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Point-driven Mathematics Teaching Studying and Intervening in Danish Classrooms

Arne Mogensen PhD Dissertation
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Point-driven Mathematics Teaching
Studying and Intervening in Danish Classrooms

Af: Arne Mogensen

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Do Danish mathematics teachers have a "point"? In this dissertation the concept of a mathematical point is clarified and defined as a statement presenting *a clearly delineated and significant mathematical content or climax*. It is examined to what extent Danish mathematics teachers in lower secondary schools emphasize such points in their teaching. Thus, 50 randomly selected mathematics teachers are filmed in one grade 8 math lesson and the dialogue investigated. The study identifies large variations and many influential components. There seems to be room for improvement.

In order to examine possibilities to strengthen the presence and role of mathematical points in teaching two intervention studies are conducted.
First a focus group of 5 of the original 50 teachers from each school are offered peer coaching by the researcher. This study indicates that different teachers appreciate peer coaching to individual desire and need, and the effect can be significant to the teachers' communication in the classroom.
Also, a lesson study is conducted with all 18 mathematics teachers at one school. It is shown that the group of all mathematics teachers at one school with a limited use of resources can be supported in significant changes to a point-oriented mathematics teaching. The teachers emphasized joint planning of study lessons, and they regarded the peer coaching after each of these lessons as valuable.

The studies with the two teacher groups indicate different opportunities and challenges, and it is recommended that all schools appoint and support a *mathematics tutor* with responsibility for such to offer or arrange for peer coaching and guidance, including that mathematic teachers in *teams* arrange systematically peer coaching as a *lesson study*. In mathematics teacher education is also recommended that the practice preparation, practice teaching and assessment are organized as a *lesson study* with focus on mathematical points.

Arne Mogensen, 2011

Point-driven Mathematics Teaching
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PhD Dissertation
Arne Mogensen, June 2011

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Abstract

The dissertation poses three questions:

1. To what extent, how and why do teachers *articulate* mathematical points in Danish mathematics teaching?
2. In what way can the occurrence and role of mathematical points be *strengthened* in mathematics teaching practice?
3. To what extent and by which means can mathematics teachers be *supported* in point-driven mathematics teaching?

The TIMSS Video Studies (1005 and 1999) among others indicated several parameters to quality in mathematics teaching. And a Danish report on the future matematics in primary and lower secondary schools (Niss et al, 2006) recommended an emphasis on mathematical points.
There is not found any Danish or international research which examines the extent and role of points in mathematics teaching. But referred in the dissertation are several other research projects, partly similar to mine because of parameters to the mathematic teaching researched or because of similar forms of intervention. Methods and results are compared with the current research.

A mathematical point is clarified and defined as *a statement presenting a clearly delineated and significant mathematical content or climax.* And a didactic point is defined as *a mathematical point, the teacher has judged particularly important to the student.*
The methods are outlined for a quantitative study of 50 mathematics teachers in grade 8 at public municipality owned schools representatively and randomly selected from Denmark's five regions. For each participant the data consists of a video recording from one mathematics lesson, a questionnaire, a brief researcher memo and copy of teaching material. Mathematical points are articulated by teachers in both common classroom communication and in conversations with students individually or in groups. Also students articulate points. Four types are identified:

- Justification, definition or other reference to a mathematical *concept*
- Justification, development or demonstration of a specific rule or *method*
- Inductively or deductively based mathematical *result* or conclusion
- Assessment or *interpretation* of a mathematical result, model or definition.

Do Danish mathematics teachers then have a "point"? 58% of the lessons contained one or more points. In 44% of the lessons the teachers articulated a point to the whole class and in 30% there was a point from the teacher to individual students or groups. In 52% of the lessons one or more students articulated a point. The extent is measured both as a *quota* of the lesson length and the *number* of references in each lesson. Many points did *not* seem planned or guiding the lesson and nearly half of the 50 lessons were entirely *without* points articulated by the teacher. The four types of points are not made equally often: *Procedural* points are the most common and *result* points are rare.

Points are mostly articulated by teachers having mathematics as a major subject or teaching in large schools. There is not found distinct differences due to gender, teaching experience or mathematics textbooks.

Two intervention studies are conducted to examine possibilities to strengthen the presence and the role of points in mathematics teaching:

- First with a focus group of 5 of the original 50 teachers from each school in a period of peer coaching with me as researcher.
- Next with all 18 mathematics teachers at one large school in a *lesson study* course which included a short course and where I then as researcher also was the coordinator.

The studies with the two teacher groups indicate different opportunities and challenges:

- The study with single teachers identified the ability to adjust peer coaching to individual desire and need. There are differences in teachers' personal background knowledge, experience and attitude (*beliefs*), but also in the local school culture. The study showed that different teachers appreciate a targeted individual peer coaching, and the effect can be significant to the teachers' communication in the classroom. Most evident was an increase in teachers' *elicitation* dialogue in the investigated grade 9 classes.
- The *lesson study* showed that mathematics teachers at one school with a limited use of resources can be supported in significant changes to a point-oriented mathematics teaching. The teachers put an emphasis on joint planning of study lessons, and they regarded the peer coaching after each of these lessons as valuable. Some emphasized the room for diversity and the structured communication. Professional coaching in an open peer environment was thus recommended, but it should be noted that the school's leadership and the 18 mathematics teachers all were positive in advance. The study showed that teachers' willingness to act with professionalism at the same school can build a bridge across any "teaching gaps" between peers.

It is *not* shown whether there is a lasting effect of the two forms of intervention. But there is some new knowledge. The research has shown two ways in which interventions with modest resources can support teaching without change in the overall framework. But both forms required an introduction to the idea and importance of points.
In light of this new knowledge, it is recommended that all schools appoint and support a *math tutor* with responsibility for such to offer or arrange for peer coaching and guidance, including that mathematic teachers in subject *teams* arrange systematically peer coaching as *lesson study*. In mathematics teacher education is also recommended that the practice preparation, practise teaching and assessment are organized as *lesson study* with focus on mathematical points.

Resumé

Afhandlingen stiller tre spørgsmål:

1. I hvilket omfang, hvordan og hvorfor fremhæver matematiklærere matematiske pointer?
2. Hvordan kan man styrke forekomst og rolle for matematiske pointer i undervisningen?
3. I hvilket omfang og med hvilke midler kan lærere støttes i pointestyret matematikundervisning?

Argumenter for en pointe-styret matematikundervisning hentes bl.a. fra observationer i TIMSS Video Studies (1995 og 1999). Også i rapporten *Fremtidens matematik i folkeskolen* (Niss et al, 2006) blev det anbefalet, at der fokuseres på matematiske pointer i matematikundervisningen.
Der er ikke fundet dansk eller udenlandsk forskning, der også undersøger omfang og rolle for pointer i matematikundervisning, men der refereres i afhandlingen til udvalgt forskning, der minder om min forskning. Enten fordi der afdækkes lignende parametre i matematiklærerarbejdet, eller fordi der afprøves forskellige former for intervention. Metoder og resultater holdes op mod den aktuelle forskning.

En matematisk pointe er præciseret og defineret som *en præsentation af et klart, afgrænset og betydende matematisk indhold eller resultat.* Og en didaktisk pointe som *en matematisk pointe, læreren har vurderet særlig betydningsfuld for eleven.*
Der redegøres for metoden i en undersøgelse af 50 matematiklærere på 8. klassetrin i en kommunal skole, repræsentativt og tilfældigt udvalgt i Danmarks fem regioner. For hver deltager består datamaterialet af en videooptagelse fra én tilfældig matematiktime, et spørgeskema, et kort forsker-memo og kopi af undervisningsmateriale.
Matematiske pointer formuleres af lærere både i klasseundervisning og i samtaler med eleverne enkeltvis eller i grupper. Og elever formulerer også pointer. Der er identificeret fire typer:

- Begrundelse, definition eller anden reference til et matematisk *begreb*
- Begrundelse, udvikling eller demonstration af en bestemt regel eller *metode*
- Induktivt eller deduktivt baseret matematisk *resultat* eller konklusion
- Vurdering eller *fortolkning* af et matematisk resultat, model eller definition.

Har danske matematiklærere så en "pointe"? 58 % af lektionerne indeholdt en eller flere pointer. I 44 % formulerede læreren en pointe for hele klassen og i 30 % var der en pointe fra læreren til enkeltelever eller grupper. I 52 % af lektionerne formulerede en eller flere elever en pointe. Omfanget er opgjort både som *andel* af lektionens længde og som *antal* af referencer i den enkelte lektion. Mange pointer virkede *ikke* planlagte eller styrende for lektionen, og næsten halvdelen af de 50 lektioner var helt *uden* pointer formuleret af læreren. De fire slags pointer formuleres ikke lige ofte: *Metode*-pointer er mest almindelige og *resultat*-pointer forekommer sjældent.

Pointer formuleres oftest af linjefagsuddannede lærere og på store skoler. Der er ikke fundet tydelige forskelle pga. køn, undervisningserfaring eller matematikbogssystem.

To interventionsstudier er gennemført for at undersøge, hvordan man kan styrke forekomsten af og den rolle pointer har i matematikundervisning:

- Først med en fokusgruppe på 5 af de oprindelige 50 lærere fra hver sin skole i en periode med kollegial sparring med mig som samtidig forsker.
- Dernæst med alle 18 matematiklærere på én skole i et *lesson study* forløb, der indeholdt et kort kursus, og hvor jeg som forsker derefter deltog som tovholder.

Studierne med de to lærergrupper peger på forskellige muligheder og udfordringer:

- I studiet med enkeltlærerne peges på muligheden for at afpasse kollegial sparring til den enkeltes ønske og behov. Der er forskelle i læreres personlige baggrund som viden, erfaring og holdning (*beliefs*), men også i den lokale skolekultur. Studiet viste, at forskellige lærere værdsætter en målrettet individuel kollegial sparring, og effekten kan være betydelig på lærernes kommunikation i klassen. Tydeligst var en forøgelse af læreres ”lokke-dialog” (*elicitation*) i de undersøgte 9. klasser.
- I et *lesson study* forløb er det vist, at matematiklærere på én skole med en begrænset brug af ressourcer kan støttes i betydelige ændringer mod en pointe-orienteret matematikundervisning. Lærerne vægtede den fælles planlægning af studielektioner, og de anså den kollegiale sparring efter hver af disse lektioner for værdifuld. Nogle understregede pladsen til forskellighed og den velstrukturerede samtale. Faglig sparring i et åbent kollegialt miljø blev således anbefalet, men det skal bemærkes at skolens ledelse og de 18 matematiklærere alle var positive på forhånd. Studiet viste, at læreres vilje til at handle med professionalisme på den samme skole kan bygge bro på tværs af eventuelle ”*teaching gaps*" mellem fagkolleger.

Det er *ikke* vist, om der er en holdbar effekt af de to former for intervention. Men der er tilvejebragt nogen ny viden. Forskningen har vist to måder, hvorpå intervention med beskedne midler kan støtte lærere uden ændring i de overordnede rammer. Men begge former krævede en introduktion til ideen og vigtigheden af pointer.
På baggrund af den nye viden anbefales det, at alle skoler udnævner og understøtter en *matematik-vejleder* med ansvar for bl.a. at tilbyde eller arrangere kollegial sparring og vejledning, herunder at matematiklærere i *fagteam* sætter kollegial sparring i system som *lesson study*. I matematiklæreruddannelsen anbefales ligeledes, at praktikforberedelse, praktikundervisning og evaluering tilrettelægges som *lesson study* med fokus på matematiske pointer.

Preface and Acknowledgements

This research project has been made possible by support from several sources. The Research Agency (Forskningsstyrelsen), Roskilde University and VIA University College have contributed equally to my PhD grant for three years. Part of this was a grant from NAVIMAT (Nationalt Videnscenter for Matematik).

Professor Mogens Niss at Roskilde University has been an excellent supervisor. His generous ideas and questions were invaluable for my research from the initial ideas developed in a governmental commission to articulation of research questions and research design for an adequate analysis and report. I enjoyed the steady demand for clarity and consequence. Thank you, Mogens.

My research also depended on mathematics teachers opening their classrooms and I am indebted to 50 colleagues teaching grade 8 all over Denmark. I hope that some of the ideas developed and presented in this dissertation will support mathematics teachers in developing teaching quality. Also thanks to the school administrators and students for letting me in. Without such openness, researchers would be limited to qualified estimation.

Special thanks should go to the seven mathematics teachers who volunteered as a focus group and invested their time in writing and talking with me on several occasions. And thanks a lot to the school where administration and all 18 mathematics teachers were enthusiastically involved in a lesson study project.

During the three years of research I was lucky enough to participate in three seminars of the Nordic Graduate School of Mathematics Education. The fellow doctoral students and visiting supervisors always made these seminars invaluable for getting peer feedback on your research in progress. Barbro Grevholm was a key person to these NoGSME seminars; thank you for always encouraging me, Barbro.

I've met many more people whom I ought to thank. Kaori Myuachi followed me for almost two weeks at the Nagasaki University in 2008. Without her simultaneous translation of what was going on in lesson study in mathematics classrooms in schools and at the university my outcome would certainly have been limited. Jo Boaler led a fantastic seminar at the University of Sussex in 2009, and I benefitted from the very thorough feedback on a research paper. David Clarke was kind to let me stay many weeks in 2010 at the International Centre for Classroom Research at Melbourne University. The talks I had with him and his colleagues clearly challenged my research and analysis.

Finally I owe many thanks to my teacher training colleagues in Denmark. At the VIA University College in Aarhus they have been prompting me for observations and results in a friendly manner. And I was offered possibilities for feedback and to give talks on my research in NAVIMAT and SeMat (Association of mathematics teachers at teacher training colleges).

Claire Neesham has kindly assisted me with English proofreading. And finally: Thank you Lone; my patient and supportive wife.

Contents

1 Context

The main reason for conducting the research reported in this dissertation is the acknowledged low level of mathematics education in Danish compulsory primary and lower secondary schools as evidenced by, for example, international comparisons of resources and student outcomes, especially those relating to East Asia as documented by PISA (Mejding, 2004; Egelund, 2007; Egelund, 2010; OECD, 2004, 2007 and 2010).

As an experienced decision maker and participant in Danish mathematics teaching I feel some responsibility for suggesting changes that could help improve the situation, which may be enacted without the need for major policy decisions or financial allocations.

Following a description of the current Danish situation, I expand upon the motives outlined above and also point to several parameters that indicate why not everything in Danish mathematics teaching is as it should or could be. Besides being a study of 50 teachers the research also include two intervention studies. One study was action research with a focus group of single teachers from five schools and the other a lesson study with 18 mathematics teacher at one school.

1.1 Current Danish situation

Like in other countries the Danish educational authorities stress the importance of quality in teaching and learning in general, including mathematics. To this end teachers are required to follow certain guidelines, e.g.:

- Since 1993 the Folkeskolen Act has requested *formative assessment to be incorporated* in teaching at all grade levels (Folkeskoleloven, 2010):

"**§13 Section 2**. Part of the teaching assessment should focus on the students' acquisition of knowledge and skills in subjects compared to the steps and endpoints, see § 10. The assessment will form the basis for the future guidance of the individual student and for further planning and organization of teaching, see § 18, and for informing parents about student learning outcomes. **§18**. The organization of teaching, including the choice of teaching approach and work, methods, teaching materials and content, must in all subjects meet the main school goal, goals for the subjects and topics and	"**§13 Stk. 2**. Som led i undervisningen skal der løbende foretages evaluering af elevernes udbytte heraf, herunder af elevens tilegnelse af kundskaber og færdigheder i fag og emner set i forhold til trin- og slutmål, jf. § 10. Evalueringen skal danne grundlag for vejledning af den enkelte elev og for den videre planlægning og tilrettelæggelse af undervisningen, jf. § 18, og for underretning af forældrene om elevens udbytte af undervisningen. **§ 18.** Undervisningens tilrettelæggelse, herunder valg af undervisnings- og arbejdsformer, metoder, undervisningsmidler og stofudvælgelse, skal i alle fag leve op til folkeskolens formål, mål for fag samt emner

also attend to the individual student's needs and requirements".	og varieres, så den svarer til den enkelte elevs behov og forudsætninger."

- Since 2006/07 a *Departmental order of student plans in schools* specifies the need for *student plans* for all grade levels (Elevplaner, 2009). The plans must be given to the parents at least once a year: *§3 Section 2. The student plan must include details of the agreed follow-up on the results of ongoing assessment, including follow-up of results from the tests mentioned in section 3 in certain subjects and grade levels and any agreements on parental and student involvement to assist the student in reaching the stated learning objectives*" (own translation).

Reports from various commissions and agencies also present suggestions for improvements in schools, e.g.:

- In 2004 an OECD-report (Mortimore et al, 2004) recommended a more conscious and systematic integration of formative assessment into school subjects to Danish educational authorities. As a result of that the initiative "Assessment Culture" (www.evaluering.uvm.dk) was introduced since 2006 with a variety of assessment tools recommended to teachers, all exemplified with descriptions of mathematics teaching for grade levels 1-10. This has lead teachers to turn their attention to the construction and establishment of adaptive and IT-based mandatory national tests in selected school subjects. Mathematics is tested in grades 3 and 6.

- A Danish Commission Report (Niss & Jensen, 2002) suggests six necessary *competencies for mathematics teachers besides the purely mathematical ones*. They are not a surprise however the juxtaposition may be. The listing below combines didactic skills with ability to engage actively in professional collaboration with parties in the school and a personal, prospective reflection and development (Georgiev et al, 2008):

Curriculum competence, i.e. to estimate and work out a curriculum	• To read, analyze and relate to current and future frames • To describe and carry out (intermediate) goals in given framework
Teaching competence, i.e. to plan, organize and practice teaching	• To keep track and interact with students, establish rich teaching and learning situations • To find, evaluate and produce teaching aids and materials • To substantiate and discuss teaching content,

	form and perspectives with students • To motivate and inspire students to commitment to math-activities
Learning uncovering competence, i.e. to uncover and interpret students' learning as well as their view and attitude to mathematics	• To understand the cognitive and affective backgrounds for mathematical learning of the individual student
Evaluation competence, i.e. to uncover, estimate and characterize the students' mathematical outcome and competences	• To choose, construct and make use of a wide spectrum of evaluation tools. Both formative (continuous) and summative (final) assessment
Collaboration competence, i.e. to collaborate with colleagues and others about teaching and framework	• To combine competences when addressing mathematical, pedagogical and didactic problems • To collaborate with parents, administration and authorities on teaching framework
Development competence, i.e. to further develop one's competences as a mathematics teacher	• To reflect on one's own teaching and identify areas for development • To choose, arrange and assess suitable activities, e.g. further education, conferences, collegial initiatives • To keep up-to-date, eventually writing articles or teaching material

- In 2006 a group asked by the Danish Minister of Education to report on *Future mathematics in primary and lower secondary* school (Niss et al., 2006) proposed a number of *actions for improving mathematics teaching*. Most of these have been acted upon to varying extent.

- In 2006 the Danish Evaluation Institute stated in a review of *mathematics at the middle school level* (*Matematik på grundskolens mellemtrin*, 2006*)* that teachers in schools seldom *use the written ministerial curriculum guidelines as a planning tool.* Use of these guidelines was recommended to be increased and in 2009 this recommendation was strengthened with changes to the curriculum guidelines (*Fælles Mål 2009*, 2009).

Finally Danish *teacher education* seems under pressure. New acts and regulations are implemented before it is possible to assimilate prior initiatives. Also pressure on funding for institutions offering teacher education are making these establishments dependent exclusively upon student numbers and exam pass rates.

In this turbulent field several University Colleges are seeking partnerships with universities and other suitable institutions, partly to maintain quality and partly to cut costs. The institutional landscape may very well change over the coming years.

- Since 2007, the *Teacher Education Act* (*Læreruddannelsesloven*, 2009) has specified the doubling of the amount of time spent on teaching mathematics in the teacher education. The course content is a combination of mathematics, mathematics didactics and pedagogical content knowledge for teaching mathematics and now offers the possibility for student teachers to specialize in either primary or lower secondary mathematics teaching.
- Only a few teachers with this improved grounding in mathematics teaching are currently employed at schools, but this will of course change in years to come. In any case the mathematics teacher education has been clearly improved since 1991 (*Undervisning og kvalifikationer*, 1992; *Rapport fra arbejdsgruppen om efteruddannelse af lærere og skoleledere*, 2006; *Undersøgelse af linjefagsdækningen i folkeskolen*, 2009, p. 11).

Proportion of mathematics teachers with mathematics as a major subject or similar competence (according to school principal's decision):

	Arithmetic/Mathematics 1991	Mathematics 2006	Mathematics 2009
Major subject	29%	57%	63%
Similar competence	18%	23%	26%

Mathematics as a major subject among mathematics teachers at grade levels 1-10:

Grade level 2009	1	2	3	4	5	6	7	8	9	10
Major subject	55%	53%	55%	57%	61%	61%	76%	77%	78%	81%
Similar competence	30%	32%	32%	31%	29%	28%	19%	18%	17%	15%

In 2010 the Danish government presented the goal outlined below (among others) for the next 10 years (DANMARK 2020, 2010):

3. Danish schoolchildren should be among the best in the world *In 2020 Danish students must be in the top five internationally for reading, mathematics and science as measured by the regular, comparable PISA studies and with regard to English compared to non-English speaking countries….*	***3. Danske skolebørn skal være blandt de dygtigste i verden*** *I 2020 skal danske skolebørn være i top fem internationalt – både for så vidt angår læsning, matematik og naturfag målt ved de regelmæssige, sammenlignelige PISA undersøgelser og for så vidt angår engelsk målt i forhold til ikke-engelsktalende*

... Since 2001 the government has undertaken a number of initiatives to strengthen schools. We have introduced mandatory national goals for education and more lessons in the subjects Danish, mathematics, physics / chemistry, English, history and PE. We have strengthened the professional level and reformed teacher education. In Denmark we have the preconditions for really good schools. We are among the communities spending most money per student. We have a skilled and dedicated corps of teachers, educators and school leaders. We have students who are happy with - and good at teamwork, and are happy to go to school. But there are things we must do better. Today, the Danish students' academic skills when they leave school are only average compared to other countries. The talented Danish students are not in the international top. ... At the same time we must maintain all the qualities that characterize Danish students: Curiosity, independence, cooperation and desire to learn. Professionalism is not just grind. It is practice. Just like no one expects athletes to become excellent in their sport without some hard training, there's no reason to expect that students will be as excellent as possible without practicing.	*lande. ... Regeringen har siden 2001 gennemført en række initiativer for at styrke folkeskolen. Vi har indført bindende nationale mål for undervisningen og flere timer i fagene dansk, matematik, fysik/kemi, engelsk, historie samt idræt. Vi har styrket det faglige niveau og reformeret læreruddannelsen. I Danmark har vi forudsætningerne for en rigtig god folkeskole. Vi er et af de samfund, der bruger flest penge pr. elev. Vi har et dygtigt og engageret korps af lærere, pædagoger og skoleledere. Vi har skoleelever, som er glade for – og gode til – at samarbejde, og som er glade for at gå i skole. Men der er også ting, vi skal gøre bedre. I dag er danske elevers faglige færdigheder, når de forlader folkeskolen, kun gennemsnitlige set i forhold til andre lande. De dygtigste danske elever er ikke i top set i forhold til udlandet. ... Samtidig skal vi fastholde alle de kvaliteter, der kendetegner danske skolelever: Nysgerrighed, selvstændighed, samarbejde og lyst til at lære. Faglighed er ikke bare terperi. Det er øvelse. Præcis ligesom ingen forventer, at sportsfolk kan blive fantastiske til deres sportsgren uden at træne hårdt, er der ingen grund til at forvente, at elever bliver så dygtige som muligt uden at øve sig.*

Everyone in Denmark would have little argument with this statement. But many Danish *teachers* often express discontent with the "endless suggestions for change" without extra resources. The school easily becomes a battle ground for political statements, blaming teachers for not keeping high (enough) standards, teaching too few lessons a week, not coping with the number of students misbehaving or having sufficient skills to teach students with immigrant background (eventually of second generation) with poor knowledge of the Danish language. Many subject teachers, mathematics included, have also shifted their identity from being professional subject teachers towards considering themselves mainly as general or class teachers.

1.2 Motives

I consider myself a rather experienced "member" of the Danish mathematics community:

- Between 1976-87 I taught mathematics in grades 1-9, and since 1987 I have been involved with Danish teacher education (and in-service training). This has provided me with an insight into the shifting challenges and conditions for mathematics teaching in primary and lower secondary schools.
- Between 1995-2009 I co-wrote a series of Danish mathematics textbooks for grades 1-7. This required the publishers to adopt a necessary and new approach with regards supportive materials for teachers. These included printed matter (books, posters, copy sheets) with a variety of tasks, computer software and teacher guides.
- From 1998 until 2004 I led an action research program on assessment tools, especially the roles of working and presentation portfolios, which included 26 Danish mathematics teachers in 2002. Then from 2003-06 I led another action research program involving more than 60 teachers in challenging gifted students in "un-streamed" classes (Mogensen, 2011). Both programs were followed-up with books in Danish (Mogensen & Abildgaard, 1999; Mogensen, 2005; Mogensen, 2006).
- From 2006 until 2010 I was involved in a national assessment scheme in Denmark. This included writing texts on evaluation tools for the official website (www.evaluering.uvm.dk) and the development of items for the yearly adaptive, IT-based and mandatory national tests in mathematics for all students in grade 3 and 6.
- I am the national chair for the external examiners in math-teacher education to primary and lower secondary schools (In Danish: Censorformandskabet for læreruddannelsen, www.laerercensor.dk/index.php). This means I read and analyze all reports from the examinations of future primary and lower secondary teachers in mathematics. I am also required to submit an annual report with my recommendations to the Minister of Education and the University Colleges of Education. I was also a member of the commission charged with the task of preparing a national action plan by the Minister of Education: *Future mathematics in Danish primary and lower secondary school* (Niss et al., 2006). And I was chair of the group which made the most recent revisions to the curriculum for Danish mathematics teacher education for primary and lower secondary schools (*Læreruddannelsesloven*, 2009).

These activities have made me aware and kept me up-to-date with the current situation of mathematics teaching and learning at primary and lower secondary level in Denmark. It also suggests some responsibility, which I accept as a consequence of my experience, knowledge and personal networks. This responsibility of course includes the presenting of research findings, but also to draw possible conclusions in the form of concrete recommendations that can be implemented without extensive changes in teachers' work conditions or economy.

1.3 Uncertainties

I know of many proficient mathematics teachers. And I am convinced that most mathematics teachers are doing their best under the local circumstances. But my own observations and along with those reported by others, suggest that improvements could be made in the way mathematics is typically taught in Danish schools.

For many years student teachers at my University College have reported on the *differences among individual guidance teachers in the way they show, suggest and expect student teachers to prepare mathematics lessons* in schools. From my personal experience of leading in-service courses, I also became aware that many primary and lower secondary teachers often do not make lesson plans. Also, if such plans exist, they are often based on headlines from the students' *mathematics text book*. It is also my impression, that text books play a very important role as *the* dominant (or almost only) source for tasks in mathematics lessons.

The type of questions for peer support uploaded onto IT-based conference forums also suggest that some teachers lack *sufficient mathematical knowledge for teaching* mathematics. There also seems to be a growing recognition that it is becoming more difficult to *reach all students* in a class. There are a variety of reasons for this – not all related to subject matter. Many teachers have classes, which include a growing number of immigrant students who have a poor mastery of the Danish language. Also, classroom management has become a cause for concern in recent years.

The economic situation has also had an impact on mathematics teaching. A growing number of mathematically weak students are being kept in normal classes. There is a reluctance in some authorities to buy new books, activities and software. Also some mathematics teachers have misgivings about relying on the results of mandatory national or local assessment schemes to inform their mathematics teaching.

The role of teachers' *beliefs* in explaining classroom management has been documented widely (Leder, Pehkonen and Törner, 2002). But as they also are conditions and formatting the mathematics teaching these understandings need consideration in my present research. Teachers' beliefs and attitudes may be seen as products of teacher education and teaching community socialization and therefore hard to change. Such knowledge also informs discussion of the Danish results (section 9.4.3).

A focus on teachers' professional routines, such as ensuring that the mathematics content taught is of a high standard as validated by curriculum guidelines may help address some of these issues. Also mathematics teaching should benefit from the steps taken to improve teachers' pedagogical content knowledge. But as mentioned above, the teachers who have followed the new education are yet to fill a significant number of jobs in the classroom.

To improve the situation now with the given circumstances, any suggestions for improvements need to be cost and time effective – as well as tempting to the professionals! It is reasonable to expect mathematics teachers in Danish primary and lower secondary schools to have solid experience in setting goals for their mathematics teaching – possibly for groups of students. A mutual high quality planning and administering of mathematics lessons might not become more demanding, but certainly more interesting to many teachers. And when addressing the issue of mixed ability classes, the teachers may benefit from schools arranging situations that would enable the mathematics teachers to share their concerns and offer advice to colleagues, which would be inspiring to everybody involved.

2 The idea of points

My personal experience and the information collected in Denmark and elsewhere, partly described above, has convinced me, that one especially important condition for quality teaching – and thus learning – is the selection and presentation of a mathematical content with a very clear purpose and correspondingly selected means.

An example of this is lesson study. This is an approach that originated in Japan, and offers a way to plan mathematics lessons through peer discussions. In the TIMSS Video Study from 1995 (*TIMSS Videotape Classroom Study,* 1999; Stigler & Hiebert, 1999), researchers documented and described a Japanese "script" for mathematics teaching. In fact, German, Japanese and American grade 8 mathematics teachers turn out to have their own special (and therefore national) features when they are observed from "outside".

According to this research Japanese mathematics teachers most often present one central problem or challenge to appropriately stimulate students' interest and desire for exploration. In class discussions and during summing up the teacher highlights this important thing. This is often done on the blackboard, where the lesson topic has been written along with questions and answers discussed along the way. In this way the teacher ensures, for example, that students have taken note of an important concept or a good technique. The Japanese teachers take great care to emphasize mathematical ideas, and they expect that mathematical concepts alone are enough to stimulate students' natural curiosity (Stigler & Hiebert, 1999 p. 90). They arrange lessons with a clear beginning, middle and end. And they lead a focused common conversation throughout the lesson.

Generally the TIMSS Video Studies have provided knowledge of the variety of teaching practices seen in many different countries, the importance of a clear purpose and link between different lesson segments and the role of the mathematics teacher's way of communicating. An extended description of such Video Study findings will be presented later (3.4.3 and 3.4.4).

Every three years PISA questionnaires are used to collect information on parameters like class sizes, teachers' educational background, school resources, teachers support for students', the general commitment of teachers and students (Mejding, 2004, p.181) or student attitudes to science teaching (Egelund, 2007 p. 99). Over the years Denmark has participated in these surveys.

For the past 30 years, however, there has been no updated systematic overview (Foss Hansen, 1980) of how the Danish mathematics teaching in primary and lower secondary schools is organized and carried out in practice. Such knowledge would be very relevant and a useful basis for making informed decisions.

Presently politicians and educational planners must rely on episodic knowledge gained in different parts of the country or firm beliefs based on their own experience or schooling. We lack updated knowledge on how mathematics teaching is actually designed by mathematics teachers.

The planning by many mathematics teachers may be dictated by working through a textbook, which in correspondence with curriculum guidelines is the lone option for formulating goals of lessons and courses. In fact Danish didacticians assume the dominant mode of instruction to be textbook-driven. This may still provide clear purpose and content for students, but we don't know. Research on mathematics teaching practice is needed to qualify such assumptions, and such research could offer suggestions for change if needed. Lesson studies may open the way to teachers being able to use each other's ideas and strengths a little better.

2.1 Danish recommendations

It requires confidence and a good mathematical overview to break with traditional, textbook-driven instruction. The report *Future Mathematics in primary and lower secondary school* (Niss et al., 2006) recommended mathematics teaching to be point-driven (author's translation and emphasis):

The main mathematical goals, the goals in lessons and the didactical points, the teacher focuses on, should be driving the planning, while textbooks and other materials and activities are means to achieve this goal. The goals of (a sequence of) lessons may be understood as the detailed learning- and competence-related outcome that the lessons aim to provide. The didactical point of a lessons may be understood as the climax, the moment of sharpened insight (the "aha" experience) with respect to an academic concept or result, a mathematical method or technique or when noticing the relationship between different concepts or themes, eventually stressed in the final summing up. However such goal and point-driven teaching requires the very existence and availability of didactical carefully prepared and varied teaching materials, not the least for the teacher to reach all students with appropriate challenges. … **Recommendation Va: Mathematics teaching should be organized into sequences focusing on mathematical goals and didactic points**	Det er de overordnede faglige mål, målene for de enkelte undervisningsforløb samt de didaktiske pointer læreren fokuserer på, der skal være styrende for planlægningen, mens lærebogen, samt andre materialer og aktiviteter, er midler til at nå dette mål. Ved målene for et undervisningsforløb forstår vi det nærmere lærings- og kompetencemæssige udbytte som det pågældende undervisningsforløb sigter mod at bibringe eleverne. Ved en didaktisk pointe for et forløb forstår vi det klimaks i form af en tilspidset indsigt (”aha-oplevelse”) i et fagligt begreb eller resultat, en faglig metode eller teknik, i sammenhængen mellem forskellige begreber eller emner m.v. som tilstræbes ved afslutningen af forløbet. En sådan mål- og pointestyret undervisning fordrer dog i høj grad eksistens af og adgang til didaktisk gennemarbejdede og varierede undervisningsmaterialer, ikke mindst for at læreren kan nå alle elever med de rette udfordringer. … **Anbefaling Va: Matematikundervisningen skal tilrettelægges i forløb der fokuserer på fagpædagogiske mål og fagdidaktiske pointer**

Addressees: Mathematics teachers and the schools' educational management.	Adressater: Matematiklærere og skolernes pædagogiske ledelse.

If it aint broken don't fix it! Why change anything? These may be sensible statements. But some qualities in lower secondary student *learning* of mathematics are measured regularly and by many found unsatisfying low – even above a PISA average.
And assuming teaching quality is correlated to student learning *and* the presence of articulated points an analysis of mathematics *teaching* seems needed before possible recommendations or decisions. Among current suggestions for improving *conditions* to teaching are fewer students in class or 2-teacher arrangements. This may work well in combination with e.g. manageable goals, books and peers. Regrettably however, we lack knowledge about how mathematics teachers would profit by this.

This recommendation above implies a need for a further investigation of the actual occurrence of points in Danish mathematics teaching. Impressions alone won't do, we have to know the degree of presence or absence of mathematics didactic points to suggest and decide on proper actions.
The Danish report gives additional reasons for the recommendation:

International studies suggest that there may be significant gains in having the teacher build courses and lessons around subject oriented climaxes. Work on organizing teaching of explicit goal- and point-driven mathematics courses may profit from sharing knowledge with colleague mathematics teachers, and could thus enhance both teaching and the professional identity and culture at the school.	Internationale studier peger på, at der kan være betydelige gevinster at hente ved at lade undervisningsforløb og lektioner være bygget op omkring et af læreren planlagt fagligt klimaks. Arbejdet med organisering af undervisningen i eksplicitte mål- og pointestyrede forløb vil kunne trække på de potentialer, der findes hos den enkelte og hos faglærerteamet, og vil dermed kunne styrke både undervisningen og den faglige identitet og kultur på skolen. …

The Danish report finally links the effect of points as the controlling or guiding factor in mathematics teaching depending on the *way or place* in which they appear in teaching. This calls for a discussion of the different roles such points may have at the beginning of a lesson as part of or supplementary to a proposed goal or content, during the lesson as one or more climaxes or at the end in a summing up of the lesson. As points are viewed as crucial to sequences of mathematics teaching by many mathematics educators, this proposes an analysis of the kind of sequences actually seen in singular or connected mathematics lessons.
Clearly, there are various reasons to investigate the *occurrence of points* as seen in Danish mathematics teaching. Points may differ in *the way they appear*, are said or shown. They may be presented by the teacher only, or in certain types of classroom communication.

2.2 Research questions

Many questions keep popping up: What drives the mathematics lessons – i.e. gives or offers direction and level? Are points in lessons planned for and expected, or perhaps more like a pleasant surprise? Are there even other aspects, not expected or realized beforehand that relates to quality teaching? Do mathematics teachers e.g. have other or higher priorities? What is researchable at all?

I decided upon three research questions:

RQ1: To what extent, how and why do teachers *articulate* mathematical points in Danish mathematics teaching?
RQ2: In what way can the occurrence and role of mathematical points be *strengthened* in mathematics teaching practice?
RQ3: To what extent and by which means can mathematics teachers be *supported* in *point-driven* mathematics teaching?

There are of course several ways to design research on such matters. Does it e.g. indicate a quantitative and / or a qualitative approach? My choice is based on the considerations outlined above and of the time available. It should be noted that I carried out the research for this project on my own. The research questions could refer to all grade levels, but my research will have a focus at lower secondary mathematics teaching.

Regarding RQ1
So what are points in a mathematics teaching context?
Points contribute to a *description* of actual mathematics teaching. Contexts and frequency of points in Danish mathematics teaching will be researched. As will how points are made in oral, written or visual uses of mathematical language including representation, symbols, formulae, models etc.?
Will points be more often articulated in lessons taught by the better educated teachers, younger ones, teachers taking part in collaborative efforts at school, in larger cities or with fewer students in class? These are some of the hypotheses put forward in debates on qualitative change in teaching and learning. Assuming that some Danish mathematics lessons have observable points and some have not, the study also asks why this is the case. My hypothesis is, that teachers with mathematics as a main subject will present more point-driven teaching than teachers without this background.

Regarding RQ2
Points become of *normative* interest to many parties, when convinced that they are drivers and / or indicators of mathematical insight. The role of points as intermediates or goals should be considered. What or where are the obstacles and the ideal conditions for an increase of point occurrence? What does it take to make teachers plan and execute mathematics teaching leading to articulation of more points? It is my hypothesis that all mathematics teachers consider mathematical points as crucial statements but that many teachers rarely articulate such points to students by other means than by mathematical textbook presentations.

Regarding RQ3
Peer support of various kinds may facilitate change inside given conditions. In some countries teacher education, a certain organization or tradition seems to ensure a high standard in student learning according to international comparisons being made. One such example is the Asian lesson study tradition. I assume that many teachers will appreciate peer support as a supplement to e.g. that of textbooks and teacher guides in letting mathematical points steer their teaching – whether this is by by articulation or as underlying premises.

Findings leading to actual suggestions will be directed at the whole education system, but with a focus on lower secondary mathematics teaching.

- The mathematics *teacher* community may expect to be informed in curriculum guidelines of the newest and relevant research findings applicable in teaching.
- Mathematics *teacher educators* are regularly seeking to include recommended research findings in teacher education.
- And *mathematics research* communities constantly exchange and compare national and international findings.

3 Previous research

There are a vast number of reports, research papers and books on lower secondary mathematics teaching both in Denmark and elsewhere. Some are monitoring studies with a focus on the description, interpretation and evaluation of current practice, and some are intervention studies which aim to explore the effects of deliberate changes in practice.

The survey of previous research has more goals: One is to situate my research in the scientific landscape of parameters similar to my point notion in mathematics teaching. Has my research questions been researched by others, or what reminds of my research in scope? And are there methods and findings which should affect my research design?

Ideally I would restrict my search using the criteria outlined here:

- Research about the extent, the “hows” and the “whys” of articulated points connected to concepts, algorithms, results or interpretations in mathematics teaching – preferably at the lower secondary grade level.
- Research about the possibility to strengthen the occurrence and role of mathematical points in mathematics teaching.
- Intervention studies about support for mathematics teachers in point-driven teaching; in-service initiatives; intervention or action research and teacher education.

Such criteria are closely connected to my three RQ’s. But as points do have several different meanings a search cannot be restricted to one such keyword as points. Also research on other issues in mathematics teaching may offer methods or findings absolutely relevant to consider in my research. Therefore the research found and referred to below is on somewhat wider issues and done by consulting various sources.

I have been looking through the multiple sources listed below, e.g. reports from large (quantitative) international research studies, selected surveys of research on mathematics teaching, international handbooks, peer-reviewed journals, international conferences and international article databases. The research I found most relevant to my own is presented below following a short overview of my search procedures:

Research surveys

Two research surveys:

- An international survey on mathematical education (Lerman & Tsatsaroni, 2004)
- An international survey on mathematics teacher education (Adler, Ball, Krainer, Lin & Novotna, 2004).

International handbooks

- Bishop, A.J.; Clements, M.A., Keitel, C.; Kilpatrick, J. and Leung, F.K.S. (Eds.) (2003). *Second International Handbook of Mathematics Education*. Kluwer Academic Publishers.

- Lester, F. K. (Ed.) (2007). *Second Handbook of Research on Mathematics Teaching and Learning*. Information Age Publishing.
- Sullivan, P., Tirosh, D., Krainer, K., Jaworski, B. and Wood, T. (Eds.) (2008). *The international handbook of mathematics teacher education*. Sense Publishers.

Conference proceedings

Proceedings from the ICME-conferences I have attended since 1992 have been searched, three of them by access to websites: ICME-7 1992 (Canada), ICME-8 1996 (Spain) http://mathforum.org/mathed/seville.html), ICME-9 2000 (Japan), ICME-10 2004 (Denmark, www.icme10.dk), ICME-11 2008 (Mexico, http://icme11.org).

Two more conference proceedings have been selected partly for representativeness, partly because I also attended them myself:

Proceedings from PME-30 2006 (http://class.pedf.cuni.cz/pme30) and CERME-6 2009 (www.inrp.fr/editions/editions-electroniques/cerme6) are reviewed.

I have not, however, included the extensive range of other PME's (Conferences of the International Group for the Psychology of Mathematics Education) or CERME's (Conferences of European Research in Mathematics Education) in my research.

Extensive International comparisons like FIMSS, SIMSS, TIMSS and PISA

For the TIMSS (1995) and TIMSS-R (1999) I refer to the final reports on classroom studies (sections 3.4.3 and 3.4.4). For the PISA studies I refer to the International and the Danish reports 2003, 2006 and 2009 (section 3.4.5).

Journals

ESM (Educational Studies in Mathematics) is searched since 1999, and ZDM (The International Journal on Mathematics Education) since 1997, both by using Springer Link. JRME (Journal for Research in Mathematics Education) is searched since 1999 by looking through issues at the NCTM website.

The Nordic journals MONA (Mathematics and Science Education: The Journal for teachers, facilitators and researchers) and NOMAD (Nordisk Matematikk Didaktikk, Nordic Studies in Mathematics Education) are searched by looking through all copies back to 2005 and 2000 respectively.

Search by keywords

Electronic access is used for several article bases.

The keywords below were used alone or in combinations to narrow the search:

a) Mathematical point / idea
b) Mathematics classroom research
c) Mathematics teachers' questioning, listening, responding, elicitation, communication
d) Phase in / structure of mathematics lesson
e) CK / Mathematical Content Knowledge
f) PCK / Pedagogical Content Knowledge

g) Intervention study / action research in mathematics teaching.

The findings most relevant to the actual Danish research are separated according to their relation to teaching practice, quality parameters and intervention, that is partly corresponding to my three research questions and also whether they are in Danish or international studies.

3.1 Research surveys

At ICME-10 Lerman and Tsatsaroni presented a survey showing the different perspectives, positions and approaches in *mathematics education* research (Lerman and Tsatsaroni, 2004). This was based upon an analysis of a representative sample of the papers in the Proceedings of the International Group for the Psychology of Mathematics Education (PME) for the past 12 years, and of two journals: ESM (Educational Studies in Mathematics) and JRME (Journal for Research in Mathematics Education). Among their findings is a notable increase in the use of qualitative methods in research papers (though not in PME). They also report that researchers most often are teacher educators, research in mathematics education is "*almost exclusively around the school"* and they note the absence of "public intellectuals". Adler et al (Adler, Ball, Krainer, Lin & Novotna, 2004; Niss (Ed.), 2004) in a plenary presentation at ICME-10 referred the work of Survey Team 3 on research in *mathematics teacher education* 1999-2003. The trends noted were:

- Small scale qualitative research is dominating.
- Teacher educators study the teachers they work with.
- Countries with English as the national language dominate the research.

Among the important questions still not answered the authors mention issues relating to how teachers learn outside "reform" contexts or the "scaling up" of programmes.

Such reviews do not necessarily affect my present research. But it *is* interesting to note, whether you are in the main stream or paving new ground on content or methods. E.g. my research uses *both* quantitative *and* qualitative methods in searching for answers to the RQ's. And I will discuss the challenge of scaling up at the end of this thesis (section 14.3).

At the ICME-11 in Mexico 2008 a topic study group offered an overview of research on classroom practice. My research also involves "*organizing observations, means of observation, ways of describing and recognizing teaching-learning phenomena*" (http://tsg.icme11.org/tsg/show/25) to understand various practices in different didactical systems and the theories supporting these practices, and how such descriptions may help the development of points-based teaching. So, for example, Brousseau's contribution (section 3.3.3) on observation made me look for patterns in teacher reactions. Both as observed in experienced teachers, and also as suggested in action research. Findings from my actual research are also presented and discussed in ways adopted by this topic study group. E.g.:

- Analytical accounts of empirical lessons based on observations of classroom practices (chapters 6-7, 13).
- Perspectives (theoretical, socio-cultural, political) informing different classroom practices and their analysis (chapters 9, 14).

At the ICMI centennial symposium in 2008 researchers presented overviews and research reports relevant to my own project. Boaler (Boaler, 2008a) referred to some projects that successfully bridged the gap between research and practice. I found the reference to Mei and Yan's (Mei & Yan, 2005) Singapore project particularly inspiring as an example of the impact research findings may have on new approaches to teaching, learning and assessment in schools. As I see some parallels with my own research as a less extensive, yet careful examination of the practice of Danish mathematics teachers I hope it might lead to similarly effective proposals (chapter 14).

Grevholm and Ball (Grevholm, B. & Ball, D. B., 2008) covered methodological issues in studies of mathematics teacher education. Larger studies as the *Learning Communities of Mathematics* (section 3.3.2) and COACTIV (section 3.3.4) are seen as informative about their variations on methods in action research, which I have also chosen for my research. Even (Even, 2008) reports on a somewhat similar attempt to address professional development for high school teachers in the Israeli two-year MANOR program. Such methods and experiences of involving mathematics teachers in professional development with ongoing support from experts and other members of the community underly my intervention studies in the present research (chapters 10-12).

Skott (Skott, 2008) explains problems of implementation approaches by considering conditions in the social context. Such mechanisms are discussed (section 9.4) and confirmed in my findings also (section 10.4). The intervention studies relating to RQ3 are therefore to be imbedded in the school culture (sections 13.4, 13.5).

The ICMI Study on the Professional Education and Development of Teachers of Mathematics (Even & Ball, 2009) is another overview of practices and programs for prospective and practicing teachers of mathematics. Matos et al's "Mathematics Teachers' Professional Development: Processes of Learning in and from Practice" (Matos et al, 2009) also seems of particular relevance to my research as they claim that the move from a training model to a practice-based model for the professional development of mathematics teachers reflects a significant change in our understanding of learning. I consider this to be in line with the view on necessary competencies for mathematics teachers to engage in professional collaboration, as already referred to in 1.1 (Niss & Jensen, 2002).

3.1.1 International handbooks

The Second International Handbook on Mathematical Education is in two volumes (Bishop, 2003). And the second part has a section on "professional practice in

mathematics education” which seems especially relevant to my research questions. Over seven chapters researchers discuss developments in research and practice that seem to aim at reducing the gap between research recommendations and classroom practice. This does not exactly fit with my RQ’s, but an intervention to challenge and change mathematics teaching may utilize the ideas and experience of others.
I found one contribution especially interesting: Tirosh and Graeber discuss the “*Challenging and changing mathematics teaching classroom practices*” (Tirosh & Graeber, 2003) by focusing on two forceful factors in teacher change: (1) values and beliefs like “mathematics-for-all” and (2) technological advances as the introduction of calculators and computers. They also discuss crucial dimensions of planned effort to change classroom practice: an organizational approach as top-down or bottom-up, and the nature of professional development as e.g. the components required for proficient teaching following the *Adding It Up* publication (Kilpatrick et al, 2001, p. 216):

- “*Conceptual understanding of the core knowledge required for mathematics teaching*
- *Fluency in carrying out basic instructional routines*
- *Strategic competence in planning effective instruction and solving problems that arise underway*
- *Adaptive reasoning in justifying and explaining one’s instructional practices and in reflecting on those practices so as to improve them*
- *Productive disposition towards mathematics, teaching, learning and the improvement of practice*.”

In my definition below of a mathematical *point* (chapter 4) subject proficiency is regarded somewhat differently than listed above. A clearly delineated, significant mathematical content will include mathematical concepts, but procedures, results and interpretations are considered equally important.

The *Second Handbook of Research on Mathematics Teaching and Learning* is in two volumes (Lester, 2007). Here Schoenfeld (Schoenfeld, 2007) reminds of trustworthiness, generality and importance as three very important criteria for quality research. Such considerations are followed in sections on validity (section 8.9), credibility (8.10) and discussion on actual research findings (chapter 13).
Hill et al (Hill, Sleep, Lewis, and Ball, 2007) presents important views and findings on mathematical knowledge for teaching. Some of these U.S. findings are also referred to by other references in this handbook (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep & Ball, 2008, section 5.9.2).
Philipp (Philipp, 2007) presents an overview of research on teachers’ beliefs and affects, which I refer to in a section on school culture (section 9.4) The review by Franke et al (Franke, Kazemi & Battey, 2007) mentions projects and understandings you may illustrate by other sources, e.g. the role of tasks in the QUASAR project (section 3.6.3) or classroom discussions as they are prompted by teachers’ elicitation

(sections 6.7.1 and 10.5). And Tate and Rousseau (Tate & Rousseau, 2007) discuss the challenge of scaling up promising designs, which is also relevant (section 14.3).

The International Handbook of Mathematics Teacher Education is in four volumes (Sullivan et al, 2008). Stacey (Stacey, 2008) gives an overview of recommended discipline knowledge for secondary mathematics teachers in several countries as consisting of:

- Knowledge of the content of mathematics
- Experience of doing mathematics
- Knowledge about mathematics as a discipline
- Knowing how to learn mathematics.

She concludes that there is a lack of research in how gaps in teacher knowledge can be overcome. This is regrettable as I expect the absence of *points* to be partly explained by teachers' lack of knowledge. Therefore it becomes even more important to demonstrate a possible compensating intervention (RQ3).
Graeber and Tirosh (Graeber & Tirosh, 2008) describe pedagogical content knowledge (PCK) as originally suggested by Shulman as "*the particular form of content knowledge that embodies the aspect of content most germane to its teachability*" (p. 118) and suggest further research on the relations between different definitions of PCK and "quality teaching" in mathematics. They ask about the validity of proposed components of PCK for teaching mathematics and ask which components of PCK are actually *best* considered in the education of prospective teachers, and which are *best* learned in later professional development, or on the job. My RQ3 is on needs and peer support of already practicing mathematics teachers. The "*best*" practice will be left for the participating teachers to decide, as the extent and nature of support needs to be managed by the teachers involved.
Many researchers offer advice for colleagues entering research. One relevant example is Brown (Brown, 2000), who by examining research on effective teaching found that an ideal empirical study should combine iteratively both quantitative and qualitative methods. *"Small-scale mainly qualitative studies lead to insights which can be tested on a wider scale; patterns in large-scale quantitative data suggest associations which can be explored and better understood by case-studies, and so on."* (p. 105).
In the present research quantitative and qualitative methods are combined in answering RQ1. The search for "how" patterns requires more types of data from many teachers, and to answer the "why" a subsequent analysis demands a closer look at reasons by considering comments from teachers and interviewing teachers about their understandings and beliefs. The RQ2 and RQ3 will be dealt by qualitative approaches only.

Research projects based on observations and possibly even video recordings of classroom teaching are becoming more frequent. Video makes it possible for the researcher to witness teaching modes and strategies in the actual setting. Thus teaching routines and competences may be evaluated by suitable coding of transcripts and actions.

But as pointed out by Staub (Staub, 2007) the methodologies used for the purpose of identifying practices and patterns for describing classroom cultures vary a great deal. He poses three fundamental methodological questions:

1. *What are the units of analysis to guide the sampling practices recorded for the purpose of identifying patterns (lessons or sequences of lessons)?*
2. *Based on what unit of analysis are the video recordings of teaching to be analyzed (whole lesson, episodes or sequences of lessons)?*
3. *What conceptual resources are to be used to arrive at reliable and practically useful descriptions of practice?* (p. 320)

My research questions require different strategies, which will be clearly described in the following chapters. My RQ1 requires one whole lesson as the sampling unit but visits to many classrooms to identify patterns. For RQ2 and RQ3 it is episodes of lessons that are considered in the analysis and stimulated recall and more lessons by the same teacher are observed to look for possible changes.

3.2 Research on teaching practice in Denmark

3.2.1 Lesson structure

The latest only large scale investigation of Danish mathematics teaching routines was published in 1980 (Hansen, 1980). This was based on two-hour long interviews with each of four experienced mathematics teachers (picked by the headmaster) in a randomly chosen school in each of 18 municipalities surrounding Copenhagen. 71 teachers teaching grade 1-10 were involved. The data from these interviews were on teachers' experience, organization, teaching strategies and materials. There were no observations of mathematics lessons. The report is mainly on the variation in organization of lessons, and among the findings were, that mathematics teachers mainly organized traditional classroom teaching in phases as: (1) Correction of homework, (2) Presentation of new content and (3) Individual seatwork.

Three patterns were dominant:

- Type 1, e.g. 15 % of classes involved, mainly had two phases: One very long phase of teacher presentation and one very short individual seatwork phase. Correction of homework was absent or addressed briefly at the beginning of lessons.
- Type 2, e.g. 56 % of classes involved, mainly had the same two phases as type 1, but they were more equal in time. These lessons could also include a shorter phase for correction.
- Type 3, e.g. 10 % of classes involved, mainly had two phases: One very short introduction to new content and one very long individual seatwork phase.

Group work and experimental work was rare. Classes were kept working on the same tasks. When dialogue occurred during phase 2 it was between single students and the teacher. Teachers used a "check-dialogue", where possibilities to "invent answers

and use guesses were sparse, as they will only have a reasonable possibility in a progressive dialogue:
"*.. presentation is often ended* before all students have had time to assimilate the information, meaning that students have few possibilities to reach their potential. *Moreover failing to provide all students with a common, relevant background which may inform later teaching will give students trouble in structuring their new knowledge, as they are only provided with a few opportunities to verbalize this new knowledge."* (p. 63, own translation).
Organization of teaching is dependent upon the textbook used by the teacher. And the length of time spent on a topic is decided by teachers' estimation of the importance and complexity of the mathematical content.
The occurrence of important mathematical ideas was *not* registered in this relatively old study.
I shall later describe such phases and patterns, as they became evident during video recording and coding of lessons in my present research (sections 8.2-4). Even though an overview of lesson structure was not the main objective of my research, it is relevant to the 'why' discussions addressed in RQ1 (13.1.3).

3.2.2 Mathematics at Danish Middle school level

In 2006, seven schools participated in a process of self-evaluation, preparation of written reports and visits from EVA, The Danish Evaluation Institute. *(Matematik på grundskolens mellemtrin*, 2006). And two nationwide questionnaire surveys were carried out among mathematics teachers in Middle school grades and among school principals.
In total 918 schools participated in the study. At all the schools the principal and one or two teachers were invited to participate. Of the 1267 teachers who received the questionnaire, 63% (793) answered. The low return rate seems to be due correspondence being lost as the forms were sent by researchers to the principals who were then asked to forward them to teachers. Despite the low response rate the study, however, still seems to provide a partially representative picture of how school leaders and teachers work to develop students' mathematical skills.
The report concluded that, "*many teachers seem to lack tools that can help them to continue to develop and professionalise their mathematics teaching, and this development task is to a too high degree left to the individual teacher*." (p. 9, own translation).
The survey partly bases such conclusions on mathematics teachers' views on the subject and conclusions drawn from the questionnaires:
"Firstly, the study shows that the view on mathematical competence found in the formal curriculum guidelines, generally have not reached a wider dissemination among teachers in Middle schools as part of their teaching practice, let alone as an explicit dimension in their overall vision of the profession or in their view on learning. This conclusion is supported by the fact that the planning of teaching practice to a much higher degree is content-driven rather than objective-driven

(refers to Section 5.1.2). In this context it is also striking that among the teachers at the self-evaluating schools there are very few references to the general changes and demands of the school Act of 1993 and its amendments in 2003, including the importance of these changes for the mathematical and pedagogical practices relevant to mathematics teaching. It confirms the impression that changes in the formal framework for teaching are seldom sufficient in themselves to change pedagogical practice in classrooms." (p. 23, own translation).

The questionnaire contained specific questions on e.g. *Lesson planning and implementation*. And findings were presented on these key areas (Chapter 5, p. 37-62):

• *Common Goals* and other sources for teaching planning • Importance of yearly plans to practice • Ways of organizing and working • Differentiation • Educational evaluation and on-going assessment • Student influence	• *Fælles Mål* og andre kilder til planlægning af undervisningen • Årsplanernes betydning i praksis • Organiserings- og arbejdsformer • Differentiering • Undervisningsevaluering og løbende evaluering • Elevindflydelse

The mathematics teachers were asked to what extent their selected textbooks were important in the preparation and implementation of teaching.
"*Three quarters of the teachers answered that the textbook has a significant influence on the sequence of topics in the teaching. Several teachers suggest that there are a number of benefits to following a textbook system. Teachers argue that progression and coherence in teaching is created by following the book's chapters. .... Finally, teachers often believe that using a particular textbook automatically ensures that their teaching meets the requirements of the Common Goals ....*" (p. 40, own translation).

According to the survey mathematics teachers use various ways of organization and work in their teaching, which is borne out by the self-evaluation reports and impressions from visits. Classroom instruction is preferred for the presentation of new content, homework review and common class dialogues. The report concludes on organization practices:
"*… pair-work is the most preferred kind of organization together with individual work and classroom teaching. Virtually all the teachers indicate that they use these organizational forms to some extent (and in some instances to a significant extent). Group work is used in lesser extent (41%) ...*" (p. 45, own translation).

The responding mathematics teachers had questionnaires with questions like Question 39: *To what extent do you see the need for more efforts to develop* *mathematics teaching at the school?* (In Danish: *I hvilken grad vurderer du, at der er behov for flere indsatser for at udvikle matematikundervisningen på mellemtrinnet? (Arbejdet med at udvikle elevernes matematikkompetencer på folkeskolens mellemtrin*, 2006, Appendix 2, p. 29*)*. Answers show, that teachers see a need for peer support (N=768):

	To a high degree	To some extent	Not much	Not at all
By the mathematics teacher team	35 %	51 %	11 %	2 %
By school management	10 %	38 %	42 %	10 %
By school board	4 %	14 %	48 %	32 %

3.3 Research on teaching practice in other countries

As my research is confined to a national setting, it is of obvious interest to review similar attempts to report on a national characteristic in other countries. The criteria are still a focus on teacher's points in planning and mathematics teaching preferably at the lower secondary level.

3.3.1 Mathematics Education Research Group, MERG (Norway)

Cestari (Cestari, 2004) reports on the MERG project in Norway, where an inquiry / analytical modus was introduced among teacher students during the course: *Teaching and Learning.* The model has been developed over a 10-year period with these aims:

1. To produce knowledge (for entering dialogues with students on arguments, challenges, analysis, evaluation, etc.) directly linked to teaching and learning practice
2. To relate personal experiences in learning and teaching of mathematics to theoretical questions presented in literature
3. To analyse activity related to the context where professional practices take place.

Mathematics lessons have been observed and analysed – since 2004 using video. Cestari demonstrates in a microanalysis example how *"complex questions can be answered in a limited period of time. In this second fragment, for example, the question has four elements, including the solution and the approaches the pupil used in order to find the solution. Secondly, they can learn how there are elements of different types embedded in the same question. Thirdly, they can learn how pupils display the different reasoning procedures they use in order to find the solution to the problem, if they are asked to do so. The procedures used to solve the problem posed by the teacher became visible, thus offering possibilities for analysis, evaluation and, if necessary, re-examination"* (p. 5).

Classrooms become an object of study for these mathematics student teachers, who in this project are supposed to identify possibilities for new ways to present problems,

to explain, and to give feed-back and thus hopefully be better prepared to their own future in teaching.
Many other studies have focused on communication between teacher and students in qualitative settings. Bjuland, Cestari and Borgersen (Bjuland, Cestari & Borgersen, 2009) identify certain communicative strategies used by one mathematics teacher for sixth grade students. Riesbeck (Riesbeck, 2009) uses discourse analysis in her research on teaching Middle school students. She found words like *speak, think and write* to reinforce students' consciousness and participation in mathematical thinking. Observations on teachers' elicitation are discussed in sections 6.7.1, 10.5 and 12.3.

3.3.2 Learning Communities of Mathematics (Norway)

Jaworski, who for a while was working at the Agder University in Norway describes (Jaworski, 2006) a collaborative model of school teachers and university educators engaging in research / inquiry to explore the design and implementation of tasks, problems and activity in classrooms. The educators' role was to use inquiry as a tool to enable mathematics teachers to develop their teaching. Twelve didacticians and 40 teachers in eight schools took part. Analyzing data from meetings with the didacticians "*to conceptualise our roles in workshops and schools, highlights tensions, the addressing of which affords considerable learning opportunity about the practices of creating inquiry communities with teachers."* (p. 206). The Norwegian experience shows that inquiry communities are not so easy to establish but may provide a basis for critical alignment of the different social conditions in which teachers work.

3.3.3 Didactic Contracts (France)

Brousseau's observation of classroom practices carried out at the COREM (Center for Observation and Research on the Teaching of Mathematics) is well known. Some classroom practices are explained as effects of a didactical contract between teacher and students. In an ICME-11 paper (Brousseau, 2008) he underlines the importance of researchers' conscience of effects of observation itself on the practices of teaching. For example when observing the teachers' strategies as they perceived or noted an insufficient or failed attempt to teach some determined piece of knowledge. Teachers' spontaneous responses to the failure of a didactic attempt may vary and the COREM observation of practices and discussions with teachers made it possible to reveal not only isolated decisions but also genuine rhetorical strategies. Brousseau considers them to be effects resulting from contradictory requirements of the didactic contract.
Observation of practices and discussions with teachers revealed these rhetorical strategies (p. 3):

- The teacher can put an error on trial by comparing the wrong answer to students' prior knowledge in order to get the right answer with reference to the justification.
- The teacher tries to explain the error with one or more "causes", acknowledging the need for new teaching.
- The teacher puts all parties in the situation on trial including him/herself.

But the teacher may also resume the failed teaching by several methods (p. 3):

1. Restart the issue in its original form, but now freed of redundant information.
2. "Decomposition of the question" as in the case of a teacher breaking up a mode of resolving the issue into steps, so providing a new set of "simpler" issues.
3. The "Topaze" effect, which is when the teacher requires a formally correct answer but achieves it by illegitimate means (e.g. by whispering the answer to the student).
4. The "Jourdain" effect, which is when the teacher accepts an answer that the student has achieved and justified "illegally", but also accepts and validates these means.
5. "Misuse of analogy" when the teacher gets the right answers from the students by mentioning a "similar" problem and the responding student reproduces a solution not because s/he sees it fits the problem, but because it is consistent with the given model.
6. "Meta didactic slippage" when the teacher explains his/her explanation.
7. "Didactic permeability" when the teacher is so eager to explain things to the students that s/he introduces knowledge, concepts and language that is only poorly understood by the students.

Student errors are not coded in my research, but teachers' *elicitation* (section 6.7.1) and *hints* (section 6.7.2) are coded to identify challenges to mathematics teachers. The methods identified by Brousseau offer a fine mesh for a possible further analysis of teacher reactions and capabilities.

3.3.4 Cognitive Activation in the Classroom, COACTIV (Germany)

A goal of this German project was to "*conceptualize the professional knowledge of secondary teachers and on this basis, to construct reliable tests*" (Krauss et al, 2008; Kunter et al, 2007).

Questionnaires and tests on Content Knowledge (CK) and Pedagogical Content Knowledge (PCK) were constructed and used for those mathematics teachers, whose classes participated in PISA 2003. Also parallel questionnaires on lesson attributes as classroom management, cognitive activation and individual learning support were administered. 35 open-ended items were administered to approximately 200 secondary mathematics teachers in two rounds. 43% of these teachers taught at Gymnasiums, and therefore had a more content-focused education.

Among the findings were, that PCK profits from a solid base of CK, although such correlation was less strong among teachers in the NGY group (not teaching at gymnasiums). Also no positive correlation was found between years of teaching practice and the two knowledge categories. Therefore teacher training is assumed to be the crucial core to these categories of knowledge.

This result will be interesting to compare with possible patterns in my research on the occurrence of points related to education background and years of teaching practice (section 8.7).

3.4 Comparative research on teaching practice

3.4.1 FIMS and SIMS

Many international studies have focused on mathematics classroom research and student outcome at lower secondary grade levels.

Since 1964 the IEA (International Association for the Evaluation of Educational Achievement) has carried out international research on teaching and learning. The First and the Second international Mathematics Study, FIMS and SIMS, were conducted at age 13 and pre-university level in 1964 and 1980-82. Denmark did not take part in FIMS and SIMS.

3.4.2 TIMSS 1995 and 1999

TIMSS (Third respectively Trends in International Mathematics and Science Study) involved three student populations in the participating countries. In 1995 these were:

- POP1: 9-year old students (the two grade levels containing most students. Denmark did not take part in this).
- POP 2: 13-year old students (in Denmark the two grade levels were grade 6 and 7, in most other countries this was grade 7 and 8. Denmark included 2500 more grade 8 students and 2500 more grade 9 students in the national testing used to put the international test into perspective. In total 514 classes with teachers and principals from 168 schools in Denmark were involved (Weng, 1996).
- POP 3: The final year of secondary education (in Denmark at gymnasium, commercial- and technical schools).

In 1999 TIMSS was repeated (therefore also known as TIMSS-R) at grade 8 (Denmark was not included).

The data from TIMSS are meant to provide answers to general questions on the intended, the implemented and the attained curriculum (student outcome). The data comes from questionnaires sent to experts, principals, teachers and students. Curriculum guides, textbooks and other curricular materials are analyzed. Students are tested, and data on the demographic and economic resources are also collected. Teaching is NOT observed, but the so-called TIMSS Video Studies from 1995 and 1999 did offer such observation. These are described below (section 3.4.4).

In 1995 the questionnaires showed that most Danish mathematics teachers did not believe in the idea of giving mathematically weak students more skill training during lessons. 70% of the Danish mathematics teachers found it important to understand how mathematics is applied in everyday life and 60% that students should be able to justify their solutions (Weng, 1996 p. 152).

As another example of the type of results offered by TIMSS I attach an excerpt from: "*What Activities Do Students Do in Their Mathematics Lessons?"* (TIMSS 1999, 2000, p. 210):

Exhibit 6.10 presents a profile of the activities most commonly encountered in mathematics classes around the world, as reported by mathematics teachers. The two predominant activities, accounting for nearly half of class time on average, were teacher lecture (23 percent of class time) and teacher-guided student practice (22 percent).

Exhibit 6.10 Time Spent on Various Activities in Mathematics Class

	Average Percentage of Class Time Spent in a Typical Month of Lessons							
	Administrative Tasks	Homework Review	Lecture-Style Presentation by Teacher	Teacher-Guided Student Practice	Re-teaching and Clarification of Content/ Procedures	Student Independent Practice	Tests and Quizzes	Other
Australia	4 (0.4)	9 (0.5)	19 (1.3)	22 (0.9)	12 (0.6)	22 (1.3)	8 (0.3)	3 (0.4)
Belgium (Flemish)	4 (0.3)	7 (0.4)	24 (1.1)	29 (1.0)	10 (0.4)	14 (0.9)	10 (0.3)	2 (0.4)
Bulgaria	2 (0.4)	7 (0.4)	37 (1.7)	18 (1.1)	10 (0.5)	14 (1.3)	12 (0.6)	1 (0.3)
Canada	r 5 (0.2)	r 14 (0.4)	r 20 (0.9)	r 18 (0.8)	r 10 (0.3)	r 20 (0.7)	r 10 (0.3)	r 3 (0.6)
Chile	6 (0.7)	14 (0.6)	24 (1.2)	18 (0.9)	19 (0.8)	8 (0.5)	12 (0.6)	3 (0.5)
Chinese Taipei	3 (0.6)	12 (0.5)	39 (1.3)	15 (0.5)	11 (0.6)	9 (0.5)	10 (0.5)	2 (0.4)
Cyprus	r 3 (0.4)	r 21 (0.8)	r 17 (1.0)	r 25 (1.0)	r 12 (0.5)	r 10 (1.0)	r 9 (0.7)	r 2 (0.3)
Czech Republic	3 (0.3)	5 (0.4)	23 (0.7)	29 (1.2)	10 (0.5)	19 (1.0)	9 (0.6)	3 (0.4)
England	s 3 (0.2)	s 6 (0.5)	s 18 (0.9)	s 27 (1.2)	s 11 (0.4)	s 24 (1.5)	s 8 (0.4)	s 3 (0.7)
Finland	2 (0.3)	16 (0.6)	15 (0.7)	25 (1.1)	10 (0.4)	24 (1.4)	7 (0.3)	2 (0.3)
Hong Kong, SAR	5 (0.7)	12 (0.7)	32 (1.6)	18 (0.8)	8 (0.4)	14 (0.8)	8 (0.4)	3 (0.4)
Hungary	2 (0.2)	11 (0.5)	14 (0.7)	29 (1.0)	13 (0.5)	15 (0.7)	9 (0.4)	3 (0.4)
Indonesia	7 (0.5)	15 (1.2)	11 (1.0)	24 (1.3)	13 (0.6)	15 (0.8)	16 (0.9)	4 (0.4)
Iran, Islamic Rep.	6 (0.9)	19 (2.6)	25 (2.4)	21 (2.6)	22 (2.6)	16 (2.8)	22 (2.6)	9 (1.2)
Israel	r 4 (0.6)	r 15 (0.8)	r 19 (0.8)	r 21 (1.2)	r 14 (0.8)	r 22 (1.1)	r 10 (0.5)	r 3 (0.5)
Italy	2 (0.2)	14 (0.5)	25 (0.7)	22 (0.7)	13 (0.4)	12 (0.5)	12 (0.5)	1 (0.2)
Japan	2 (0.5)	5 (0.4)	34 (1.6)	26 (1.3)	16 (0.9)	9 (0.7)	7 (0.5)	2 (0.3)

As mentioned Denmark has not been part of TIMMS since 1995. But there were TIMSS performed by IEA in 2003 and in 2007 (grades 4 and 8) suggesting that a four yearly cycle has been re-established.

3.4.3 TIMSS Videotape Classroom Study

The TIMSS videotape classroom studies include:

1995	TIMSS Video Study	Third International Mathematics and Science Study	45 countries in TIMSS The supplementary Video Classroom Study in 3 countries (Germany, Japan and US). 1 lesson from each of 50-100 randomly chosen grade 8 mathematics classrooms per country.
1999	TIMSS-R Video Study	Trends in International Mathematics and Science Study	38 countries in TIMSS-R The supplementary Video Classroom Study in 7 countries (638 lessons). 1 lesson from each of 50-140 randomly chosen grade 8 mathematics classrooms per country.

Each of these studies will be dealt with below:

The TIMSS Videotape Classroom Study in Germany, Japan and U.S. in 1994-95 was one of two studies with a focus on these three counties to complement the main TIMSS study. It was the first study ever to gather videotaped record of what actually happens in a nationally representative sample of mathematics classrooms. Reported in 1999 the study had four goals *(TIMSS Videotape Classroom Study*, 1999, p. 1):

1. *"Provide a rich source of information regarding what goes on inside eighth-grade mathematics classes in the three countries.*
2. *Develop objective observational measures of classroom instruction to serve as valid quantitative indicators, at a national level, of teaching practices in the three countries.*
3. *Compare actual mathematics teaching methods in the United States and the other countries with those recommended in current reform documents and with teachers' perceptions of those recommendations.*
4. *Assess the feasibility of applying videotape methodology in future wider-scale national and international surveys of classroom instructional practices."*

Among the findings were national differences on:

- How lessons are structured and delivered
- What kind of mathematics is presented in the lesson
- What kind of mathematical thinking students are engaged in during lessons.

This documentation, and the observation that teaching was culturally influenced, has since been much debated and challenged.

The excerpt below indicates the methods used to construct suitable codes for the TIMSS video transcripts (*TIMSS Videotape Classroom Study*, 1999, p. 32).

"First-Pass Coding and the Sampling Study

Next, we divided all transcripts into utterances, which was the smallest unit of analysis used for describing discourse. An utterance was defined as a sentence or phrase that serves a single goal or function. Generally, utterances are small and most often correspond to a single turn in a classroom conversation. Utterances were then coded into 12 mutually exclusive categories.

Six of the categories were used to code teacher utterances: Elicitation, Direction, Information, Uptake, Teacher Response, and Provide Answer. Five categories were applied to student utterances: Response, Student Elicitation, Student Information, Student Direction, and Student Uptake. One category, Other, could be applied to both teacher and student utterances. Elicitations were further subdivided into five mutually exclusive categories: Content, Metacognitive, Interactional, Evaluation, and Other. And Content Elicitations were subcategorized as well.

Definitions of each of these categories will be presented later, together with the results.

Although all lessons were coded with the first-pass categories in the lesson transcripts, we decided to enter only a sample of the codes into the computer for preliminary analysis.

Thirty codes were sampled from each lesson according to the following procedure. First, three time points were randomly selected from each lesson. Starting with the last time point sampled, we found the first code in the transcript to occur after the sampled time. From this point, we took the first 10 consecutive codes, excluding Other, that occurred during public talk. If private talk was encountered before 10 codes were found, we continued to sample after the period of private talk. If the end of the lesson was encountered before 10 codes were found, we sampled upward from the time point until 10 codes were found. The same procedure was repeated for the second and first of the three time points. In those cases, if working down in the lesson led us to overlap with codes sampled from a later time point, we reversed and sampled upward from the selected time point."

There are some similarities but also many differences between the coding of the TIMSS material and my present research, where all points in each lesson will count. The transcribed dialogue is divided into natural chunks of communication, most often consisting of several teacher utterances including answers or repeated elicitation for answers provided by students. The same code may then be used and counted as one reference only, until the teacher or students change focus or move on to something new.

Even when I refer to TIMSS mainly for the research design there are several other differences that will further complicate comparisons with my research.

- Most important, the TIMSS Video Study was of another magnitude according to the number of aspects covered and the statistical calculations included. It was vast in manpower and had corresponding resources.

  My research is performed by one person only, who alone handles contacts to schools, the videotaping and the subsequent transcription and analysis.
- The schools selected for the TIMSS Video Study were a subset of the schools already selected for the main TIMSS Study. They also represented a national aspect to what was otherwise an established multi-national research program. The response rate after eventual replacements by another school was a modest 66% in the U.S., but 96 % in Japan and 87 % in Germany.

  The percentage of schools responding affirmative in my research is lower. This aspect will of course be dealt with in a later paragraph on the Danish stratification (section 5.3).
- According to "Information given to U.S. Teachers Prior to Videotaping" *(TIMSS Videotape Classroom Study.* 1999 p. 142) teachers were offered "*300 $ to your school when you have completed the videotaping and returned the questionnaire. Use of these funds is at the discretion of your principal, in consultation with you. We also will be happy to send you a copy of the videotape we make in your classroom.*"

  In my research the schools will be offered anonymous feedback, once the research results are published. This will be an excerpt from the dissertation or a subsequent publication. Copies of videos, money or other compensation are not involved.

3.4.4 The TIMSS-R Video Classroom Study

This study performed in 1999 (*Teaching Mathematics in Seven Countries*, 2003) was based on videotaping in seven countries including the U.S., whose authorities initiated and financed the major part of the study. The other six countries were selected as high-achieving compared to the U.S. in the TIMSS 1995 assessment. In total the TIMSS-R 1999 Video Study sample included 638 eighth-grade mathematics lessons (www.lessonlab.com/TIMMS/sampling.htm):

Country	Number of lessons
Australia	87
Czech Republic	100
Hong Kong SAR	100
Japan*	50
Netherlands	78
Switzerland	140
U.S.	83

- The Japanese mathematics data collected for the TIMSS 1995 Video Study were re-analyzed using 1999 methodology as part of the TIMSS-R 1999 Video Study.

