projections* of B.

2.2. LEMMA. *Let* B *be an elliptic self-adjoint operator of non-negative order. Then its spectral projections* $P_{\pm}$ *are pseudodifferential operators of order zero and their principal symbols* $p_{\pm}(y,\eta)$ *are the orthogonal projections onto the direct sum of the eigenspaces of the homomorphism*

$$\sigma_L(B)(y,\eta) : E_y \to E_y , \quad y \in Y , \quad \eta \in SY_y ,$$

corresponding to the positive (resp. negative) eigenvalues. Here $\sigma_L(B)$ denotes the principal symbol of $B$.

The following equalities hold:

$$P_+ + P_- = \mathrm{id} , \qquad (P_+ - P_-)^2 = \mathrm{id},$$

$$p_+ + p_- = \mathrm{id} , \qquad (p_+ - p_-)^2 = \mathrm{id},$$

$$p_+ - p_- = \sigma_L(\hat{B}) .$$

PROOF. We start with the construction of the resolvent of $\hat{B} - \lambda$ for

$$\lambda \in \mathbb{C} \setminus \mathrm{spec}(\hat{B}) .$$

In local coordinates $(U,\kappa)$ we have a well defined full symbol

$$\sigma(\hat{B} - \lambda) \sim \sum_{j \geq 0} \sigma_{-j}(\hat{B} - \lambda)$$

where

$$\sigma_0(\hat{B} - \lambda) = \sigma_L(\hat{B}) - \lambda$$

and

$$\sigma_{-j}(\hat{B} - \lambda) = \sigma_{-j}(B) \quad \text{for} \quad j \geq 1.$$

By the standard formal procedure given e.g. in [33,ch.1] or [35,ch.2] we construct a symbol of the parametrix

$$\rho\ (y, \eta\ \lambda) \sim \sum_{j \geq 0} \rho_{-j}(y, \eta, \lambda) .$$

If $f$ is a section of $E$ with support in $U$ we write for $y \in U$

$$P_{\pm} f(y) = \int_{\Gamma_{\pm}} d\lambda \int_{\mathbb{R}^n} e^{iy\eta} \rho(y,\eta,\lambda) \hat{f}(\eta) d\eta .$$

Since

$$\|P_{\pm} f(y)\| \leqslant \frac{1}{2\pi} \int_{\Gamma_{\pm}} d\lambda \int_{R^n} \|\rho(y,\eta,\lambda) \hat{f}(\eta)\| d\eta$$

$$\leqslant C \|\hat{f}\|_{L^1}$$

we can interchange the order of integration and we obtain

$$(P_{\pm} f)(y) = \int_{R^n} e^{iy\eta} \{\frac{1}{2\pi i} \int_{\Gamma_{\pm}} \rho(y,\eta,\lambda) d\lambda\} \hat{f}(\eta) d\eta .$$

Hence $P_{\pm}$ is a pseudodifferential operator and its full symbol is given by the integral in $\{\ldots\}$. More precisely, the full symbol $\tilde{\rho}_{\pm}$ of $P_{\pm}$ is of the form

$$\tilde{\rho}_{\pm} \sim \sum_{j \geqq 0} \tilde{\rho}_{-j}$$

where

$$\tilde{\rho}_{-j}(y,\eta) := \frac{1}{2\pi i} \int_{\Gamma_{\pm}} \rho_{-j}(y,\eta,\lambda) d\eta .$$

It is standard that the $\tilde{c}_{-j}$ are homogeneous of order $-j$ if the symbols $\sigma_j(\hat{B} - \lambda)$ are such.

The principal symbol of $P_{\pm}$ is given by

$$p_{\pm}(y,\eta) := \frac{1}{2\pi i} \int_{\Gamma_{\pm}} (\sigma_L(\hat{B}) - \lambda)^{-1} d\lambda$$

and it is obvious that it is a projection onto the suitable subspace of $L^2(Y;E)$.

For the principal symbol $\hat{\sigma}$ of $\hat{B}$ one finds

$$\hat{\sigma} = (\sigma_L(B)^2)^{-\frac{1}{2}} \circ \sigma_L(B) .$$

So we have

$$\hat{\sigma}^2 = \mathrm{id} \quad \text{and} \quad \hat{\sigma} p_{\pm} = p_{\pm}\hat{\sigma}.$$

From that we get

$$\hat{\sigma} = p_+ - p_- . \quad \square$$

2.3. LEMMA. *Let $B$ be as above an elliptic self-adjoint operator of non-negative order. If the principal symbol $p_+$ (or $p_-$) of its spectral projection $P_+$ (or $P_-$) vanishes then $B$ is half-bounded as operator in $L^2(Y;E)$.*

NOTE. - For an operator $B$ of order zero the conclusion is in fact that the essential spectrum of $B$ lies on one half line of $\mathbb{R}$ only, i.e. $B$ is essentially positive (or essentially negative). This is the case if and only if the principal symbol of $B$ is positive (or negative).

PROOF. Let the principal symbol $p_+$ of the pseudodifferential operator $P_+$ vanish. Then $P_+$ is a pseudodifferential operator of order $-1$ and hence compact as operator in $L^2$. As a projection it has only the eigenvalues $\{0,1\}$. Since $P_+$ is compact the multiplicity of the eigenvalue 1 has to be finite, hence it must have finite dimensional range. This proves that $B$ has only a finite number of positive eigenvalues and hence

$$\langle Bu,u\rangle \leqq \max_{j\in\mathbb{Z}} \lambda_j \langle u,u\rangle$$

for all $u \in L^2(Y;E)$.

If $p_-$ vanishes we have similarily only a finite number of negative eigenvalues and we get

$$<Bu,u> \gtrless \min_{j \in Z} \lambda_j <u,u>. \quad \square$$

Lemma 2.3 shows that $p_\pm$ provides a "measure" for the spectral asymmetry of B. More generally, let $E_\pm$ denote the bundles over SY which consist of the ranges of $p_+$ and $p_-$ . What's happening if the bundles $E_+$ and $E_-$ are pull backs from vector bundles $F_+$, $F_-$ over Y by the projection

$$\pi : SY \to Y,$$

i.e.

$$E_\pm := \text{image } p_\pm = \pi^*(F_\pm)?$$

This is the case when $p_\pm$ are functions of $y \in Y$ alone. Then the situation is more complicated than in lemma 2.3. However we find

$$B = p_+Bp_+ + p_-Bp + (p_+Bp + p_-Bp_+),$$

where the last term

$$p_+ Bp_- + p_- Bp_+$$

is an operator of order $-1$.

Hence up to compact operators the operator B is then a direct sum of half-bounded (essentially positive or essentially negative) operators

$$B \sim \begin{pmatrix} B|F_+ & 0 \\ 0 & B|F_- \end{pmatrix} : C^\infty(Y;E) \to C^\infty(Y;E)$$

with $E \simeq F_+ \oplus F_-$ .

So we see that $B$ is non-trivial from our topological point of view if $p_+$ defines as its image a vector bundle over $SY$ which is not a lifting from $Y$. Then $B$ has infinitely many eigenvalues on both sides of $0$ and it can not be reduced to the sum of half-bounded (essentially positive or essentially negative) operators.

<u>NOTE.</u> - In general the range bundle $E_+$ of $p_+$ is a vector bundle only over a connected component of $SY$ since the dimension of the range of $p_+$ may change when we pass to another component.

We want now to give a topological formulation of our results:

The bundle $E_+$ defines an element of the group $K(SY)$. Neglecting the above discussed topologically trivial case we get in fact an element of the group

$$K(SY)/\pi^*K(Y) \simeq K^{-1}(TY).$$

This last isomorphism can be derived (cf. [ 4, §3] from the Gysin sequence

$$\begin{tikzcd} K^*(TY) \arrow[rr] & & K^*(Y) \cong K^*(BY) \arrow[dl] \\ & K^*(SY) \arrow[ul, "\partial"] & \end{tikzcd}$$

and it follows that the element $\partial[E_+]$ is equal to the stable symbol class defined in section 1.

This proves the following

2.4. PROPOSITION. *The stable symbol class*

$$[\tilde{\sigma}_B] \in K^{-1}(TY)$$

*of an elliptic self-adjoint operator* $B$ *of non-negative order vanishes if and only if the operator decomposes into the sum of half-bounded (essentially positive or essentially negative) operators. This is the case if the principal symbol* $p_\pm$ *of the spectral projection* $P_\pm$ *of* $B$ *depends (modulo deformations within the class of idempotent symbols) only of* $y \in Y$. *If in the contrary* $[\tilde{\sigma}_B] \neq 0$, *then we have*

$$\dim \text{image } P_+ = \infty = \dim \text{image } P_- .$$

2.5. EXAMPLES. (a) Let us consider again the operator

$$-i\frac{d}{dx} : C^\infty(S^1) \rightarrow C^\infty(S^1),$$

cf. above 1.21(d). It provides the simplest example of an operator which is non-trivial from our point of view.

The operator has

$$\{j, e^{ijx}\}_{j \in \mathbb{Z}}$$

as spectral decomposition. The corresponding spectral projections are constructed in [27,ch. XVI]. The symbol of $-i\frac{d}{dx}$ is

$$\sigma(y,\eta) = \eta, \quad y \in S^1, \quad \eta \in S(S^{-1})_y.$$

Apparently we have

$$S(S^1) \simeq S^0 \times S^1,$$

i.e. the cotangent sphere bundle consists of two connected components. It turns out that the bundle $E_+$

is equal to the trivial line bundle over one copy of $S^1$ and zero over the other:

$$E_+ \simeq (S^1_+ \times \mathbb{C}) \cup (S^1_- \times \{0\}).$$

So it is not a true vector bundle and in any case it is not a lifting from $S^1$.