The research had the following objectives (*Teaching Mathematics in Seven Countries*, 2003, p. 1-2):

- *"To develop objective, observational measures of classroom instruction to serve as appropriate quantitative indicators of teaching practices in each country;*
- *To compare teaching practices among countries and identify similar or different lesson features across countries; and*
- *To describe patterns of teaching practices within each country."*

The report also states, that *"building on the interest generated by the TIMSS 1995 Video Study, the TIMSS 1999 Video Study had a final objective regarding effective use of the information:*

- *To develop methods for communicating the results of the study, through written reports and video cases, for both research and professional development purposes."*

According to the report, the research provided answers for questions as (p. 11-12):

- *"What were the goals for the lesson?*
- *How were the lessons divided among activities that focused on review, introducing new material, and practicing new material?*
- *How was the classroom organized in terms of whole-class discussion and individual student work?*
- *What mathematical topics were covered in the lessons?*
- *What kinds of mathematical reasoning were encouraged by the problems presented?"*

Like the TIMSS Video Study from 1995, the later TIMSS-R Video Study included detailed findings on mathematical content and the organization (structure) of mathematics lessons. Especially interesting compared to my current research is the *"objective, observational measures of classroom instruction"* and patterns of teaching practice.

The TIMSS report discusses:

- "*The Purpose of Different Lesson Segments*" (p. 49)
- "*How Mathematics is Related Over the Lesson*" (p. 76)
- "*Opportunities to talk*" (p. 107).

Such foci are also relevant to the significance of "*points*" in mathematics teaching. But it is not the prime intention to gather similar statistics for Denmark. Lesson segments for e.g. repetition and introduction of new content will be identified by length of phases in % of lesson length (section 8.2). Points articulated by students and teacher's elicitation for this will also be dealt with (sections 6.6 and 6.7.1).
The TIMSS Video Studies also focused on the processes of instruction and mathematics teachers' views on reforms. This is in line with my RQ1: How and why do mathematics teachers articulate mathematical point(s)? Is mathematics teaching in Denmark organized as recommended "into sequences focusing on mathematical goals and didactical points"?

The TIMSS-R Video Classroom Study also to some extent modified "national scripts" as described by the TIMSS Video Study in 1995. It showed, that mathematics teachers in countries with high achievement teach in a variety of ways, and not necessarily the "Japanese way" (*Teaching Mathematics in Seven Countries*, 2003, p.149-150):
"One thing is clear however: the countries that show high levels of achievement on TIMSS do not all use teaching methods that combine and emphasize features in the same way. Different methods of mathematics teaching can be associated with high scores on international achievement tests. …The comparison between Japan and Hong Kong SAR is especially instructive …
Given that students in both Japan and Hong Kong SAR have performed well on international achievement tests such as TIMSS, it is interesting that their instructional practices lie on the opposite ends of these dimensions.":

TABLE 6.2 Similarities and differences between eighth-grade mathematics lessons in Japan and Hong Kong SAR on selected variables: 1995 and 1999

Lesson variable	**Japan**	**Hong Kong SAR**
Reviewing	24 % of lesson time	24 % of lesson time
New content	76 % of lesson time	76 % of lesson time
Introducing new content	60 % of lesson time	39 % of lesson time
Practicing new content	16 % of lesson time	37 % of lesson time
Problems	Making connections (54 % of problems)	Using procedures (84 % of problems)
Private work	Repeating procedures (28 % of work time, Table 5.13)	Practicing procedures (81 % of work time)

SOURCE: U.S. Department of Education, National Center for Education Statistics, Third International Mathematics and Science Study (TIMSS), Video Study, 1999.

"If the learning goal for students is high performance on assessments of mathematics, the findings of this study suggest that there is no single method that mathematics teachers in relatively high-achieving countries use to achieve that goal. Different methods of mathematics teaching were found in different high-achieving countries. This conclusion suggests that informed choices of which teaching methods to use will require more detailed descriptions of learning goals than simply high performance on international tests. A particular country might have specific learning goals that are highly valued ... and for which particular methods of teaching may be better aligned than others. The results of this study make it clear that an international comparison of teaching, even among mostly high-achieving countries, cannot, by itself, yield a clear answer to the question of which method of mathematics teaching may be best to implement in a given country."

According to the TIMSS-R report and also suggested by Leung (Leung, 2006) this research is very reliable. It is the first video-survey with a PPS sampling method, e.g. every student in the actual population has a chance to be selected, and definitely not a case study. Two cameras are used: one pointing at the teacher and one directed at the class. Researchers from every participating country spent three years developing the 45 codes used in 7 coding passes for each lesson. The reliability of a code is proven to be at least 85 %.
A common feature is the high achievement of students in the Asian countries. And it is interesting to look for patterns of similarity in the quantitative analysis of the lessons. Some examples are (Leung, 2006):

1. Teacher talk
   Teachers in all of the countries talked more than students, at a ratio of at least 5:1 words, respectively. The number of words to every student varies. In Hong Kong teachers use 16 times as many words as the students (37 students per class), in the U.S. 8 times as many as the students. Mathematics teachers use more than

5000 words on average per lesson (of 50 minutes), but students in Hong Kong, for example, only used 640 (Figure 5.14 p. 109).

2. Content

   On average 75 % of the lesson time is spent on new content. Hong Kong and Japan gave the most time to new content. Thematic lessons (with problems related to a single topic) amount to 94% in Japan compared to 35% in the Czech Republic (Figure 4.8 p. 80).

   In East Asia the mathematics problems are typically unrelated to real-life with this being the case for 89% of the Japanese sample, while the percentage for the Netherlands is 42 % (Figure 5.1 p. 85). At least 90 % of lessons in all the countries made use of a textbook or worksheet of some kind.

3. Problem handling

   In all seven countries grade 8 mathematics was often taught through problem solving. On average at least 80 % of lesson time was devoted to solving mathematics problems. But in East Asia the mathematical problems are more complicated when measured as the number of steps needed to solve a problem. 84 % of problems in Japan are of medium or high complexity (Figure 4.1 p. 71).

   The fraction of problems including at least one proof is 39 % in Japan compared to 0 % in the U.S. and The Netherlands (Figure 4.4 p. 74). In Hong Kong there is deductive reasoning in 15% of lessons. In Australia this is 0 % and in Switzerland 10 %.

4. Organization

   Eighth-grade mathematics lessons in all seven countries were organized to include some public, whole-class work and some private, individual or small-group work. During the time that students worked privately, the most common pattern across the countries was for students to work individually, rather than in pairs or groups.

The TIMSS-R Video Study offers a general rating of lessons (p. 195) based on: (1) Coherence, (2) Presentation, (3) Student engagement and (4) Overall quality. Hong Kong lessons scored top marks in all categories! And the U.S. lessons were always in the bottom.

Leung investigated more country background variables in Hong Kong (China), Singapore, Japan, Korea and Taiwan for explanations of the East Asian high score. In a talk he suggested the following (Leung, 2008a):

- Parents are well educated in Korea and Japan, but not elsewhere in East Asia.
- Large class-sizes with 35-40 students are common.
- Instructional time is approximately 140 hours a year. The international average is 123. 12-14% of time is for student work.
- Only Singapore has a lot of homework, the rest is below the international average.
- All students lack self-confidence in the East Asian countries. Students (Singapore excluded) did not value mathematics, neither did they enjoy it.

One may wonder from these comments and the fact that, to Westerners classroom teaching in East Asia seems backward and traditional, how students from East Asian countries perform so well in international studies? Based on research in the five countries visited Leung suggested that this had something to do with East Asian cultural characteristics and beliefs (Leung, 2008b):

- The Confucian heritage culture is an examination culture (since 587 A.D.). Also, on the one hand students may say that they hate math, while on the other they are proud of their scores.
- Practice makes perfect! The constructivism strategy may even lead to laziness. In Asian countries the extrinsic motivation to honor our parents and do well as a race also counts.
- The philosophy is pragmatic: we don't come to school for stories, teachers are hesitant to adopt new ideas. Asian teachers don't tell why, but offer procedures for efficiency!

The implications and recommendations to colleagues responsible for mathematics curricula or teaching seem to be to understand and value one's own cultural values, whatever they may be.

This is in line with the view on culture and mathematics teaching style presented by Li (Li, 2006). He states the importance of routine or manipulative practice seen from a Chinese perspective. Imitation and memorization is seen as "*the genetic place of mathematical thinking and the foundation of concept formation*" (p. 130). But "practice makes perfect" has more to it in the Chinese wording than in the English: *"It means both familiarize with and be proficient with*" (p.132). And Li presents more examples from a videotaped Shanghai classroom to illustrate the routines and manipulations of an experienced teacher. It would be interesting to compare this with the teaching routines and eventually the role of homework in Danish classrooms.

3.4.5 PISA 2000, 2003, 2006, 2009

Denmark was included in the triennial PISA's (The Programme for International Student Assessment) initiated by the OECD in 1997. Data for the year 2000 (32 countries), 2003 (41 countries), 2006 (57 countries) and 2009 (65 countries) tests show only small changes in the mathematical competences of Danish 15-16 year old students: Danish grade 8 students perform just above the OECD average. The research is on the numeracy of students, and not on the mathematics teachers' teaching competence. But every 9th year, mathematics is the focus subject of PISA. This was the case in 2003 and will be again in 2012. In PISA students are given a two-part written test and a questionnaire during a three-hour session (Egelund, 2010 p. 189). And student achievement is placed into six proficiency levels with Level 6 as the highest and Level 1 as the lowest. Since the focus is mainly numeracy the relevance to my project is the fact that the data refer to the age group under study.
In 2009 17.0 % of Danish students were performing below or at Level 1 (Egelund, 2010, p. 92).

Students who score below level 1 are *not* able to show routinely the most basic type of knowledge and skills that PISA seeks to measure. Such students have serious difficulties in using mathematical literacy as a tool to advance their knowledge and skills in other areas. Students at level 1 are able to answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations.
In 2009 2.5 % of Danish students were performing at level 6. Students at level 6 are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understanding along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations.

Among the conclusions in the Danish PISA 2006 report were suggestions on how to increase the (relatively few compared to the OECD average) number of high achieving mathematics students and decrease the number of low achieving mathematics students (Egelund, 2007, p. 188). The 2009 report repeated the recommendation to focus on the marginal student groups, gender differences and the use of (oral) language in an early effort to especially primary school students.
Even though each participating school completed a questionnaire about their school that included demographic characteristics and an assessment of the quality of the learning environment at school PISA 2009 has no specific information on the mathematics teachers, their views and teaching practices, as the findings are based on student test results only.

The results of the international studies continue to attract interest as the number of participating countries increases. The results also contribute to an absolutely relevant, but also often quite heated, national debate on educational traditions, costs and outcomes.

The PISA 2009 ranking by mean score is given in this table, where 500 was the OECD average established in 2003 (OECD, 2010, p. 136):

Mean	Comparison country
600	Shanghai-China
562	Singapore
555	Hong Kong-China
546	Korea
543	Chinese Taipei
541	Finland
536	Liechtenstein
534	Switzerland
529	Japan
527	Canada
526	Netherlands
525	Macao-China
519	New Zealand
515	Belgium
514	Australia
513	Germany
512	Estonia
507	Iceland
503	Denmark
501	Slovenia
498	Norway
497	France
497	Slovak Republic
496	Austria
495	Poland
494	Sweden
493	Czech Republic
492	United Kingdom
490	Hungary
489	Luxembourg
487	United States

Such tables do *not* indicate the precise mathematical literacy of Danish students, or the development over three-year periods. Also, as more countries take part in the PISA research, the rankings alone may be misleading with respect to national progress.

PISA does not provide information about the organization and execution of mathematics teaching in the participating countries, but countries positioned significantly higher in the ranking list may still be studied and considered for ideas. I shall return to the lesson study tradition of Asian countries later (sections 3.6.7 and 13.5).

3.4.6 International Project on Mathematical Attainment, IPMA

The aim of IPMA 1998-2006 (www.cimt.plymouth.ac.uk/projects/ipma/; Burghes, 2000) was to monitor and share good practice in all aspects of primary mathematics teaching and learning and make both national and international recommendations on e.g. teaching style and sequencing of topics.

The methodology involved testing the students, collecting data using student and teacher surveys, interviewing the students and teachers, and classroom observations of mathematics lessons in Brazil, China, Czech Republic, England, Finland, Holland, Hungary, Ireland, Japan, Poland, Russia, Singapore, South Africa, Ukraine, USA, and Vietnam.

Each country coordinator wrote chapters on their progress and findings for the final monograph and has contributed to a DVD showing clips of what they consider good teaching practice. Among the IPMA key recommendations were:

- *Teachers should have high expectations of what pupils can achieve and clear objectives for lessons*
- *Lessons should have pace and variety…*
- *Most activities should have an introduction through whole-class, interactive teaching, followed by individual or paired practice, then whole-class interactive review, discussion and correction of mistakes.*

My focus on *points* as drivers to mathematics teaching is not in contradiction to these recommendations. But the project is also interesting because of its impact. In Singapore e.g. a cohort of 856 grade one students were followed for five years. Kaur, Lee & Fwe (Kaur, Lee & Fwe, 2004) state that this longitudinal study has shed light on some of the strengths and weaknesses of teaching and learning of mathematics in Singapore's primary schools. The primary mathematics curriculum seems well structured and generally well learnt by a majority of students. But they see a need for a differential curriculum in school to cater to the needs of pupils with different mathematical abilities and urgency to develop a more innovative hands-on pedagogy that promotes understanding. As reported by Mei & Yan (Mei & Yan, 2006), this has later led to changes in the Singapore curriculum. I discuss a possible impact of my own research in chapter 14.

3.4.7 Kassel Project (Germany - UK)

The aim of the Kassel project 1993-2003 was to provide relevant data for e.g.:

- comparison of mathematics curricula in participating countries
- evaluation of the effectiveness of different approaches to teaching mathematics
- evaluation of self-based schemes of work compared with traditional teacher-led methods. (http://www.cimt.plymouth.ac.uk/projects/kassel/)

The project grew out of collaborative work between the CIMT (Center for Innovation in Mathematics Teaching) in the School of Education at the University of Exeter in England and the Mathematics Education Group at Kassel University in Germany. It has since been joined by 13 more countries.

The research was based on a longitudinal study of representative samples of students in the participating countries. According to the project website, this was the first extensive comparative study in Mathematics, based on monitoring the progress of individual pupils.

The project monitored the progress of cohorts of pupils (over a two or three year period) in schools which are as representative of each country as possible in type and location. The cohorts ranged from complete year groups in some schools to selected classes in other schools. Data were collected by student tests to assess mathematical potential and a series of questionnaires for teachers and student were used to select schools / classes of particular interest. These were observed so that individual recommendations could be made in each country.

Some conclusions were formulated by Blum and Kaiser (Blum & Kaiser, 2004, p. 1):

"The study shows that, as already assumed, the different didactic approaches to the structure and design of mathematics lessons as well as the differences of the underlying educational philosophies have a strong impact on mathematical performance of students in the lower secondary level as well as on their further development of performance. Thus, the study shows that German students achieved better results in certain areas than English students, while in other areas the situation was reversed.
On the whole it became obvious that German students performed well especially in solving problems for which they could apply trained and intensively taught and practised algorithms from arithmetic and algebra. However, in solving geometry and advanced non-routine problems for which a certain degree of autonomy is demanded, they performed distinctly worse. Especially they experienced great difficulties in treating more complex applied problems, particularly if they were asked for justification, which generally they ignored. Furthermore, they showed only little motivation and stamina in dealing with non-algorithmic problems for which non-calculative ways of solving were asked.
…
Thus, the typical spiral-type structured lessons enable English students to cope with more complex mathematical problems much earlier than German students. In most cases these performance deficits vanished during the students' further development. Furthermore, the typical German formulation of curricular goals for each year that gives only little space to individual choice of focus, leads to distinctly more homogeneous performances of German students. This means that, on the one hand, better performing students are supported only a little, while on the other hand, weaker students are more assisted than in England."

Such results are interesting to compare with PISA 2009, where the rankings of Germany and England differed in scores. In my research the extent and type of points being stated is not associated with the type of student assignments. The data would make it possible though, as assignments are registered. But this would require a continued research effort.

3.5 Research on quality parameters in mathematics teaching

3.5.1 IC-model (Denmark)

Alrø and Skånstrøm (Alrø & Skånstrøm, 2000) report a case study which followed up on their IC Model (Inquiry Cooperation) developed in earlier work to help analyse the role of dialogue in the learning of mathematics. The researchers observed two students as they worked on a problem, and looked at their patterns of communication as they made progress and when they got stuck. They observed the role of teacher interference in this process.
Alrø and Skovsmose understand dialogue as part of an inquiry process that includes risk-taking and maintains equality between parties (Alrø and Skovsmose, 2002).

Their model for quality communication in the mathematics classroom includes various dialogic acts such as "*getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging and evaluating"* (Alrø and Skovsmose, 2004, p. 47). There is no absolutism to this, as the teacher has a responsibility with a regard to all students, and many situations may force the teacher to break away from dialogue. But intention and reflection are discussed and emphasized as crucial to student engagement and a reference to the role of mathematics textbooks and the asymmetrical relationship between teacher and students are made. The notion of "bureaucratic absolutism" is suggested for this.
Their idea of inviting students into a "landscape of investigation" represents a totally different approach to teaching and tasks as the one most often found in my research. Especially the role of the mathematics textbook is challenged (section 9.3).

3.5.2 Teaching to the Mathematical Point (US)

The research by Sleep (Sleep, 2009) has investigated the teaching style of 17 pre-service elementary teachers teaching by observation of mathematics lessons (in grades 1-5) and interviews about mathematics and teaching backgrounds both immediately before and after the observed lesson. She is building on findings from the LMT-project (Learning Mathematics for Teaching), which since 2003 has developed tests to measure student teachers' MKT (Mathematical Knowledge for Teaching):
"*We use a series of five case studies and associated quantitative data to detail how MKT is associated with the mathematical quality of instruction. Although there is a significant, strong, and positive association between levels of MKT and the mathematical quality of instruction, we also find that there are a number of important factors that mediate this relationship, either supporting or hindering teachers' use of knowledge in practice."*
(Hill, Blunk, Charalambous, Lewis, Phelps, Sleep & Ball, 2008, p. 430).

The sample used by Sleep is not a random sample of student teachers at this university as they were selected from a larger group (N=30) to ensure variation in their MKT (Mathematical Knowledge for Teaching). Sleep defines a mathematical point "*to include the mathematical goals for an activity, as well as the connection between activity and its goals*" (p. 13) and teaching to the mathematical point is conceptualized as "*being composed of three different types of work: articulating the mathematical point; orienting the instrumental activity; and steering the instruction toward the mathematical point*" (p. 175). The first two are also named as mathematical purposing by Sleep.
The framework developed in this project is a series of lists of what to do when unpacking the mathematical terrain, specifying mathematical goals for student learning (p. 209), examining instructional activities and the mathematical point of its details in order to orient the instructional activity (p. 211). Even if the Sleep definition of a point is different from mine, I consider this work a relevant suggestion

on how to strengthen the occurrence and role of mathematical points as asked in my RQ2.

3.5.3 Learners Perspective Study - LPS

Other researchers have conducted large-scale video studies since TIMSS.

1999+	LPS	Learners Perspective Study	12 countries Sequences of 10 lessons from each of 3 “well-taught” grade 8 mathematics classrooms in each country.

The Learners Perspective Study (LPS) was designed in 1999 to examine practices in grade 8 classrooms in four countries: Australia, Germany, Japan and the U.S. The research was meant to supplement reported national norms of student achievement and teaching practices like the TIMSS Video Studies but with an even more in-depth analysis. More countries have since joined the LPS group and the purpose has been “reinterpreted and expanded”.
In a LPS video study of well-taught (by local decision) mathematics classrooms in twelve countries (Australia, The Czech Republic, Germany, Hong Kong and mainland China, Israel, Korea, Japan, The Philippines, Singapore, South Africa, Sweden and the U.S.) a group of researchers from this LPS community presented an analysis of practices and meaning in grade 8 classrooms in these 12 countries. Their research questions are outlined in *The Insider’s perspective* (Clarke, Keitel & Shimizu, 2006) and *Making connections* (Clarke, Emanuelsson, Jablonka & Mok, 2006):

1. *Within the classrooms studied in each country, is there evidence of a coherent body of student practice(s), and to what extent are these practices culturally-specific?*
2. *What are the antecedent and the consequent conditions and actions (particularly learner actions) associated with teacher practices identified in earlier studies as culturally specific and nationally characteristic?*
3. *To what extent does an individual teacher employ a variety of pedagogical approaches in the course of teaching a lesson sequence?*
4. *What degree of similarity or difference (both locally and internationally) can be found in the learner (and teacher) practices occurring in classrooms identified by the local education community as constituting sites of competent teaching practice?*
5. *To what extent are teacher and learner practices in a mutually supportive relationship?*
6. *To what extent are particular documented teacher and learner practices associated with student construction of valued social and mathematical meanings?*

> 7. *What are the implications for teacher education and the organisation of schools of the identification of those teacher and learner practices that appear to be consistent with the realisation of local goals (and those which are not)?*

As three teachers in each county were followed for 10 consecutive lessons, a consistent description of a recommendable tradition from each country was hoped to be achieved. Of course this should not be interpreted as a national characteristic.

The study had the potential to relate the findings of "cultural teaching scripts" from the TIMSS Video Studies to certain teacher actions and student practices. Eventually the analysis might consider teaching practice as a condition and / or a result of the interaction between students and teachers in the classroom.

The LPS study does not offer national characteristics based on large samples. Respondents teaching grade 8 are selected in a quite *different* way, but then followed for 10 lessons.

In this way it is also not comparable to the expected answers to my first research question. But I regard the LPS-project as a very important source of information and consideration for my project because of its focus on teacher practices in a qualitative way, as the LPS data has been analyzed for cross-national characteristics. Certain lesson events have been the focus of such later analysis of the LPS data, showing that rich data may be reconsidered through new lenses.

- The *matome* ("summing up" of lesson activity and points) is very relevant to my research (section 6.3.8). Shimizu found similar events (with variations in form) identifiable across data sets from six countries and also reported that Japanese teachers usually initiate this lesson event and students perceive it as significant to their learning (Shimizu, 2004).
- Different forms of guidance *kikan-shido* ("between-desks-instruction") by teachers in mathematics classrooms are found across six countries (Clarke, 2004; O'Keefe, Xu & Clarke, 2006).
- "*Oral interactivity*" e.g. "utterances that would be viewed as public from the "student's perspective" were reported by Clarke and Xu (Clarke & Xu, 2008). The use of three cameras for each lesson (teacher camera, student camera, whole class camera), means three types of oral interaction could be identified (p. 966):
  - *whole class interactions, involving utterances for which the audience was all or most of the class, including the teacher;*
  - *teacher-student interactions, involving utterances exchanged between the teacher and any student or student group, not intended to be audible to the whole class; and*
  - *student-student interaction, involving utterances between students, not intended to be audible to the whole class.*

These are examples of the qualitative investigations in certain specific areas made possible by a large database of quantitatively sampled data. But when it comes to the actual coding procedures, the methods used for analysis are *comparable* to those used in research for the study outlined in this dissertation. Studiocode used in the LPS is a Mac program with options just like those in the NVivo software used for this study.

A study visit to ICCR (International Centre for Classroom Research) at the University of Melbourne in 2010 gave me an opportunity to discuss both some analytical challenges and choices, when you code video transcripts, but also to gain a critical review from (more) experienced colleagues in the field. I was familiar with the design and some of the outcomes described by Clarke, Keitel and Shimizu. And in other contexts I have met members of the LPS group several times, so I knew what LPS was all about. I have discussed with Clarke and colleagues at the ICCR whether e.g. lesson plans actually have to be written down to give the sufficient guidance to teaching or whether students may be well aware of lesson goals without these being spelled out in teaching.

3.6 Research on intervention in Mathematics Teaching

Two smaller research projects were presented at the PME-30 conference in 2006 with a focus on teachers' "mathematical knowledge for teaching", MKT. They are included in my review because of the identification of MKT as one of more decisive parameters in the interaction between mathematics teacher and students. This is further discussed in section 9.5:

Seago and Goldsmith (Seago & Goldsmith, 2006) report on 49 teachers' participation in the project: "Turning to the Evidence" on professional development of MKT. They report on three teachers showing gains in different types of learning: unpacking the mathematics, choice and use of mathematical representations and finally evaluation competence in a possible detailed and later analysis.

Powell and Hanna (Powell & Hanna, 2006) report on a three-year project: "Informal Mathematics Learning Project" (IML) investigating how three pairs of mathematics teachers facilitate IML sessions and attend to students' ideas and reasoning. The research notes three categories of knowledge from which teachers interact with students: Teachers' knowledge of mathematics, their (epistemological) awareness of the students' existing and evolving knowledge and finally their MKT.

My present research will go into the coding and analysis of various lesson events in some detail, but the projects mentioned provide larger and more in depth examples.

3.6.1 Classroom Discourse (US)

Lappan and Theule-Lubienski (Lappan and Theule-Lubienski, 1992) focus on the importance of classroom discourse and communities of collaborative reflective practice and consider orchestrating classroom discourse as one of four aspects of teaching. Classroom discourse is perceived as important according to PSTM (Professional Standards for Teaching Mathematics, 1991), and could be used to craft a framework for teacher education. In this context, classroom discourse is seen as

"the ways of representing, thinking, talking, agreeing, and disagreeing" (p. 36). Since 1988 researchers at Michigan State University have studied pre-service teachers' beliefs on how one should teach mathematics to children, including the classroom discourse. Findings from following a subset of the original 24 students through their first three years of teaching highlighted the complexities that new teachers face in attempting to create environments for learning mathematics in which children engage. The research concluded that an intervention study *did* develop disciplinary knowledge and a disposition to engage in mathematical inquiry or sense-making, *but* that it was insufficient in overcoming deeply-held beliefs about how students should learn mathematics and what was important for them to know. Such beliefs cannot be challenged, examined or reconstructed solely in the pre-service phase of teacher education. The researchers therefore suggested professional development programs based on the tenets that teachers *need* to see a new approach work in their own classrooms before their beliefs change.
This is in line with my results described below (sections 10.4 and 12.5).

3.6.2 IMPACT (US)

Campbell reports (Campbell, 1996) on possible changes in elementary mathematics teachers from a constructivist perspective in the project IMPACT involving researchers from the University of Maryland and teachers from Montgomery County. The enhancement model involved (a) a summer in-service program for all teachers of mathematics, (b) an on-site mathematics specialist at each school, (c) manipulative materials for each classroom and (d) teacher planning and instructional problem solving during a common grade-level planning time each week.
A survey of teachers' beliefs and confidence was carried out at three schools to characterize the rationale that might be guiding teachers' actions. Classroom observations provided information on teacher's actual conduct. Two schools with a total of 93 teachers were supported in a change toward implementing a more reform-based perspective, and one school acted as a "comparable-site". The results showed that 12 teachers made no real change in their instruction, 19 moved considerably beyond routinized practice and direct instruction, but did not pursue the meaning of students' explanations. 25 teachers evidenced instructional changes consistent with a constructivist perspective and in 37 classrooms the instruction was supportive of students' construction of knowledge and attentive to mathematics.
The research concluded that individual teachers should not be left alone: supportive atmosphere, along with mathematical *and* pedagogical knowledge were clearly important. Finally the IMPACT project seems to demonstrate the potential of applying research to instructional decision-making across whole schools.
I find such findings very relevant to the discussion of longitudinal effects of intervention projects found successful in the short run (section 10.5). Also when considering recommendations to scaling up (section 14.3).

3.6.3 QUASAR (US)

Silver (Silver, 1992) recalls his vision of "*mathematical classrooms as places rich in communication of and about mathematical ideas, places in which justification and verification were emphasized, and places in which teachers and students engaged in authentic forms of mathematical practice*" and he mentions two projects with empirical evidence of the power of teachers and resource partners (e.g. university teacher educators) creating communities of reflective collaboration as they develop new forms of instructional practise. One such project is QUASAR, where mathematics teachers from urban middle schools and resource partners from the University of Pittsburgh's Learning Research and Development Center have used common meeting times to plan instruction, to visit each other's classes or to watch videotapes of each other's teaching, to reflect on their individual and collective pedagogical practices and to discuss the work of their students. Data indicated that QUASAR schools build teachers' capacity to improve the quality of their mathematics instruction and the ideas have since been expanded in another project: COMET. The suggestion and need for common meeting time is repeated in my intervention study with a group of teachers in a lesson study approach (sections 12.5 and 13.5).

3.6.4 RADIATE (US)

Cooney reports this study of pre-service secondary teachers progressing through their program at the University of Georgia and into their first year of teaching (Cooney, 1999). The RADIATE project (Research and Development Initiatives Applied to Teacher Education) 1993-97 was a qualitative study showing teacher students' (in)ability or (un)willingness to reflect on possible interpretations of teacher education activities.

Cooney found two kinds of characteristic connectedness among the teacher students:

- Naïve connectionism, in which the importance the students attach to basic skills creates tension with their core belief, that mathematics should be made interesting by distinct connections between mathematics and the real world.
- Reflective connectionism, when various positions are successfully weaved into a coherent set of beliefs.

The project showed how one may conceptualize differences in beliefs about mathematics and one may "*influence and stimulate teacher's reflective thinking about their own beliefs about mathematical and pedagogical situation"* (p. 184) by introducing an element of doubt into the teaching combined with some preparation for different classroom situations through cooperative learning. Beliefs are further discussed below (sections 9.4.3 and 10.2).

3.6.5 MINERVA (Portugal)

Krainer (Krainer, 1996) indicated the different and conflicting understandings of how to improve mathematics education in different countries through in-service education due to varying conditions such as class size and teacher education. The Portuguese

MINERVA project is one example. This project ran between 1985 and 1994 with an aim to promote the use of information technology into Portuguese primary and secondary education. Ponte (Ponte, 1994) described the driving force of the project as its nodes:

"located in higher education establishments and composed of university professors and seconded teachers from different teaching levels. ... Regardless of their own specific activities, all the nodes, generally speaking, carried out a set of common activities. These ranged from the provision of equipment and software to schools, training and support for teachers wanting to use IT, organization of meetings, seminars and conferences, production of material, preparation of publications and, in general, curriculum development activities…. Teacher training became one of MINERVA's most important activities and for this alone a separate report could be written…

In many cases, it was not easy for MINERVA to fit in with the schools' activities and to appeal to the great majority of teachers. In some schools, the project remained closed on itself only available to a certain "elite". In others, it was a source of controversy between teachers. But there is no doubt at all that the schools were tied up after the drastic cut of the teachers release time. The teams simply collapsed.…

Firstly, MINERVA's design should have been limited from the beginning. It simply was too big. It should have been less ambitious and shorter. Its historical role could only have been what it was – a project of dissemination, a seed-bed, a launching of new views. To go beyond this, another type of project would have been needed, with more specific aims and with less ambitions of covering the national territory: projects of software development, projects of training/intervention directed by certain schools, projects of curriculum development in certain areas, and all this solidly based on educational research."

The long quotation is a reminder that challenges met in large-scale implementations of new content, tools or strategies for teaching may be totally different from the small-scale experience.

3.6.6 SINUS (Germany)

This Program: Steigerung der Effizienz des mathematisch naturwissenschaftlichen Unterrichts (Increasing the efficiency of mathematics and science teaching) (Ostermeier et al, 2010; Blum, 2004) has since 1998 aimed to increase the effectiveness of mathematics and science teaching by changing the culture of problem and task posing and in general changing the culture in German mathematics teaching. This is related to my research as a move towards more point driven mathematics teaching may require a similar change in the Danish mathematics teaching culture.

During 1998-2003 SINUS involved 180 schools from all over Germany. These were organized into 30 sets with 6 schools each. Each of the 30 sets was called a "model project". The grades involved were 5-10.

There were two guiding principles:

- Changing mathematics teaching requires changing to “a new culture of tasks”. Mathematical competencies are acquired and advanced by appropriate activities when completing tasks and solving problems. Mathematics teachers have to select or construct appropriate tasks and implement them in the classroom.
- Developing teachers’ professionalism requires a “new culture of systematic collaboration”. Teachers even ought to cooperate across institutions.

Schools in the program had to choose at least two modules to work on. Modules outline central aspects of a problem area and provide examples of how to overcome identified shortcomings. They also help to categorize the documentation of processes and products (developed units, materials, etc.) and provide a shared language to facilitate communication about science and mathematics teaching. The choice of a system of modules also makes professional development adjustable to the specific local situation and challenges in the participating schools. These modules provided the framework for the work of the participating teachers.

The program introduces processes of quality development at the school level. The teachers are encouraged to set their specific working goals, to develop new materials or modify existing approaches, and to engage in self-evaluation methods that are easily applied to their classroom teaching. The program’s leading principle is cooperation and collaboration on different levels, especially between the participating teachers.

Teachers’ work is supplemented by support from science and mathematics educators and through research on learning and instruction. Teachers working on modules have access to scientifically-based materials and worked-out examples referring to the modules. Also offered are various possibilities for consultation and in-service training.

SINUS is regarded (also by politicians) as the most successful educational program in Germany ever. This is why SINUS was continued and considerably extended between 2003–07. In this second phase of scaling-up over 1,700 schools were involved. And from 2007, it was the federal states’ responsibility to use the built infrastructure and competencies of networks, facilitators, and teachers, and to further disseminate the SINUS approach to more schools. The central question for this enterprise is how to disseminate experiences and processes – not only products and developed materials – to a larger group of schools and teachers. The key elements of the program (cooperative development of classroom teaching, framed by modules) are retained.

The SINUS project has now ended, but a new project based on similar ideas, Mathematik Anders Machen, MAM (www.mathematik-anders-machen.de) supported by funding from a telephone company seems to be successfully involving teachers in developing course packages like those developed by SINUS.

3.6.7 Lesson study

This originally Japanese tradition of a peer sparring through lesson study is spreading through literature, networks and conferences from Asia to Europe and the U.S. (Isoda, Stephens, Ohara & Miyakawa, 2007; Fernandez & Yoshida, 2004). Lesson study is a professional development process in which teachers systematically examine their own practices. The aim is to make their teaching more effective, and the core of lesson study is a group of teachers collaborating on a small number of study lessons.

The lesson-study format invites collegial, professional dialogue that can be directly translated into clear points and a perhaps better class discussion during teaching. But it requires mutual respect and responsiveness to accommodate not only the teacher but also students' own explanations on many levels. And it requires knowledge about different approaches to the same mathematical topic.

In 2008 I spent 10 days at the University of Nagasaki discussing experiences and recommendations with Japanese colleagues and participating in lesson observations at local schools (section 11.2.1). Japanese mathematics teachers first encounter this way of collaborative working in their University education during teaching practice, and teachers are allotted time and resources for common meetings and classroom observations. I will later describe one Japanese lesson study process in more detail (section 11.2) and discuss whether this is a possibility in a Danish context (chapter 12).

4 Concepts and terms

As described in chapter 2, the idea of points was among the recommendations made by a Danish committee on the future of mathematics in primary and lower secondary school (Niss et al., 2006).

4.1 The Point concept

The word point is common in every-day English and Danish language (and presumably in many other languages), in combinations of words and collocations. To secure a sensible interface to the everyday meaning I find it necessary to start with a thorough etymological explanation. This may lead to a manageable definition in the mathematics didactic context.

4.2 Etymology and collocations

The English noun *point* corresponds to the Danish noun *pointe*. They both have their origin in French, which in Danish also decides the pronunciation. According to the Merriam-Webster dictionary (2004, p. 957) point is partly a prick, dot, moment, from Latin punctum, which is neutrum of punctus, and partly a sharp end, from Latin puncta, the feminine form of punctus.

Several meanings are listed in this dictionary, e.g.:

a) the individual detail as an item or a distinguishing detail
b) the most important essential in a discussion or matter
c) cogency.

According to the Oxford dictionary (2005, p. 1162) meanings of the noun are categorized as:

a) opinion/fact b) main idea c) purpose d) detail e) quality f) time	g) place h) direction i) in competition j) measurement k) punctuation l) sharp end	m) land n) of light/color o) for electricity p) in ballet q) on railway tracks r) size of letters

In many languages a point has an *everyday* meaning of importance. Still point(s) may come in many settings, e.g. in a conclusion, a moral, a lesson, a solution or as the final surprise, e.g. a punch line in a joke.

A point may be a brief version of the essential meaning of something. It can refer to meaning or significance, i.e. the message intended, expressed or signified. The decisive point is the bottom line in a discussion, the most important point is the crux of the matter, an especially persuasive point helping to support an argument or discussion may be called the talking point. And if points are absent, the word pointless may be used, indicating no meaning or purpose, like in a pointless journey.

Dictionaries and lexica list several examples of the word “point” in collocations. E.g.: (www.thefreedictionary.com/point):

- Distinct condition or degree: *finally reached the point of exhaustion.*
- A specific moment in time: *At this point, we are ready to proceed*.
- An objective or purpose to be reached or achieved, or one that is worth reaching or achieving: What *is the point of discussing this issue further?*
- The major idea or essential part of a concept or narrative: *You have missed the whole point of the novel.*
- A significant, outstanding, or effective idea, argument, or suggestion: *Your point is well taken.*

Expressions like: “Get to the point”, “He forgot his point” and “I did not understand her point”, suggest that points are crucial to oral, interpersonal communication. But also written material as jokes and papers may have points.

I am going to use the concept of didactic *points* suggested above (section 2.1) as crucial mathematical ideas in a lesson. These may be main ideas as well, but in mathematics individual important concepts or procedures may be details in a longer teaching sequence, thus not necessarily fitting the idea of one main or big idea.

4.3 Definitions

In mathematics and mathematics teaching I suggest points to indicate similar pointed ideas or climaxes. A *mathematical point* is in principle independent of teaching - but imbedded in subject content or processes. A “pointed” or *precise insight* into a key issue in a text or a mathematical process reflects the French origin of the word “point”. A mathematical point may also indicate the *climax* in a series of thoughts or processes leading to a *culmination*. I apply this definition:

A mathematical point is a statement presenting a clearly delineated significant mathematical content or climax.

A mathematical point presents a *delineated content or climax*, sufficiently easy to handle, e.g. to determine whether something is included or not and to apply the point in contexts. And a mathematical point is *significant* as having or likely to have influence or effect on teaching and learning of mathematics. These are a priori characteristics or properties seen as important by e.g. mathematicians and people involved in curricular decisions.

If a *mathematical* point drives the teaching it is also a *didactical* point whether articulated or not. So there may be a difference between the mathematical points and the ones put forward in teaching. When it comes to didactical points, the articulating teacher or student is the one to decide. This is explicated in curriculum guidelines and teacher guides for student text books.

The following clusters are expected to offer mathematical points:

Concepts

A concept may be defined as an abstract or generic idea generalized in a definition from particular instances or occurrences. Examples from mathematics are the number, the function or the probability concept, the concept of five, a linear function or a sample space. The word or the visualization is not to be mistaken for the concept content itself, but is a necessary tool to grasp and use a concept distinct from others. Therefore competence with symbols, formalism and representations becomes crucial for the handling of a concept and indicates more levels of concept understanding and mastery.

A *concept* is a mathematical point when it is significant, i.e. has or is likely to have influence or effect and when it is clearly delineated by definition, symbol or application what is and what is not then a mathematical concept. Some mathematical concepts are superior to others and thus more significant. This may be visualized by their position in a concept hierarchy. The *number* concept is e.g. more important than the *number five* concept because of generality.

Are there different kinds of concepts? I won't go into a discussion of mathematical axioms, objects and structures, or concepts in set and number theory versus concepts to geometry. But certainly one could discuss concept categories.

A quick look at the Danish curriculum to grade 8 suggests a *function* as a significant concept because of generality. It is the common name to a cluster of "child" concepts, as there are several types of functions – each associated with algebraic expressions like formulas, solutions and graphical images. *Common Goals 2009* (*Fælles Mål 2009: Matematik*, 2009) among other things states that students must become able to:

- distinguish between definitions and sentences, between individual cases and generalizations and apply this insight to explore and engage in dialogue about various mathematical concepts and range restriction (thinking competence)
- understand and use variables and symbols, including when rules and relationships are to be displayed, and to translate between everyday language and symbols (symbol processing competence).

Common Goals also describe concepts by mathematical topics where students must be able to e.g.:

- understand and apply formulas and mathematical expressions involving variables
- understand and apply the concept of percent.

The *linear equation* is an *example* of such a pointed mathematical concept e.g. a mathematical point exemplifying the concept of a linear function by an equation. The same applies to the definition of the area of a unit square and other rectangles.

Procedures

Procedure and method are seen as synonyms in this context. A mathematical procedure may be defined as a specified sequence of steps, decisions, calculations and/or operations, that when undertaken in a regular, definite order produces a mathematical result, product or outcome. In other words, performing a procedure has a distinct aim.

A *procedure* is a mathematical point when it is significant, i.e. has or is likely to have influence or effect because of possible application and clearly delineates the steps that must be taken to apply it. *Common Goals 2009* (*Fælles Mål 2009: Matematik*, 2009) among other things states that students must:

- know various tools, including ICT, their strengths and limitations, and how to use them appropriately, e.g. in investigation of mathematical relationships, calculations and for presentations (aids and tools competence).

Common Goals also describe procedures by mathematical topics where students must be able to e.g.:

- calculate with fractions, e.g. when solving equations and algebraic problems
- perform simple geometric calculations, including using Pythagoras' theorem
- use IT to draw, in investigations, calculations and reasoning regarding geometrical shapes.

Competencies in problem-tackling, modeling, reasoning, various aids- and tools become crucial when introducing more levels of procedural skills and mastery. The method of choosing and calculating a set of (x, y) values to graph a linear function is an example of a pointed mathematical procedure, i.e. a procedural point.

Results

A mathematical result may be defined as an outcome obtained by mathematical investigation including calculation. Formulas and theorems developed are examples of mathematical results. Results also include validation of a procedure.

A *result* is a mathematical point when it is significant by being necessary or useful in further study as a formula, a theorem or a procedure valuable in itself, and as this result clearly delineated by conditions and circumstances. *Common Goals 2009* (*Fælles Mål 2009: Matematik*, 2009) among other things states that students must become able to:

- devise, implement, understand and evaluate oral and written mathematical reasoning and work with simple proofs (reasoning competence).

Common Goals also describe results by mathematical topics where students must e.g.:

- work with sequences and changes in order to investigate, systematize and generalize
- know the calculation hierarchy and justify and apply calculation rules.

Culminating thought processes, algorithms and general, not simple calculations are considered as results. The formula of area of a triangle and the Pythagorean Theorem are examples of pointed mathematical results, i.e. result points.

Interpretations

A mathematical interpretation may be defined as adding of meaning to mathematical concepts, models or parameters or the explanation of a result. The interpretation may be offered by example, application or comparison of representations. Competencies in mathematical thinking and communication become crucial and indicate more levels of interpretative skills and mastery.

An *interpretation* is a mathematical point when it is <u>significant</u>, i.e. has or is likely to have influence or effect as an important model, result or comparison of representations and clearly <u>delineates</u> this understanding. *Common Goals 2009* (*Fælles Mål 2009: Matematik,* 2009) among other things states that students must become able to:

- establish, define and solve both purely academic and applied mathematical problems and evaluate solutions, e.g. to generalize results (problem handling competence).
- decode, use, and select appropriately between different forms of representation and being able to see their relations (representation competence).

Common Goals also describe this by mathematical topics where students must be able to e.g.:

- investigate, describe and assess relationships between a drawing and the object drawn
- apply statistical concepts to the description, analysis and interpretation of data
- perform and interpret experiments in which randomness and chance are included.

The interpretation of $y = ax + b$ as the algebraic expression corresponding to a line with the slope a and the y-axis intercept at (0, b) is an example of a pointed mathematical interpretation, i.e. an interpretation point.

These four types of mathematical points articulated by a teacher or by students are the primary ones researched in this study:

- A *conceptual* point as the teacher or a student presents and comments on a mathematical concept by definition, symbol or application.
- A *procedural* point as the teacher or a student presents and comments on a rule or a method in an application or example.
- A *result* point as the teacher or a student develops, presents and comments on a mathematical result like a formula, theorem or procedure.
- An *interpretation* point as the teacher or a student interprets a model or a result or compares representations.

In a teaching context, the driving and possibly articulated mathematical points depend on the goal and other deciding framework *and* the teacher. These points become *didactic points* as they are connected to the intention, planning and execution of mathematics teaching. Hence they are also dependent upon students / learners being present.

The goal of mathematics teaching whether by explanation, communication or activity would often be for students to hear, see or experience one or more mathematical points to develop their mathematical insight and competence.

Points may also present decisive and thus critical stages in a study and learning “process” – perhaps as appropriate "intermediates”. Therefore points may be

characteristic of *certain moments* in mathematics lessons and / or “guiding” *the way* mathematics is being taught.
Teachers are the decisive actors here. Points, suggested by the mathematical content as pointed insights or culminations in processes, are sort of filtered or versioned when chosen and made by mathematics teachers – or their students.

I apply this definition in the mathematics teaching context:

A didactic point is a mathematical point, the teacher has judged particularly important to the student's insight and understanding.

4.4 Point-supported or point-driven?

I am convinced, that mathematics teaching would benefit from being ("strategically") designed with points in mind and with attention to recognized and anticipated hurdles, i.e. including reflection on how they are overcome. This is the Danish recommendation referred above (section 2.1), and it is inspired by the lesson-study experience in Japan and elsewhere, which I shall discuss in a later section dealing with RQ3 (section 11.2).
Which function do points then have in a didactic context in mathematics? Points have to be recognized, though not necessarily by the teacher, in order for the teaching to be considered fruitful. When mathematical points are recognized and implemented in teaching they may *support* the teachers in

- *arguing* rather than postulating or demonstrating
- *focusing*, i.e. keeping an eye on the main goal
- *structuring* systematically towards a point (conclusion)
- *criticizing*, i.e. evaluating and discussing own and students' reasoning.

By *point-driven* mathematics teaching I understand teaching designed with planned points as triggers for insight to be identified, constructed and / or demonstrated. Points may be waiting to be articulated at a distance, possibly guiding the teaching as a beacon leads the traveler. The type of points will then depend on the mathematical content, the lesson structure or the teacher’s beliefs.
A point may also be an intermediate result, such as a concept or a technique that has a value in itself, but also is needed or useful in continued work on mathematical (higher-order) concepts and skills.
Are points then the process-oriented drivers or bearers of the teaching or merely to be seen as the results? Reliable knowledge about type and frequency of such points in Danish mathematics teaching is expected to provide the answers. Assuming that some Danish mathematics lessons have points and some have not, why is it so? And is not having a point a problem?

4.4.1 When are points being “made”?

One may ask whether points are always made by the teacher and possibly repeated by students. Are points communicated to the whole class, to groups - or to individual students? And are some points even articulated by students?
When observing mathematics classrooms for points whether you are a learner or a researcher, the attention and expectation may shift during phases in the lesson:

<u>At the start of the lesson</u>

- How is the goal to this lesson mentioned / listed / quoted?
- Is this also a point?
- Does the teacher refer to previous lesson(s) or knowledge?

<u>During the lesson</u>

- Is whole class discussion planned, predicted or surprising?
- If something is established as important, is this a point then?
- Is seatwork announced by more than an activity list?
- Are points mainly articulated in common class discussion, in group work or to / with individual students?

<u>By the end of the lesson</u>

- Is there a final summing up on important mathematical findings?
- Do the final remarks exclusively consist of homework instructions and / or announcement of prospective activities?

4.4.2 How are points being made?

Points that frame and / or lead the activity in a lesson may represent a climax or "staging" of a longer course (end of chain of thought or elucidation of a complex, possibly open problem area). Teaching is then supposedly "strategically" planned with such points in mind and with attention to anticipation and recognition of hurdles and consideration of how they may be bypassed.
A point may be connected to new insight / understanding (a mathematical competence) to students. It could be the very goal of one or more lessons. Often the point then is a result of an instructive guiding with careful progression, possibly culminating as students themselves solve tasks leading to this insight.
Mathematical points may be articulated in oral, written or visual forms of mathematical language including e.g. representation, symbols, formulas, models etc. And points *may* occur in the teaching of:

A concept
A concept has to be presented clearly delineated and significant to be a didactic point.
Example: A parabola in itself is not a didactic point, but it’s a point, when students get to know or see that this curve describes the trajectory of all throws in a gravitational field without air resistance. Or, as it once happened to me: to discover that a variation of b in the expression $y = ax^2 + bx + c$ ($a, b, c \in R$, $a \neq 0$) made the

graphic image of the parabola vertex move in a (similar) parabola in a coordinate system.
A concept in which a wider scope is (suddenly) acknowledged may also be a point:
Example: It is reasonable (necessary) that $a^0 = 1$ and $a^{-2} = 1/ a^2$. When you demonstrate and / or realize why, this is also a point.

A procedure
Students often (fortunately) experience methods, which work well. It may be a “how to” use of isometric paper, procedures for solving equations or graphing of functions. The satisfying feeling, when you suddenly realize the powerful tool in your hand, is a heady point – of an affective nature. When students realize the *usability* of techniques in different contexts, the technique works as a driving point.
The purpose with an activity (formulated by a teacher) may well be to get students trained, or to learn a new technique such as being able to produce a table to a function in a (x,y)-scheme, or simply to do well in an exit test. These will be considered as driving points. The point may be articulated by students as a *sudden understanding of procedures that work*, when / if some students say or think: Aha!

A result
A "nice" and useful result also has the quality of a didactic point. *Example:* GCD is part of a formula for the number of “crossings” of a straight line in a n x m grid. This property may be a point to the few students, but everyone in a class may be led this way by coloring the 4- and the 7-table in a 100-panel, and then looking for crossings in a 4 x 7 and a 4 x 8 grid.
Example: The medians in any triangle have a common intersection point and the medians of an arbitrary cardboard triangle intersect in the center of gravity.
The first of these latter points must await a number of arguments rather demanding of lower secondary students or possibly a number of experiments with a dynamic geometry program like *GeoGebra*, where the visual "wholesale simulation" is really convincing (but of course no proof). The second point may be experienced with construction in cardboard, and then the use of scissors and a needle.
One counter example to “kill" a common assumption may also be didactic point.
Example: When you square a real number you don’t always get a bigger number as demonstrated by $0.5^2 < 0.5$ or when you divide one number with another, the quotient may become larger than the dividend, e.g. 30: 0.5 = 60.

An interpretation
E.g. a teacher may ask students to draw graphs of 10 linear functions. This activity is not a point. But the activity may drive / guide students to discover the role of the parameters a and b in the expression y = ax + b. To realize that b always is the y-axis intercept or that a is the slope of the graph is a didactic point.
Example: When describing a data set a statistical measure as the arithmetic mean may be interpreted as an acceptable approximation.

5 Methodology in a study of 50 teachers

My first research question is:

RQ1:	**To what extent, how and why do teachers articulate mathematical points in Danish mathematics teaching?**

So far, the didactic use of mathematical points in mathematics teaching has not been investigated in Denmark, and I have not seen any similar research on points.
My method to research this question is to choose schools, which are representative of Danish mathematics teaching and a random mathematics teacher of grade 8 from each of these schools. Grade 8 is a relevant choice as this is the common grade level chosen for the TIMSS and LPS studies. In Denmark most students leave lower secondary school after grade 9, but grade 8 is not supposed to be affected by special training or focus on this coming formal assessment.

This research is also a *snapshot* of Danish mathematics teaching strategies as of today. Just one lesson per class / teacher is studied and patterns or diversities in the occurrence of didactic points are being searched. Not being able to watch a series of lessons by the same teacher does, however, demand that care be taken when drawing conclusions concerning *single* teachers. There may be more random events in single lessons affecting a teacher's choice, which is why larger projects often combine the stratified sample with an in-depth analysis based on more, possibly consecutive lessons from a few teachers. I refer to such methodology in the LPS video study (section 3.5.3) and the LMT (section 5.9.2).

To compensate for such arbitrariness a follow-up study is performed with a focus group of teachers from the first part of my study in relation to RQ2 (chapter 10), and finally a longer study of possible effects of a certain kind of intervention in relation to my RQ3 (chapter 12).

5.1 Mixed methods

To obtain an answer several methods are combined in the collection of data. Also care is taken in the method design. The sampling of mathematics teachers for the research is intended to meet scientific standards sufficiently convincing for providing evidence in answering the research questions in section 2.2. To ensure this and also make some comparison possible to partly similar national or international research I have considered multiple sources of information in a systematic inquiry.

5.2 Framing conditions (grade 8, non private)

In this research grade 8 mathematics classrooms are chosen for the reasons mentioned above. The first research question requires data from classrooms and teachers. And I expected observations in classrooms to show the presence of different sorts of mathematical points fitting the definition.

RQ1 can be answered in part by statistics showing the frequency of use of mathematical points in teaching. Some points may then be connected to concepts, some to procedures and some to results or interpretations. And teachers may express the importance in classroom communication – possibly in overview, underlining or summing up. Other classroom observations may show an *absence* of points being made, which then may indirectly inform on the “why” part of my question. Priorities in these lessons may be deduced from the relative weight of different phases in the lesson like homework review or seatwork.

Most children in Denmark attend one school from grade K-9 (mixed primary and lower secondary). In 2008 approximately 510 Danish schools having these grade levels were private (but still state subsidized). A further 260 private schools were boarding schools (in Danish: Efterskoler). And 1,542 schools were public municipality owned (Uni-C: Statistik & Analyse, 2008). In 2008 15 % of students in grade 8 went to private schools and 85 % to the public municipality owned schools.

Some private schools have a special emphasis on e.g. religion, music, sport or high-level teaching in preparation for further studies. Students in private schools may come from other or certain parts of the country, as some of these schools are boarding schools. To limit the number of significant parameters I decided to confine the research to public municipal schools with grade 8 students. The students who attend these schools almost exclusively live in the district around the school.

5.3 Stratification, sampling

To assess the use of points among Danish mathematics teachers the sampling of classes has to be stratified to meet current standards and to make comparisons possible to similar observations and analyses in other countries. Besides the choice of municipality owned schools described above a geographic spread was attempted which, respected the varying density of grade 8 students in a small country like Denmark, and incorporated both rural and urban settings. But decisions on school size were not taken beforehand, the random sample later showed large variations in school size.

The relative weight of student population in the 5 regions of Denmark is reflected in the sample. Eventually, the schools that took part in the study represented 41 out of the 96 Danish municipalities. A minimum number of 50 schools was chosen to ensure trustworthiness comparable to international research such as the TIMSS Video Studies – 50 was the minimum number of schools in each country in the first TIMSS Video Study.

Contact and correspondence with 106 Danish schools was necessary in order to arrange for 50 acceptances representing geographical variety. It also included presentation pamphlets, forms for parents’ acceptance, organization of mail, telephone calls, a research design to cope with eventual declinations, etc.

Danish grade 8 students in the 5 regions, October 1, 2006 (www.statistikbanken.dk):

Regions	Municipal schools	Private schools	Private in %	Grade 8 in total	*) Target number	**Schools asked**	**Schools appointed**
Capital region	14 267	3 306	19%	17 573	12.9	**39**	**11**
Zealand region	8 577	1 512	15%	10 089	7.8	**16**	**7**
South region	12 917	2 164	14%	15 081	11.7	**17**	**10**
Central region	13 005	1 963	13%	14 968	11.8	**27**	**16**
North region	6 463	805	11%	7 268	5.9	**7**	**6**
In total	55 229	9 750	15%	64 979	50.0	**106**	**50**

*) This column shows the target number of schools in regions to keep representativeness.

In each municipality all public schools with grade 8 students were listed alphabetically in a spreadsheet file and schools for the study were selected by using the random function. In the four largest cities 2-5 schools were selected, otherwise only 1 school per municipality was chosen. In the case of a school declining the invitation to participate in the research, the next school in the alphabetically sorted list was asked.
At each school the grade 8 class named 8.a or similar alphabetically first was then chosen. The schools which couldn't meet this criterion, were rejected apart from a very few cases where the principal guaranteed arbitrariness.

5.4 Communication to and from schools, bias due to rejections

Once selected by the sampling procedure all schools were sent a letter with the same wording addressed to the principal with a presentation and invitation to be part of the research project (Appendix A). But there were many necessary variations in the way actual communication to and from the chosen schools developed.

- The letter of invitation consisted of a presentation of the project both for the school management and the grade 8 mathematics teacher (of the grade 8a class or equivalent) as well as an acceptance sheet with a calendar on which the teacher could indicate when it was suitable to visit the class. This sheet also included the possibility to tick a box saying: *No thank you* to inform me of rejection and the possible reason. Finally a draft sheet was enclosed for informing students and parents of the study and that the school's permission had been requested.
- In many cases this letter was lost or forgotten once received at the school. After leaving a reasonable time, if there was no reply I followed up the introductory letter with a phone call to the school office.
  If the letter seemed to be lost, I offered to send another one or the same information by e-mail. The latter was often preferred, as it is easy to send it on. If the letter had been passed on to a teacher or another person, I excused my impatience and prepared for a further wait.

If the school already had decided to reject my request, I often tried to convince the school (a secretary or the principal or another leading person) of the interesting opportunity and the importance of contributing to a valid “picture”. Sometimes the school reconsidered.

- Rejections came by letter or by e-mail. E-mails were always friendly answered by me with a thanks anyway *and* a regret.
- All correspondence with the 106 schools is kept. In many cases contact included more letters, e-mails and up to 4-5 telephone conversations before reaching the person in charge or the mathematics teacher.

To investigate possible bias, the reasons for school refusal were categorized. Categories of reasons from 56 declining schools were:

Teacher not willing	Too busy and the like	Disciplinary Problem	Other, or not stated
24	12	7	13

Does this represent a risk of bias in the data?
There is a risk of drawing too rosy a picture, since it is likely that a significant number of teachers who were not willing to participate in the survey, were less self-confident teachers than those who participated. To investigate if there were regional differences, I split the known reasons across the 5 regions of Denmark in the table below:

Region	Schools asked	Accept	Declination	T. not willing	Too busy and the like	Disciplinary problem	Other, or not stated
Capital	39	11	28	10	8	4	6
Zealand	16	7	9	6	1		2
South	17	10	7	1	2	1	3
Central	27	16	11	6	1	2	2
North	7	6	1	1			
Total	106	50	56	24	12	7	13

In the Capital region (e.g. Copenhagen and surroundings) the number of declinations is especially high. One explanation might be that mathematics teachers in the capital area feel under more pressure regarding official obligations and inappropriate student behavior than those in the rest of Denmark.
If this is the case, the video observations may present a more positive situation than is actually the case in the capital area and it is unlikely to be the case that point’s driven mathematics teaching is more prevalent in the schools that declined the invitation to take part in the study.

5.5 Video manual (technical, the layout, copying, storage…)

As described in *The TIMSS Videotape Classroom Study (TIMSS Videotape Classroom Study, 1999, p. 15 ff)* and the *Data Collection Manual for the TIMSS-R Video Study* (*TIMSS-R Video Study. Data Collection Manual*) two basic principles were followed in the present Danish research of 50 grade 8 classrooms, which used one camera:

1. *Document the visual perspective (attention) of an ideal student*.

The camera will always follow the teacher like an observant, ideal student. Focus is sometimes shifted to students (peers) for communication requiring the attention of the teacher and rest of the class.

2. *Document the teacher*.

Everything the teacher is doing to instruct the class should be captured on video. If the teacher moves around or leaves the classroom, the video camera is demounted and the teacher followed around with a handheld camera. If moving to another room, e.g. when using computers or if the class is taken outdoors, the teacher is also followed and the teacher's messages / dialogue caught on tape.

The camera was mounted on a tripod for rotation and zooming at the start of every lesson. The position taken will depend on available room. The preferred camera position is a 1/3 of the room’s distance from the blackboard or teacher’s desk and with the back to light and windows. This will make it possible to swivel from a frame of the teacher at the board towards most of the class in one motion.
The recording is started when the lesson begins as signalled by the bell or time. Even if this is before the teacher arrives in class or starts teaching. The recording is ended at the end of the lesson as marked by the teacher.

Before lesson start the mathematics teacher was given short information to pass on to the class. Most teachers did use this information, although some had already informed the class about the study. The information sheet read as follows:

Information for grade 8	**Information til 8. klasse**
1. A is from the Teacher Education College (in Aarhus). 2. He records the whole lesson on video (until the bell). 3. Pretend he is not here – even if he walks around. 4. Please do not say hello – or ask him about anything – until afterwards. 5. It is the intention to film a completely normal mathematics lesson. 6. To get good ideas.	1. A er fra lærerseminariet (i Århus). 2. Han optager hele timen på video (indtil det ringer). 3. Lad som om han ikke er her – også selv om han går rundt. 4. Altså ikke sige hej – eller spørge ham om noget – før bagefter. 5. Det er meningen at filme en helt normal matematik-time. 6. For at få gode idéer.

Prior to visiting the schools, I had some training in filming using a video camera. This took place at my University College in Aarhus, Denmark following the same routines as expected in TIMSS-R Video Study (*TIMSS-R Video Study, Data Collection Manual*).

For this and the handling of digital video data I had the valuable assistance of a skilled media consultant working full time as an advisory teacher to teachers and students at the college. E.g. handling and storage of data was discussed – including the use of bar codes for all materials / artefacts picked up at the 50 schools. The final decision was to label tapes and digitize these and any printed material supplied with filenames including the name of the school. Schools were also assigned numbers 1-50 chronologically.

5.6 Data types

When visiting schools I would take:

- All written correspondence, notes from telephone conversations, printouts from the school's website and often the weekly timetable for the class in question
- Driving directions
- Questionnaire sheet for the teacher
- Camera, tripod, extra camera, extension cord and batteries recharged daily.

As some schools demanded inland travel with ferries or an overnight stay, the routines also demanded reservations and expense accounting.

A complete set of data from every class consists of:

- Video recording of mathematics teaching in one whole lesson with the camera following the teacher
- Collected sample of the teaching material (copies)
- Filled out questionnaire from the mathematics teacher
- Memo written by the researcher, i.e. Arne Mogensen.

5.6.1 Video

During the actual recordings the question on lesson length came up, which was a surprise. In Danish schools quite a lot of mathematics lessons now are organized in 90 minutes units, which is the equivalent of having two "normal" lessons in a row. In all circumstances, the recording was made for "only" 45-50 minutes at the appointed time. In most cases this was from the very beginning of the lesson(s), and in the case of "double-lessons", the recording was stopped at the latest after 50 minutes or at a natural break after 45 minutes.

In some cases the teacher planned the double-lesson to change content and organization midway as I left the classroom. Other class took a short break at the midway point. Of the 50 mathematics lessons studied, 18 were part of "double-lessons". For those lessons the recorded lesson time is marked with a D in the table below on lesson phases (section 8.2).

Some recordings were a bit shorter than the anticipated 45 minutes. This happened, when the mathematics teacher broke off the lesson a bit before time was up.
All lessons were digitally taped on DV-mini tapes, and I always took along an extra video camera – although it was never necessary to use the reserve. The tapes were copied as Windows Movie Maker files onto my computer with a regular backup on an external disc.

5.6.2 Questionnaire

Any presence of an external observer will someway affect the teacher and the students. Therefore care has to be taken to ensure maximum normality when observing.
Validity is supported by the use of a teacher questionnaire (Appendix B), teaching plans and teaching materials to triangulate data collection. The purpose of this procedure is to find information valuable for valid understanding and interpretating video data. Even though a short questionnaire was announced in the initial letter, the document was kept to one page to avoid informants dismissing the sheet as too time consuming.
Beside factual information about educational background and seniority as a teacher, which provided valuable statistics on the sample of teachers for a "casebook" (section 8.1), the questionnaire included:

- *Goal*: Did the teacher import a goal from a teacher's guide or did he / she have his / her own? Did the teacher have a written lesson-plan and how detailed was it? The teacher was asked shortly to describe her/his own goals for the lesson and the ones surrounding it to indicate if the lesson was to be seen as a stand-alone lesson or one in a series.
- *Content*: The video tapes show the mathematical topic and the teaching material used, but the teacher was also asked whether this related to a textbook, and if so, which page in the book or their own material distributed to all or some of the students.
- *Assessment*: The teacher is asked whether assessment was included in the lesson, and when the students were last assessed in mathematics.

The questionnaire for the mathematics teacher was given to the teacher immediately before or after the lesson to avoid any effect on the teaching. It was in a folder with a stamped and addressed envelope. In one instance the teacher found time for answering it while students were working. In all other cases the questionnaires were later returned by mail. Quite a few teachers had to be prompted for answers – even twice. The response rate was 90 % (45 out of 50). The questionnaires were scanned and digitally stored using NVivo software.

5.6.3 Memos

During the 50 visits to schools I gained many impressions and much information, even before and after the mathematics lesson. As a researcher and an experienced former teacher I tried to keep a low profile when invited to express views on the teaching I had observed and registered. But later, after the lesson, I was often invited to discuss conditions and results with the participating teacher or colleagues. I decided to go into such communication since it wouldn't affect the recorded video data, but might still influence the questionnaire. In practice this was also difficult to avoid. At some schools I had an (even long) conversation with the school principal or the "pedagogical leader". At some schools I was also shown around the premises. But in most schools I was welcomed briefly and offered a cup of coffee in the staffroom and then only had contact with the mathematics teacher in question.
Whatever the circumstances, I felt able to pose any possible question. At some schools management or teachers were very interested in getting some feedback on the findings. As mentioned earlier all schools were offered the opportunity to receive a copy of an edited version of the PhD dissertation or of a book that may result from the research.

Immediately after each recording I wrote a memo on the supplementary information I came upon in the visit. This was very different from school to school. As note-taking easily becomes very formal and as I was busy enough preparing and coping with the video camera I wrote down this short memo *immediately* after each visit in order to remember. This was normally based on notes scribbled in my car even before leaving the school parking lot.
I did not carry out proper interviews during these visits, but some questions became almost standard. Beforehand I assumed some categories to be particularly pertinent to understand the teachers' ability to let points drive the teaching. Such categories were prepared in a template on my computer and this may of course have framed my curiosity. Below is an example of a memo:

<table>
<tr><td>School</td><td>No. 7</td><td>Date</td><td>January 17th, 2009</td></tr>
<tr><td colspan="4">Relevant information from previous correspondence
None.</td></tr>
<tr><td colspan="4">Relevant information from contact to school management on date
The school has been focusing on "learning styles" for 3-4 years.
Colleagues in the staffroom seemed conscious of the qualities in this focus.
The mathematics teacher actually met Rita Dunn *).</td></tr>
<tr><td colspan="4">Statements from students or colleague teachers
Three teachers at the school were certified by Rita Dunn (to become a "Certified Learning Styles Trainer" normally requires a 5 day course in New York).
Two of them have now left the school for better positions!</td></tr>
</table>

Socio-economics / bilingual students 1-2 students did not look “ethnic-Danish”, but there are no (language) problems. One very weak student has homework especially assigned, and some gifted students are working “ahead” – but with the same mathematics textbook. When they want, they may work with other materials (even in other subjects) during mathematics lessons.
Student attitude Really good. It’s the conscious decision of the mathematics teacher to let students organize themselves in small groups. E.g. a group of girls settled in the common area outside the classroom and worked together around one large table listening to music from a “ghetto blaster”. They were allowed to sit on chair backs or windowsills.
Girls: 13 **Boys**: 6 The class has 23 students, only 19 were present.
Routines Students do get homework approximately every 3 weeks, but according to the teacher this is never corrected! The teacher uses www.evalueringssystem.dk for assessment. Homework: page 88, exercise 1-3.

*) Rita Dunn (St. John's University, New York) worked with the concept of "Learning Styles". The theory focuses on what factors affect different people's learning. Dunn takes the approach that everyone has their own individual way of learning. Each person has certain preferences, through which they learn best. Dunn’s research recommends classroom practice allowing individual students to pursue different learning styles.

5.6.4 Teaching materials

Materials referred to in the lesson were also scanned using NVivo 8.
E.g. this could be:

- Pages from a mathematics textbook
- Other printed matter
- Teacher’s own material
- ICT printouts.

When teaching was based on mathematics textbooks with which I was familiar, I just noted the page numbers and tasks referred to. I later scanned the pages from my own collection of textbooks or the one in the college and stored the files in my computer. For other printed matter, I asked for a copy. This was of course also the case, when teachers used their own materials.
At some schools computers were used during the mathematics lesson that was video recorded. It is not directly registered, to what extent this was done and the types of program that were used. Besides spreadsheets and dynamic geometry software I noticed computers used for self-assessment (Able – Almost able, school no. 17), reporting (Jing, school no. 30) and data search on the Internet (Google, school no. 50).

5.7 Verbal reports as data – “grounded” or not

Coding refers to the analytical process in which data are categorized to facilitate analysis. Some data are quantitative such as a tick box or information in questionnaires on seniority or educational background. And some are qualitative such as transcripts.
Corbin and Strauss (Corbin & Strauss, 2008) suggest looking systematically at qualitative data as transcripts of interviews or protocols of observations with the aim of generating a “grounded” theory. Grounded theory thus combines research with a pragmatic theory of action and some methodological guidelines, inductively generating new theoretical ideas or hypotheses based on the data. Corbin and Strauss suggest the analysis of interviews to be divided into phases starting with an *open coding* to identify relevant categories (nodes).
In the open coding testimony is reviewed to identify ideas and to code data material:
In a second *axial coding* the categories may be refined, developed and interrelated. Relationships between categories (nodes) may be investigated and connections between them may be created (in node trees). Correlations may be visualized in a chart or in a model. Finally a *selective coding* may be done selecting the central category which relates the best to all other categories. One of the codes may be considered as the central phenomenon. Around this central phenomenon a narrative may be constructed that brings together most elements (nodes) in the analysis. The selective coding is then systematically related to a "master" node to other nodes.

My expectation and intention was to follow this approach. Developing the coding categories finally decided upon was quite a long process. Gradually and before the data collection began, I also decided to identify more phenomena in lessons besides the identification and statistics on points. I used NVivo software for coding, which defines codes as nodes organized in trees.
Many important boundary conditions may be described by quantitative data such as age, sex, experience etc. Such data was collected in questionnaires by the participating teachers, while information on school size was accessible at school or municipality based homepages.

5.8 A point “being made” by students

The first research question is focusing on the mathematics teacher. But students also may present didactic points in response to teachers’ questioning the whole class or in communication to groups or individual students during seatwork.
According to Ericsson and Simon (Ericsson & Simon, 1993) focused thinking requires the carriage of registered thinking (heeded thoughts) from short-term memory (STM) to long-term memory (LTM):*”information that is heeded during performance of a task is the information that is reportable, and the information that is reported is information that is heeded”* (p. 167).
An understood / realized point supports memory and the possible repetition of the line of thought (deduction). Point-driven or point-containing teaching thus becomes a

(sequence of) lesson(s), where points are guiding the teaching and / or seen / heard / learned / encountered by the students. Points could frame and lead the activity (its goal, content, methodology, organization) or represent appropriate highlights or stages to a longer course (chain of thought or explanation of a complex, possibly open problem).

Teachers may be expected to invite two types of verbal reporting representing cognitive processes in learning (Ericsson & Simon, 1993, p. 16):

a. *“Concurrent verbal reports – talk aloud and think aloud reports”*.
b. *“The retrospective report”* immediately after solving a problem may be at least partly found in the short-term memory, but otherwise based in the long-term memory.

This also suggests different levels of verbalization (Ericsson & Simon, 1993, p. 79ff):

- *“Vocalization of covert articulatory or oral encodings”*
  The internal speech of a student to himself will not be caught by the video recording, but on invitation by the teacher students might inform on their thinking this way.
  E.g. A teacher may ask as Krutetskii did in his research (Krutetskii, 1976):
  Think aloud, while you solve the task.
- *“Description or rather explanation of the thought content.”*
  No new information need to be brought to attention of the student, but the restructuring of current information is demanded as the student has to explicate or label information already held in another format.
  E.g. A teacher may ask a student who has stalled: What are you thinking?
- When a student has *“to explain own thought processes or thoughts”*.
  This demands lining to former thoughts and previous information.
  E.g.: A teacher may ask: How were you thinking, as you calculated 23 x 12?

Knowledge of such different ways to ask may be the basis of mathematics teachers’ elicitation of students’ points in communication to and with a whole class, groups or individual students. Key questions and other interview skills may show proficient teaching even if points are not immediately being made in the conversation. Several lessons in the research contain dialogues between teacher and individual students where the teacher invites verbalization of student thoughts. Teachers’ use of elicitation will be discussed more fully in section 6.7.1.

5.9 Possible classroom research coding categories

I have been transcribing parts of each lesson with an emphasis on possible points and the lesson layout. It is quite obvious that extreme care has to be taken when deciding to code the transcripts for the occurrence of points. As one person (I) is expected to handle all coding, the question of reliability is different from a situation with more

coders. Still, the codes should be described so precisely, that a consistent and accurate use is ensured.
Transcription and coding of *everything* said in the videos is impossible in a project this size (a PhD study). But the registration of occurrence and the categorization of points is crucial. The amount of time a code is applied is not considered more decisive than the actual occurrence – on the contrary. The extent of points in lesson length or as a frequency is discussed in sections 8.5 and 8.6.

When coding categories are not explicitly decided upon before looking through the data, grounded theory may be an idea to apply. In the present research I had several ideas about what to look for beforehand, but wanted to reserve the possibility to change views in a repeated passing of collected data.
In several research projects referred to in the overview above (chapter 3) transcripts of instants in mathematics classrooms have also been coded. And as my present research focuses on the teacher, I choose to investigate the extensive range of codes used by TIMSS Videotape Classroom Studies and LMT on teacher utterances in order to minimize misinterpretation of data.

5.9.1 TIMSS Videotape Study

The aims and scope of TIMSS Videotape Classroom Studies described in chapter 3 are much wider and larger and than the Danish research in question. But the interest in mathematics teaching routines and beliefs are common. Therefore it is interesting to note which codes were used. Codes selected for the 1995 study were (*TIMSS Videotape Classroom Study*, 1999, p. 104):

Categories used for first-pass coding of utterances during public discourse:

Category	***Description***
Elicitation	A teacher utterance intended to elicit an immediate communicative response from student(s), including both verbal and non-verbal responses.
Information	A teacher utterance intended to provide information to the student(s). Does not require communicative or physical response from students.
Direction	A teacher utterance intended to cause students to perform some physical or mental activity. When the utterance is intended for future activities, it is coded as Information even if the linguistic form of the utterance is a directive.
Uptake	A teacher utterance made in response to student verbal or physical responses. It may be evaluative comments such as “Correct,” “Good,” or “No,” repetition of student response, or reformulation of student response. Uptake is intended only for the respondent, and when it is clear that the utterance is intended for the entire class, it is coded as Information instead of Uptake.
Response	A student utterance made in response to an elicitation or direction.

Student Elicitation	A student utterance intended to elicit an immediate communicative response from the teacher or from other students.
Student Information	A student utterance not intended to elicit any immediate response from teacher or from other students.
Student Direction	A student utterance intended to cause the teacher or other students to perform immediately some physical/mental activity.
Student Uptake	A student utterance intended to acknowledge or evaluate another student's response.
Teacher Response	A teacher utterance made in response to a student elicitation. Provide Answer PA A teacher utterance intended to provide the answer to the teacher's own elicitation.
Other	An utterance that does not fit into any of the above categories or that is not intelligible.

Elicitations were further subdivided into five mutually exclusive categories:

Category	***Description***
Content Elicitation	An elicitation that requests information directly concerned with mathematics, mathematical operations, or the lesson itself. Such elicitations may request the student to supply a quantity, identify a geometric shape, explain a mathematical procedure, define some mathematical term, or evaluate a mathematical answer, among other things.
Meta-cognitive Elicitation	An elicitation designed to determine a student's current state of mind or level of understanding. These types of elicitations are often used to assess student progress as well as student understanding.
Interactional Elicitation	An elicitation that requests a student to modify his/her behavior, to acknowledge his/her participation in some current activity, to recall specific classroom procedures or rules, or to gain students' attention.
Evaluation Elicitation	An elicitation that requests a student or students to evaluate another student's answer, response, etc. Generally, the evaluation of responses is a role taken by the teacher, but on occasion, the teacher may turn that role over to a student or students.
Other Elicitation	An elicitation that does not fit into any of the above categories, including all forms of conversational repair. When an elicitation occurs in the middle of a student's long response, it could be coded as Other Elicitation when it is obvious that the teacher does not intend to terminate the response but to clarify a part of response.

And Content Elicitations were subcategorized as well.

Category	***Description***
Yes/No	Any content elicitation that requests a simple yes or no response from student(s).

Name/State	Any content elicitation that requests a relatively short response, such as vocabulary, numbers, formulas, a single rule, an answer to some mathematical operation, etc. Also, an elicitation that requests a student to read a response (from a notebook, book of formulae, etc.) or that requests a student to choose among alternatives.
Describe /Explain	Any elicitation that requests description of a mathematical object (rather than its label), explanation of a generated solution method (rather than an answer), or a reason why something is true or not true.

My main reason for not simply adopting the coding methods from TIMSS Videotape Classroom Studies in the Danish study is lack of manpower. In this project coding and scope is concentrated on the idea and recommendations of points. The data sampling makes it possible to also report on teaching styles, the role of homework, phases in lessons etc. Such items will be referred to in the extent it is found useful to describe teachers' choices and possibilities when teaching grade 8 mathematics with points in mind.

5.9.2 LMT – Learning Mathematics for Teaching

Also in the Learning Mathematics for Teaching project (LMT) many codes have been used for analyzing mathematics teaching ability and strategy. This project based at the University of Michigan has since 2003 been focused on the relation between teacher's mathematical knowledge and their mathematics teaching (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep & Ball, 2008).
The research project is based on the assumption that teacher knowledge works through instruction to cause student learning. But this relationship is not simple and easily understood. The LMT researchers stress, that larger video studies in mathematics classrooms until now have not compared the practice of "higher-knowledge and lower-knowledge" teachers. There are many suggestions regarding how certain knowledge matters for teaching. *"But with sample sizes of one teacher per study in many cases, and few objective measures of teachers' mathematical knowledge, generalizations have to date been limited."* (p. 431).
Six elements are considered crucial in the LMT project to the *quality* of teachers' instruction.

Three of these are due to expectations in the "deficit" literature on teachers' lack of mathematical understanding quoted by Hill et al. (Hill et al, 2008,
p. 437):

1. Mathematical errors
2. Inappropriate response by misinterpretation or no response to student misunderstanding
3. Connecting classroom practice to important mathematical ideas and procedures.

Two further themes are identified from the "affordance" literature on teachers' strong mathematical understanding:

4. Rich mathematical representations, explanations and justifications
5. Appropriate response to students' mathematical utterances and misunderstandings.

And one final category stems from the exploratory phase of the LMT project:

6. Accurate mathematical language and clearly conveyed mathematical ideas.

The LMT video recordings are coded in detail and connected to the quality parameters mentioned above. As part of the validation study nine lessons from each of 10 teachers were collected and coded using a rubric with codes in the following categories:

- Mathematical errors (9 subcodes)
- Use of mathematical and general language to describe mathematical ideas (4 subcodes)
- NOT doing mathematics (3 subcodes)
- Presence of "rich" mathematics – justification, explanation, multiple representations, explicit development (7 subcodes).

The final LMT video coding tool, the "Quality of Mathematics in Instruction" instrument, was developed to allow observers to evaluate the quality of the mathematics in instruction. 10 specially selected teachers from grades 2-6 were filmed three times and encoding of video very extensive with 83 codes divided into five sections:

1. Instructional formats and content
2. Knowledge of mathematical terrain of enacted lesson
3. Use of mathematics with students
4. Mathematical features of the curriculum and the teacher's guide
5. Use of mathematics to teach equitably.

Compared to my research the sections 2 and 3 in the LMT coding tool seem of obvious relevance. My coding of points articulated by the mathematics teacher will provide a measure of teacher proficiency by extent and variation in length and frequency of points-containing communication. This is *not* the same quality measure for teaching as the LMT instrument, but manageable in a smaller project, and due to the rather wide categories of points (child nodes, sections 6.4, 6.5 and 6.6) with an estimated sufficient validity and credibility (sections 8.9 and 8.10).

5.9.3 Mathematics Classroom practice in Finland and Iceland

Several research studies based on videotaped math lessons are presently being carried out in the Nordic countries. Most are rather small as restricted to 1-2 classrooms, thus eventually followed for a longer period. But one larger project is

reported by Savola (Savola, 2008) who in 2007 collected data from Finland and Iceland in two lessons with each of 10 randomly chosen math teachers of 14-15 year-olds in each country.
The research questions were on the scope of video-based methods of lesson-structure analysis. The intention was to juxtapose this method against those used in the TIMSS and LPS studies with their emphasis on the purpose, content and task structure (in TIMSS) and the interaction of function and form elements with "lesson events" as Kikan-Shido (section 3.5.3 on LPS).

The coding categories in the first pass coding are rather wide (p. 150):

- Review
- Introducing New Content"
- Practicing/Applying
- Other

But in the second coding pass, the categories are further divided into subcategories based on classroom interaction. E.g. the category: *Introducing New Content* was divided in five:

- Teacher presents a lesson, intermittent questions
- Teacher elicits responses from the class by asking a series of connected questions
- Students work on new type, teacher helps
- Students copy text
- Students read the book.

The results are statistics on time allotted to different kinds of activity, mathematical or not. It also shows some differences between teachers in the two countries. E.g. "*Finnish teachers spent 63% of the time presenting in front without involving the students for more than an occasional question. For Iceland, the percentage is 80.*"
(p. 169)
There are no references in the report on this project to the concept of "points" as the important mathematical messages in the lessons. In this way the project is very different from my Danish one. But some of the considerations when deciding on coding are similar to the thoughts and choices to be made by myself.

5.9.4 Pilot video

I made a pilot recording (60 minutes) at a local school in June 2008 in order to develop a field routine, and eventually make some last minute corrections to the setup, including the questionnaire for teachers. I knew this teacher and the school management beforehand and the purpose was solely to tune the research apparatus and try out a pilot-version of papers:

- Letter for the school principal
- Letter asking permission from parents
- Keywords for teacher's presentation of project to class
- Teacher questionnaire.

The pilot study was valuable for logistical and practical reasons. Familiarity with the technical operation of video equipment allows for better forecasting of development in teachers' dialogue with class or individual students and a similar focus with close-up or wide shots. When lessons started the camera was placed on a tripod in a position to cover the whole class, but also what may be going on at the teacher's desk or blackboard. During seatwork the camera was handheld and followed the teacher as closely as possible without disturbing.

It was instructive to record video and not so easy when the teacher started moving around. The video was later copied to a DVD to facilitate searching for episodes. I also collected a copy of the teacher's teaching material and the filled out teacher questionnaire. Everything in the pilot study including helpful suggestions from the teacher indicated that the text could still be improved.

I found it difficult afterwards to code utterances in the pilot video, as the teacher provided many kinds of information and entered into different kinds of communication with class and individual students. Remarks from students were repeated, reformulated or taken up in teacher utterances. There was also more teacher-student dialogue than expected as the teacher shifted among the roles of informing, rhetorical, leading and controlling.

The questionnaire was working pretty much as was hoped for. Fields in the query form did not leave room for longer writing but seemed to be understood as intended and were completed by the teacher.

5.10 Ethics, anonymity, code of conduct – actual visits

According to Hitchcock and Hughes (Hitchcock & Hughes, 1995) visits at schools and subsequent writing of memos is very much dependent upon the researcher (p. 122). Emotions, values, beliefs and expectations may distort or hide the perspective and ideas of the informants. Engagement and professionalism should carry through, and it is important as an external researcher always to keep an observant, positive attitude to everything.

Beside the informants, i.e. participating teachers and students, several other people are framing the research:

- The school offices are staffed by informed personnel very much aware of and knowing about what is actually happening at a school and also often informed about teachers' whereabouts and attitudes. Some are even efficient and necessary "gatekeepers" capable of hindering or advancing the admittance of an external observer.
- The principal or other persons in the school management are sometimes "only" passing on invitations, not wanting or able to interfere with the decision of teachers. More seldom principals are so sure of teachers' acceptance, that they express an acceptance beforehand. In some cases the mathematics teachers subsequently declined.
- The staffroom is where you meet teaching colleagues, who can supply valuable information about the school profile.

Hitchcock and Hughes stress (p. 124) that as a visiting researcher you don’t arrive unannounced at a school. Being a former mathematics teacher in lower secondary school myself, the familiarity with a school community may be both an advantage and a disadvantage. Also observations may be open or hidden. Informant reliability and response validity is an issue (p. 126). Not everyone at school is probably considered as an equally valid, reliable and competent informant by the school itself, but my sampling design copes with this. Students could also act as informants – the social setting can produce "noise", this may also go into researcher memos.
Arriving at the school, I always “checked in” at the school office. Occasionally I was greeted welcome by the school principal or a substitute. At some schools the principal invited me to longer discussion about school development in general and local priorities. At two schools (15 and 18) this meant I was a bit late for the classes and sort of jumped in lessons already begun.
In one case my arrival was a surprise, the teacher who was a (young) substitute without any teacher education was not asked, and not aware of the arrangement. She was talked into teaching an extra lesson by the principal, but I kept to the routine, as substitute teachers are also a part of the everyday routines in Danish schools.
When leaving schools I also “popped in” at the office to say goodbye, often it was only at point that I met the principal – and often only briefly.

Schools are granted anonymity in the research. Therefore each school is given a number in the subsequent analysis. All schools are also promised feedback when the outcome of the research is reported in a PhD dissertation. This will perhaps be in form of a book in Danish sent to all participating schools.

6 Coding

With the NVivo software one can assign a specific, predefined code for a time stamped extract from a video file, either by checking the transcribed dialogue or by checking on a timeline interval. Digitized "takes" from the lesson and all other digitized data in this research are filed in this software environment. Codes for points and other foci of interest are organized into a tree structure corresponding to subdivision of a general term.

6.1 Coding categories

There are different categories of codes. According to NVivo these are named *Parent nodes*. And each parent node may be further sub-divided into a number of *Child nodes*.
Two codes are on linked to the *start up* of the actual lesson (section 6.2).
These will inform on the time it takes to get started with the actual mathematics teaching,

Parent codes	***Child codes***
Start up	Arrival
	Information on video-taping

Eight codes are on various *activities* during the lesson (section 6.3)
These may be different kinds of information, handling of homework or student seatwork. This may also cover interruptions in the lesson after starting.

Parent codes	***Child codes***
Information	Announcement of lesson content
	Non-mathematical information
	Organization of lesson
Homework	Collecting or returning homework
	Correction of homework on class
	New homework
Instants addressing points	Overview (repetition)
	Summing up

And 15 codes are related to mathematical *points* in actual content (sections 6.4-7).
My focus is point-driven teaching. This is why most effort has been laid down in coding teachers' points throughout the lesson. This means that the codes are associated with different teacher aims. I have chosen these to cover the spectrum:

- Justification, definition or other reference to a mathematical concept
- Justification, development or demonstration of a precise mathematical method
- Inductively or deductively based mathematical result or conclusion.

- Assessment and interpretation of a mathematical result.

It is tempting to work with many codes, because so much is happening in every lesson that one may want to remember. Conversely, it may spoil your overview as multiple codes easily occur simultaneously or partially overlap.

Parent codes	***Child codes***
Points by teacher	Conceptual point
	Procedural point
	Result point
	Interpretation point
Points during seatwork	Conceptual point in seatwork
	Procedural point in seatwork
	Result point in seatwork
	Interpretation point in seatwork
Points by student	Student conceptual point
	Student procedural point
	Student result point
	Student interpretation point
Instants addressing points	Elicitation
	Hint
	Missed point

Different codes for a point are attached depending on the point being articulated in whole class teaching or while the teacher guides individual students and groups. Accordingly points may be coded when made by students in dialogues with the teacher or others.

If the coding only identifies mathematical points, the research may turn out to describe some mathematics lessons as point free. But mathematics teachers pose many questions in order to get students to inform, give reasons or conclude. Such "fishing trips" do not always end with students expressing a proper and desired point. Often the teacher will reformulate a student's opinion or simply self provide the answer.

Some teachers may continue to reformulate their questions with more and more clues until the students through guidance or eager guessing finally get a "hit". Other teachers may be reluctant to give students answers to questions like: *Is this the correct answer?* Such instances also give evidence that the teacher certainly plans for points to be made. Therefore I consider it reasonable to include such "elicitation" questions and "hints" in the coding.

Finally coding of missed opportunities to articulate points are coded. There may of course be more reasons to leave points un-articulated. Perhaps the teacher intends to have students articulate the point with further support, or the teacher feels it is too early. But it could also be due to the teacher ignoring, overlooking or forgetting the possibility.

Each code is described and exemplified in the same way in the sections below.
Each code is presented by a definition, i.e. a short description to follow when coding for the actual child node. A table shows the structure in five phases according to time stamps of an example lesson identified by school number and then 1 or 2 excerpts from this transcribed lesson with communication coded as the actual child node.
The excerpt is described by timestamp (mm:ss) from start of the videotape. In transcripts of a dialogue corresponding to a selected type of point T indicates the teacher, S a student.
After each excerpt the credibility of the actual coding is discussed and I seek to account for uncertainty and choices. For point codes the summed four types of teacher to class points are shown as a percentage of the lesson length.

6.2 Start up

Video recording is always started when the lesson is to begin whether this is indicated by a bell ringing or by the time. Sometimes not all the students have arrived in the classroom or the teacher is not quite ready to start teaching. Most classes will also wonder about the presence of a video photographer, and the teachers were asked to shortly explain or to remind the class about the study. Therefore codes are assigned for time slots on arrival and video information.
The description below decides when to start and finish coding for *Start up*:

Parent code	***Child codes***	***Description***
Start up	Student and/or teacher arrival in room	Students and teacher enter classroom, find seats and unpack
	Video information	Teacher informs on purpose and practice of the video-taping

6.2.1 Arrival

Mathematics is normally taught in the same classroom where the class has several other subject lessons. Most often the students were outside classrooms during the break, eventually arriving from the “playing ground”, from home if this was the first lesson of the day or from another room in school (e.g. for P.E. or physics).

An example: Lesson 29 Small school in South region Female teacher, seniority 0-4 years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:45 0:00-2:45	6:29 2:45-9:14	6:04 9:14-14:13, 16:15-17:20	27:34 14:13-16:15, 17:20-42:52	2:59 42:52-45:51	45:51

(0:00 - 2:45):

(T sitting at her desk talking to some students standing in front. The students are in class and ready). T: Are we missing someone today? S: Nah. T: So everyone is present! Then I just need to hear about this note on communal eating: I, Se, Si, and R (the names were on the blackboard): Did you bring it? S: No. T: So you have to hand it to N on Monday. S: Why are you not here? T: Because I've got a day off on Monday. But on Tuesday I'm here. (The conversation continues on the “welfare day "and a not too good organization of the sale of juice and cake the other day). T: What is it J? S: Can I print this in the recess on history? T: You may, I am teaching you the next lesson too, so you can just make it. Well M, you just put the note down in ... S: A, I have not (inaudible). T: Well, then you must bring it on Wednesday.	(L sidder ved katederet og taler med nogle elever, der står foran. Eleverne er i klassen og parate). L: Mangler vi nogen i dag? E: Næh. L: Alle mand og damer er her! Så skal jeg lige høre, den her seddel om fællesspisning: Se, Si og R (navnene stod på tavlen): har I den med? E: Nej. L: Så er I nødt til at aflevere den til N på mandag. E: Hvorfor er du her ikke? L: Fordi jeg har fået en fridag på mandag, men på tirsdag er jeg her. (Samtalen fortsætter om ”trivselsdagen” og en uheldig organisering af salg af saft og kage forleden). L: Hvad siger J? E: Må jeg gerne printe den her ud i frikvarteret om historie? L: Det må du gerne, jeg skal jo have dig i næste time også, så det kan du lige nå. Nå M, får du lige pakket den seddel ned i … E: A, jeg har ikke (uhørligt). L: Nå, men så må du tage den med på onsdag.

I have found no difficulty in applying the codes for *Arrival*.

6.2.2 Video information

Many teachers chose to read aloud the short information I always brought on a sheet of paper in large lettering (section 5.5). A few times this was not done at all, as the class seemed previously instructed. The mathematics lesson was then begun immediately without any comment on the circumstances.

An example: Lesson 32 Large school in Capital region Male teacher, seniority information missing.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:59 0:00-2:59		45:56 2:59-48:55			48:55D

(1:51-2:22):

T: Well, I've brought Arne with me [Hi Arne]. He is from the teacher training college in Aarhus, and he is going to video tape this lesson. It is ONLY 45 minutes: it is only the first lesson - then he will leave again. And he will record that on video, and you must pretend he is not here. It is always easy to pretend that someone who stands with camera is not there (ironically)! So there's nothing to it really - this is just an ordinary lesson as usual. So there is not more to it. [No]. Ok.	L: Godt, jeg har taget Arne med [Hej Arne]. Han er fra lærerseminariet i Århus, og han optager lige den her time. Det er KUN 45 minutter: det er kun den første time, så damper han af igen. Og det optager han på video, og I skal lade som om han ikke er her. Det er jo altid nemt at lade som om at folk, der står med kamera ikke er der (ironisk)! Så der er ikke rigtig noget - det her det er bare en ganske almindelig time som vi plejer. Så er der ikke mere i det. [Nej]. Ok.

I have found no difficulty in applying the codes for *Video information*.

6.3 Activities

Three sorts of activities are coded, and described below: Information, homework and some specific instants.

Some activities are presentation of various kinds of *information*. Mathematics teachers provide lots of information for students on the mathematics content during most of the mathematics lessons. But the information code in this context is reserved for announcements given by the teacher to the students before or after the actual mathematics teaching.

Three types of information are coded. The description below decides when to start and finish coding for *Announcement of lesson content*, *Non mathematical information* and *Organization of lesson*:

Parent code	*Child codes*	*Description*
Information	Announcement of lesson content	Teacher gives/requires information on lesson topic, method, connection or test
	Non-mathematical information	Teacher gives/requires non-mathematical information, e.g. field-trips, absence
	Organization of lesson	Teacher informs on organization of the lesson and place of seatwork for individuals/groups

6.3.1 Announcement of lesson content

An example: Lesson 33 Large school in South region Male teacher, seniority 15+ years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
3:00 0:00-3:00		16:20 3:00-19:20	28:13 19:20-47:33		47:33D

(4:50-5:35 + 19:20–20:20):

T: Yes, we are almost ready. Yesterday we started – or the day before yesterday – we started the section on percentages. And we had some talk about it. And we worked with some of the exercises on page 88-89. And I was well aware that we were pushed for time to achieve the tasks. But we agreed that we would now review the theory on percentages. J, you can both listen and ….? S: Yes T: You can? That's good. … T: There is a more detailed description in the book. <u>And you can read it yourselves</u>. You will only need what it says on the board here. I want you to look now at the exercises starting on page 92. And you know you still need to do some exercises on pages 88-89.	L: Ja, så er vi ved at være klar. I går startede vi jo – eller i forgårs – startede vi på det afsnit med procent. Og vi havde sådan lige en snak om, hvad det var for noget. Og vi arbejdede med nogle af de øvelser, der var på side 88-89. Og jeg var godt klar over, at det var sådan lige i underkanten med tid til at nå de opgaver. Men vi blev enige om, at vi i dag ville gennemgå teorien i procent. J, kan du både lytte og ..? E: Ja L: Det kan du? Det er godt.… L: Det står sådan mere udførligt beskrevet i bogen. <u>Og det kan I selv læse.</u> Det er bare det, der står på tavlen her, I skal bruge. Det der er meningen, I skal lave nu, det er at I skal kigge på de øvelser, startende på side 92. Og I ved godt, at I mangler nogle øvelser

Now you start with these, quietly. Coming up and getting help if you need it. Ok? S: On page 92? (There is knocking on the door, and another teacher comes in to deliver two books to a couple of boys: J and D. The books are placed in the teacher's drawer).	omme på side 88-89. Nu går I i gang med dem her, stille og roligt. Kommer op og får hjælp, hvis I har brug for det. Ok? E: På side 92? (Det banker på døren, og en anden lærer kommer ind for at aflevere to bøger til et par drenge: J og D. Bøgerne lægges i katederskuffen).

I have found no difficulty in applying the codes for *Announcement of lesson content*. But the underlined sentences may show that a distinction between this code and the one on *Organization of lesson* can be difficult. I chose *not* to code the excerpt as Organization of lesson because I understand it as teacher's encouragement to student reflection before asking for help.

6.3.2 Non mathematical information

An example: Lesson 10 Small school in Central region Male teacher, seniority 5-9 years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
6:47 0:00-6:47		8:02 6:47-14:49	31:12 14:49-46:01		46:01

(30:55-31:51 + 43:40-44:41):

(T walks between groups of students in the hallway and gets stopped at the librarian's desk by a student from class who wants to borrow video camera during the holidays). S: A! T: What do you say? S: May we take a camera home during the holidays? T: For what? S: It's because we do the project work I've told you about, and we will then return it after the holidays. We will (inaudible) and be filming a lot. But then would have to	(L går mellem elevgrupperne på gangen og bliver stoppet ved bibliotekaren af en elev fra klassen, der gerne vil låne videokamera i ferien). E: A! L: Hvad siger du? E: Kan vi låne et kamera hjem i ferien? L: Til hvad? E: Det er fordi vi laver projektopgave, som jeg har snakket om, og så vil vi så aflevere det efter ferien. Vi skal (uhørligt) og filme en hel masse. Men så skal vi så låne et videokamera i

borrow a video camera during the holidays. And we could then (inaudible) or what? T: Yeah, uhh – is it about getting it on Friday? S: Yes, just on Friday [Yes]. … (T is in a final round among the groups). S: We're finished. T: (inaudible) S: I don't have the time for that. T: No. But I will say it can wait until after the project assignment. S: What? T: It's not until you are have finished your project assignment. You have no more mathematics lessons before you finish your project work anyway. S: When is it we should be finished with it? We have not been told yet. T: The project? [Yes], it is in week 8. On Friday – no tomorrow, we actually do have a mathematics class tomorrow, but that's ok. After the winter break. On Friday, you have winter break. And then project assignment is in Week 8 that is the week after the winter break. Presentation will be on Monday the following week again. And then there's normal teaching. You will even have practicing student teachers. S: Will those we had be back then? T: Yes.	ferien. Og det skal vi så gøre, når vi (uhørligt) eller hvad? L: Ja, øhh - men er det ikke noget med, at I får det på fredag? E: Ja, bare på fredag [Ja]. … (L er på en sidste runde mellem grupperne). E: Vi er færdige. L: (uhørligt) E: Det har jeg ikke tid til. L: Nej. Men jeg vil godt være så large at sige, at det kan vente til efter projektopgaven E: Hvad? L: Det er til når du er færdig med din projektopgave. Du har ikke flere matematiktimer før du er færdig med din projektopgave alligevel. E: Hvornår er det vi skal være færdige med den? Det har vi ikke fået at vide endnu. L: Projektopgaven? [Ja]. Det er jo i uge 8. På fredag – nej i morgen, så vi har faktisk en matematiktime i morgen, men det er ok. Efter vinterferien. På fredag får I vinterferie. Og så er det projektopgave i uge 8, altså ugen efter vinterferien. Fremlæggelse om mandagen i ugen efter igen. Og så er der normal undervisning. Oven i købet får i praktikanter. E: Kommer dem der igen så? L: Ja

I have found no difficulty in applying the codes for *Non-mathematical information*. The latter part of the excerpt is also coded as *Organization of lesson* described below.

6.3.3 Organization of lesson

An example: Lesson 27 Large school in Capital region Male teacher, seniority 5-9 years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
4:16 0:00-4:16	2:14 4:16-6:30	1:38 6:30-8:08	37:00 8:08-45:08		45:08D

(8:08-9:20):

T: And we covered that last time. And this is also what we do in these first 20 minutes where we work forward on these tasks that everybody should be sitting and doing right now. And it's just like last time: We have done a different number of tasks. Some have come to task 3 and others to task 6. It does not matter. We work, and I come around. And of course I will come around to questions, but also if you finish completely, then I have something new to begin. We will stop only after these 20 minutes. S: May we to talk together? T: You may talk together. This is also why you sit together 2 and 2 with the neighbor. I do not know, N – you do not sit next to someone, but it's also (inaudible) S: (inaudible) T: Well, that's good enough: you're there. It's – K, could you sit down next to N, because then we sit 2 by 2 S: (inaudible) T: No, you're also – can you move over here? Are there any questions about these first 15-20 minutes? S: Do you provide help? T: Yes, yes of course. I'll come around.	L: Og det tog vi sidste gang. Og det er også det vi gør her de første 20 minutter, hvor vi regner fremad i de her opgaver i det, som alle gerne skulle sidde med nu. Og det er præcis lige som sidste gang: Vi sidder og er nået forskelligt. Nogle er nået til opgave 3 og andre til opgave 6. Det er sådan set ligegyldigt. Vi arbejder, og jeg kommer rundt. Og selvfølgelig kommer jeg rundt til spørgsmål, men det er også sådan, at bliver man helt færdig, så har jeg noget nyt, man kan gå i gang med. Så vi stopper først efter de der 20 minutter. E: Må vi gerne snakke sammen? L: Man må gerne tale sammen, det er også derfor i sidder sammen 2 og 2 med sidemand. Jeg ved ikke, N – du sidder ikke ved siden af nogen, men det er også (uhørligt) E: (uhørligt) L: Nå, det er godt nok: du sidder der. Det er – K, kan du ikke sætte dig ned ved siden af N, fordi så sidder vi 2 og 2. E: (uhørligt) L: Nej, du sidder også – kan du ikke sætte dig herover? Er der nogle spørgsmål til de her første 15-20 minutter? E: Må man få hjælp af dig? L: Ja, ja selvfølgelig. Jeg kommer rundt.

I have found no difficulty in applying the codes for information on *Organization of lesson*.

Homework, or *homework assignment*, refers to tasks assigned to students by their teachers to be completed mostly outside of class, and derives its name from the fact that most students do the majority of such work at home.
The description below decides when to start and finish coding for *Homework*:

Parent code	***Child codes***	***Description***
Homework	Collecting or returning homework	Teacher collects or returns seen homework, tests etc. to students
	Correction of homework on class	Students are involved in whole-class correction of homework
	New homework	Teacher assigns new homework to class or individual students

6.3.4 Collecting or returning homework

An example: Lesson 23
Small school in Capital region
Female teacher, seniority 5-9 years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:35 0:00-0:35	6:02 0:35-6:37	26:38 6:37-10:33, 12:33-35:15	17:15 10:33-12:33, 35:15-50:30	0:16 50:30-50:36	50:36

(3:35-6:37):

T: This was a recap of past lessons (pointing to the agenda on the blackboard). Now I would like your homework. And one from each group, the one with the first (inaudible). S: Do you also want the task or just the paper. T: I do HAVE the task, so I don’t. Just what you've done. S1: Ok (because of rattle with chairs and paper, the rest is inaudible). S2: K, mine is slightly curly, because I put it	L: Det var opsamling på den sidste tid (peger på dagsordenen på tavlen). Så vil jeg gerne have jeres blækregning. Og en fra hver gruppe, den med det første (uhørligt). E: Skal du også have opgaven eller kun papiret. L: Jeg HAR opgaven, så det skal jeg ikke. Bare det, som du har lavet. E1: Ok. (der er nu lidt skramlen med stole og papir, så noget er uhørligt …) E2: K, min er lidt krøllet, for jeg kom til at

in the folder. S3: Well, I did not do the last one. (T receives papers from all students) ... S4: Will we get a grade? T: You will ... (the teacher counts 13 sheets). T: Well, how many are we today? 1, 2, 3 ... 13, so everyone has returned. Beautiful.	lægge den i mappen. E3: Jeg har altså ikke lavet den sidste. (L modtager papirer fra alle elever) … E4: Får vi karakter for den? L: Det får I. … (læreren tæller 13 ark op). L: Nå, hvor mange er vi i dag? 1, 2, 3, ...13, så alle har afleveret. Smukt.

I have found no difficulty in applying the codes for *Collecting or returning homework.*

6.3.5 Correction of homework

An example: Lesson 25 Medium school in North region Female teacher, seniority 15+ years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:45 0:00-2:45		6:50 2:45-9:35	36:35 9:35-46:10	1:00 46:10-47:10	47:10

(4:00-6:12):

T: You have two tasks for today. One is about rectangles, the other is about squares. What do you know about squares? H? S1: All four angles are equal. T: Yes. Do you know more? S2: The sides have equal size. T: Yes. Do you now know something new: What you found out by doing the task? All? S3: All squares are similar. T: All squares are similar yes ... They all have corresponding angles equal. They're just in different sizes. T: What did you discover about the rectangles? Have you made the assignments N?	L: I har to opgaver for til i dag. Den ene handler om rektangler, den anden handler om kvadrater. Hvad ved I om kvadrater? H? E1: At alle fire vinkler er lige store. L: Yes. Hvad ved I mere? E2: At siderne er lige store. L: Ja. Ved I så noget nyt: Det I fandt ud af ved at lave den opgave? Alle? E3: Alle kvadrater er ligedannede L: Alle kvadrater er ligedannede, ja. … De har jo alle sammen ensliggende vinkler lige store. De er bare i forskellig størrelse. L: Hvad fandt I så ud af med rektanglerne? Har du lavet opgaverne N? E: Hvad er det for en opgave?

S: Which problem is it? T: It is problem 6 and 7. S: Yes, I have. T: That's good! Now you can help responding. B? S: (inaudible). T: Give me an example of two rectangles NOT similar. Give me the length and the width ... S: Width 1 and length 3. T: I simply did not hear that? S: Width 1 and length 3. T: And the other? (A student arrives in the class) S: Width 2 and length 5. T: And why are they not similar? S: 5 and 3 do not have the same scale as 1 and 2. T: As 1 and 2, yes. Can you create a third rectangle that is similar to one of those we have here? Or is someone else able to? D? S: 1 and 3, and 2 and 6 T: Yes (drawing the third rectangle). What is the scale there? J? S: 1: 2	L: Det er opgave 6 og 7. E: Ja, det har jeg. L: Det er godt! Så kan du også være med til at svare. B? E: (uhørligt). L: Giv mig et eksempel på to rektangler, der IKKE er ligedannede. Du giver mig længde og bredde ... E: Bredde 1 og længde 3. L: Det hørte jeg simpelthen ikke? E: Bredde 1 og længde 3. L: Og den anden? (en elev ankommer her til klassen) E: Bredde 2 og længde 5. L: Og hvorfor er de ikke ligedannede? E: 5 og 3 har ikke samme målestoksforhold som 1 og 2. L: Som 1 og 2, ja. Kan du så lave et tredje rektangel, som er ligedannet med ét af dem, vi har her? Eller er der en anden, der kan? D? E: 1 og 3 og 2 og 6. L: Yes (tegner det tredje rektangel). Hvad er målestoksforholdet der? J? E: 1: 2.

I have found no difficulty in applying the codes for *Correcting of homework*. In some cases this dialogue is also coded to e.g. *Procedural point*, *Result point* or *Elicitation*. The excerpt above is also coded as *Elicitation* as the teacher invites points by asking students for their new findings in a focused and oriented way.

6.3.6 New homework

An example: Lesson 26 Small school in North region Male teacher, seniority 15+ years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
3:29 0:00-3:29	2:07 3:29-5:36		43:32 5:36-49:08		49:08D

(5:36-7:40):

T: And then, as I promised. One of you asked for that. It is this homework. You may sit and work with it now. I'll be there to ask, etc. The sheets are stapled and there should be one answer sheet. And the answer sheet, it looks like this. And this is important to remember: It must be returned - together with what you write. On A4 sheets. S1: How much of it must be done? T: What do you say? S1: How much of it should we do? T: Everything. Work on doing everything. S2: You know there's an incredible amount of work, don't you?? T: Yes. That's why you get to work on it today. S3: When should it be returned? S4: Still the 6th (date April 6th). T: That's it. And who did not receive it? (T hands out answer sheets to more students). Yes. So … E5: I do not have a booklet for answers! T: Well, that's a good answer! But, uh - you may of course work on any A4. S4: But I don't get it! T: No, it was not a good answer, but a good comment, I mean. But now you will start. And this is what the lesson is all about. And if there is room out here, some may – now don't get out all of you. S: I want to be out there! T: But do stay here in the library. Don't go any further!	L: Så var det, som jeg havde lovet. Og det var der en, der spurgte til. Det er det her hjemmeregning. Det kan I sidde og arbejde med nu. Jeg skal nok være der til at kunne spørge om, osv. Og der er et hæftet sammen med opgaverne, og så skal der være ét svarark. Og svararket, det ser sådan her ud. Og det er altså vigtigt at huske: Det skal afleveres, sammen med det, I skriver. På A4 ark. E1: Hvor meget af det er det, der skal være lavet? L: Hvad siger du? E1: Hvor meget af det skal vi lave? L: Det hele. Arbejd på at lave det hele. E2: Du ved godt, der er helt vildt meget ikke også? L: Jo. Det er også derfor, I får lov til at arbejde med det i dag. E3: Hvornår er det til? E4: Den 6. stadigvæk (datoen 6/4) L: Sådan. Og hvem er det, der ikke har fået det? (L deler svarark ud til flere elever). Yes. Så ... E5: Jeg har ikke et afleveringshæfte! L: Nej, det er et godt svar! Men, øh - men du kan jo arbejde på noget A4. E4: Men jeg fatter det ikke! L: Nej, det var ikke et godt svar, men en god kommentar mener jeg. Men nu går I i gang med det her. Og det er det, den time her skal gå på. Og hvis der er ledigt herude, så kan nogen - I skal ikke smutte derud alle sammen. E: Jeg skal derud! L: Men bliv her i biblioteket. I går ikke længere!

I have found no difficulty in applying the codes for assigning: *New homework*. But this excerpt has also been coded as: *Organization of lesson*, as this information on amount, deadline and answer sheets is interwoven with the homework assignment and because the students may work on the assignment in this lesson.

Two kinds of *Instants* in the lessons are seen as especially point-addressing activities and therefore specifically coded:

Parent code	***Child codes***	***Description***
Instants addressing points	Overview (repetition)	Teacher refers or repeated already taught mathematical content, concepts, rules, techniques
	Summing up	Teacher revises or sums up the lesson so far

6.3.7 Overview

An example: Lesson 34
Small school in Central region
Male teacher, seniority 15+ years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:44 0:00-0:44		3:56 0:44-4:40	44:49 4:40-49:29		49:29

(2:35 – 3:28):

T: There you have of course seen before that we have - when we do statistics - that we go in and do it here with the dice, right. And if we make it up here it's smart to let it run itself (L shows at the Smartboard a "dynamic" dice you just touch). No one expects you to throw 100 times! You can do with 10 throws and then do some statistics on that. And then try to see if you can use those words. S: R, we should not make those sheets (inaudible)? T: Yes, but as I also told you lately - now I don't know whether you were here last lesson, the day we started? S: I was. T: Because then I also told that I am going to start up here every time, 10 minutes to	L: Der har I jo set før, at vi har - når vi laver statistik - at vi går ind og laver det her med terningen ikke også. Og hvis vi laver den her oppe er det jo smart at lade den køre selv (L viser på Smartboard en "dynamisk" terning, der blot trykkes på). Der er ingen, der siger I skal bruge 100 slag! I kan lave 10 slag af dem, og se lave lidt statistik over det. Og så prøve at se, om I kan finde ud af de der ord. E: R, skal vi ikke lave de der ark (uhørligt)? L: Jo, men jeg fortalte jer også sidst - nu ved jeg ikke om du var her sidste gang, den dag, vi startede? E: Det var jeg. L: For da fortalte jeg også, at jeg starter heroppe hver gang, 10 minutter, med at gennemgå nogle af de ting, det handler om

review some of the things it's about [But]. Yes, but that's what I say: You MAY do those sheets, but you do not slavishly have to do them, like doing No. 1, No. 2, No. 3 S: Slavishly? T: You may, if e.g. you with task 3, for example you, say we don't understand something then you may like to make something more of it. It could be perspective drawing or statistics or anything like that.	[Men]. Jo, men det er det, jeg siger: I må GERNE lave de der ark, men I behøver ikke slavisk at lave, det betyder lave nr. 1, nr. 2, nr. 3. E: Slavisk? L: I må gerne, hvis I fx i opgave 3 siger det her forstår vi ikke, så må I gerne lave noget mere af det. Det kunne fx være perspektivtegning eller statistik eller noget i den retning.

The teacher reminds the class of a previously taught technique to simulate the roll of a dice on the computer. The information in the excerpt is also coded as *Announcement of lesson content*. And immediately before and after the teacher was informing students on *Organization of lesson*.

6.3.8 Summing up

An example: Lesson 37 Medium, school in Central region Male teacher, seniority 15+ years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
1:30 0:00-1:30	14:53 1:30-16:23	9:47 16:23-21:40, 32:10-36:40	19:10 21:40-32:10, 36:40-45:20	0:48 45:20- 46:08	46:08D

(45:20-46:08):

T: Quite a lot of you have found out – we sum up eventually, then we will have a break – J do follow – a lot of you have found out that it really went OK with 3, 4, 5 (indicates side lengths in right triangles). It also went really well with 6, 8, 10 – but when we had to use the 30 (matchsticks) then it went wrong ... We will look to that in the next lesson, so leave them on the table (first part of double lesson ends here).	L: Rigtig mange af jer har fundet ud af – vi samler lige op her til sidst, så holder vi frikvarter - rigtig mange af jer – J følg lige med – rigtig mange af jer har fundet ud af, det gik rigtig fint med 3,4,5 indikerer sidelængder i retvinklede trekanter). Det gik rigtig fint med 6,8,10 – men når vi skulle til at bruge de 30 (tændstikker) så gik det galt ... Det ser vi på i næste time, så lad dem ligge på bordet (første del af dobbelttime slutter her).

This is the first half of a double-lesson. The teacher is *Summing up* the activity and findings up until the break. After this excerpt and the break then the class will continue this investigation of possible side lengths in right angle triangles using matchsticks.

6.4 Teacher to class points

Points as important mathematical ideas may be articulated by the teacher or by the students. Both parties may even be stating points in the same conversation. And points may be articulated by the teacher or a student in whole class teaching or in guidance of groups or individual students. I use the *same* description for each type of point, independent of the sender or the organization.

Below examples are given of the four different types of *teacher to class* points. In section 6.6 similar examples will be given on student points.

Parent code	***Child codes***	***Description***
Teacher to class	Conceptual point	Teacher presents and comments on a mathematical concept using definition, symbol or application
	Procedural point	Teacher presents and comments on a rule or a method in an application or example
	Result point	Teacher develops or presents and comments on a mathematical result such as a formula, theorem or procedure
	Interpretation point	Teacher interprets a model or a result or compares representations

It has not been difficult to decide when to start and finish coding for *Teacher to class* points. But as mathematical communication in grade 8 teaching often was a mixture of concepts, procedures, results and interpretation the following examples are chosen to illustrate both the very clear situations not leaving the coder in doubt – and communication, where coding choices may be challenged. I have sought to explain my reasons for decisions, when the rather wide code descriptions seem open to more interpretations.

To each of the following examples a group of data has been made from these sources:

- Coding of “teacher to class” points in % of total lesson length. When more codes are applied to the same excerpts the total is adjusted to indicate the proportion of the total lesson length with expressed teacher points.
- Teacher’s short indication of lesson goal from questionnaire.
- And finally the researcher’s memo immediately after visit to school.

6.4.1 Teacher conceptual point

An example: Lesson 35F Large school in Central region Female teacher, seniority 5-9 years.

CODES (in % of lesson)	Teacher → class				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Equation of line	1.9	15.0		8.6	20.4

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
4:00 0:00-4:00	5:00 4:00-6:00, 22:00-25:00	16:00 6:00-22:00	24.21 25:00-49:21		49:21

(19:35-20:32):

T: M? S: This calculation, is this, what you call the calculation of a straight line or what? Or has it got a special name? T: So this one, as I always write it: f (x) = ax + b (the teacher writes this on the blackboard) this is the FORMULA for a straight line. It's like having a formula for – what did we use last week as an example? The formula for a circle, or whatever (showing perimeter with her hands). If we have the diameter times pi. That formula cannot be used for anything until we know what the diameter is. With this formula here (pointing again at f (x) = ax + b), we cannot draw a straight line from that before we know about a and b. We need to have the values of a and b to draw the line.	L: M? E: Den der udregning der, kalder man den udregning af en ret linje eller hvad? Eller har den et specielt navn? L: Altså den her, som jeg skriver i tide og utide, den her f(x) = ax + b (skriver det på tavlen), det er FORMLEN for en ret linje. Det er ligesom hvis du har formlen for, hvad brugte vi i sidste uge som eksempel? Formlen for en cirkel, eller hvad (viser omkredsen med hænderne). Hvis vi har diameteren gange pi. Den der formel, den kan vi ikke bruge til noget før vi ved, hvad diameteren den er. Den her formel (peger igen på f(x) = ax + b), den kan vi heller ikke tegne en ret linje ud fra, før vi ved hvad a og b er. Vi er nødt til at have a og b-værdier for at kunne tegne den rette linje.

QUESTIONNAIRE (of teacher)

Goal of previous lesson: To continue the work on slope number.
Goal of lesson: That b in y = ax + b is the intersection point between graph and y-axis.
Goal of next lesson: To work on certain functions on teacher's handout.

MEMO (by researcher)

The topic of the course (which includes several lessons) is graphs. T uses own "overhead" in the review, and she has some slightly "superficial" procedural points in her review of linear functions and their graphical image.
It is only marginally explained how a and b are particular numbers to the graph. T does not have mathematics as a main subject and less than 7 years of service (- 2 x maternity).
Students ask quite a lot, and T expresses in the questionnaire, that students are "very good to come and ask if they do it right."

This explanation of a formula is clearly a presentation and a comment on a mathematical concept by symbol or application. The teacher makes a *conceptual point*.
Before and after this *conceptual point* the teacher articulates *procedural points* on graphing of the function y = ax + b. The teacher also interprets the formula in response to student's questions. Therefore this excerpt is also coded as an *interpretation point*.

An example of a less "pure" conceptual point:

An example: Lesson 46
Large school in Zealand region
Male teacher, seniority information missing.

CODES (in % of lesson)	Teacher → class				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Total
Equation of line	7.2	5.8		2.3	13.0

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
4:15 0:00-4:15		33:32 4:15-37:47	13:13 37:47-51:00		51:00D

(23:47-26:18):

T: So I could well imagine that we try to look at a cumulated frequency of it. ... Cumulated frequency from 7 to 9? (one student has trouble reading the small numbers from a traffic count on the Smartboard). T: It says 421. You should move closer. 421. S: Plus 198. T: What about the next: 9 to 11? And now we are talking about cumulated frequency. Yes? S: 616, no 19. T: 619, it is not there? S: No, if you add with one of them. T: So you added. You've said cumulated frequency is both those driving from 7 to 9 and those driving from? S: 9 to 11. T: Exactly. And you get the number? S: 619. T: (inaudible) P, this was absolutely correct. Is anyone able to assist me with the next number here? [Yes]. M? S: 835. T: Now you just suggest a number. But I have not figured out how you arrive at such a number? S: I used my calculator, and added the two numbers together: 216 to 619. T. Yes, exactly. Ok. S: Yes and the next is 1079. L: Should we let M run the rest of the numbers? [Yes]. What did you say? S: 1079 [And?] And 1537. Am I clever or not? T: You are very clever M, I know that. And this is what we've learned so far.	L: Så jeg kunne godt tænke mig, at vi prøve at kigge på en opsummeret hyppighed af det. ... Opsummeret hyppighed fra 7 til 9? (en elev har besvær med at læse de små tal fra en trafiktælling på Smartboardet). L: Der står 421. Så skal du sætte dig nærmere. 421. E: Plus 198. L: Hvad med den næste, fra 9 til 11? Og nu taler vi om opsummeret hyppighed. Yes? E: 616, nej 19. L: 619, det står der da ikke nogen steder? Elev: Nej, det er hvis man plusser med en af dem. L: Så du har plusset. Du har sagt opsummeret hyppighed er både dem, der kører fra 7 til 9 og dem, der kører fra? E: 9 til 11. L: Lige præcis. Og du får et tal, der hedder? E: 619. L: (Uhørligt) P, det var fuldstændig rigtigt. Er der nogen, der kan hjælpe mig med det næste tal her? [Ja]. M? E: 835. L: Nu siger du bare sådan et tal. Men jeg har ikke regnet ud, hvordan kommer du til sådan et tal? E: Jeg har brugt min lommeregner, og så plusser jeg de to tal sammen: 216 med 619. L. Ja, lige præcis. Ok. E: Ja og den næste, det er 1079. L: Skal vi lade M køre resten af tallene? [Ja]. Hvad sagde du? E: 1079 [Og?] og 1537. Er jeg ikke bare klog. L: Du er meget klog M, det ved jeg godt. Og det er det, vi har lært indtil videre.

In the class presentation the teacher elicits the students' understanding of cumulated frequency in an application on the Smartboard. The frequency concept is further developed by exemplification in the following transcript from the same lesson.

26:18-27:48):

T: Now there is some mysterious word (pointing to the table shown at the Smartboard), which is called *frequency*. Did any of you encounter the word frequency before? No one? Does anyone know it from physics? S: In the radio. T: Are any of you playing music? ... There is something with oscillations, which we refer to as frequency. How deep (lowers hand) and how high (raises hand) a note is. A high note, are we then in a high frequency or a low frequency? If you have not heard about it you may just guess! Very high notes have a very high frequency! That is, there are many, many oscillations (now waving a vertical palm from side to side) within 1 second. This is physics – we'll return to that. I do have some of you in physics as well (information probably in honor of the video recording, AM).	L: Nu står der et eller andet mystisk ord (peger på skemaet), som hedder *frekvens*. Er der nogen af jer, der er stødt på ordet frekvens før? Det er der ikke? Er der nogen af jer, der har hørt det i fysik? E: I radioen. L: Er der nogen af jer, der spiller musik? … der er noget med nogle svingninger, som man omtaler som frekvens. Hvor dyb (sænker hånden) og hvor høj (hæver hånden) en tone er. En høj tone, er vi så i en høj frekvens eller en lav frekvens? Hvis I ikke har hørt om det, så kan I bare gætte! Meget høje toner er en meget høj frekvens! Dvs. der er mange, mange svingninger (vifter nu en lodret håndflade fra side til side) i løbet af 1 sekund. Det er så fysik - det vender vi tilbage til. Jeg har nogle af jer i fysik også (vist mest til ære for videooptagelsen, AM).

This latter excerpt is also coded as an *Interpretation point*.

QUESTIONNAIRE (of teacher)

This was unfortunately not received.

MEMO (by researcher)

We were in the block of secondary school classes – i.e. there are four grade 8 classes = 80 students, but they are not broken down into the standard classes (8.a is the weakest group of 22). T also teaches physics / chemistry and sports to this grade. The school has Smartboards everywhere, as there are no old-fashioned blackboards. The rooms are very small, but there is a room for group work everywhere and in specific areas, wireless network and the school also has many portable notebook computers available to students. The class was working from 37:47 on two assignments (368 and 370) from copied material. T gets his ideas from various sources, and often also uses small tests. I am following T on the whole second floor of the school as the students spread around for working outside the tiny classroom. There are only a few discussions with students, as most seem to be able to handle the tasks on their own.

This is a teacher's introduction of the concept of cumulated frequency. The excerpt is coded as a *Conceptual point*. The concept cumulative frequency is only prepared though, and not properly defined in the excerpt. On the other hand students are proposing the procedure followed in calculations. The excerpt is therefore also coded as a *Student procedural point*.

The two excerpts from lesson 46 show how difficult coding suddenly becomes. The students are not very active on their own, but reacting on the elicitation of the teacher dominating (and coded) the next 10 minutes.
The shift in context from an actual daily traffic count of cars during passing the school to frequency in sound and physics is marking a shift from concept formation to concept and terminology interpretation. I made a choice, which could have been done differently. Therefore this lesson became the one with the second longest period (in total 4:01) coded as conceptual point.

6.4.2 Teacher procedural point

An example: Lesson 7 Medium school in South region Male teacher, seniority 15+ years.

CODES (in % of lesson)	Teacher → class				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Fractions	10.9	$\boxed{16.2}$		14.7	30.9

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:10 0:00-2:10		22:10 2:10-24:24	19: 38 24:24-44:00		44:00

(17:15-19:35+19:35-24:24):

T: I think I'll give an example on percentage calculations before you have to work. I have a class here called 8a and another class called 8b (writes): 8a 8b ♂ 10 4 ♀ 15 12 T: Do you grasp what I've written on the blackboard? [Yes]. Now the intelligent	L: Jeg tror lige, jeg vil give et eksempel med procentregning som det sidste inden I skal arbejde. Jeg har en klasse her, der hedder 8a og en klasse, der hedder 8b (skriver): 8a 8b ♂ 10 4 ♀ 15 12 L: Er I med på hvad jeg har skrevet på tavlen? [Ja]. Nu kommer det intelligente

question: In which class are there more girls? ... S1: 8.a T: There is still a raised hand? I asked which class has more girls? S2 (the very active A): If you consider how many there are in the classes, counting the percentage, it's b. There are 5 more there, and 8 in the second. … T: I have got two answers, and I may NOT settle with myself, which of you I find correct. S2: It depends on the (inaudible). Do you calculate the percentage then ... T: There are a few then who have seen the light. So now I write up something, which usually is not in mathematics books: Percentage has to do with being relative. S: What does it mean relative? T: It means in relation to. ... It depends on. ... But if we compare with how many are in the class we can try to figure it out in %. Percent has something to do with fractions, I hope you agree on that? [Yes]. So I start with a fraction line, later to become a decimal number that will ultimately turn into a percentage. Watch out now: what number do I put on that fraction line? (The relative proportion of girls in the two classes is now calculated in class conversation) S: 15/25 ... 60%. T: Now I shall keep quiet, and in a while you will tell me the % there are girls in the b-class ...	spørgsmål: I hvilken klasse er der flest piger? … E1: 8.a L: Der er stadigvæk en hånd oppe? Jeg spurgte, hvilken klasse er der flest piger i? E2 (den meget aktive A): Hvis man tager efter hvor mange der er i klasserne, ja regner procentdelen ud, er det jo b. Der er jo kun 5 mere der og der er 8 i den anden. … L: Jeg har altså fået to svar, og jeg kan altså IKKE afgøre med mig selv, hvem af jer jeg skal give ret. E2: Det kommer jo an på (uhørligt). Regner du procenten ud så ... L: Det er der altså nogle, der har gennemskuet. Så nu skriver jeg noget op, der normalt ikke står i matematikbøger: Procent det har noget at gøre med at det er relativ. E: Hvad betyder relativ? L: Det betyder i forhold til … det kommer jo an på … Men hvis vi sammenligner det med, hvor mange der er i klassen, så kan vi prøve at regne det ud i %. Procent, det har noget med brøk at gøre, kan vi ikke blive enige om det? [Jo]. Så jeg starter med en brøkstreg, som senere skal blive til en decimalbrøk, som senere skal blive til et procenttal. Pas på nu: hvilket tal kan jeg sætte på den brøkstreg? (Den relative andel af piger i de to klasser beregnes nu i klassesamtalen) E: 15/25 … 60%. L: Nu tier jeg stille lidt, og så om lidt så svarer I på, hvor mange % der er piger i b-klassen …

QUESTIONNAIRE (of teacher)

Goal of previous lesson:	Probability
Goal of this lesson:	Fraction - decimal numbers – percent – permille – position system – relative
Goal of next lesson:	Percentage increase, percentage decrease

MEMO (by researcher) Look in section 5.7.3 for the original memo-format.

The school has worked with learning styles for 3-4 years. Colleagues learn to be very conscious about the qualities of this focus. One very weak student gets special tasks and some very able work "ahead" - but with the same textbooks. When they want to, they may instead do something else (even for another subject)! The student atmosphere is really good. It is the teacher's conscious choice that the students organize themselves into small groups. I.e. a large group of girls was in the common area working together around a large table with their own music from the "ghetto blaster". Some sat on chairs and some on the window sills. The class had homework, but this was never corrected! The teacher uses www.evalueringssystem.dk

The teacher presents and comments on a method to decide which class has more girls, and the excerpt is therefore coded as a *procedural* point.
In the latter part of the excerpt the "more" is discussed by the teacher as an absolute or relative concept. This is therefore also coded as both a *conceptual* point and as *elicitation*. The procedure is still developed by the teacher in a dialogue initiated by the teacher while eliciting contributions from the students.

Another example: Lesson 18 Medium school in North region Male teacher, seniority 15+ years.

CODES (in % of lesson)	Teacher → class				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Pythagoras		8.1			8.1

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
		5:35 0:00-5:35	28:18 5:35-33:53	0:57 33:53-34:50	34:50

(0:43-3.32):

T: If I imagine I need to chalk up a football field. I've mentioned this to you before. And so I want to have these angles out here to be correct (indicating the 4 corners in a drawing on the board). It is an advantage especially when someone is going to play a game. So I could use Pythagoras. I could calculate	L: Hvis nu jeg forestiller mig, jeg skal ud og kridte en fodboldbane op. Det har jeg såmænd nævnt før for jer. Og så vil jeg gerne have de her vinkler herude til at være rette (indikerer de 4 hjørner på en tegning på tavlen). Det er en fordel, især hvis der er nogle, der skal spille kamp. Så kunne jeg jo

how far it should be across (draws a diagonal) using the formula. I could put in 50 and 100. Then I can figure out that this one, it must be 111 meters. And I can use that you see. If I now take 3 pieces of string. Put a stick in here, and then I lay out 100 meters of string between these. Then a string of 50 m in here, and a string of 111 m down here. Where the 2 strings meet together is exactly where the angle is right. I can use the strings again you see. I can move this over here ... (T shows how to sustain the other corners the same way). T: I can assure you that it is NOT at all easy to make these right angles. Some of you might be thinking: Well, yes – I'll just take this (grabs a big triangle from the Board kit) and then stand and aim at the farther end of the field. I then make a hit 100 meters up in the other end and then ... it might be done. But you don't need many degrees of error in a field of 100 m to create a big difference at the other end. If nothing else, the opponents – the ones you play against who have lost – they will find if there is an error! S: But don't you use a laser thing today? T: But you could do that. Yes, you could easily do that. One could easily position a laser thing. But not many football clubs have a laser to use for such things. S: Are they expensive? T: ... if you buy them expensive enough you can get a laser that can send out beams perpendicular to each other ...	bruge Pythagoras. Jeg kunne jo regne ud, hvor langt der skal være her tværs over (tegner en diagonal) ved hjælp af formlen. Jeg kunne sætte 50 ind, 100 ind. Så kan jeg regne ud, at den her ovre, den skal være 111 meter. Og det kan jeg jo bruge. Hvis jeg nu laver mig 3 stykker snor. Sætter en pind i hernede, og så lægger jeg 100 m snor herudaf. Så sætter jeg en snor i her på 50 m, og en snor i hernede på 111. Der hvor de 2 når sammen, det er lige nøjagtig der, hvor den her vinkel bliver ret. De snore kan jeg jo bruge igen jo. Jeg kan jo flytte den her over ... (L viser at man kan sikre de andre hjørner på samme vis). L: Jeg skal altså lige hilse og sige, at det er IKKE nemt at lave de her vinkler rette. Heller ikke selv om der er nogle der tænker: Nå, ja – jeg tager da bare den her (tager en stor tegnetrekant fra Tavlesættet) og så står jeg og sigter derudad. Så rammer jeg den der 100 m oppe i den anden ende og så … Det kan godt lade sig gøre. Men der skal altså ikke ret mange graders fejl til på sådan en fodboldbane på 100 m før der er stor forskel i den anden ende. Om ikke andet, så kan modstanderne, ikke også –dem, der har tabt, de kan i hvert fald finde ud af, at der var fejl i! E: Men bruger man ikke en laserting i dag? L: Men det kunne man godt gøre. Ja, det kunne man kunne sagtens. Man kunne sagtens stille en laser-ting op. Men der er ikke ret mange fodboldklubber, der har en laser lige til at bruge til den slags. E: Er de dyre? L: ... hvis man køber dem dyre nok, så kan man få en laser, der kan sende stråler ud vinkelret på hinanden ...

QUESTIONNAIRE (of teacher)

Goal of previous lesson: Introduction of Pythagoras.
Goal of this lesson: Work with Pythagoras, exercises.
Goal of next lesson: Complete work on Pythagoras. Possibly construct outdoor triangles.

MEMO (by researcher)

The deputy school principal kindly met me. The agreement was reconfirmed in the morning (at my request), as there was no reaction to my suggestion to postpone the session by a week, since the school had trouble with the first date of March 2. due to a "bridging" arrangement. He is a mathematics teacher himself and very interested in the project (and results). He has previously done developmental work with a wellknown Danish mathematics didactician and a colleague from the Mathematics Teachers Association. He was happy with my little "black" book about gifted students.
No visible socio-economic or 2-language challenge. The student atmosphere was good! Two girls strolled around – but were apparently working. The teacher has a "natural authority", but according to himself, he has also insisted hard! He has only had the class for ¾ year. It is composed of 26 students from several smaller schools in the municipality!
The school is the largest in the municipality, built in several stages – a bit confusing. They are waiting to begin a building and renovation project costing 50 million DKK. They have a large forge in the schoolyard, and their cooperation with a technical school is well known by others (e.g. my wife). A Danish professor is apparently also carrying out an evaluation project at the school (according to her – I did not hear about it). In the schoolyard there were several large concrete objects (cube, pyramid and a truncated cone).
The teaching was just starting as I came to the classroom. There is no break between sessions, and the deputy principal had informed me, that I would be picked up in the staff room (this did not happen). But I turned on the camera as soon as I was in class. Total recording is approximately 35 minutes only, as the lesson also stopped 5 minutes early. Students simply packed away (perhaps being used to it) here in the late seventh lesson. T had 3-4 returns of homework during grade 8, so this is *not* a routine.

This teacher articulates a procedural point as he presents a method to allocate right angles on a football field by using the Pythagorean Theorem. He refers to this example as used before (by him) in this class. The enthusiastic explanation referring to own experience seems to engage students, who pose questions on accessibility and price to lasers and not to the long explanation.

6.4.3 Teacher result point

An example: Lesson 21
Small school in Central region
Female teacher, seniority 15+ years.

CODES (in % of lesson)	Teacher → class				Adjusted total
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	
Powers		11.7	5.1	4.3	19.2

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:10 0:00-2:10	5:50 2:10-8:00	19:50 8:00-27:50	17:38 27:50-45:28		45:28

(2:10-3:08 + 3:08-3:18 + 3:18-4:30):

T: And then you must start to correct what you had to do in the book on page 97 And it is powers, we are working on. And before we correct, we will just repeat (writes $a^2 \cdot a^3$ on the blackboard). S: Do we write this down? T: No! How is it we multiply two such powers with one another, M? S1: Hmm, add them. T: Yes, what do I add? (T shows a + between the exponents). S1: Yes the powers. S2: The exponents. T: We say 3 + 2, and this equals a^5, yes (this is written also on the board). If instead I had to divide the two powers? (T writes $a^3 : a^2$ on the blackboard). What about that, F? S: Then you must say 3 minus 2. T: Yes. … T: If I have a power that looked like this, a^0, N? S: It is equal to 1. T: It is 1, a^0 is 1. … T: And then you – if I now have one with a negative exponent, how can I change that? S: I do not grasp that one! T: No, it's also why I ask. J? S: I wonder if it is 0.0 something? T: Yes, but it is not so easy when it says a. You could do that, if I had written 10 to minus 3, then you were right. If I had written 10 to minus 3, what is the result?	L: Og så skal I starte med at I retter det, vi har haft for i bogen på side 97. Og det er potens, vi er i gang med. Og inden vi retter, så skal vi lige repetere (skriver $a^2 \cdot a^3$ på tavlen). E: Skal vi skrive det der ned? L: Nej! Hvordan er det vi ganger sådan to potenser med hinanden, M? E1: Øh, plusser dem. L: Ja, hvad plusser jeg? (L viser et + mellem eksponenterne). E1: Ja potenserne. E2: Eksponenterne. L: Vi siger 3 + 2, og det er så a^5, ja (dette skrives også på tavlen). Hvis jeg nu i stedet for havde skullet dividere to potenser? (L skriver $a^3 : a^2$ på tavlen). Hvad så F? E: Så skal man sige 3 minus 2. L: Ja. … L: Hvis jeg nu har en potens, der så sådan her ud, a^0, N? E: Det er lig med 1. L: Det er 1, a^0 er 1. … L: Og så skal man - hvis jeg nu har én med en negativ eksponent, hvordan kan jeg så skrive det om? E: Jeg kan ikke finde ud af den! L: Nej, det er også derfor jeg spørger. J? E: Jeg tænker på, om det er 0,0 et eller andet? L: Ja, men den er nu ikke så nemt, når der står a. Det kunne du gøre, hvis jeg havde skrevet 10 i minus 3, så havde du ret. Hvis jeg havde skrevet 10 i minus 3, hvad giver det så?

S: Then it's 0.001. T: Yes. But a^{-3}, that's not so easy, I suppose. Do you have a suggestion? S: If you have a number, you can put it as a fraction? T: Yes, you change it into a fraction, and then what? S: Then it is it like 3 and then a fraction line, and then a in the denominator. T: So it will be positive, when it moves down in the denominator. And if I had a number, one would just have to calculate that. Those are the few rules that we have had until now and which we will need – and as you hopefully have used.	E: Det er så 0,001. L: Ja. Men a^{-3}, det er jo ikke lige så nemt, vel. Har du et forslag? E: Hvis du har et tal, så kan man sætte det som brøk? L: Ja, man laver det om til en brøk, og hvad så? E: Så er det sådan 3 og så en brøkstreg og så a i nævneren. L: Så bliver det positivt, når det rykker ned i nævneren. Og hvis jeg havde et tal, så skulle man bare til at regne dét ud. Så det er lige de der få regler, som vi indtil nu har haft, og som vi skal bruge – og som I forhåbentlig har brugt.

QUESTIONNAIRE (of teacher)

Goal of previous lesson:	Calculation rules to powers and the graph of second degree equations.
Goal of lesson:	Calculation rules to powers and graph of hyperbola.
Goal of next lesson:	Calculation rules for powers

MEMO (by researcher)

Assignments every 3 week (to be presented to teacher in nice writing). They are corrected, of course! T told of a second class, where she brought the notebooks home once a week (so students have two books) – all to be corrected. T: "How else would you do it?" It is obviously a well prepared teacher with the classical "virtues" T has mathematics as one of her main subjects and 40 years experience. The course (several sessions) is on powers and the goal in this particular lesson is that students learn power rules and graphing of the hyperbola. Rules for calculation with powers are also the goal of the previous and the next lesson.

This *Result point* is partly coded (the first excerpt only) as a *Procedural point* and a *Missed point* (the two next excerpts). The latter may seem odd: A point and then not a point!

But the missed point is a conceptual one, as the teacher misses an opportunity to articulate why a^0 being 1 and a^{-3} being 1/ a^3 are reasonable. The observer is left in doubt whether the students have been taught multiplication and division rules to powers by generalizing from number examples and now have forgotten the rationale behind – or if the case is a set of rules to be learned by rote only.

Another example: Lesson 28 Large school in Capital region Male teacher, seniority 5-9 years.

CODES (in % of lesson)	Teacher → class				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Powers		5.6	4.7		10.3

There are more points from the teacher in this lesson.
Besides the teacher, students offer points in 9.5% of this lesson.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:52 0:00-0:52		16:58 0:52-15:20, 32:55-35:25	21:08 15:20-32:55, 40:05-43:38	4:40 Break + 35:25-40:05	43:38

(9:00-10:41 + 10:41-11:01):

T: What if it is – now let me see again, what was I thinking? S: J, what does it mean, "exact"? T: Well it just means that we arrive at the exact number. Let's consider an example like this one? (T writes 10^3: 10^3) If we apply our rule from before: What should one do, if you had to divide the two powers with each other? … What did we do up here? [Minus] Yes, it was minus. Now we have here, what does it say? [10^3] Yes, 10^3. So if we use that, we should say 3 minus 3, what is 3 minus 3? [0] T: Yes. 10^0 it says. But like we did on Monday, what is the result if we had to find the exact value? What would it give? It is perhaps not quite logically, A? S: Isn't it just 1? T: How did you find out? It is absolutely right.	L: Hvad nu, hvis det er - nu skal jeg lige se engang, hvad var det jeg tænkte på? E: J, hvad betyder "eksakte"? L: Jamen det betyder bare, at vi kommer frem til det præcise tal. Hvis det nu er at vi fx vil have en – det kunne godt være sådan én her? (L skriver 10^3 : 10^3). Hvis vi nu skal bruge vores regel fra før: Hvad var det man skulle gøre, når man skulle dividere to potenser med hinanden? … Hvad var det, der blev gjort heroppe? [Minus] Ja, det blev minus. Nu har vi her, hvad står der heroppe? [10^3] Ja, 10^3. Så hvis vi nu tog her, så skulle vi sige 3 minus 3, hvad giver 3 minus 3? [0] L: Ja. 10^0 står der nu. Men vi havde ikke om i mandags, hvad er resultatet af det, hvad vi skulle finde ud af den eksakte værdi af det? Hvad ville det give? Det er måske ikke helt logisk, A? E: Er det ikke bare 1. L: Hvordan fandt du ud af det? Det er fuldstændig rigtigt.

S: It should be 0, ... T: Well but it's true. If you just look here: How much is this, if we were to calculate 10^3 they are of course up here. What is it calculated? The precise number, if we were to convert it? [1000] Yes, and down here? [It is also 1000]. Yes. 1000 divided by 1000 [1] So the same number divided by itself [1], it will always end up with 1. … T: So when you have a power, what's it called, where it is raised to 0, then it will always be 1, though it may seem illogical. So it will be equal to 1. What would you say? S: But couldn't we not just say that where it says 10^3 (unclear) then you know there are 3 zeros, and then you (unclear)? T: Yes, yes fine.	E: Det skal være 0, ... L: Jamen det er rigtigt nok. Hvis I lige kigger her: Hvor meget står der her, hvis vi skulle lave de 10^3, de står jo heroppe. Hvad er det i tal? Det præcise tal, hvis vi skulle omregne det? [1000] Ja, og hernede? [Det er også 1000]. Ja. 1000 divideret med 1000 [1] altså det samme tal divideret med sig selv [1] det vil altid give 1. …. L: Så når man har en potens, hvad hedder det, hvor, hvad hedder det, den står i 0, så vil det altid være 1, selvom det virker, måske lidt ulogisk umiddelbart. Så vil det altid give 1. Hvad vil du sige? E: Jamen kan man ikke også bare sige, at der hvor der står 10^3 (uklart) så ved man der står 3 nuller, og så kan man (uklart)? L: Jo, jo fint.

QUESTIONNAIRE (of teacher)

Goal of previous lesson:	Powers – why use powers?
Goal of lesson:	Rules to Powers – for use in task of today
Goal of next lesson:	New course

MEMO (by researcher)

T used the electronic board in his review. Some clear points in the review of rules for calculation with powers including a justification for $10^0 = 1$ and $10^{-2} = 0.01$.
Also possible point-containing clips were captured with guidance of individual students.
After 35 minutes there was a 2-minute break, after which T for the final 5 minutes of recording distributed copies with the "calculation hierarchy". Many students seemed to find it difficult to understand, what e.g. was meant by a "part" in an algebraic expression.

This excerpt showing a teacher's *result point* for the whole class is found during correction of homework and is coded as a *student result point* as well.

6.4.4 Teacher interpretation point

An example: Lesson 41
Medium school in Capital region
Female teacher, seniority information missing.

CODES (in % of lesson)	Teacher → class				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Total
Algebra, area				**7.7**	7.7

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
3:23 0:00-3:23	29:37 3:23-21:20, 25:20-37:00	12:02 37:00-49:02	4:00 21:20-25:20		49:02D

(33:13-37:00):

T: There are a few of you who say it is something that is "inside". But area is more – like a plane. So you must be so very (careful?), because what do you mean by "inside"? Because I do not know what you think. But in any case I think of that as box-like: how much can be inside the box? And that's the [cm^3]. Yes .... S1: So I would just say that the area it's like the "shell on the outside" ... S2: Circumference that is what is around. S1: It's a bit hard to explain. T: Yes. A has got something there, it's that using the word as a "plane", i.e. how big it is this plane, it is … (T writes on the blackboard: Area: Size of a plane.) T: When we calculate something, an area, which unit will we get then? Which unit do we use? P? S: It's just the one that has a tiny 2. I do not know quite what one calls it [Yes]. (T writes on the blackboard: Unity e.g. cm^2). S2: Squared. S1: Yes, what do you call it? T: Square centimeters or square meters (writes m^2). S1: Well, yes. S3: km^2 (L writes this and put a circle around the tiny 2-digit).	L: Der er nogle stykker af jer, som siger, at det er noget der er ”inden i ”. Areal, det er jo bare – altså det er jo én flade. Så man skal så være meget (forsigtig?), for hvad mener man med "indeni"? For jeg ved ikke, hvad I tænker. Men jeg tænker i hvert fald sådan ligesom kasse-agtigt: Hvor meget kan der være inden i kassen? Og det er jo det, der er [cm^3]. Ja.... E1: Så jeg ville lidt sige, at arealet det er jo lidt "skallen udenpå"... E2: Omkredsen det er det, der er rundt om. E1: Det er lidt svært at forklare. L: Ja. A har fat i noget, det er det der med at bruge ordet måske en "flade", altså hvor stor den der flade, den er.. (L skriver på tavlen: Areal: Størrelsen på en flade.) L: Når vi regner noget ud, et areal ud, hvilken enhed får vi så? Hvilken enhed bruger vi? P? E: Det er bare den der, hvor der står et lille 2-tal. Jeg ved ikke helt, hvad man lige kalder den [Ja]. (L skriver på tavlen: Enhed Fx cm^2). E2: I anden E1: Ja, hvad kalder man den? L: Kvadratcentimeter eller kvadratmeter (skriver m^2). E1: Nå, ja. E3: km^2 (L skriver dette og sætter en cirkel om det lille 2-tal).

T: In fact we have, you may put it, 2 numbers which we multiply together. As with equations, we have (T writes) $a \cdot a = a^2$. So we have here one can say cm times cm = cm^2 (T writes this on the blackboard). So depending on how many cm it is (unclear), it becomes squared.	L: Vi har egentlig, kan man sige, 2 tal vi ganger med hinanden. Ligesom med ligninger, vi har haft (L skriver) $a \cdot a = a^2$. Så har vi her kan man jo sige cm gange cm = cm^2 (L skriver det på tavlen). Så efter hvor mange cm det er (uklart), så bliver det så i anden.

QUESTIONNAIRE (of teacher)

This is unfortunately not received.

MEMO (by researcher)

The school secretary was waiting for me, and served coffee in the staff room. She brought a greeting from T, who expected me to locate the empty classroom, but I waited for T in the staff room instead. Shortly before the lesson start, I was nevertheless uncertain, and a colleague showed me to the class. Thus the video begins with the lesson immediately begun. Students were informed. And T got the "package" with the questionnaire - and this time a copy of my textbook to grade 7 as thanks for the help. I noticed no difference due to socio-economics or 2-languages. The student atmosphere is good, students were really sweet, but many had not done their homework or brought their books! This did not seem to bother the teacher. The lesson was review of homework (functions), brainstorm for area calculation. I noticed very few guiding points, if any. At the end of the lesson students were asked about the area of different shapes in their book, including oval lakes covered by grids. There was no good explanation from T for this method, although the class is thrown out to determine the number of squares at 4 different sub-divisions of the underlying grid. To the class this seems like a strenuous exercise, possibly without anyone seeing the point of it.

This excerpt is considered and coded as an *Interpretation point* made by the teacher. It is also coded as a *Student interpretation point*, since students contribute to the understanding of notation.

Some interpretation points are simultaneously also other types of points.

An example: Lesson 13F Medium school in Central region Male teacher, seniority 15+.

CODES (in % of lesson)	Teacher → class				Adjusted total
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	
Perspective drawing	4.1	13.3		4.2	17.4

This *interpretation* point is also coded as a *procedural* point.

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:33 0:00-0:33	0:30 9:30-10:00	8:57 0:33-9:30	32:55 10:00-42:55	0:51 42:55-43:46	43:46

(6:15-8:05):

T: Yes – well I won't do that (draw a house as suggested by one student) I try to draw. (T outlines a simple geometrical figure on the blackboard). A vanishing point, yes. D, would you sit up please! A vanishing point, yes. And then I do what? S: Make a house. T: Then I shall draw a house, if this was a house – right? [Yes] (L continues drawing on the board). T: And this is our front plane, right? What happens to the lines that are parallel? Which must go through the vanishing point? What happens to those lines? The other lines that were here? (T points at the blackboard). S: They pass through the vanishing point. T: Yes (T draws depth lines through the same vanishing point). Now there are rules on how, where to stop with the length. We'll get back to that in grade 9. (But) we can stop one here, run down, drag out ... shall we say this is it. This is a perspective drawing. How is it with perspective drawing? Which of these three ways of drawing resemble most the real situation? (inaudible) A? S: Perspective. T: Yes, yes perspective drawing yes. Now some of you were not present last Friday. What was it we located – what is the vanishing point? What is it really? What does it correspond to, when I draw this? We brought a digital camera up here on Friday! (Inaudible).	L: Ja – det gør jeg så ikke (tegner et hus som nævnt af en elev) jeg prøver at tegne. (L skitserer en enkel geometrisk figur på tavlen). Et forsvindingspunkt, ja. D, prøv lige at sætte dig helt op! Et forsvindingspunkt, ja. Og så skal jeg hvad? E: Lave et hus. L: Så skal jeg tegne et hus, hvis det var et hus - ikke også [Ja] (L tegner videre på tavlen). L: Og det er så vores frontplan, ikke også? Hvad sker med de linjer, der er parallelle? Som så skal gå gennem forsvindingspunktet? Hvad sker der med de linjer? De andre linjer, der var herpå? (L peger på tavlen). E: De går gennem forsvindingspunktet. L: Ja (L tegner dybdelinjer gennem samme forsvindingspunkt). Så findes der regler for, hvordan, hvor man skal stoppe med længden. Det vender vi tilbage til i 9. klasse. (Men) vi kan stoppe én her, køre ned, køre ud ... skal vi sige dét. Det er en perspektivisk tegning. Hvordan er det med perspektivisk tegning? Hvilken af de her tre tegneformer ligner mest om den virkelige situation? (uhørligt) A? E: Perspektiv. L: Ja, perspektivisk tegning ja. Nu var der en del af jer, der ikke var her i fredags. Hvordan var det vi fik placeret – hvad er forsvindingspunktet? Hvad er det i virkeligheden? Hvad svarer det til, når jeg nu tegner det her? Vi havde et digitalt kamera med herop i fredags! (uhørligt).

What is the vanishing point? What does it correspond to? K? S: The eye height, where you observe. T: Yes, that is the observer stays at a completely different place (points to the blackboard) on the line that points perpendicular to this, right? Do you understand? [Yes].	Hvad er forsvindingspunktet? Hvad svarer forsvindingspunktet til? K? E: Øjenhøjden, der hvor man observerer. L: Ja, dvs. den der observerer her, står et helt andet sted (peger på tavlen) på den linje, der peger vinkelret ind, ikke også. Er I med? [Ja].

QUESTIONNAIRE (of teacher)

Goal of previous lesson:	Primarily to understand the "vanishing point".
Goal of lesson:	To make drawings of an individually chosen concrete model build by centicubes or some simple wooden blocks by 3 different methods.
Goal of next lesson:	Not yet decided.

MEMO (by researcher)

> The textbook was not in use during this lesson. The topic was different ways of drawing (of different wooden models, T had brought). T has mathematics as a main subject and 31 years of teaching experience.
> Selected in the previous and in this lesson is a course on drawing principles (working sketches, isometric drawing and perspective drawing).
> T starts with a very careful instruction (approximately at the 9 minutes mark) with many points.
> The students are very hard-working – almost no disturbance. When T comes around, most have a question - otherwise, he asks. But there is no forest of waiting fingers.

The excerpt above is coded as both an *Interpretation point* and a *Procedural point*. The teacher demonstrates a method for perspective drawing to the students while he interprets certain lines in his drawing as front lines or depth lines. After this excerpt the teaching turns to careful instruction on the three available types of drawing paper for the students to choose. This teacher talk is on procedure only.

6.5 Teacher to student points

Only 5 of the 50 lessons did *not* have seatwork with students working on their own or in smaller groups on tasks of various kinds. In 16 lessons this phase was longer than 30 minutes (more statistics on this in section 8.2). During seatwork teachers have the possibility to guide, help or ask students in groups or individually and even articulate points. Below is an example of each of four types given:

Parent code	***Child codes***	***Description***
Teacher → student	Teacher → student conceptual point	Teacher presents and comments on a mathematical concept by definition, symbol or application
	Teacher → student procedural point	Teacher presents and comments on a rule or a method in an application or example
	Teacher → student result point	Teacher develops, presents or comments on a mathematical result like a formula, theorem or procedure
	Teacher → student interpretation point	Teacher interprets a model or a result or compares representations

6.5.1 Teacher → student conceptual point

An example: Lesson 26
Small school in North region
Male teacher, seniority 15+ years.