Now we have

$$K^{-1}(TS^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

since

$$K^*(TS^1) \simeq K^*(S^1 \times [-1,1]/S^1 \times \{-1,1\}) \simeq K^*(S^1 \times S^1).$$

We want to construct the stable symbol class of the operator $-i\frac{d}{dx}$ . Following the formulas of section 1 we define the suspension of the symbol of $-i\frac{d}{dx}$ :

$$\tilde{\sigma}(y,\eta,t) := \begin{cases} \cos t + i\,\eta\,\sin t & 0 \leq t \leq \pi \\ \\ e^{it} & \pi \leq t \leq \pi \end{cases}$$

Hence we find

$$\tilde{E} := \underset{t}{S^1} \times \underset{y}{S^1} \times \underset{\eta}{[-1,1]} \times \mathbb{C} \underset{\tilde{\sigma}(y,\eta,t)|S^1 \times SS^1}{\cup} S^1 \times S^1 \times [-1,1] \times \mathbb{C}.$$

For $\eta = -1$ the suspension $\tilde{\sigma}(y, -1, t)$ has vanishing degree, hence we have a trivial clutching at one end. So we may assume

$$\tilde{\sigma}(y, -1, t) = \text{id}.$$

However, for $\eta = 1$ we have

$$\tilde{\sigma}(y, 1, t) = e^{it}$$

which proves that $\tilde{E}$ is a non-trivial bundle over $S^1 \times S^1 \times S^1$. The difference class

$$[\tilde{E}] - [S^1 \times S^1 \times S^1 \times \mathbb{C}]$$

is an element of $K(S^1 \times S^1 \times S^1)$ equal 0 for $t = 0$. So this gives an element of $K^{-1}(S^1 \times S^1)$.

Moreover, we can describe also the Chern character of this element. For this we define a connection on $\tilde{E}$ by the formula

$$D(t, y, \eta) := d + r(\eta)\, i\, dt$$

where $r$ is a smooth real function equal 1 near 1 and equal 0 near $-1$. (We choose such an $r$ in order to avoid singularities at the ends). Then we have

$$\mathrm{ch}(D) = \frac{1}{2\pi}\, dr \wedge dt,$$

so it gives in $H^1(S^1 \times S^1; \mathbb{R})$ an element defined by the form $[dr]$.

(b) The most important example of a non-trivial self-adjoint operator is the boundary signature operator. It is described in all details in [ 3, §4]. We recall only the definition of

$$B : \Omega(Y) \to \Omega(Y)$$

where $\Omega(Y)$ is the graded algebra of all differential forms on a $(2k-1)$-dimensional manifold $Y$. One defines

$$B\varphi := (-1)^{k+q+1} (\varepsilon * d - d*)\varphi$$

where $*$ is the Hodge operator and $\varphi$ is either a $2q$-form (then we take $\varepsilon = 1$) or a $(2q-1)$-form (then we take $\varepsilon = -1$).

The operator $B$ preserves the parity of the forms and commutes with

$$\varphi \mapsto (-1)^q * \varphi ,$$

so

$$B = B^{ev} \oplus B^{odd}$$

and $B^{ev}$ is isomorphic to $B^{odd}$.

The operator $-i\frac{d}{dx}$ from example 2.5(a) is in fact the operator $B^{ev}$ on $S^1$.

It is proved in [ 4, §4] that the stable symbol class $[\sigma_L(B^{ev})]$ of $B^{ev}$ is a generator of $K^{-1}(TY)$. To stress the dependance alone of the principal symbol we write here $[\sigma_L(B^{ev})]$ instead of $[\tilde{\sigma}_{B^{ev}}]$ which was the notation used in section 1. More precisely, $K^{-1}(TY)$ is a module over $K(Y)$ and $[\sigma_L(B^{ev})]$ is a module generator after tensoring with the rationals.

A detailed discussion of these symbols is done in [19] from a slightly different point of view.

## 3. A minimal classifying space for K-theory.

In the preceding sections we met operators of the form

$$S = P_+ - P_-$$

where $P_+, P_-$ are projections on infinite dimensional subspaces $H_+, H_-$ of a Hilbert space H. The remarkable fact is that S belongs to the topologically non-trivial component $\hat{F}_*$ of the space $\hat{F}$ of self-adjoint Fredholm operators whereas $P_+$ and $P_-$ are each on its own topologically uninteresting. From classical operator theory it's elementary (cf. [21]) that in fact each element in $\hat{F}_*$ can be written in such a way or more generally as the difference between two positive operators. Since by proposition 1.2 $\hat{F}_*$ is a classifying space for the functor $K^{-1}$ one can find a representation for each element in $K^{-1}(P)$ by a family of such elementary differences parametrized by the compact space P. In 1.21(b) we have seen an example for such a family with non-vanishing spectral flow.

Now we want to develop a theory of operators which can be written as such differences or as unitary "perturbations" $P_+ - g\,P_-$ of such differences. This leads us out of the category of self-adjoint Fredholm operators back to Fredholm operators with non-vanishing index and to characterizing a family $\{B_t\}_{t \in S^1}$ of self-adjoint Fredholm operators parametrized by $t \in S^1$ by one single Fredholm operator of the form

$P_+ - g\,P_-$. This method of "desuspension" provides an additional link between the geometry of $\hat{F}_*$ and K-theory. The following commutative diagram illuminates the relation between "desuspension" and the technically less advanced method of "suspension" which is given by the map

$$\alpha : \hat{F}_* \to \Omega\, F$$

of proposition 1.2:

$$\begin{tikzcd}[column sep=huge]
& \pi_2(F) \arrow[dd, phantom, "\wr\quad \text{periodicity}"] \\
\pi_1(\hat{F}_*) \arrow[ur, "\text{suspension}"] \arrow[dr, "\text{desuspension}"'] & \\
& \pi_0(F)
\end{tikzcd}$$

Before explaining this in detail (in §4 and with applications to partial differential equations in §5 along the line indicated in 1.21(d)) we give now a systematic treatment of the class of operators which are candidates for the "desuspension".

Let $H$ be a complex separable Hilbert space with complementary projections $P$ and $Q$, i.e.

$$P = P^* = P^2, \quad Q = Q^* = Q^2, \quad P + Q = \mathrm{id},$$

$$PQ = 0 = QP.$$

We assume that the images $PH$ and $QH$ are both infinite dimensional. Let $\mathcal{B}(H)$, $\mathcal{K}(H)$ and $GL(H)$ denote the algebra of bounded operators on $H$, the ideal of compact operators and the group of invertible operators respectively.

Recall the characterization of the space $F(H)$ of Fredholm operators by

$$\begin{aligned} B \in F(H) &\Leftrightarrow \dim \ker B < \infty \\ &\qquad \text{and } \dim \operatorname{coker} B < \infty \\ &\Leftrightarrow \exists\, R \in \mathcal{B}(H)\ BR-\mathrm{id},\ RB-\mathrm{id} \in K(H), \end{aligned}$$

The operator $R$ is called a parametrix for $B$.

From a topological point of view the space $F(H)$ is very interesting since it is, as mentioned above, a classifying space for the functor $K$. Moreover many integer-valued invariants of topology, differential geometry and algebraic geometry as the Euler characteristic, the Hirzebruch signature, the arithmetic genus, some fixed point numbers can be written down explicitly as the index of certain elliptic differential operators canonically associated to the geometric problems (see [12, ch.III]). However, for many concrete topological calculations the space $F(H)$ is too large and not comprehensible. It contains the group $GL(H)$ which of course is nice for calculations but topologically uninteresting because it is contractible by Kuiper's theorem (see [12, I.8]). So, keeping in mind the above representation of Fredholm operators by the invertible elements in the Calkin algebra $\mathcal{B}(H)/K(H)$ one will search a candidate for another representation of the space $F(H)$ among the topological subgroups of $GL(H)$.

This leads us to a definition given by Bojarski [ 9 ] in the context of the analysis of Cauchy data for elliptic partial differential equations which looks quite different from the problem of studying K-theory and the geometry of Fredholm operators. (The inherent relations between these two contexts will be explained below in §5).

<u>3.1. DEFINITION.</u> Let $P, Q$ be complementary projections onto a Hilbert space H. Then $S := P-Q$ is an involution, i.e. $S^2 = Id$ with $S = Id$ on $PH$ and $S = -Id$ on $QH$. We define some spaces of *essentially S-commutative operators*:

$$\mathcal{B}_S := \{B \in \mathcal{B}(H) \mid BS - SB \in \mathcal{K}(H)\}$$

and

$$GL_S := GL(H) \cap \mathcal{B}_S.$$

From the definition it is clear that $\mathcal{B}_S$ is closed under addition and also under composition since

$$(BCS - SBC) - (BSC - BSC) \in \mathcal{K}(H)$$

if

$$CS - SC, \quad SB - BS \in \mathcal{K}(H).$$

Since $P$ and $Q$ are orthogonal projections we get further

$$B^*S - SB^* \in \mathcal{K}(H) \quad \text{if} \quad BS - SB \in \mathcal{K}(H).$$

Since the ideal $\mathcal{K}(H)$ is closed in $\mathcal{B}(H)$ it follows that $\mathcal{B}_S$ is complete and hence a $C^*$-algebra. It is obvious that $GL_S$ is a topological group.

3.2. LEMMA. *For each* $B \in \mathcal{B}_S$ *we have also commutativity with* $P$ *and* $Q$ *modulo compact operators, i.e.*

$$PB - BP \;,\quad QB - BQ \;\in\; K(H).$$

PROOF. We have

$$\begin{aligned} SB - BS &= (P-Q)B - B(P-Q) \\ &= (P-Q)B(P+Q) - (P+Q)B(P-Q) \\ &= 2(PBQ - PBP) \\ &= 2(PBQ(P+Q) - (P+Q)QBP) \\ &= 2(PB - BP) \quad \text{and similarly} \\ &= 2(BQ - QB). \quad \square \end{aligned}$$

The following proposition was observed already by Bojarski (l.c.):

3.3. PROPOSITION. *Let* $P, Q, S$ *be as above. Then we have for each* $g \in GL_S$:

$$Pg-Q,\; gP-Q,\; P-Qg,\; P-gQ \;\in\; F(H).$$

PROOF. We show e.g. that $P - Qg^{-1}$ is a parametrix for $P - gQ$. In fact

$$(P - gQ)\;(P - Qg^{-1}) - \mathrm{id} \;=\; P - g\,Qg^{-1} - P - Q$$

and

$$gQg^{-1} - Q \;=\; (gQ - Qg)g^{-1} \;\in\; K(H).$$

Similarly one finds the parametrices for the other operators. $\square$

Note that proposition 3.3 doesn't remain true for arbitrary $g \in GL(H)$. Take for $g$ e.g. the involution which interchanges the images $PH$ and $QH$. Then the space

$$\ker P - gQ = QH$$

is of infinite dimension.

The kernel and cokernel of the Fredholm operators of proposition 3.3 are given by "twisting" the complementary subspaces $PH$ and $PQ$ by the automorphism $g$. More precisely we have

3.4. LEMMA. *For* $P, Q, S$ *as above and* $g \in GL_S$ *we have*

$$\ker P-gQ = \{u \in H \mid Pu = g\,Q\,u\}$$
$$\simeq \{w \in QH \mid g\,w \in PH\}$$

and

$$\operatorname{coker} P-gQ \simeq \{u \in QH \mid g^* u \in PH\},$$

hence

$$\operatorname{index} P-gQ = \operatorname{tr}\,(g^{-1}Pg - P).$$

PROOF. The first equality is obvious. The determination of coker $P-gQ$ follows from

$$\operatorname{coker} P-gQ \simeq \ker P-Qg^*. \quad \square$$

NOTE. If $g$ is unitary one gets

$$\operatorname{coker} P-gQ \simeq \{u \in QH \mid g^{-1}u \in PH\}$$
$$= \{u \in PH \mid gu \in QH\}.$$

From proposition 3.3 and lemma 3.4 it follows that $GL_S$ has an interesting topological structure:

3.5. THEOREM. *Let* $P, Q$ *be complementary projections and* $S = P - Q$ *as above. Then the topological group* $GL_S$ *is homotopically equivalent to the space* $\mathcal{F}(H)$ *of Fredholm operators in* $H$ *(in fact to* $\mathcal{F}(QH)$*). In particular it is a classifying space for the functor* $K$.

PROOF. We consider the surjection of Banach spaces

$$T : \mathcal{B}_S \rightarrow \mathcal{B}(QH)$$

given by

$$TB := Q\,B\,Q.$$

For each $g \in GL_S$ the operator $Tg$ is a Fredholm operator (on the space $QH$) since $Qg^{-1}Q$ is a parametrix for $Tg = QgQ$. Let

$$T' : GL_S \rightarrow \mathcal{F}(QH)$$

be the restriction of $T$ onto $GL_S$.

$T'$ is surjective, too: Let $F$ be a Fredholm operator on $QH$. We choose an isomorphism $h$ of $PQ\oplus$ ker $F$ onto $PH\oplus$ coker $F$. Let $D$ be the orthogonal complement of ker $F$ in $QH$. Then

$$h \oplus F|D : PH \oplus \ker F \oplus D \rightarrow PH \oplus \operatorname{coker} F \oplus \operatorname{image} F$$

defines an isomorphism from $H$ onto $H$ which commutes with $P - Q$, hence

$$h \oplus F|D \in GL_S$$

and

$$T(h \oplus F | D) = F$$

More precisely, we get

$$T^{-1}(F) \cap GL_S \approx \{h | \ h \ \text{is an isomorphism of} \ PH \oplus \ker F \ \text{onto} \ Ph \oplus \operatorname{coker} F\}.$$

This last space is homeomorphic to $GL(H)$. So we know that

$$T' : GL_S \to F(QH)$$

is a fibration with fibre $GL(H)$.

We are now going to show that this is a principal fibre bundle.

Recall the theorem of Bartle and Graves (see e.g. [7, p.86]) which states that each surjective continuous linear operator from one Fréchet space onto another Fréchet space possesses a right (not necessarily linear) continuous inverse. Hence we have a continuous map

$$R : \mathcal{B}(QH) \to \mathcal{B}_S$$

such that

$$TR = \mathrm{id} | \mathcal{B}(QH).$$

Now let $g \in GL_S$. Then $Tg$ has a neighbourhood $U$ in $F(QH)$ such that there exists a map

$$r : U \to GL_S$$

with the property

$$Tr = \mathrm{id}|U.$$

Take e.g.

$$rF := g - R\,Tg + RF$$

if $\| Tg - F \|$ is small.

This proves the existence of local sections and hence that $T' : GL_S \to \mathcal{F}(QH)$ is a principal fibre bundle with the long exact homotopy sequence (see e.g. [34, §17 ])

$$\ldots \to \pi_j(GL(H)) \to \pi_j(GL_S) \xrightarrow{T'} \pi_j(\mathcal{F}(QH)) \to \pi_{j-1}(GL(H)) \to \ldots$$

Since the fibre $GL(H)$ is contractible by Kuiper's theorem we obtain that the projection $T'$ is a week homotopy equivalence and so by lemma 1.3 a homotopy equivalence. □

3.6. COROLLARY. *The homotopy groups of* $GL_S$ *are*

$$\pi_j(GL_S) = \begin{cases} \mathbb{Z} & \text{for } j \text{ even} \\ 0 & \text{for } j \text{ odd} \end{cases} .$$

PROOF. By the preceding theorem this is an immediate consequence of the well known fact (Atiyah-Jänich and Bott isomorphy) that

$$\pi_j(\mathcal{F}) \simeq K(S^j) \simeq K(\mathbb{R}^j) = \begin{cases} \mathbb{Z} & \text{for } j \text{ even} \\ 0 & \text{for } j \text{ odd} \end{cases} . \quad \square$$

3.7. EXAMPLE. Let $H$ be a complex separable Hilbert space with an orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$ and let $P$ be the projection onto the space generated by $\{e_k\}_{k \geq 0}$ and $Q$ the complementary projection. Let $g$ be the shift operator defined by

$$g\, e_k := e_{k+1}.$$

Then $P - gQ$ is a Fredholm operator with

$$\text{index } P - gQ = 1,$$

since

$$(P - gQ)(e_k) = \begin{cases} e_k & \text{if} \quad k \geqslant 0 \\ e_{k+1} & \phantom{\text{if}} \quad k < 0 \end{cases}$$

hence

$$\ker Pg - Q = \operatorname{span}\{e_0 + e_{-1}\} \quad \text{and} \quad \operatorname{coker} Pg - Q = \{0\}.$$

## 4. Spectral flow as the index of a single operator.

Let $Y$ be a closed smooth Riemannian manifold and $E$ a smooth Hermitian vector bundle over $Y$. Let $\{B_t\}_{t \in I}$ be a smooth family of elliptic self-adjoint operators of non-negative order acting on sections of $E$. We make the following assumptions:

(I) All $B_t$ have the same principal symbol $\sigma_o$.

(II) There exists a unitary automorphism $g$ of the bundle $E$ ($g$ induces the identity in the basis $Y$) such that

$$B_1 = g^{-1} B_o g$$

where

$$(gu)(y) := g(y)(u(y)), \quad u \in C^\infty E, \; y \in Y,$$

defines a zero order operator which will be denoted with the same letter $g$.

Our aim is to construct one single elliptic pseudodifferential operator on $Y$ which depends only on $B_o$ and $g$ and which gives the same topological information (in its index) as the family $\{B_t\}_{t \in I}$ (in its spectral flow).

First we want to comment upon the assumptions made above. Note that the space $\hat{E\ell\ell}_{\sigma_o}$ of all elliptic self-adjoint operators of the same order acting on sections of a fixed Hermitian bundle and having the same principal symbol $\sigma_o$ is convex. This is an important difference to the case of the space $\hat{F}$ of <u>all</u> self-adjoint Fredholm operators which has the topological non-trivial connected component $\hat{F}_*$ as seen above in § 1. Hence every true periodic family in $\hat{E\ell\ell}_{\sigma_o}$ could be deformed in the constant family. In order to get a topologically meaningful family it is necessary to admit that $B_1$ is different from $B_0$. Assumption (II) then admits to reconnect $B_1$ with $B_0 = B_2$ in the whole space $\hat{F}$ via

$$B_{t+1} := g_t^{-1} B_0 g_t \ , \quad t \in [0,1]$$

where $\{g_t\}_{t\in I}$ is a retraction of $g = g_o$ to $id = g_1$ in the contractible group $\mathcal{U}(L^2E)$ of all unitary operators on the Hilbert space $L^2E$ of square Lebesgue integrable sections of $E$. Of course, it is not granted that one can choose a retraction of the operator $g$ to the identity within the subgroup of invertible pseudodifferential operators or even in the group of automorphisms of the vector bundle $E$. In general the path $\{B_{t+1}\}_{t\in I}$ will leave the convex space $\hat{E\ell\ell}_{\sigma_o}$. This explains why we can get a topological non-trivial family by connecting $B_0$ and $B_1$. It is obvious that the spectral flow of the family $\{B_{2t}\}_{t\in I}$ depends only on the first half of the path. Clearly the choice of the retraction $\{g_t\}$ doesn't inflict on the spectrum and hence has no influence at all on the spectral flow. The situation may be illustrated by the following picture: (For a more general discussion of the geometry of $\hat{F}$ compare [18] where it is shown that in fact a certain subspace of involutions can serve as classifying space for $K^{-1}$):

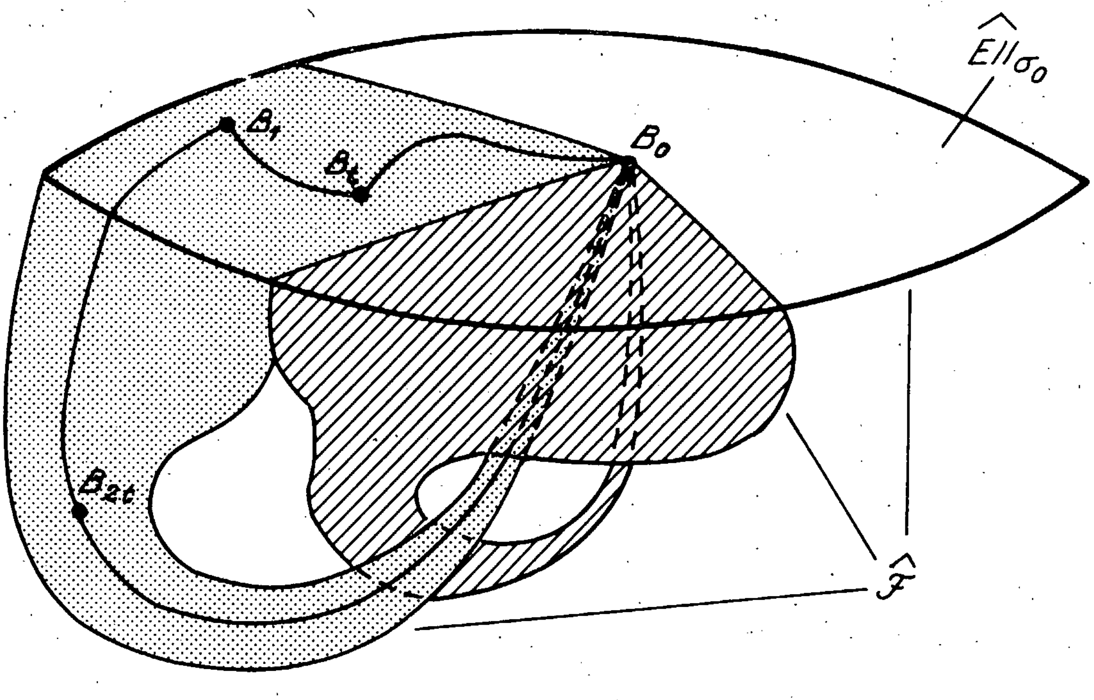


fig. 6

Moreover it turns out that the spectral flow of a smooth family of elliptic self-adjoint operators depends only on the location of the start and end operators $B_0$ and $B_1$. So we can without loss of generality assume that all operators between $B_0$ and $B_1$ have the same principal symbol (assumption (I)).