CODES (in % of lesson)	Teacher → student				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Problem solving	10.5				10.5

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
3:29 0:00-3:29	2:07 3:29-5:36		43:32 5:36-49:08		49:08D

(19:00-24:10):

(T helps a student with scale, walks to the blackboard with the student seated in front. T writes: 1:150). T: Do you remember what this means? (T says “drawing” and writes the first letter of this then says “reality” and writes the first letter of that beneath the scale.) T: What does it mean that it was drawn in	(L hjælper en elev med målestoksforhold, går til tavlen, med eleven siddende foran. L skriver: 1:150). L: Kan du huske, hvad det betyder? (L siger først tegning og skriver T, derpå virkelighed og der skrives V under målestoksforholdet.) L: Hvad betyder det, at den er tegnet i

1 to 150? (The student is silent). You usually say: In the drawing this much represents that much in reality. S: Yes T: Do you remember some of that? S: No! T: Not at all? Well, then I must help you (returns to the board). This means that 1 cm is equal to, right – now I write is equivalent to, for it is not equal to, there is no such thing, it is equal to 150 cm, that is out in reality ...	1 til 150? (Eleven er tavs). Så plejer man her at sige: På tegningen svarer så meget til så meget i virkeligheden. E: Ja L Kan du huske noget af det? E: Nej! L: Slet ikke? Nå, så må jeg hjælpe dig (vender tilbage til tavlen). Det betyder, at 1cm den svarer til, ikke også – nu skriver jeg svarer til, for den er ikke lige med, det er der ikke noget der hedder, den svarer til 150 cm, altså ude i virkeligheden …

This lesson was also given above as an example of *New homework* (section 6.3.6). The goal of the lesson was simply stated as to solve problem tasks in the teacher questionnaire. The researcher memo states, the "*class was hard working. The students have had a project week, so it was the first "regular" math lesson for a long time. T was happy that I did not need to see them together in class. Some students also spread immediately into two groups in the library when they were allowed to work with their homework assignments during this first lesson (of a double lesson). The lesson is held as a homework café.*"
The actual excerpt shows a demand for ideas and time for individual guidance as the student in question is lacking fundamental knowledge of the mathematics content. Only a part of the transcript is shown, but the dialogue continues for 5 minutes, during this period the teacher is not available to help other students.

6.5.2 Teacher → student procedural point

An example: Lesson 2 Medium size school in Central region Female teacher, seniority 15+ years.

CODES (in % of lesson)	Teacher → student				Adjusted total
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	
Coordinate system		3.8		3.8	3.8

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:22 0:00-2:22	5:11 2:22-7:35	6:05 7:35-13:40	35:58 13:40-49:38		49:38

(38:45-40:31):

T: If I place an eraser here, can you move it in parallel? S: Is it not just (pointing with the pencil)? T: Do it. Take the eraser and slide it in parallel. S: That is, what I don’t understand. T: Well (inaudible). Now look for what I do: I just push it: Push, push, push, push. I did not turn it or anything. I only put it there and then push, push, push. I can push it up (shows that), I can push it down. I push. S: OK. Is it not (inaudible)? T: Yes, I have pushed it up. And when I now leave you, and you must sit with the next. Then the next thing there: there you must turn. That was what we did up at the blackboard. Look: Now I'll take and turn it (T shows this). OK? And the last one, there you need to reflect. There you go over to the other side (T turns the eraser upside down). S: I can do that. T: You can. But it is those 3 you need to get a hold on!	L: Hvis jeg nu lægger et viskelæder her, kan du så parallelforskyde det? E: Er det ikke bare (peger med blyanten)? L: Gør det. Tag ved viskelæderet og parallelforskyd det. E: Det er det, jeg kan ikke forstå. L: Nå (uhørligt). Kig nu efter, hvad jeg gør: Jeg skubber det bare: Skub, skub, skub, skub. Jeg har ikke drejet det eller noget som helst. Jeg har lagt det der, og så skub, skub, skub. Jeg kan skubbe det opad (viser det), jeg kan skubbe det nedad. Jeg skubber. E: OK. Er det ikke (uhørligt)? L: Ja, jeg har skubbet det op. Og når jeg nu går fra dig, og du skal sidde med de næste. Så den næste der: der skal du dreje. Det var det, vi gjorde oppe ved tavlen. Prøv at se: nu tager jeg og drejer det (L viser dette). OK? Og den sidste, der skal du spejle. Der skal du over på den anden side (L vender viskelæderet). E: Det kan jeg godt. L: Det kan du godt. Men det er de 3, du skal have hold på!

The student in this excerpt is not fluent in the Danish language and has difficulty understanding the mathematical words connected with transformations as slide, turn and flip in the coordinate system. The excerpt is also coded as a *Teacher → student* interpretation point.

6.5.3 Teacher → student result point

No lessons were coded for this type of point.

6.5.4 Teacher → student interpretation point

An example: Lesson 14 Large school in Capital region Male teacher, seniority 0-4 years.

CODES (in % of lesson)	Teacher → student				Adjusted total
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	
Perspective drawing		15.3		2.0	17.3

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
		31:50 0:00-31:50	14:52 31:50-46:42		46:42

(35:50-36:46):

S: Is it silly to have a horizon line fairly high up on paper? T: The higher up, the more elongated the house will just seem. The more depth you will see in the house, right. If it is closer, then you will see it more from the side. S: It is the future, teacher. It is the future! T: Try drawing the same house again here, but then you just place the vanishing point *differently*. So, for example, you see it from the front, right. As is the case here or maybe just a little closer, so you see the house from a different angle. ...	E: Er det dumt at have en horisontlinje rimeligt højt oppe på papiret? L Jo højere oppe, jo mere aflangt vil huset ligesom virke. Jo mere i dybden vil du se huset, ikke også. Hvis det er tættere på, så vil du se det mere fra siden. E: Det er fremtiden, lærer. Det er fremtiden! L: Prøv at tegne det samme hus igen herovre, men hvor du ligesom har forsvindingspunktet et *andet* sted. Så du fx ser det forfra, ikke også. Ligesom det er tilfældet her, eller måske bare lidt tættere på, så man ser huset fra en anden vinkel. ...

This excerpt on interpretation also contains a suggestion or direction from the teacher. This is *not* considered detailed enough to be labeled as a Teacher → student procedural point though.

6.6 Student points

Below examples are given of the four different types of *student* points.

Parent code	***Child codes***	***Description***
Student	Student conceptual point	Student presents and comments on a mathematical concept by definition, symbol or application
	Student procedural point	Student presents and comments on a rule or a method in an application or example

	Student result point	Student develops, presents or comments on a mathematical result like a formula, theorem or procedure
	Student interpretation point	Student interprets a model or a result or compares representations

It has not been found difficult to decide when to start and finish coding for *Student* points. But as with *Teacher to class* points the following examples are chosen to illustrate both the very clear situations not leaving the coder in doubt – and communication, where coding choices may be challenged. I have sought to explain my reasons for decision, when the rather wide code descriptions seem open to more interpretations.

6.6.1 Student conceptual point

An example: Lesson 3 Small school in Central region Male teacher, seniority 10-14 years.

CODES (in % of lesson)	Student				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Algebra, area	2.4		3.9		6.3

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:59 0:00-2:59		45:56 2:59-48:55			48:55D

(10:17-12.00)

T: I'll just take a few examples to set the stage for what comes following in the book, right. First I take one like this (T draws a square in the blackboard pane). What shape do we have here? ... E? S: A square. T. A square. What is the special thing in a square, E? St: It is equal on all sides.	L: Jeg tager lige nogle eksempler, for at lægge op til det, der kommer følgende i bogen, ikke også. Først så tager jeg lige sådan én som den her (L tegner et kvadrat i tavlens rudenet). Hvilken figur har vi her? … E? E: Et kvadrat. L. Et kvadrat. Hvad er det specielle ved et kvadrat, E? Elev: Den er lige på alle sider.

T: It is equal on all sides (pointing at one of the square corners). And what do you mean when you say it is equal? Em? S: Two and two they are equal. T. Yes, the lengths of the sides are… equal, yes. Is there anything more you can say about the square there? S? S: The angles in the corner are 90 degrees. T: They are 90 degrees ... here too (T pointing at several angles). Then we have a square when they are 90 degrees. Good.	L: Den er lige på alle sider (peger på et af kvadratets hjørner). Og hvad mener du med, når du siger, at den er lige? Em? E: To og to er de lige. L. Ja, længden af siderne er ... lige, ja. Er der noget mere, man kan sige om kvadratet der? S? E: Vinklerne i hjørnet er 90 grader. L: De er 90 grader … her også (L peger på flere vinkler). Så har vi et kvadrat, når de er 90 grader. Godt.

QUESTIONNAIRE (of teacher)

Goal of previous lesson:	Reduction and calculation rules with x and y
Goal of lesson:	Repetition of quadratic, rectangle, calculation of perimeter and areal, also using letters. New content: Generalization, higher abstraction level.
Goal of next lesson:	Calculation rules, addition and multiplication

MEMO (by researcher)

The headmaster has emailed a few times in acknowledgment from me – and also promised to give T message about my choice of date (I never had an e-mail address for him).
I waited half an hour before class (8:30) at the school office where I was immediately offered coffee by the school secretary. The pedagogical leader M took me to the room that was empty of students – so I could arrange the camera in peace. The class was in another room than normally in their first morning lesson 8:15-9, and should change and continue with mathematics at 9:00.
I then spoke about half an hour to M on the research and my own background. He is a trained teacher with a master's degree in sociology – already had been to 3 schools, this time now as a leader. He was pretty taken with the project, might even go into research himself. I gave him a set of my 2 books on "gifted students" that I had in the bag. I've decided to present these as a possible “receipt” after school visits. Of course never before teaching.
All students were 2-language speaking (Iraq, Afghanistan, Turkey, etc.). But only one of them had difficulty in Danish. The eager class worked zealously and marked with a show of hands. The teacher immediately went to work with teaching without me having to introduce myself. The class was clearly directed in advance for one of them said as she passed me, "Well, no – we must of course not talk to him." There were missing 3 of the 15 students today, according to T. No points articulated by the teacher but quite some elicitation to invite students to present algebraic expressions, reduction, formulas for area in squares and rectangles (i.e.: 4y respectively. L x W). Calculating the circumference leads to the need for parentheses (distributive law). Motif is consolidation, no direct reference to own or authors' objective. It is a student response that determines the teacher's point-focus, which then is merely indirect.

This excerpt is coded as a *Student conceptual point* and also as *Elicitation*. The following dialogue is coded as a student result point. The teacher plays a very active role during this lesson in getting the students to word their knowledge and findings. A similar example is:

An example: Lesson 32 Large school in Capital region Male teacher, seniority information missing.

CODES (in % of lesson)	Student				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Powers	6.1	7.1			13.2

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:59 0:00-2:59		45:56 2:59-48:55			48:55D

(2:59-5.58):

T: Well, You've read on these powers, p. 85, 86, 87 and 88 ... You did read it, didn't you? Who did not read? ... Have you made the tasks too? All 20? I told you to spread a little on them ... T: What is a power? S: It is a number that you can make smaller. T: A? S: A number having a tiny number in the upper corner. T: (repeats it). What does the tiny number in the corner mean? S: You have to multiply the other number with itself as many times as indicated. ... T: For instance, if we take the number 4 – did you just have German? Lovely, it's a lovely language – for example, take 4^3, what does it mean? S: That you multiply 4 by itself 3 times. T (writes $4^3 = 4 \cdot 4 \cdot 4$): What does it give? E: 12 L: I didn't hear that! A?	L: Nå, I har læst på det der potens, s. 85, 86, 87 og 88 … Det har I læst på, har I ikke? Hvem har ikke læst? … Har I lavet opgaverne også? Alle 20? Jeg sagde jo, I ku´ sprede jer lidt på dem … L: Hvad er en potens? E: Det er et tal, man kan gøre mindre. L: A? E: Et tal, hvor der er et lille tal oppe i hjørnet. L: (gentager det). Hvad betyder det lille tal oppe i hjørnet? E: Man skal gange det andet tal med sig selv så mange gange som det er angivet. … L: Hvis fx vi tager tallet 4 – er det tysk I har haft? Dejligt, det er et dejligt sprog – tager fx 4^3, så betyder det? E: At man ganger 4 med sig selv 3 gange. L (skriver $4^3 = 4 \cdot 4 \cdot 4$): Hvad giver det? E: 12 L: Det hørte jeg slet ikke! A?

Student: 64 ... L: That, the big number – do you know what it is called? Pupil: Root! ...	Elev: 64 … L: Det der, det store af tallene - ved I hvad det kaldes? Elev: Rod! …

QUESTIONNAIRE (of teacher)
This was unfortunately not received.

MEMO (by researcher)

Accept was made very quickly by the Deputy Headmaster J, who this day caught T in the staff room. I said good morning to the Headmaster, otherwise none. Colleagues helped me to coffee. T was school yard inspecting during the recess before so we then walked together to the small classroom 5 minutes after the bell! The camera was started immediately when arriving in class.
None of the approximately 19 students were clearly non-ethnic Danish, the atmosphere was good and students were very attentive. They had fun with T's fresh style.
Twice along the way T loudly disapproved the textbook. It was poorly written, one ought to tell the authors, etc. The one he himself had in school was according to T much better – but he hung on this, which the school had chosen. Quite remarkable considering T's obvious lack of knowledge on reasons for power calculation rules.
It follows from the videotape, that rules for multiplication and division of power with the same root after T's view has to be memorized. There was not a shred of justification.

This excerpt from an overview dialogue is coded as a *Student conceptual point* and as *Elicitation*.

6.6.2 Student procedural point

An example: Lesson 38
Medium size school in Capital region
Female teacher, seniority 15+ years.

CODES (in % of lesson)	Student				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Reduction		2.3			2.3

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:45 0:00-0:45	24:12 0:45-3:00, 10:55-26:40, 42:30-48:42	9:30 3:00-10.55, 26:40-28:15	14:45 28:15-42:30	2:58 48:42-51:30	51:30D

(3:00-4.10):

T: These fractions, which are more or less a trouble. ... Today we'll just – before you get the next 10-minute tasks – talk about how we add these fractions. And if we look at a task like this (writes on blackboard 2/9 + 4/9), then what? Follow along here, H! ... What is it that you think first? S: We just move over, right? So it is 6/9. T: Yes, why? S: They have common denominator. T: Yes, it is 6/9 (T writes this). Can I do anything about it? S? S: Yes, you can say 2/3. T: I can say 2/3, because 3 divides in both.	L: De her brøker, som der er større eller mindre problemer med. … I dag så vil vi lige, før I får de næste 10-minutters opgaver, snakke lidt om, hvordan vi plusser de her brøker. Og hvis vi kigger på en opgave som den her (skriver på tavlen 2/9 + 4/9), hvad så? Følg med heroppe, H! … Hvad er det så, man tænker først? E: Vi flytter bare over, ikke? Så det bliver 6/9. L: Ja, hvorfor? E: De har fælles nævner. L: Ja, det bliver 6/9 (L skriver dette). Kan jeg gøre noget ved det? S? E: Ja, du kan sige 2/3. L: Jeg kan sige 2/3, fordi 3 går op i begge.

QUESTIONNAIRE (of teacher)

Goal of previous lesson: Reduction + fraction
Goal of lesson: Reduction + fraction + number understanding
Goal of next lesson: Reduction + fraction

MEMO (by researcher)

For some years this school has not followed the “normal” formation of classes in secondary schools. The entire cohort (i.e. here the grade 8) is divided in different ways 1-2 times a year. It may be after a request from one language teachers (at level) or otherwise. T do not think such criteria were used at the current formation. The mathematics teacher shares the class teacher function with 2 colleagues for this grade level.
Students sit at group tables as usually 5 "families". This evening there would be school-home gathering, and here the parents would also be grouped in these families! T considers this as really good.
The school secretary had mediated the agreement, and I didn’t see the headmaster this morning. The secretary kindly showed me in the staff room with a coffee machine!

Some of the other teachers asked who I was and I felt free to ask one of them about this "paradise". Yes, to the extent that it is too much, was her reaction. In the municipality this school was held completely free of 2-language speaking students, who then were concentrated on other schools. It means that it is *very* wanted by Danish parents in the area. She strongly deplored this. The school is otherwise relatively old (from the 50's, I think). A nice atmosphere with really focused students, without exception. 11 girls, 6 boys, 4 were absent.
Each lesson is started with a 10-minute task (I was handed a copy). They also use the "yellow book" containing similar tasks. Otherwise it was the textbook – and they got homework here later in the recording (ending at 51:30): All tasks on page 146-147. But students are expected to select only the tasks they are challenged by! The first 10 minutes were used for analysis of two tasks on fractions. The next 10 minutes on the so-called 10-minute task, and the rest of the lesson was spent solving problems in the textbook. The focus on points is merely indirect.
T acts as a gifted and interested teacher. She recently participated in a Mathematics Teachers' Day arrangement in Copenhagen which she was very happy with. 4 from the school attended. I noticed a Teacher's Guide to the textbook for grade 2 in the staff room, where I am one of the authors!

The excerpt shows a review of calculation rules for fractions having a common denominator. The procedure is demonstrated by example, but without a justification. No students ask for that, the class seems to know how to handle the procedure. After this excerpt, the teacher goes on with fractions having different denominators.

An example: Lesson 20
Large school in Central region
Male teacher, seniority 15+ years.

CODES (in % of lesson)	Student				Adjusted total
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	
Area		$\boxed{11.4}$	9.7		11.4

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
2:10 0:00-2:10	5:50 2:10-8:00	19:50 8:00-27:50	17:38 27:50-45:28		45:28

(15:00-17:02)

T: What if it's a pie that we have? (T draws this on the board) a "pie piece" ... what do we do then? N? S: But is it not like a triangle? T: Well, there's not much triangle there – it's an arc you see. S: So, to make such a line across (S indicates the chord). T: Yeah? S: Then you find the triangle area. T: It could be done. But what about the part out there (circular section)? And the bigger pie piece you want the larger arc it is out there? The area of complete circle … yes, M? S: Well we find out how large a part it is of the circle, and then we can say pi times r^2 and then divided by – that is if it is for example a sixth, then we must divide by 6 T: Ok, that would be correct. So if it was a sixth of the circle, we took there (T writes on the blackboard), that would be correct. But if we took any random angle – we must do something else? K? S: It should be the area … T: Yes, we obviously have to know the angle (T now himself writing the formula) i.e. 1 degree and then multiplied by the number of degrees. Then we have the area of a pie, a pie piece.	L: Hvad nu, hvis det er et cirkeludsnit, vi har? (tegner det på tavlen) Et "lagkagestykke" ... hvad gør vi så? N? E: Jamen er det så ikke ligesom en trekant? L: Åh, der er jo ikke meget trekant der – det er jo en bue. E: Så laver man sådan en streg over (E indikerer korden). L: Jah? E: Så finder man trekantens areal. L: Det kunne man godt. Man hvad så med det derude (cirkelafsnittet)? Og jo større lagkagestykke, du gerne vil have, jo større bue bliver der derude? Arealet af hele cirklen ... ja, M? E: Altså vi finder ud af, hvor stor en del det er af cirklen, og så kan vi sige pi gange r^2 og så divideret med – altså hvis der fx er en sjettedel, så skal vi dividere med 6. L: Ok, det ville være rigtigt. Altså hvis det var en sjettedel af cirklen, vi tog der (L skriver på tavlen), det ville være rigtigt. Men hvis vi nu tog en eller anden tilfældig vinkel – så må vi gøre noget andet? K? E: Det skulle være arealet … L: Ja, vi skal selvfølgelig kende vinklen (L skriver nu selv formlen) dvs. 1 grad og så ganget med antal grader. Så har vi arealet af et cirkeludsnit, et lagkagestykke.

QUESTIONNAIRE (of teacher)

Goal of previous lesson: Was before week 7 (the winter break).
Goal of lesson: Area of geometrical shapes and conversion m → cm.
Goal of next lesson: Areal of complex shapes.

MEMO (by researcher)

I already met T in the school parking lot, so there was time for small talk. He and the class were ready. T deplored the frequent "fragmentation" of teaching periods due to "bridging" project progress, etc. He considered the students quite good (in all three grade 8 classes) – compared to the school (with students up to grade 7) he was appointed to in another city 1 ½ years ago (in a well-off neighborhood).
Nice students, usually there is 24 in the class.
After class, I had a long conversation in the hallway with the Deputy J, which I once had on

mathematics course! He told about an (unfortunately capsized) attempt to set up deals for the academically talented. He got a lot on his back when he suggested that the funds allocated to special education in the municipality also could be allocated for gifted students. Some students were moved at the time being by parents for private schools (in a larger city) ... J wrote a good application for pool funds to particular efforts, but this was rejected - after which the municipality also withdrew its expected support. So now he sought to establish an offer based in the youth school.
L has not answered the question on line subject, but he has 28 years seniority. The lesson offers many points, especially from T. Some of them indirectly and perhaps not as clear to the students as observers.

The coding of this excerpt from an overview session was a quite complicated. Besides being a *Student procedural* point it is also coded as a *Student result point*, but the student result $A = \pi r^2/6$ is elicitated (se section 6.7.1) by the teacher for the general result $A = \pi r^2/360\times v$.
The teacher plays a necessary and active role in processing the dialogue. Therefore the excerpt is also coded as a *Teacher procedural point*.

6.6.3 Student result point

An example: Lesson 12
Large school in Central region
Male teacher, seniority 15+ years.

CODES (in % of lesson)	Student				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Polar coordinates	14.8		$\boxed{2.0}$	14.8	16.8

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:40 0:00-0:40			43:30 0:40-44:10	1:18 44:10-45:18	45:18

(28:20-29:15):

T: How many cm^2 are one m^2? What do you have to multiply together to get that? S1: To get a square? It's just 100 times 100. S2 (sitting in outerwear and not really taking part in discussions): I have found it: It is 9.	L: Hvor mange cm^2 er én m^2? Hvad skal I gange med hinanden for at få det? E1: Altså for at få en kvadrat? Det er jo bare 100 gange 100. E2 (der sidder i overtøj og ikke rigtigt tager del i overvejelserne): Jeg har fundet ud af

(S shows his calculator – have possibly just typed 9. But the suggestion relates to a former question in class: How many sheets of A3 does it take to cover 1 square meter?) T: N, N – you say 100 times 100 How many cm^2 is it? S3: It's damn 1000, right? T: How many cm^2 is it, 100 times 100? S1: It is 10 000 ... L: 10 000 cm^2 is 1 times 1 meter, 1 m^2. Now how do you then find out, how much a A3 sheet is? (T indicates the sides of the sheet and leaves the group to do some thinking on their own).	det: Det er 9. (E viser sin lommeregner - har muligvis netop tastet 9. Men dette svar er tænkt til et tidligere spørgsmål i klassen: Hvor mange A3 ark skal der til en kvadratmeter?) L: N, N – du siger 100 gange 100. Hvor mange cm^2 er det? E3: Det er sgu da 1000 ikke? L: Hvor mange cm^2 er det, 100 gange 100? E1: Det er 10 000. … L: 10 000 cm^2 det er 1 gange 1 meter, 1 m^2. Hvad så for at finde ud af, hvad ét A3 ark fylder? (L peger rundt langs kanten af arket og efterlader gruppen til at tænke videre selv).

QUESTIONNAIRE (of teacher)

Goal of previous lesson:	Work with the coordinate system.
Goal of lesson:	Read and understand a long text. Create a different coordinate system than the usual, cooperation.
Goal of next lesson:	Finish the work, talk about the outcome.

MEMO (by researcher)

I had coffee in the staff room where I happen to know several of the teachers from a project on gifted students. Also I was welcomed by the headmaster and the pedagogical leader (who reminded me, that I actually had taught her mathematics at the college).
A few students looked like they had a different ethnic background, but there were no language problems. In this town some students are "bussed" from a closed school to others.
The student atmosphere is good, hard-working – but also fluctuating concentrations of individual students. Noise level a little high in this lesson because of the "practical" work with design and testing of a paper shape (formed by a number of A3) with area of one square meter. Students were asked to investigate the necessary number.
4 girls, 12 boys: 16 in total. Usually around 20 according to T, but the 4 was absent due to practice in youth education. T do not have mathematics as a line subject, but indicates pleasure and desire. The course is on academic reading and design of another coordinate system (the polar) than the usual. This lesson‘s goal is to "Read and understand a long text, produce a different coordinate system than the usual, cooperation".
The lesson also develops this way, since T quickly starts the work – then switches his attention between the groups – and always asks openly about the progress. In this

way, the lesson seems rather unusual. However, T repeatedly adds this question: How many sheets of A3 cover one square meter? At the end of the lesson, some of the groups present a more or less robust response to this question, but T abstains at first from "correcting". Instead, he demands explanations on uncertainty and method.

The excerpt is quite typical for the whole lesson, as this teacher involves himself in dialogues with single students and groups all the time, but never gives the answer himself before being pressed by very convincing argument from the students.
I found no difficulty in coding this excerpt as a *Student result point*. This is a code used only three times in the entire transcripts of the 50 lessons.

6.6.4 Student interpretation point

An example: Lesson 22 Small school in South region Female teacher, seniority 5-9 years.

CODES (in % of lesson)	Student				Adjusted
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	total
Statistics				1.7	1.7

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:55 0:00-0:55	1:45 0:55-2:40	5:15 2:40-7:55	36:06 7:55-44:01		44:01

(16:55-17.40):

S (conversation continued from 16:00 and reading aloud from task): ... has been ill for *maximum* 2 days! Then you probably do not include those, who have *not* been sick? T: Try to read: In *maximum* two days! S: Yes! So it is 1 to 2 of course really. T: You are right there (T is silent and looking for a while at the text). I really think, it is meant so you should include 0. But I do understand that it is poorly formulated (unclear?). S: Yes ...	E (samtale fortsat fra 16:00 og læser højt fra opgaven): ... har været syg i *højest* 2 dage! Så skal man vel ikke have dem med, som *ikke* har været syge? L: Prøv at læse: I *højest* to dage! E: Ja! Så er det fra 1 til 2 jo egentlig. L: Det har du da ret i (L tier og ser længe på teksten). Jeg tror altså, den er tænkt så du skal tage 0 med. Men jeg kan godt forstå, at den er dårligt formuleret (uklart?). E: Ja ...

QUESTIONNAIRE (of teacher)

Goal of previous lesson: Common repetition of concepts, then practice.
Goal of lesson: Practicing statistics.
Goal of next lesson: Test for the whole class.

MEMO (by researcher)

The management was not present, but in the staff room they were well informed. Coffee and when T arrived after eating with her class, a little information. L has had a class only 1½ months as 6 weekly lessons overtime. This is because of a long-term ill teacher, and it had been quite hard to adjust the class to another teachers' style! Now it went (slightly) better. They will have a test tomorrow in statistics, so the lesson today will be used to prepare well to deal with key concepts and techniques. It also appeared in the lesson to be a very clear goal from the teacher.
None 2-speaking students. The atmosphere is good – students are largely independent, but with large differences in mathematics competence. Therefore, most of T's time was for dialogues and explanations to 2-3 students. It is a large class with about 26 students.
No highly visible routines, but T informed that the class is handing in homework every Wednesday. This is also seen in the beginning of the lesson. It is often an exam paper, a whole set then divided into 5 individual assignments.
About half of the students submitted their tasks in Excel files on the school intranet. T is also IT responsible at the school! Afterwards we talked a bit about talented students – and the difficulty in differentiation. She seems a highly competent teacher, but she is also captured by a time-consuming weak group with needs (and expectations) for private lessons by "teacher's desk". Finally the lesson was dissolved in some turmoil, students packing up 10 minutes before, etc. Some sat in "clusters" in a common area. This cannot be for the sake of peace, for certainly there was none!?

The excerpt has clearly been found as an example of *Student interpretation point* as an interpretation of convention. It's debatable though, if the interpretation is correct. But the student expresses such doubts, that the coding has been decided upon. The excerpt is an example of the challenge to many students in actually reading and understanding texts with mathematical content.
The next example shows an even more complicated dialogue between one student and his fellow classmates and their teacher. This student has clearly understood a way to handle time information in minutes and seconds when changing to a decimal notation, but struggles in his explanation.

An example: Lesson 24
Medium size school in Central region
Male teacher, seniority 15+ years.

CODES (in % of lesson)	Student				Adjusted total
Content	**Conceptual**	**Procedural**	**Result**	**Interpretation**	
Area and time				11.9	

STRUCTURE of lesson in minutes

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
4:00 0:00-4:00	18:10 4:00-22:10	25:14 22:10-47:34		2:16 47:34-49:50	49:50D

In the excerpt below a student interprets 33 minutes as 0.55 hour based on an understanding of 30 minutes as half an hour and 33×100/60 = 50. The student's interpretation is preceded by a previous accepted calculation of 3 t 45 min = 3.75, but difficult to follow though.

(38:07-39.35):

T: R? S1: This can also be the two 3 digits, if the number really indicates a 5. The two 5's then is 55 because 3 has to be 5. 30 divided by 10 is the 3, which then will be the 5. So we might as well say the 30 will be the 50. Then is 55. S2: There, I went cold! S3: No R, do take that one more time. T: R he's really quite developed. When is it, why is it that you have to learn this? Why is it so damned important, frankly, that you grasp this about time? Why must you be able to calculate time as decimal numbers? When is it you use that? S: We have exactly used it in math. T: When is it you use it? When is it that you have to change time to decimal numbers? When is it smart? S: In mathematics homework. T: Yes, there are many of you who have errors when you meet it in tests, and in mathematics homework you have errors in it. S: Please stop asking me. T: Well, I have asked you many times. T, welcome (?) what did you say? S: When we do mathematics project.	L: R? E1: Det kan også være de to 3 taller, hvis 3-tallet det er jo et 5 tal. De to 5-taller det er så 55, fordi 3, det er jo 5. 30 divideret med 10, det er jo de 3, som der så vil være de 5. Så kan vi lige så godt sige de 30 vil være 50. Så er det 55. E2: Dér gik jeg helt kold! E3: Nej R, prøv lige at tage den én gang til. L: R han er egentlig udviklet nok. Hvornår er det, hvorfor er det, I skal lære det her? Hvorfor er det så forbandet vigtigt rent ud sagt, at I kan det her med tiden? Hvorfor skal I kunne lave tiden om til decimaltal? Hvornår er det, I bruger det? E: Det har vi lige brugt i matematik. L: Hvornår er det, I bruger det? Hvornår er det, I skal kunne lave tid om til decimaltal? Hvornår er det smart? E: I matematik aflevering. L: Ja, der er mange af jer der har fejl når I har det i prøver, og i matematik aflevering har I fejl i den. E: Hold nu op med at spørge mig. L: Jeg har da spurgt dig mange gange. T, velkommen til (?) hvad siger du? Elev: Når vi skal lave matematik projekt.

T: He is very smart over here, he is totally – well met! B? S: If ever you find out how much, if you are now working and must find out the time spent. T: You could do that, I'm sure you think about percentages of something.	L: Han er simpelthen kvik herovre, han er helt totalt – vel mødt! B? E: Hvis man nu skal finde ud af hvor meget, hvis man nu arbejder og skal finde ud af den tid, man bruger. L: Det kunne man godt, jeg er sikker på du tænker på procentdele af et eller andet.

QUESTIONNAIRE (of teacher)

Goal of previous lesson: Repetition before mathematics project.
Goal of lesson: Repetition of e.g. $cm^3 \to m^3$, time and equation systems.
Goal of next lesson: Equation systems.

MEMO (by researcher)

Agreement on the date was postponed due to project work, but it turned out that the project work already *had* been in mathematics – so actually it was quite unnecessary. But the result was now that the lesson on the video is the first more normal for T in a month.
A hoarse (!) headmaster met me, sourced coffee and showed the way to the classroom and back in the staff room.
T has one skills test per month and homework assignments on problems to be handed in regularly.
None 2-language students, a good atmosphere – students seemed quite focused even though T considered them restless and talkative (it is the "paw" that counts!). Around 21 students were present. They apparently have finished their textbook book (Sigma), and now run on copies. Also in the new topic of equations presented in end of the lesson (the first of a double lesson)

This excerpt is one of two longer *Student interpretation points* in this lesson. The excerpt is also coded as a *Missed point* (see section 6.7.3) as the teacher refrains from giving a proper explanation to the class. It is clear from the video, that very few fellow students are able to follow this student's explanation. Such ignoring may be done consciously but could also indicate a teacher lacking ideas for better illustration. The teacher chooses to continue teaching in another direction after just stressing the importance of such calculation skills.

6.7 Point addressing instants or episodes

During most lessons the mathematics teachers' communication is addressing a possible point by elicitation, but sometimes without points being articulated by teacher or students. Teachers may even direct their questions or suggestions very clearly towards a mathematical point, i.e. give a hint. But obvious and possible points may also be overlooked deliberately or simply missed by the teacher.

To register such instants or episodes coding was done for the three different types of communication described below:

Parent code	***Child codes***	***Description***
Instant or episode	Elicitation	Teacher poses question(s), possibly is in dialogue with class or single students, expecting contribution(s) or answer(s) from student(s).
	Hint	Teacher presents a hint, an idea or guides in a certain direction.
	Missed point	Teacher overlooks or does not use an obvious opportunity to state a point.

6.7.1 Elicitation

This is the kind of mathematical questions intended to provoke an answer from students.

Therefore it does not include the teachers' instructions (a TIMSS Videotape Classroom Study category is *directions* to perform some physical or mental activity), repetitions (uptake) or response (5.9.1).

Elicitation might be of different kinds as shown by these few examples, also underlined in the example from lesson no. 44 below. Such categories are present in the TIMSS coding schemes:

- Content oriented: *What is a sample space?*
- Metacognitive: *Think of what M just said?*
- Interactional: *Close the book. Now use you own words!*
  Try to say a bit more?
- Evaluation oriented: *We will wait for everyone to be with us in this one.*

An example: Lesson 44
Large school in Capital region
Male teacher, seniority 15+ years.

The excerpt shows teacher's *Elicitation* in a lesson without any teacher points. The first part is also coded as a *Student conceptual point*, but heavily teacher-directed.

CODES (in % of lesson)	Teacher → class				Adjusted Total
Content	Conceptual	Interpretation	Procedural	Result	
Probability					0.0

STRUCTURE (of lesson) in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/ groups	Common summing up or messages	Total lesson time
1:30 0:00-1:30	1:38 1:30-3:08	26:22 3:08-29:30	10:24 29:30-35:22, 37:06-41:38	5:06 35:22-37:06, 41:38-45:10	45:10

(3:50-5:01 + 5:01-9:22):

T: <u>What is a sample space?</u> (T is waiting, app. 7 students marking). There are many who has a suggestion. What does V say? S: Number of possible. T: Number of possible? Yes, anyone can say otherwise? Supply? C? J? S: It is the set of possible outcomes (more laughs, because he reads it from the book). T: <u>Close the book. Now use your own words!</u> S: The amount of, no it's how many options you have. E.g. if you throw a dice, how many different options are there – for a number. T: Ok. You explain by an example, right? [Yes]. T: You say all the possibilities. Ok, yes. We're talking about the quantity – perhaps because we tend to write it up as a set, right? S: Yes. -------------------------------- L. OK, now C gave one example. Could you come up with another? Try to come up with another example of a sample space. There are many who have a suggestion. Excellent. M, what do you say? Student: Cards? T: Card games? Yes, what is a sample space really, if you … <u>Try to say a bit more?</u> S: It is, if you want to pull an ace (inaudible). T: So now I hear you talking about probability? [Yes]. What is the sample space, if (fetches a bunch of playing cards). Pull a card, pull a card! Ok, what did you get? (S displays the card)	L: <u>Hvad er et udfaldsrum?</u> (L venter, ca. 7 elever markerer). Det er der mange, der har et bud på. Hvad siger V? E: Antal mulige. L: Antal mulige? Ja, nogen der kan sige noget andet? Supplere? C? J? E: Det er mængden af mulige udfald (flere griner, fordi han læser op). L: <u>Prøv at lukke bogen. Så sig det med dine egne ord!</u> E: Mængden, nej det er hvor mange muligheder du har. Fx hvis det er, du slår med en terning, hvor mange forskellige muligheder der så er – for hvad tal det bliver. L: Ok. Så du forklarer det lidt ved at komme med et eksempel, ikke? [Ja]. Du siger alle de muligheder, der er. Ok, ja. Vi snakker om mængden – måske fordi vi plejer at skrive det op som en mængde, ikke? E: Ja. --------------------------------- L. OK, nu kom C med et eksempel. Kunne man komme med et andet eksempel. Prøv at komme med et andet eksempel på et udfaldsrum. Der er mange, der har et bud, dejligt. M, hvad siger du? E: Kort? L: Kortspil? Ja, hvad er et udfaldsrum egentligt, hvis man ... <u>Prøv at sige noget mere?</u> E: Det er sådan, hvis du skal trække et es (uhørligt). L: Så nu begynder du at tale om sandsynlighed, kan jeg høre? [Ja]. Hvad er udfaldsrummet, hvis (henter en bunke spillekort) Træk et kort, træk et kort! Ok,

T: Ok, you pulled a pawn of hearts. Ok. It was one outcome, right? How many outcomes are there in the sample space – how many possible? You drew the pawn of hearts, yes? What more could you take? S: (inaudible). Yes, club something, right? How many possibilities are there? S2 (another): Me! ... S: 52 T: 52 The experiment is called: Take a card from the deck, and write down the kind of card. Then there are 52 outcomes. Ok, good. T: What if I say: M, take a card – and I am interested in whether it is a club? (L lets a girl pull a card, while everyone is thinking). T: No, that's not a good one. T: What is sample space, as I say to M: We will see if you can pull a club? ... T: Think of it what M just said, and what C said ... it had something to do with the possible, what is possible? S: And sets! And elementary events! (Laughs) T: He flings some fine expressions! What do you say, Ma? S: 52. T: Now please take, just try, hmm, yes. Let's just try again (a girl pulls a card - a hearts). S: Was it a club? T: No. What else could she have pulled? S: A diamond... T: What will you then say the outcome space is ..., J? S: Isn't it 4? T: What are these 4? S: Clubs and diamonds and ... T: Clubs, spades, diamonds, and - you thought of hearts I suppose? [Yes!]. T: In this way there are 4, and this is what I think many of you were confused by in the problems for today.	hvad trak du? (E viser kortet) L: Ok, du trak en hjerter bonde. Ok. Det var ét udfald, ikke? Hvor mange udfald er der i udfaldsrummet – hvor mange mulige? Du trak hjerter bonden, ikke. Hvad kunne du mere tage? E: (uhørligt). L: Jaeh, klør et eller andet, ikke? Hvor mange muligheder er der? E2 (anden elev): Her! … E: 52. L: 52. Eksperimentet hedder: Tag et kort fra kortbunken, og skriv hvad det er for et kort. Så er der 52 udfald. Ok, godt. L: Hvad nu, hvis jeg siger: M, træk et kort – og jeg er interesseret i, om det er en klør? (L lader en pige trække et kort, mens der tænkes). L: Nej, den var ikke god. L: Hvad er udfaldsrummet, når jeg siger til M: Vi skal se, om du kan trække en klør? … L: Prøv at tænke på det, som M sagde og hvad C sagde … det var noget med det mulige, hvad er der muligt? E: Og mængder! Og elementar hændelser! (der grines) L: Han slynger om sig med fine udtryk! .. Hvad siger Ma? E: 52. L: Nu tag, prøv lige, mm, ja. Prøv, lad os lige prøve igen (en pige trækker igen et kort - en hjerter). E: Var det en klør? L: Nej. Hvad kunne hun mere have trukket? E: En ruder … L: Hvad vil I så sige udfaldsrummet er …, J? E: Er det ikke 4? L: Hvad er det for nogle 4? E: Klør og ruder og … L: Klør, spar, ruder og - du tænkte nok på hjerter?]Ja!]. L: På den måde er der 4, og det er det jeg tror, mange af jer blev forvirrede af. I de her opgaver. Nogle af de lektier, I havde for til i

It read: Draw a card and then jot down the color. (T walks to a student desk). Pick a card and tell me what color it is. What color is it (is repeated)? S: Black. T: Black. What is then the outcome space, if I ask: Which COLOUR did N pick? (T looks around - waits) ... <u>we will wait for everyone to be with us in this one.</u> He picks a black card, what are the options, L? S: Isn't it 26 out of 52? T: I ask what color is it? He pulled a black. Now you pick one (the student picks a card). What was it? S: It was red. T: Can you pull another. S: This was black. T: Ok, how many possibilities are there, do you think? (T waits) You said, this was red, and that was black, right. You can pick one more. ... No, you're thinking of probabilities now? S: Yes. T: What could it be, what color would it be? S: Red and black. T: Yes, red or black. So the sample space consists of red and black. You cannot pick a green card or a pink card.	dag. For der stod: Træk et kort og så notér farven. (L går ned til en elev) Træk lige et kort og så fortæl mig, hvad farve det er. Hvilken FARVE er det (gentages)? E: Sort. L: Sort. Hvad er så udfaldsrummet, hvis jeg spørger: Hvilken FARVE trækker N? (L ser sig omkring - venter) … <u>vi skal lige have de sidste med også</u>. Han trækker et sort kort, hvad muligheder er der, L? E: Er det ikke 26 ud af 52? L: Jeg spørger, hvad farve er det? Han trak et sort. Så trækker du lige et (eleven trækker). Hvad var det? E: Det var rødt. L: Kan du trække et andet. E: Det var sort. L: Ok, hvor mange muligheder er der så, tror du? (L venter) Du sagde, dét var rødt, dét var sort, ikke. Bare træk ét til. … Nej, du tænker på sandsynligheder nu? E: Ja. L: Hvad kunne det blive, hvilken farve kunne det blive? E: Rødt og sort. L: Ja, rødt eller sort. Så udfaldsrummet består af rød og sort. Du kan ikke trække et grønt kort eller et pink kort.

The excerpt above from lesson no. 44 is an example of a lesson *without* any points from the teacher in whole-class teaching. But the lesson does contain quite a lot of elicitation from the teacher (more than shown in the excerpt) and some conceptual and procedural points being made by the students. Therefore this lesson is a counter-example to a too hasty conclusion like labeling half the Danish mathematics teaching as point-free.
A further discussion of recommendations (Franke et al, 2007) and research findings on elicitation is provided later (8.5 and 13.4-5).

6.7.2 Hint

An example: Lesson 39 Medium school in Zealand region Female teacher, seniority 15+ years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
3:57 0:00-3:57	38:41 3:57-42:38	0:50 42:38-43:28	6:22 43:28-49:50		49:50D

(14:20-16:08):

T: OK. Why is it now we get to 103 down here with the summed frequencies? The ideal would be that we landed on 100. Why are we beyond that? What may be the explanation? S: Because it is an odd number? T: Well maybe we should try to look at rates over here. When we look at a fraction as that one – or a fraction like this one – in decimals – what has then happened here? How many of those here (inaudible) makes us land on something skew down here (pointing at the 103 %)? S: There are multiplied by 100? T: Yes it's true, it's true. But by doing that, what happens? S: Does the 8/35 not result in 32, since you will say 8 times 4? To me it does anyway. Because 4 times 25 gives 100. T: Yes. Then this means, not everything here is quite as it should be. That may be the reason. Simply that some calculations are wrong! But there is indeed one more thing! S: And sometimes, you make a fault when you convert from 50 and the sorts. T: Yes, indeed. Yes and then there is also something about rounding the numbers! If you consistently round up all the time without looking to the decimals, then you will end having too much in the end (inaudible). So the art is to mingle a little in order to land at 100, or like 99.9 %. Or perhaps slightly above. So well enough. But the procedure – this is how you must do it! Then there are a	L: OK. Hvorfor er det nu vi lander på 103 hernede med den summerede frekvens? Altså det optimale ville jo være, at vi landede på 100. Hvorfor er det, vi overskrider det? Hvad er dét, der kan være begrundelsen? E: Fordi det er et ulige tal? L: Ja måske skal vi prøve at kigge på frekvenserne herovre. Når nu vi kigger på sådan en brøk som den her – eller sådan en brøk som den her – i decimaltal – hvad er der så sket her? Hvor mange af de her (uklart) gør, at vi lander på noget skævt hernede (peger på de 103 %)? E: Der er ganget med 100? L: Ja, det er rigtigt, det er rigtigt. Men i forbindelse med, at der er ganget, hvad sker der så? E: Giver det der med 8/35 ikke 32, i og med at man siger 8 gange 4? Det gør jeg i hvert fald. For 4 gange 25 det giver 100. L: Ja. Så det vil sige, der er måske også noget her der ikke er som det skal være. Dét kan være grunden. At der simpelthen er nogle regnefejl. Men der er faktisk også en ting til! E: Og nogle gange, så regner man så forkert, i det her at man omregner fra det der 50 og sådan noget. L: Ja, jo. Ja og så er der også noget afrunding i det! Hvis man konsekvent runder op hele tiden, uden sådan at skele til decimalerne, så ender man altså med, at det bliver for meget til sidst (uklart). Så kunsten er at mingelere lidt, så vi lander på 100, eller de 99,9 %. Eller evt. lidt over. Så godt nok. Men fremgangsmåden – det er sådan I skal gøre det! Så er der lidt

few tiny mistakes, but let it go.	småtterier, men lad nu det ligge.

The teacher didn't quite succeed with her *Hint* in this excerpt, but an attempt is there. Perhaps the students did not guess the teacher's thoughts because their attention was occupied by a faulty calculation only discovered later by the teacher. The actual hint is also coded as a teacher's *Procedural point.*

6.7.3 Missed point

The LMT choice to code teachers' "*inappropriate response by misinterpretation or no response to student misunderstanding*" (section 5.9.2) has no equivalent code in the present research. But I wanted to have an impression of the amount of missed points, if / when the teacher overlooks or does not use an obvious opportunity to state a point.

I am aware, that obvious means obvious to the researcher here. The teachers' action may be a conscious choice in that situation not to disturb the student line of thought. But it may also be due to ignorance, the teacher simply overlooking or not knowing about important points in a given context.

An example: Lesson 9 Small school in North region Male teacher, seniority 0-4 years.

STRUCTURE of lesson in minutes:

Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total lesson time
0:35 0:00-0:35	2:20 0:35-2:55		40:52 2:55-42:02, 42:20-44:05	0:18 42:02-42:20	44:05

(28:10-29:50):

(T at the boys' group that gets a result to assignment 11e on reduction of $2x^2y^2 - x^2 - y^2$ not fitting the answers in the back of the book). S1: It says $2x^2y^2 - 13$ Why is that the result? It says that it is the same. S2: It should not. .. It ought to be $1x^2y^2$, right? T: The same result? S1: Yes because the results list, states that this one is the best.	(L ved drengegruppe, der får et resultat i opgave 11e om reduktion af $2x^2y^2 - x^2 - y^2$, der ikke passer med facitlisten bag i bogen). Elev1: Der står $2x^2y^2 - 13$. Hvorfor giver det det? Der står at det er det samme. E2: Det burde det ikke. .. Burde det ikke give $1x^2y^2$? L: At det giver det samme? E1: Ja for i facitlisten står der, at det der er bedst.

S2: The result surely must be x^2y^2? T: Ok? Let us see. What does it mean that you have $2x^2$. What does it mean? S: That you have 2x times 2x. T: Yes. Then you multiply by y^2 also [Yes]. That is, now it says what? There are a multiplication sign between these things. What is it that brings together parts and what is it that separates parts? S: x and y is the bringing together and plus or minus separates. T: Yes. When then this is brought together into one part, it says something different there. It is like if this meant "pears". The number called x^2y^2. That number is called only x^2. [Mmm]. It's like as if they were looking at bananas and deducting oranges and raisins. S: Oh. Well then I understand it. T: It is named quite different. S: So I guess that I have grasped it. S2: So nothing will be changed. T: Nothing will be changed. S: Oh. T: It's pretty smart as it correlates.	E2: Facit må da være x^2y^2? L: Ok? Lad os prøve at se. Hvad betyder det, du har $2x^2$. Hvad betyder det? E: At du har 2x gange 2x. L: Ja. Så har du gange y^2 også [Ja]. Dvs., så står der hvad? Der står gange mellem de her ting. Hvad er det der samler led, og hvad er det der skiller led? E: x og y er det, der samler og plus og minus deler. L: Yes. Når det her så er samlet til ét led, så står der noget forskelligt der. Det er det samme som at, hvis der står "pærer" der. Det der tal, det hedder x^2y^2. Det der tal hedder kun x^2. [Mmm]. Det er jo lidt det samme, som her prøver de at tage bananer og trække appelsiner og rosiner fra. E: Nåh. Jamen så forstår jeg det. L: Det hedder jo noget forskelligt. E: Så tror jeg nok, at jeg har fattet det. E2: Så der bliver ikke lavet noget om. L: Der bliver ikke lavet noget om. E: Nåh. L: Det er ret smart, som det hænger sammen.

One gets a feeling of a teacher not on top of the mathematical content in this dialogue. Of course students may pose questions requiring a teacher to familiarize with the task or the context. But in this case the help offered seems to be a reference to some obscure rules learned by heart earlier in the class. The student in this dialogue declares that she understands, but the observer is left in serious doubt.

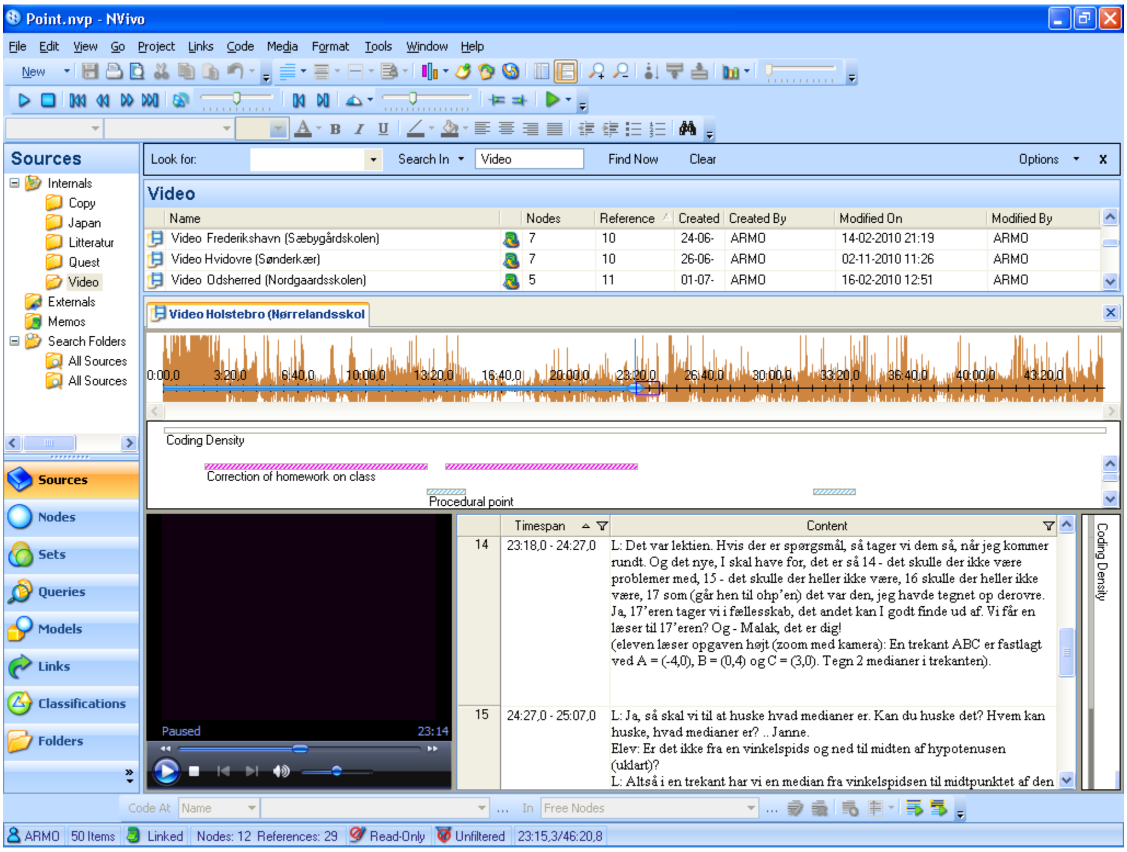
7 NVivo

7.1 Coding procedure

The NVivo software for analysis of qualitative data categorizes data as sources. In this study the sources are arranged in 50 cases each consisting of all data from one teacher linked together, i.e. a video recording with a transcript by timestamps, a filled out teacher questionnaire and a researcher memo. For many cases teaching material is also coded as a source.

The possible codes are organized in a tree structure, and families of codes are named tree nodes. A definition of each code decided upon is stored in the program and may be called upon during coding.

Each video and transcript is coded as described by the examples and statistics above. The software screen can show lists on sources or codes. In working with a source e.g. a video recording codes become visible on the screen as colored strips that can also show the simultaneous occurrence of more codes. When coding the screen will combine lists and an actual window showing the video and transcript aligned:



7.2 Possible extracts

The possible extracts are statistics:

- Casebook as in 8.1 below
- Source summaries
- Node summaries
- Charts of coding for a source
- Charts of coding for a node.

Statistics may be exported as Excel-files for reports. NVivo also shows code intensity as multiple codes overlap in single sources such as a transcribed video.

The casebook on the next page presents information from the questionnaires to be used later in cross-references when looking for patterns in points being made. Schools with a student numbers of between 400 and 600 are labeled medium sized school. Others either small or large.

8 Cumulated findings

8.1 Casebook to 50 lessons

Lesson	Students 2010	Region	Seniority	Content	Gender	Math a major subject	Teaching material
1	691	Central	0-4	Semester test	Female	Yes	Matematrix
2	415	Central	15 +	Coordinate system	Female	No	Faktor
3	305	Central	10-14	Algebra, area	Male	Yes	Faktor
4	229	Central	5-9	Fractions, skills	Male	Yes	Andet
5	944	Capital	0-4	Equations	Female	Yes	Faktor
6	620	Zealand	?	Negative numbers	Female	No	Faktor
7	480	South	15 +	Fractions	Male	Yes	Matematrix
8	491	North	0-4	Currency	Male	Yes	Matematiktak
9	92	North	0-4	Equations	Male	Yes	Faktor
10	269	Central	5-9	Probability	Male	No	Matematrix
11	444	South	15 +	Reduction	Male	No	Matematik i 8.
12	671	Central	15 +	Polar coordinates	Male	No	Faktor
13	486	Central	15 +	Perspective drawing	Male	Yes	Matematrix
14	858	Capital	0-4	Perspective drawing	Male	Yes	Matematrix
15	516	South	10-14	Arithmetic	Female	?	Matematiktak
16	415	South	0-4	Use of formulas	Female	Yes	Matematiktak
17	404	South	15 +	Arithmetic	Female	No	Faktor
18	584	North	15 +	Pythagoras	Male	Yes	Matematiktak
19	728	Capital	0-4	Arithmetic	Male	?	Faktor
20	778	Central	15 +	Area	Male	Yes	Faktor
21	381	Central	15 +	Powers	Female	Yes	Faktor
22	296	South	5-9	Statistics	Female	Yes	Faktor
23	308	Capital	5-9	Equations	Female	No	Faktor
24	467	Central	15 +	Area and time	Male	?	Sigma
25	426	North	15 +	Quadrangles	Female	?	Matematiktak
26	262	North	15 +	Problem solving	Male	No	Matematiktak
27	680	Zealand	5-9	Calculation hierarchy	Male	?	Matematiktak
28	656	Capital	5-9	Powers	Male	Yes	Faktor
29	379	South	0-4	Equations	Female	?	Matematiktak
30	379	South	15 +	IT Competence	Female	?	Matematrix
31	434	Central	5-9	Repetition	Male	No	Matematiktak
32	715	Capital	?	Powers	Male	?	Faktor
33	616	South	15 +	Percentages	Male	Yes	Matematrix
34	396	Central	15 +	Individual practice	Male	?	Matematrix
35	614	Central	5-9	Equation of line	Female	No	Matematiktak
36	416	Central	?	Speed	Male	?	?
37	563	Central	15 +	Pythagoras	Male	No	Matematiktak
38	465	Capital	15 +	Reduction	Female	Yes	Matematiktak
39	437	Zealand	15 +	Statistics	Female	?	Matematiktak
40	621	North	5-9	Decode information	Male	Yes	Matematik i 8.

41	495	Capital	?	Algebra, area	Female	?	Matematiktak
42	419	Capital	5-9	Volume – project	Female	Yes	Matematiktak
43	713	Zealand	15 +	Angles, compass	Male	Yes	Faktor
44	601	Capital	15 +	Probability	Male	Yes	Faktor
45	761	South	15 +	Equation of line	Female	?	Matematrix
46	731	Zealand	?	Statistics	Male	?	?
47	422	Capital	15 +	Decode information	Male	?	Matematrix
48	351	Zealand	15 +	Perspective drawing	Female	No	Matematik i 8.
49	584	Zealand	10-14	Equation of line	Female	Yes	Faktor
50	699	Central	10-14	IT competencies	Male	Yes	Flexmat

Region		Seniority		Gender		Major subject	
Capital region	11	0-4	8	Female	21	Yes	23
Zealand	7	5-9	10	Male	29	No	12
South region	10	10-14	4			Not known	15
Central region	16	15 +	23				
North region	6	Not known	5				

8.2 Lesson structure – phases and weight

Lessons graphically:

Lesson	Arrival, unpacking, messages	Teacher led correction of homework, return of tasks or a test	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
1	25:15 0:00-25:15	16:35 25:15-41:50	2:43 41:50-44:33	6:42 44:33-51:15		51:15D
2	2:22 0:00-2:22	5:11 2:22-7:35	6:05 7:35-13:40	35:58 13:40-49:38		49:38
3	3:35 0:00-3:35	5:15 3:35-8:50	14:14 8:50-23:04	23:18 23:04-46:22		46:22
4	2:00 0:00-2:00	11:32 2:00-2:32+ 12:18-23:18	17:05 2:32-12:18, 23:18-25:07, 28:35-34:05	15:25 25:07-28:35, 34:05-46:02	0:18 46:02-46:20	46:20
5	4:24 0:00-4:24	44:58 4:24-49:22				49:22D
6	1:50 0:00-1:50	17:25 1:50-19:15	22:32 19:15-41:47			41:47
7	2:10 0:00-2:10		22:10 2:10-24:24	19: 38 24:24-44:00		44:00
8	4:08 0:00-4:08	10:12 4:08-14:20		27:50 14:20-42:10	1:22 42:10-43:32	43:32
9	0:35 0:00-0:35	2:20 0:35-2:55		40:52 2:55-42:02, 42:20-44:05	0:18 42:02-42:20	44:05

10	6:47 0:00-6:47		8:02 6:47-14:49	31:12 14:49-46:01		46:01
11	3:09 0:00-3:09	19:46 3:09-22:55	3:05 22:55-26:00	22:21 26:00-48:21		48:21D
12	0:40 0:00-0:40			43:30 0:40-44:10	1:18 44:10-45:18	45:18
13	0:33 0:00-0:33	0:30 9:30-10:00	8:57 0:33-9:30	32:55 10:00-42:55	0:51 42:55-43:46	43:46
14			31:50 0:00-31:50	14:52 31:50-46:42		46:42
15	1:28 0:00-1:28	43:48 1:28-45:16				45:16D
16	0:35 0:00-0:35	1:46 0:35-2:21	7:34 2:21-9:55	35:50 9:55-45:45		45:45
17		25:50 0:00-25:50		17:57 25:50-43:47	1:39 43:47-45:26	45:26D
18			5:35 0:00-5:35	28:18 5:35-33:53	0:57 33:53-34:50	34:50
19	9:40 0:00-9:40	25:10 9:40-34:50		15:08 34:50-49:58		49:58D
20	3:03 0:00-3:03		17:52 3:03-20:55	25:23 20:55-46:18	0:35 46:18-46:53	46:53
20	2:10 0:00-2:10	5:50 2:10-8:00	19:50 8:00-27:50	17:38 27:50-45:28		45:28
22	0:55 0:00-0:55	1:45 0:55-2:40	5:15 2:40-7:55	36:06 7:55-44:01		44:01
23	0:35 0:00-0:35	6:02 0:35-6:37	26:38 6:37-10:33, 12:33-35:15	17:15 10:33-12:33, 35:15-50:30	0:16 50:30-50:36	50:36
24	4:00 0:00-4:00	18:10 4:00-22:10	25:14 22:10-47:34		2:16 47:34-49:50	49:50D
25	2:45 0:00-2:45		6:50 2:45-9:35	36:35 9:35-46:10	1:00 46:10-47:10	47:10
26	3:29 0:00-3:29	2:07 3:29-5:36		43:32 5:36-49:08		49:08D
27	4:16 0:00-4:16	2:14 4:16-6:30	1:38 6:30-8:08	37:00 8:08-45:08		45:08D
28	0:52 0:00-0:52		16:58 0:52-15:20, 32:55-35:25	21:08 15:20-32:55, 40:05-43:38	4:40 Break! 35:25-40:05	43:38D
29	2:45 0:00-2:45	6:29 2:45-9:14	6:04 9:14-14:13, 16:15-17:20	27:34 14:13-16:15, 17:20-42:52	2:59 42:52-45:51	45:51
30			0:30 0:00-0:30	26:50 0:30-27:20	(27:20- 28:38=1:18, T explains photographer)	28:38
31	1:50 0:00-1:50	1:08 1:50-2:58	7:14 2:58-10:12	35:08 10:12-45:20		45:20

32	2:59 0:00-2:59		45:56 2:59-48:55			48:55D
33	3:00 0:00-3:00		16:20 3:00-19:20	28:13 19:20-47:33		47:33D
34	0:44 0:00-0:44		3:56 0:44-4:40	44:49 4:40-49:29		49:29
35	4:00 0:00-4:00	5:00 4:00-6:00, 22:00-25:00	16:00 6:00-22:00	24:21 25:00-49:21		49:21
36	2:06 0:00-2:06	18:42 2:06-20:48		23:49 20:48-43:37		43:37
37	1:30 0:00-1:30	14:53 1:30-16:23	9:47 16:23-21:40, 32:10-36:40	19:10 21:40-32:10, 36:40-45:20	0:48 45:20-46:08	46:08D
38	0:45 0:00-0:45	24:12 0:45-3:00, 10:55-26:40, 42:30-48:42	9:30 3:00-10.55, 26:40-28:15	14:45 28:15-42:30	2:58 48:42-51:30	51:30D
39	3:57 0:00-3:57	38:41 3:57-42:38	0:50 42:38-43:28	6:22 43:28-49:50		49:50D
40	3:15 0:00-3:15		1:33 3:15-4:48	22:12 4:48-27:00	12:11 27:00-39:11	39:11
41	3:23 0:00-3:23	29:37 3:23-21:20, 25:20-37:00	12:02 37:00-49:02	4:00 21:20-25:20		49:02D
42	2:33 0:00-2.33		2:27 2.33-5:00	39:09 5:00-44:09	1:03 44:09-45:12	45:12
43	1:05 0:00-1:05	6:33 13:42-20:15	27:41 1:08-8:30, 20:15-40:34	5:12 8:30-13:42		40:34D
44	1:30 0:00-1:30	1:38 1:30-3:08	26:22 3:08-29:30	10:24 29:30-35:22, 37:06-41:38	5:06 35:22-37:06, 41:38-45:10	45:10
45	6:16 0:00-6:16			41:20 6:16-47:36		47:36
46	4:15 0:00-4:15		33:32 4:15-37:47	13:13 37:47-51:00		51:00D
47		9:04 0:00-9:04		50:03 9:04-59:07	0:41 59:07-59:48	59:48
48	2:50 0:00-2:50	16:00 2:50-18:50	5:10 18:50-24:00	21:04 24:00-45:04		45:04
49		2:19 1:52-4:11	1:52 0:00-1:52	41:45 4:11-45:56		45:56
50	1:16 0:00-1.16		6:15 1:16-7:33	37:16 7:33-44:49		44:49

1 Arrival, unpacking, messages	2 Teacher led correction of homework, return of tasks or a test	3 Teacher repeats or presents new content to whole class	4 Seatwork or investigations individual/groups	5 Common summing up or messages

Time in minutes	Ad 1) Arrival
0	6
]0; 1]	9
]1; 2]	8
]2; 3]	10
]3; 4]	9
]4; 5]	4
More than 5	4
I alt	50

Time in minutes	Ad 2) Homework
0	17
]0; 5]	10
]5; 10]	7
]10; 15]	3
]15; 20]	6
]20; 25]	1
More than 25	6
I alt	50

Time in minutes	Ad 3) New content
0	11
]0; 5]	9
]5; 10]	13
]10; 15]	2
]15; 20]	6
]20; 30]	6
More than 30	3
I alt	50

Time in minutes	Ad 4) Seatwork
0	5
]0; 5]	1
]5; 10]	3
]10; 15]	5
]15; 20]	6
]20; 30]	14
More than 30	16
I alt	50

Time in minutes	Ad 5) Summing up
0	27
]0; 1]	9
]1; 2]	5
]2; 3]	3
]3; 4]	0
]4; 5]	1
More than 5	5
I alt	50

8.3 Lesson codes – occurrence and relative weight

Regarding Arrival (example in section 6.2.1)
In 42 lessons this code was used during the start of the lesson. Most time spent for this was 7:25 at school no. 19 (0:00-7:25), where the teacher had a schoolyard inspection and misunderstood the date of recording.

Regarding Video information (example in section 6.2.2)
In 22 lessons information on video is given during the start of the lesson. Most time spent for this was 3:18 at school no. 3 (5:11-8:29).

Regarding Announcement of lesson content (example in section 6.3.1)
In 30 lessons announcement of lesson content was placed mainly in the first part of the lesson. Most time spent for this was 18:42 at school no. 3 (11:50-25:15 + 34:18-36:48 + 45:00-47:47).

Regarding Non-mathematical information (example in section 6.3.2)
In 12 lessons information on non-mathematical content seems spread through the lessons. Most time spent for this was 10:55 at school no. 43 (20:15-22:50 + 24:20-32:40).

Regarding Organization of lesson (example in section 6.3.3)
In 14 lessons this information on organization is often placed before individual work on new tasks. Most time spent for this was 5:15 at school no. 22 (2:40-7:55).

Regarding Collecting or returning homework (example in section 6.3.4)
In 10 lessons homework is collected or returned and always at the start of the lesson. Most time spent for this was 5:32 at school no. 24 (7:48-13:20).

Regarding Correction of homework (example in section 6.3.5)
Two lessons (5 and 39) are special cases, as most of these are spent correcting homework. In 20 other lessons homework is corrected during first half of the lesson. Most time spent for this was 41:54 at school no. 5 (5:44-25:40 + 27:26-49:24).

Regarding New homework (example in section 6.3.6)
One lesson (21) is a special case, as 24% of the lesson is spent explaining new homework. In 20 other lessons homework is given and mainly confirmed at the end of the lesson. Most time spent for this was 11:00 at school no. 21 (8:00-10:30 + 19:20-27:50).

Regarding Overview (example in section 6.3.7)
In a total of 13 lessons of the 50 an excerpt has been coded for *Overview*.
Two lessons are special cases (20 and 33), as 28% and 26% respectively of these lessons are coded as overview. Most time spent for this was 13:09 at school no. 20 (7:46-20:55).

Regarding Summing up (example in section 6.3.8)
Only 4 of the 50 mathematics lessons ended this way by the teacher summing up the outcome. Most time spent for this was 1:44 at school no. 44 (35:22-37:06).
The summing up is considered an extremely important part of a mathematics lesson in the Japanese lesson study tradition. I shall later return to the reasons for this and the response from Danish teachers (sections 11.2 and 13.3).

Regarding Teacher conceptual points (example in section 6.4.1)
A total of 7 lessons of the 50 recorded contained conceptual points given by the teacher to the whole class. This is an astonishing low number considering the importance of concepts to mathematics. I will return to this worry later. Most time spent for this was 3:41 at school no. 46 (23:47-27:28).

Regarding Teacher procedural points (see example in section 6.4.2)
A total of 18 lessons of the 50 recorded contained *Procedural points* made by the teacher to the whole class. In other words this type of point is the most frequently made by Danish mathematics teachers. Most time spent for this was 7:09 at school no. 7 (17:15-24:24).

Regarding Teacher result points (see example in section 6.4.3)
A total of only 3 lessons of the 50 contained result points made by the teacher to the whole class. Most time spent for this was 2:20 at school no. 21 (2:10-4:30).

Regarding Teacher interpretation points (see example in section 6.4.4)
A total of 13 lessons of the 50 recorded contained interpretation points made by the teacher to the whole class. Most time spent for this was 6:30 at school no. 7 (30:10-36:40).

Regarding Teacher → student conceptual point (see example in section 6.5.1)
A total of (only) 1 lesson was coded with conceptual points by the teacher to groups or individual students. The time spent for this was 5:10 at school no. 26 (19:00-24:10).

Regarding Teacher → student procedural point (see example in section 6.5.2)
A total of 12 lessons were coded with procedural points by the teacher to groups or individual students. Most time spent for this was 7:10 at school no. 14 (36:46-43:56).

Regarding Teacher → student result point (see example in section 6.5.3)
No lessons were coded with result points by the teacher to groups or individual students.

Regarding Teacher → student interpretation point (see example in section 6.5.4)
A total of 7 lessons were coded with interpretation points by the teacher to groups or individual students. Most time spent for this was 5:10 at school no. 26 (19:00-24:10).

Regarding Student conceptual point (see example in section 6.6.1)
A total of 12 lessons contained conceptual points made by students. Most time spent for this was 6:42 at school no. 12 (36:18-43:00).

Regarding Student procedural point (see example in section 6.6.2)
A total of 14 lessons contained procedural points made by students. Most time spent for this was 15:21 at school no. 46 (8:00-20:50 + 23:47-26:18).

Regarding Student result point (see example in section 6.6.3)
A total of 4 lessons only contained result points made by students. Most time spent for this was 1:41 at school no. 28 (9:00-10:41).

Regarding Student interpretation point (see example in section 6.6.4)
A total of 9 lessons contained interpretation points made by students. Most time spent for this was 10:23 at school no. 39 (3:57-14:20).

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21 lessons didn’t show any point as they are defined by the teacher to the whole class or to students in seatwork. But some lessons may still be characterized by a substantial number of elicitation questions from the teacher inviting students to articulate one or more mathematical points.

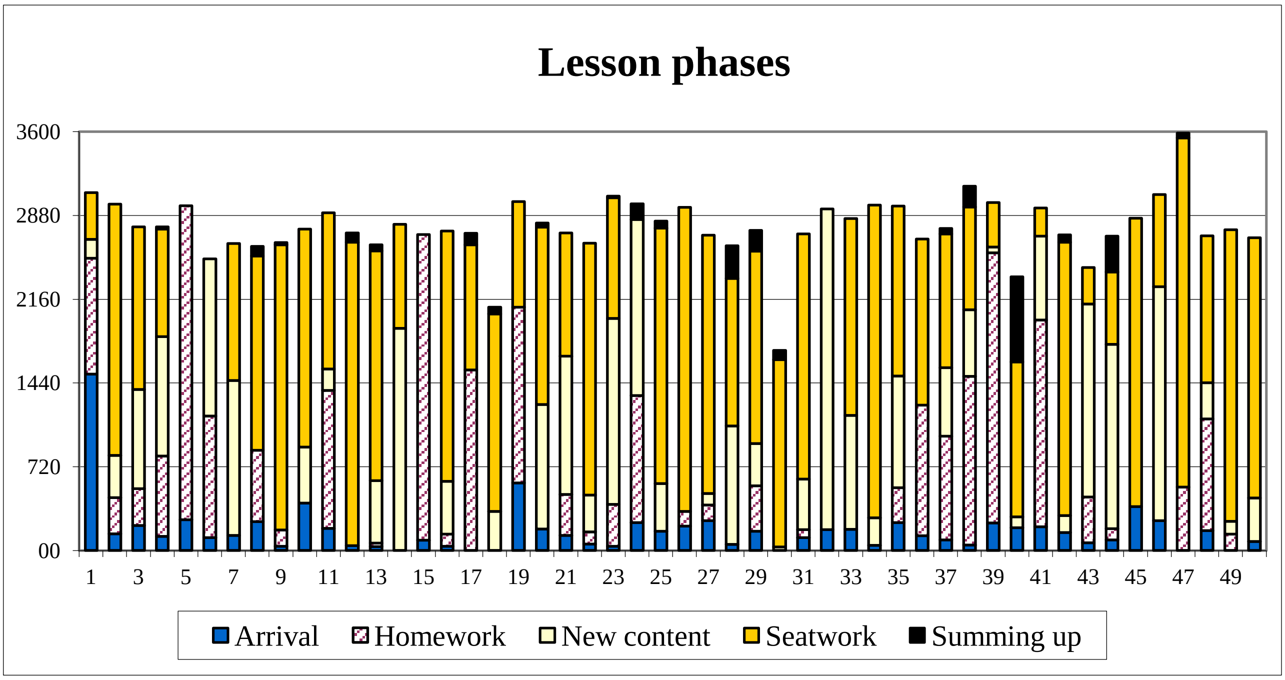
Regarding Elicitation (see example in section 6.7.1)
A total of 38 lessons contained dialogues coded as elicitation. Most time spent for this was 18:27 at school no. 44 (3:50-10:36 + 14:10-23:40 + 25:50-28:01).

Regarding Hint (see example in section 6.7.2)
A total of 11 lessons were coded with hints. This might perfectly well be seen differently by another coder as the distinction between elicitation and hint is somewhat subtle. Most time spent for this was 15:33 at school no. 18 (17:22-33:05).

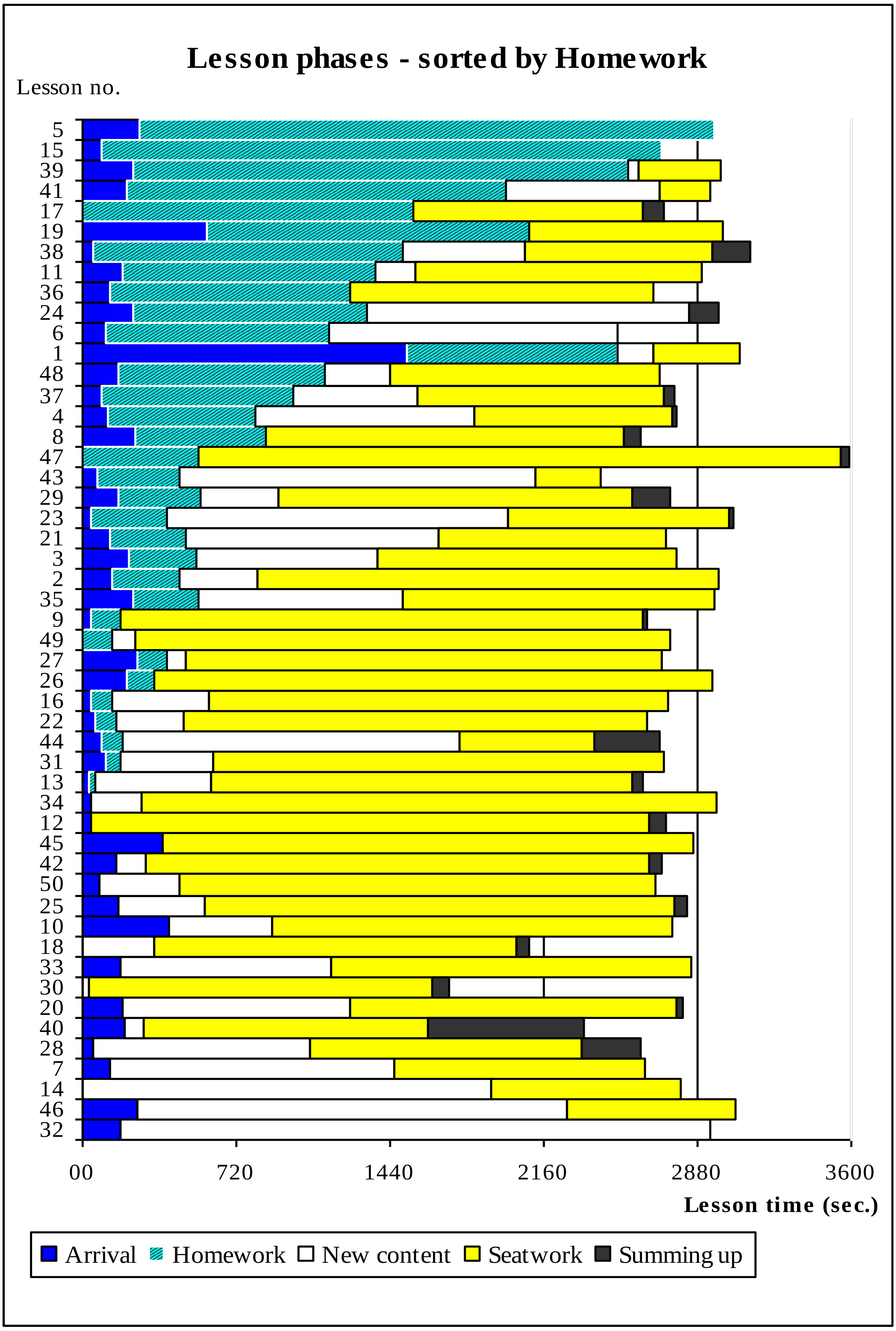
Regarding Missed point (see example in section 6.7.3)
A total 16 lessons were coded for a missed point. In one rather extreme case (lesson 10), 14:31 (32% of the lesson length) is coded for missed points (7:30-11:35 + 14:28-14:49 + 16:55-27:00).

8.4 Lesson phases graphically

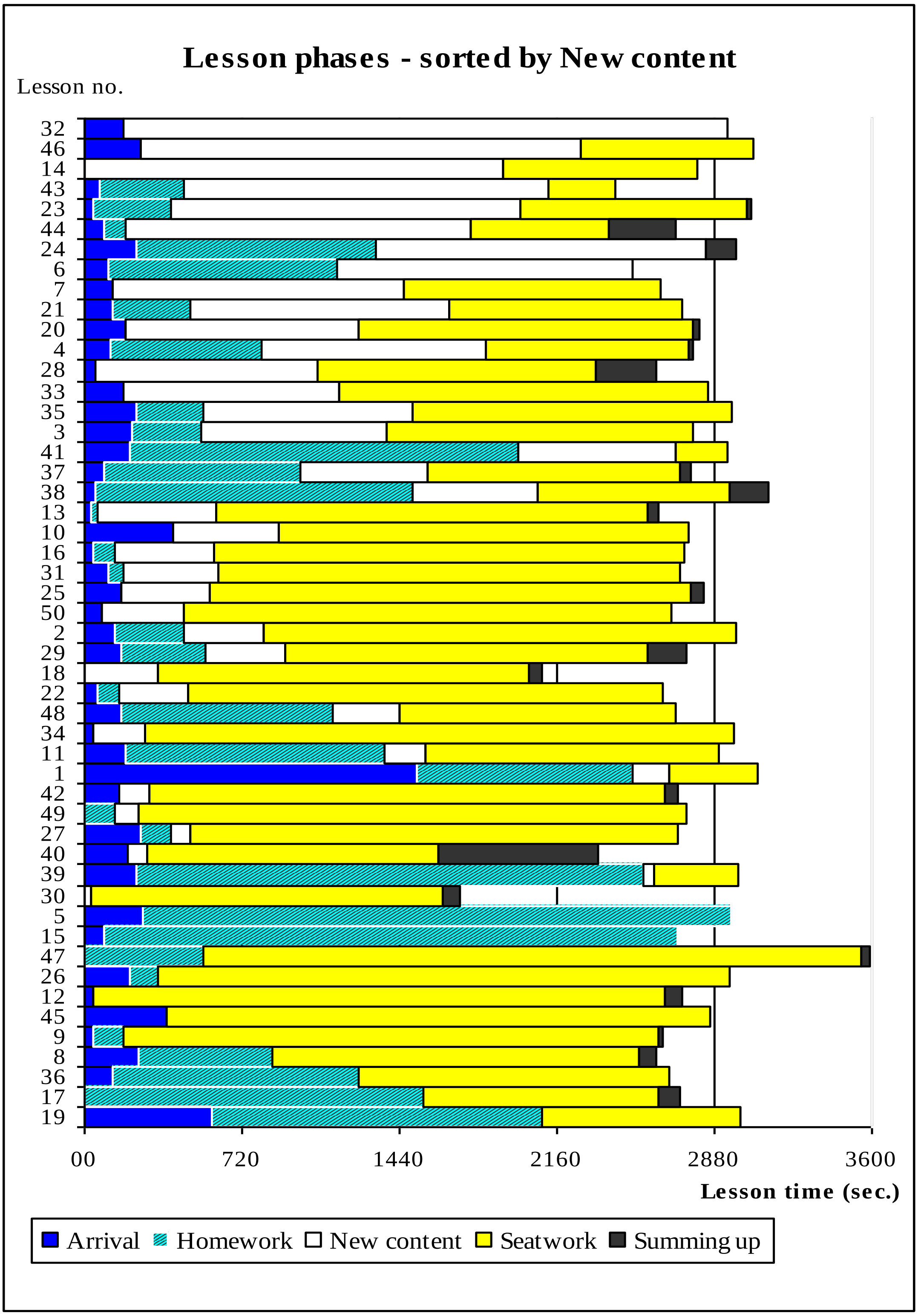
There are more ways to illustrate the phases graphically. Below all lessons are sorted by number and the 5 phases already described above (8.2) stacked vertically.



For another overview phases may be sorted. When lessons are sorted according to length of *Homework* phase as below, you immediately notice the large variation in length of this phase, but also the many lessons, which don't relate to homework:



Lessons may also be sorted according to length of the *New content* phase. Below you may notice a large variation in length of this phase, and again the many lessons, where teachers don't present new content:



8.5 Number of references

NVivo facilitates an overview by providing lists of references. Below is an overview of the reference distribution of codes on all nodes. In 10 lessons the code for *Collecting or returning homework* has e.g. been used for a total of 13 references. This code is therefore applied to more than one lesson sequence in at least one of the 50 lessons (in fact this was done in two).

Codes	**No. of sources**	**No. of references**
Start up		
Arrival	42	55
Video information	22	22
Information		
Announcement of lesson content	30	84
Non-mathematical information	12	26
Organization of lesson	14	19
Homework		
Collecting or returning homework	10	13
Correction of homework from class	22	105
New homework	21	34
Instants addressing points		
Overview (repetition)	13	23
Summing up	4	4
Points by teacher		
Conceptual point	7	12
Procedural point	18	38
Result point	3	7
Interpretation point	13	21
Points during seatwork		
Conceptual point in seatwork	1	1
Procedural point in seatwork	12	18
Result point in seatwork	0	0
Interpretation point in seatwork	7	10
Points by student		
Student conceptual point	12	16
Student procedural point	14	20
Student result point	4	7
Student interpretation point	9	11
Instants addressing points		
Elicitation	38	110
Hint	11	17
Missed point	16	31

Below is another overview of the reference distribution for the 50 lessons transcripts. The table shows the number of different nodes being coded for each case. And the relatively high number of references show, that some codes are used several times for that lesson.

A high frequency (large number of references) may be regarded as a quantitative measure of quality. But as some codes are for arrival and non-mathematical information, this would be a too hasty conclusion.

Lesson	Nodes	References
1	8	23
2	12	29
3	8	22
4	9	23
5	6	17
6	5	15
7	11	24
8	4	6
9	6	9
10	12	25
11	7	12
12F	7	7
13F	11	19
14	9	17
15	6	7
16	3	6
17	1	1
18	8	14
19	8	19
20F	14	32
21F	11	25
22	10	22
23	7	10
24	8	15
25	6	16

Lesson	Nodes	References
26	10	15
27	8	9
28	11	20
29	8	15
30	2	3
31	3	4
32	6	12
33	6	13
34	3	8
35F	12	19
36	3	6
37	5	13
38	9	14
39	10	21
40	7	10
41	7	12
42	**8**	**17**
43	5	15
44	8	21
45	3	4
46	7	10
47	3	3
48	5	11
49	5	8
50F	4	6

For each case a *Coding Summary Report* will link the actual nodes with names and coverage.

You may even decide to have a printout of the corresponding transcript excerpts. In the case of lesson no. 42 above the report finds 17 references. They are distributed this way:

An example: Lesson 42 Medium size school in Capital region Female teacher, seniority 5-9 years.

Nodes	References	Coverage
Announcement of lesson content	5	13,4 %
Organization of lesson	3	6,6 %
Elicitation	2	19,2 %
Hint	2	18,3 %
Student interpretation point	2	11,4 %
Student procedural point	1	8,4 %
Procedural point in seatwork	1	3,8 %
Arrival	1	3,4 %

And this report shows the two references on *Elicitation*:

Reference 1, coverage 15,9 %	Timestamp 24:00-31:10
S: Is this a good enough sketch? (S hands T his paper) T: Yes, but a sketch – it is also seen from where and from where? S: In front of, from front end and ... T: What is the difference between in front of and from front end? S: Well, excuse me! ... Top down! T: That's right and the last one? (Occasionally T has a conversation with another student to find an approximate solution on a calculator) S: How was it again from the front ... T (borrows a model and indicates several times on it): From the front, right? From the side. Top down! S: So only this should be visible on the drawing? Ok!	E: Er det her en god nok arbejdstegning? (E rækker L sit papir) L: Jo, men en arbejdstegning – det er jo også set fra hvad og fra hvad? E: Foran, forfra og ... L: Hvad er forskellen på foran og forfra? E: Nå, undskyld! ... Oppefra og ned! L: Det er rigtigt, og hvad er det sidste? (Indimellem har L en samtale med anden elev om at tilnærme en løsning på lommeregner) E: Hvordan var det med forfra ... L (låner en figur og peger hver gang på den): Forfra, ikke? Fra siden. Oppefra! E: Så det er kun lige den her, der skal ses på tegningen? Ok!
Reference 2, coverage 3,3 %	**Timestamp 38:50-40:20**
T: You decide! Think of a Toblerone [Yes]. A Toblerone's cylindrical, it's just one – no, it is not a cylinder (T shows a triangular base	L: Det bestemmer du! Tænk på en Toblerone [Ja]. En Toblerone er jo cylinderformet, det er bare sådan én – nej, det er ikke en cylinder

with her hands). ... But YOU decide the way your model should look ... do you understand what I mean? [No]. Now that you have your triangle down here on the end, right? Which you stand on. Don't you then ...	(L viser en trekantet grundflade med hænderne). ... men DU bestemmer jo, hvordan din skal se ud ... forstår du, hvad jeg mener? [Nej]. Når nu du har din trekant her nede i enden, ikke? Som du står på. Kan du så ikke ....

Many references

Some lessons contain more references to the <u>same</u> type of points, but 1 is the typical number:

Teacher→ class points			Teacher → student points			Student points		
Conceptual point			**Conceptual point**			**Conceptual point**		
Lesson	References	Coverage	Lesson	References	Coverage	Lesson	References	Coverage
7	1	10.9%	26	1	10.5%	12F	1	14.8%
46	3	7.2%				41	1	11.7%
13F	3	4.1%				44	3	10.5%
2F	3	3.6%				32	1	6.1%
35F	1	1.9%				1	1	5.9%
19	1	0.9%				2F	3	3.6%
50F	1	0.7%				28	1	2.8%
						25	1	2.6%
						3	1	2.4%
						19	1	1.9%
						23	1	0.6%
						35F	1	0.4%
Procedural point			**Procedural point**			**Procedural point**		
Lesson	References	Coverage	Lesson	References	Coverage	Lesson	References	Coverage
20F	3	16.6%	14	1	15.3%	46	3	30.1%
7	2	16.2%	18	2	8.5%	39	1	20.9%
35F	4	15.0%	9	2	9.2%	20F	3	11.4%
13F	4	13.3%	13F	1	7.6%	42	1	8.4%
38	1	12.1%	50F	1	5.7%	32	1	7.1%
21F	2	11.7%	4	3	4.9%	28	2	6.7%
29	2	10.9%	22	2	4.4%	44	1	4.8%
14	1	10.9%	2F	1	3.8%	27	1	3.6%
6	2	9.8%	42	1	3.8%	33	1	3.3%
5	2	9.5%	20F	1	3.2%	4	2	3.2%
39	1	9.2%	49	1	2.8%	49	1	3.1%
50F	3	8.8%	12F	1	2.0%	10	1	3.1%
2F	3	8.5%				35F	1	2.5%
18	1	8.1%				38	1	2.3%
10	1	6.3%						
46	1	5.8%						
28	1	5.6%						
4	1	1.5%						
Result point			**Result point**			**Result point**		
Lesson	References	Coverage	Lesson	References	Coverage	Lesson	References	Coverage
20F	2	6.0%	None			20F	3	9.7%
21F	3	5.1%				28	1	3.9%
28	1	4.7%				3	2	3.9%
						12F	1	2.0%

Teacher→ class points			Teacher → student points			Student points		
Interpretation point			Interpretation point			Interpretation point		
Lesson	References	Coverage	Lesson	References	Coverage	Lesson	References	Coverage
7	4	14.7%	26	1	10.5%	39	1	20.8%
14	2	10.7%	22	3	7.6%	12F	1	14.8%
35F	4	8.6%	2F	1	3.8%	24	2	11.9%
41	1	7.7%	11	1	3.3%	42	2	11.4%
20F	2	5.9%	10	1	3.1%	41	1	7.7%
21F	1	4.3%	14	1	2.0%	7	1	3.6%
13F	1	4.2%	13F	1	1.4%	14	1	3.3%
23	1	3.3%				22	1	1.7%
38	2	3.3%				35F	1	0.4%
46	1	2.3%						
2F	1	1.6%						
40	1	1.6%						
29	1	0.9%						

The overview above indicates that procedural points by far are the most common in Danish grade 8 mathematics teaching. The main reason for making points seems to be to instruct students in problem solving procedures. This is in possible accordance with a process-oriented curriculum, stressing the importance of students' knowing not only how to handle a mathematical problem, but also to be able to explain their working.

Also striking is the complete absence of result points in the teachers' communication with individual students or groups. Even in whole class communication only three teachers presented mathematical results. And two of these lessons were among the three where students were articulating result points.

8.6 Frequencies of points made by teacher or students

Several other statistics may be drawn from the data once the transcripts have been coded. This can be done on the whole material or lesson-wise. Below I will focus on the “how and why” of the points as asked in RQ1.

Teacher to class points in 22 lessons in % of lesson length and number of references:

Lesson	Mathematics content	*Conceptual*	*Procedural*	*Result*	*Inter-pretation*	*Adjusted total*	T →class points
7	**Fractions**	**10.9**	**16.2**		**14.7**	**30.9**	**7**
20F	**Area**		**16.6**	**6.0**	**5.9**	**28.5**	**7**
35F	**Equation of line**	**1.9**	**15.0**		**8.6**	**20.4**	**2**
21F	**Powers**		**11.7**	**5.1**	**4.3**	**19.2**	**4**
13F	**Persp. drawing**	**4.1**	**13.3**		**4.2**	**17.4**	**8**
38	**Reduction**		**12.1**		**3.3**	**15.4**	**3**
14	**Persp. drawing**		**10.9**		**10.7**	**13.9**	**3**
46	**Statistics**	**7.2**	**5.8**		**2.3**	**13.0**	**5**
2F	**Coord. system**	**3.6**	**8.5**		**1.6**	**12.1**	**6**
29	**Equations**		**10.9**		**0.9**	**11.8**	**3**
28	**Powers**		**5.6**	**4.7**		**10.3**	**2**
6	Negative numbers		9.8			9.8	2
5	Equations		9.5			9.5	2
39	Statistics		9.2			9.2	1
50F	IT competencies	0.7	8.8			8.8	2
18	Pythagoras		8.1			8.1	1
41	Algebra, area				7.7	7.7	1
10	Probability		6.3			6.3	1
23	Equations				3.3	3.3	1
40	Decode information				1.6	1.6	1
4	Fractions, skills		1.5			1.5	1
19	Arithmetic	0.9				0.9	1
1	Semester test						
3	Algebra, area						
8	Currency						
9	Equations						
11	Reduction						
12F	Polar coordinates						
15	Arithmetic						
16	Use of formulas						
17	Arithmetic						
22	Statistics						
24	Area and time						
25	Quadrangles						
26	Problem solving						
27	Calcul. hierarchy						
30	IT Competence						
31	Repetition						

32	Powers						
33	Percentages						
34	Individual practice						
36	Speed						
37	Pythagoras						
42	Volume – project						
43	Angles, compass						
44	Probability						
45	Equation of line						
47	Decode information						
48	Persp. drawing						
49	Equation of line						

The statistics above are referring to *whole class teaching*. F indicates the group of teachers, who later were asked and accepted to be in a focus group for a seminar and subsequent video recordings. The column to the right indicates the number of references from the actual lessons.

- 11 lessons (bold letters) have more than 10% of the lesson length characterized by points being made by the teacher in communication to the whole class.
- 28 lessons (shaded lines) have no common "Teacher to class" points at all.

Different types of points may appear simultaneously, in such cases adequate adjustments to percentages have been done to show the ratio of lessons, where points are showing.
Mathematics teachers also present *points to individuals and groups*. Teachers' points to individuals and groups during seatwork or investigations have been dealt with similarly in this research. *Points expressed by students* in response to or in dialogue with the teacher have also been coded in this way.
The following two tables show the cumulated results.

Teachers' points to groups or individual students in 15 lessons in % of lesson length and number of references:

Lesson	**Mathematics content**	***Conceptual***	***Procedural***	***Result***	***Inter-pretation***	***Adjusted total***	**T → stu. points**
7	Fractions						
20F	Area		3.2			3.2	1
35F	Equation of line						
21F	Powers						
13F	Persp. drawing		7.6		1.4	9.0	2
38	Reduction						
14	**Persp. drawing**		**15.3**		**2.0**	**17.3**	**2**
46	Statistics						
2F	Coord. system		3.8		3.8	3.8	2
29	Equations						

28	Powers						
6	Negative numbers						
5	Equations						
39	Statistics						
50F	IT competencies		5.7			5.7	1
18	Pythagoras		8.5			8.5	2
41	Algebra, area						
10	Probability				3.1	3.1	1
23	Equations						
40	Decode information						
4	Fractions, skills		4.9			4.9	3
19	Arithmetic						
1	Semester test						
3	Algebra, area						
8	Currency						
9	Equations		9,2			9.2	2
11	Reduction				3.3	3.3	1
12F	Polar coordinates		2.0			2,0	1
15	Arithmetic						
16	Use of formulas						
17	Arithmetic						
22	**Statistics**		**4.4**		**7.6**	**10.1**	**5**
24	Area and time						
25	Quadrangles						
26	**Problem solving**	**10.5**			**10.5**	**10.5**	1
27	Calcul. hierarchy						
30	IT Competence						
31	Repetition						
32	Powers						
33	Percentages						
34	Individual practice						
36	Speed						
37	Pythagoras						
42	Volume – project		3.8			3.8	1
43	Angles, compass						
44	Probability						
45	Equation of line						
47	Decode information						
48	Persp. drawing						
49	Equation of line		2.8			2.8	1

- 21 lessons (shaded lines) are completely without any points made by the teacher.
- 35 lessons are without any teacher points being made during guidance of individual students or groups.
- 3 lessons (bold letters) have more than 10% of lesson length characterized by points being made by the teacher in communication with groups or single students.

Some teachers plan and orchestrate lessons involving students more actively in communication than others. Therefore student articulated points were also coded. The cumulated statistics are in the table below, where totals indicate student points in % of lesson time adjusted in lessons where several types appear simultaneously:

Student points in 26 lessons in % of lesson length and number of references:

Lesson	**Mathematics content**	***Conceptual***	***Procedural***	***Result***	***Interpretation***	***Adjusted total***	Student points
7	Fractions				3.6	3.6	1
20F	**Area**		**11.4**	**9.7**		**11.4**	6
35F	Equation of line	0.4	2.5		0.4	2.8	3
21F	Powers						
13F	Persp. drawing						
38	Reduction		2.3			2.3	1
14	Persp. drawing				3.3	3.3	1
46	**Statistics**		**30.1**			**30.1**	3
2F	Coord. system	3.6				3.6	3
29	Equations						
28	**Powers**	**2.8**	**6.7**	**3.9**		**13.4**	4
6	Negative numbers						
5	Equations						
39	**Statistics**		**20.8**		**20.8**	**20.8**	2
50F	IT competencies						
18	Pythagoras						
41	**Algebra, area**	**11.7**			**7.7**	**19.4**	2
10	Probability		3.1			3.1	1
23	Equations	0.6				0.6	1
40	Decode information						
4	Fractions, skills		3.2			3.2	2
19	Arithmetic	1.9				1.9	1
1	Semester test	5.9				5.9	1
3	Algebra, area	2.4		3.9		6.3	3
8	Currency						
9	Equations						
11	Reduction						
12F	**Polar coordinates**	**14.8**		**2.0**	**14.8**	**16.8**	3
15	Arithmetic						
16	Use of formulas						
17	Arithmetic						
22	Statistics				1.7	1.7	1
24	**Area and time**				**11.9**	**11.9**	2
25	Quadrangles	2.6				2.6	1
26	Problem solving						
27	Calcul. hierarchy		3.6			3.6	1
30	IT Competence						
31	Repetition						
32	**Powers**	**6.1**	**7.1**			**13.2**	2

33	Percentages		3.3			3.3	1
34	Individual practice						
36	Speed						
37	Pythagoras						
42	**Volume – project**		**8.4**		**11.4**	**11.4**	3
43	Angles, compass						
44	**Probability**	**10.5**	**4.8**			**15.3**	4
45	Equation of line						
47	Decode information						
48	Persp. drawing						
49	Equation of line		3.1			3.1	1

- 10 lessons (bold letters) have points being made by students in communication to or with the mathematics teacher for more than 10% of the lesson length.
- 24 lessons are without any student points being made in communication with the teacher.
- 13 lessons (shaded) don't contain any points at all, neither from the mathematics teacher nor the students.

8.7 Cross tabulations

As the casebook contains information on school size, geographical region, teacher gender, teacher seniority and whether mathematics is one of the teacher's major subjects a few cross tabulations are made to discover possible correlations.
The average of articulated points in teachers' communication to the class, groups or individual students (in % of total lesson length) has been calculated for lessons matching chosen parameters. All points, students' include, are added and sums adjusted for overlaps (Appendix C).
This should only indicate patterns for further analysis, as the number of lessons is relatively small and several lessons do not contribute points.
Results were:

School size	Points in % of lesson length	Lessons	L. containing points
Small (< 400)	7.5	12	9 ≈ 75 %
Medium	9.6	20	12 ≈ 60 %
Large (> 600)	14.2	18	16 ≈ 89 %

This suggests a possible correlation between school size and the occurrence of points in % of lesson length. At large schools with more than 600 students there was more classroom communication that contained articulated points, and only 2 lessons out of 18 at these large schools were without points.
There may be reasons for such a pattern, not *dependent* on school size. E.g. more teachers at large schools may have mathematics as a major subject, whereas single teachers at smaller schools may be hired to cover a broader range of subjects. Or as

Ferguson and Brown describe it (Ferguson and Brown, 2000, p. 149), “*student quality in a school or district can affect which teachers choose to apply there.*” Also locations in larger cities, where schools are often larger, may attract more teacher candidates, thus increasing the standards of those appointed.

Gender	Points in % of lesson length	Lessons	L. containing points
Female	9.3	21	15 ≈ 71 %
Male	11.8	29	22 ≈ 76 %
Average	10.8	50	37 ≈ 74 %

The difference between lessons taught by female and male teachers with respect to point formulation is presumably so small (2.5 percent points of lessons length), that no correlation should be presumed. One might be interested in differences here according to seniority, but the data are too few for that. One young male teacher (lesson 14) includes an especially high number of points in his lesson 14: 32.5 %, and this influences the gender statistics quite a lot.
Without his lesson, the male statistics would decrease to 11.1 %.

Seniority	Points in % of lesson length	Lessons	L. containing points
0-4 years	9.2	8	6 ≈ 75 %
5-14 years	8.6	14	12 ≈ 86 %
15+ years	11.5	23	15 ≈ 65 %
No information	16.1	5	4 ≈ 80 %

The difference between lessons by novice and experienced teachers with respect to point articulation is also so little (2.3 percent points), that no correlation should be presumed. But the data suggest, that lessons by teachers with medium seniority (5-14 years) most often (86 %) contain points, while teachers with more experience and presumably an older educational background only articulate or elicit points in 65 % of their lessons.
The high score among the lessons lacking information on seniority is due to one lesson (no. 46) containing points in 38.2 % of the lesson length. This teacher was experienced but did not indicate how experienced. In total five lessons lack information on teacher seniority due to five teachers not returning questionnaires as described in section 5.6.2.
One hypothesis not investigated is that more senior teachers – with their experience of what works and what doesn’t work in the classroom – tend to put a priority on procedural points, if any.

Major subject	Points in % of lesson length	Lessons	L. containing points

Yes	13.6	23	20 ≈ 87 %
No	7.7	12	8 ≈ 67 %
Information missing	8.9	15	9 ≈ 60 %

The difference between lessons by teachers having mathematics as one of their major subjects and those who have not with respect to point articulation seems so distinct (5.9 percent point of lesson length), that correlation may be presumed. I have no way of finding the educational background of teachers leaving this rubric information missing in the questionnaire. 10 of the 15 in fact did return the questionnaire, but left out an answer to this question.

One other deciding parameter might be the textbooks used in mathematics teaching. It is often assumed, and also confirmed by the questionnaires in this research, that Danish mathematics teachers use textbooks almost all the time. Therefore it would be sensible to assume that the points being articulated by the teacher in a common class communication – or the points driving the mathematics teaching in this respect – are points implicit or explicit in the mathematics textbook.
The table below shows the actual textbook choice compared with registration of the points in % of actual lesson length:

Lesson	**Mathematics content**	T → class	T → student	Student	Teaching material
7	**Fractions**	**30.9**		3.6	Matematrix
20F	**Area**	**28.5**	3.2	**11.4**	Faktor
35F	**Equation of line**	**20.4**		2.8	Matematiktak
21F	**Powers**	**19.2**			Faktor
13F	**Perspective drawing**	**17.4**	9.0		Matematrix
38	**Reduction**	**15.4**		2.3	Matematiktak
14	**Perspective drawing**	**13.9**	**17.3**	3.3	Matematrix
46	**Statistics**	**13.0**		**30.1**	?
2F	**Coordinate system**	**12.1**	3.8	3.6	Faktor
29	**Equations**	**11.8**			Matematiktak
28	**Powers**	**10.3**		**13.4**	Faktor
6	Negative numbers	**9.8**			Faktor
5	Equations	**9.5**			Faktor
39	Statistics	**9.2**		**20.8**	Matematiktak
50F	IT competencies	**8.8**	5.7		Flexmat
18	Pythagoras	**8.1**	8.5		Matematiktak
41	Algebra, area	**7.7**		**19.4**	Matematiktak
10	Probability	**6.3**	3.1	3.1	Matematrix
23	Equations	**3.3**		0.6	Faktor
40	Decode information	**1.6**			Matematik i 8.
4	Fractions, skills	**1.5**		3.2	Andet

19	Arithmetic	**0.9**	4.9	1,9	Faktor
1	Semester test			5.9	Matematrix
3	Algebra, area			6.3	Faktor
8	Currency				Matematiktak
9	Equations				Faktor
11	Reduction		9.2		Matematik i 8.
12F	Polar coordinates		3.3	**16.8**	Faktor
15	Arithmetic		2.0		Matematiktak
16	Use of formulas				Matematiktak
17	Arithmetic				Faktor
22	Statistics		10.1	1.7	Faktor
24	Area and time			**11.9**	Sigma
25	Quadrangles			2.6	Matematiktak
26	Problem solving				Matematiktak
27	Calculation hierarchy		**10.5**	3.6	Matematiktak
30	IT Competence (Jing)				Matematrix
31	Repetition				Matematiktak
32	Powers			**13.2**	Faktor
33	Percentages			3.3	Matematrix
34	Individual practice				Matematrix
36	Speed				?
37	Pythagoras				Matematiktak
42	Volume – project		3,8	**11,4**	Matematiktak
43	Angles, compass				Faktor
44	Probability			**15.3**	Faktor
45	Equation of line				Matematrix
47	Decode information				Matematrix
48	Perspective drawing				Matematik i 8.
49	Equation of line		2.8	3.1	Faktor

Three series of mathematics textbooks are predominant:

- Faktor (1. edition 1992)
- Matematiktak (1. edition 1996)
- Matematrix (1. edition 2001)

The newest of these series most frequently refers to the current curriculum and the corresponding concept of mathematical competencies. This is explicitly stated in the teacher guide and also confirmed by reading the textbook. Older textbooks tend to cope with priorities and demands from curricula with new editions.

These are some variations in the % of communication containing points depending on the textbook. Cumulated statistics are:

Textbook	No. of lessons	Average T → class points in % of lesson	Average T → student points n % of lesson	Average student points in % of lesson	Total
Faktor	17	5.9	1.8	4.6	12.2
Matematiktak	15	4.8	1.7	4.2	10.7
Matematrix	10	6.9	2.9	1.9	11.7
Other	8	2.8	1.7	5.0	9.4

The 13 lessons showing *no* points of any kind at all are represented like this:

Textbook	No. of lessons
Faktor	2
Matematiktak	5
Matematrix	4
Other	2

It is a bit surprising, that four of the 10 users of the most modern textbook, Matematrix do not implement the strongly emphasized priorities of the textbook in their teaching practice to a greater extent. The responsibility is, however, the individual teacher's, and I do not consider the found differences significant.

8.8 Shortcomings?

Although it was very exciting to register and document the extent to which *points* could be said to drive the teaching – or at least were present – it was my (skeptical) guess before analysis of the 50 sets of Danish data that points would not be identifiable to a great extent.
Though many mathematics teachers elicitate in their communication with classes and individual students, there are not as many points in the mathematics teaching observed, as one might have hoped for, when convinced of their importance in mathematics teaching and learning. 56% of the mathematics lessons in this research were without any teacher to class points and 42% of the lessons did not have any teacher points at all.
48% of the lessons did not have any student points in communication with the teacher. The points observed are typically quite local to an instance in the lesson and could seldom be observed as a "driver" for one complete lesson.

Before the research I feared that Danish mathematics teachers may be missing the point. And the analysis shows this may in fact be so:

1. Goals are often *invisible* to observers in typical grade 8 mathematics lessons.
2. Most mathematics teachers are *not preparing* a detailed lesson plan.

3. Many mathematics teachers seem to look upon lessons as "*stand alone" lessons*. Lessons are seldom presented to the students as *fitting in a series* of units striving for the same explicit goal.
4. Many teachers do not organize lessons and take roles to cope with *students' different needs*. All students are most often given the same closed tasks, and the teachers are not prioritizing common communication in the classroom.
5. Most mathematics teachers do not consciously and regularly use *differential assessment tools* in class.

It is not possible to confirm or refute these findings based on a coding of points alone. The questionnaire and other sampled material like lesson plans and my memos after the visit are needed for the subsequent review here.

Re 1)
When asked in the questionnaire for the most important thing, the teacher wants the students to learn very few teachers write more than a series of keywords or a single sentence. The mathematical goal of most mathematics lessons is expressed as a mathematical concept or topic, e.g.: Fractions, Coordinates or two equations with two unknowns.
The lesson goal is most often formulated to the students in terms of activity lists, e.g. a number of tasks to be solved. And that goal is of course very clear to students. But unfortunately it has no reference to *Common Goals 2009* or mathematical competencies.

Re 2)
The questionnaire also suggests teachers should classify their lesson plan as rather general, fairly detailed or very specific. Most teachers tick off that they produced a rather general plan. Everyone was prompted, but very few teachers enclose a more detailed lesson plan. Most mathematics teachers do not seem to prepare any written at least moderately detailed lesson plan. The few plans shown and given to the researcher are mostly activity lists and occasionally self-designed assignments.

Re 3)
It is customary to indicate in the questionnaire that the lesson builds on the previous and/or is continued in the next lesson. 18 of the recorded lessons were one of two in a double lesson (usually as the first part). This was not a guarantee of coherence as often the two parts of the lesson had different programs, perhaps for variation's sake. The connection between lessons was normally stated as a common or familiar topic, not in terms of a common goal.
Observed lessons are most often based on texts and assignments from a textbook. In the textbook a series of lessons that strive for the same goal may be suggested and this may even be stated explicitly in the teacher manual. But in the teacher questionnaires the common connection between lessons are described by

mathematical topics solely. Mathematics teachers seldom referred to connections between lessons while teaching apart from indicating to continue in the textbook.

Re 4)
Many teachers arrange for students' seatwork individually or in groups. In such periods some teachers use the majority of their time in guiding students with special needs. Hardly any mathematics lessons ended with the teacher summing up the results achieved or reminding students of a "red thread" in lesson activities. Most lessons ended with the bell ringing and everyone suddenly realizing, that time was up.

Re 5)
Most teachers were modest about themselves when making statements on evaluation and assessment in the questionnaires. It appears from the recorded lessons that it is normal to have homework wholly or partially corrected at the beginning of the lesson. Often there are specific tasks on a weekly basis or less frequently. This is often written in the upper corner of the class blackboard with a delivery date. The tasks reviewed and corrected by the teacher may be returned with a grading. Less often mentioned is a small test – on one occasion it was videotaped (lesson 15). There are also examples of students taking a test for the first 20 minutes and then managing a self-evaluation tool which enables the students to place themselves on a target with three rings: NOT ABLE (= outer ring), ALMOST ABLE and ABLE (= bulls-eye) and they can then discuss their placing with their teacher (lesson 17). Students then choose to work on topics for which they believe they need more practice through more exercises.

8.9 Validity

The stratification procedure is designed to ensure randomness and equal distribution of Danish mathematics teaching to grade 8. But the encoding of video recordings of mathematics teaching from the many schools will be entirely dependent on the researcher's decisions. How did I ensure the maximum validity and reliability?

Validity of encoding of points in the video and the subsequent interpretation can be enhanced in several ways.

- *Triangulation* may be done by using different methods on the same data or by sampling different data on the same informants. This study uses the latter, as there is a usable video tape for all 50 mathematics teachers in the sample as well as supplementary completed questionnaires (45 respondents), a memo by the researcher and teaching material from actual lessons.
- *Respondent validation* may be done by inviting the teachers to react on selected data or even the tentative result in order to refine them in light of the reactions of the teachers. This is done with a focus group of seven of the 50 teachers and is described below (chapter 10).

But qualitative methods still imply risks of imposing researcher's assumptions, hypotheses and wishes for patterns in the data and thus could overlook important results. Silverman suggests five more strategies to ensure validity (Silverman, 2009, p. 278):

"The refutability" principle"

This imposes an attempt to refute initial assumptions about data to achieve objectivity. The researcher should resist easy conclusions just because of some evidence leading in interesting directions. Instead one should subject the evidence to every possible test.

In the context of some of the excerpts above I have described such dilemmas and choices when coding teacher or student lines.

"The constant comparative method"

This is a suggestion for a preliminary testing of each hypothesis in at least one other situation. But it may also be done by inspecting and comparing all the data fragments that arise in a single case.

I have preferred the last approach and combined data from several sources. This becomes more evident in my later research with a focus group (chapter 10).

"Doing comprehensive data treatment"

When using quantitative methods like a survey research one may be satisfied if nearly all data supports your hypothesis. In a qualitative approach one should not be satisfied until a generalization fits every bit of relevant data. This is also discussed below, as I search for explanations on the rather different data from the focus group teachers, who took part in a common seminar and were offered similar peer coaching (section 10.5).

"Searching for deviant cases"

If possible one may include and discuss cases that do not fit into the pattern searching further for explanations not available in the original data.

“Making appropriate tabulation"

When quantitative data make sense use them in a mixed-methods design. Simple counting techniques apply in the search of points in mathematics lessons. E.g. searching words as “why”, “because” or “excellent” in transcripts. This is not done, but several other quantitative measures have been taken. E.g. the lesson structure in phases and their relative weight (section 8.2), point articulation measured in lesson time and as a number of references (section 8.5), the extent of point articulation is divided between teachers or students, between whole class communication and teachers offering individual guiding.

8.10 Credibility

According to Silverman detailed data presentations which make minimal inferences are always “*preferable to researchers' presentation of their own high-inference of data*” (Silverman, 2009, p. 287). In the current research data are gathered by questionnaires distributed after video recordings and an overview of these data presented in a casebook (section 8.1). Also the tables showing phases in lessons based on transcripts are very reliable.

Creswell recommends one or more strategies to ensure the credibility and trustworthiness of results (Creswell, 2008, p. 191):

1. ”Triangulation”

I commented on this above, and the suggestion is again, that if themes are established based on converging several sources of data from informants, then this will add to the validity of the study.

2. ”In member checks”

By checking parts of reports or specific descriptions with informants one may determine whether they find it accurate. This may also include a follow-up interview providing participants with an opportunity to comment. This is done in the focus group (chapter 10).

3. ”Rich, thick description”

Detailed descriptions of settings makes results become more realistic and richer thus adding to the validity.

4. *”Clarifying researcher bias”*

Self-reflection may create an open and honest narrative. The quality of qualitative research is raised when researchers are frank when commenting about their interpretation being shaped by their own background and motivation. I do not consider this necessary in this case. Ethics are considered in section 5.10 and these discussions will be followed-up in relation to the findings below (especially in sections 9.4, 10.5 and 12.7).

5. *”Negative case analysis”*

Real life in schools is composed of different perspectives that do not always combine smoothly. Discussing contrary information adds to the credibility.

6. *”Prolonged engagement and persistent observation in the field”*

To develop in-depth understanding of phenomena you will have to spend time and study in detail. Researchers familiar with the field as former teachers may have quicker access to accurate and valid findings. But there is also the risk of bias due to the researcher’s former experience and own beliefs.

7. *”Using peer review or debriefing”*

Peer debriefing may be by a supervisor or by attending a presentation followed by a discussion with colleagues at a seminar. In this case the actual research decisions on definitions, coding categories and coding of selected excerpts were discussed on several occasions with my supervisor. Also the design and some decisions were presented to and discussed with colleagues at my University College, at two international conferences (CERME 6 and NORMA 11) and at three summer schools in the Nordic Graduate School.

8. ”External audits”

Review with questions to papers on the project may enhance accuracy, so that the account will resonate with other people.

8.11 And now what next?

My analysis shows that quite a lot of the communication in Danish mathematics lessons for grade 8 classes contains didactic points, even if this in no way to a dominating extent.

Many mathematics teachers do not have *any* points in their teaching in the single lessons observed. These lessons seem guided by activity and task lists, possibly supported by a joint review by the teacher on how to work individually in the rest of the lesson. Maybe the picture would be different, if these teachers were followed for a period or at least for some consecutive lessons. This is one of the perennial issues associated with taking a snapshot of teaching as in the current research.

Nevertheless, it is recommended as referred to in section 2.1, that mathematics teaching should focus on didactic points. In 21 of the 50 lessons the teacher did not do this (section 8.6) neither in whole class teaching, nor when talking to groups or individual students.

In a second phase of this project I researched ways to possibly facilitate and increase a point-driven design by offering teachers' some peer response. To be realistic this must be done without any major change of conditions, preferably seen as a tempting change in preparation and practice. The fact that some teachers did present points in their teaching shows that this is fully possible.

What are the barriers and obstacles for point-driven mathematics teaching seen or met by some mathematics teachers? And what kinds of invitations, stimulants or structures are required to ensure that mathematical points are planned for *and* articulated in mathematics teaching?

The next chapter will consider these questions, among others.

9 Influential components when planning for points

The research described in the chapters above provides an analytical mapping of didactic points articulated in Danish mathematics teaching for grade 8. I consider the how in my first research question answered as far as the research design adopted allows it:

RQ1:	**To what extent, how and why do teachers *articulate* mathematical point(s) in Danish mathematics teaching?**

The reason "*why*" may be given by the teachers observed if asked directly. I had very few chances to do that, as I didn't get involved in this kind of discussion before recording the lessons, and there was little time for talk once the lesson and recording were finished. The questionnaire did not introduce the notion of a mathematical point, and the questions for the "most important thing, you want the students to learn" was most often stated in rubrics of mathematical content areas. Only a few of my researcher memos refer stated and relevant priorities in planning and teaching by teachers (section 9.4.4).

A few lessons have notable number of simultaneous and different types of points. If a high frequency of articulated points is seen, I consider this a special feature of the lesson, which might contribute to consistency in student learning. This may be planned by the teacher, and / or trigged by the mathematical topic, the organization, the textbook, the students – or something else? Statistics are provided (section 8.6) by a listing of lesson points as concepts, interpretations, methods and results.

When points are planned for, foreseen or established underway the influential parameters may be subject tradition, teacher's academic background, excellent textbooks, students' and / or parent's expectations. Alternatively it may reflect a less controlled or less reflected use of points.

A quite different way to seek an answer is of course to look for causes for the absence of points. RQ1 then becomes a "*why not*" question. This leads to several new questions and considerations. How could one interpret the absence of points? These 50 teachers did not know that I was searching for points. And teachers may go for other crucial moments or issues in mathematics teaching?

9.1 Lesson content

Students in grade 8 almost always were given the *same* problems to solve. Due to their different speed students did not always work with the same mathematics problems, though.

In some lessons one or a few students worked on *special* assignments because they lacked knowledge to keep up with the rest of the class. Points made in such teaching were then confined to the teachers' individual guidance.

In whole class teaching some points may have been left unstated by the teacher due to stress in a situation with video recording, lacking knowledge or overview of topic

content or due to the need to ensure flow in the lesson and / or not to disturb students who have a hard time coping with a merely superficial understanding.
It’s tempting to compare *different* teachers teaching the *same* content or topic. E.g.:

- lessons 5, 9, 23 and 29 were on “*equations*”
- lessons 21, 28 and 32 on “*powers*”
- lessons 13, 14 and 48 were on “*perspective drawing*”
- lessons 22, 39 and 46 on “*statistics*”.

The first presentation of a mathematical topic in a lesson sequence would invite for points to be driving the following progression or being stated underway – also by students. But some topics were previously introduced in the grade 8 classrooms observed, and therefore points in actual lessons might be connected to previously introduced procedures, results or interpretations. Such *different* agendas in a sequence of lessons on the same topic, but with *different* classes and *different* teaching materials, make it hard to draw conclusions on content as a parameter with respect to points.

9.2 Lesson organization

The necessary attention to create and maintain a peaceful atmosphere may overrule teachers’ attention to the subject. Among the relevant questions are:

- How long does it take for the *mathematics* teaching to begin in lessons?
- What temporal emphasis is given to non-mathematical information such as term tests?
- How do mathematics teachers organize students' individual work, pair work or work in larger groups?
- Which routines were observed regarding homework?
- Were there any interruptions in the lesson? If so, what kind (students arriving late, other teachers who came in …)?
- Were lessons held in a "regular” classroom with the door closed?
- Did students’ mood, ethnicity, gender, noise, anxiety, bullying or room size have any importance?

In sections 6.2 and 6.3 examples and statistics are given for *some* of these issues. The 50 lessons were all coded for an arrival phase (example in section 6.2.1 and statistics in sections 8.2 and 8.3), information on the video recording (example in section 6.2.2 – this was almost always short, and often not given at all as students knew beforehand), announcement of lesson content, non-mathematical information and finally the organization of lesson.
The teacher may plan points beforehand without them being articulated in the lesson. They may be unspoken aims, or they may be clearly stated stepping-stones along the way. It’s obvious that certain lesson organizations may either hinder or provide opportunities for teachers to articulate or elicit points in whole class teaching.

Some teachers spent a lot of time dealing with homework. Examples and statistics are given on the collecting or returning of homework, the correction of homework and whether new homework is assigned. Dealing with homework is an important routine for many teachers, and many points are articulated in this context. I consider such points planned and intended.

Some teachers were quick at arranging seatwork, and if points were articulated, this had to be done in later communication among teacher and students in groups or individually. Seatwork is often announced to achieve more routine and practice. Therefore it should neither be a surprise nor cause for worry when 35 lessons of the 50 observed were without any stated teacher to student points in seatwork.

9.3 Mathematical textbooks

Do teachers have to depend on – or decide to confine themselves to the clear sequencing of content by following – textbooks? Maybe textbooks give the necessary guarantee to a safe progression that students' and parents' accept. The mathematics textbooks seem to play an important role creating the relationship between one lesson and the next. Many teachers did not state any other link than this, perhaps because the textbook worked as *the* script for lesson activities. Teacher (and students) count on being in capable hands following the progression of a textbook, which often is part of a system of materials for teaching and learning including copy sheets, a concept book, task collections and a teacher manual. I compared the choice of textbooks to the use of points in the lessons observed in section 8.7.

Only three of the 50 teachers did not indicate the use of a regular textbook system. Teacher questionnaires indicate the page numbers of textbooks referred to in the lessons observed. And video-tapes and transcripts tell exactly on the extent and nature of reference to the textbook.

In many of the observed lessons, the textbook is referred to in *all* or *more* of these phases:

- Correction of homework
- Teacher presentation
- Seatwork individually or in groups.

Because of time I have not been able to make a closer analysis of e.g. the degree of "loyalty" to textbook suggestions. This might certainly provide interesting results though. Some teachers seem to follow the textbook slavishly from page 1 to the end, while other teachers seem to switch such texts and assignments with their own. But very few of the 50 teachers relied solely on their own teaching material or combined ideas from other materials, colleagues, peers or the internet in the lessons observed.

9.4 School culture

The research of Boaler (Boaler, 2002) shows the influence of school culture and politics on student outcome. The American "mathematics war" is one prominent example (Boaler, 2008b). Schön describes this as a crisis of confidence in professional knowledge (Schön, 1983), when "*practitioners are frequently embroiled in conflicts of values, goals, purposes, and interest*" (p. 17).

9.4.1 School management

What kind of attention did the school management give the project (in contacts before and during visit)? Generally, school leadership and teachers were intrigued by the prospects of feedback on the project. I also promised this in the invitation to participate in the project.

There were several examples of school management interest among the 50 schools visited in this research. In the researcher memos I wrote immediately after each visit, a fixed headline was: *Relevant information from contacts with school administration on day*.

Below are some examples:

Lesson 47	
Both the school principal and T reported on the school's special arrangement. … The school has had a new structure as of August – after many signs of crisis: Teachers became sick in a row, students went to other schools. The management took the initiative and sought extra funding from the municipality, 3 x 1 million DKR, for different actions. … Now they have 30 to 60 minute modules, rolling morning and lunch breaks, mandatory exercise, a cheap breakfast catering for students and staff from 7:30 am, free fruit ... There is still a challenge in retaining students. Enrolment this year was 40 from the west and 5 from the beach side.	Både skoleleder og L berettede om skolens særlige ordning. … Skolen har haft ny struktur siden august – efter mange krisetegn: Lærerne gik ned på stribe, eleverne forsvandt til andre skoler. Ledelsen tog initiativet og søgte kommunen om 3 x 1 mio. kr. til forskellige tiltag. … Nu har de 30 og 60 minutters moduler, rullende formiddags og frokostpauser, obligatorisk motion, en billig morgenmadskantine for elever og personale fra 7:30 gratis frugt… Der er stadig udfordring i at holde på eleverne. Indskrivning i år gav 40 fra vest og 5 fra strand-siden.
Lesson 30	
I spoke to both the deputy principal and the principal J. The latter conversation was so long that the lesson with T had begun (without any warning bell, as there was no break before the lesson, and nobody came into the staff room, and also because the school principal was very eager to explain inclusion issues). He and the school board chairman had been	Jeg talte med både viceskoleleder og skoleleder J. Med den sidste desværre så længe, at timen med L var startet (uden varsel, da der ikke var pause før lektionen, og ingen kom på lærerværelset, og også fordi skolelederen var vældigt optaget af inklusions-problematikken). Han og skolebestyrelsens formand havde inviteret til åbent møde om dette på skolen i

invited for an open meeting on this subject at the school tonight.	aften.

Lacking contact with the school management does not necessary indicate lack of interest in research results. Managers are busy people. But I met with a large variety of ways of handling my research request from the first letter to the school (and its neglect, rejection or confirmation) to the termination of a visit, which sometimes ended with obvious curiosity about my perception of the mathematics teaching just observed.

9.4.2 Teacher communities

A community may simply be defined as a group of interacting people living or working in a common location. The group may be organized around common values and attributed with social cohesion within a shared location, generally in social units larger than a household. Mathematics teachers at the same school form such a community but do not necessarily share *all* attitudes, values, goals, and practices.
In my conversations with teachers, we often talked about possible professional peer collaboration in schools. When cooperation occurs, it can be planning of teaching at a very general level as annual plans, or simply a coordinated choice of mathematics textbooks. At a detailed level it may be on a selected topic or a computer program. It is not my impression that detailed cooperation is particularly widespread – not even between mathematics teachers at same grade level. But this was not a question in the teacher questionnaire.
Few teachers mentioned peer efforts in my interviews. One teacher (T35) had experienced common teaching for more classes by one other teacher before my visit and also referred to such experiences later when she joined my focus group the year after the first video recording (section 10.3).

Taped interview before take 2 in grade 9 (December 1, 2009):

T: We've had 3 x 2 lessons, where all three grade 9 classes have been together... where we can sit all together ...
I: Who taught these?
T: J did that [a colleague?] yes. And then he – and then we worked with reductions at various levels. So for some, if there is a minus-bracket, it is really difficult for them. And then there are others who can multiply more brackets with each other. We then worked with, 4 lessons in the multi-room all classes with each other and we had three teachers.
The first two times we noticed that the girls I used to have *working*, they worked. The girls I have that are *not working*, they did not work! And then there is this boy group, who never used to bother to work – they actually did *work*! So more students were working than usual. But it might just be because of the novelty, because when we got to the third double-lesson, where we had a short talk on equations, the interest was like dropping again.

Another teacher (T13) had previously, but unsuccessfully, tried to establish a shared knowledge base on mathematics teaching with colleagues. They had now given up. He also commented on this in a taped interview before take 2 in grade 9 (December 7, 2009) when he joined the focus group (10.3).

From my researcher memo:

> T had some years ago been a coordinator for the mathematics group at the school and then tried to bring together colleagues for didactic discussions, exchanging ideas etc. There is still a folder from then and what it contains are 3 proposals for working sessions, all from T. So he was rightfully somewhat disillusioned and not inclined to to believe in peer, professional dialogue (again).

But peer efforts to establish or improve points by giving direction to mathematical teaching were never traced in the observed 50 lessons. Why is this not more widespread?

The frames, the work conditions are not too rewarding:

- In many countries new public management ideas have been introduced by the state or municipal administration of schools. Teaching may then be regarded as production of knowledge partly comparable with other kinds of production, where the quality can be controlled and expenses watched carefully. For economic reasons national testing is suggested on issues easily tested – and by some with a demand for publication of results. This may lead to bias in teaching and is also considered by many teachers as an attack on their professionalism. Some teachers see a risk to evaluate teachers in relation to student scores.
- Public education *is* quite expensive, and the economic conditions are scrutinized and regularly worsened by the municipalities, which in Denmark are owners of most primary and lower secondary schools. Most teachers and school managements see this as systematic pressure. New initiatives have to be free – or result in savings elsewhere. It’s considered difficult to provide means for in-service or continued education for teachers.
- Many teachers feel unfairly blamed in political and public debates, as challenging students and families are often overlooked as decisive conditions to teaching. Any teacher would want students to pass exams. But it requires something very special to establish a feeling of dedication to the job under any condition. So many teachers only do the necessary.

Perhaps these are reasons for points not being so distinct in research observations. I shall discuss the concept of "belief" and some mechanisms, which may influence this.

9.4.3 Beliefs

Routines seen in video recordings and interviews referred to in memos show some of the mathematics teachers' attitudes or beliefs to what mathematics teaching is, what it should lead to and how to perform it.

In an overview of Mathematics teacher’s beliefs and affect Philipp (Philipp, 2007) suggests this definition of beliefs (p. 259): *“Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one’s view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes.”*

In the research literature there is no general agreement on a definition and the extensive report of Philipp discusses beliefs versus the constructs of values and knowledge.
Because of complexity some researchers may find inconsistency between teachers’ expressed beliefs and their teaching practice (Philipp, 2007, p. 271). Some teachers see themselves as team players in two possible teams:

- One team consists of the colleagues, who share the teaching responsibility to the same class of students. This team may support the mathematics teacher, from a holistic point of view, in relation to a class of students and other colleagues teaching this class.
- Another team is formed by all teachers, who teach the same subject: mathematics to all or to many groups of students at the school in question. This team may support the individual mathematics teaching in higher mathematics and didactic proficiency levels by peer sparring or coaching.

Philip finds (Philipp, 2007, p. 263) ”*the emergence of sociocultural and participatory theories of learning*” as one occurrence affecting research in the area of “belief” since the first of these handbooks on research in mathematics teaching and learning in 1992. Skott suggests a belief-practice relationship should take a social stance. In a case study (Skott, 2009) this is illustrated by a novice teacher who is invited to comment on dilemmas over differentiation between investigative and routine tasks in a stimulated recall session. Yackel and Cobb (Yackel & Cobb, 1996; Cobb & Bauersfeld, 1995) also remind of a significant challenge to sociomathematical norms in the classroom, when teachers’ introduction, negotiation and establishing of how to “do” mathematics in the classroom is met with student beliefs, values and needs. E.g.: Students are obliged to explain and justify their reasoning. And the teacher’s job is to pose questions and offer grading.
Lave suggest two units of analysis of teacher actions, as she regards teachers as members of communities of practice: An arena and a setting (Lave, 1988; Lave &

Wenger, 1991). A school or a grade 8 classroom is an arena for mathematics teaching like a supermarket is an arena for shopping that is an external and rather stable context, identifiable but beyond control. But such arenas are conceived differently by different teachers. Teaching conditions depend on the setting: School management, teacher colleagues and student population all have implications for what is possible for what mathematics teachers can do in their classrooms. "*Because a social order and the experience of it mutually entail one another, there are limits on both the obdurate and malleable aspects of every context"* (Lave, 1988, p. 151).
Also Lerman (Lerman, 2002) in his review of contributions on teacher's beliefs stresses the influence of social settings.
In the next section (10.2), I shall return to some teachers' own description of teacher identity and beliefs, when they were interviewed as a focus group.

9.4.4 Professionalism

Teachers internalize the mantra that every student should profit from teaching that fits their *own* level.
You cannot as a teacher ignore weak, late, lazy or disrespectful students. You will have to cover the curriculum anyway and prepare students to pass the final exam (in Denmark in grade 9). This is expected by the school, the students and the parents.
My researcher memos offer examples of teachers coping with such challenges by careful preparation of progression (points), cooperative learning in groups, taking into account students' different learning styles, having a critical approach to the actual textbook, designing lesson structure to cope with differences in student capabilities, individual guiding by homework correction and by adopting student oriented assessment tools (portfolio):

Lesson 25 This lesson was an example of a teacher's correction of homework in section 6.3.5.	
T seemed very well prepared – and as seen in the video with a specific idea of looking at the points together in class BEFORE individual seatwork.	L virkede meget velforberedt – og som det ses på videoen med en bestemt forestilling om at se på pointerne sammen i klassen, FØR der regnes individuelt.
Lesson 23 This lesson was also an example of teachers' homework collection in section 6.3.4.	
T is a very energetic teacher with an unfinished study from university, and thus *not* teacher trained. Where possible she produces her own teaching materials, often by "copy paste". She strongly supports "cooperative learning", which she announced I would see	L er en meget energisk underviser med et uafsluttet studium på universitetet, altså *ikke* læreruddannet. Hun laver helst selv sine undervisningsmaterialer, ofte gennem "saksning". Hun går varmt ind for "cooperative learning", som hun annoncerede jeg ville kunne se

later in the lesson when some tasks become pair or group work. An example of her assessment methods is also referred to at the start of the lesson (where one student acts as a director and one as a secretary).	senere i lektionen, hvor noget så bliver par- eller gruppe-arbejde. Et eksempel på hendes evalueringsmetoder omtales også i lektionens start (hvor en elev er direktør og en sekretær).
Lesson 7 This lesson was also an example of a teacher procedural point in section 6.4.2.	
The school has worked with learning styles for 3-4 years. It is the teacher's conscious choice that students organize themselves into small groups. E.g. a large group of girls went to the common area where they worked together around a large table with music from a "ghetto blaster! They are allowed to sit on chair backs or the windowsills.	Skolen har arbejdet med læringsstile i 3-4 år. Det er lærerens bevidste valg, at eleverne selv organiserer sig i små grupper. Fx gik en stor gruppe piger ud i fællesarealet, hvor de arbejdede sammen omkring et stort bord med egen musik fra ”ghettoblaster! De må gerne sidde på stoleryg eller i vindueskarme.
Lesson 32 This lesson was also an example of teacher information on the video in section 6.2.2.	
T was loudly dissatisfied with the mathematics book. It was poorly written, one ought to tell the author, etc. The one he had as a student in school was, according to T much better – but he was stuck with this, which the school had chosen. It’s quite remarkable as T had an obvious lack of knowledge on logic to power calculation rules.	L var højlydt utilfreds med matematikbogen. Den var dårligt skrevet, det burde man fortælle forfatteren, etc. Den han selv havde haft i skolen var ifølge L meget bedre – men han hang på denne, som skolen jo havde valgt. Det sættes lidt i relief af L’s manglende viden om begrundelser for potens-regneregler.
Lesson 10 This lesson was also an example of teachers’ non-mathematical information in section 6.3.2.	
But it was uphill with many students. Also at the end of the lesson. The explanations were not the best, T went on to say "Isn’t that correct?” in his reviews, and students were almost forced to endorse him.	Men det var op ad bakke med mange elever. Også ved timens slutning. Forklaringerne var ikke de allerbedste, L blev ved med at sige ”Er det ikke rigtigt” i sine gennemgange, og eleverne var næsten tvunget til at bekræfte ham.
Lesson 34 This lesson was also an example of a teacher giving an overview in section 6.3.7.	
T does not see any problem in some students	L synes det er helt uproblematisk at lade

covering more ground than others. "They would also like to."	nogle elever nå mere end andre. ”Det vil de også gerne.”
Lesson 37 This lesson was also an example of a teacher’s summing up in section 6.3.8.	
T has a form that certainly does not condemn or give one way instruction. There is a remarkable calmness and attention! Also, while T clarifies incorrect answers.	L har en form, der bestemt ikke fordømmer eller meddeler. Der er udtalt ro og opmærksomhed! Også mens L afklarer fejlsvar.
Lesson 21 This lesson was also an example of a teacher result point in section 6.4.3.	
Assignments every 3 week (in ink). They are corrected of course! T told of a second class where she took home exercise books once a week (the students then have two books) – all to be corrected. "How else could you assess?" It's obviously a well prepared teacher with the classic "virtues".	Afleveringsopgaver hver 3. uge (indskrevet). De rettes, naturligvis! L fortalte om en anden klasse, hvor hun havde hefterne med hjem hver uge (eleverne har så to hæfter) – alt rettes. ”Hvordan skulle man ellers?” Det er tydeligvis en velforberedt lærer med de klassiske ”dyder”.
Lesson 44 This lesson was also an example of elicitation in section 6.7.1.	
T is a well-informed teacher, started diploma study (mathematics supervisor module). In this connection, he studied the applied assessment methods among his mathematics colleagues and found, that the sample is limited to “MG-tests” and own self-made mini-tests. Would advocate the use of a portfolio strategy in his school.	L er en velorienteret lærer, i gang med diplomuddannelse (matematikvejleder-modul). I den forbindelse har han undersøgt de anvendte evalueringsformer blandt sine matematik-kolleger og fundet, at udvalget er begrænset til MG-prøver og egne, selvlavede miniprøver. Ville arbejde for porteføljetænkningen på sin skole

These memos should be seen as a complement to videos, transcripts and questionnaires. Teachers prepare and act differently in their classrooms to differentiate their teaching. And my memos and the various transcripts do not contradict the impression of beliefs and attitudes that partly determines their teaching actions.

9.4.5 Routines

A routine may refer to a sequence of normative, standardized actions or procedures that are followed regularly, often repetitiously. I do *not* consider routines as points in themselves, but some procedural points in the research may be *evidence* of teacher know-how.

The role of professional routines is treated by Rösken, Hoechsmann and Törner (Rösken, Hoechsmann & Törner, 2008). In this review on mathematics teacher routines they stress that some researchers consider routines to be *knowledge-in-action* due to the action being evident to observers:
"*Weite Bereiche des Lehrerhandelns, insbesondere wenn es sich um kommunikatives Handeln in Kontext der Schulklasse oder der Schule handelt, basieren auf praktischen Wissen und Können (knowledge in action). Dieses Wissen ist erfahrungsbasiert, in spezifische Kontexte eingebettet und auf konkrete Problemstellungen bezogen. Es manifestiert sich als Können des professionellen Experten*." (Baumert & Kunter, 2006, p. 483)

Routines may show as *knowledge-in practice* being based on recommendation or own experience.
"*Thus teachers need mathematical knowledge in ways that equip them to navigate these complex mathematical transactions flexibly and sensitively with diverse students in real lessons. Not providing this undermines and makes hollow efforts to prepare high-quality teachers who can reach all students, teach in multicultural settings, and work in environments that make teaching and learning difficult. Despite frequently heard exhortations to teach all students, many teachers are unable to hear students flexibly, represent ideas in multiple ways, connect content to contexts effectively, and think about things in way other than their own."* (Ball & Bass, 2000, p. 94)

I do not consider knowledge-in-action and knowledge-in-practice as described by the quotes above as being very different. Teachers always act in practice, but the wordings above suggest different starting points.
One example is the beginning of a new topic. Some teachers are convinced or prefer to present an overview or a key example themselves. Very often this is done by using a section in a textbook, and the teachers restrict themselves to points, that they read aloud. Other teachers let the students experiment and reflect on a sample problem before establishing a whole class discussion.
Such routines may be decided by mathematics teachers' conviction of quality and gains in *cooperative learning* or a design of teaching allowing for students individual *learning styles*. Routines may therefore promote student points. Knowledge for teaching is also discussed below (section 9.6.1).

In my research some routines seem to format organization. Among the excerpts coded for mathematical information are these examples of mathematics teachers' work expectations on the kind of work (role fulfillment) to be done by students. The routines are therefore parts of didactic contracts (section 3.3.3) as they are understood by students and teachers: Delivery obligations, deadlines postponements, grading, tests, attention and hard work are all components:

Lesson 42: (2:52)	
T: If there are some who HAVE made their homework assignments, then they obviously may deliver that today. S: You should just grab the collar of D! T: Yes, I'll just grab him afterwards. S: Why? T: Because you believe the B-class has had an extended deadline. S: Again? S: Well, have we not already had extension three times. T (smiling): It IS of course, until tomorrow. S: But if there is extension, then I would really like to ...? T: Then I shall come and tell you during N's lesson …	L: Hvis der er nogle, der HAR lavet deres blækregning, så må de selvfølgelig gerne aflevere den i dag. E: Du skal lige tage fat i kraven på D! L: Ja, jeg skal nok lige tage fat i ham bagefter. E: Hvorfor? L: Fordi I mener, B-klassen har fået udsættelse. E: Igen? E: Jamen har vi ikke allerede fået udsættelse tre gange. L (smiler): Den ER selvfølgelig til i morgen. E: Jamen hvad nu hvis nu der er udsættelse, så vil jeg meget gerne ...? L: Så kommer jeg ind og siger det til jer i N's time …
Lesson 23: (6:37)	
T: So, now I want you to be quiet. So do stop that annoying silliness. Ladies and gentlemen. Are you ready with your pencils? S: Yes. T: To sit and write. Now we start a new topic: Equations. S: How should we write it up? S: We do get our grades back – for the homework assignments? T: Thanks! S: We DO get our assignments back? T: You want grades for that? S: Yes! T: Is *that* what you ask for? S: And when do we get it back? T: ... but usually it is (inaudible, students talk ...). Honestly, why don't you quieten down? I know its Friday afternoon, I also know we have this one EVERY Friday. You are frivolous, you only think about the weekend, but you have to shut up. If I can, I will of course have it ready on Monday.	L: Så, nu må I gerne være stille. Så hold nu op med det der irriterende fjolleri. Mine damer og herrer. Er I klar med jeres blyanter? E: Ja. L: Til at sidde og skrive. Nu skal vi i gang med et nyt emne: Ligninger. E: Hvor skal vi skrive det op? E: Vi får vores karakterer tilbage – i blækregning? L: Tak! E: Vi FÅR vores opgaver tilbage? L: Du vil gerne have karakterer for den der? E: Ja! L: Er det *dét*, du spørger om? E: Og hvornår får vi den tilbage? L: … men som regel plejer det at være (uhørligt, eleverne snakker …). Helt ærligt, hvorfor tier I ikke stille? Jeg ved godt det er fredag eftermiddag, jeg ved også godt, vi har den her hver ENESTE fredag. I er kulrede, I tænker kun på weekenden, men I er nødt til at klappe kajen. Hvis jeg kan, har jeg den selvfølgelig færdig til på mandag.
Lesson 15: (1:28)	

T: But do try and listen, today we will do as we usually do: You hand in your assignments. Has anyone not delivered yet? No? It's just you I've talked to, N. And then you will have a proficiency test. And then, at the end of the lesson we continue work in the textbook, right. And I'd like you to work together. Do remember to help each other. S: And the textbook? T: Yes, we do that at the end of this double ... it is skill set 13, we must do today. Do take all those skill sets out (a student arrives). Good morning, do you bring your assignment for me, or what? You've made it at least? Well, then we will just see who is the first to finish, M. .. Yes, it's number 13 all right. N, do you have mine? S: No, that's why I asked for an extra. T: Well I have one. ... You may work together, like you usually do, right. ... It is 13, right – no that one is number 12.	L: Men prøv og hør her, i dag der gør vi ligesom vi plejer: I får afleveret jeres afleveringsopgaver. Er der nogen, der ikke har afleveret endnu? Nej? Det er kun lige dig, jeg har snakket med, N. Og så ellers får I en færdighedstest. Og så, i slutningen af timen, der arbejder vi videre i matematikbogen, ikke også. Og jeg vil gerne have, I arbejder sammen. Husk nu lige, at hjælpe hinanden. E: Og matematikbogen? L: Ja, det gør vi i slutningen af den her dobbelt… det er prøvesæt 13, vi skal lave i dag. Tag lige de der færdighedssæt der frem (en elev ankommer). Godmorgen, har du aflevering med til mig, eller hvad? Du har lavet den i hvert fald? Nå, men så må vi lige se, hvem der bliver først færdige, M. .. Ja, det er 13'eren. N, har du min? E: Nej, det var derfor jeg spurgte efter en ekstra. L: Godt jeg har én. … I må gerne arbejde sammen, ligesom I plejer, ikke også. ... Det er 13'eren – nej den der, det er 12'eren.
Lesson 19: (17:20)	
T: How do you calculate the volume of a box? S1: Isn't it height times width times baseline? S2: What if there are brackets? T: Then you've got half a point. S: Shut up, half a point! ...	L: Hvordan regner man rumfang ud af en kasse? E1: Er det ikke højde gange bredde gange grundlinje? E2: Hvad hvis der er parentes om? L: Så har du fået et halvt point. Elev: Hold kæft, et halvt point! …
Lesson 29: (13:30)	
T: Mind, in number 4 you must first – like up here – pick the x's first – and then afterwards calculate just as usual. Number 5 is almost the same. And number 6, there comes some with a few half x's in it, but it really does not matter ...	L: Prøv, i 4'eren der skal I først – ligesom heroppe – samle x'erne først – og så bagefter regne ligesom I plejer. 5'eren er lidt det samme. Og 6'eren, der kommer der nogle med nogen halve x'er ind, men det gør altså ikke noget …

The next examples are on instructive routines to students. The teacher's function is that of an employer, students are assigned workloads fitted to capability and time and expected not to doubt the appropriateness. This may also be expected to be a routine instruction for the teacher:

Lesson 28: (14:46)	
T: You are actually ready now to, what's it called, to make the final assignments. And I would like everyone to reach at least task 17. If you are further than that, one can easily get further than that, because the tasks are mostly a routine to use exponents when multiplying and dividing. L, do you listen? Because otherwise do go out! S: I heard it all right. T: What is it called: To practice, what's it called, using multiplication and division in relation to powers.	L: I er faktisk klar til nu, hvad hedder det, at lave de sidste opgaver. Og jeg så gerne, at alle kunne nå i hvert fald at komme minimum til opgave 17. Hvis man er længere end det, man kan sagtens komme længere end det, fordi opgaverne er mest en rutine i, at man lige brugte det her med at bruge eksponenterne når der skal ganges og når der skal divideres. L, hører du efter? Fordi ellers så gå ud! E: Jeg hørte det godt. L: Hvad hedder det: At træne, hvad hedder det, med at bruge gange og dividere i forhold til potenser.
Lesson 4: (16:00)	
S: Are they all like these? The others, are they like these? T: It IS a sequel, they always gets a little different ... They become a little harder than those. S: Such slave labor! T: Yes – it takes that to become good! Is it not correct? E: Yes.	E: Er alle sammen ligesom de her? Altså de andre, er de også ligesom de her? L: Det ER en fortsættelse, de bliver hele tiden lidt anderledes … De bliver lidt sværere end dem der. E: Sikke noget slavearbejde! L: Ja - det skal til for at blive god! Er det ikke rigtigt? E: Jo.

These dialogues are signs of unspoken contracts between teacher and student on divided responsibilities, power and mutual expectations. The organization formatted does make point presenting less necessary to keep lesson flow but also quite impossible. At least in these excerpts motivation seems to some extent secured by grading and discipline.

9.4.6 Student behavior

Colleagues may have differing views on quality teaching and school management may promote or hinder certain teaching styles and priorities by expressing awareness of e.g. parent pressure or test results. But a short note on lower secondary student behavior is also needed to complete a picture on teaching conditions. Students don’t always sit quietly listening to and answering questions from the mathematics teacher. Some of them become noisy and obstructive in periods.

Lesson 19: (14:35)	
T: M! S: This is insane, I always get the blame! T: Yes, but was it not you? S: No. T: Well I thought it was M. If it were you, do you bother to leave it. If you know who it is ... S: It was not me.	L: M! E: Det er sindssygt, jeg får altid skylden! L: Ja, men var det ikke dig? E: Nej. L: Jamen jeg troede, det var M. Hvis det var dig, gider du så ikke lade være. Hvis du ved, hvem det er … E: Det var ikke mig.
Lesson 5: (2:55)	
T: Well, your homework was so far I remember… S: It was for Monday! T: I did not say it was till today. S: I could not solve them! (It is written on the board that it is to Monday) T: Exercise 8 to 19, page 38. Go ahead! And it is as usual: You are allowed to have music, but in the ears – and not so it disturbs others – and then just show your paw, if there is anything. S: May I just go and get my headset? T: Your what? S: Will you also bring mine? It's in my coat pocket.	L: Godt, jeres lektier det var, så hut jeg visker… E: Det var til på mandag! L: Jeg har ikke sagt, det var til i dag. E: Dem kunne jeg ikke finde ud af! (Det ses på tavlen, at det er til mandag) L: Opgave 8 til 19, side 38. Gå i gang! Og det er som sædvanlig: I må ha' lov til at have musik, men i ørene – og ikke så I forstyrrer andre – og så ellers op med labben, hvis der er noget. E: Må jeg godt lige gå ud og hente mit headset? L: Dit hvad? E: Kan du så ikke lige tage mit med? Det ligger i min frakkelomme.

Teachers' potential interest in a classroom culture with a shared sense of purpose and justification of mathematical content and points regularly gets challenged by such needs for framing of student work and discipline.

9.5 Mathematics curriculum on teacher or student dialogue

The mathematics curriculum guidelines (*Fælles Mål 2009*, 2009) state binding goals for grade levels 3, 6, 9 and 10 and include a 50+ page teacher's guide to mathematics teaching. The importance of mathematical communication between teacher and students and among students in and about *mathematics* is mentioned several times. There are in total 88 occurrences of the word *dialogue* in the document.
In a paragraph on working methods (in Danish: Arbejdsmåder) it says (p. 3):

Paragraph 2. *The teaching is organized, so that students independently and through <u>dialogue</u> and cooperation with others can learn, that the work with mathematics requires and promotes creative action, and that mathematics contains tools for problem solving, reasoning and communication.*	*Stk. 2.* *Undervisningen tilrettelægges, så eleverne selvstændigt og gennem <u>dialog</u> og samarbejde med andre kan erfare, at arbejdet med matematik fordrer og fremmer kreativ virksomhed, og at matematik rummer redskaber til problemløsning, argumentation og kommunikation.*

And at the end of grade 9 on mathematical competencies (p. 9):

The teaching should lead towards students having acquired knowledge and skills that enable them to … • *engage in <u>dialogue</u> and express themselves orally and in writing about mathematical issues in different ways and with some technical precision, and interpret others' mathematical communication ...*	*Undervisningen skal lede frem mod, at eleverne har tilegnet sig kundskaber og færdigheder, der sætter dem i stand til at …* • *indgå i <u>dialog</u> samt udtrykke sig mundtligt og skriftligt om matematikholdige anliggender på forskellige måder og med en vis faglig præcision, samt fortolke andres matematiske kommunikation …*

In the guidelines to teachers on students' needs the document also comments on possible mathematics teacher attitudes, e.g. the following excerpt on the teacher's own curiosity (p. 50):

The teacher may choose his approach to mathematics teaching and his view on students. The teacher may choose to consider his class as a group of students who are hopelessly behind, or as a group of students with potential to develop mathematical skills. *The teacher must give students the courage to test their ideas by taking the others in the class into the "on-road thoughts" – e.g.: the thoughts that are about to become clear –*	*Læreren kan vælge sin holdning til matematikundervisning og vælge sit syn på eleverne. Læreren kan vælge at betragte sin klasse som en flok elever, der er håbløst bagud, eller som en flok elever med potentiale til at udvikle matematiske kompetencer.* *Læreren må give eleverne modet til at prøve deres idéer af, at tage de andre i klassen med ind i "på-vej-tankerne" – altså de tanker, som er på vej til at blive klare – de tanker,*

the ideas that are clearer when they are said aloud. *The teacher may consider himself as someone who also is in a learning process, with goals that include learning more about students' learning and approach to mathematics, but perhaps also sometimes a question is raised in a mathematics class, that students and teacher together can investigate. The teacher can display a curiosity in relation to students' thinking and learning so clearly that it arouses their curiosity in relation to learn more math. In a way a kind of "contagious curiosity" becomes the focal point.*	*som står klarere, når de bliver sagt højt.* *Læreren kan betragte sig selv som én, der også er i en læringsproces, hvor hensigten bl.a. er at lære mere om elevernes læring og tilgang til matematik, men måske rejses der også nogle gange spørgsmål i en matematiktime, som elever og lærer sammen kan undersøge.* *Læreren kan vise en nysgerrighed i forhold til elevernes tænkning og læring så tydeligt, at det vækker deres nysgerrighed i forhold til at lære endnu mere matematik. Det bliver på en måde en form for "smittende nysgerrighed", der bliver omdrejningspunktet.*

The mathematics curriculum guidelines are in a book of almost 100 pages in total And some teachers teach many subjects with similar comprehensive guides. Because of that and the lack of systematic introduction in courses or peer supported discussion groups, some teachers probably don't quite get the support and frames offered here. Therefore inspiration and shifts in teaching tradition towards a clearer point driven mathematics teaching will have a hard time.

Above (sections 9.2-3) I exemplified and discussed textbooks, lesson's organization and content, the school culture with management and teacher communities, teachers' beliefs, professionalism and routines and finally student behavior as influencing mathematics teaching. But to make mathematical points the teachers of course first and foremost must have significant, relevant knowledge about mathematics and mathematics teaching.

9.6 Teacher knowledge

Teacher knowledge plays an important role in teacher actions and several researchers have conceptualized components in such knowledge. The common view seems to be an acceptance that teaching competence *not only* depends on subject knowledge. There is a spectrum of components deciding teacher professionalism, also with respect to mathematics teaching. It would be interesting in another study to take a closer look at the various kinds of teaching knowledge observed in this research, as this would probably explain some of the variations among individual teachers.

Ma (Ma, 1999) uses teacher interviews to show systematic differences in understanding among U.S. and Chinese teachers. The biggest problem for U.S. teachers according to this research was that many of them were uncertain about the reasons for presenting a mathematical concept in a certain way. Although the U.S. teachers performed mathematical problems correctly they were more uncertain about their own competence than their Chinese counterparts. And the Chinese teachers offered both more stable answers to the various tasks and alternative ways to disseminate insight to students.