Finally it should be noticed that we meet in our main application, the general linear conjugation problem (see §5), only such families which satisfy the assumptions (I) and (II).

We are going to prove the following

4.1. THEOREM. *Let $\{B_t\}$ be a smooth family of elliptic self-adjoint operators of non-negative order acting on the sections of a Hermitian bundle* E *over a closed manifold* Y *and having the same principal symbol* $\sigma_0$. Let g be an automorphism of E such that

$$B_1 = g^{-1} B_o g.$$

Let $\{e_j\}_{j\in\mathbb{Z}}$ be a spectral decomposition of the Hilbert space $L^2(Y;E)$, i.e. an orthonormal basis of $L^2(Y)$ consisting of eigenfunctions $e_j$ of $B_0$:

$$B_0 e_j = \lambda j e_j .$$

Let $P_\pm$ be the projections onto the subspaces

$$\text{span } \{e_j \mid \lambda_j \geq 0\}$$

and

$$\text{span } \{e_j \mid \lambda_j < 0\}.$$

Then $P_+ - gP_-$ is an elliptic pseudodifferential operator of order zero over $Y$ and we have

$$\text{sf } \{B_t\}_{t\in I} = \text{index } P_+ - gP_-.$$

PROOF. For simplicity we consider only families of order zero: If $\{B_t\}_{t\in I}$ is a family of order $m$ then the family

$$\{B_t'\}_{t\in I} := \{(\text{id} + B_t^2)^{-1/2} \circ B_t\}_{t\in I}$$

has the same spectral flow and the same principal symbol over the cotangent sphere bundle $SY$ as $\{B_t\}_{t\in I}$.

The operators $P_\pm$ were discussed in §2.

We assume that

$$\dim \text{ image } P_+ = \infty = \dim \text{ image } P_-$$

because otherwise $[\tilde{\sigma}_{B_o}] = 0$ by lemma 2.2, hence both

$$\text{sf } \{B_t\}_{t\in I} = 0$$

and obviously index $P_+ - gP_- = 0$

Now we set

$$S := P_+ - P_- .$$

We show that the family $\{B_t\}_{t\in I}$ can be deformed into the family

$$\{\tilde{B}_t\}_{t\in I} := \{(1-t)S + t(g^{-1}Sg)\}_{t\in I}$$

without changing the spectral flow.

Since $\hat{E\ell\ell}_{\sigma_o}$ is convex we may assume that $\{B_t\}_{t\in I}$ has already the form

$$B_t = (1-t)B_0 + tB_1 , \quad t \in I.$$

Note that the retraction of the original path $\{B_t\}_{t\in I}$ onto the straight line connecting $B_0$ and $B_1$ doesn't change the spectral flow since it is a homotopy invariant by proposition 1.12.

By lemma 2.2 we get for the principal symbol of $S$

$$\sigma_S = \sigma_{B_o} = \sigma_o ,$$

hence the operator $S$ belongs to the same convex space $\hat{E\ell\ell}_{\sigma_o}$. Let $\{C_s\}_{s\in I}$ be the straight line connecting $C_0 := B_0$ with $C_1 := S$. Then the "parallel move"

$$C_{s,t} := (1-t)C_s + t(g^{-1}C_s\, g) , \quad s, t \in I$$

defines a homotopy between the family $\{B_t\}_{t\in I}$ (and its "invisible" closing $\{B_{t+1}\} = \{g_t^{-1}B_o\, g_t\}$) and the family $\{\tilde{B}_t\}_{t\in I}$ (and its "invisible" closing $\{\tilde{B}_{t+1}\} = \{g_t^{-1}\, S\, g_t\}$).

Thus we have

$$\mathrm{sf}\,\{\tilde{B}_t\}_{t\in I} = \mathrm{sf}\,\{B_t\}_{t\in I}.$$

Since $\tilde{B}_0$ and $\tilde{B}_1$ have the same principal symbol there difference is an operator of order $-1$ and hence it is compact. This proves that

$$g \in GL_S ,$$

i.e. $Sg - gS \in K(L^2(Y;E))$, since

$$\tilde{B}_1 - \tilde{B}_0 = g^{-1}Sg - S - g^{-1}(Sg - gS).$$

So we can apply lemma 3.4 and we get for $H := L^2(Y;E)$

$$\begin{aligned}\text{index } P_+ - gP_- &= \dim \{w \in P_-H \mid gw \in P_+H\}\\ &\quad -\dim \{w \in P_+H \mid gw \in P_-H\}.\end{aligned}$$

We determine the discrete spectrum of

$$\tilde{B}_t = (1-t)S + t\, g^{-1}Sg$$

by $\{-1, 1, 1-2t, 2t-1\}$ from the elementary calculation

$$\tilde{B}_t v = \begin{cases} v & v \in P_+H \text{ and } gv \in P_+H \\ -v & v \in P_-H \text{ and } gv \in P_-H \\ & \quad \text{if} \\ (1-2t)v & v \in P_+H \text{ and } gv \in P_-H \\ (2t-1)v & v \in P_-H \text{ and } gv \in P_+H \end{cases}$$

This gives the following graph of the spectrum of the family $\{\tilde{B}_t\}$

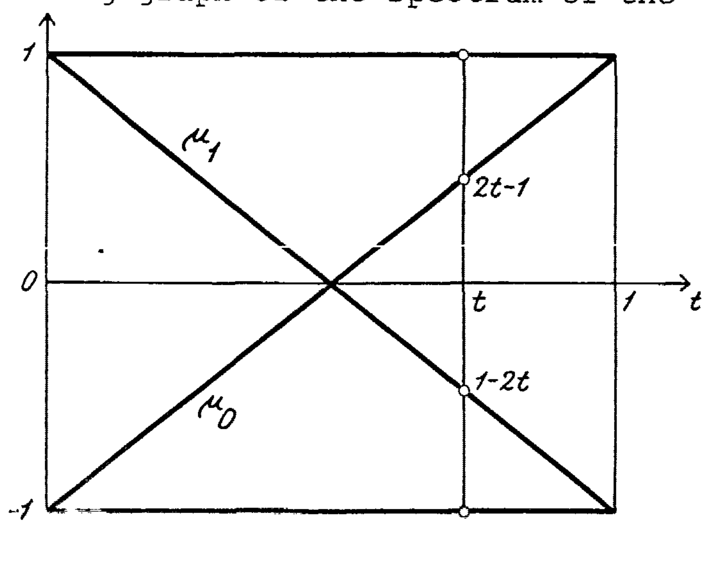


fig. 7

where $\mu_0$, $\mu_1$ are parametrizing the eigenvalues $2t-1$ and $1-2t$ with multiplicity $m_0$ and $m_1$, hence

$$\mathrm{sf}\,\{\tilde{B}_t\} = m_0 - m_1 .$$

As seen above we have

$$m_0 = \dim \{v \in P_- H \mid gv \in P_+ H\} = \dim \ker(P_+ - gP_-)$$

and

$$m_1 = \dim \{v \in P_+ H \mid gv \in P_- H\} = \dim \operatorname{coker}(P_+ - gP_-)$$

which proves the theorem. □

4.2. EXAMPLE. Recall the example given in 1.21.(c) consisting of the family

$$\{B_t\}_{t\in I} = \{-i \frac{d}{dx} + t\}_{t \in I}$$

of ordinary differential operators acting on $C^\infty(S^1)$ with $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. For $u \in C^\infty(S^1)$ we have

$$\begin{aligned} B_1(u)(x) &= -i \frac{du}{dx} + u \\ &= -ie^{-ix} \frac{d}{dx} (e^{ix}u)(x) \\ &= (e^{-ix} B_o e^{ix} u)(x) , \end{aligned}$$

hence

$$B_1 = g^{-1} B_o g$$

where

$$g : S^1 \to GL(\mathbb{C},1)$$

with

$$g(x) := e^{ix}.$$

For $B_o$ we find the spectral projections

$$P_+ := \text{projection onto } \operatorname{span}\{e^{ikx}\}_{k \geq o}$$

and

$P_-$ : = projection onto $\operatorname{span}\{e^{ikx}\}_{k<o}$

since $\{e^{ikx}\}_{k \in \mathbb{Z}}$ are the eigenfunctions of $B_o$.

Since multiplication with $g$ defines the (positive) shift operator on $L^2(S^1)$ with respect to the chosen orthonormal basis we are back in the situation of example 3.7 and we obtain

$$\ker(P_+ - g\,P_-) = \{c(e^{-ix}+1) \mid c \in \mathbb{C}\}$$

and

$$\operatorname{coker}(P_+ - g\,P_-) \simeq \{0\},$$

hence

$$\operatorname{index}\,(P_+ - g\,P_-) = 1$$

in concordance with $\operatorname{sf}\{B_t\} = 1$.