Knowledge of typical student errors and difficulties were also mapped in the German COACTIV project (see section 3.3.4) (Krauss et al, 2008) that found, that German teachers of the highest track Gymnasium (“GY-teachers”) outperform other secondary teachers (“NGY-teachers”) both in CK (Content Knowledge) and in PCK (Pedagogical Content Knowledge). But after statistically controlling for CK, NGY-teachers slightly outperform GY-teachers in PCK!

In several in-service courses for Danish mathematics teachers I have presented a map like below of “dark spots” to be revisited for varying explanations:

Minus times minus is plus *)	To find a percentage you simply multiply by the number and divide by 100.
You divide with a fraction by multiplying with the inverse?	You can’t divide by 0.

*) This was also a task to teachers on instruction in a COACTIV test in PCK.

Most mathematics teachers are secure and confident with such procedural knowledge and may therefore ignore or forget that it is not evident at all to students, and especially *why* this is the case. There is a risk that such knowledge is being transmitted as facts to be simply memorized by students. It is part of teachers’ necessary mathematical knowledge for teaching that such findings not only are self-evident, but that teachers have more and varied explanations at hand for each case.

9.6.1 Knowledge for teaching (Shulman and followers)

Shulman suggests a focus on the subject matter in his paper on knowledge growth (Shulman, 1986):
“*What we miss are questions about the content of the lessons taught, the questions asked, and the explanations offered*” (p. 8).
He suggested three categories of content knowledge, which have been accepted and later elaborated by many other researchers:

a) *Content knowledge* (CK) is on understanding *that* something is the way it is, *why* it is so and why “a given topic is particularly *central* to a discipline” (p. 9).
b) *Pedagogical Content Knowledge* (PCK) is for example knowledge of “*the most useful forms of representations*” when teaching and “an understanding of what makes the learning of a specific topic easy or difficult” (p. 9).
c) *Curricular Knowledge* involves knowledge of (alternative) materials for teaching a subject to a given grade level.

Shulman also suggested three forms of teacher knowledge;

- *Propositional knowledge*, which may correspond to (1) principles based on empirical findings of what is useful, (2) practical experience as "*Never smile until Christmas*" (p. 11) or (3) norms and values on fairness, equity etc.
- *Case knowledge*, which is knowledge of cases being "*instances of a larger class*" (p. 11) and useful in illustrations of e.g. concepts or procedures.
- *Strategic knowledge*, which is knowledge of contradictions or "*particular situations or problems … where principles collide or no simple solution is possible*" (p. 13).

In section 5.9.2 on the LMT project (Learning Mathematics for Teaching) I refer to similar elements, which in that project are found crucial to quality in teachers' instruction. Two of these are:

- Connecting classroom practice to important mathematical ideas or procedures.
- Rich mathematical representations, explanations and justifications.

These parts of mathematical knowledge are represented in the PCK concept as described by Shulman. And in my research teacher knowledge on these elements seems visible as parameters for success or failure when points are being made. The propositional knowledge is e.g. demonstrated in the mathematics teaching seen in lesson 7 (section 6.4.2, learning styles) and lesson 12 (section 6.6.3, cooperative learning).

Another suggestion to map the School Mathematical Knowledge (SMK) and the Pedagogical Content Knowledge (PCK) of teacher students has resulted in a four unit model: A Knowledge Quartet (Rowland & Turner, 2007). This framework is based on a study of 24 video-taped lessons, taught by novice teachers. The four units are:

1. *Foundation*: Knowledge, beliefs and understanding of trainees and acquired in preparation for the teacher role.
2. *Transformation*: Knowledge-in-action as demonstrated in planning to teach and in the actual teaching.
3. *Connection*: A combination of choices and decisions made for the more or less discrete parts of mathematical content.
4. *Contingency*: Ability to act in unexpected classroom events, when you react with sense"on your feet".

The connection of choices and decisions on mathematical content is seen here as one unit in lesson planning and execution, and it is meant to direct attention to the subject dimension of classroom practice. Such a connection is hopefully supported by the mathematics *textbook* as textbooks seem to influence the teachers' decisions a lot.
The extent and type of points driving the teaching will still depend on the importance of such statements considered by the *teacher*. When teachers sequence mathematical

content for instruction they plan and steer for progression towards students' skill or a pointed insight in mathematics, and points will have to be articulated.
I consider such models constructive to an ongoing exchange of viewpoints among didacticians, being co-responsible for the current curriculum in Danish teacher education, which is referred to below.

9.6.2 The KOM report and the Danish Teacher Education Act

I referred to the Danish KOM-report in chapter 1 (Niss & Jensen, 2002) where these six necessary *competencies for mathematics teachers besides the purely mathematical* ones are suggested:

1. *Curriculum* competence, i.e. to estimate and work out a curriculum
2. *Teaching* competence, i.e. to plan, organize and practice teaching
3. *Learning uncovering* competence, i.e. to uncover and interpret students' learning as well as their view and attitude to mathematics
4. *Evaluation* competence, i.e. to uncover, estimate and characterize the students' mathematical outcome and competencies
5. *Cooperation* competence, i.e. to collaborate with colleagues and others about teaching and framework
6. *Development* competence, i.e. to further develop one's competences as a mathematics teacher.

Among the necessary competencies for mathematics teachers recommended in the report, I consider the "*Teaching competence*", i.e. to plan, organize and practice teaching as *the* competence most directly connected to knowledge for teaching in the Shulman sense. It may seem a somewhat reductionist view on teaching to stick to one out of six advocated teacher competences, as competences in relation to e.g. curricula and assessment of course are also crucial. But teaching competence must in any case be a goal for the Danish Teacher Education Act. The Danish Teacher Education Act to lower secondary mathematics teachers states on such knowledge for student teachers (*Læreruddannelsesloven*, 2009):

<table>
<tr><th colspan="2">Among goals to the first half of the mathematics teacher education to grade 4-10 are</th></tr>
<tr><td>… that the teacher student acquires
• competence to understand and avail themselves of appropriate forms of representation; link between everyday language and technical language, decode, translate and treat symbol containing statements with awareness of the special role effective symbol processing plays in mathematics, and to apply and assess it in a professional</td><td>… at den studerende opnår
• kompetence til at forstå og betjene sig af hensigtsmæssige repræsentationsformer; knytte forbindelse mellem hverdagssprog og fagsprog; afkode, oversætte og behandle symbolholdige udsagn med bevidsthed om den særlige rolle, effektiv symbolbehandling spiller i matematikken, samt anvende og vurdere it i en faglig og pædagogisk sammenhæng,</td></tr>
</table>

and educational context, • competence to grasp, analyze and assess the framework and modalities for mathematics nationally and locally, and to devise and justify learning and teaching goals, • competence to justify, plan and implement mathematics teaching in interaction with students; find, assess and develop teaching resources for mathematics teaching, motivate and inspire students to engagement in mathematical activity • competence to identify, assess and characterize the students' mathematics academic achievement and competencies; to be familiar with a wide range of tools for assessment in mathematics with knowledge of validity and reliability, and identify students' learning strategies and attitudes towards mathematics on the progression and differentiation in teaching, ...	• kompetence til at kunne sætte sig ind i, analysere og vurdere rammer og bestemmelser for faget matematik nationalt og lokalt samt udforme og begrunde lærings- og undervisningsmål, • kompetence til at kunne begrunde, planlægge og gennemføre matematik-undervisning i samspil med eleverne; finde, bedømme og udvikle undervisningsmidler til matematikundervisning samt motivere og inspirere elever til engagement i matematisk aktivitet, • kompetence til at kunne afdække, vurdere og karakterisere elevernes matematikfaglige udbytte og kompetencer; være fortrolig med et bredt udvalg af redskaber til evaluering i matematik med kendskab til validitet og reliabilitet, samt afdække elevers læringsstrategier og holdninger til matematikfaget med henblik på progression og differentiering i undervisningen, …

The excerpts show an increased emphasis compared to former teacher education acts on the categories of content knowledge as emphasized by Shulman (section 9.6.1): teachers' knowledge for teaching as curricular knowledge, pedagogical content knowledge (PCK) and mathematical content knowledge.

The first teacher graduates following this curriculum from 2008 will finish their four-year education in 2012. If the goals as exemplified above are attained the teachers should be prepared for a point driven or point supported mathematics teaching, as the one advocated by the present research.

10 Intervention study with focus group of teachers from five schools

This research does not say anything about the correlation between point-concentration (in some measurable sense of duration, frequency, variability) and student benefits. It is not longitudinal or a learning study designed to measure such effect.
When convinced of the importance of points in mathematics teaching, one has to investigate ways to strengthen both the occurrence and the role. And the communication between teacher and class or individual students may show whether there is consistency in student learning. The 50 lessons described in previous chapters offer many examples of mathematical points presented in a way that are meant to promote or provide evidence of student understanding.

Changing the occurrence and role of points for the better may depend on increased frequency and / or stronger relations between points and what else is happening in the classroom. My research question 2 was:

RQ2:	**In what way can the occurrence and role of mathematical points be *strengthened* in mathematics teaching practice?**

10.1 Points to promote or provide

One could look for lessons showing teachers' presentation of points such as these:

1. Examples of connections or relationships in individual sessions between different *representations* of a concept (lesson 37)
2. Examples of different *interpretations* of a result (lesson 23)
3. Examples of different *uses* of same method (lesson 28)
4. Examples of different *methods* to solve the same problem (lesson 24)
5. Examples of confirming consistency in the students' understanding through dialogue (lesson 7).

Example 1, lesson 37 (18:22 + 36:40): Different *representations* of a concept	
T: And this we shall start with (refers to textbook, page 101: the Pythagorean theorem). And we will start it quite differently ... We'll go out in the nice weather. S: Yes! T: And here is a piece of string for each group. It is bound together at the end – to make a circle (T draws a circle on the	L: Og den skal vi starte med (refererer til bogens side 101: Pythagoras' sætning). Og den skal vi starte helt anderledes … Vi skal ud i det gode vejr. E: Yes! L: Og her er en snor til hver gruppe. Den er bundet sammen i enden - en rund snor (L tegner en cirkel på tavlen), og der er 1, 2, 3, …, 12 knuder.

blackboard), and there are 1, 2, 3, ..., 12 nodes. And in the group you must form a triangle of it. Now I've made the shape of a circle, but I want it in the shape of a triangle. But out at the tips, there must be a knot ... And I have one more request: The triangle *must* be what is called a right angle triangle. S: No! T: Why not? S: Because it's stupid! T: Ha, ha. What does it mean that it must be right? ... (later, back in class) T: If it had been raining, then I would probably have said that we don't bother to go outside ... then you would have tried to build triangles with matches. And then I would have given you 12 matches and you would have had to make a triangle. But now I want you to make a similar form, not with 12 but with 24 matches. (All groups easily find out how to double side-lengths) and finally – just before the break when recording is stopped, each group gets 30 matches. Now a respectable challenge).	Og så skal I i gruppen få lavet en trekant ud af den her. Nu har jeg lavet den i form som en cirkel, men jeg vil gerne have den i form som en trekant. Men ude i spidserne, der *skal* være en knude … Og så er der et krav mere: Trekanten *skal* være det, der hedder retvinklet. E: Neej! L: Hvorfor ikke det? E: Fordi det er dumt! L: Ha, ha. Hvad vil det sige, at den skal være retvinklet? … (senere, tilbage i klassen) L: Hvis det havde været regnvejr, så havde I nok sagt at så gider vi ikke gå udenfor ...så skulle I have prøvet at bygge trekanter med tændstikker. Og så havde I fået 12 tændstikker og skullet lave en trekant. Men nu vil jeg gerne have jer til at lave en tilsvarende figur, ikke med 12 men med 24 tændstikker. (Alle grupper finder let ud af at fordoble sidelængderne og endelig – lige før pausen hvor optagelsen afbrydes) får hver gruppe 30 tændstikker. Nu en respektabel udfordring).
Example 2, lesson 23 (14:55): Different *interpretations* of a result	
T: But now all of you know what an equation is. It's something with an x, it is something with an unknown. Do you play cards, H? Ordinary playing cards, poker – don't you use ordinary cards there? [Yes]. A deck of cards, how many cards is that? [54 / / 52 / / 56 ...]. Who is the closest? ... 52 yes. Then there are some other cards too. There are always some extra cards in the deck [Jokers!] What do the jokers do? (T asks the class to think of the unknown as a joker in the deck, it could also be with other symbols).	L: Men I ved altså alle sammen, hvad en ligning er. Det er noget med x, det er noget med en ubekendt. Spiller du kort, H? Helt almindelige spillekort, poker bruger man ikke almindelige kort der? [Jo]. I spillekort, der har man, hvor mange kort? [54 // 52 // 56...]. Hvem kommer tættest på? ... 52 ja. Så er der nogle andre kort også. Der ligger altid nogle ekstra kort i [Jokere!] Hvad kan jokerne? (L beder klassen tænke på den ubekendte som en joker i et spil kort, det kunne også være med andre betegnelser).
Example 3, lesson 28 (9:00): Different *uses* of same method	
T: What if it is – now let me see again, what	L: Hvad nu, hvis det er – nu skal jeg lige se

was I thinking? S: J, what does it mean, "exact"? T: Well it just means that we arrive at the exact number. Say we wanted a – this might well be the case? (Writes 10^3: 10^3) If we apply the rule from before: What should one do, if you had to divide the two powers with one another? ... What did we do up here? [Minus] Yes, it was minus. Now we have here, what does it say? [10^3] Yes, 10^3. So if we took this, we should say 3 minus 3, what is 3 minus 3? [0] T: Yes. 10^0 it says. But we did not on Monday, what is the result if we had to find the exact value of it? What would it give? It is perhaps not quite logical, T? S: Is it not just 1. T: How did you find out? It is absolutely correct. S: It must be 0. T: Well it's true. If you look right here: How much is this, if we were to take the 10^3 written up here. What is it as a number? The exact number, if we were to convert it? [1000] Yes, and down here? [It is also 1000]. Yes. 1000 divided by 1000 [1] Thus, the same number divided by itself [1] will always end up with 1.	engang, hvad var det jeg tænkte på? Elev: J, hvad betyder "eksakte"? L: Jamen det betyder bare, at vi kommer frem til det præcise tal. Hvis vi nu fx vil have en – det kunne være sådan én her? (Skriver 10^3: 10^3) Hvis vi nu skal bruge vores regel fra før: Hvad var det man skulle gøre, når man skulle dividere to potenser med hinanden? … Hvad var det, der blev gjort heroppe? [Minus] Ja, det blev minus. Nu har vi her, hvad står der heroppe? [10^3] Ja, 10^3. Så hvis vi nu tog her, så skulle vi sige 3 minus 3, hvad giver 3 minus 3? [0] L: Ja. 10^0 står der nu. Men vi havde ikke om i mandags, hvad er resultatet af det, hvad vi skulle finde ud af den eksakte værdi af det? Hvad ville det give? Det er måske ikke helt logisk, T? E: Er det ikke bare 1. L: Hvordan fandt du ud af det? Det er fuldstændig rigtigt. E: Det skal være 0. L: Jamen det er rigtigt nok. Hvis I lige kigger her: Hvor meget står der her, hvis vi skulle lave de 10^3, de står jo heroppe. Hvad er det i tal? Det præcise tal, hvis vi skulle omregne det? [1000] Ja, og hernede? [Det er også 1000]. Ja. 1000 divideret med 1000 [1]. Altså det samme tal divideret med sig selv [1] det vil altid give 1.
Example 4, lesson 24 (23:18): Different *methods* to solve the same problem	
T: Listen: If we just go from one unit (T writes cm^2 on the blackboard). What is there on the blackboard? Raise your paws! S: Square centimeters. T: What does it mean ... S: It is the area of something. T: (Repeats) And what is an area? R? S: It is two lengths, um a length to the second. It's two lengths *multiplied* together. T: Just try to put it right, I know what you mean. ... T: Now I've got a number here. I will change	L: Prøv at høre her: Hvis vi lige går fra en enhed (L skriver cm^2 på tavlen). Hvad står der oppe på tavlen? Op med labberne! Elev: Kvadratcentimeter. L: Hvad betyder det … E: Det er arealet af en eller anden ting. L: (Gentager) Og hvad er et areal? R? E: Det er to længder, øh en længde i anden. Det er to længder *ganget* med hinanden. L: Prøv lige at formulere det rigtigt, jeg ved godt hvad du mener. … L: Nu har jeg fået et tal her. Det vil jeg have

it into m^2 (writes $cm^2 \rightarrow m^2$). Now comes the question: My almost favorite student H: How do I ... ? S: If we must make it 10 times larger, we must multiply by 10 squared ... S: Times by 100 / / Divide by 100 / / squared ... T: Why is it exactly by 100, we must divide? ... (T continues with $cm^3 \rightarrow km^3$). Good morning TH, The Earth calls TH ... T: Well (T writes :100 :100 :100). What am I at now? S: Now you are on meters. T: That's better ... You will simply be halfway hanged, if you make mistakes in this –again! It was scary to see that almost all of the you fell for it.	lavet om til m^2 (skriver $cm^2 \rightarrow m^2$). Nu kommer spørgsmålet: Min næsten yndlingselev H: Hvordan kommer jeg. …? E: Hvis vi skal gøre det 10 gange større, skal vi gange med 10 i anden … E: Gange med 100 // Dividere med 100 // I anden … L: Hvorfor er det lige 100, vi skal dividere med? … (L fortsætter med $cm^3 \rightarrow km^3$). Godmorgen TH, Jorden kalder TH … L: Altså (L skriver :100 :100 :100).. Hvad er jeg så på nu? E: Så er du på meter. L: Det var bedre … I bliver simpelthen halvvejs hængt, hvis I laver fejl i det her – igen! Det var uhyggeligt at se, at næsten alle sammen faldt i.
Example 5, lesson 7 (30:10): Confirming consistency in the students' understanding through dialogue	
S: That, it's ridiculous when you cannot compare? T: Probably, let's look at it: "What is the most, 10% – what is that? What is the most: 10% of your weight or 15% of your height?" I really don't know. That's very strange. S: You obviously cannot compare when it is kg and cm? T: That's interesting. It was nice, if it was something that was comparable. S: Yes exactly! But otherwise, then it's weight, because weight is more than cm. T: No, you are *more* centimeters tall than you are kilograms heavy. I hope. S: Yeah ... (laughs) T: I won't ask about your weight! You know, I think here – it's the first task of the section, they should be regarded as relatively easy ...	E: Det der, det er jo åndssvagt, når man ikke kan sammenligne? L: Sandsynligvis, skal vi se på den: "Hvad er størst, 10 % – hvad for noget? Hvad er størst: 10 % af din vægt eller 15 % af din højde?" Det ved jeg da ikke. Det var da noget mærkeligt noget. E: Man kan jo ikke sammenligne når det er kg og cm? L: Det er interessant. Det havde været rart, at det var noget, der var sammenligneligt. E: Ja nemlig! Men ellers så er det jo vægt, for vægt er jo større end cm. L: Nej, du er *flere* cm høj end du er kilogram tung. Håber jeg da. E. Joh.. (ler) L: Jeg skal lade være med at spørge til din vægt! Ved I hvad, jeg tror den her – det er jo den første opgave i afsnittet, de skal betragtes som forholdsvis lette ...

These five dialogues were different ways for teachers to articulate a mathematical point.

Mathematics teachers' readiness requires subject knowledge and perhaps quite a bit of knowledge based on teaching experience to cope with the various student's questions and answers. When teachers' elicitation succeeds in mathematical points being understood by students – or even presented by students, it may be because of competencies such as the six referred to above.

Above I have listed different components influencing the occurrence of mathematical points:
Lesson content (section 9.1); lesson organization (section 9.2); mathematical textbooks (section 9.3); school culture (section 9.4: management, communities, beliefs, professionalism, routines and student behavior); mathematics curriculum (section 9.5: its affect on teacher or student dialogue); and teacher knowledge (section 9.6: PCK, KOM report and the Danish Teacher Education Act).
For the handling of research questions 2 and 3, I now intend to concentrate on one component: the school culture, i.e.: the possible support of management and peers (section 9.4). This chosen effort is researched in two different ways in order to consider the possibilities for strengthening the occurrence and role of mathematical points in mathematics teaching practice.

a) A group of teachers from the 50 teacher cohort are invited to a seminar on points and invited to focused interviews, before subsequent classroom observations. This is a kind of individual sparring between me, as a proficient mathematics teacher colleague, and each individual focus teacher. Design and outcome of this research is described in the rest of chapter 10 (sections 10.2-5).
b) In one school all 18 mathematics teachers for grades 1-9 are invited to join a lesson study oriented scenario on points, where I will take part both as an external instructor and as the observing researcher.
   The method is thus a school based sparring among colleagues teaching at the same school. Design and outcome of this research is described in chapter 12 (sections 12.1-7).

I expect this research involving action research with two groups of teachers to help provide some answers to my research question 3. Data will be collected by a mix of methods as in the survey study, i.e.: video recordings, questionnaires and memos. In both settings the teachers will also be interviewed.

RQ3: How can mathematics teachers be supported in a point-driven mathematics instruction through peer networks?

10.2 Need for a focus group

The results of the research so far are based on and evidenced by the observations of 50 mathematics lessons at grade 8. And it is shown that there is considerable variation in mathematics teaching approaches, when the presence of points is used as the sole indicator.

It is already described (chapter 8) how didactic points in different ways and to a mostly modest extent are articulated in the 50 observed lessons. The expectations expressed as regards academic level, differentiation and evaluation and mentioned in chapter 1, seem only to a limited extent fulfilled in the 50 lessons observed.

Some explanations may be found by asking teachers to give explicit reasons for actions. This was only occasionally done in the research among the 50 teachers, where information had to be deduced from observations, transcripts, researcher memos and rather short teacher questionnaires. A more systematic investigation of e.g. teachers' beliefs, attitudes or knowledge would require a new sub-study with this in mind.

Since such an effort should explore opportunities for developing an even better approach to mathematics teaching, it is natural to pursue the very clear recommendation in *Future mathematics* (Niss et al., 2006) and *The Teaching Gap* (Stigler & Hiebert, 1999), that didactic points should be clearly driving or guiding the mathematics teaching. But one observed lesson obviously implies some uncertainty in the assessment of the individual teacher's options and personal choices, so for a more valid study of teachers' opportunities for change and development, I decided to invite a selection of the 50 teachers to participate in a focus group.

For pragmatic reasons, as I planned a joint session and more video recordings at each school, this invitation was sent only to those schools in the Central region. All 16 teachers within this region from the original 50 teacher cohort were considered. They were from 12 different municipalities, as five of the teachers were from one large city (T1, T3, T12, T31 and T37). Three teachers among the total of 16 had previously refused to participate in any follow-up research (T10, T24 and T36). Two teachers (T34 and T37) were deliberately not invited as I found them so talented and experienced that they and I would hardly see much progress in point driven teaching even if they joined a focus group.
I thus invited a total of 11 teachers. The written invitation to the school principal and the mathematics teacher is attached as Appendix D. It was a still a large number to include for the involvement I planned, but one teacher never answered, one had moved to another school and two teachers declined. So, eventually there were seven teachers who wanted to participate in focus group activities.

The table below shows the 16 teachers from the original 50 teacher cohort in the Central region, of whom 11 teachers were invited and 7 accepted the invitation to join the focus group:

Teacher	No in first questionnaire	Questionnaire never returned	Too clever!	No thank you	No answer	Teacher moved	**OK**
1				X			
2							**X**
3					X		
4				X			
10	X						
12							**X**
13							**X**
20							**X**
21							**X**
24	X						
31						X	
34			X				
35							**X**
36		X					
37			X				
50							**X**

The seven teachers who accepted are located both in cities and small towns and vary in seniority, gender and educational background (see table in 10.2). They taught grade 8 in the school year 2008-09 and all taught the same class now in grade 9 in 2009-10. For two teachers (T2 and T20), the grade 9 class was mixed with other students as well, but still from same school.

I intended to use the focus group in different ways:

- First by arranging a "stimulated recall" interview with each of the seven focus teachers at a common seminar, where these teachers are shown and prompted for comments on excerpts from their recorded grade 8 lessons (take 1). At the same seminar the concept and scope of points is to be presented.
- Appointments are made for video recordings of future point driven mathematics lessons. Peer coaching and support is offered by me before the second video recording (take 2, December - January 2009/10) including a short interview and a questionnaire.
- Finally an interview and a third lesson is recorded (take 3, April 2010) after some months without any contact. The video recordings are coded and compared with respect to the occurrence of points and teacher elicitation to see if there is any change.

Beside the focus group research I arranged to research a lesson study intervention to increase the occurrence of mathematical points with all 18 mathematics teachers at one school. The school is not considered special in any way, but management and teachers expressed interest in my research. It is the school of one of the teachers from the original 50 teacher cohort and the focus group (T12). I intended to use this opportunity in several ways:

- First by presenting the idea of a lesson study strategy with an overarching goal and the importance of mathematical points to the school management and teachers.
- Next by organizing more rounds of lesson study in three groups of six teachers, who prepare, perform and evaluate each others' teaching in common. I intended to take part in this as both a processor and a researcher.
- Two grade 8 teachers, not previously involved in this research, are video recorded in one mathematics lesson before the course and one lesson after the course. The tapes are coded and compared with respect to the occurrence of points and teacher elicitation to see if there is any change.
- Finally the course is evaluated both collectively and by individual questionnaires.

The design and outcome of the two intervention studies are described in separate chapters below. I have been able to answer my research question 3, e.g.: by demonstrating ways of successful intervention by taking advantage of teachers' professional curiosity and ambition, as well as the idea of proficient peer support.
This may demand a deliberate strategy from teachers and schools regarding making up a mathematics tutor function for one proficient teacher at each school. Some Danish schools already have this position. The advantage would then be that peer support may be arranged on short notice and in a cost effective manner, and within the context and frames for mathematics teacher preparation, teaching and evaluation.
It is of course extremely interesting if a link can be shown between peer support and the extent and way points are being made. A successful use of points is not guaranteed by their mere presence. I register the occurrence and the type of mathematical points being made. In the initial research among 50 teachers they *may* be planned for by the mathematics text book or occur spontaneously. In the intervention studies points will be planned for by the teachers. But I still do not measure the quality of the mathematics teaching or the students' learning outcome.

10.3 Focus group seminar with "stimulated recall interviews"

A letter was sent to the school management inviting the participation of mathematics teachers in a focus group (Appendix D), and the seven teachers who accepted participated in a six-hour seminar at the VIA University College in Aarhus, Denmark in November 2009.
The seminar followed the plan below:

The project up to date: Actual Danish situation, recommendations. Research questions, data collection and processing and the need for a "focus group". Individual browsing of 2 clips of approximately 5 minutes in total from “own” video. Review in plenary of video clips, a semi structured interview of the teacher and a short debate. About *Lesson Study* as this is practiced in Japan. Points in mathematics teaching: On the various types and their role in teaching? Can we create change? A possible agreement on return visits at your schools (video recording in grade 9): Purpose and dates.

Two clips from each lesson were selected as one “successful” and one more “debatable” clip.

- A successful clip is meant to be an excerpt showing teachers’ conscious intentions leading towards a point in the classroom. Except for T50 these clips were all coded for one or more types of point. By including such clips I hoped to establish and maintain a constructive atmosphere allowing frank exchange of views.
- A “debatable” clip is meant to be an excerpt, where teacher actions may be chosen differently by others – or when students behave unexpectedly or inappropriately.

Lesson	**Seniority in years**	**Gender**	**Major subject**	**School size (students)**	**Town size**	**Successful clip**	**Debatable clip**
2	15+	Female	No	415	City	12:18-14:49	28:35-30:30
12	15+	Male	No	671	Large	27:10-29:15	0:40-4:40
13	15+	Male	Yes	486	Small	1:53-4:10	22:20-24:10
20	15+	Male	Yes	778	City	4:16-7:46	40:00-41:30
21	15+	Female	Yes	381	Small	2:10-5:08	29:00-30:15
35	5-9	Female	No	614	Small	6:00-8:20	18:52-22:00
50	10-14	Male	Yes	699	City	3:37-7:33	37:00-37:55

After morning coffee I told the teachers about my research methods so far without being too explicit on my special focus on mathematical points.

Each teacher were presented with a DVD copy of their own lesson and was then requested to review two excerpts of their teaching and prepare some comments for my later interview.

In an open plenary the semi-structured interview guide below was followed for the interview with each teacher individually after everyone had watched the relevant excerpts together:

1. Was this a "standard lesson" - or was it special? 2. Was there anything in the selected clips you see as your teacher routine? 3. Were there any surprises? 4. Is there anything you would prefer to have done differently? 5. How do you consider a certain structure in your teaching? 6. Examples of math-teacher-actions that you value highly? 7. Examples of beliefs that affects your choices? 8. Do you consider mathematics an important part of your teacher identity – or how do you assess it?	1. Var det en ”standard-time” – eller var den speciel? 2. Var der noget i de udvalgte klip, du vil kalde din lærerrutine? 3. Var der noget, der overraskede? 4. Er der noget, du godt ville have gjort anderledes? 5. Hvordan vægter du en bestemt struktur i din undervisning? 6. Eksempler på matematiklærer-handlinger, du vægter højt? 7. Eksempler på overbevisning, der styrer dine valg? 8. Er matematik en vigtig del af din lærer-identitet – eller hvordan vurderer du det?

The teachers’ responses were taped and transcribed, and I have chosen excerpts that illuminate teachers' reasons for choice – and thus also connect to beliefs and attitudes (as defined in section 9.4.3). T2 refers to the teacher in lesson 2.

1. Was it a "standard lesson" - or was it special?	
T2 (Topic: Coordinate system) For sure it was a standard lesson, as I was correcting homework. We began with some fussy (Christmas) calendar tasks, and then we hear some homework, they get something new to do, and they can sit and work during the rest of the lesson, while I go around giving a little help. And in the end, they will be given some homework. This is standard.	**L2** (Emne: Koordinat-systemet) Det var klart en standard-time, det var jo overhøring af lektie, vi indledte med noget julekalender-halløj, og så hører vi noget lektie, de får noget nyt for, og de kan sidde og arbejde i timens løb og jeg går rundt og hjælper lidt. Og de får nogle lektier for til slut. Det er standard.
T12 (Topic: Polar coordinates) Well, the book is split up to have some tasks – first, there is some theory, and then some tasks. And I will typically review the theory with the students. And then after each section, chapter – then there's a theme, or some theme pages. And depending on how much time we have, then I can go in. And we just got there, around this theme. And I thought it was obvious to ask them to do it, because	**L12** (Emne: Polære koordinater) Altså, nu Faktor er jo delt op, så der er nogle opgaver – først er der noget teori, så er der nogle opgaver. Og dem vil jeg typisk gennemgå med eleverne, teorien. Og så efter hvert afsnit, kapitel – så er der et tema, nogle tema-sider. Og det er alt efter hvor meget tid, vi har, så kan jeg godt gå ind. Og der er vi lige nået hertil, omkring det tema. Og der syntes jeg det var oplagt, og sætte dem til at lave det,

we had time for that. It might not always be possible to have plenty of time for things like that.	og fordi vi havde tid til det. Det kan godt være svært at have rigtig meget tid til sådan nogle ting der.
T13 (Topic: Perspective drawing) Yeah it is indeed – but it is also random. You asked to visit a random lesson, so it's a series of lessons already begun. We have been engaged in digital cameras and other things to see perspectives. Then you dive in one of these lessons – as they often are, one might say.	**L13** (Emne: Perspektivtegning) Jah det er det jo – også, men det er også tilfældigt. Du bad jo om at komme til en tilfældig time, så det er jo et forløb, der er påbegyndt. Vi har været i gang med digital-kameraer og andet, for at se perspektiver. Så du dukker ind i en time, som er – som de mange gange er, kan man sige.
T20 (Topic: Area of polygons) Yes, I think so, I think so. Review of some content I know, they rummage around in. Which they find it difficult to get an overview of – and which they have had before.	**L20** (Emne: Polygoners areal) Ja, det tror jeg nok, det tror jeg nok. Gennemgang af noget stof, som jeg ved, de roder rundt i. Har svært ved at få overblik i – og som de har haft før.
T21 (Topic: Powers) Yes, you can trust that (laughs)! Yes, this is precisely the way it is, we always start to pick up, yes – every week we have a plan of what we are doing in all the lessons, and the tasks they must do to every day. And so we gather up. Maybe there is some theory, we must review – and then otherwise they get started.	**L21** (Emne: Potens) Ja, det kan du tro, det var (ler)! Ja, det er lige sådan det er, at vi altid starter med at samle op, ja – vi har for hver uge en plan over, hvad vi laver i alle timer, og de opgaver, de skal lave til hver gang. Og så samler vi op. Måske er der noget teori, vi skal gennemgå – og så skal de ellers selv i gang.
T35 (Topic: Line equation) Well, I think, I think it was. It is first year teaching grade 8 in mathematics, so it is not more typical than that. But yes, it is the way I use to teach: I present some theory and they do some tasks. But this was really a repetition of something, we have had previously.	**L35** (Emne: Linjens ligning) Jamen det tror jeg, det tror jeg det var. Det er første gang, jeg underviser 8. klasse i matematik, så mere typisk er den heller ikke end at det, ja det er dén måde, jeg bruger til at undervise efter: Jeg laver noget teori eller de laver nogle opgaver. Men det her, det var så repetition af noget, vi har haft tidligere.
T50 (Topic: ICT competence) Well, it was part of a standard course, I would say. [It is a good answer, then I have to ask, if you might repeat a lesson like this?] I would *certainly* do that. You see, I have been a class-teacher to these students	**L50** (Emne: IKT kompetence) Altså, det er en del af et standardforløb, vil jeg sige.[Det er et godt svar, så er jeg nødt til at spørge, om du kan finde på det her igen?] Det kunne jeg *sagtens* fordi, altså den der klasse, den har jeg været

since grade 1. So I know them very well, all these students. And there have been massive problems with some of them, and now they have chosen (inaudible) with the supervisor and things like that – and many of them, including B – also have had problems and started to think, that they must move away from home to youth flats and things like that.	klasselærer for lige siden i første klasse. Så jeg kender dem meget godt, alle de her elever. Og der har været massive problemer med nogle af dem, og nu har de så valgt (uklart) med vejleder og sådan nogle ting - og mange af dem, bl.a. B som også har været ude i problemer, er begyndt at tænke på, at de skal flytte hjemmefra i ungdomsboliger og sådan noget.

All teachers confirm that the lessons in question are as they often are. More teachers justify this by referring to the structure including review or “pick up” of difficult content. Some teachers also by referring to the mathematical textbook: theory first, and then some tasks.

2. Was there anything in the selected clips you see as your teacher routine?	
T2 (Topic: Coordinate system) Yes, when it's something with drawings, I often make it myself on a squared sheet – and copy this onto a transparency. Sometimes I draw it on the squared area on the blackboard, it depends on the type. Because, it gets too messy otherwise – we must, there is too much idle time, students sitting there waiting if you do it carefully. So I would say, in grade 8, I prefer mainly to do it using transparencies. In grade 7 and also further down in grade 6, we will do it more in slow motion. At that time it is new. Now it’s not for the first time anymore, it is perhaps the third or fourth time. So then it is a different situation. **T12** (Topic: Polar coordinates) I would very much like the students to talk and find solutions. And then I am more of a consultant. I would also very much like them to learn the theory. Sometimes, they need a little help on the road. But I prefer, they do it on their own. They do have the “*Concept book*”, they can consult. And they work in groups, so they may be able to help each other. At this school we also use "*cooperative*	**L2** (Emne: Koordinat-systemet) Ja, når det er noget med tegninger, så har jeg ofte lavet den selv på et kvadreret ark – og kørt det over på en transparent. Nogle gange laver jeg det også på det kvadrerede felt på tavlen, det kommer an på, hvad det er for noget. Fordi, det bliver for sjusket ellers – man skal, der er for megen spildtid, de sidder der og venter hvis man gør det omhyggeligt. Så jeg vil sige: i 8. klasse, der vil jeg tage det nok mere på transparent. I 7. og også længere nede i 6., der tager vi det jo i slowmotion, der er det indlæring, der er det første gang. Det her, det er jo ikke første gang, det er måske tredje eller fjerde gang. Og så er det en ny situation. **L12** (Emne: Polære koordinater) Jeg vil meget gerne have eleverne til at snakke, og finde løsningerne. Og så er jeg mere en konsulent. Jeg vil også meget gerne have dem til at indlære teorien. Nogle gange, så skal de hjælpes lidt på vej. Men jeg vil meget gerne have dem til at gøre det selv. Og de har jo *Begrebsbogen*, som de kan slå op i. De sidder så i grupper, så kan de hjælpe hinanden. På skolen, der bruger vi også ”*cooperative*

learning" and this is a way to communicate together in such groups. And you don’t really ask the teacher in these situations, before everyone in the group more or less has given up [No, ok]. I find this very very, very good: They are doing the talking, instead of me.	*learning*”, og det er sådan en måde at kommunikere sammen på i denne her gruppe der. Og man kan faktisk ikke spørge læreren i nogen situationer, før alle i gruppen har givet op, mere eller mindre [Nej, ok]. Det synes jeg er rigtig, rigtig godt: så er det dem, der snakker i stedet for mig.
T13 (Topic: Perspective drawing) [In this clip, is it you standing at the blackboard repeating something, and ...] Yes, that's actually what I do here. It is not a wider review, but a brief intro to what will happen in the lesson. And this is part of the routine, I practice that regularly: Summing up what happened previously. Some students in the class also need such clarification. So that is the reason.	**L13** (Emne: Perspektivtegning) [I klippet her, er det så fx det, at du står ved tavlen og repeterer nogle ting, og …] Ja, det er jo det, jeg gør. Det er ikke spørgsmålet om en større gennemgang, men en kort intro til det, der skal ske i timen. Og det er også en del af rutinen, det gør jeg en del i: Samler op på, hvad der tidligere er sket. Også fordi, der er en række elever i klassen, der har behov for lige at få klaring, ikke. Så det er tanken med det.
T20 (Topic: Area of polygons) It's probably something in using what students say. And if it is correct, then it's fine. And if it is wrong, then we have to – preferably with their help – find out what is the correct answer. I think I use that.	**L20** (Emne: Polygoners areal) Det er vel et eller andet at bruge det, eleverne siger. Og hvis det er rigtigt, så er det jo fint. Og er det forkert, så må vi jo – helst ved deres hjælp – finde ud af, hvad der så er rigtigt. Dét synes jeg, jeg bruger.
T21 (Topic: Powers) [… Are you never in doubt that this is the best way for these students?] Yes, you may well be, but the fact is – this is the best way for me. Because I am like many others, I prefer peace and order, and when I can see all the students and see what they are doing. And therefore I don’t feel comfortable with – ok, they can sit and work in groups and the like, but I do not want them to sit outside the room because I need to be able to watch what they do.	**L21** (Emne: Potens) [… Er du aldrig i tvivl om, at det er den bedste måde for de her elever?] Jo, det kan man jo godt være, men det er jo sådan, jeg er ikke også – og det er den bedste måde for mig. Fordi jeg har det jo lige som visse andre, at jeg eksisterer bedst, når der er ro og orden, og når jeg kan se alle eleverne og kan se, hvad de laver. Og derfor har jeg det ikke godt med – altså de kan godt sidde og arbejde med det i grupper og sådan nogle ting, men jeg kan ikke have dem til at sidde uden for lokalet, for jeg kan holde opsyn med, hvad de laver.
T35 (Topic: Line equation)	**L35** (Emne: Linjens ligning)

[This is your routine, to be very well prepared with materials for transparencies. Or such a good overview for common conversation?] Yes [At least you had that in this lesson] Well it really is. But not all lessons are like this one. Following this one, we might as well have two lessons, where they only do tasks. So it is not always so much theory. **T50** (Topic: ICT competence) Yes, at least I catch – I try to anticipate and use what is in the mind of these students at the moment. I like to put what we do in mathematics into a context that has something to do with these kids.	[Det er det, der er din rutine, at være vældig godt forberedt med materialer til transparents, eller sådan nogle gode oversigter til fælles samtale?] Ja [Det havde du i hvert fald i denne her time], jamen dét er det. Fordi, men det er jo ikke alle timer, der er på dén her måde. Ud fra det her, der kan vi godt have to timer, hvor det så bare er opgaver, de laver. Så det er ikke så meget teori hver eneste gang. **L50** (Emne: IKT kompetence) Ja, jeg griber i hvert fald – jeg forsøger på at gribe det, der er i klassen lige p.t. til at bruge til noget. Jeg kan godt lide, at det vi laver i matematik, det bliver sat ind i en kontekst, som et eller andet sted har med de børn at gøre.

Most teachers confirm that the clips show one of their routines. There are certain roles to ensure correctness (T2, T35), to involve students with different needs and abilities (T12, T13 T20 and T50). But also by creating and maintaining frames for student work (T21).

3. Were there any surprises?	
T12 (Topic: Polar coordinates) Well, I'm more pleased with the last clip than I am with the first clip. Because the first clip is a little messy, I think. And this is because - there's always something that has happened in advance. I told them what we would be doing in the next lesson. They know what we need to do. And they also know that you come for the visit. I might have worded something in writing about the A3 and A4 sheets? To make them realize this was what they were working on. Some groups are very good at organizing themselves. And some would rather sit and glue sheets together before even beginning to figure out something more. So this is the reason for my remark, because I think it was a great idea to find out how much is inside a square meter. Perhaps I should have written	**L12** (Emne: Polære koordinater) Altså, jeg er mere glad for det sidste klip end jeg er for det første klip. Fordi det første klip er lidt rodet, synes jeg selv. Og det er det, for – der er jo noget forud. Altså, jeg har fortalt dem, hvad vi skal næste gang. Det kan de jo se, at det er det vi skal. Og de ved også, at du kommer på besøg. Jeg skulle måske have formuleret et eller andet på skrift omkring det her med A3-arket og A4-arket? Sådan at de vidste, at det var det, de skulle. Fordi der er nogle grupper, der er gode til at gå i gang med det. Men der er også nogle, der hellere vil sidde og klistre det her sammen, før de går i gang med at tænke over det andet. Så den der henskudte bemærkning, fordi jeg synes det var en rigtig god idé at finde ud af, hvor meget der er på en kvadratmeter. Den skulle jeg måske have skrevet eller

or said that. This is not indicated in the task.	formuleret. For det står ikke i opgaven.
T13 (Topic: Perspective drawing) I cannot say, even as I wasn't able to hear everything (because of noise in the room), I think I could remember what I said in the lesson (laughs), so that much of routine perhaps there is. There is nothing surprising in it. I couldn't say that.	**L13** (Emne: Perspektivtegning) Det kan jeg ikke sige, altså selv om jeg ikke kunne høre det før (pga. støj i lokalet), så tror jeg nok jeg kunne erindre, hvad jeg siger (ler), så meget rutine er der måske i det, ikke også. Der er ikke noget, der er overraskende i det. Det synes jeg ikke, jeg kan sige.
T20 (Topic: Area of polygons) How much chaos there is (laughs). [You think there is much chaos?] Yes, I think so. But it also pleases me to recognize K, the big boy sitting in front – fiddling around with all sorts, yet he is completely following the lesson! [He was very quick – one also saw that during the lesson]. Yes [He has not missed anything] No.	**L20** (Emne: Polygoners areal) Hvor meget kaos, der er (ler). [Du synes, der er meget kaos?] Ja, det synes jeg. Men det kan også glæde mig, at K – ham den der store dreng, der så sidder foran – så sidder og roder rundt med alle mulige, alligevel er han jo med! [Han var meget hurtigt med – det kan man også se i resten af lektionen]. Ja [han har ikke mistet noget som helst]. Nej.
T21 (Topic: Powers) Yes, it's always terrible to see yourself (laughs). [But apart from that?] Apart from that, no, I do not feel surprised.	**L21** (Emne: Potens) Ja, det er altid skrækkeligt at se sig selv jo (ler). [Men bortset fra det?] Bortset fra det, nej så synes jeg ikke, jeg bliver overrasket.
T35 (Topic: Line equation) No, no – not really. But I also know I am in a different place now, than I was at that time. Since, I have not – I've been teaching for seven years now, but I have had two maternity leaves and time off for poor health, so I've not been teaching for that many years really. And just in this half year since the video recording, I can see that I have become much more confident in the things I do.	**L35** (Emne: Linjens ligning) Nej, nej – det gør jeg egentlig ikke. Men jeg ved også jeg er et andet sted nu end jeg var dér. Altså, fordi, jeg har ikke – nu har jeg godt nok været uddannet i syv år efterhånden, men jeg har holdt to barsler og en sygdomsperiode undervejs, så jeg har ikke været lærer så længe egentlig. Og bare på dét halve år kan jeg se, at jeg er blevet langt mere sikker i de ting, jeg gør.
T50 (Topic: ICT competence) No, not really. So, this is my way of organizing things because there are also – every student does not sit inside the class right here and now. There is a system running, where those students, who I know can sit elsewhere and work without me having to look at them all the time, they may as well be allowed to leave the room. And	**L50** (Emne: IKT kompetence) Nej, egentlig ikke. Altså, det er min måde at organisere tingene på fordi, der er også – alle børn sidder jo ikke inde i klassen lige nu og her, og der kører sådan et system med, at de børn, som jeg *ved* kan sidde andre steder og arbejde uden at jeg behøver at kigge på dem hele tiden, de må gerne få lov til at forlade lokalet. Og de børn, som *ikke* kan finde ud af

kids who cannot administer working elsewhere, they stay inside the classroom. It is what I [Yes] and then I look out to watch elsewhere a couple of times. [Yes, I know - I followed you in every step]. Because we are a little scattered and things like that, and the computers are not always running. There may be a few start-up problems we have to cope with.	at arbejde andre steder, de sidder så inde i lokalet. Det er så hvad jeg [Ja] og så er jeg så lige ude og kigge nogle gange. [Ja, det ved jeg – jeg fulgte jo bagefter hver eneste gang]. Fordi vi ligger lidt spredt og sådan noget, og det er ikke altid computerne, de virker. Der kan være nogle opstarts-problemer, som vi lige skal have overstået.

Some teachers react by commenting on a relative mess (T12), chaos (T20) or uncertainty (T35) in “their” clips. But everyone accepts that the chosen clips represent part of their teaching.

4. Is there anything you would prefer to have done differently?	
T2 (Topic: Coordinate system) Well, it' depends on the students, it is both feedback from students, you know: Where are they? How thoroughly should they have it? How many times must it be told, and what did they really get hold of? And it has to do with the students we actually take? I got these students in grade 7, and that is to say that I'm going to have to put some of my standards in place.	**L2** (Emne: Koordinat-system) Jamen, det kommer jo an på eleverne, det er jo både feedback fra eleverne. Hvor er de henne? Hvor grundigt skal de have det? Hvor mange gange skal det siges, og hvad har de egentlig fat i? Og der er vi jo inde i det der med, hvilke elever overtager vi? De her elever, dem fik jeg i 7., og det vil altså sige, der skal jeg ind og have sat nogle af mine standarder på plads.
T12 (Topic: Polar coordinates) No, the most important is, that they get something written down and that they sit and work together, that is communicate on this task. I think this is actually the most important.	**L12** (Emne: Polære koordinater) Nej jeg synes det er vigtigst, at de får skrevet noget ned, og at de sidder og arbejder sammen, altså kommunikerer sammen om den her opgave. Det synes jeg faktisk er det allervigtigste.
T13 (Topic: Perspective drawing) No, we have been around, as I said – we've been around, what can I say – we were outside the classroom and trying to recognize some of these things they are about to work with. Then obviously it is important that it becomes real for them. As they are later put to work with some of the spatial shapes.	**L13** (Emne: Perspektivtegning) Nej vi er altså omkring, som jeg sagde – vi har været omkring, hvad skal man sige – det udenfor lokalet og så prøve på at genkende nogle af de her ting, som de skal til at arbejde med. Og så er det selvfølgelig vigtigt, at det også bliver konkret for dem. At de skal til at arbejde med nogle af de rumlige figurer der.
T20 (Topic: Area of polygons)	**L20** (Emne: Polygoners areal)

No, I do not think so. Yes, I would have used an overhead projector – but we do not have one at school, so [(laughs) What?] Yes, unfortunately. The budget does not allow that, at least not in every classroom. So I've given up [You perform with chalk then?] That I have to – no, we just got this one there (an electronic board) in another room, but that is another matter.	Nej, det tror jeg ikke. Jo, jeg ville have brugt en overhead – men det har vi ikke på skolen, så det [(ler) What?] ja, beklageligvis. Det siger budgettet ikke til, i hvert fald ikke ret mange steder. Så det har jeg opgivet [så du klarer dig med kridt?] det bliver jeg nødt til – nej vi har lige fået sådan en der (en elektronisk tavle) i en anden klasse, men det er en anden sag.
T21 ((Topic: Powers) Well, firstly I think – before you even say anything as a teacher – then there must be absolutely quiet. Then you are sure, that everyone will hear. If only half the class listens to what you say, there is not much point in that. They must focus on where it happens. And then I also think that it is very important that before you get them going, then you must be sure that everyone knows what to start with. So they have an opportunity to get started, so you do not suddenly have a, or immediately have 10 fingers, because so many cannot be dealt with at the same time. Students should feel confident, that they can get started. I think that is important. And then I think it is important that they all feel I've been around them in every lesson. That they have been in contact with me, in one way or another. [Yes, do you always succeed?] I do not know, but at least I try.	**L21** (Emne: Potens) Ja, altså for det første, så synes jeg jo altså – inden man selv siger noget – så skal der være helt ro. Så man er sikker på, alle har hørt det. For, hvis kun halvdelen hører, hvad man siger, det er der jo ikke megen idé i. Og de skal have fokus på, hvor det sker. Og så synes jeg også, at det er meget vigtigt, at inden man sætter dem i gang, så skal man være sikker på at alle ved, hvad de skal i gang med. Sådan så de har en mulighed for at komme i gang, sådan så man ikke pludselig har et, eller med det samme har 10 fingre, fordi så langt kan man jo ikke komme på én gang. Så eleverne føler, de kan komme i gang. Dét synes jeg er vigtigt. Og så synes jeg det er vigtigt, at de alle sammen føler jeg har været omkring dem i hver time. Altså, de har haft kontakt med mig på én eller anden måde. [Ja, lykkes det altid?] Dét ved jeg ikke, men det prøver jeg i alt fald.
T35 (Topic: Line equation) […it's the three girls, who in turn ask something. And they declare, although perhaps not quite true, that they really do not understand your explanations. … Should you then "rewind", or return to them for later help - or what are you thinking about this challenge?] Ok, this was A, who had not been around, when this was explained the day before – or in the previous lessons. That she did not understand it, I quite saw that. But she	**L35** (Emne: Linjens ligning) [… det er nemlig tre piger, der på skift spørger om noget. Og de erklærer jo, selv om det måske nok ikke er helt rigtigt, at de ikke forstår det. … Skal man så spole helt tilbage, eller gemme dem til sidst eller hvad tænker du om sådan noget?] Altså, nu var der lige A, hun havde ikke været der, da vi gennemgik tingene før – eller de foregående timer. At hun ikke forstod det, det var jeg helt med på. Men det må hun selv tage med mig lidt senere.

must take this up with me a little later. We should not spend the common class time on that.	Det skulle vi så ikke bruge hele klassens tid på.
T50 (Topic: ICT competence) [If I may be a bit cheeky, and I think I dare in such a closed circle here, one could say that you do not teach much mathematics? You seem to teach how to retrieve information?] Yes, right here … Yes, and how the spreadsheet works. And what we can use it for and so on.	**L50** (Emne: IKT kompetence) [Hvis man er en lille smule fræk, og det tør jeg jo godt være i sådan en lukket kreds her, så kan man sige, du underviser jo ikke ret meget i matematik? Altså, du underviser i, hvordan man henter oplysninger.] Ja, lige her … Ja og så sådan lidt med hvordan regnearket, det fungerer. Og hvad vi kan bruge det til og sådan.

Most teachers are not surprised by what they see. Some reflect on the necessary "standards" (T2, T12 and T13) required for such steady work by students as shown in the clips. Some teachers reflect on the differentiation in order to reach all students. Hence comments that you may have to take care in teacher led common presentation sessions requiring silence and attention (T21) or postpone individual guidance (T35) until later.

5. How do you consider a certain structure in your teaching?	
T2 Well I do not know, but I mentioned, when they have a coordinate system, then they should not put a blob because: what can you read in a blob? Is it a grid point or it is not a grid point! It depends on what kind of students, I get. What should I be starting with? I may as well give an example from today – yesterday, if you want that too, because I've discovered that the new class I had last year in grade 7, they had quite some trouble multiplying two two-digit numbers, and a two-digit number by a three-digit number. And now I've realized that there is a point when they should start unlearning that they apply crosses instead of 10s and 100's of zeroes? I did that yesterday. So it'll depend on the students. And that is what I call unlearning some of what they already HAVE, and then introducing what you yourself think is more mathematically correct.	**L2** Ja det ved jeg ikke, men jeg var selv inde på det der med, at når de har et koordinatsystem, så skal de ikke sætte en klat, fordi: hvad kan du aflæse i en klat? Er det et gitterpunkt eller er det ikke et gitterpunkt. Det kommer så an på, hvad det er for nogle elever, jeg får. Hvad skal jeg i gang med? Jeg kan også godt give et eksempel lige fra i dag – i går, hvis du vil have det med, for der jeg har opdaget at den nye klasse jeg havde sidste år i 7., de havde bøvl med at gange to tocifrede tal, og et tocifret tal med et trecifret tal. Og nu har jeg set flere gange, hvornår skal de begynde at aflære, at de skal sætte krydser i stedet for 10'er-nul og 100'er-nul? Det gjorde jeg i går. Så det kommer jo an på eleverne. Og det er det, jeg kalder, at vi jo skal aflære det, de HAR haft, og så ind med det, som man selv mener, er mere matematisk korrekt.

T12 This is typically when they've had some homework. Then I always ask whether they have any questions about it. And it is also within this "*cooperative learning*" that you can communicate in pairs as you discuss the homework, you have done. And then you sit and help each other with how things are done. And then ,when you have done this for 5-10 minutes, then I ask: Are there any questions now? And then, often they have actually solved them in pairs or in the 4-person groups they are in. Before I get going. It can get pretty boring for those who are talented and must listen to me, doing a review at the board – but I do it sometimes if I have a feeling that this they simply do not understood. And here I give them a lot of time to themselves in this group. And I actually give them homework before going out into these groups. That, they must do before the next lesson. And we do not really always sum up anything at the end.	**L12** Det er typisk, når de har haft et eller andet for. Så spørger jeg altid, om de har nogle spørgsmål til det. Og det ligger også i det her "*cooperative learning*", at man kan kommunikere to og to som man nu sidder omkring de hjemmeopgaver, man nu har lavet. Og så sidder man og hjælper hinanden med hvordan man nu har løst opgaverne. Og så, når man nu har gjort det i 5-10 minutter, så spørger jeg: Er der så nogle spørgsmål nu? Og så, tit så har de faktisk løst dem to og to eller i den 4-mandsgruppe, der nu er. Før jeg kommer på banen. For det kan godt blive ret kedeligt for dem, som er dygtige og skal høre på, at jeg står og gennemgår noget ved tavlen – men jeg gør det nogle gange, hvis jeg har en fornemmelse af, at det har de simpelthen ikke forstået. Og så er det så jeg, jeg overlader rigtig meget af tiden til dem selv i den her gruppe her. Og jeg giver dem faktisk lektier for, før de går ud i de her grupper. Altså, de skal lave herfra og hertil til næste gang. Og så er det faktisk ikke sikkert, vi samler op på noget til sidst.
T13 You can add that the lessons often start with some students having some questions about the tasks. This goes without saying. But I would like to postpone these, because it is not always the same number. Students should start their work, and then I talk to the few who might have some problems.	**L13** Man kan lige sige, at timerne ofte indledes med, at der er nogle der har nogle spørgsmål til opgaverne. Men det siger sig selv. Men dem vil jeg gerne tage efterfølgende, for det er jo ikke altid det samme antal. Så eleverne skal i gang med deres arbejde, og så samtaler jeg med de få, der måtte have nogle problemer.
T20 [If one visited you again, would you then experience the same structure in your ...?]. I might do that, yes. [How about such unstructured action, as leaving the classroom – or going to the computer room?] We could certainly do that… Yes, yes. [It does not indicate that you won't do	**L20** [Hvis man besøgte dig igen, ville man så nok opleve den samme struktur i din ...?] Det kunne jeg godt finde på, ja. [Dvs. sådan noget ustruktureret noget, med at gå udenfor fx eller gå i edb-rummet?] Det kunne vi sagtens finde på! Ja, ja. [Det dækker ikke over sådan nogle, altså at du ikke gør det?]

that?] No. There's some outside space opportunities, which means that if some students feel their best there – and they should be allowed to… So if this is called unstructured, then it is unstructured! I would not call it that (laughs).	Nej. Der er jo nogle plads-muligheder udenfor, som gør at der er nogle elever, der befinder sig bedst der – og det skal de have lov til. Så hvis det kaldes ustruktureret, så er det ustruktureret! Det ville jeg ikke kalde det (ler).
T21 [Some students are more demanding for the teacher than others…] It is quite clear! ... Yes, he's such – he is really a very good student, but terribly messy, unstructured. [I noticed that you even wrote in his notebook!] Yes! (laughs) [You do that occasionally?] Oh yes, believe me (laughs)! Well, because he should have as little as possible in his head, I know that – this is why he never will get it written down. So he cannot progress, unless he has this information. He is such a one who constantly needs help: what do I do? But he has it in his head.	**L21** [Nogle af eleverne er jo mere krævende for underviseren end andre …] Det er helt klart! … Ja han er sådan – han er egentlig en meget dygtig elev, men frygtelig rodet, ustruktureret. [Jeg lagde mærke til, at du på et tidspunkt *skrev* i hans hæfte!] Ja! (ler) [Det kan du altså godt finde på?] Ja, det kan du tro (ler)! Ja, altså fordi han skal have så lidt som muligt i hovedet, det ved jeg – så derfor det vil han aldrig få skrevet ned. Så kan han ikke komme videre, hvis han ikke har de oplysninger. Han er sådan én, der hele tiden skal have hjælp til, hvordan gør jeg? Men han har det oppe i hovedet.
T50 No, not in a mathematics context. Well, now this is a part where the tasks are very open. And I'm also doing courses in which the tasks are rather closed. Such as now, as I just started drawing shapes and stuff. And there are, after all – these are rather closed tasks. Yet the way that we must come down and draw. They must draw the house back home and such. [Yeah, so if I came down to you again, then I might see such a review and correct sequence?] You might, we also need occasionally to follow up or such. And I also need this myself, personally and constantly to follow up on: where do we go? Did they learn what I wanted them to learn? You could get into my classes, and then see that we are training skills: multiplication and decimal numbers and such.	**L50** Nej, ikke i matematiksammenhæng. Altså, nu er det her så en del, hvor opgaverne jo er meget åbne. Og jeg kører jo også forløb, hvor opgaverne er meget lukkede. Altså fx så, nu er jeg også lige i gang med tegneformer og sådan noget. Og der er, jo – det er meget lukkede opgaver. Men alligevel på den måde, at vi skal ned og tegne. De skal tegne huset derhjemme og sådan. [Ja, så hvis jeg kom ned til dig igen, så kunne jeg godt komme til at se sådan en gennemgå og rette-sekvens?] Det kunne du godt, for det har vi også brug for en gang imellem, at følge op på eller sådan. Og det har jeg også brug for selv, personligt og hele tiden følge op på: hvor er vi henne? Har de lært det, jeg gerne vil have, de skal lære? Man kan godt komme ind i mine timer, og så se at vi sidder og arbejder med færdighedsregning: multiplikation og decimaltal og sådan.

This question on structure made one teacher reflect on mathematical structures such as when teaching a certain multiplication algorithm (T2). But the teachers generally reserve their option to refer to variations in structure relating to homework (T12), student questions (T13, T21), needs for computer access or space (T20) and the mathematical topic in question (T50).

6. Examples of math-teacher-actions that you value highly?	
T2 I think it is important to use a mathematical language [it means that one must be correct – or what lies in the word?] No, it means: that it stands perpendicular to each other, e.g. Presently we are teaching the coordinate system and then we’ll use the words that belong to that. When we multiply, we know that we have one factor, and we have two factors. And then you don’t say the number you multiply with, you say the factor. They learn the mathematical words that belong to the various concepts we work with. So the mathematical language, I favor that. **T12** [Can you completely avoid standing at the blackboard to demonstrate or teach something new? You never do that?] Yes, I do. I do it in, now I teach grade 9, and there are some new areas which they must pass. There I review the theory with them. And again it is this "*cooperative learning*": They can sit and read some theory together in the group, and so try to understand it. And if this is something they do not understand, they put up their hand in the group and then you go to help them. But again, if I have a feeling that this is a problem to the whole class, I might as well do it in whole class teaching instead of repeating it say 4 or 5 times. So that's the way I do it. It’s simply communication, they should express as much as possible – and work as much as possible themselves. That I am very committed to! And it may well of course – there is a possibility –	**L2** Jeg synes, det er vigtigt at bruge et matematisk sprog [betyder det, at man skal være korrekt – eller hvad ligger der i det ord?] Nej, det betyder: at det står vinkelret på hinanden, fx Nu har vi om koordinatsystemet her, og så bruge de ord, der hører med. Når vi ganger, så ved vi at vi har én faktor, vi har to faktorer. Og så siger man altså ikke det tal man gange med, man siger en faktor. De lærer de matematiske ord, der hører til de forskellige begreber vi arbejder med. Altså det matematiske sprog, det går jeg ind for. **L12** [Kan du helt undgå at stå ved tavlen og demonstrere eller undervise i et eller andet nyt? Gør du aldrig det?] Jo, det gør jeg. Jeg gør det i de, altså nu har jeg 9. klasse, det er jo blevet 9. klasse den her, og der er en del nye områder, som de skal igennem. Der gennemgår jeg selvfølgelig teorien sammen med dem. Og det er igen det her ”*cooperative learning*”: de kan jo sidde og læse teorien sammen i gruppen og så prøve på at forstå det. Og hvis der så er noget, de ikke forstår, så rækker de hånden op i gruppen, og så går man ned og hjælper. Og igen, hvis jeg har en fornemmelse af, at det er over hele linjen, de spørger om det samme, så kan jeg lige så godt gøre det samlet i stedet for, hvor jeg siger det 4 eller 5 gange. Så det er den måde, jeg gør det på. Altså, det er simpelthen kommunikation, og de skal sige så meget som muligt selv – og arbejde så meget som muligt selv. Det går jeg meget ind for! Og

because sometimes it can be hard to know where they all are – and whether the talented help the more weak along. In this case it is my job to try and discover it!	det kan selvfølgelig godt – der er en mulighed for – for nogle gange kan det godt være svært, at vide, hvor de helt er – og om de dygtige får de svage med. Og det er så min opgave, at prøve og opdage det!
T13 [Are you not using much of the common time to correct or review tasks? Is this the way it should be understood?] Namely, if there is one – this can quickly be determined – if there is a common problem, it will be presented while we are together. But often there are very few, and so the others should continue on their course.	**L13** [Bruger du ikke ret meget af den fælles tid på at rette eller gennemgå opgaver? Er det sådan, det skal forstås?] Det skal forstås sådan, at hvis der er et – det kan man hurtigt lokalisere – hvis det er et fælles problem, så bliver det præsenteret mens vi er sammen. Men ofte, så er det jo få, og så skal de andre videre i deres forløb.
T20 [In the last clip I have caught you in a group where you are observing for a while. Is this also a characteristic …?] Yes because, I know there is great diversity in this group, and there is this one “leader-boy” who has all the answers, and sometimes needs to be challenged to see other answers. And those – I’ve got to make sure that the weaker students, they do not get left behind. That is why I've looked into the group discussion at some stage, taking the lead to tell them that there are other options than the one of J. I think they should know that. This is probably why I did so. [The group, did they form it themselves?] Yes. [And this is how it works: You sit together, or in larger groups if you prefer that?] Often, yes.	**L20** [I det sidste klip har jeg fanget dig i en gruppe, hvor du meget observerer et stykke tid. Er det også karakteristisk …?] Ja, fordi det er, jeg ved der er stor spredning i den gruppe, og der sidder sådan en ”fører-dreng”, som har svarene, og indimellem skal udfordres til at der også er andre svar. Og dem – jeg skal jo sikre mig, at de svage elever, de ikke bliver hægtet af. Derfor går jeg jo så ind på et eller andet tidspunkt og tager noget styring for at fortælle dem, at der er altså andre løsningsmuligheder end J, han kommer med. Og det synes jeg, de skal vide. Jeg tror, det er derfor, jeg har gjort sådan. [Den gruppe, har de selv dannet den?] Ja. [Og sådan fungerer det: Man sidder sammen, eller i større grupper, hvis man synes at man gerne vil det?] Tit, ja.
T21 [I also noticed, that you at some point asked students to read the task aloud – even for the whole class. Is this something you prefer doing – because you consider it necessary to …?]. Yes, there are some weak readers in the class. I have a few, who are good at arithmetic, but very poor readers. And this means, if they cannot read the task,	**L21** [Jeg bed også mærke i, at du på et tidspunkt beder dem læse opgaven højt selv – også for klassen. Er det noget, du gerne gør. Altså fordi det er noget, du tror, er nødvendigt for…?] Ja, for der er nogle dårlige læsere derinde. Jeg har et par stykker derinde, som er gode regnere, men meget dårlige læsere. Og det betyder, hvis ikke de kan læse

they won’t even get started. **T35** Well, I think I put much emphasis on mathematics when we count and construct drawings, then it is a precision subject. So, it must therefore be precise, we should not have such a number as around something. If it is 3 cm, it is therefore precisely 3 cm. It is not just around 3. It should be exact.	opgaven, så kan de jo ikke komme i gang. **L35** Jamen, jeg tror jeg lægger meget vægt på, at matematik når vi regner og tegner, så er det et præcisionsfag. Altså, det skal altså være præcist, vi skal ikke have sådan nogle, hvad hedder det: afkortelser, der sådan er: Nå, sådan cirka – eller hvis det er 3 cm, så er det altså 3 cm. Det er ikke sådan lige omkring 3. At det skal være præcist, det vi laver.

The mathematics teacher actions mentioned may be very much affected by the short clips. Highly valued were presenting and demanding a correct mathematical language (T2, T35), theoretical introductions and necessary summing up (T12, T13), interventions and help based on observation (T20, T21).

7. Examples of beliefs that affects your choices?	
T13 [Is there any – is there such a belief behind your choices, so one could say I'm driven by this and this belief when I am a teacher of mathematics?] Yes, but there are many things at stake in teaching situations. So, I believe, I prefer to feel good, you might say. That is, my teaching is somewhat influenced by the fact that there are some things that must be successful. And also considering that I am working the best in non-chaotic situations, one could say. I do, and I also think the students then get the most out of it. But after a very brief intro to a number of things, I consider it important for students to get started with their work. And then through conversation with each one of them I can get them to move forward in the project. **T35** [Can one see your personality? Does it color your teacher role do you think?] Yes, to some extent it does. I'm probably	**L13** [Er der nogen – er der sådan en holdning bag dine valg, så man kan sige, jeg er styret af dén og dén overbevisning, når jeg er lærer i matematik?] Ja, men der er mange ting, der er på spil i undervisningssituationer. Altså, jeg har det, jeg skal selv have det godt, kan man sige. Det vil så sige, min undervisning er også lidt påvirket af, at der er nogle ting, der skal lykkes. Og så også den opfattelse, at jeg jo fungerer bedst i ikke-kaos, kan man sige. Det gør jeg selv, og jeg tror også eleverne mange gange får mere ud af det. Men efter meget kort intro til en række ting, så synes jeg det er vigtigt, eleverne kommer i gang med deres arbejde. Og jeg så via samtale med de enkelte kommer videre i projektet. **L35** [Kan man se din personlighed? Farver den igennem synes du, i lærerrollen?] Ja, det gør den lidt, det gør den. Jeg er nok

extremely caring and (laughs) I am also a class-teacher for these students. And this (unclear) next question, on mathematics, it is an important part [Of your teacher-identity?] Yes, but I am first and foremost a teacher! And I will primarily be that to them, and make them feel good. If they don’t feel good, they will not learn anything. So, of course I am a mathematics teacher, but I am first and foremost class-teacher.	ekstremt omsorgsfuld og (ler) jeg er også klasselærer for dem her. Og det (uklart) dit næste spørgsmål her, om matematik, det er en vigtig del [af din lærer-identitet?] Ja, men jeg er først og fremmest lærer! Og jeg skal først og fremmest være det for dem, og få dem til at have det godt, fordi hvis ikke de har det godt, så kan de ikke lære noget. Såh, jo jeg er matematik-lærer, men jeg er først og fremmest klasse-lærer.
T50 I like that – I think this – my teaching is of course based on mathematics, right. Some of the content I think they should learn. But it also draws on what is happening to them right now. And it also assumes that I would like to – I have the principle that now it is *me* who is paid to be here – therefore I will not do anything! It is the students who must work and they must – and at the same time, it will also be good that they develop their language. Sometimes it is the mathematical jargon, but other times it turns up in their everyday language. That's what they use to think with, I assume. So mathematics must also be inserted into their thinking. I think the language is the key to that.	**L50** Jeg kan godt lide, at – jeg synes, det – min undervisning den tager selvfølgelig udgangspunkt i matematik, ikke. Noget af det stof, som jeg synes de skal lære. Men det tager også udgangspunkt i, hvad der rører dem lige nu. Og så tager det også udgangspunkt i, at jeg vil godt have – jeg har det princip, at nu er det *mig*, der får løn for at være her – så derfor skal jeg ikke lave noget! Det er eleverne, der skal arbejde, og de skal – og samtidigt med, så vil jeg også godt have, at de får sat sprog på. Og det er nogle gange det matematiske fagsprog, men andre gange så tager det altså også udgangspunkt i deres hverdagssprog. Det er den måde, jeg mener, det er det de bruger til at tænke med. Så matematikken skal også sættes ind i deres tankegang. Derfor mener jeg sproget det er så indgangsvinklen til det.

These examples show teachers awareness of expectations in the subject. As a mathematics teacher you have a responsibility to succeed in teaching the subject (T13, T50) and the students need to feel good when doing mathematics (T35).

8. Do you consider mathematics an important part of your teacher identity – or how do you assess it?	
T2 Yes, I am a mathematics teacher! And I am a mathematician. And one of the reasons that I do not have mathematics as my main subject	**L2** Ja, jeg er matematiklærer! Og jeg er matematiker. Og én af grundene til, at jeg ikke har matematik som linjefag, det er fordi

is – because in ancient times, when I went to teacher training college – I considered it a waste of time. I would spend my time at the college better by taking some other main subjects, which developed me personally. Because the mathematics I actually did rather well. Later it turned out, to teach the "extended" courses we had in the 70’s, there were some other old mathematics teachers who said: No-no! So I therefore took the relevant courses [Yes]. Then they could not decline it anymore. Now they all have been pensioned off (laughs).	i tidernes morgen, da jeg gik på seminariet, da syntes jeg det var tidsspilde. Jeg ville bruge min tid på seminariet bedre ved at tage nogle andre linjefag, der udviklede mig personligt. Fordi matematik, det kunne jeg faktisk godt. Senere viste det sig så, at for ar få de der ”udvidede” hold, som vi havde der i 70’erne, så var der jo nogle andre gamle matematiklærere, der sagde: Nej-nej! Så tog jeg så de kurser, der var relevante [ja]. Så kunne de så ikke komme med det mere – nu er de så også gået af (ler)
T12 I am a science-teacher! So geography, biology, I am very interested in those subjects. Mathematics as well of course [You perceive mathematics here than as part of the natural sciences?] Yes.	**L12** Jeg er naturfags-lærer! Altså geografi, biologi går jeg også meget op i. Og så matematik selvfølgelig [og du opfatter matematik her som end del af naturfagene?] Ja.
T13 Yes, but we can say that it has become that, because I have the number of lessons I have. And that is also good. But if you must know where my main interest lies, it lies in civics and community oriented content. And I also think now I can see – I also teach geography – that there is a consistency in the subjects. And of course I use that, because I have students for the same thing – all three subjects.	**L13** Ja, men det kan man så sige, at det er blevet, fordi jeg har det antal timer, jeg har. Og det er også godt. Men hvis man skal sige, hvor den største interesse ligger, den ligger omkring samfundsfag og det samfundsorienterede. Og der synes jeg også efterhånden jeg kan se – og jeg har også geografi – at der er sammenhænge i fagene. Og det udnytter jeg selvfølgelig, fordi jeg har eleverne til det samme – alle tre fag.
T20 Yes, I do. But it can be made use of, as you also said (a former teacher interviewed in the focus group) in geography, and history also of course.	**L20** Ja, det synes jeg. Men det kan bruges, som du også sagde (en tidligere interviewet lærer i fokusgruppen) i geografi, og historie for så vidt også jo.
T21 Yes! I am, I think so! [Really?] Yes, I think (laughs)! [What IS it to be mathematics teacher, then?] Yes, I do not know. But it is because one has had many lessons in the subject and always, you know ... [Yes, but does it provide such a specific	**L21** Ja! Det er jeg, det tror jeg nok! [Med store bogstaver?] Ja, det tror jeg (ler)! [Hvad ER det at være matematik-lærer, så?] Ja, det ved jeg ikke. Men det er jo fordi man altid har haft mange timer i faget og altid, ikke også ....

security, or a particular clout or do you just feel comfortable with the role?] You probably feel comfortable with a subject, where you consider yourself in total control. And then there's probably something with – it has certainly been this way for many years – that some subjects are more prestigious than others. It is rarely, that parents ask their children about how they are doing in Christianity, you see. But they always ask, how is x in mathematics and how is it in Danish. In that way, there is more prestige in such a subject than there is in many others.	[Ja, men giver det sådan en særlig sikkerhed, eller en særlig pondus eller man føler sig godt tilpas med den rolle?] Man føler sig vel godt tilpas med et fag, som man synes man er herre over. Og så er der vel et eller andet med – sådan har det i hvert fald været gennem mange år, at nogle fag er mere prestigefyldte end andre. Det er altså sjældent, at nogle forældre de spørger deres børn om hvordan det går dem i kristendom, ikke også. Men de spørger altid, hvordan går det dem i matematik og hvordan går det dem i dansk. Så på dén måde, så er der mere prestige om sådan et fag, end der er om så mange andre fag.
T35 We had shared class-teacher functions to start with, the Danish teacher and me. But then the Danish teacher got sick, and also quit his job, so now it's me that [So you are the anchor to this group of students].	**L35** Vi havde delt klasselærerfunktion til at starte med, dansklæreren og jeg. Men så er dansklæreren blevet sygemeldt, og har også sagt sit job op, så nu er det mig, der [det er dig, der er det faste holdpunkt for den her gruppe elever].
T50 Well, I'm probably more a natural science-teacher, I think [Yes, one more – do you include mathematics in that crowd?] Yes as science, yes.	**L50** Jamen, jeg er nok mest naturfags-lærer, tror jeg [ja, én mere – hører matematik med i den flok der] ja som naturfag, ja.

The waters divide here. Some teachers declare themselves mathematics teachers (T2, T20 and T21). One sees himself primarily as a science-teacher, mathematics included (T12, T50) and another one the combination with civics and geography teaching (T13) as identity formatting. Finally one teacher feels more generally responsible to these students being also their class teacher (T35). The relatively small number of seven teachers certainly does not justify too many generalizations. Five of the seven teachers are very experienced with a seniority of 15+ years. The gender is well mixed (three female and four male). The formal mathematics background is also mixed as three of the seven did not have mathematics as a line subject. One took a course to compensate.

The stimulated recall interviews quoted above indicate more varying mathematics teacher routines, mathematics teacher beliefs and feelings about teacher identity.
Generally the teachers were pleased to see the excerpts. They all commented on the clips as quite typical of their teaching, and comments on the debatable clips were

rather few. Perhaps because they were asked in the interview to comment on possible "surprises" and not on the "discussable":

- T2 recognized her *standards* as she keeps insisting on precise language and drawings also by the students.
- T12 regretted the lacking *precision* in his initial presentation of the lesson assignment, as this led to some confusion in the student (cooperative learning) groups.
- T13 accepted an occasional need for a common review but *emphasized guiding individual students.*
- T20 found the lesson too chaotic when reviewing the excerpt, but emphasized his wish to have *students discovering* relationships.
- T21 did not like to see herself but was satisfied by her expressed demands for *student silence and attention*.
- T35 estimated that she (half a year later) was *now much more confident in her teacher actions* seen in the clip.
- T50 accepted that his teaching was oriented also towards *students' extra-mathematical* needs.