Now we explain a topological formula for the index of the operator $P_+ - g\,P_-$. Its principal symbol $\tilde{\sigma}$ on the cotangent sphere bundle $SY$ is given by $p_+ - g\,p_-$ or in matrix form

$$G = \begin{pmatrix} \mathrm{id} & 0 \\ & \\ 0 & -g \end{pmatrix} : E_+ \oplus E_- \to E_+ \oplus E_-,$$

where $E_+ \oplus E_-$ is the decomposition of the bundle $\pi^*E$, $\pi : SY \to Y$, defined in section 2. This gives us over

$$\textstyle\sum Y := BY \cup_{SY} BY$$

(cf. [4, §6]) a bundle

$$\textstyle\sum E := \pi^*_{BY}(E) \cup_{\hat{\sigma}} \pi^*_{BY}(E),$$

hence

$$[\hat{\sigma}] = [\sum E] - [\pi^*_{\sum Y}(E)],$$

where $\pi_{BY}$ and $\pi_{\sum Y}$ are the corresponding projections onto $Y$ and $[\hat{\sigma}]$ denotes the symbol class of the elliptic 0-order operator $P_+ - gP_-$ which belongs to

$$K(TY) \simeq K(BY/SY) \simeq K(\sum Y \setminus BY).$$

We want to determine the Chern character of this element. So, let $D_1$ be a connection on $\pi^*_{BY}(E)$. We are going to modify $D_1$ such that it becomes suitable for our calculations.

First we choose a connection $D_2$ on $\pi^*_{SY}(E)$ of the form

$$D_2 = D_+ \oplus D_-$$

where $D_\pm$ is a connection on $E_\pm$, and we join $D_1|SY$ (more precisely the pull-back of $D_1$ under the embedding $\pi^*_{SY}(E) \to \pi^*_{BY}(E)$) with $D_2$ by the family

$$\{D_{1,t}\}_{t\in I} := \{(1-r(t))\, D_1 + r(t)\, D_2\}_{t\in I}$$

with a smoothing function $r$.

Now we are able to construct a suitable connection on the one part of $\sum E$ lying over one copy of $BY$ (which we will denote by $BY_+$). We put for $\eta \in BY_+$

$$D_4 := \begin{cases} D_1 & |\eta| < \frac{1}{2} \\ D_{1,4|\eta|-2} \quad \text{for} & \frac{1}{2} \leq |\eta| \leq 3/4 \\ D_2 & 3/4 < |\eta| \leq 1 \end{cases}$$

On the other copy we must perform one more modification. We take the family

$$\{D_{3,t}\} := \{(1-r(t))\, D_2 + r(t)(D_+ \oplus g \circ D_- \circ g^{-1})\}$$

and define on $\pi^*_{BY_-}(E)$ a connection $D_4$ for $\eta \in BY_-$ by

$$D_4 := \begin{cases} D_1 & |\eta| < 1/2 \\ D_{1,4|\eta|-2} \quad \text{for} & 1/2 \leq |\eta| \leq 3/4 \\ D_{3,4|\eta|-3} & 3/4 < |\eta| \leq 1 \end{cases}$$

Since these constructions are consistent with the identifications defining $\sum E$, the pair $(D_3, D_4)$ defines[x)] a connection $\tilde{D}$ on $\sum E$. On $\pi^*_{\sum Y}(E)$ we fix the connection $\tilde{\tilde{D}}$ equal $D_3$ over each copy of $BY$.

From the chosen form of $\tilde{D}$ and $\tilde{\tilde{D}}$ we can conclude immediately

4.3. LEMMA. *The Chern character*

$$ch([\sum E] - [\pi^*_{\sum Y}(E)]) = ch(\tilde{D}) - ch(\tilde{\tilde{D}})$$

*is given by a form which vanishes outside of*

$$E_- | \{\eta | \eta \in BY_- \text{ and } 3/4 \leq |\eta| \leq 1\}.$$

4.4. COROLLARY. *Let* $P_\pm$ *be the spectral projections of an elliptic self-adjoint operator of non-negative order acting on sections of a Hermitian bundle* $E$ *over a closed Riemannian* $(n-1)$*-manifold* $Y$ *with characteristic bundle* $E_-$

---

x) One has to reverse the sign of the normal derivative on the second copy of the disc bundle when one glues the two connections $D_3$, $D_4$ together. However, this doesn't inflict on the Chern character.

and let $g$ be an automorphism of $E$. Then

$$\operatorname{index} P_+ - gP_- = (-1)^n \int_{SY} ch[E_-, g]\ \pi^*_{SY}\ T(Y),$$

where $T(Y)$ the Todd class of $Y$ and $[E_-, g]$ the difference bundle in $K^{-1}(SY)$ defined by the natural identifications (as in the note after definition 1.7).

PROOF. After elementary calculations we obtain the following formula for the Chern character of the symbol class

$$[\hat{\sigma}] = [\textstyle\sum E] - [\pi^*_{\sum Y}(E)]$$

of the operator $P_+ - gP_-$ :

$$ch[\hat{\sigma}] = \sum_{k=1}^{\infty} \frac{1}{(2\pi i)^k k!} \{r(t)^{k-1}(1-r(t))^{k-1}\, dr \wedge tr(\omega^{2k-1}) + r(t)^k(1-r(t))^k\, tr(\omega^{2k})\}$$

where $\omega = g^{-1}dg$.

By the preceding lemma these cohomology classes have compact support contained in $[3/4, 1] \times SY$, hence $ch([\hat{\sigma}])$ defines an element in $H^{odd}(SY;R)$, in fact

$$ch([\hat{\sigma}]) = ch[E_-;g].$$

It follows by the Atiyah-Singer index theorem

$$\begin{aligned} \operatorname{index} P_+ - gP_- &= (-1)^n \int_{\sum Y} ch[\hat{\sigma}]\ \pi^*_{\sum Y}(T(Y)) \\ &= (-1)^n \int_{[3/4,1]\times SY} ch[\hat{\sigma}]\ \pi^*_{\sum Y}(T(Y)) \\ &= (-1)^n \int_{SY} ch[E_-;g]\ \pi^*_{SY}(T(Y)). \ \square \end{aligned}$$

## 5. The general linear conjugation problem.

Now we turn back to the situation described in the introduction where we have a smooth manifold $X$ and a smooth submanifold $Y$ which divides $X$ into two parts $X_+$, $X_-$. We consider an operator

$$A : C^\infty(X;E) \rightarrow C^\infty(X;F)$$

acting between sections of smooth vector bundles $E$ and $F$ over $X$ which splits near $Y$, i.e. which has over a tubular neighborhood $N$ of $Y$ in $X$ the form

$$\rho \circ (\frac{\partial}{\partial t} + B)$$

where

$$\rho : E|Y \rightarrow F|Y$$

is a fixed bundle isomorphism and

$$B : C^\infty(Y;E|Y) \rightarrow C^\infty(Y;E|Y)$$

is a self-adjoint elliptic operator. (We fix a Riemannian structure on $Y$ and a Hermitian structure on $E$ and $F$ which provide the means for the necessary "parallel transport" of the tangent vectors and sections over $N$).

5.1. DEFINITION. Let

$$g : E|Y \rightarrow E|Y$$

be an unitary automorphism of $E|Y$ (inducing the identity in the base space $Y$, i.e. mapping $E_y$ onto $E_y$ for each $y \in Y$) such that

$$g \circ \sigma_B \circ g^{-1} = \sigma_B , \qquad (5.1)$$

where $\sigma_B$ denotes the pricipal symbol of $B$.

Then we have

$$g_F \circ \sigma_A = \sigma_A \circ g_E \tag{5.2}$$

where

$$g_E := g \quad \text{and} \quad g_F := \rho\, g\, \rho^{-1}$$

the corresponding automorphism on $F|Y$.

We define the glued vector bundles
$E^g := E|X_- \cup_g E|X_+$ and $F^g := F|X_- \cup_{g_F} F|X_+$.
Then the principal symbol $\sigma_A$ of $A$ gives us a new symbol

$$\sigma_A{}^g : \pi^*(E^g) \to \pi^*(F^g),$$

by

$$\sigma_A{}^g(x,\xi)v := \sigma_A(x,\xi)v;\ x\in X, \xi\in T_xX, v\in E_x^g = E_x.$$

Here

$$\pi : SX \to X$$

denotes the canonical projection. Note that $\sigma_A{}^g$ has the same values as $\sigma_A$ but it operates on another bundle.

Now we take any operator $A^g$ with the principal symbol $\sigma_A{}^g$ and investigate the value of the difference

$$\mu(g,A) := \text{index } A^g - \text{index } A.$$

We call this The General Linear Conjugation Problem.

NOTE. - In our situation we can define $A^g$ directly by

$$A^g := \begin{cases} A & \\ & \text{on} \\ \rho\left(\frac{\partial}{\partial t} + B_t\right) & \end{cases} \begin{matrix} X \setminus N_- \\ \\ N_- \end{matrix}$$

where

$$N_- := N \cap X_-$$

and

$$B_t := r(t)\ g^{-1}\ B\ g\ + (1-r(t))\ B$$

with a smooth real function $r$ equal $i$ near $i \in \{0,1\}$.

This problem was formulated by B. Bojarski in lectures given (since 1976) in Bielefeld, Darmstadt, and Tbilisi (cf. [9] and [10]). Its name is due to the strong interconnections between this problem and the Classical Riemann-Hilbert Problem which however might be more subtle from the analytical point of view since it deals with conjugated pairs of "local" solutions on the both halves (and hence with "serious" singularities over $Y$, cf. [32]) whereas our General Linear Conjugation Problem deals with "truely global" solutions of $A^g$ though not of $A$.

5.2. EXAMPLES. (a) The Classical Riemann-Hilbert Problem. - Let $X$ be the two-sphere

$$X := S^2 \simeq \mathbb{C} \cup \{\infty\},$$

$$Y := S^1,$$

hence

$$X_\pm := \{ z \mid |z| \gtrless 1 \}.$$

Let $g : Y \to \mathbb{C} \setminus \{0\}$ be a $C^\infty$-map.

We are looking for functions $\phi$ on $X \setminus Y$ such that

(I) $\frac{\partial \phi}{\partial \bar{z}} = 0$ on $\operatorname{int}(X_\pm)$,

(II) $\phi_+(\infty) = 0$,

(III) $\phi_\pm(z) := \lim\limits_{z_\pm \to z} \phi(z_\pm)$ exists for each $z \in Y$

where $z_\pm$ denotes a sequence of points in $\operatorname{int}(X_\pm)$ approaching $z$ and $\phi_\pm$ belongs to $L^2(S^1)$,

(IV) $\phi_+(z) = g(z)\ \phi_-(z)$ for almost all $z \in Y$.

This classical problem was posed by Hilbert (in modified form already by Riemann) and subsequently solved in whole generality by F.Noether, Vekua, Bojarski et al. (cf. [26]).
The crucial step in all approaches to this problem is the analysis near the dividing contour $Y$, actually the reduction of the (differential) conjugation problem in two dimensions to an (integro-differential) problem over the contour $Y$, i.e. in one dimension. In our simplest case one has to consider the spaces $H_\pm$ of the functions on $Y$ which can appear as limits of functions holomorphic in the outer (inner) region. One gets (see also [30, §1]).

$$H_+ = \{\sum_{k\geq o} a_k z^k\} \quad \text{and} \quad H_- = \{\sum_{k<o} a_k z^k)$$

and it turns out that the solutions $\phi$ are in one-one correspondence with the limit functions

$$\phi_+ \in H_+ \cap g\, H_-.$$

Let

$$\deg g > o.$$

Then $H_+ \oplus gH_-$ span the whole $L^2(S^1)$ and

$$\dim H_+ \cap g\, H_- = \deg g.$$

Bojarski's starting point was the goal to understand better the relations between the different integer valued invariants and indices involved.

(b) <u>The "Heat Equation" on the Torus.</u> -
Now let $X$ be the torus $T^2$ which is parametrized as $I \times S^1/\{0,1\} \times S^1$

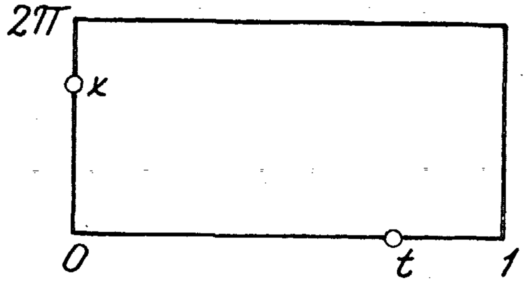

fig. 8

let

$$Y = \{(0,x)\} \simeq S^1$$

and let $E$ be the trivial complex line bundle over $T^2$. Let $g$ be the automorphism of $E|Y$ given by

$$g(x) := e^{ix} \quad , \quad x \in S^1 .$$

Then the bundle $E^g$ is defined by

$$E^g = I \times S^1 \times \mathbb{C}/\sim \quad \text{with} \quad (1,x,z) \sim (0,x,e^{-ix}z),$$

hence

$$C^\infty(T^2;E^g) \simeq \{f \in C^\infty(I \times S^1) \mid f(1,x) = e^{-ix} f(0,x)\}.$$

(Y doesn't divide X into two parts. However, this doesn't make principal problems for our theory since the normal bundle of $Y$ in $X$ is trivial).

Let us analyze the situation of example 1.21(d) further:

$$A^g = \frac{\partial}{\partial t} - i\frac{\partial}{\partial x} + r(t) \quad : \quad C^\infty(T^2;E^g) \rightarrow C^\infty(T^2;E^g),$$

where $r(t)$ is again a smoothing function equal $i$ near $i \in \{0,1\}$ and

$$A := \frac{\partial}{\partial t} - i\frac{\partial}{\partial x} : C^\infty(T^2) \rightarrow C^\infty(T^2).$$

Over the whole of $T^2$ the operator $A$ splits into $\frac{\partial}{\partial t} + B$ with

$$B = -i\frac{d}{dx} .$$

Obviously we have

$$\text{index } A = 0$$

and we derived earlier by theorem 1.19

$$\text{index } A^g = sf\{B_t\} = 1,$$

where

$$B_t := -i\frac{\partial}{\partial x} + r(t) .$$

Then the spaces

$$H_\pm := \text{span } \{e^{ikx}\}_{k \gtrless 0}$$

consist of those elements in $L^2(Y) = L^2(\{0\} \times S^1)$ which can be extended to bounded solutions $u_\pm$ of the equation

$$\frac{\partial u_\pm}{\partial t} - i\frac{\partial u_\pm}{\partial x} = 0$$

over the cylinder $\mathbb{R}_\pm \times S^1$. In fact, one has a natural separation of variables, hence

$$u_\pm(t,x) = \sum_{k \gtrless 0} a_k \, e^{-kt} \, e^{ikx}.$$

Note that the space $H_\pm$ coincides with the space of eigenfunctions with non-negative (negative) eigenvalues.

The kernel of the glued operator $A^g$ is characterized again by those elements in $L^2(Y)$ which belong to $H_+$ and to

$$gH_- = H_- \oplus \text{constants} ,$$

hence

$$\dim \ker A^g = \dim(H_+ \cap gH_-) = 1 .$$

Similarily, the range of $A^g$ is characterized by $H_+ \oplus g H \simeq L^2(S^1)$,

hence

$$\dim \text{coker } A^g = 0.$$

Actually, we find for the formal adjoint operator $(A^g)^*$

$$\ker (A^g)^* \simeq \ker A^{g^{-1}} \simeq H_+ \cap g^{-1} H_- = \{0\},$$

hence once more

$$\dim \operatorname{coker} A^g = 0$$

and

$$\operatorname{index} A^g = \operatorname{index} A^g - \operatorname{index} A = 1.$$

(c) <u>The Signature Operator over $S^{2m}$ with Coefficients in an Auxiliary Bundle.</u> - Recall the definition of the generalized signature operator $D_V$ of a Hermitian bundle $V$ over a closed oriented Riemannian manifold $X$ of dimension $2m$ (cf [12, III.4.D] or [27, IV.9]):

$$D_V := (d_V + d_V^*) | \Omega_V^+ : \Omega_V^+ \to \Omega_V^- ,$$

where $\Omega_V^{\pm}$ denote the $\pm 1$-eigenspaces of the involution

$$\tau := i^{p(p-1)+m} * : \Omega_V^p \to \Omega_V^p , \quad p \geq 0 ,$$

$$\Omega_V^p := C^\infty(X; \Lambda^p(TX) \oplus V)$$

$$d_V(u \otimes v) := du \otimes v + (-1)^p u \wedge \nabla v , \quad u \in \Omega^p, \ v \in C^\infty(X;V)$$

and $\nabla_V$ the connection of $V$. If $V$ is the trivial complex line bundle we are back in the situation of just the standard signature operator

$$D_X : \Omega_X^+ \to \Omega_X^-$$

and we obtain for the principal symbols

$$\sigma_{D_V} = \sigma_{D_X} \otimes \mathrm{id}_V$$

Let $Y$ be a closed submanifold dividing $X$ into $X_+$ and $X_-$. We suppose that the bundle $V$ is obtained by clutching the trivial bundles over the two components by a map

$$g : Y \rightarrow U(N) ,$$

i.e.

$$V \simeq X_+ \times \mathbb{C}^N \cup_g X_- \times \mathbb{C}^N$$

and so

$$D_V = A^g$$

where

$$A := D_{X \times \mathbb{C}^N} \simeq D_X \otimes \mathrm{id}_{\mathbb{C}^N} \simeq ND_X .$$

Since the signature operator splits near $Y$ (cf.[ 3 , p.63])[x)]

$$D_X = \rho \circ (\frac{\partial}{\partial t} + B_o)$$

where $B_o$ is the boundary signature operator of example 2.5(b), we have the following integers to look at:

(1) index $A$ = $N \times$ index $D_X = N \cdot \mathrm{sign}(X)$

(2) index $A^g$ = index $D_V$

(3) $\mathrm{sf}\{B_t\}$ = index $P_+ - g\, P_-$

where $\{B_t\}_{t\in I}$ is a family of self-adjoint elliptic operators connecting $B_o$ and $g^{-1}B_o g$ and $P_\pm$ the spectral projections of $B_o$.

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x) if we choose such a Riemannian metric on $X$ which is the product metric on $Y \times I \simeq N$.

From the definition it follows

$$\begin{aligned}
\mu(g,A) &= \text{index } A^g - \text{index } A \\
&= \text{index } \{\tfrac{\partial}{\partial t} + B_t\} \quad (\text{operator on } Y \times S^1) \\
&= \mathrm{sf}\{B_t\} \quad (\text{by theorem 1.19}) \\
&= \text{index } P_+ - g\, P_- \quad (\text{by theorem 4.1}) \\
&= \int_{SY} \mathrm{ch}[E_-;g]\,\mathcal{T}(Y) \quad (\text{by corollary 4.4})
\end{aligned}$$

where $E_-$ the characteristic bundle of $B_o$, i.e. the range bundle of the principal symbol of $P_-$.

A simple exercise in K-theory shows

$$[E_-;g] = [\tilde{\sigma}_{B_o}]\,[Y \times C^N;\, g]$$

where the multiplication operates in

$$K^{-1}(TY) \otimes K^{-1}(TY) \rightarrow K(TY) \simeq K^{-1}(SY),$$

hence

$$\mu(g,A) = \int_{SY} \mathrm{ch}[\tilde{\sigma}_{B_o}]\ \mathrm{ch}\ [Y \times \mathbb{C}^N;g]\ \mathcal{T}(Y)$$

Now let $X$ be the 2m-sphere with the (2m-1)-sphere $Y$ dividing $X$ into two discs $X_\pm$. Then we have to evaluate the cup product

$$\mathrm{ch}[\tilde{\sigma}_{B_o}]\ \mathrm{ch}[Y \times C^N;g]\ \tau(Y)$$

on the fundamental cycle $[TS^{2m-1}]$.

Since

$$TS^{2m-1} \simeq S^{2m-1} \times R^{2m-1}$$

we get

$$K^{-1}(TS^{2m-1}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Let $\alpha$ be a generator of the first component and $\gamma$ of the second. It was shown in [27, XV.7] that

$$[\tilde{\sigma}_{B_o}] = 2^{m-1}\,\alpha.$$

Moreover we have

$$T(Y) = 1$$

and

$$[Y \times \mathbb{C}^N; g] = k\gamma$$

if $k$ is the generalized "winding number" of $g$, i.e.

$$[g] = k\ x$$

where $x$ is the generator of $\pi_{2m-1}(U(N))$, $N$ sufficiently large. Since the Chern character maps generators of the K-groups into generators of the cohomology we get in this example

$$\mu(g,A) = k\,2^{m-1}$$

in concordance with

$$\operatorname{sign}(S^{2m}) = 0$$

and

$$\operatorname{index} D_V = k\,2^{m-1}$$

as shown in [27, XV.7].

After these examples one should expect that the difference

$$\mu(g,A) = \operatorname{index} A^g - \operatorname{index} A$$

depends only on the principal symbol of $B$ and on the automorphism $g$, i.e. only on objects living on $Y$. This is the case. However, before we prove our formula for $\mu(g,A)$ we define one further concept which is useful for our considerations.