One teacher (T12) feels convinced that students benefit from a cooperative learning approach to lesson organization, i.e. he intends to let the students do the talking. His classroom becomes a bit noisy, and dialogue between teacher and students tends to be more or less "Socratic". Another teacher (T21) expects and demands a quiet classroom and carefully instructs students to avoid queues building up. One teacher (T35) describes herself as caring and puts emphasis in a "feel good" atmosphere. One teacher (T2) has worked hard for years to put some "standards" in place. But her conviction on priorities does not hinder her in using feedback from students.
Two teachers (T13, T21) both stress the importance of lessons not being too chaotic and associate this with their own need for calmness and concentration.

The stated beliefs and attitudes of the teachers impose different teacher roles and varying possibilities for their students to make conceptual, procedural, result or interpretation points.
Very few of the teachers visited in the first part of the present research referred to peer support in planning or evaluation of teaching. One teacher regretted this and sort of had given up after trying to establish some cooperation (T13).

Points were *not* a distinct issue to these teachers in my interview questions and their answers. Please note that the teachers reacted to the clips just shown. Especially question 6 though on mathematics teacher actions highly valued offered an obvious possibility to mention points. Two focus teachers chose to mention a precise language (T2, T35), which is certainly crucial and connected e.g. to mathematical concepts and reasoning. Another (T12) also emphasized students' taking part in classroom communication. But more common was the weight on teachers' responsibility to lesson organization and flow when meeting different students' prior knowledge and needs.

By the end of the seminar each focus teacher was made aware of my emphasis on points. In my talk I defined a "point" as *a mathematical idea (result, statement, method, ..), the teacher has judged particularly important for the students'(s insight, understanding, application, ...).* And these research questions were presented:

- How much, how and why do mathematics teachers formulate mathematical points?
- To what extent and how can mathematics teachers be supported in point-driven mathematics teaching by peer networks?

I also presented the different types of points already described above in chapter 6. And I suggested various situations, where one might expect points to be articulated in a lesson. Finally I invited a common debate on how to create change:

- Would *I* see something different if I observed the teaching of you focus teachers again?
- Can you plan for a lesson to have one or more points?
- Is it possible for us to partake in peer coaching via email and / or a chat in the staff room, for example, the week before?

I offered peer help in preparation of some coming mathematics lessons to evaluate the potential in such peer support. Peer coaching may qualify the alleged use of the *Common Goal* (*Fælles Mål*, 2009) to frame management and actual planning. Will it also be possible to support conscious teachers in a point-driven mathematics teaching by peer efforts? This is researched in the focus group.

10.4 Sparring with focus teachers December 2009 - January 2010

A way to raise consciousness on clear goals and careful design of single lessons might be to engage in mutual planning with colleagues. One may wonder why such collaboration absent among mathematics teachers. Has this to do with the subject characteristics, teacher knowledge or beliefs, or does the extensive use of mathematics textbooks perhaps make the teachers regard collegiate sparring or reflection as superfluous.

After the seminar invitations to attend peer coaching and to partake in video recordings 2 and 3 in grade 9 were sent to school managements and teachers. These include a letter for parents to get permission to record video and eventually use clips or pictures in teaching or publications. Also a teacher questionnaire including questions on points was produced for the take 2 recording (Appendix E).

All participants signed agreements on my return visit for the second video recording (take 2) and potential sparring before this during December 2009 to January 2010. The class would be the *same* students as videotaped earlier, but now in grade 9.
Further it was the agreement that I could come to the schools one last time for video recording (take 3) in April 2010. The assumption here is of course that peer sparring among mathematics teachers should not be recommended unless there is some observable effect of the joint reflection in the November seminar (section 10.2) and my individual sparring with teachers before the second recording of a mathematics lesson.

It turned out that the 7 teachers had very different energies, needs and desires to be active in a sparring with me. The table below shows the extent, and the following examples of mail tell about different levels of teachers' planning. When I had the chance, I reminded each teacher in the focus group that it was the didactic "points" that had my main interest.

Teacher	Sparring with researcher on content	Take 2	Take 3
T2	AM mail 17/11: Medians and center of gravity T mail 11/12: Order and program AM mail 14/12: Order and A4 paper T mail 15/1: Samples, MathCad and program AM mail 20/1: LIX	20/11 2009	22/4 2010
T12	AM mail 2/12, 3/12 and 14/12 AM at meeting at school 16/12 Mail 16/12: Pythagoras and GeoGebra software	30/11 2009	12/4 2010
T13	AM mai1 10/12: GeoGebra and triangles T mail 5/1: How much at a time? AM mail 5/1: Here we go AM mail 10/1: Squares in Networks T mail 18/1: The progression etc. AM mail 20/1: Pending	7/12 2009	8/4 2010
T20	T mail 7/12: Quadratic equation? AM mail 9/12: The parameter significance T mail 13/12: Parabola considerations AM mail 14/1: Two straight lines ideas T mail 14/1: Sounds exciting	27/1 2010	14/4 2010
T21	AM letter 8/12: Differentiation, red thread ... T letter 20/12: Thanks + Cancellation due to surgery.	11/12 2009	Declined
T35	AM mail 15/12: GeoGebra software and triangles	1/12 2009	19/4 2010

	T mail 17/1: Trigonometry and objective AM mail 20/1: The mathematics team and peer sparring		
T50	AM mail 17/12 + post (as he does not respond): GeoGebra software and triangles	16/12 2009	No confirmation

The many e-mails included suggestions on topics, structure and points. The following excerpts are made to demonstrate the large variation in extent and success rate, when “inducing” the idea of points.

<table>
<tr><th colspan="2">T2: E-mails</th></tr>
<tr><td>20 /1 2010
Hi M
Thanks for your mail of 151 2010 with another update on the work of weeks 3-5.
I'm <u>very</u> impressed that you work with MathCad. For the ambitious and talented, who will continue on to high school next year, it's definitely a good idea.
But I am by now almost convinced that it is too difficult for everyone in a school. …
Regarding LIX you maybe know the pages I wrote for a grade 6 textbook on "Readability"?
I attach a work card I made back then (Lix.doc). There are probably better ideas – and you are welcome to elaborate. There are many links to sites on the Internet that offer ways to calculate the lix index. …
Many greetings, AM</td><td>20/1 2010
Hej M
Tak for din mail af 15/1 2010 med nok en opdatering om arbejdet i uge 3-5.
Jeg er <u>meget</u> imponeret over, at I arbejder med MathCad. For de ambitiøse og dygtige, der vil fortsætte i gymnasiet næste år, er det absolut en god idé. Men selv er jeg efterhånden ved at mene, at det er for svært for alle i en folkeskole. …
Vedrørende LIX kender du måske de sider, jeg skrev til Faktor 6 om ”Læsbarhed”? …
Jeg vedhæfter et arbejdskort, jeg lavede dengang (Lix.doc). Der er nok bedre ideer – og du er selvfølgelig velkommen til at brodere videre. Der er mange links til steder på Internettet, der tilbyder at beregne lix. …
Mange hilsner AM</td></tr>
<tr><td>31/1 2010
Hello
Thanks for the quick reply. Yes, I have always points on my mind. I have become more conscious of this after you presented the concept. Over the years, I have of course often had to use the idea to repeat a problem with own examples for several consecutive sessions, simply because the textbook steps too quickly forward and leave the students with a feeling that the subject has not had time to settle …</td><td>31/1 2010
Hej
Tak for hurtigt svar. Ja, jeg har hele tiden pointerne i baghovedet. Jeg er blevet mere bevidst om dette, efter at du satte begreb på. I årenes løb har jeg naturligvis ofte været nødt til at bruge den ide, at gentage et problem med egne eksempler i flere fortløbende lektioner, simpelthen fordi lærebøgerne går hurtigt videre og eleverne sidder med en fornemmelse af, at emnet ikke har nået at bundfælde sig …</td></tr>
</table>

Sincerely, MK	Venlig hilsen, MK

T12: E-mails	
Hi P Thanks for your mail with the notice of a new topic. Proofs are exciting, but hard for students in grade 9. I have seen, the textbook authors themselves note this in their thin little teacher guide. So the tasks are more about to show through examples that the relationships are correct. In the "*Common Goals II*" the aim is probably a bit higher. On page 9 of the book it says in the section on *Mathematical competencies* that students after grade 9 must be able to "*devise, implement, understand and evaluate oral and written mathematical reasoning and work with simple proofs*." If there is time you may: 1) Let each student draw a large triangle on thick cardboard (e.g. from a cardboard box). In this triangle all 3 medians are drawn, and it is cut out. With a needle (pins are good), students now must find the point that can be used as shaft … It turns out (of course) to be the median intersection. But why, one might well ask in class. Yes each median divides the triangle area … 2) In a dynamic geometry program as … the free GeoGebra one may construct one or two arbitrary triangles. … In one triangle, one could for example show that the perpendicular bisectors of the three sides <u>always</u> intersect at the same point, and this is always the center of the <u>circumscribed</u> circle (and ask the class: Why is this so?). No matter how much you change the triangle by "pull" on the corners. 3) In the second triangle, one could show that the bisector lines of the three	Hej P Tak for din mail med varslet om nyt emne. Bevisførelse er spændende, men svært for elever i 9. klasse. Det har jeg set, Faktor-forfatterne selv noterer i deres lidt tynde lærervejledning. Så opgaverne handler mere om at vise gennem eksempler, at sammenhængene er gode nok. I *Fælles Mål II* stiles nok en smule højere. På side 9 i hæfteudgaven står der under *Matematiske kompetencer*, at eleverne efter 9. klasse skal være i stand til, at "*udtænke, gennemføre, forstå og vurdere mundtlige og skriftlige matematiske ræsonnementer og arbejde med enkle beviser*." Er der tid kan man måske: 1) Lade hver elev tegne en stor trekant på tykt pap (fx fra en papkasse) I trekanten tegnes alle 3 medianer, og den klippes så ud. Med en nål (nipsenåle er gode) skal eleverne nu finde det sted, der kan bruges som aksel … Det viser sig (selvfølgelig) at være medianernes skæringspunkt. Men hvorfor, kunne man så spørge i klassen. Ja hver median deler jo trekantens areal … 2) I et dynamisk geometriprogram som … det gratis GeoGebra kan man konstruere en eller to vilkårlige trekanter. … I den ene trekant kunne man fx vise, at midtnormalerne <u>altid</u> skærer hinanden i samme punkt, og at det altid er centrum for den <u>omskrevne</u> cirkel (og spørge klassen: hvorfor mon?). Altså ligegyldigt hvor meget man ændrer trekanten ved at "trække" den i hjørnerne. 3) I den anden trekant kunne man vise, at vinkelhalveringslinjerne også skærer

angles also intersect at one point, and this is always the center of the triangle's inscribed circle (and ask the class: Why is it so?). 4) Searching the Web with "GeoGebra" you'll find dynamic examples that e.g. demonstrate Pythagoras. … 5) Finally, there is a great Norwegian website, you ought to try. Here you can also choose e.g. Pythagoras and look at one of the visual proofs … Well – these are just some ideas. … Many greetings AM	hinanden i ét punkt, og at det altid er centrum for trekantens indskrevne cirkel (og spørge klassen: hvorfor mon?). 4) Søger du på nettet med ”Geogebra” kan du finde dynamiske eksempler, der demonstrerer fx Pythagoras. …. 5) Endelig er der et fantastisk, norsk websted, du næsten selv bør prøve. Her kan man også vælge fx Pythagoras og se på et af de visuelle beviser … Nå – det er løse idéer. … Mange hilsner AM

T13: E-mails	
22/11 2009 Dear A! Thanks for last time. The accompanying DVD was immediately demonstrated for my wife when I got home, as if I had not seen enough! The performance in teaching, I found decent enough (everything else would have surprised me somewhat). Moreover, not a bad idea to be filmed during the first year as a teacher - and probably even later! Now it is Monday, December 7 at 10 am. Where I last time "just" had to teach a normal lesson, I sense that you would like to see a more organized lesson containing certain elements. It has given / gives rise to numerous speculations on what I should work on. It has always been central to my teaching to draw conclusions along the way, either by doing it myself or by having the students do it for me, usually at my invitation. What I in the current lesson could wish is that students to an even larger extent are able on their own to find the "point". Topic: Working with graphs in modeling of "real" events. How can graphs be used e.g. politically?	22/11 2009 Kære A! Tak for sidst. Den medfølgende dvd skulle straks demonstreres for min hustru, da jeg kom hjem, som om jeg ikke havde set nok! Præstationen i undervisningen kunne jeg godt være bekendt, synes jeg (andet ville også have overrasket mig noget). I øvrigt ikke en dårlig ide at lade sig filme i løbet af de første år som lærer – og nok også senere! Nu gælder det mandag den 7. dec. kl.10. Hvor jeg sidste gang ”blot” skulle gennemføre en normal lektion, fornemmer jeg, at du gerne ser en mere tilrettelagt lektion, indeholdende bestemte elementer. Det har givet/giver anledning til en række spekulationer om hvad, jeg skal arbejde med. Det har altid i min undervisning været centralt undervejs at konkludere og pointere, gerne fra elever, som oftest på min opfordring. Det jeg i den aktuelle lektion kunne ønske er, at eleverne i endnu højere grad er i stand til på egen hånd at finde ”pointen”. Emne: Arbejde med grafer i forbindelse med modellering af ”virkelige” hændelser. Hvordan kan grafer bruges f.eks.

Manipulation, opinion, etc. How can graphs illustrate everyday events? And preferably more situations. ... At the end of the lesson I hope that students are able to summarize / conclude! Do you have relevant input to this, I would like to know. Is the topic choice not OK and / or the task misunderstood, then let me know this. Sincerely, PG	politisk? Manipulation, meningsdannere mm. Hvordan kan grafer illustrere almindelige hverdagsbegivenheder? Og gerne flere situationer. … Jeg håber på afslutningsvis i timen, at elever er i stand til at sammenfatte / konkludere! Har du relevante indspark til dette, modtager jeg gerne disse. Er emnevalget ikke OK og/eller opgaven misforstået, lad mig da dette vide. Venlig hilsen, PG
23/11 2009 Dear P Thanks for your mail and related "speculation”. You are absolutely right that of course I also hope to see / hear some points when we meet next time. One can easily find the inspiration for "graphs" in many places. One idea that has worked well for me is a Norwegian booklet from 1995 about "diagnostic teaching" (Brekke, 2002). Or rather: diagnostic tasks, which then tell the teacher more than just: whether the students are doing it right or wrong …. It's good to get wiser by, but it is only the pages 11-15 in the file above that are about graphs. Perhaps they can inspire you? One can copy 1-2 figures and then let students discuss what they think and see and as you propose, present it to the rest. Or just use the figures in a class discussion. AM	23/11 2009 Kære P Tak for din mail og tilhørende "spekulation”. Du har helt ret i, at jeg selvfølgelig håber også at se/høre nogle pointer, når vi ses næste gang. Man kan nok finde inspiration til "grafer" mange steder. Én idé, der har fungeret godt for mig, er et norsk hæfte fra 1995 om "Diagnostisk undervisning" (Brekke, 2002). Eller rettere: diagnostiske opgaver, der altså fortæller læreren mere end blot: om eleven regner rigtigt eller forkert … Det er godt at få forstand af, men det er kun hæftets s. 11-15 i filen herover, der er om grafer. Måske kan de inspirere dig? Man kan jo sakse 1-2 figurer og så lade elever drøfte det, de mener og se og som du lægger op til, præsentere det for resten. Eller bare bruge figurerne i en klassesamtale. AM
9/1 2010 Dear A Happy New Year! Thanks for the sketch to a GeoGebra course. Inspired by this sketch and other material, I enclose my proposal for a course. Changes, ideas and tips are welcome … Many greetings P	9/1 2010 Kære A Godt nytår! Tak for skitsen til et GeoGebra-forløb. Inspireret af denne skitse og andet materiale sender jeg hermed mit forslag til et forløb. Ændringer, ideer og tips modtages gerne … Mange hilsner P

18/1 2010 Hi A Now we had the first lessons in GeoGebra. It turned out that all mastered this. More had to get the formula collection to repeat names of the forms (fine). My spontaneous creativity with respect to new tasks was tested. It was among other things: How many equilateral triangles will be used to form a “turn"? Squares? Point: Is this possible without drawing? Some could reason in calculations! - How is the leather of a ball constructed? - Unfolding of various forms? - Patterns as such! The lessons showed that I might move a little on the progression (read: speed) in the plan or maybe even better, find relevant mathematics tasks where the use of GeoGebra gives a clear application sense. … Students find the program elegant and expect quite something of the upcoming lessons (I think), some could not wait to use the program’s drawing function (which is our current topic.) … Sincerely, P	18/1 2010 Hej A Så havde vi de første lektioner i GeoGebra. Det viste sig hurtigt, at det kunne alle mestre. Flere skulle hente formelsamling el. a. for at repetere navne på figurerne (fint). Min spontane kreativitet mht. til nye opgaver blev sat på prøve. Det blev blandt andet til: Hvor mange ligesidede trekanter der skal bruges til at danne en "omgang"? Kvadrater? Pointe: Er det muligt uden at tegne? Enkelte kunne ræsonnere sig til beregninger! - Hvordan er læderet på en bold konstrueret? - Udfoldninger af forskellige figurer? - Mønstre som sådan! Timerne viste, at jeg måske kan rykke lidt på progressionen (læs: tempoet) i planen eller måske endnu bedre, finde relevante matematikopgaver, hvor anvendelsen af GeoGebra giver en tydelig anvendelsesmæssig mening. … Eleverne finder programmet elegant og venter sig en del af de kommende lektioner (tror jeg), nogle kunne ikke vente med at anvende programmet til funktionstegning (som er vort aktuelle emne). … Venligst, P
31/1 2010 Hi A Status: My plan is followed. Students are getting a little dead in the program. I find "real" mathematics problems, which then are solved using GeoGebra. That helps! Points are found and debated, but it might be a little artificial if the students are only the messengers. But findings in collaboration with students are probably also points? I'm trying. … Regards P	31/1 2010 Hej A Status: Min plan er fulgt. Eleverne er ved at gå lidt død i programmet. Jeg finder "rigtige" matematikopgaver, som så løses ved hjælp af GeoGebra. Det hjælper! Pointer findes og debatteres, men det kan godt blive lidt kunstigt, hvis eleverne alene skal være budbringere. Men konklusioner i samarbejde med elever er vel også pointer? Jeg forsøger. … Hilsen P

<table>
<tr><th colspan="2">T20: E-mails</th></tr>
<tr><td>7/12
Hi A
I try to be in time and the reason is: we have a mock exam next week, where grade 9 gets the assignment from last summer. In this there is a bit about drawing a parabola. Therefore they will just try to make a "herringbone" tabeling of three parabolas beforehand.
THEN, I will review the solution of a quadratic equation on January 12 and find the vertex. I would like to review it by presenting 1-2 numerical examples. Then take the general formula; similar with respect to the vertex.
What do you think of this in relation to the reflection (of any kind!), that can be done?
Regards K</td><td>7/12
Hej A
Jeg prøver at være lidt i god tid og baggrunden er: vi har terminsprøve i næste uge, hvor 9. klasse får opgaven fra i sommer. I den er der ganske lidt om at tegne en parabel. Derfor skal de lige prøve at lave ”et sildeben” på tre parabler inden.
SÅ vil jeg gennemgå om løsning af en andengradsligning d. 12.januar og finde toppunkt. Jeg kunne tænke mig at gennemgå det, ved at gå fra 1 til 2 tal-eksempler igennem. Derefter tage den generelle formel; tilsvarende mht. toppunkt. Hvad siger du til det i forhold til de overvejelser (af enhver art!), der kan gøres?
Mvh K</td></tr>
<tr><td>9/12 2009
Hi K
Thanks for the mail with a "warning" about the quadratic equation …
It's probably one of the quite abstract topics for students in grade 9, and although your textbook "saves" proof of formulae to vertex and roots to grade 10, it is hard enough. It sounds like a good idea to count on a few numerical examples, so my "good advice" (which you do not need to listen to) is to make a "wholesale simulation" of important parameters in parabola functions of the type: $y = ax^2 + bx + c$.
1. But you could start by recalling the function of a straight line graph:
$y = ax + b$, where a is the line slope and b indicates the line's intersection with the y-axis.
I would for example ask students:
<u>Why</u> is it now that a indicates the slope?
(What happens if a is increased extensively, e.g. doubled ...)
<u>Why</u> is it now that b indicates where the line intersects the y-axis?</td><td>9/12 2009
Hej K
Tak for mailen med et ”varsel” om andengradsligningen …
Det er vel et af de lidt abstrakte emner for elever i 9. klasse, og selv om Faktor ”gemmer” udledning af toppunktsformel og rødder til 10., er det svært nok. Det lyder som en god idé at regne et par tal eksempler, så mine ”gode råd” (som du jo slet ikke behøver lytte til) går på at ”en gros-simulere” parametrenes betydning i funktions-forskriften for en parabel: $y = ax^2 + bx + c$.
1. Men man kunne starte med at minde om funktions-forskriften for en ret linje:
$y = ax + b$, hvor a er linjens hældnings-tal og b angiver linjens skæring med y-aksen.
Jeg ville fx spørge eleverne:
<u>Hvorfor</u> er det nu, at a et tal for hældningen?
(hvad sker der, hvis a gøres større, fx fordobles ...)
<u>Hvorfor</u> er det nu, at b viser hvor linjen</td></tr>
</table>

(What happens if b e.g. is increased by 5) And I would appreciate different explanations WITH justifications! 2. If you then look at the parameters in the functional expression of a parabola: $y = ax^2 + bx + c$, it is natural to give a sense of parametric significance by demonstrating or using a computer program where you can "fiddle" with one of them at once. This *can* be done in Excel – but there are mini-programs that are tailored for such use. 3. On the electronic Skolekom-conference "Mathematics FSK" there was a suitable link (it uses the free program GeoGebra but you do not need to know anything about traces for the use). You should at least see it yourself, I think ;-) … 4. With (better?) time, students could probably get something out of "playing" afterwards. But then I'd demand that they also wrote 2 lines of what each parameter means for the parabola-graph. NB: It is actually only by this kind of software that I myself have discovered that b moves the parabola along a (second) parabolic-shaped trajectory! 5. Finally you might consider relating parabolas to everyday life. That is e.g.: Fountains (garden watering) Cannonballs (throw parabolas), e.g. with the delightful little program: http://phet.colorado.edu/simulations/sims.php?sim=Projectile_Motion (you also can throw pianos, cars, etc ...) Dishes for signals (it's almost physics ...). I do not know whether this is completely inappropriate. No one can achieve everything, but must go shopping. But it is not a secret that I am looking forward to hearing your and your students' academic "points". Therefore, the reasoning and	skærer y-aksen? (hvad sker der, hvis b gøres fx 5 større) Og jeg ville sætte pris på forskellige forklaringer MED begrundelser! 2. Hvis man så derefter ser på parametrene i funktions-forskriften for en parabel: $y = ax^2 + bx + c$, er det oplagt at give en fornemmelse af parametrenes betydning ved at demonstrere eller bruge et computer-program, hvor man kan ”pille” ved én ad gangen. Det *kan* laves i Excel – men der er mini-programmer, der er skræddersyede til det. 3. På Skolekom-konferencen ”matematik FSK” har der været et link, der ser fint ud (det bruger gratis-programmet GeoGebra, som man ikke behøver at vide spor om for anvendelsen). Du bør i hvert fald selv se det, synes jeg ;-) …. 4. Med god (bedre?) tid, kunne eleverne nok have udbytte af selv at ”lege” bagefter. Men så ville jeg forlange, at de også skrev 2 linjer om, hvad HVER parameter betyder for parabel-billedet. NB: Det er faktisk kun med den slags software, at jeg selv har opdaget, at b flytter selve parablen rundt langs en (anden) parabel-formet bane! 5. Endelig kunne du overveje at fortælle lidt om parabler i hverdagen. Dvs. fx: Springvand (havevanding) Kanonkugler (kasteparabler), fx med det herlige lille program: http://phet.colorado.edu/simulations/sims.php?sim=Projectile_Motion (man kan også kaste med flygler, biler, etc…) Paraboler (det er jo næsten fysik …). Jeg aner jo ikke, om det her er helt på tværs. Ingen kan jo ikke nå alt, men må shoppe. Men det er jo ikke spor hemmeligt, at jeg lytter meget efter dine og elevernes faglige ”pointer”. Derfor er ræsonnement og begrundelser

justifications are important. For example, I am also excited that the "lesson" might have a summing up controlled by you. This is some of what we see recommended: the "red thread" made visible to the students here as well. See you - and Merry Christmas in the meantime! Regards A	vigtige. Fx er jeg også spændt på, at "lektionen" kan have en afrunding styret af dig. Det er noget af det, vi ser anbefalet: altså at den røde tråd gøres synlig for eleverne også her. På gensyn – og glædelig jul i mellemtiden! Mvh A
13/12 2009 Hi A Thank you for your consideration and help. Lovely. I have now been in the Mathematics Bank (a Danish mathematics link), it is really nice. And the students shall try it next. There are mock exams now with the task from this summer with golf = parabola. Therefore, all our three classes had drawing exercises with parabolas last week, when I took also a starting point in the straight line. So I also do this in January, but then I imagine myself showing a parabola in wire and an eyelet, which I could then mount onto the blackboard with tape (on the y-axis). Then you can easily simulate the parabola moving around – I hope. Merry Christmas to you and yours. Regards K	13/12 2009 Hej A Tak for dine overvejelser og hjælp. Dejligt. Jeg har nu været inde på Matematikbanken, det er rigtig fint. Og det skal eleverne prøve efterfølgende. Der er terminsprøve nu med opgaven fra i sommer med golf = parabel. Derfor havde alle vore tre klasser tegneøvelser med parabler i sidste uge, her tog jeg også udgangspunkt i den rette linje. Så det gør jeg også til januar, men så kunne jeg forestille mig at møde op en parabel i ståltråd samt et øsken, som jeg sætter fast på tavlen med tape (på y-aksen). Så kan man jo nemt illudere at parablen flyttes rundt – håber jeg.. Glædelig jul til dig og dine. Mvh K
14/1 2010 Hi K Thanks for last time! … And you've already announced that it is about 2 straight lines, i.e. their intersection. I've just read about a Norwegian school, where a teacher worked with an idea like the one I attach on 4 work cards (Hundeland, 2009). I think that the cards have been cut apart and possibly printed somewhat larger, and then students (preferably in pairs) all started with card no. 1. Each "group" or so can only get card 2 when they finish card 1, etc. The "work	14/1 2010 Hej K Tak for sidst! … Og du har jo varslet, at det handler om 2 rette linjer, dvs. deres skæring. Jeg har netop læst om en norsk skole, hvor man har arbejdet med en idé, som den jeg vedhæfter på 4 arbejdskort (Hundeland, 2009). Jeg tror, at kortene så har været klippet fra hinanden og muligvis trykt noget større, og så eleverne (gerne parvist) alle er startet med kort 1. Hver "gruppe" eller kan så først hente kort 2, når de er færdige med kort 1, osv. "Arbejdskort

card" method suggests that you follow up on "points" in a class discussion. For example, one could then have a transparency with a pre-printed coordinate system and pens in different colors. Of course this can also be created on the blackboard ... I have no idea whether it might be of use or inspire you to get students started. The cards are good for variation. And there is probably no one who says that everybody must finish everything before summing up. In Norway 2 lessons in total were allotted to these 4 work cards with a follow-up. Let me hear what you have planned , if it can be written down before we meet. Greetings AM	metoden" lægger op til, at man følger op på "pointerne" i en klassesamtale. Fx kunne man så have en transparent med et fortrykt koordinatsystem og penne i forskellig farve. Det kan selvfølgelig også laves på tavlen ... Jeg aner ikke, om det muligvis kan bruges eller inspirere til at få elever i gang. Kortene er jo gode til at differentiere med. Og der er vel ingen, der siger at alle skal nå det hele før man samler op. I Norge var der sat 2 timer af i alt til disse 4 arbejdskort med opfølgning. Lad mig høre, hvad du får planlagt, hvis det da kan skrives ned, før vi ses. Mange hilsner AM
14/1 2010 Hi A It sounds exciting, I will look into it, you'll get a plan. Great to get ideas from you - thank you. Regards K	14/1 2010 Hej A Det lyder spændende, det ser jeg mere på, du skal nok få en plan. Dejligt at få ideer af dig - tak skal du have. Mvh K
21/1 2010 Dear A I think it's a really good idea to let the students themselves work. So, after an introduction by me: $y = ax + b$. (All will know I hope that this is a straight line, but really to have their fingers on it, some students need that!) For other values one can also write like this $x + y = 7$. The students work in pairs, but such that each student works with a private-mm paper. Then card 1. Points: conversation about: how to draw the line / slope (Is it a line or is it points?), compare with $y =-x + 7$ Card 2. Now the line comes! Summing up / points before the end of the lesson. Card 3. Might be reached in this lesson, otherwise the next lesson Card 4. Next lesson	21/1 2010 Kære A Jeg synes det er en rigtig god ide at lade eleverne selv arbejde. Så efter en indledning af mig: $y = ax + b$. (Alle ved jo nok at det er en ret linje, men rigtig at have fingrene i det, har nogle af eleverne brug for!) Med andre værdier kan man også skrive sådan $x + y = 7$. Det arbejder eleverne med to og to, men sådan at hver elev arbejder med eget mm-papir. Derefter kort 1. Pointerne: samtale om: hvordan tegnes linjen / hældning (Er det en linje eller er det punkter?), sammenligning med $y = -x + 7$ Kort 2. Nu kommer linjen så!! Opsamling/pointer før timens slutning. Kort 3. Kan måske nås i den time, ellers næste time Kort 4. Næste time.

Feel free to comment. Regards K	Du må gerne kommentere Mvh K
27/1 2010 Dear K Thanks for last time. It was a really good lesson, I think, where I among other things was pleased that you picked up along the way and in the end. As I pondered at the last moment, it is probably NOT like ALL having a grip on the issues, although it seemed like that. You will of course find out one day – if not before, then when you /they do a "review" ... Regards A	27/1 2010 Kære K Tak for sidst. Det var en rigtig god lektion, synes jeg, hvor det bl.a. glædede mig, at du samlede op undervejs og til sidst. Som jeg overvejede på falderebet, er det nok IKKE sikkert at ALLE har styr på sagerne, selv om det altså virkede sådan. Det finder du selvfølgelig ud af en dag – om ikke før, så når du/I skal "repetere" ... Mvh A

T21: E-mails	
Aarhus, December 8, 2009 Dear EK Yes, thanks for last time. I first got hold of your letter today with weekly schedule and task today Tuesday – but I hasten to reply as promised. … My own experiences with reduction and fractions in school are old and I have to be careful not to show a (too) better understanding. I believe you can learn very much by heart, and probably understand a part of it. But the last is a challenge. You could say that my "challenges" to you on this occasion are of several types (and those you can all easily reject). I will certainly respect that, partly because the students now resume a course after another organization, where there was differentiation. *Differentiation* So I'm excited to attend your lesson, where I obviously hope that the students are invited to give (some of) the explanations. If a substantial part of the lesson is controlled by you, it's difficult to give	Århus, 8. december 2009 Kære EK Ja selv tak for sidst. Jeg fik først fat i dit brev i dag med ugeplan og opgavekopi i dag tirsdag – men jeg iler med en replik som lovet. … Mine egne erfaringer med reduktion og brøkregning i folkeskolen er jo gamle og jeg skal passe på med ikke at være (for) bedrevidende. Jeg tror man kan lære ganske meget udenad, og sikkert også forstå en del af det. Men det sidste er en udfordring. Man kan sige, mine ”udfordringer” til dig ved denne lejlighed er af flere slags (og dem kan du sagtens alle afvise). Det har jeg bestemt stor respekt for, bl.a. fordi eleverne jo nu genoptager et forløb efter en anden organisering, hvor der har været differentieret. *Differentiering* Så jeg er spændt på at overvære din lektion, hvor jeg selvfølgelig håber på, at eleverne inviteres til at give (nogle af) forklaringerne. Hvis en væsentlig del af lektionen styres af dig, er det svært at tage

consideration to all. But I remember after all that you had students in your grade 8 class who might need extra (another) easier or harder material? *Concretization* You are now probably long <u>past</u> the stage, where you could show using colored plastic circles on a transparency that 2/4 equals 4/8 etc. But there are many games … that may be used to illustrate fractions. E.g. dominoes. Perhaps you have such material at you school, perhaps not? *Supplement* Your textbook has of course a lot on fractions, and there are plenty of (additional) materials in other books and on the Internet. I do not know if you could demonstrate, or refer to some of that. At some schools it is still very difficult to link up, or you do not have access from a computer in the actual classroom. E.g. (I have just searched Google on fractions and selected the third and the first hit) … *The red thread* Maybe you remember that I told you about the Japanese teachers, that they always sum up lessons by reminding the students about the “red thread” in the process. They call this phase "*matome*" in "*lesson study*", so this is a known "trick" which they consider very important. It is probably also a simple and effective focus, which certainly supports the students' long-term memory. Would you consider doing that on Friday? Sincerely, AM	hensyn til alle. Men jeg erindrer jo, at der i din 8. klasse da *var* elever, der *kunne* have behov for ekstra (et andet) lettere eller sværere materiale? *Konkretisering* I er nu nok for længst <u>forbi</u> det stade, hvor man med farvede plastcirkler på transparent kan vise, at 2/4 er det samme som 4/8 etc. Men der findes jo mange spil … der kan bruges til at konkretisere brøkregning. Fx domino. Måske har I den slags på skolen, måske ikke? *Supplement* Jeres bog har selvfølgelig en del om brøker, og der er masser af (supplerende) materiale rundt i andre bøger og på internettet. Jeg ved ikke, om du kunne finde på at demonstrere, eller henvise nogle til dét. På nogle skoler er det stadig meget besværligt at koble op, eller man har ikke adgang fra en computer i egen klasse. Fx (Jeg har blot søgt i Google på brøkregning og valgt det 3. og det 1. ”hit”) … *Den røde tråd* Måske husker du, at jeg fortalte om de japanske lærere, at de *altid* afrunder lektioner med at minde om ”den røde tråd” i forløbet. De kalder denne fase for ”*matome*” i ”*lesson study*”, så det er et kendt ”trick”, som de vurderer meget vigtigt. Det er vel også en enkel og effektiv fokusering, der i hvert fald støtter elevernes langtidshukommelse. Kunne du finde på det på fredag? Med venlig hilsen AM

T35. E-mails	
15/12 2009 Hi A	15/12 2009 Hej A

Naw, I have not forgotten you, but you're right - the stacks on the desk get higher and higher. In return, I booked the computer room for January and will then see whether or not *GeoGebra* is already downloaded. I will look through the material (and send the forms) and try to make myself familiar with the program – and then you'll probably get questions from me ;-) Merry Christmas from MB	Næhh, jeg har såmænd ikke glemt dig, men du har ret – stakkene på skrivebordet bliver højere og højere. Til gengæld HAR jeg booket edb-lokale til januar og vil så undersøge om ikke *GeoGebra* allerede er hentet. Jeg kikker materialet igennem (og sender skemaerne) og prøver at sætte mig ind i programmet – så skal du nok få spørgsmål fra mig ;-) Glædelig jul fra MB
4/1 2010 Hi A and Happy New Year. I teach class 9a for a single lesson this week (they have mock exams) where I expect to go through a lot of features in GeoGebra. I'm going to plan in detail throughout tomorrow (have quite some time) so then the questions will probably tumble in ;-) - Just so you know I have not forgotten you! Greetings MB	4/1 2010 Hej A og godt nytår. Jeg har 9.a i en enkelt time denne uge (de har terminsprøver) hvor jeg regner med at gennemgå en masse funktioner i GeoGebra. Jeg har tænkt mig at detailplanlægge hele forløbet i morgen (har nogenlunde god tid) så der vælter spørgsmålene nok ind ;-) - Bare lige så du ved jeg ikke har glemt dig! Mvh MB
5/1 2010 Hi A The first question is: How much should I introduce in each lesson? 1. lesson I imagine that I must limit myself and introduce no more than max 5-7 features before they have to fiddle with it themselves. (Grid, points, polygon drawing, perpendicular bisector to opposite sides, bisectors to angles, moving a point, marking intersections). And then give them 4-8 specific tasks (which are posted on the student Intranet so that they themselves must find them there) with drawing of shapes. Comments – I'll be happy ;-) M	5/1 2010 Hej A Det første spørgsmål der trænger sig på er: Hvor meget skal jeg introducere på en gang? 1. time Jeg forestiller mig at jeg skal begrænse mig og ikke vise mere end max 5-7 funktioner før de skal rode med det selv. (Gitter, punkter, polygontegning, midtnormaler, vinkelhalveringslinjer, flytning af punkt, markering af skæringspunkt). Og så give dem 4-8 konkrete opgaver (som er lagt på elev intranet så de selv skal finde dem der) med tegning af figurer. Kommentarer – meget gerne ;-) M
17/1 2010 Hi A Now we have made all sorts of triangles,	17/1 2010 Hej A Nu har vi lavet alle mulige trekanter,

squares and heights, medians, angle bisectors and perpendicular bisectors to lines. Both in GeoGebra and on paper. (Students were extremely excited about GeoGebra and have asked if they can deliver drawings with the problem assignments which were drawn by this program – of course they may.) The academic focus has been on bisectors to lines and angles and the related circles, since it is not an area we have made so much of before. We have briefly touched upon Pythagoras, but we made much of that in grade 8, so calculations of side lengths in a right triangle, THAT they can ;-) I have extended the course by one week (minimum), … if we immerse ourselves in sine, cosine and tangent it will require just a little extra. Week 4 will certainly go in as well. The school already has the book you recommend (New trigonometry) and tomorrow I'm going to start by defining sinus from the unit circle and that is the academic focus (copy page 66 and possibly 67 for the quick ones). Otherwise, I can inform you that many colleagues, myself included, attended a background course on assessment at the school in spring 2009, and we have begun also to share with the students what is the goal of a lesson, a course or an academic theme. Of course, students should know why we do what we do, I just have not been good at communicating it to them (the same applies for many of my colleagues). It is not ALL lessons I do / manage to do this, but in the beginning of a course they get a list of what it is expected of them by the end of the course. I think immediately that it fits well with your "Academic Focus". Tuesday we have a pedagogical afternoon, where professionalism is on the program – both subject proficiency,	firkanter og højder, medianer, vinkelhalveringslinjer og midtnormaler. Både i GeoGebra og på papir. (Eleverne var yderst begejstrede for GeoGebra og har spurgt om de må aflevere tegninger med problemregningerne der var tegnet i dette program – selvfølgelig må de det.) Det faglige fokus har ligget på midtnormaler, vinkelhalveringslinjer og de dertil hørende cirkler, da det ikke er et område vi har gjort så meget ud af før. Vi har kort berørt Pythagoras, men det har vi til gengæld gjort meget ud af i 8. klasse, så sideberegninger i en retvinklet trekant, DET kan de ;-) Jeg har udvidet forløbet med en uge (foreløbig), … hvis vi skal fordybe os i sinus, cosinus og tangens kræver det lige lidt ekstra. Så uge 4 er helt sikkert også med. Skolen har allerede bogen du anbefaler (Ny trigonometri) og jeg har tænkt mig at vi i morgen starter med at få defineret sinus ud fra enhedscirklen og at det er det faglige fokus i morgen (kopiside 66 muligvis 67 til de hurtige). Ellers kan jeg fortælle at jeg, ikke kun i matematik, og med baggrund i kursusforløb om evaluering vi havde på skolen i foråret 2009, er begyndt også at indvie eleverne i hvad der er målet med en time, forløb eller fagligt emne. For selvfølgelig skal eleverne vide hvorfor vi gør som vi gør, jeg har bare ikke været god til at formidle det til dem (det samme gælder for mange af mine kolleger). Det er ikke i ALLE timer jeg gør / når det, men i starten af et forløb får de en oversigt over hvad det er det forventes at de kan efter den kommende undervisning. Jeg synes umiddelbart at det harmonerer godt med dit "Faglige fokus". Tirsdag har vi pædagogisk eftermiddag, hvor faglighed er på programmet – både fag-fagligheden, lærerfagligheden og alle andre

teacher professionalism and all other variants of the term. That's the current situation here in town. Mvh. MB	varianter af begrebet. Sådan ser tingenes tilstand ud her i byen. Mvh. MB
Hello M Thanks for your mail 17/1 2010. I do realize you have gained speed! And I am both impressed and delighted to read about your intention to also tell the students WHAT the point is in a lesson's activities. Then the academic points should be expected – and perhaps even more obvious? … I find it exciting that you arrange a pedagogical afternoon on professionalism. This is promising for an "opening" of cooperation in e.g. a mathematics teachers' team. I suppose you know that I've written a book with ideas for topics to be addressed here. But initially it can of course also "just" be a little more systematic peer sparring between those of you who teach parallel classes. I look forward to hearing more about your eventual decisions ;-) Many greetings, A	Hej M Tak for din mail 17/1 2010. Jeg skal love for at I har fart på! Og jeg er både imponeret og glad over at læse om dit fortsæt om også at fortælle eleverne, HVAD meningen er med en times aktiviteter. Så må de faglige pointer vel være ventet – og måske endnu mere tydelige?… Jeg synes, det er spændende, at I afholder en pædagogisk eftermiddag om faglighed. Det lover godt for en ”åbning” af samarbejdet i fx et matematik fagteam. Du ved nok, at jeg har skrevet en bog med idéer til emner, man her kunne tage op. Men i første omgang kan det jo også ”bare” være en lidt mere systematisk kollegial sparring mellem jer, der underviser parallelklasserne. Jeg glæder mig til at høre lidt mere om en evt. landing på beslutninger ;-) Mange hilsner, A

T50: E-mails	
Dear B …. And I'm curious as to whether you would like some peer coaching in January? I.e. I just want to remind about the forms and envelopes, you got ;-). Either way, I attach materials on a GeoGebra-course for general encouragement. And if you have other plans, you should of course just ignore it. But otherwise it's probably the last time to book some computers! And / or a room! And you must have GeoGebra installed! And if some of it should be linked to a trigonometry-course, I will be happy to	Kære B … Og så er jeg spændt på, om du kunne tænke dig en kollegial sparring i januar? Dvs. jeg minder lige om de skemaer og kuverter, du fik ;-). Uanset hvad, så vedhæfter jeg materiale om et GeoGebra-forløb til almindelig opmuntring. Og hvis du har andre planer, skal du jo bare ignorere det. Men ellers er det nok i sidste øjeblik tid for at reservere nogle computere! Og / eller et lokale! Og man skal have GeoGebra installeret! Og hvis noget af det skal hænge sammen med et trigonometri-forløb, går jeg også gerne ind i en fælles

offer you some sparring about that. In any case I wish you pleasure with Christmas and the preparations. There must of course be time for that too. Greetings A	sparring om det. Jeg vil under alle omstændigheder ønske dig god fornøjelse med julen og forberedelserne. Der skal jo helst være tid til dét også. Mange hilsner A

For some of the seven teachers in the focus group the e-mails referred to above show an increasing degree of trust and confidence in the relation and the outcome as the communication keeps evolving. But there are significant differences:

T2	This teacher was very active indeed. Constantly interested in developing her own teaching, and very aware of her style.
T12	This younger teacher had his very own teaching style, and often referred to his belief in cooperative learning. This sort of hindered a more progressive guidance during lessons.
T13	This teacher was among the most committed in the focus group. He seemed very proficient but absolutely willing to make points more visible.
T20	This elder teacher seemed a bit hesitant to new ways of teaching, thought of going to pension after this final year of teaching. But now welcomed new ideas and certainly also tried them out in his classroom.
T21	This more mature "old-school" classic teacher was not too inviting – like "you may try, but you probably cannot change me". She later left the focus-group because of surgery.
T35	This younger teacher was very active and busy doing other things. She developed very much during our contact, eagerly taking up and trying out suggestions.
T50	This younger teacher never responded in dialogue. He was busy writing science textbooks and later left the focus group.

The peer sparring was between a practicing teacher and a teacher / researcher colleague from outside the actual school, namely me. This was a condition, but it also brought a deeper knowledge of subject and actual teaching content within reach of the colleagues responsible. Under such circumstances the e-mail correspondence above documents changes in the mind of some of these focus teachers.

To register if such changes also are implemented in teaching, I met the seven teachers again and made a second video tape immediately after the correspondence, and finally a third one with five of them several months later and without any intermediate correspondence on content. These video recordings (take 2 and 3) are analyzed and findings presented below.

10.5 Round 2 and 3 for focus teachers

For *each* of the seven focus teachers (only five though were active and involved in take 3) there follows a description consisting of:

1. **Data of actual teacher T2, T12, T13, T20, T21, T35 and T50 and school**
   Numbers refer to the numbering among the first 50 lessons recorded.
2. **Statistics for codes on points and elicitation in % of lesson length for take 1, 2 and 3**
   These are fitted in a table for an overview of possible changes in the occurrence of four types of points from take 1 to takes 2 and 3. The extent of teacher elicitation is included to indicate the degree of the teacher's active involvement in choreography of common class conversation.
3. **Structure of lesson takes 1-3 in minutes and also shown as bar chart**
   Time is indicated in the (mm:ss) format to give an overview of lesson structure. The table also shows how some teachers jump back and forth between various stages in the lesson where assignments alternate with class discussion. A stacked bar chart is made to indicate the respective length of phases.
4. **Interview with teacher before the take 2 lesson**
   Right before the lesson I interviewed teachers on their planning, which in many cases was clearly influenced by my sparring. But they all rightfully claimed teachers' ownership and responsibility. For T2 this is not an interview, but a short researcher memo as I forgot to bring a tape recorder to this first take 2 lesson.
5. **Excerpt of take 2 lesson**
   One excerpt is selected to show the kind of points and / or elicitation observed in take 2. A short analysis is attached to that.
6. **Excerpt of take 3 lesson**
   One excerpt is selected to show the kind of points and / or elicitation observed in take 3. A short analysis is attached to that.
7. **Interview with teacher before or after the take 3 lesson**
   After a period of no contact, I revisited five of the seven focus teachers in April 2010. One had declined due to illness (back surgery, T21) and one had not responded, despite several emails and a letter – probably mostly due to a lack of time (he was writing science textbooks = busy), because I've since heard from his editor that he had told her about his involvement.
   At each of these five visits a brief conversation was tape recorded immediately before or after the actual lesson on the goal, the teacher previously had indicated. That is also about the points that were considered important. The conversation did *not* follow any interview guide, it was quite informal and took place rapidly at break time – I may not always have even mentioned the word point.
8. **Discussion of results**
   A final assessment of my process of peer sparring for this one teacher.

T2: Female teacher, seniority 15+ years
Medium sized school in Middle region

CODES (in %) of lesson length (Topic: Coordinate system)

TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class	3.6	8.5		1.6	1.6
Teacher → Student		3.8		3.8	
Student → Teacher	3.6				

CODES (in %) of lesson length (Topic: Repetition + Pythagorean Theorem)

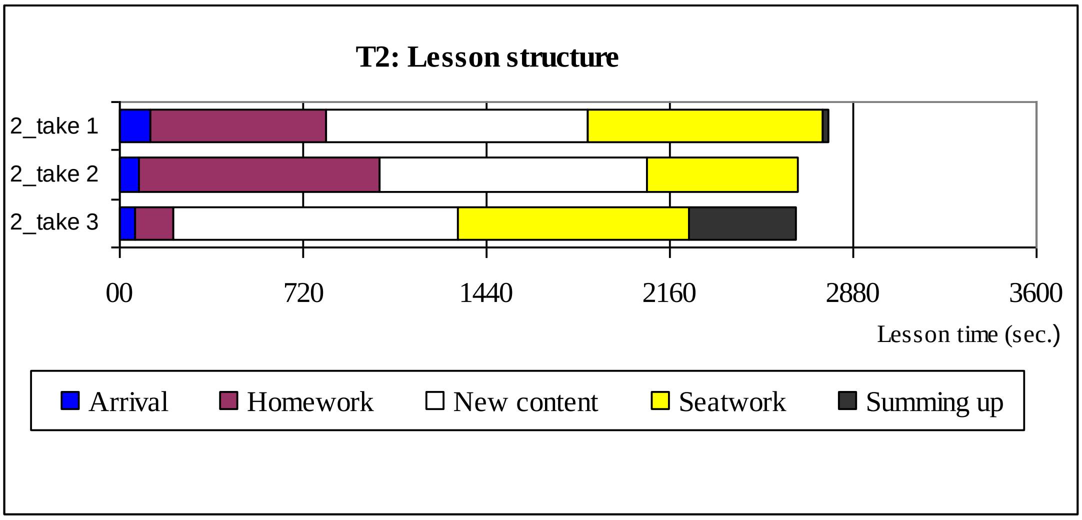
TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		23.3	9.6	20.7	14.9
Teacher → Student					
Student → Teacher	0.6	20.9			

CODES (in %) of lesson length (Topic: Economy + Repetition on probability)

TAKE 3	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		2.5			22.5
Teacher → Student					
Student → Teacher	6.9	6.1			

T2: Lesson structure

Teacher 2	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	2:00 0:00-2:00	11:32 2:00-2:32, 12:18-23:18	17:05 2:32-12:18, 23:18-25:07, 28:35-34:05	15:25 25:07-28:35, 34:05-46:02	0:18 46:02-46:20	46:20
Take 2	1:12 0:00-1:12	15:48 1:12-17:00	17:33 17:00-23:38, 28:56-34:50, 39:24-44:25	9:52 23:38-28:56, 34:50-39:24		44:25
Take 3	1:00 0:00-1:00	2:32 3:51-6:23	18:38 1:00-3:51, 6:23-22:10	15:06 22:10-28:28, 35:30-44:18	7:02 28:28-35:30	44:18



T2: Researcher memo from take 2 visit	
The first video (a DVD was given to each focus teacher at the seminar) had made T seriously consider her language. She thought now that she was unclear, particularly because of the many half finished sentences, inserted phrases, etc., as shown by the transcript. She intended to work with that.	Den første videooptagelse (en DVD var givet til hver fokuslærer at the seminar) havde fået L til alvorligt at overveje sit sprog. Hun syntes nu, at hun ofte formulerede sig uklart, bl.a. pga. mange halve sætninger, indskud etc., som det også fremgår af transskriberingen. Det agtede hun at arbejde med.

T2: Excerpt of take 2 lesson (5:58) Topic: Repetition + Pythagorean Theorem	
S: "Show that the graphs y = 2x + 1 and y = 4x – 3 has (2, 5) as the intersection point." T: There it says <u>*show*</u> in contrast to the next task. What is written there, H? S: There it says <u>*prove*</u> rather than show. T: So we need to get that there is a difference between <u>*showing*</u> something and <u>*proving*</u> something. Ok? Have you also done this in your tasks? S: No. I have *shown* it twice. (Other students interfere: I have also done, etc ...) T: Well, let's take the first one. It is to <u>*show*</u> it. M, how did you handle it, by *showing* it? S: So, I wrote it as an equation. T: Ok. Would you say something more about it? ...	E: "Vis, at graferne y = 2x + 1 og y = 4x - 3 har (2,5) som fælles punkt." L: Der står altså <u>*vis*</u> i modsætning til næste opgave. Hvad står der dér, H? E: Der står <u>*bevis*</u> i stedet for vis. L: Så der skal vi have fat i, at der er forskel på at <u>*vise*</u> noget, og <u>*bevise*</u> noget. Ok? Har I også gjort det i jeres opgaveløsning? E: Nej. Jeg er kommet til at *vise* det to gange. (Andre elever blander sig: Det har jeg også gjort, etc.…) L: NÅ, nu tager vi lige den første. Den der med at <u>*vise*</u> det. M, hvordan har du grebet den an, med at *vise* det? E: Så har jeg stillet det op som en ligning. L: Ok. Vil du prøve at sige noget mere om det? ….

S: Then I've taken (S writes the equation 2x + 1 = 4x - 3 and solves it the algebraic way by "swapping" the parts: x = 2). And then you see that it is y is equal to 2x + 1 (now inserts the found x value in both expressions around the equal sign and gets 5 every time). T: And then you get (T reaching for the student's chalk) (x, y) = (2, 5). Others who did it that way? What do you say (a student has marked)? S: In task 10 it says prove! [You did that?] Yes. S (who was at the blackboard): Was it not also what I should do? T: You know M, I think what you did is really nice (thank you) and if you have done it that way, then you actually did solve it by *proving* (task 10).	E: Så har jeg taget (E skriver ligningen 2x + 1 = 4x - 3 og løser den algebraisk ved at ”bytte rundt” på leddene: x = 2). Og så kan man jo se, at det er y, der er lig med 2x + 1 (indsætter så den fundne x-værdi i begge udtryk omkring lighedstegnet og får hver gang 5). L: Og så får du (L rækker ud efter elevens kridt) (x, y) = (2, 5). Var der andre, der har gjort det på den måde? Hvad siger du (en elev har markeret)? E: I 10’eren, der står bevis! [Dér har du gjort det?] Ja. E (der var ved tavlen): Var det ikke også det, jeg skulle? L: Ved I hvad M, jeg synes det der, det er rigtig flot (tak), og hvis I har gjort det på den måde, så har I faktisk fået løst den der med at *bevise* (opgave 10).

The announced topic for take 2 was Repetition + Pythagorean Theorem, the latter not seen in the excerpt above though. The excerpt is coded as a *student procedural point*. The teacher is very careful to make sure, that students’ language and use of concepts and procedures is always very correct. She does this in dialogues with single students invited to present their solutions to the class. Everyone else seems very observant and are actively supplying information when asked.

T2: Excerpt of take 3 lesson (14:46) Topic: Economy + Repetition on probability	
T: Sample space? Who will explain what a sample space is? Try looking in your formula collection, it says what a sample space is … J? S: For instance, if you have a dice, a regular dice with 6 sides. There the sample space is 6 because there are 6 different possible outcomes. T: Yes. So the sample space, it's all the possibilities. Do you all agree? (T writes: "- all possibilities" on the blackboard). T: Event? Who would then attempt to find out what kind of a word that is? S: Is it not impossible? T: Yes, you may talk about an impossible event. But let's get into what an event is? H?	L: Udfaldsrum? Hvem vil forklare, hvad et udfaldsrum er? Prøv at se i formelsamlingen, der står hvad et udfaldsrum er … J? E: Fx, hvis man har en terning, en almindelig terning med 6 sider. Der er udfaldsrummet 6 fordi der er 6 forskellige mulige udfald. L: Ja. Altså udfaldsrummet, det er alle de muligheder, der er. Er alle med på det? (L skriver: ” - samtlige muligheder” på tavlen). L: Hændelse? Hvem vil så prøve at finde ud af, hvad det er for et ord? E: Er det ikke umulig? L: Ja, men kan snakke om en umulig hændelse. Men lad os lige få fat i, hvad hændelse er for noget? H? E (med skelen til bogen): En hændelse kan

S (with an eye in the book): An event can be one or more outcomes. T: It can be *one* of the outcomes, one of the outcomes we're after. And if it is an *impossible* event, what is it then? So, now J mentioned the dice thing. There I will say that to throw 7 with a dice, that I would call an impossible event. It cannot be done. SS: It is because there are not 7 sides! / No, it does not apply if it is 3-sided!	være et eller flere udfald. L: Det kan være ét af udfaldene, et af de udfald vi er ude efter. Og, hvis det er en *umulig* hændelse, hvad er det så? Altså, nu nævnede J den der med terningen. Der vil jeg sige, at få 7 med en terning, dét ville jeg kalde en umulig hændelse. Det kan ikke lade sig gøre. EE: Det er fordi der ikke er 7 sider! / Nej, det gælder ikke, hvis den er 3-kantet!

The topic for a part of take 3 was "economy". Here the teacher asked students for keywords on this and wrote these on the blackboard. In a dialogue form the teacher then asked students for definitions and examples. Then a few tasks were discussed in detail – one of these from homework. Finally the teacher offered repetition of concepts and procedures from probability as shown in the excerpt. This topic was suggested by students as the final exam in grade 9 was coming close.

The excerpt above is coded both as teacher *elicitation* and as a *student conceptual point*. The teacher reminds of concepts as sample space, outcome and event in dialogue with students, successfully asking students to suggest the definitions and explanations.

T2: Interview of teacher immediately after the take 3 lesson:	
I: I would ask you what points you hope to present, and then I may also ask you about how it evolved, now that the lesson has been taught? T: Well, I did intend to make a point in students' practice to read the task texts, so they get to answer the task, the author asks for. And then I'd try to (develop) their vocabulary, thus their concepts, transactions, those balances and all that stuff, so they understand these words, we juggle in mathematics. That was what I was looking for. I: Yes. It went well? T: I think they – I just think they held back a little, because they knew very well the words I was looking for.	I: Jeg ville spørge dig om, hvad for nogle pointer du håber du får frem, og så kan jeg jo også spørge dig om, hvordan det gik, nu hvor timen er gået? L: Altså, jeg ville jo have de pointer frem med, at de skal øve sig i at læse ind til med opgave stilleren, så de får svaret på det, opgave-stilleren beder om. Og så ville jeg forsøge at (udvikle) deres ordforråd, altså deres begreber, bevægelser, de der saldoer, restbeløb og alt det der, at de forstår, hvad er det for nogle ord, vi jonglerer med i matematik. Det var det, jeg var ude efter. I: Ja. Gik det godt? L: Jeg synes, de – jeg tror bare, de holdt sig lidt tilbage, for de kunne jo godt de ord, jeg var ude efter.

In this short interview the teacher's understanding of points in the lesson seems to be the important *methodological* goal for the lesson, which then is to be accurate on reading and explaining. The observation and the lesson transcript also show several *mathematical* points on concepts and procedures, or as the teacher states: "vocabulary, thus their concepts, transactions ... and all that stuff".

T2: Discussion of results

There is a distinct increase from take 1 to take 3 in the lesson time coded for elicitation by this teacher. This may be due to the peer suggestion to make distinct mathematical points. But such change may also be affected by the change in topic and the repetition mode, and then not necessarily by increased awareness of the importance of points.

The teacher seems to follow her initial strategy of a dominant role in classroom discourse. She does a lot of talking, repeats student answers, and demands and presents correctness when presenting the mathematics. Points on procedure are prominently presented in all three video recordings.

T12: Male teacher, seniority 15+ years
Large school in Central region

CODES (in %) of lesson length (Topic: Polar coordinates)

TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class					
Teacher → Student		2.0			
Student → Teacher	14.8		2.0	14.8	

CODES (in %) of lesson length (Topic: Proof: Gravity center in triangle)

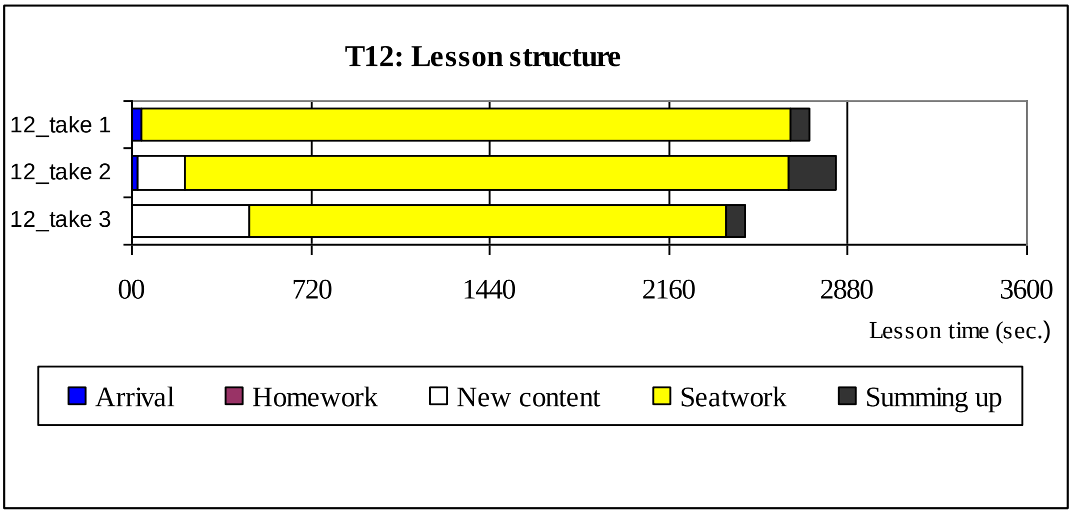
TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class					19.2
Teacher → Student					
Student → Teacher				10.1	

CODES (in %) of lesson length (Topic: Linear function)

TAKE 3	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class					35.2
Teacher → Student					
Student → Teacher				3.2	

T12: Lesson structure

Teacher 12	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	0:40 0:00-0:40			43:30 0:40-44:10	1:18 44:10-45:18	45:18
Take 2	0:25 0:00-0:25		3:08 0:25-3:33	40:25 3:33-43:58	3:13 43:58-47:11	47:11
Take 3			7:53 0:00-7:53	31:59 7:53-39:52	1:14 39:52.41:06	41:06



This teacher seems to stick to the relative weight on phases in lesson planning and teaching. As referred to earlier this teacher often expressed his conviction of the strength of “*cooperative learning*”. In the classroom students are always encouraged to discuss their questions in working groups with other students before involving the teacher. This also implies the rather few points expressed were on interpretation in takes 2 and 3. Very distinct is the increase in elicitation codes. I estimate this as a possible effect of the sparring and outside observance.

T12: Interview of teacher immediately before the take 2 lesson:	
T: What students must do is draw a triangle on a large piece of cardboard and then they shall find the midpoint. They must discuss this in the "cooperative learning" groups they are in, how to find the midpoint. They have been through this before, but it's exciting if they remember. I: That word "midpoint", don’t you think that many will ask: What's the deal?	L: Det eleverne de skal, det er at de skal tegne en trekant på et stort stykke pap og så skal de finde midtpunktet. Det skal de diskutere i de ”cooperative learning” grupper, de sidder i., hvordan de finder midtpunktet. De har været igennem det før, men det er spændende om de kan huske det. I: Det der ord ”midtpunkt”, tror du ikke mange vil spørge: Hvad er det for noget?

T: Yes I think so, but *I* have made a triangle in advance. I will not show them the lines. But I show them how – where I have stuck the needle in – but I will not show them lines. I: It's a really good idea, I think. It is very smart, that there is balance. T: And then I show that how it can work as a propeller slowly, gravitationally too. I: It's actually a great idea, I think. T: Well, but I hope that there are some, because we HAVE talked about these lines that go through a triangle: The heights and bisector lines and middle normals and medians. We have talked about this before and they have spent much time on it, we'll see. I: They may well come up with many proposals for what lines there could be. But they might be tracked into, that it could be something like that? T: No, they are not. Because I only told them that we have started with "proofs" now. I: Would you let them "run to the end of line" if they now start to draw some other lines? T: Yes, I think actually I would, because – they should also discuss with each other. Because I will go around and be a consultant. I get them started and I show them the triangle I've made – and then we will see.	L: Jo det tror jeg, men jeg har lavet en trekant i forvejen. Jeg viser dem ikke linjerne. Men jeg viser dem, hvordan – der hvor jeg har stukket nålen ind – men jeg vil ikke vise dem linjerne. I: Det er en virkelig god idé, tror jeg. Det er meget smart, at der er balance. L: Og så viser jeg, at hvordan den kan virke som en propel, langsomt tyngdemæssigt også. I: Det er faktisk rigtig godt tænkt, tror jeg. L: Såh, men jeg håber på, at der er nogle, fordi vi HAR snakket om de her linjer som går gennem en trekant: højder og vinkelhalveringslinjer og midtnormaler og medianer. Det har vi snakket om før, og de har brugt megen tid på det, nu må vi se. I: De kan jo godt komme med mange forslag til, hvad for nogle linjer der kunne være. Men de er måske sporet ind på, at det kunne være noget i den retning? L: Nej, det er de ikke. For jeg har kun fortalt dem, at vi er gået i gang med ”bevisførelse” der. I: Vil du lade dem ”løbe linen ud”, hvis de nu begynder at tegne nogle andre linjer? L: Ja, det tror jeg faktisk jeg vil, fordi – de skal også diskutere med hinanden. For jeg vil gå rundt og være konsulent. Jeg sætter dem i gang og jeg viser dem den trekant, jeg har lavet – og så må vi se.

T12: Excerpt of take 2 lesson (35:40) Topic: Proof: Gravity center in triangle	
T: Why is there a balance? You must just explain that to me by these lines there. You're right, you're right! S: These are the triangles you have drawn inside. From one half of one side to the opposite tip. Then it ends with them being split equally. No, that? T: Now I just ask you: If I draw from the midpoint here and over there, how is this	L: Hvorfor er der ligevægt? Det skal du lige kunne forklare mig med de streger der. Du har ret, du har ret! E: Det er de trekanter, man har tegnet indeni. Fra halvdelen af en af siderne til den modsatte spids. Så ender det med, de bliver delt lige. Nej, det? L: Nu spørger jeg så bare: Hvis jeg tegner fra midtpunktet her og over, hvordan er dét areal

area in proportion to that area then? S: The same. T: Yes. ... Are there other lines which mean that we can say there is a balance? S: It's that you have – well all the lines here, they in fact divide it into halves. So, this does and that does, and so it is all the way through – that is, I do not know what you mean. What's that you ask for? T: Are there any triangles in the triangle, which have equal size? S: There are, because when you just divide into halves, there are still two triangles. T: (laughs) That's enough for me!	i forhold til dét areal så? E: Ens. L: Yes. ... Er der andre linjer som gør, at vi kan sige der er ligevægt? E: Det er jo, at du har – altså alle de linjer her, de deler den jo faktisk op i halvdele. Altså, den gør, og den gør, og sådan er det bare hele vejen igennem – altså, jeg ved ikke helt hvad du mener. Hvad er det nu, du spørger om? L: Er der trekanter i trekanten, som er lige store? E: Det er der da, for når du bare deler den halvt over, så er der stadigvæk to trekanter. L: (ler) Det er nok for mig!

This excerpt is coded as *teacher elicitation* and a *student interpretation point*. By rephrasing his question the teacher either tests for a (more) convincing reasoning or at least tries to get this student to express himself more accurately. The teacher's final comment is delivered with a smile and a happy attitude conforming that this student is working in the right direction.

T12: Excerpt of take 3 lesson (28:03) Topic: Linear function	
S: P, I need some help! [Yes?] Because, I cannot figure out the task. It is difficult. T: Why is it difficult? What is it that you draw there? You drew 2 lines that are parallel. [Yes]. And do you think it may just not be correct? S: It is because they well never meet each other! It's something I do not have … T: But why is it true that? S: It *is* correct, they will never meet. T: No. Why do they not? If you look at the two, why do they not? S: Because, there is no common intersection. T: Because they have the same slope! S: But it says we must find the intersection. And I cannot. T: No, they're also a little cruel to you. They only challenge you C!	E: P, jeg har brug for hjælp! [Ja?] Fordi, jeg kan ikke finde ud af den opgave. Den er besværlig. L: Hvorfor er den besværlig? Hvad er det, du har tegnet der? Du har tegnet 2 linjer, som er parallelle. [Ja]. Og det synes du, det kan bare ikke passe? E: Det er fordi, de kan aldrig nå sammen! Det er noget, jeg ikke selv har.. L: Men hvorfor er det rigtigt, det der? E: Det *er* rigtigt, de rammer aldrig hinanden. L: Nej. Hvorfor gør de ikke det? Hvis du kigger på de to, hvorfor gør de så ikke det? E: Fordi, der er ikke er nogen fælles skæring. L: Fordi, de har samme hældning! E: Men der står, vi skal finde skæringspunktet. Og det kan jeg jo ikke. L: Nej, der er de jo lidt onde ved dig. De udfordrer dig bare, C!

The excerpt is coded as teacher *elicitation* only. In this discussion with one student, the teacher elicitates an explanation from the student based on slopes of the graphs to the two functions in the assignment: y = 3x + 4 and y = 3x - 2. The student is left with a short explanation from the teacher which is possibly not fully understood and accepted by the student though. This encouragement to seek solutions through communication within the group seems to reflect teacher's preference for cooperative learning as an efficient teaching strategy.

T12: Interview of teacher immediately after the take 3 lesson:	
I: Which points did you intend to present in this lesson, and how do you think it evolved? T: What I wanted was to have them understand this function properly, which is y = ax + b. That they know what gradient (slope) is, and that they know to see the intersection of the y-axis. It is simply the most important thing for me because I could see in their skills test, that they simply had not mastered it. And then that they initially draw this and that *some* of them can learn to understand, to figure it out. There are also some that are *completely* blank. But (laughing) it's a challenge, I think. But it's rare we get, into a problem set, two equations with two unknowns that we must solve. I: ... it is a kind of repetition mode, we are in, right? T: Yes, it is basic repetition of the function, which I started today. Because they learned it in grade 7 first. [Yes, they probably did]. But they simply do not remember it [No]. It's so far away, so… I: Do you have any good advice? What to do next time you get the chance? Is it just the conditions? T: Well I think you'll have to time things better and organize. And I actually think I've done a lot. But it is not enough! They often forget if it is a year ago. So it is when you have a skill test that takes it up, then you just have to repeat it.	I: Hvad for nogle pointer ville du gerne have frem i den her time, og hvordan synes du det gik? L: Det jeg gerne ville have frem, det er at de forstår den der funktionsforskrift, som er y = ax + b. At de ved, hvad hældningstal er, og at de ved hvad skæringen med y-aksen er. Det er simpelthen det vigtigste for mig fordi jeg kunne se i deres færdigheds prøve, at de simpelthen ikke havde styr på det. Og så, at de i første omgang får det tegnet, og så at *nogle* af dem, som kan lære at forstå, at regne den ud. Der er også nogle, der er *helt* blanke. Men (ler) det er en udfordring, synes jeg. Men det er jo sjældent vi får, i en problemregning, to ligninger med to ubekendte, som vi skal løse. I: … det er en slags repetitions-mode, vi er røget ind i, ikke? L: Jo, det er grundlæggende repetition af den funktionsforskrift, som jeg startede med i dag. For det har de jo lært i 7. klasse, i første omgang. [Ja, det har de jo nok] Men de kan simpelthen ikke huske det [Nej]. Det er så langt væk, så… I: Har du et godt råd? Hvad skal man gøre næste gang, man får chancen. Er det bare vilkårene? L: Jamen jeg tror man bliver nødt til at time og tilrettelægge. Og det synes jeg faktisk, jeg har gjort meget ud af. Men det er ikke nok! De glemmer det, når der er gået måske et år siden de har set det sidst. Så er det, når man har en færdighedsprøve der tager det op, så må man lige repetere det.

This seems very much oriented towards the coming national exam at the end of grade 9. The teacher has a short summing up in all three lessons. The excerpt below also illustrates this:

T12: Excerpt of take 3 lesson (39:52) Topic: Linear function	
T (looks at the watch): Well, time is nearly up. I would really like to have shown you how to find this intersection in a different way. But we simply cannot today because I noticed you need to work a bit on placing it in a coordinate system in the first place. This is important! So next time that is tomorrow, we will take and solve it in a different way than by drawing. You can simply calculate the solution. S: I knew that!	L (ser lige på uret): Nå, tiden er ved at være gået. Jeg ville egentlig gerne have vist jer, hvordan man kunne løse det her skæringspunkt på en anden måde. Men det når vi simpelthen ikke i dag, for jeg kan se, I har brug for at arbejde lidt med at putte det her ind i et koordinatsystem i første omgang. Det er vigtigt! Så næste gang, og det vil sige i morgen, der tager vi og løser det på en anden måde end at tegne det. Man kan nemlig regne det ud. E: Det vidste jeg godt!

Really it’s more a teaser on tomorrow’s lesson than a recap of points from today. But the teacher has the attention of his students, and gets a nice reaction from one of them.

T12: Discussion of results

There is again a distinct increase in the lesson time coded for elicitation from take 1 to take 3 for this teacher. This may be due to the peer suggestion to make distinct mathematical points.

The teacher seems to follow his initial strategy of teaching through cooperative learning in all three lessons, but the increase in elicitation may be interpreted as an even more conscious elicitation for student points.

There is not much common discussion, and almost all teaching is done in an elicitating dialogue between the teacher and single students or groups. The teacher seems clearly oriented towards distinctive lesson goals in takes 2 and 3.

In take 2, the task was actually suggested by the sparring researcher, and many students ended up being able to argue for medians intersecting in the triangle gravity center.

In take 3 the tasks were from the textbook culminating in a teaser as in the chosen excerpt. The teacher chooses to accept the textbook formulation, which may be a bit debatable.

T13: Male teacher, seniority 15+ years
Medium sized school in Central region

CODES (in %) of lesson length (Topic: Perspective drawing)

TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class	4.1	13.3		4.2	7.7
Teacher → Student		7.6		1.4	
Student → Teacher					

CODES (in %) of lesson length (Topic: Interpretation of graphs)

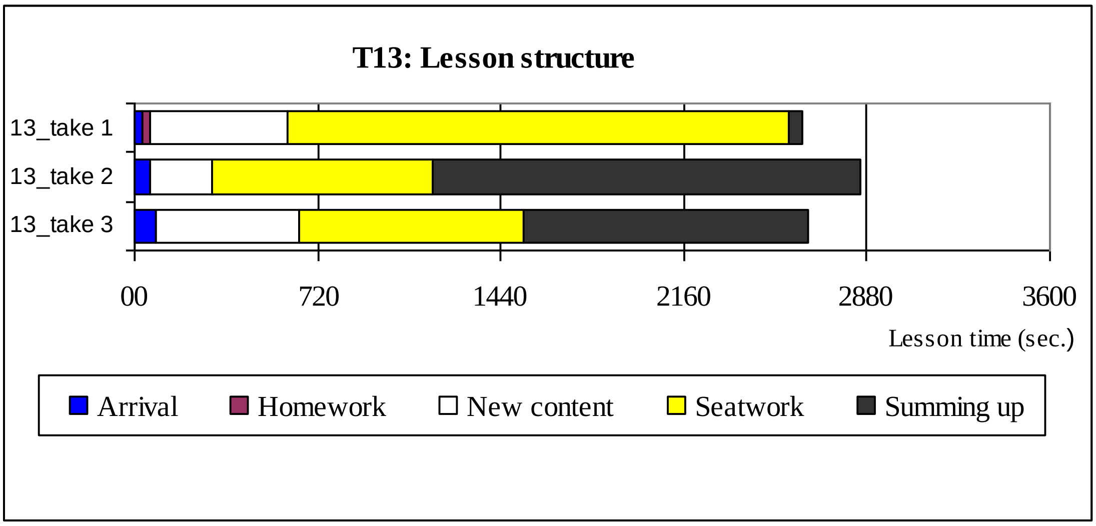
TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class			1.3		16.2
Teacher → Student					
Student → Teacher				47.5	

CODES (in %) of lesson length (Topic: Construction of regular polygons)

TAKE 3	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		5.5			30.5
Teacher → Student					
Student → Teacher	6.8	3.1	15.0		

T13: Lesson structure

Teacher	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	0:33 0:00-0:33	0:30 9:30-10:00	8:57 0:33-9:30	32:55 10:00-42:55	0:51 42:55-43:46	43:46
Take 2	1:03 0:00-1:03		4:02 1:03-5:05	14:28 5:05-8:56, 16:38-27:15	28:00 8:56-16:38, 27:15-47:33	47:33D
Take 3	1:25 0:00-1:25		9:25 1:25-10:50	14:43 10:50-22:29, 27:26-30:30	18:39 22:29-27:26, 30:30-44:12	44:12



T13: Interview of teacher immediately before the take 2 lesson:	
T: But Arne, you've heard that I will do some work with graphs and you sent me to a Norwegian link which I already mentioned. And on that link there are some graphs. And there I'm going to let the students work with them – I just translated them into Danish. So what is new? It is that they should describe the context in writing. And then I'll have them not telling, but reading aloud what they have written. To see if they can, partly interpret it correctly, but also to see if they can express themselves in writing. For it is a skill that they do not practice too much. This may give rise to some "talk" which I do not foresee how it will develop, but we will see. So it is the idea that they then must do several different activities, but not all. Some have to make some and others some other things. And this is especially to interpret some finished graphs that are made.	L: Men Arne, du har jo hørt at jeg vil arbejde noget med grafer og du har sendt mig videre til et norsk link, som jeg har været inde på. Og på det link, der er der også nogle grafer. Og der har jeg tænkt mig at ville lade eleverne arbejde med dem – jeg har lige omformet dem til noget dansk. Det, der så bliver det nye, det er, at de skal beskrive den sammenhæng, der er i den her situation skriftligt. Og så vil jeg have at de bagefter ikke fortælle, men læse op hvad de har skrevet. For at se om de kan, dels fortolke den rigtigt, men også se om de kan formulere sig skriftligt. For det er jo en færdighed, de ikke gør så meget i. Det giver måske anledning til nogle ”snakker”, som jeg ikke kan overskue, hvad det bliver, men det må vi se. Så er tanken, at de dernæst skal lave flere forskellige aktiviteter, men ikke alle. Nogle skal lave nogle, og andre nogle andre ting. Og det er at de bl.a. skal tolke på nogle færdige grafer, som er lavet.