5.3. DEFINITION. Let $Z$ be a Banach space and $H_1$, $H_2$ closed subspaces of $Z$. We call $(H_1,H_2)$ a Fredholm pair of subspaces of $Z$ if $H_1 + H_2$ is closed and

$$\dim H_1 \cap H_2 < \infty$$

and

$$\dim (Z/H_1 + H_2) < \infty .$$

Then the index of the pair is defined by

$$\operatorname{index}(H_1,H_2) := \dim H_1 \cap H_2 - \dim(Z/H_1 + H_2).$$

This notion was introduced by Kato [ ,IV.4.1] in an attempt to extend the stability properties of Fredholm operators from the case of bounded operators to the case of closed unbounded ones.

There are many examples of Fredholm pairs. Of course any Fredholm operator

$$A : H \to H'$$

defines a Fredholm pair of subspaces $H_1,H_2$ of $H \times H'$ by

$H_1 := H \times \{0\}$ and $H_2 := \operatorname{graph}(A)$

and one has

$$\operatorname{index} A = \operatorname{index} (H_1,H_2).$$

Another example is provided by the trivial Fredholm pair of eigenspaces

$$(\operatorname{image} P_+(B), \operatorname{image} P_-(B))$$

where $P_\pm(B)$ are the spectral projections of an elliptic self-adjoint operator $B$ as in §2.

We will now describe a more important example:

5.4. DEFINITION. Let $A$ be an elliptic first order operator over a closed manifold $X$ which is divided by $Y$ into two parts. We define the *spaces of Cauchy data* by

$$H_{\pm}(A) := \{u_{\pm}|Y \mid u_{\pm} \in C^{\infty}(X_{\pm}, E|X_{\pm}) \text{ and } Au_{\pm} = 0 \text{ on } X_{\pm}\}$$

NOTE. - There exist naturally defined projections $P_{\pm}(A)$ of $C^{\infty}(Y; E|Y)$ onto the spaces $H_{\pm}(A)$ which are pseudodifferential operators of 0-th order. Unfortunately, the proof of this fact is long and contains many technical difficulties, cf.[15], [22, ch.II], [27, ch.XVII] and [30]. In a recent paper [8] Birman and Solomyak announced a new simpler proof of these facts. But the details are not yet published.

It turns out that the principal symbols of the "Calderon projections" $P_{\pm}(A)$ are equal to $p_{\pm}$, the principal symbols of the spectral projections $P_{\pm}(B)$, if $A$ splits near $Y$ into $\frac{\partial}{\partial t} + B$ with $B$ self-adjoint. Hence the two kinds of projections differ only by a compact operator.

Bojarski [9] noticed the following

5.5. LEMMA. *Let $A$ be an elliptic first order differential operator over a closed manifold $X$ acting on sections of a bundle $E$ and let $Y$ be a submanifold dividing $X$ into two parts. Then the spaces of Cauchy* data $H_\pm(A)$ are Fredholm pairs of subspaces of $L^2(Y;\ E|Y)$.

PROOF. We consider the elliptic operator of 0-th order $P_+(A) - P_-(A)$. Since

$$\sigma_L(P_+(A) - P_-(A))^2 = \mathrm{id}$$

we get

$$\mathrm{index}(P_+(A) - P_-(A)) = 0.$$

Actually it suffices to know that the operator $P_+(A) - P_-(A)$ is elliptic, hence

$$\dim \ker (P_+(A) - P_-(A)) < \infty.$$

Then we obtain

$$\dim H_+(A) \cap H_-(A) < \infty$$

and

$$\dim\left(L^2(Y;\ E|Y) / H_+(A) + H_-(A)\right) < \infty$$

since

$$\begin{aligned}\ker(P_+(A) - P_-(A)) = {} & \{f \in C^\infty(Y;E) \mid P_+f = P_-f\} \cup \\ & \{f \in C^\infty(Y;E) \mid f \text{ is orthogonal} \\ & \text{to the space } H_+(A) + H_-(A) \text{ in} \\ & L^2(Y;\ E|Y)\}. \quad \square\end{aligned}$$

NOTE. - A natural and still unsolved problem by Bojarski is whether the two indices

$$\text{index } A \quad \text{and} \quad \text{index}(H_+(A),\ H_-(A))$$

coincide. Recall that the unique continuation property (UCP) holds for an elliptic differential operator $A$ if there are no non-trivial functions in the kernel of $A$ with support contained in a true subset of $X$. Many sufficient conditions for UCP are known, e.g. (cf. [11])

(a) $\dim X = 1$ (Lipschitz),

(b) real coefficients and order of $A$ equal 2 and fibre dimension of $E$ equal 1 (Aronszajn and Cordes),

(c) analytic coefficients (Holmgren),

(d) no multiple characteristics, i.e. no multiple roots $\tau \in \mathbb{C}$ of the principal symbol $\sigma_A(x, v + \tau w)$ for $x \in X$ and $v, w \in T_xX$ (Carleman),

(e) $A = A_1 \circ A_2$ modulo differential operators of lower order, where the order of $A_1$ and $A_2$ is equal 2 and the fibre dimension equal 1 (Mizohata),

(f) $A = \Delta^{m-1} \circ L$ modulo differential operators of order $[3m/2]$ where $L$ is of order 2 (Protter).

If UCP holds for $A$ we have

$$\ker A \simeq H_+(A) \cap H_-(A).$$

However, a famous counterexample by Pliś [29] shows that UCP doesn't hold for all elliptic differential operators.

5.6. DEFINITION. We define the Pliš spaces

$$\ker_{\pm} A := \{u \in \ker A \mid \operatorname{supp} u \subset X_{\pm}\}$$

and the Pliš defect

$$\ell_{\pm}(A) := \dim \ker_{\pm} A.$$

From the definition it follows

$$\dim \ker A = \ell_{+}(A) + \dim H_{+}(A) \cap H_{-}(A) + \ell_{-}(A)$$

hence

$$\begin{aligned} \operatorname{index} A = (&\ell_{+}(A) + \dim H_{+}(A) \cap H_{-}(A) + \ell_{-}(A)) \\ &-(\ell_{+}(A^*) + \dim H_{+}(A^*) \cap H_{-}(A^*) + \ell_{-}(A^*)). \end{aligned}$$

Now, if

$$\ell_{\pm}(A^*) = \ell_{\pm}(A)$$

and if

$$(H_{+}(A) + H_{-}(A))^{\perp} \simeq H_{+}(A^*) \cap H_{-}(A^*),$$

we can conclude

$$\operatorname{index} A = \operatorname{index}(H_{+}(A), H_{-}(A)).$$

However, it is by no means clear under which circumstances UCP for $A$ implies UCP for $A^*$, cf. [11].

Nevertheless we can use the Fredholm pair formalism to investigate the invariant $\mu(g,A)$. Without loss of generality we may suppose that the collar neighbourhood $N$ of $Y$ in $X$ could be chosen in such a way that

$$N \cap \operatorname{supp} u = \emptyset$$

for all $u \in \ker_{+}A \cup \ker_{+}A^*$.

Since $A^g$ is obtained from $A$ only by deformation over $N_-$ we can show

$$\text{index } A^g - \text{index } A = \{\dim H_+(A) \cap H_-(A^g) - \dim H_+(A) \cap H_-(A)\} - \{\dim H_+(A^*) \cap H_-(A^{g^*}) - \dim H_+(A^*) \cap H_-(A^*)\}$$

and we obtain

$$\mu(g,A) = \text{index}(H_+(A^g), H_-(A^g)) - \text{index}(H_+(A), H_-(A)).$$

Since

$$H_+(A^g) = H_+(A)$$

one could obtain under the assumption

$$H_-(A^g) = gH_-(A)$$

(for which we have no plan nor idea of proof) the nice formula

$$\begin{aligned} \mu(g,A) &= \operatorname{tr}(g\,P_-(A) - P_-(A)g) \\ &= \text{index}(P_+(A) - g\,P_-(A)) \\ &= \text{index}(P_+(B) - g\,P_-(B)). \end{aligned}$$

In the following we prove directly the last formula since the concept of Calderon projections is not as elementary as the spectral projections and since not yet all steps in the preceding argumentation could be completely clarified. Moreover, the final formula for $\mu(g,A)$ is sufficient for all applications given in this paper.

5.7. THEOREM. Let $A$ be a first order elliptic operator acting between sections of Hermitian vector bundles $E$, $F$ over a closed Riemannian manifold $X$ which splits into

$$A = \rho \circ \left(\frac{\partial}{\partial t} + B\right)$$

near a dividing submanifold $Y$, where $B$ is a self-adjoint elliptic operator over $Y$ and $\rho$ a bundle isomorphism, and let $g$ be a unitary automorphism of $E|Y$ compatible with $B$, i.e. satisfying the condition

$$g \circ \sigma_L(B) \circ g^{-1} = \sigma_L(B)$$

where $\sigma_L(B)$ is the principal symbol of $B$.
Then we have

$$\mu(g,A) = \mathrm{sf}\{B_t\}_{t \in I}$$

where $\{B_t\}$ is a smooth family of elliptic self-adjoint operators over $Y$ connecting

$$B_0 := B \quad \text{and} \quad B_1 := g^{-1} B g .$$

PROOF. Recall the "local index theorem" (cf. [31] and [2])

$$\text{index } A = \int_X \alpha(A)(x)\, dx$$

where $\alpha(A)$ is constructed from the full symbol of $A$. By the explicit definition of $A^g$ given above in the note after definition 5.1 it is clear that

$$\alpha(A)(x) = \alpha(A^g)(x) \quad \text{for} \quad x \in X \setminus N_- ,$$

hence

$$\mu(g,A) = \text{index } A^g - \text{index } A = \int_{N_-} \alpha(A^g)(x)\, dx - \int_{N_-} \alpha(A)(x)\, dx .$$

The first integral gives us the index of the operator

$$\rho \circ \left(\frac{\partial}{\partial t} - B_t\right) : C^\infty(S^1 \times Y; E^{g'}) \to C^\infty(S^1 \times Y; F^{g'})$$

where

$$E^{g'} = I \times E|Y/\sim$$

with the identification

$$(1, y, e) \sim (0, y, g^{-1}(y)e) \quad , \quad y \in Y \; , \quad e \in E_y .$$

The reason for that is that the full symbol of this operator is equal to the full symbol of $A^g$ in each point of $N_-$ (parametrized as $I \times Y$). So we have by theorem 1.19

$$\int_{N_-} \alpha(A^g)(x)dx = \operatorname{index}\left(\frac{\partial}{\partial t} - B_t\right) = sf\{B_t\}.$$

The second integral doesn't contribute to $\mu(g,A)$ since it is equal to

$$\operatorname{index}\left(\frac{\partial}{\partial t} - B\right) = sf\{B_t' = B\} = 0. \; \square$$

5.8. COROLLARY. Let $A$, $B$, and $g$ be as in theorem 5.7 and let $P_\pm(B)$ be the spectral projections of $B$. Then

$$\mu(g,A) = \operatorname{index}(P_+(B) - g\, P_-(B)).$$

PROOF. By theorem 4.1.

As mentioned above we could express our result in the language of the Calderon projections $P_\pm(A)$, too. Since the differences $P_\pm(B) - P_\pm(A)$ are compact, we obtain

5.9. COROLLARY. *Let* $A$ *and* $g$ *be as in theorem* 5.7 *and let* $P_{\pm}(A)$ *be the Calderon projections of* $A$ *onto the spaces* $H_{\pm}(A)$ *of Cauchy data of the kernel of* $A$ *over* $Y$. *Then*

$$\mu(g,A) = \operatorname{index}(P_{+}(A) - g\, P_{-}(A)).$$

In the rest of this paragraph we present several applications of our final formula given in corollary 5.8. Our main purpose is to show how the integer $\mu(g,A)$ depends on the topology of $Y$, $g$, and $B$.

5.10. PROPOSITION. *Let* $A$, $Y$, *and* $g$ *be as in theorem* 5.7. *If* $g$ *determines a torsion element in* $K^{-1}(Y)$ *then*

$$\mu(g,A) = 0.$$

PROOF. Let $g$ determine a torsion element in $K^{-1}(Y)$. Then there exists a natural number $k$ such that

$$kg := \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & g \end{pmatrix} : k\, E|Y \to k\, E|Y$$

is homotopically equivalent to the identity.

More precisely, we have a path in the space of unitary automorphisms of $k\, E|Y$ joining $kg$ with the identity, cf. [23,p.72]. Thus, we are able to deform the family $\{k\, B_t\}$ into a family $\{C_t\}$ joining $k\, B$ with itself without changing the spectral flow, hence

$$k\ sf\{B_t\} = sf\{k\ B_t\} = sf\{C_t\} = 0$$

since $\{C_t\}$ is contractible to the constant family. □

5.11. PROPOSITION. *Let* $A$, $Y$, *and* $g$ *be as in theorem* 5.7. *Let* $A$ *admit local elliptic boundary conditions on* $Y$, *then:*

$$\mu(g,A) = 0.$$

PROOF. Since $A$ splits near $Y$ it admits local elliptic boundary conditions on $Y$ (in the sense of Shapiro-Lopatinski) if and only if $p_+$, the principal symbol of the spectral projection $P_+(B)$, can be deformed into a projection onto a bundle over $SY$ which is a pull back of a bundle over $Y$ (cf. [1], [12], [15], [30]).

Since

$$P_-(B) = id - P_+(B)$$

the symbol of $P_+(B) - g\,P_-(B)$ then becomes a matrix function of $y \in Y$ alone, hence gives the trivial element of $K(TY)$. □

NOTE. - Let

$$\{p_t(y,\xi)\}_{t \in I}\ ,\quad y \in Y\ ,\quad \xi \in T_y Y$$

be a continuous family of projection symbols such that

$$p_o(y,\xi) = p_+(y,\xi) \text{ and } p_1(y,\xi) = p_1(y).$$

Then we can not expect that $g$ leads to an automorphism of the bundle

$$V := \text{image } p_1$$

since in general

$$p_1 g \neq g\, p_1 .$$

Thus we must change $g$ continuously by a continuous family $\{g_t\}$ of automorphisms of

$$V_t := \text{image } p_t$$

leading to a $g_1$ which commutes with $p_1$.

5.12. PROPOSITION. *Let* $A$, $Y$, $B$ *and* $g$ *be as in theorem* 5.7 *and let the principal symbol of* $B$ *determine a torsion element in* $K^{-1}(TY)$. *Then:*

$$\mu(g,A) = 0 .$$

PROOF. The assumption about $B$ means that for some $k$, $N$ the symbol

$$k\, \sigma_L(B) \otimes \text{id} : \pi^*(k\, E|Y) \otimes \mathbb{C}^N \circlearrowleft$$

can be deformed into a symbol

$$v = \tilde{p}_+ - \tilde{p}_-$$

where $\tilde{p}_\pm$ are projections onto some bundles lifted from $Y$. Thus we get the equalities

$$\begin{aligned} k\, \mathrm{sf}\{B_t\} &= \mathrm{sf}\{k\, B_t \otimes \text{id}\} \\ &= \text{index}(k\, P_+(B) \otimes \text{id} - g\, k\, P_-(B) \otimes \text{id}) \\ &= \text{t-index}\, [\tilde{p}_- ; g] \\ &= 0 \end{aligned}$$

where

$$\text{t-index} : K^{-1}(SY) \to \mathbb{Z}$$

denotes the homomorphism given by

$$[\sigma] \mapsto \int_{SY} \mathrm{ch}[\sigma]\, \pi^*_{SY}\, \mathcal{T}(Y) . \quad \square$$

## 6. Various extensions and generalizations.

So far we presented in this paper all results in detail only in the case when $A$ splits near $Y$ into the form

$$A = \rho(\frac{\partial}{\partial t} + B) \quad , \quad B \text{ self-adjoint} \qquad (*)$$

and $g$ is a unitary automorphism of $E|Y$. Now we will indicate how one can weaken these assumptions.

6.1. Non self-adjoint $B$.+) It is easy to see that we can take for $B$ any elliptic operator of which the principal symbol $\sigma_L(B)$ has no eigenvalues on the imaginary axis and is compatible with $g$, i.e.

$$g\, \sigma_L(B)\, g^{-1} = \sigma_L(B) \ .$$

Without changing the principal symbol (hence the index neither) we can deform such a more general operator $A$ into our form $(*)$: Take the family

$$\{B_t\} := \{\tfrac{1}{2}(B + B^*) + \tfrac{1}{2}(1 - r(t))(B - B^*)\}.$$

Since $\sigma_L(B)^*$ commutes with $g$, too, when $g$ is unitary, we have

$$g\, \sigma_L(B + B^*)\, g^{-1} = \sigma_L(B + B^*).$$

Thus we can deform $A$ to an operator which on $N = I \times Y = N_- \cup N_+$ takes the form

$$A = \rho(\frac{\partial}{\partial t} + C_t)$$

where

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+) It was noted in [22, p.192] that every elliptic first order differential operator splits with $B$ not necessarily self-adjoint, see also [19, ch.II].

$$C_t := \begin{cases} B_{2t} & \text{for } 0 \leq t \leq \frac{1}{2} \\ B_{1-(2t-1)} & \phantom{\text{for }} \frac{1}{2} < t \leq 1 . \end{cases}$$

Hence in a smaller collar neighbourhood $A$ has the form (*).

In fact we don't need this deformation to the self-adjoint case since one could define a spectral flow for all families of operators of which the principal symbol has no eigenvalues on the imaginary axis. We just count the number of eigenvalues whose real parts change the sign when $t$ is going from $0$ to $1$.

6.2. Elliptic symbols over a mapping torus.

This observation is essential to the more general situation when we consider instead of $g$ a diffeomorphism

$$\Phi : E|Y \to E|Y$$

of the total spaces which is linear on the fibres though not inducing the identity but an arbitrary diffeomorphism $\varphi$ in the base. In this case we can not reduce the problem to a self-adjoint family. Then the spectral calculus is getting more advanced. Instead of $K^{-1}(TY)$ we must work with some suitable K-groups over the mapping torus $(Y \times I)^{\varphi}$. The details of this approach will appear in a separate publication by the second author.

6.3. Non-splitting symbols. Using the rather big machinery of calculus of the Boutet-de-Monvel type one can get parts of the results of this paper for operators which do not necessarily split near $Y$ ([13], [14]).

Finally, we want to present the simplest example of the situation when $g$ is not an automorphism.

6.4. EXAMPLE. Once again we consider the operator

$$B := -i\frac{d}{dx} : C^\infty(S^1) \to C^\infty(S^1).$$

Now we put

$$g(x,z) := (x+\pi, e^{ix} z) \qquad (**)$$

where $z \in \mathbf{C}$ and $x$ is a real coordinate mod $2\pi$.

We have

$$g^{-1}(x,z) = (x+\pi, e^{ix} z),$$

hence

$$\begin{aligned} g^{-1} Bg(x, f(x)) &= g^{-1} B_{|x+\pi}(x+\pi, e^{ix} f(x)) \\ &= g^{-1}(x+\pi, -ie^{ix} f'(x) + e^{ix} f(x)) \\ &= (x, -i\underbrace{(-e^{i\pi})}_{=1} f'(x) + f(x)) \\ &= (x, -if'(x) + f(x)). \end{aligned}$$

So, in fact we obtain the same operator as in the previous analyzed case (see 4.2)

$$g(x,z) := (x, e^{ix} z),$$

and we recall that the standard family $\{B_t = -i\frac{d}{dx} + t\}$

has a spectral flow equal to 1.

The only difference is the action on the eigen-functions: Let $g$ be defined as in (**) and set

$$u_k(x) := e^{ikx}.$$

Then we have

$$\begin{aligned}(x, (g\, u_k)(x)) &= (x, e^{i(x-\pi)}\, e^{ik(x-\pi)}) \\ &= (x, e^{i(k+1)x}\, e^{-i(k+1)\pi}) \\ &= (x, (-1)^{k+1}\, u_{k+1}(x))\end{aligned}$$

hence

$$g\, u_k = (-1)^{k+1}\, u_{k+1} .$$

6.5. EXAMPLE. More generally one could compute the operator $g^{-1}Bg$ for an arbitrary first order elliptic differential operator as an exercise.
It turns out that $g^{-1}Bg$ has the same first order part as $B$ and only the zero order terms (which are bundle morphisms) differ. So, even if $g$ is not an automorphism, we can join $B$ with $g^{-1}Bg$ with a family of the type considered in this paper, although several technical problems arise.

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ISSN 0106-6242