T13: Excerpt of take 2 lesson (8:56) Topic: Interpretation of graphs	
T: A, what have you written? S: I have written that for the first hour, 2 km are walked ... T: I will not – we will comment on it together in a moment. S, what have you written? S: I have written that it takes almost 2 hours to walk 5 km, and then there is a break ... T: Is this the experience others have? There must be other descriptions. J, you have one? S: The trip is 5 hours and the person is no more than 5 km from home … T: Ah, surely we should be able to get a second, it's almost boring this. Yes ... S: I do not know if it is different. But there are 5 km to your destination ...	L: A, hvad har du skrevet? E: Jeg har skrevet, at den første time, der bliver der gået 2 km ... L: Nu skal jeg ikke – vi kommenterer på det samlet lige om lidt. S, hvad har du skrevet? E: Jeg har skrevet, at det næsten tager 2 timer for at gå 5 km, og så bliver der holdt en pause… L: Er det den oplevelse andre har? Der må jo være andre beskrivelser. J, har du én? E: Turen den er 5 timer og personen er højst 5 km væk hjemmefra … L: Ah, vi må da kunne få en anden, det er da næsten kedeligt, det her. Ja… E: Jeg ved ikke, om den er anderledes. Men der er 5 km til destinationen …

T: Yes. Are they similar the explanations we've got? SS: Yes. T: Is there no one who is going to say something different. KA, that was nice? S: Yes. For starters, he walks briskly. He seems to have a lot of energy ... S: I would say it takes less time to go home, because the slope is steeper on the outward journey than it is on the way home. ... T: So you're using the slope to determine if it is one or the other?	L: Ja. Ligner de hinanden de forklaringer, vi har fået? EE: Ja. L: Er der slet ikke en, der er lidt frisk til at mene noget andet. KA, det var dejligt? E: Ja. Til at starte med, da han går derudaf. Da har han sådan set masser af energi … E: Jeg vil sige, det tager kortere tid at gå hjem, for hældningen den er større på udturen end den er på hjemturen. … L: Så du bruger altså hældningen til at afgøre, om det er det ene eller det andet?

The excerpt is coded as teacher *elicitation* and *student interpretation point*. The students are invited to interpret a graph and eagerly do so, also proficiently. The lesson is therefore coded as having quite a few student interpretation points without the teacher always eliciting them.

The lesson assignment was one suggested by the sparring researcher as a task well known for its diagnostic qualities from a Norwegian project (Brekke, 2002, p. 12) and the teacher administered the idea to fully exploit the diagnostic properties.

T13: Interview of teacher immediately before the take 3 lesson:	
T: Yes we are close to the end of a school year, and this means that some areas must be re-captured. Before the compulsory exams. And what we intend to work on today, is polygons and regular polygons. And the students' task is to work on their own initiative, hopefully to be able to construct different polygons in order to discover: how to create regular polygons, to see whether this knowledge is present. If it is not, we'll provide it along the way. And in doing so, they must decide on degrees, angle sums and similar things. And then I hope that the ultimate goals are met that some are able to develop a formula that applies to an *arbitrary* polygon. I: Yes, what kind of formula is it? T: Yes, but it is: What is the angle sum for the n^{th} polygon? [Ok]. So that you can go in and work with variables instead. [Yes]. And the work with triangles, squares, pentagons and hexagons would lead to some	T: Ja vi nærmer os jo afslutningen på et skoleår, og det vil sige at der I hvert fald er nogle områder, vi skal have fanget. Inden de der nødvendige prøver. Og det vi skal arbejde med I dag, det er polygoner, og regulære polygoner. Og elevernes opgave det bliver af egen drift forhåbentlig at kunne konstruere forskellige polygoner for derved at opdage: Hvordan kan man lave regulære polygoner, for at se om den viden er til stede. For er den ikke det, så må vi jo give den undervejs. Og i den forbindelse, så skal de jo tage stilling til gradstørrelse og vinkelsum og lignende ting. Og så håber jeg jo, at endemålet det bliver at der er nogle der bliver i stand til at udvikle en formel, som gælder for en *vilkårlig* polygon. I: Ja, hvad er det for en formel? L: Ja, men det er: Hvad er vinkelsummen for den n'te polygon? [Ok]. Altså, at man kan gå ind og arbejde med variable i stedet for. [Ja].

reflections: How is it that this may be expanded? And we will have to see if this succeeds. ... And then I imagine that the results achieved along the way are noted on the blackboard. I: Is it a lesson to be ended by a summing up or will it be continued next time you teach math? T: Yes, it's always a question you can say, because you should not force anything. But I have a feeling or an assumption that the extent fits the schedule. So that we can complete this within 40-45 minutes. I will strive for that, because otherwise we cannot get hold of the necessary point, you might say. But then it must of course be in the next lesson.	Og det arbejde med trekanter, firkanter, femkanter og sekskanter skulle gerne gøre, at de gør sig nogle overvejelser over: Hvordan er det, det udbygges? Og det må vi jo så se, om det lykkes. … Og så forestiller jeg mig, at de resultater der opnås undervejs, de noteres på tavlen. I: Er det en time, der afsluttes med en opsummering, eller den bare fortsætter næste gang I har matematik? L: Ja, det er altid et spørgsmål kan man sige, for man skal jo ikke forcere noget. Men jeg har en fornemmelse af, eller en formodning om at omfanget det rækker. Altså at vi kan afvikle det her inden for 40-45 minutter. Det vil jeg tilstræbe, for ellers få vi ikke fat i den nødvendige pointe, kan man sige. Men så må det jo komme næste gang.

The wording indicates that students will be at work. The lesson content is presented as a job to do.

T13: Excerpt of take 3 lesson (34:43) Topic: Construction of regular polygons	
T: Well, we begin to approach it here maybe (pointing to the n-gon). If we just focus on angles ... Can you make a 7-gon? [Yes] In an n-gon, how many angles are there in an n-gon? S: n T: ... Who are close to a system here, which means that we are able to write a formula? Look at previous jobs. Because I would in fact like to at some point know: What is the situation if it is a 12-sided polygon we are dealing with? ... S: So every time we have said, first it was 180 times 1, representing 180 in angle sum, the next: 2 times 180 so it's 360 in angle sum. And then next 3, 4 and so all the way up L: Yes ... Can we express this using a formula? ... What is the connection between a quadrangle and the arithmetic expression, between a pentagon and the arithmetic	L: Nå, vi begynder at nærme os den her måske (peger på n-kanten). Hvis vi lige sætter fokus på vinkler… Kan I lave en 7-kant? [Ja] I en n-kant, hvor mange vinkler er der i en n-kant? E: n L: … Hvem er ved at nærme sig et system her, som gør at vi er i stand til at skrive en formel? Kig på tidligere opgaver. Fordi jeg vil nemlig gerne på et tidspunkt have at vide: Hvad nu hvis det er en 12-kant vi har fat i? … E: Altså hver gang har vi sagt, først var det 180 gange 1, altså 180 i vinkelsum, den næste: 2 gange 180 så det er 360 i vinkelsum. Og den næste 3, 4 og sådan hele vejen op L: Ja ... Kan vi udtrykke det ved en formel? … Hvilken sammenhæng er der mellem en firkant og regneudtrykket, mellem en femkant og regneudtrykket, mellem en

expression, between a hexagon and the arithmetic expression? Can you write a formula, K for this? S: But is it not just – what is it called – no you cannot. T. S? S: Yes, if you call the angle sum x, then the angular size is x/n. T: ... If we start with the angle sum, what is the angle sum as in an n-gon? Yes, what was it in a 6-gon, there it was? S: 4 times 180. T: What was it in a pentagon? S: 3 times 180. T: What do you think it is in a n-gon? Which term? G? S: n times 180. T: n times 180, I'll just try to write it for you. S: n minus 2 T: Why n minus 2? S: There are two edges to each side that cannot be used. T: This means, that for the 6-gon we did not multiply by 6, but with? [4] 2 less. ... S: (n-2).	sekskant og regneudtrykket? Kan du skrive en formel, K for det her? E: Jamen er det ikke bare, hvad er det det hedder – nej det kan man ikke. L. S? E: Ja, hvis man nu kalder vinkelsummen x, så vinkelstørrelsen det er x/n. L: … Hvis vi starter med vinkelsummen, hvad er vinkelsummen så i en n-kant? Ja, hvad var det i en 6-kant, der var det? E: 4 gange 180. L: Hvad var det i en femkant? E: 3 gange 180 L: Hvad tror I så det er i en n-kant? Hvilket udtryk? G? E: n gange 180. L: n gange 180, jeg vil lige prøve at skrive det for dig. E: n minus 2 L: Hvorfor n minus 2? E: Der er to sider ude i hver side, der ikke kan bruges. L: Dvs. 6-kanten, der gangede vi ikke med 6, men med? [4] 2 mindre. ... E: (n-2).

This excerpt is coded as a *student result point*. The students express the reasoning and the result, but the teacher is actively processing. I decided not to label this as elicitating, but it comes pretty close of course.

The summing up is framed by short essential remarks from the teacher as in the excerpt below from take 3:

T13: Excerpt of take 3 lesson (42:16):	
L: Well I think we should just slowly conclude the lesson. We will continue next time – and built up the same way – with the number of diagonals. So, what did you learn? What can you talk about now? S: The angles. L: Yes, the angles of polygons and what regular means.	L: Godt jeg tror lige vi skal så småt skal til at runde af. Vi fortsætter så næste gang – og bygget op på samme måde – med antallet af diagonaler. Godt, hvad har I fundet ud af? Hvad kan I udtale jer om nu? E: Vinklerne. L: Ja vinklerne i polygoner og hvad regulær vil sige.

The short remarks from teacher and students in the excerpt are not a summing up of *all* important points from this take 3 lesson. Time is short, but my observation and my transcript shows that many students probably would have been able to express more coherent explanations on their findings if given the chance. Now the teacher concludes and maybe my talk to the focus teachers has been too superficial in my warm recommendations for such closing remarks.

T13: Discussion of results

The seatwork phase became shorter in take 2 and take 3, while the teacher now does a common summing up in both sessions. I take this as a deliberately planned structure. The teacher's elicitation has increased from take 1 to take 3, and the students' points are substantial. All three lessons have many clear points, but there seems to be a change from the teacher expressing them and towards making the student's do this.

T20: Male teacher, seniority 15+ years
Large school in Middle region

CODES (in %) of lesson length (Topic: Area of polygons)

TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		16.6	6.0	5.9	28.5
Teacher → Student		3.2			
Student → Teacher		11.4	9.7		

CODES (in %) of lesson length (Topic: Line equation)

TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class				3.0	13.5
Teacher → Student					
Student → Teacher		3.5		9.4	

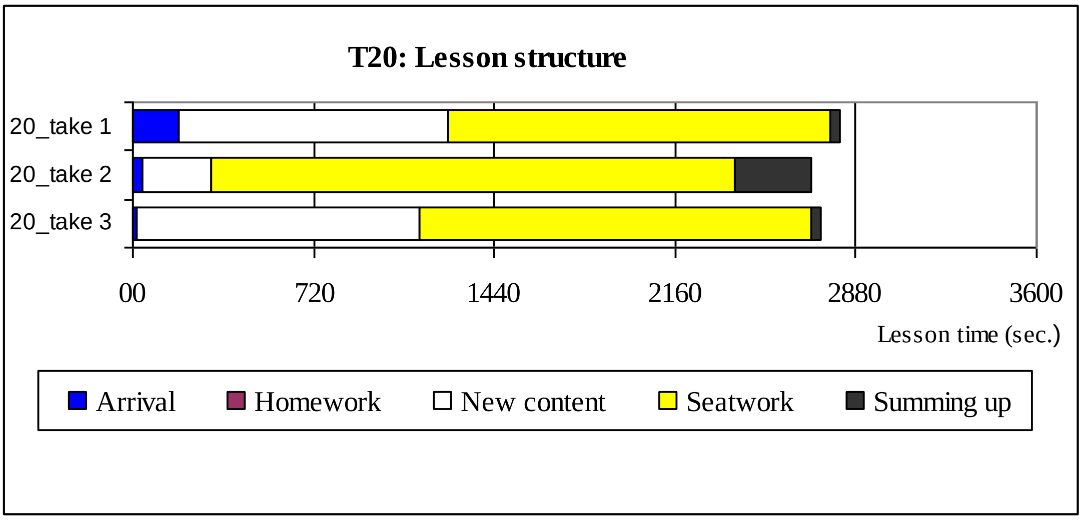
CODES (in %) of lesson length (Topic: Quadratic equation and the parabola)

TAKE 3	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		3.9		11.6	31.4
Teacher → Student		0.7			
Student → Teacher			2.6		

T20: Lesson structure

Teacher 20	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	3:03 0:00-3:03		17:52 3:03-20:55	25:23 20:55-46:18	0:35 46:18-46:53	46:53
Take 2	0:35 0:00-0:35		4:35 0:35-5:10	34:45 5:10-14:54,	5:08 14:54-16:29,	45:03

				16:29-29:57, 31:43-43:16	29:57-31:43, 43:16-45:03	
Take 3	0:14 (0:00-0:14)		18:44 (0:14-18:58)	26:03 (18:58-45:01)	0:37 (45:01-45:38)	45:38



The teacher's presentation seems short in take 2, where common summing up is done in three rounds. This structure is formatted by the teacher's use of four work cards. Each card presents a new development in a continuing investigation of linear equations and their graphs. The structure seems new to the teacher, who was tempted to try it after my sparring.

T20: Interview of teacher immediately before the take 2 lesson:	
I: What do you expect them to be able to by the end of the lesson? T: If we get through (task card 3), then they can draw some different lines. And they will also have them intersecting [Yes]. These two (T now pointing to his own "solution" on mm-paper), they do not intersect, they are parallel.... I: Working with such task cards, they didn't do that before? T: They did not! I: So you take a couple of chances? T: It, yes – and I think that is really nice by having a visit from you! Well, that [I am pleased to hear that] it surely is. And, well I've sent the one with parabolas (link for a computer program) to mathematics colleagues in the secondary, right [Yes?].	I: Hvad regner du med de kan, når timen er forbi? L: Hvis vi når til og med (arbejdskort 3), så kan de tegne nogle forskellige linjer. Og de vil sådan set også få dem til at skære hinanden [Ja]. De to (L peger nu på sin egen "løsning" på mm-papir), de skærer ikke hinanden, de bliver parallelle. ... I: Den der arbejdsform, med sådan nogle kort dér, det har de ikke prøvet før? L: Det har de ikke! I: Så du tager nogle chancer? L: Det, ja – og dét synes jeg er rigtig godt ved at have besøg af dig! Jamen, det [det er jeg da glad for at høre] det er det bestemt. Og, altså jeg har jo sendt den med parablerne (et computer program) til matematik-kollegerne på overbygningen, ikke [Ja?].

And there are more that have been linking and looking in and commenting, that this is quite smart. And on Monday the students went and used it themselves. Even with cannons! Then they were told to write two things down, that is using the slider (which varies the parameters a, b and c in the standard quadratic equation). And some have sent it to me, where I have commented on it because – well a indicates the legs pointing up, like that. Something is a bit too thin, but it ...	Og der er da flere, der har været inde og kigge på den og sige, at den er da godt nok smart. Og eleverne var inde i mandags, og bruge den selv. Også med kanoner! Da fik de besked på at skrive to ting ned, altså ved skyderen der (som varierer parametrene a, b og c i standard 2. grads ligningen). Og nogle har sendt det til mig, hvor jeg har kommenteret det, fordi – jamen a det er noget med benene opad sådan der. Noget er lige tyndt nok, men det …

The teacher is very experienced and not really in doubt about what to do and keep to in his teaching. Yet he chooses to test new ideas for content and organization in take 2, which also involves students more actively in conversations with each other and the common summing up sessions.
In the next excerpt, the teacher has a whole class discussion about the assignment below (Hundeland, 2009, p.70):

Card 2 (where x and y are decimal numbers): **x + y = 7**
1. What happens if x = 2½?
2. What if y = 4.7?
3. How many different solutions are there?
4. **Map** the solutions in the same coordinate system as before.
5. What happens if one of the numbers is 9?
6. What happens to the mapping now?
7. **Find** four new solutions where either x or y is larger than 9
8. Also **map** them.
9. **Discuss** what you have found out. **Write at least two things down**.

T20: Excerpt of take 2 lesson (29:57) Topic: Line equation	
T: Should we just take a short stop, now all have got card 3. Card 2, let’s return to that one all of you. SS: We made a (whole) essay about it here // Conclusion ... T: What do things on card 2 indicate? What have you found out? K? S: That there are many ways to make the same equation (shows with her hand a line with negative slope).	L: Skal vi lige tage et lille stop, nu har alle fået kort 3. Kort 2, skal vi lige vende tilbage til den alle sammen. EE: Vi har lavet en (hel) stil om det her // Konklusion ... L: Hvad fortæller nogle ting på kort 2? Hvad har I fundet ud af? K? E: At der findes mange måder at lave den samme ligning på (viser med hånden en linje med negativ hældning).

T: Yes. Are there infinitely many points now? S: Yes. T: Yes, and did you get some negative also [Yes]. Because if anyone didn't get them before then – M? S: If one of the numbers is larger than 7, then the other is negative. T: Right. Others? Other things N? S: If only the result is the same, then it is the same line all the way through. T: Yes. S: Then all the points are on the same line. T: And what caused that – try looking up here (on blackboard). This is because, yes, it looks like this, right (T points at x + y = 7). And those of you, now everyone has card 3, there is a number there. Then something happens. Yes, some other things you found out? F did you? S: No. T: Did all of you write down 4 solutions, it is the one called number 7? SS: Yes. T: Now you may continue with card 3.	L: Ja. Er der uendelig mange punkter nu? E: Ja. L: Ja, og fik I de negative med alle sammen [Ja]. For hvis ikke nogen havde fået dem med før, så – M? E: Hvis ét af tallene er større end 7, så er det andet negativt. L: Rigtigt. Andre? Andre ting, N? E: Hvis bare facit er det samme, så er det ens linje hele vejen igennem. L: Ja. E: Så ligger alle punkterne på samme linje. L: Og hvad skyldes det – prøv at kigge herop (på tavlen). Det skyldes jo, at den ser sådan ud, ikke (L peger på x + y = 7). Og de af jer, nu har I alle sammen fået nummer 3, der står der et tal dér. Så sker der noget. Ja, nogle andre ting, man har fundet ud af? F, har du noget? E: Nej. L: Har I alle sammen fået skrevet 4 løsninger ned, det er det der hedder nummer 7? EE: Ja. L: Så er I i gang med opgave 3.

This excerpt is coded as a *student interpretation point*. The students were asked to report on the tasks from this card 2.
As indicated in the table on lesson structure seatwork and common summing up is in three rounds also offering students opportunities to state procedural and interpretation points to teacher and class.
The take 2 lesson is ended by the teacher asking class for its opinion:

T20: Excerpt of take 2 lesson (44:38) Topic: Line equation	
T: I'd like to hear what you think of this way of working? SS: It is good. // You learn a lot. T: How about it, A? SS: It was fine. // It was so cool that (inaudible). T: Was it good to have such – some paper – in your hands? S: Yes! L: And have things written down?	L: Jeg vil gerne høre, hvad synes I om den her arbejdsform? EE: Den er god. // Man lærer en masse. L: Hvad siger A? EE: Den var fin. // Det var så fedt, at (uhørligt). L: Var det godt at have sådan noget – papir – mellem hænderne? E: Ja! L: Og fået skrevet tingene ned?

S: Yes!	S: Ja!

The teacher himself was very interested in getting experience with this type of written assignment carefully designed for progression. Here he also gets the class' assessment of an excellent working structure, though he cannot resist suggesting paper and notes to be among the quality markers.

T20: Interview of teacher immediately before the take 3 lesson:	
I: Are there any points at stake in the next lesson? T: We will repeat parabolas, right. And I will start by, like when I first time taught it, what do – now they have seen parabolas before – and then I try to practice that on the board and talk about it, and they tell me what the different terms, i.e. a, b and c stand for. And then I can also move this one (a parabola formed by wire) around the board. And they have seen this before. It should really probably go quite well [Yes]. And then there are some tasks that they can solve, so they acquire a deeper understanding of it. It's kind of what I'm doing today. [And will you make it in time?] I am quite sure. I: Is this lesson, does it connect to something that happened yesterday? T: No, no! Now it is repetition. ... This is one topic, right. Somebody wanted something and it's the choice there is. They've studied this topic for today, at least they have been told to do so. I: So in other words, if you tightened it: Then there are many parameters in the quadratic equation. Do they really, do you really think they know ... T: They've had it before … I: You hope they can explain. Do they come to say something? T: I want to ask in class: What does it mean? And last week they got an extra quiz test back, where they were to calculate the straight line equation. And *this* was difficult for them.	I: Er der nogen pointer på spil i den kommende time? L: Vi skal repetere parablen, ikke også. Og det vil jeg så starte med, ligesom da jeg gennemgik det, hvad bruger – nu har de ligesom set parabler før ikke – og så vil jeg prøve at træne det på tavlen og fortælle om, og *de* fortæller mig hvad de forskellige udtryk, altså a, b og c det står for. Og så kan jeg også flytte den her (en parabel formet af ståltråd) rundt på tavlen. Og det har de set før. Det skal såmænd nok gå stille og roligt [Ja]. Og så er der jo nogle opgaver, de kan regne, så de får endnu mere forståelse for det. Det er sådan, hvad jeg har tænkt mig i dag. [Og når du det?] Det tror jeg da bestemt. I: Er timen her, hænger den sammen med noget der skete i går? L: Nej, nej! Nu er det repetition. … Det er et emne, ikke. Nogle har ønsket noget, og det er jo så der, det valg der er. De har læst på det her til i dag, i hvert fald fået besked på at gøre det. I: Så altså, hvis man strammede den: Så er der jo mange parametre i andengradsligningen. Kan de virkelig, tror du virkelig de ved, … L: De har jo haft det før … I: Du håber, de kan forklare. Kommer de selv til at sige noget? L: Jeg vil spørge ud i klassen: Hvad betyder det? Og i sidste uge fik de en ekstra terminsprøve tilbage, hvor de skulle udregne den rette linjes ligning. Og *det* var svært for dem.

But they got some, I think, ideas as to how to go about it. And they can then hopefully use it here. ... I sometimes use this expression: How do I usually say? ... And someone might then recognize it.	Men der fik de nogle, synes jeg, fiduser til hvordan de griber det an. Og det kan man da forhåbentlig bruge her. … Jeg bruger nogen gange det her udtryk: Hvordan plejer jeg at sige? … Og nogen kan måske så genkende det.

The teacher does not announce sharp mathematical points coming up, as I prompt for that. But a repetitive approach to the parabola will of course invite several points, which were observed being articulated later in the lesson.

Excerpt of take 3 lesson (11:14) Topic: Quadratic equation and the parabola	
T: And what is it that the formula indicated in your book says? It can be found at least in my edition on page 167. It looks pretty theoretical, but that's it! (the students search in vain in their book). ... What must x be for the parabola to intersect the x-axis? It's called -b and then the square root of what is called the discriminant ... When can you extract the square root of what is there? (T indicates b^2 - 4ac)? Now there are not any numbers, so this makes it a little harder, maybe? (T waiting in vain). Yes if the number, b^2 is less than that (4ac), then what (L repeats this)? S: Then it becomes negative. T: Then it becomes negative! Then one cannot extract the square root, so there are no solutions, and it's such a type as the one up there (T points at a parabola entirely above the x-axis).	L: Og hvad er det nu den formel, der står i jeres bog, den siger? Den kan man finde i hvert fald i min udgave side 167. Den ser rigtig teoretisk ud, men sådan er dét! (eleverne kan ikke finde det). ... Hvad skal x være for at parablen skærer x-aksen? Den hedder -b og så kvadratroden af den, der hedder diskriminanten … Hvornår kan man uddrage kvadratroden af det, der står derinde? (L peger på b^2 - 4ac)? Nu er der ikke nogen tal på, så det gør det lidt vanskeligere, måske? (L afventer forgæves). Ja hvis det tal der, b^2 er mindre end den (4ac), hvad så (L gentager det)? E: Så bliver det negativt. L: Så bliver det negativt! Så kan man ikke uddrage kvadratroden, så er der ingen løsninger, og så er det sådan en type som den deroppe (L peger på en parabel, der ligger helt over x-aksen).

This excerpt is coded as a *procedural point*. It is clearly made by the teacher, who has a hard time involving students in the communication. For the following six minutes the teacher elicitates contributions from students on the solution conditions. The repetitive mode in take 3 may explain the increase in elicitation. The teacher has quite a few interpretation points at the start of the lesson, teaching the significance of a, b and c to the graphic image of the function $y = ax^2 + bx + c$, when mapped in a coordinate system.

T20: Discussion of results

In all three takes with this teacher there is no correction of homework. The take 1 and 3 activities were to some extent decided and formatted by textbook exercises. As seen in the table on lesson structure there is no interruption of seatwork once it is started. This was quite different in take 2.

The amount of elicitation is increased from take 1 to take 3. The lesson goals seem more precise and limited in take 2 and 3, where the teacher has a specified goal for the lessons. But there is no increase in student points.

T21: Female teacher, seniority 15+ years
Small school in Central Region

CODES (in %) of lesson length (Topic: Powers)

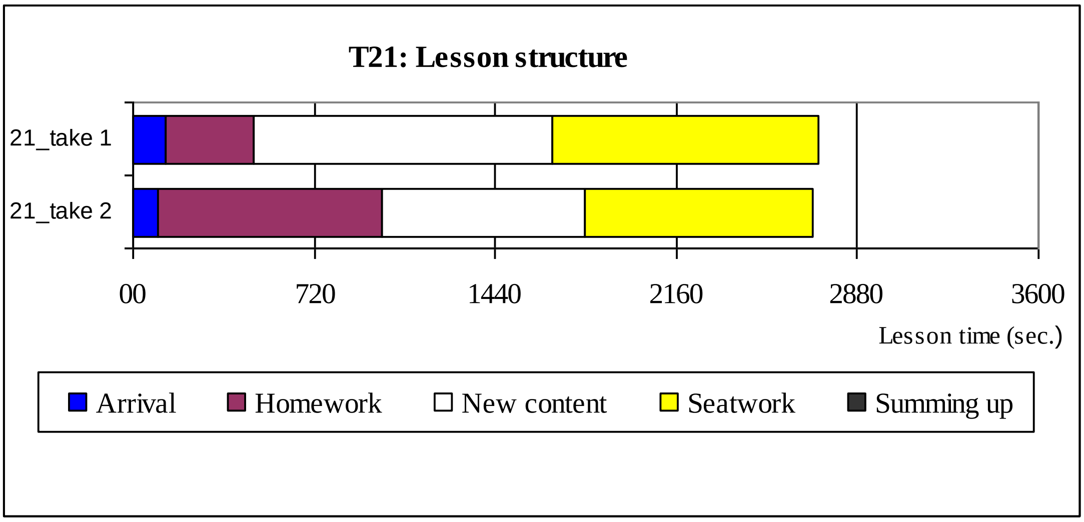
TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		11.7	5.1	4.3	6.9
Teacher → Student					
Student → Teacher					

CODES (in %) of lesson length (Topic: Fraction calculations + reduction)

TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class				2.5	16.1
Teacher → Student					
Student → Teacher		9.3			

T21: Lesson structure:

Teacher 21	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	2:10 0:00-2:10	5:50 2:10-8:00	19:50 8:00-27:50	17:38 27:50-45:28		45:28
Take 2	1:42 0:00-1:42	14:44 1:42-4:02, 5:46-11:44, 38:34-45:00	13:33 4:02-5:46, 11:44-23:33	15:01 23:33-38:34		45:00



This teacher seems unaffected by seminar and sparring. The take 2 lesson is one almost uninterrupted teacher led correction of homework and tasks given underway. There are few explanations, but a lot of corrections – also disciplinary remarks. There is an increase in lesson time coded as elicitation from take 1 though.
This teacher was regrettably unable to take part in take 3 due to illness and back surgery.

T 21 interview of teacher before take 2:	
T: And then are we going to look at moving a common factor outside the parentheses. It's something that is quite distant to them and you know by experience is always difficult [Yes]. But we will try that, and then – I usually let some students stand up and do the mathematics – on the blackboard. I: Yes – why? T: Well, firstly, they want to. And secondly, I think they look a little more closely, when classmates are up there than when I just stand and write [Yes]. And secondly also because they've been sitting at home and done the math, and then they do like to get up and show what they have [Yes]. Then there are also some who don't want to, and then I say it does not matter and then you have to.	L: Og så skal vi så til at se på det med at sætte en fælles faktor uden for en parentes. Det er jo noget, der ligger langt væk, og det er erfaringsmæssigt altid svært [Ja]. Men det vil vi prøve, og så – jeg plejer så, at jeg lader nogle elever komme op og regne – på tavlen. I: Ja – hvorfor? L: Jamen for det første, så vil de gerne. Og for det andet, så tror jeg de kigger lidt mere efter, når kammerater også er oppe, end når jeg altid står og skriver [Ja]. Og for det andet også fordi, de har jo siddet hjemme og regnet på det, og så vil de jo gerne op og vise, hvad de har [Ja]. Så er der også nogle, der ikke ret gerne vil, og så siger jeg det er også lige meget og det skal I jo så.

T21: Excerpt of take 2 lesson (4:02) Topic: Fraction calculations + reduction	
T: Now it could also be that you instead wanted to divide the two fractions with each other (T writes 1/4: 2/3). What is it then, we do? J, do you remember? S: Yes, then you reverse 2/3, so it becomes 3/2. T: Yes, and while I reverse it, what do I do then? S: Then you have to multiply. T: Yes. It will be? S: 3/8. T: Yes. If I had a whole number, which I had to divide by 2/3, so what? (T writes 5: 2/3) S: Do the same and reverse it. T: Yes, I could then do the same and reverse it. And what does it end up with? S: 15/2 T: Yes. But if there had been 2/3 : 5? Then what? Then what, K? S: I do not know. T: Nooo! S: I do not know, I cannot remember T: (Writes the answer: 2/15). You'll never forget K! Right?	L: Nu kunne det også være sådan, at man i stedet for skulle dividere de to brøker med hinanden (L skriver 1/5 : 2/3) Hvad er det så, vi gør? J, kan du huske det? Elev: Ja, så vender man 2/3, så det bliver 3/2. L: Ja, og samtidig med at jeg vender den, hvad gør jeg så? E: Så skal man gange. L: Ja. Det bliver? E: 3/8. L: Ja. Hvis jeg nu havde haft et helt tal, som jeg skulle dividere med 2/3, hvad så? (L skriver 5 : 2/3) E: Gøre det samme og vende den. L: Ja, jeg kunne da gøre det samme og vende den om. Og hvad giver det så? E: 15/2 L: Ja. Men hvis der nu havde stået 2/3 : 5? Hvad så? Hvad så, K? E: Det ved jeg ikke. L: Arj! E: Jeg ved det ikke, jeg kan ikke huske det L: (Skriver svaret: 2/15). Det glemmer du aldrig, K! Vel?

This excerpt is coded as a *student procedural point*, but also strongly elicited by the teacher. Most students give a correct method for division by fractions. But they are neither presented with nor requested any justification of the procedure.

T21: Discussion of results

There is an increase in the teacher’s elicitation from take 1 to take 2. And the students are offering points in take 2. Unfortunately this teacher was unable to participate in the take 3 research, and the teacher did not respond to my peer sparring offer before take 2. Also the lesson structure seems unaffected by the earlier seminar meeting and discussion in the focus group. The teacher described herself as a supporter of the classical mathematics teacher virtues with the teacher presenting and the students then learning by repeated practice. She joked about it, and in a pleasant peer tone explicitly declared that I would have a hard time changing her.

T35: Female teacher, seniority 5-9 years
Large school in Central region

CODES (in %) of lesson length (Topic: Line equation)

TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class	1.9	15.0		8.6	8.1
Teacher → Student					
Student → Teacher	0.4	2.5		0.4	

CODES (in %) of lesson length (Topic: Reduction and equations)

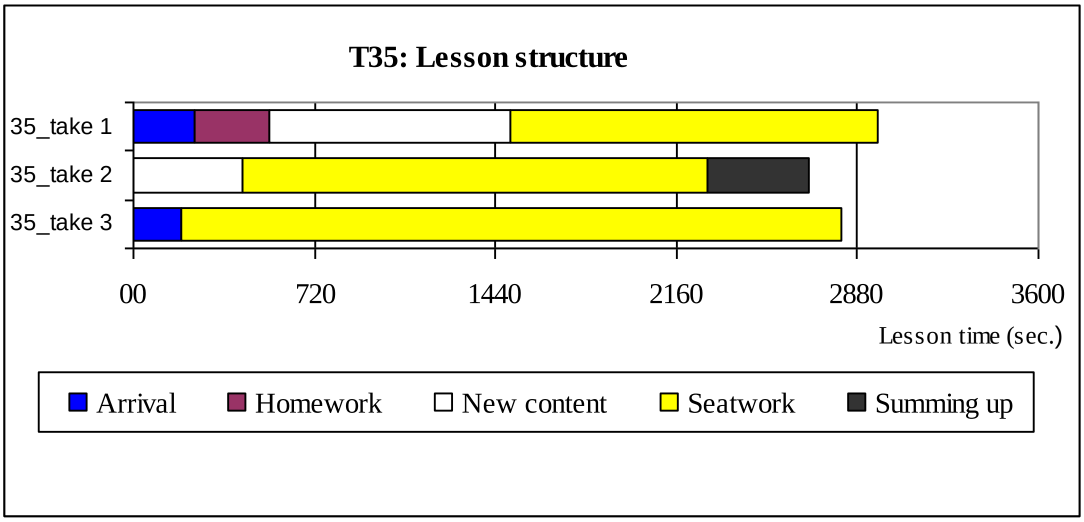
TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class		3.4			11.0
Teacher → Student					
Student → Teacher		4.6			

CODES (in %) of lesson length (Topic: Statistics)

TAKE 3	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class					38.3
Teacher → Student		5.0			
Student → Teacher					

T35: Lesson structure

Teacher 35	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	4:00 0:00-4:00	5:00 4:00-6:00, 22:00-25:00	16:00 6:00-22:00	24.21 25:00-49:21		49:21
Take 2			7:17 2:54-10:11	30:47 13:28-44:15	6:44 0:00-2:54, 10.11-13:28, 44:15-44:48	44:48
Take 3	3:10 0:00-3:10			43:46 3:10-46:56		46:56D



T35: Interview of teacher immediately before take 2:	
T: What will happen now, it's that we just need to review. I want to review this – especially how to multiply two brackets with each other. And minus-brackets, and equations: to subtract and add, to multiply and divide on both sides. These are the things I have told them several times. This is the final lesson before Christmas. … I: Well, of course it also ought to work out all right because they just had a sort of preparation or review. But it's hard for me to judge if this (an A4 sheet with tasks) it is something you hand out, and then you withdraw and collect again after 45 minutes? T: No, I do not. [It is not like that – a kind of test?] It is not a test. They do all get it and may sit and help each other. If they get stuck, I'm there. ... So this is something we would prefer not to spend more time on in grade 9 after today.	L: Det, der skal ske nu, det er at vi lige skal have repeteret. Jeg vil have repeteret, det her med – specielt hvordan man ganger to parenteser med hinanden. Og minus-parenteser, og ligninger: at trække fra og lægge til, at gange og dividere på begge sider. Det er de ting, jeg har sagt til dem flere gange. Det er den sidste time inden jul. … I: Nå, men det skulle jo gerne gå nogenlunde fordi de har lige haft en slags forberedelse eller gennemgang. Men det er jo svært for mig at vurdere, om det her (A4-arket med opgaver) det er noget du deler ud, og så trækker du dig tilbage og samler ind, når der er gået 45 minutter? L: Nej, sådan gør jeg ikke. [Det er ikke sådan - en slags prøve?] Det er ikke en prøve. De får jo alle sammen, så kan de sidde og hjælpe hinanden. Går de i stå, så er jeg der. … Så det her, det er noget vi helst ikke skal bruge mere tid på i 9. klasse efter i dag.

The lesson content was different from the one suggested and referred to by the teacher and me in previous e-mail correspondence.

In the actual lesson the class were under quite loose control, and right after a joint review with T at the board, she spent most time on two students in a tedious iteration of very few tasks. In reality it seems that T had to do the mathematics. According to

T's evaluation after the lesson especially the boys were very much influenced by my presence.

Excerpt of take 2 lesson (5:15) Topic: Reduction and equations	
T: Now there was one more thing with these reductions, we just had to look at. And I think you can, most of you by now. Say we have (x + 2) and must multiply this by (4 - x). We must multiply two brackets with each other, what do we do? (Only one student marks). I have to see some more fingers, you know that. M should just put away his paper. A, what do we do first? S: You say x times 4, x times x [Yes] and 2 times 4 and 2 times x (L shows this on the board by drawing 4 arrows between the parts). Yes. And remember that these are positive numbers and those are negative. We use the signs here.	L: Så var der én ting mere med de her reduktioner, vi lige skulle kigge på. Og det tror jeg I kan, de fleste af jer, efterhånden. Hvis vi har (x + 2) og skal gange det med (4 - x). Vi skal gange to parenteser med hinanden, hvad gør vi så? (Blot en elev markerer). Jeg skal lige se nogle flere fingre, det ved I godt. M skal lige lægge sit papir. A, hvad gør vi først? E: Man siger x gange 4, x gange x [Ja] og 2 gange 4 og 2 gange x (L viser det imens på tavlen ved at tegne 4 pile mellem leddene). Yes. Og husk at det der er positive tal og det der er negativt. Vi bruger fortegnene her.

There is no longer discussion on the topic in this lesson, only a quick reminding of rules, when parentheses with a minus sign in front are removed.
The above excerpt is coded both as teacher *elicitation* and a *student procedural point*. The student does not clearly state the procedure by handling negative signs before or in brackets however, so the teacher adds on that. But the class seems to know, as this is also demonstrated earlier in the lesson. The procedural points coded for teacher and students in the take 2 lesson are made when correcting tasks in common.

T35: Interview of teacher immediately before the take 3 lesson:	
I: Are there any points in the lesson we are about to see? T: Well, one point is that they just should pick up on some statistics. They need to remember what a median is. They must be able to convert a statistical material into a pie chart. And then there's – yes they should also be able compute average. And maximum and minimum value they must also be able to find. And that s the most important things. I: Yeah, I knew hat of course – as I saw your	I: Er der nogen pointer i den time, vi skal ind og se? L: Altså, der er den pointe, at de lige skal have samlet op på nogle ting omkring statistik. De skal lige kunne huske, hvad en median er. De skal kunne omregne, så de får lavet et cirkeldiagram for et statistisk materiale. Og så er der – ja de skal også kunne regne gennemsnit ud. Og største- og mindsteværdi skal de kunne finde. Og det er sådan de ting, som er de vigtigste. I: Ja, jeg vidste det jo godt – jeg har set din

mail about statistics. But it is a mixture of a definition for a median, and then some explanations, or how – the methods to compute different things? T: Well, they get such a specific task we have in the book [Well], where I have picked out a single question, in which they should sit and subdivide into ranges. I do not really find that task so good at that. So one simple question has been picked out, and then I have given them a few extra tasks on top ... and they will work on this for two lessons (this is the second). … I: But I've told you at some point that I register the points [Yes], if teachers have some points, also for the students to notice. That was what I was wondering about? T: Well of course, that talk has obviously resulted in some reflections *also* by me. Anyway, I have generally been thinking quite a lot about what I do. And I've also had a colleague up sometimes to observe my teaching, because I think there has been too much turmoil ... And I know I must work considerately more with discipline in the classroom.	mail om statistik. Men det er jo en blanding af en definition på en median, og så nogle forklaringer, eller hvordan – metoder til beregning af forskellige ting? L: Altså, de får sådan en konkret opgave, vi har i bogen [Nå], hvor jeg har pillet en enkelt opgave ud, for der skal de sidde og inddele i intervaller. For det synes jeg egentlig ikke opgaven er så god til. Så en enkelt opgave har jeg pillet ud, og så har jeg så givet dem et par ekstra opgaver oveni … og det skal de så arbejde med i to timer (hvor dette er den anden). … I: Men jeg har jo fortalt jer på et tidspunkt, at jeg registrerer pointerne [Ja], om lærerne har nogle pointer, som også eleverne kan høre. Det var dét, jeg tænkte på? L: Altså selvfølgelig, den snak har da selvfølgelig gjort at, det har jeg da *også* tænkt lidt. Men altså, nu har jeg i det hele taget tænkt meget over hvad jeg gør. Og jeg har også haft en kollega med oppe på et tidspunkt og overvære, fordi jeg synes der har været for meget uro ... Og jeg ved jeg skal arbejde meget med netop disciplin i klassen.

The teacher’s use of points in this interview tends to be in the everyday meaning. Probably my question also triggered that. The teacher seems challenged by quite a variation in work morale among students. She does not use one regular textbook and perhaps therefore sees a necessity of making rather fixed planning in order to cover all topics in time.

Excerpt of take 3 lesson (22.30) Topic: Statistics	
S: How do we make a cumulated curve? (L is asked by two students). T: We should have had a cumulated curve on the board from the outset, right. Do you remember what it is, cumulated – what does it mean? S: It is when you add them together. T: Yes. So a cumulated curve [Oh, no], there one might imagine you had to add something	E: Hvordan laver man en sumkurve? (L spørges af to elever). L: Vi skulle have haft en sumkurve på tavlen lige fra starten, hvad. Kan I huske hvad det der, summeret – hvad betyder det? E: Det er det der med at lægge dem sammen. L: Ja. Så en sumkurve [Åh, nej], der kunne man måske forestille sig, man skulle lægge et

together. S: Is it so? [Yes] (One student seems defeatist and yawns). T: We will take just a new sheet and then you will make such a graph drawing. S: How big should it be? (L helps with great patience one of the girls to reflect on the y-axis division)... T: Well, and you saw what cumulated meant? [No]. You were then (pointing, the girl laughs)!	eller andet sammen. E: Skal man dét? [Ja] (Den ene elev virker opgivende og gaber). L: Vi tager lige et ark mere, og så laver I sådan en graftegning. E: Hvor stor skal den være? (L hjælper med stor tålmodighed en af pigerne med at overveje y-aksens inddeling)... L: Nå, og du var helt med på, hvad summeret det betød? [Nej] Det var du da (peger, pigen ler)!

The complete lesson is devoted to seatwork in groups of two or three, and the teacher distributes the assignment on a sheet of paper. During the lesson many students are in need of teacher help because of one concept: a cumulated curve. The teacher makes an effort to elicitate student understanding of that, but this is not easily done. Some of the students are not very motivated or simply seem tired, and some lack the necessary prerequisites.

T35: Discussion of results

Seatwork becomes ever more dominating over the three lessons, but the teacher's elicitation has also increased considerately. The take 2 and take 3 lessons are both on topics previously covered by mathematics teaching to this class. The teacher states that the lessons are for repetition, and this may explain her emphasis on seatwork. The assignments however do not seem to be routine exercises for the students, since many end up needing teacher's help – simultaneously.

The extensive use of seatwork and no – or almost no – whole class communication with teacher's mathematical points, does not seem to give academic benefit to these students. The organization may have been chosen because of the expressed previous trouble with handling discipline in class.

T50: Male teacher, seniority 10-15 years: Large school in Middle Region

CODES (in %) of lesson length (Topic: ICT competence)

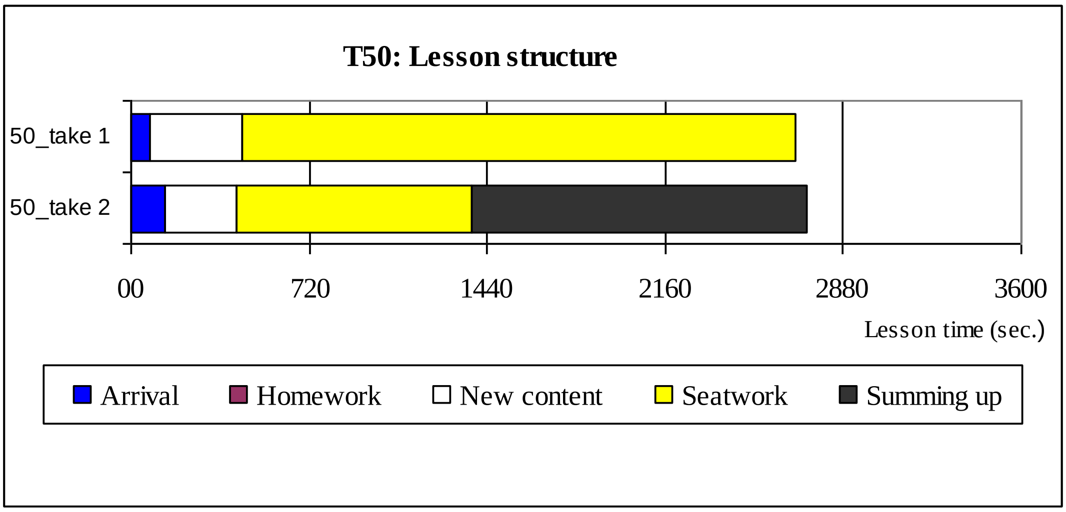
TAKE 1	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class	0.7	8.8			
Teacher → Student		5.7			
Student → Teacher					

CODES (in %) of lesson length (Topic: Perspective drawing)

TAKE 2	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class				3.6	14.3
Teacher → Student					
Student → Teacher		13.5		1.2	

T50: Lesson structure

Teacher 50	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
Take 1	1:16 0:00-1.16		6:15 1:16-7:33	37:16 7:33-44:49		44:49
Take 2	2:15 0:00-2:15		4:55 2:15-7:10	15:51 7:10-10:35, 14:20-18:09, 26:00-34:37	22:32 10:35-14:20, 18:09-26:00, 34:37-45:33	45:33



T50: Interview of teacher immediately before the take 2 lesson	
I: Should they learn something new today, or is it [Yes] more a response to something? L: No, it's more – we've been through perspective drawing before – but they're not very good at it – that is to find these vanishing points. We must have a talk about some rules for that, and then we have, if we have the time,	I: Skal de lære noget nyt i dag, eller er det [Ja] mere en opfølgning på noget? L: Nej, det er mere – vi har været igennem perspektivtegning før – men det, de ikke er ret gode til, det er at finde de her forsvindings-punkter. Vi skal have snakket noget omkring nogle regler for det, og så skal vi have, hvis vi når

then we start this placement of lampposts with the correct distance apart, so they also are using the diagonal for a checkup. I: It sounds as though it might well be a lesson with many "points". L: Well, that will be *one* point! And the point is the vanishing point and that it lies on the horizon line. I: But will it become such a teacher-led lesson of conversation and drawing on the blackboard, or? L: It's going to be that they are given a task, then they go and begin to solve the first task, and then I have – I have all the pages scanned. And then I take the drawing, they have been working with and then we shall have someone to come up and solve it. And then we proceed with the next.	det, så skal vi begynde det her med at afsætte lygtepæle i den rigtige afstand til hinanden, så de også bruger diagonalen til et tjek. I: Det lyder som om det godt kunne blive en time med mange ”pointer”. L: Jamen, det bliver én pointe! Og pointen det er forsvindingspunktet og at den ligger på horisontlinjen. I: Men kommer det til at blive sådan en lærerstyret time med samtale og tegning på tavlen, eller? L: Det kommer til at blive hvor de får en opgave, altså de tager og begynder at løse den første opgave, og så har jeg – jeg har alle siderne skannet ind. Og så tager jeg den tegning frem, de nu har siddet og arbejdet med og så skal vi have nogen op og løse den. Og så går vi videre med den næste.

Prompted for the lesson points the teacher explains the core issue to be the construction of a vanishing point and its position on the horizontal line. The planning seems guided by terms of activity and time schedule.

Excerpt of take 2 lesson: (36:15) Topic: Perspective drawing	
T: M, the next house (now switches page on the Smart Board, and M starts). S: Should I just draw a quadrangle? ... (The student draws the front and side of a house like a box). T: What is the line? (L indicates upper edge of the window at the side). S: It must fit with that (E identifies top edge of the house side). T: This one line, what should I do with it? S: It must be parallel with the other … (the lines are drawn correctly to the same vanishing point). T: But what makes the difference? (in relation to a house where the vanishing point has now been moved further to the right). S: That's the angle you view it from. T: What's the difference if we have the vanishing point far away or close?	L: M, det næste hus (skifter nu side på Smartboardet, og M går i gang). E: Skal jeg bare tegne en firkant?… (Eleven tegner husets facade og side som en kasse). L: Hvad skal den der linje? (L peger på vinduets øverste kant i sidefladen). E: Den skal passe med den der (E peger på facadens øverste kant). L: Den her linje, hvad skal jeg med den? E: Den skal være parallel med den anden … (linjerne tegnes korrekt til samme forsvindingspunkt). L: Men hvad gør forskellen? (i forhold til et hus, hvor forsvindingspunktet nu er flyttet længere mod højre). E: Det er hvilken vinkel du ser det fra.. L: Hvad er forskellen om vi har forsvindingspunktet langt væk eller tæt på? E: Jo længere ude forsvindingspunktet er, jo

S: The further away the vanishing point is, the more you see of the house.	mere kan man se af huset.

This excerpt is coded as a *student procedural point*. The most distinct change from take 1 to take 2 is the amount of elicitation. The teacher makes an effort to involve students in presentation and reasoning about their ideas. Most of the teaching in the take 2 lesson takes place in front of the Smartboard, where students and the teacher in turn are drawing lines on scanned pictures from the textbook to find vanishing points.

Discussion of results

Correspondence by mail or letters *before* the take 2 lesson was sparse, as this teacher never responded to my mail. The appointment for the take 2 was confirmed though. The teacher however did not react on my mail and letters *after* the take 2 lesson either, and therefore did not take part in a take 3 session.

The change between take 1 and 2 shows the same *increase* in teacher's elicitation as the other teachers in the focus group.

10.6 Findings

Focus group: Lesson structure in 1st take

School	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
2	2:00 0:00-2:00	11:32 2:00-2:32, 12:18-23:18	17:05 2:32-12:18 , 23:18-25:07, 28:35-34:05	15:25 25:07-28:35, 34:05-46:02	0:18 46:02-46:20	46:20
12	0:40 0:00-0:40			43:30 0:40-44:10	1:18 44:10-45:18	45:18
13	0:33 0:00-0:33	0:30 9:30-10:00	8:57 0:33-9:30	32:55 10:00-42:55	0:51 42:55-43:46	43:46
20	3:03 0:00-3:03		17:52 3:03-20:55	25:23 20:55-46:18	0:35 46:18-46:53	46:53

21	2:10 0:00-2:10	5:50 2:10-8:00	19:50 8:00-27:50	17:38 27:50-45:28		45:28
35	4:00 0:00-4:00	5:00 4:00-6:00, 22:00-25:00	16:00 6:00-22:00	24.21 25:00-49:21		49:21
50	1:16 0:00-1.16		6:15 1:16-7:33	37:16 7:33-44:49		44:49

Focus group: Lesson structure in 2nd take

School	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
2	1:12 0:00-1:12	15:48 1:12-17:00	17:33 17:00-23:38, 28:56-34:50, 39:24-44:25	9:52 23:38-28:56, 34:50-39:24		44:25
12	0:25 0:00-0:25		3:08 0:25-3:33	40:25 3:33-43:58	3:13 43:58-47:11	47:11
13	1:03 0:00-1:03		4:02 1:03-5:05	14:28 5:05-8:56, 16:38-27:15	28:00 8:56-16:38, 27:15-47:33	47:33
20	0:35 0:00-0:35		4:35 0:35-5:10	34:45 5:10-14:54, 16:29-29:57, 31:43-43:16	5:08 14:54-16:29, 29:57-31:43, 43:16-45:03	45:03
21	1:42 0:00-1:42	14:44 1:42-4:02, 5:46-11:44, 38:34-45:00	13:33 4:02-5:46, 11:44-23:33	15:01 23:33-38:34		45:00
35			7:17 2:54-10:11	30:47 13:28-44:15	6:44 0:00-2:54, 10.11-13:28, 44:15-44:48	44:48
50	2:15 0:00-2:15		4:55 2:15-7:10	15:51 7:10-10:35, 14:20-18:09, 26:00-34:37	22:32 10:35-14:20, 18:09-26:00, 34:37-45:33	45:33

The common summing up clearly has increased in length. For T13, T20, T35 and T50 this is especially different from the first recording. Also a correction of homework was not done by T35.

In total a major shift in weight of phases is seen by T13, T20, T35 and T50. T12 this time have short presentations, T21 has more emphasis on a common correction of homework. Almost no changes are noted for T2.

As the 3rd take was made in April, i.e. one month before the end of the grade 9 school year and final exam, repetition was dominant in planning. This characterized the teachers' plans.

Focus group: Lesson structure in 3nd take

School	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
2	1:00 0:00-1:00	2:32 3:51-6:23	18:38 1:00-3:51, 6:23-22:10	15:06 22:10-28:28, 35:30-44:18	7:02 28:28-35:30	44:18
12			7:53 0:00-7:53	31:59 7:53-39:52	1:14 39:52.41:06	41:06
13	1:25 0:00-1:25		9:25 1:25-10:50	14:43 10:50-22:29, 27:26-30:30	18:39 22:29-27:26, 30:30-44:12	44:12
20	0:14 0:00-0:14		18:44 0:14-18:58	26:03 18:58-45:01	0:37 45:01-45:38	45:38
21						
35	3:10 0:00-3:10			43:46 3:10-46:56		46:56
50						

A common characteristic of the focus the teachers described above (take 2 and take 3) is the *increase* in the elicitation described by percentage of lesson time. There are large variations in the presence of different points distributed across concepts, procedures, results or interpretations, but this seems to be without a uniform pattern.

The number of point nodes (with a maximum of 3 x 4 = 12 nodes), the number of point and elicitation references may also be a measure of point-driven mathematics teaching. The table below shows a rather unexpected development quite uniformly: The number of point nodes is *decreasing* in the lessons of all teachers!

Teacher	Point nodes	Point references	Elicitation references
T2_take1	6	12	1
T2_take2	5	17	7
T2_take3	3	4	6
T12_take1	4	4	0
T12_take2	1	2	4
T12_take3	1	1	9
T13_take1	5	10	2
T13_take2	2	3	7
T13_take3	3	7	7
T20_take1	7	13	5
T20_take2	3	5	5
T20_take3	4	5	7
T21_take1	3	6	2
T21_take2	2	3	4
T35_take1	6	12	2
T35_take2	2	3	3
T35_take3	1	2	4
T50_take1	3	5	0
T50_take2	2	4	3

This does not necessarily indicate that lessons in take 2 and take 3 were without points. None of the lessons were. But fewer types of points may indicate mathematics teaching with a focus only on either concepts, procedures, results or interpretations – or fewer of these.

A search for documented change reveals that points are shifted from being articulated by the teacher mainly to points being "elicited" from students. The table below cumulates the four types of points made by focus teachers in common class teaching and the point made by students.
The numbers shown are in % of lesson length:

Teacher → class points	**Take 1**	**Take 2**	**Take 3**
T2	12.1	53.6	2.5
T12			
T13	17.4	1.3	5.5
T20	28.5	3.0	15.5
T21	19.2	2.5	

T35	20.4	3.4	
T50	8.8	3.6	
Average	15.2	4.3	4.7

Student points	**Take 1**	**Take 2**	**Take 3**
T2	3.6	20.5	13.0
T12	16.8	10.1	3.2
T13		47.5	24.9
T20	11.4	12.9	2.6
T21		9.3	
T35	2.8	1.9	
T50		14.7	
Average	4.9	21.7	8.7

No general conclusion should be drawn based on such numbers. But some variations are interesting. Only with some teachers is there an increase in student points from take 1 to take 2 (T2, T13, T20, T21 and T50). But the average number of student points is considerably higher in take 2 than in take 1.

There are large variations in the amount of sparring reacted upon by the seven focus teachers. It seems reasonable to conclude, that such peer coaching (by me) at least supports some of the teachers in their use of points – though on a rather modest scale. The agenda to some extent also formats the *type* of points. When a topic containing new concepts is introduced, conceptual points may be expected to dominate. Later in a lesson sequence on the same topic the student assignments may invite procedures, results and interpretations to be the typical points. And since lessons are randomly chosen such patterns may influence the sparse statistics. In fact many take 3 lessons were on reviews because teachers wanted to prepare grade 9 for the examination in a month’s time.

Whether the effect of peer support is measurable or not when referring to the various types of data above, a remaining "problem" is that my intervention is of course not that of a colleague.
At least one of the focus teachers (T13) expressed doubt and frustration concerning colleagues' lack of willingness (or ability) to contribute to a peer support system, including extended sharing of good ideas and sparring with each other. And there is a difference between individual teachers doing it – and the situation (like Japanese lesson study) where it is part of professional mathematics’ teacher culture to work together to strengthen the organization.

The conclusion is that most focus teachers indeed were adapting some of the ideas suggested in peer coaching, and expressed more awareness of the importance of teaching leading towards mathematical points. This is documented by both interviews before take 2 and before take 3.
The effect of the peer coaching is not visible in the point coding statistics. Moreover the durability is not documented either. This may be due to variations in teaching agendas. The sample of five teachers for take 3 is a small one. And as April is only one month before the final examination in grade 9, this affects the teaching agendas as explained by some of the focus teachers.

Even considering the small sample the almost uniform increase in teacher elicitation (table below) seems unique and distinct. Along with statements from the interview sessions, I suggest this indicates a clearly more conscious approach to classroom management by the teachers.

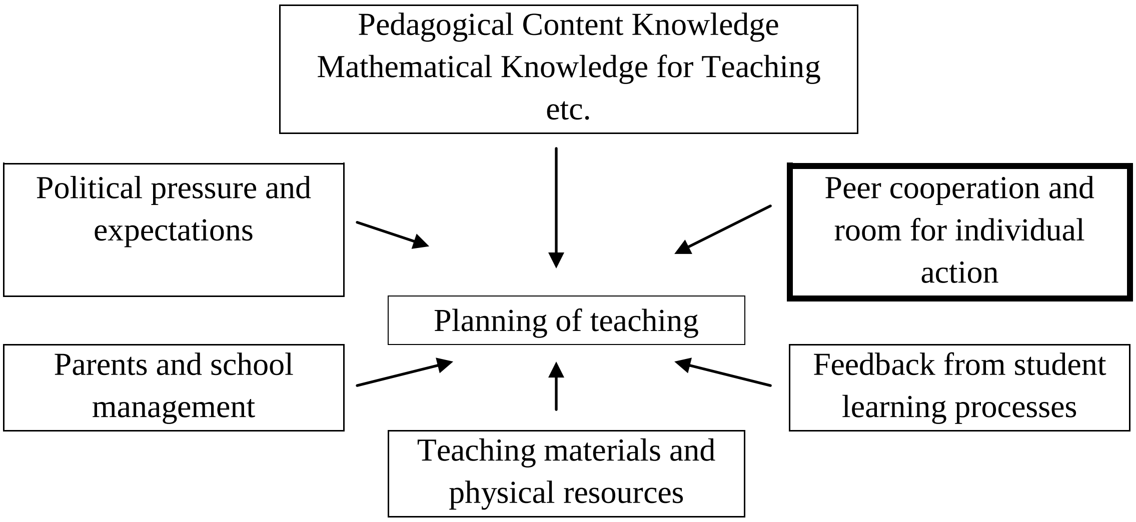
Teacher elicitation	**Take 1**	**Take 2**	**Take 3**
T2	1.6	14.9	22.5
T12		19.2	35.2
T13	7.7	16.2	30.5
T20	26.3	13.5	31.4
T21	6.9	16.1	
T35	8.1	11.0	38.3
T50		14.3	
Average	7.2	15.0	31.6

On these grounds the possibility of peer coaching by or among mathematically proficient colleagues may be considered.

11 Changing Mathematics teaching

In the present research it is already documented, that some teachers hold rather stable beliefs about the necessity of certain roles for mathematics teachers and their students.

Among the framing conditions normally referred to by lower secondary mathematics teachers asked about their reasons for action are these conditions on a macro level:



The right hand column indicates some of the competences of individual subject teachers advocated in section 1.1. Especially the *Cooperation* competence, i.e. to collaborate with colleagues and others about teaching and framework may be promising for further development. Team cooperation among mathematics teachers on choice of content and preparation of teaching may provide support for a clearer point driven or point oriented teaching than seen in the present research.

Tradition or culture among mathematics teachers may promote or hinder peer exchange of information or ideas and critique. There may be a local regularity for meeting and discussing, or such routines may be absent. The school management has a key role in such processes and may be aware or unaware of that when affording meeting time.

How could research be conducted so as to uncover the possibility for change among mathematics teachers and in everyday school culture as described above (section 9.4)? High(er) quality in mathematics teaching may depend solely on personal awareness and conviction. When teachers are convinced of a positive correlation between points as drivers and / or goals for mathematics teaching and a possible improvement in quality of teaching and learning a first step is the actual planning of lessons.

In the present research it is reasonable to investigate or construct possible explanations of the Danish situation. I do not doubt the logic in teachers actual planning and teaching. Although teachers often expressed dissatisfaction with the conditions to their teaching, I never witnessed teachers expressing doubts on what to do in their actual planning.
Maybe the idea of points is an idea far away from teachers' understanding of the important factors that affect mathematics teaching. Teachers' beliefs and knowledge may, therefore, be part of an answer to such questions.

In this project context, it will be valuable to test a peer organization in respect to point driven or led teaching, involving all (or at least many of) a school's mathematics teachers. But you don't "just do it". It requires a significant investment to "buy" working teachers' time for course participation – or simply to organize their preparation in other ways. Teachers *and* school management must be attracted by the idea.
Schools have different contracts and options here. There are various local arrangements covering teachers' preparation time. In a Danish school culture this may vary from a taxi driver mentality (where every minute is to be paid) to a professional curiosity, at some schools handled so that both parties (school and teacher) contribute (equally) to the time spent in the agreed development of teaching. And in many schools mathematics is "in line" for extra funding after a period of investment in student reading skills.

In the sections below (sections 11.2 and 12) I will describe one option, which is possibly the most obvious for increasing the preparation of clear points and perhaps also a way of spawning more focused class discussions during mathematics teaching.

11.1 Mathematics teacher teamwork in Denmark

It was a recommendation in the Danish report: Future Mathematics (Fremtidens matematik, 2006, p. 20), to seek the collaboration between mathematics teachers within schools:

Recommendation Ic: ***The professional Mathematics Teacher identity and competence should be strengthened, especially through significantly increased in-service training and through the creation of mathematics teacher teams*** ... It is recommended that each school establish a team of mathematics teachers to strengthen and develop their subject	***Anbefaling Ic:*** ***Matematiklærernes professionelle identitet og kompetence bør styrkes, frem for alt gennem væsentligt øget efter- og videreuddannelse og gennem oprettelse af matematiklærerteam*** … Det anbefales at der på hver skole etableres et team af matematiklærere for at styrke og udvikle disses fagprofessionelle identitet, og

professional identity, and create professional and financial opportunities for their work ... The rationale for creating a team of mathematics teachers in each school is to create a framework for necessary and desirable joint activities, discussions and decision making regarding mathematics teaching with an intention to promote mathematics teachers' professional identity, updating and upgrading. This will allow for the utilization of research insights and development experience in teaching practice, improvements in the selection and production of teaching materials, the initiation and implementation of peer cooperation, and planned participation in continuing education and training activities, etc.	at der skabes tjenstlige og finansielle muligheder for deres virke ... Begrundelsen for at oprette et team af matematiklærere på hver skole er at skabe en ramme for nødvendige og ønskelige fælles aktiviteter, diskussioner, og beslutningstagen vedrørende matematikundervisningen med henblik på at fremme matematiklærernes professionelle identitetsfølelse, opdatering og opgradering. Dette vil muliggøre at forskningsindsigter og udviklingserfaringer på konkret måde kan nyttiggøres i undervisningen, at udvælgelsen og frembringelsen af undervisningsmaterialer styrkes, at kollegiale samarbejdsprojekter kan iværksættes og gennemføres, og at deltagelse i efter- og videreuddannelsesaktiviteter kan planlægges, m.v.

Such interventions depend among other things on the training and appointment of peer mathematics consultants in schools. There is now a Danish Pedagogical Diploma for peer mathematics consultants, but still rather few teachers are engaged in this. I suggest this effort to establish a team of mathematics teachers to be strengthened for a number of reasons (Mogensen, 2008):

- Mathematics teacher teams are likely to match expectations for joint decisions and processes.
- Mathematics teacher teams make it easier to coordinate requirements for materials, space and other resources.
- Mathematics teacher teams help to discover and take advantage of each other's special qualities in order to better utilize special mathematics competencies in the teacher group.
- Mathematics teacher teams allow for especially highly-qualified teachers in mathematics to support subject colleagues in formulating detailed learning- and competence-related goals for teaching.
- Mathematics teacher teams offer the opportunity to support each other in teaching quality and in formulating development needs.
- Mathematics teacher teams can give a peer boost in mathematics teaching.

The KOM report (Niss & Jensen, 2002) specifies "collaborative competence" and professional “development competence" to be acquired and developed even after graduation (section 1.1).

And the Danish report on Future Mathematics states "*... the main mathematical goals, the goals in lessons and the didactical points, the teacher focuses on, should be driving the planning*" (Niss et al., 2006, p. 28, own translation). This is also characteristic of the Japanese mathematics teaching analyzed in the TIMSS Video Studies and later described by Stigler and Hiebert (Stigler & Hiebert, 1999).
The suggestion to combine collaborative efforts in a focus on mathematics didactic points in lesson planning makes it obvious to study more deeply the Japanese "lesson study" tradition.

11.2 Lesson study

My focus group coaching described in chapter 10 may be put in perspective by a general strategy called "lesson study" by the Japanese. I briefly mentioned the special Japanese tradition of peer sparring through lesson study in section 3.6.7. The tradition is described by several researchers (Isoda, Stephens, Ohara, & Miyakawa, 2007; Fernandez & Yoshida, 2004).
Lesson study is a professional development process in which Japanese teachers systematically examine their own practices. The goal is to improve teaching and learning, and the core of the lesson study is a group of teachers collaborating on a small number of study sessions.
This is systematic peer didactic support. According to Leung underlying cultural values will make it difficult to import East Asian traditions without change (Leung, 2008). Apart from money, what does it then demand in effort, organization and readiness to bring ideas from such an approach to Denmark on a large scale?

Work on a study lesson takes place in several phases:
1. Investigation and preparation, where the teachers together develop a detailed plan for the study lesson.
2. Implementation, where one teacher teaches a class in the study lesson, while other group members observe.
3. Reflection and improvement, where the group meets to discuss their observations from the lesson.
4. Repeated implementation and reflection, where a second teacher teaches a second class in a study lesson, while group members observe. Then the group will meet again and discuss their observations.

Teachers choose an overarching goal that can guide their work in all lesson study classes. Usually a school operates with the same overall goal and the same content area for several years. "*This prolonged focus is meant to provide enough time for the school to make significant progress in moving closer to attaining its chosen goal*" (Fernandez & Yoshida, 2004, p. 13). Such goals could even be suggested from educational policymakers. Every year the study lesson's objectives are refined as the group's understanding of this goal is developing in the lesson-study process. Overall lesson-study objectives could be:

- To teach students to become independent problem solvers (Stigler & Hiebert, 1999, p. 112; Lewis, 2002, p. 4).
- To encourage students to learn from each other (Fernandez & Yoshida, 2004, p. 24).
- To integrate assessment in teaching mathematics (Miyauchi, 2010).

For each study lesson, teachers can select lesson-specific goals that support the overall goal. One area can then get special attention. Examples of milestones in the lesson-study in mathematics:

- Content specific goal: "Methods to find the area of a triangle."
- The content-specific goals associated with the overall lesson-study objectives: "Students must independently discover how to find the area of a triangle."
- Identification of content-specific areas to focus on:
  "To examine how specific materials can support students' own search for a formula to triangle area."

In this way lesson study supports the idea of mathematical points. Each lesson is designed around one single goal to achieve a single goal in a topic. A (to Danish teachers very) detailed lesson plan supports the lesson-study process for all colleagues in the group in several ways: It is a *teaching tool* because it is the screenplay for the lesson's activities. It is also a *communication tool* because it tells others about the thinking of those teachers who have planned the lesson. Finally it is an *observational tool* because it will identify the points that should be seen after the lesson and a place for observers to record and share what was seen.

The mathematical point becomes clear to observers when Japanese mathematics teachers lead extensive class discussions. As the goal is to develop students' understanding of mathematical concepts and skills, the teacher is expected to facilitate mathematical discussion for students. This discussion is often called *neriage* in Japanese, which implies polishing ideas. To do this, teachers need a plan for this discussion as a part of their lesson preparation, which will anticipate the variety of statements or methods their students may present.
Another feature is the rounding off of lessons identifying the important concepts, procedures or other findings: *Matome* may be defined as "an event in which the teacher talks to the whole class to highlight *and summarize the main point of the lesson. What students engaged and discussed in the lesson is reviewed briefly in the whole-class setting and what they learned during the lesson is highlighted and summarized by the teacher*" (Shimizu, 2004, p. 4). Examples could be definition of a linear function, a method to solve one linear equation or the method of substitution by solving equation systems. This is often done by effective use of blackboard, where the lesson's topic is stated with questions and answers that are given along the way. The teacher ensures, for example, that students take note of an important concept or a good technique.

Study lessons are generally planned by groups of 4-6 teachers who primarily teach in the same (class) levels. Each group will usually perform 2-3 annual lesson-study courses, located around important school events such as festivals and competitions. Groups that work with a study lesson have a weekly meeting time, usually after school. This provides time for teachers to share their work across groups. Besides the teachers who have worked with study lesson, other teachers at the school try to come and observe and to discuss the study lessons.

Lesson study groups also share experiences and exchange ideas in other ways:

- *Reports* are often published and sold in bookstores. In Japan teachers are said to publish more than researchers (Fernandez & Yoshida, 2004, p. 212). I do not know if they are read though. A report is not just a collection of educational plans and materials. It is a reflective product that contains a discussion of motives, goals, dividends and challenges behind each lesson-study process. Support from an outside *consultant*, "invited evaluator" or "second sparring party". This person is usually an expert or researcher who has regularly been invited to advise the group. The invited consultant fulfills three purposes:
    - To provide a different perspective in responding to the group lesson-study work
    - To retrieve information about professional content, new ideas or subject objectives and
    - To share work with other lesson-study groups.
- Lesson study *Open House* allows a school to share lesson-study work with other schools, although not all schools in Japan keep an open house. The main activity in the open house is the teaching of study sessions for the invited guests (usually teachers and principals), and discussion of these lessons with them. Lesson plans are distributed to the guests along with a pamphlet describing the school and the lesson-study job you are doing. The external consultants also participate in events.
- In Japan, the rotation of teachers, both between schools and within the individual school gives teachers a good opportunity to exchange their ideas. Teachers change schools regularly, because normally, a Japanese teacher stays at the same school for no more than 10 years. Teachers also regularly change grades within the same school.
- Within each school sharing is facilitated because the teacher's desks are located in the same room. And while the students' school day ends at 3 pm., teachers stay until 5 pm. This allows for many common learning activities, lesson-study included.
- The national curriculum supports lesson-study interviews across schools. This is quite obviously also a possibility in Denmark, as the curriculum here is statewide and sufficiently detailed to support lesson study with not too detailed content-specific goals.

E.g. this goal for the Japanese lower secondary grade 2, i.e. year 8 (Mathematics program in Japan. 2000, p. 23):

For the students
- to increase their competence in calculating and transforming algebraic expressions depending on the purpose
- to learn more simultaneous linear equations with two variables, and to motivate students' ability to use such kinds of equations.

A comparable Danish quote may be this for grade 9 (Fælles Mål, 2009, p. 11):

Teaching should lead toward students acquiring knowledge and skills that enable them
- to solve equations, simple systems of equations and by inspection solve simple inequalities
- to determine solutions to equations and systems of equations graphically.

Lesson study has many forms. At each school, it may well include all teachers and be organized as content oriented study groups. Across schools, there may be a regional organization, voluntarily organized clubs and study circles, organized groups of teachers' professional associations and educational institutions or lesson-study may be part of mandatory courses for newly educated teachers. Student teachers for lower secondary in Japan get less work experience (app. 4 weeks) than Danish student teachers (app. 24 weeks), but more mandatory continuing education, including lesson-study. "*In Japan, experience with lesson study begins during a teacher's pre-service training. Practice teaching in Japan is a brief but intense time during which a research lesson is planned with many of the features found in the lesson study done with experienced teachers*" (U.S.-Japan Joint Seminar, 2002). In contrast practice teaching in Denmark takes place over a longer period of time, and seldom includes anything resembling lesson study with experienced teachers.

The vast majority of (elementary) schools for grade 1-5 and many (middle) schools for grade 6-9 (but very few upper secondary schools) in Japan conduct formal lesson-studies. But the lesson-study "mentality" is very common and often leads to informal lesson-study activities (planning, teaching, observation and reflection) of practice. It is common for Japanese teachers between lessons to follow a colleague in another class; they sit in the background and pretend they are a student.

Even though it may be difficult or impossible to import an East Asian strategy unchanged to a Western school culture, there may very well be traits to consider for Danish inspiration:

- Lesson study seems to exploit the best of the existing school culture such as teachers' professional curiosity and ambition. Most teachers could thrive in such a peer community.
- Lesson study is simply organized peer guidance within existing frameworks.
- Lesson study seems an excellent opportunity for school leadership with approval and finance to support and lead effective change.
- Lesson study activities may be arranged with short notice.
- Lesson study seems inexpensive. No teachers have to leave the school for costly courses.
  As a strategy for improvement compared with other forms of in-service courses it is very inexpensive.
- But it will probably depend on the presence or availability of a mathematically proficient peer, who enjoys such trust and respect from colleagues, to initiate and guide the process.

Which features from the Japanese lesson study tradition could work in a Danish setting? To investigate this, I decided to take advantage of a very friendly invitation from Japan.

11.2.1 Lesson Study in Nagasaki

I did studies at the Nagasaki University, Japan for a few weeks in April 2008 witnessing and taking part in study lessons at schools. I was of course totally dependent on my extremely helpful Japanese colleague, who simultaneously translated almost everything.
At this university teacher students may study videotaped lessons from elementary schools. The recordings I saw, were all taken with cameras fixed on a tripod and with the possibility to zoom in on the mathematics teacher, the blackboard or to turn and focus on a sheet of paper (handout) on some students’ desk.
With a Japanese colleague I discussed the correspondence in Japanese teacher education between teaching practice of one month during year 3 and the lesson study as it is performed in schools. In teaching practice the regular teacher observes the lessons of every individual teacher student, but before this preparation is made. The teacher student presents a teaching plan to the regular teacher, who will comment and perhaps suggest changes or supplementary ideas. Then the lesson is “played” with fellow students acting as pupils (Mo gi Jyguo = Trial lesson) before the class is taught. This is close to the lesson study practice, as practiced by professional teachers.
The schools attached to the university hold an annual conference inviting teachers from the prefecture for a presentation of several lesson studies (grade 2-6) and after that some discussion.

In Nagasaki I observed lessons in grade 2 and 4 at the Elementary school attached to Nagasaki University (by Mr. Okubo and Ms Takeyama) and in grade 7 and 8 at a

Junior high school (by Mr. Yamashito and Mr. Yamamoto), The two Elementary school teachers later met for a discussion with myself and two university colleagues. Some actions and priorities seem significantly different from Danish teaching practice. I therefore insert some notes from my observations:

Grade 2

- 35 students and some disturbance.
  7 more teachers present, with notepads – some also with cameras.
- The teacher started the lesson in a funny way by inviting one student to have a secret look in the teacher's folder without telling anybody else. This clearly aroused curiosity among students.
- Every student was then shown and excited by the 35 pictures now shown – and the individual birthday information they were about to be given to investigate.
- The teacher unfolds a long paper strip explaining the goal for the lesson. It is framed in red and put in the center of the blackboard: *Let's investigate the birthdays.*
- In the quite heated suggestion round, the teacher reminds some students by pointing to 3 sentences written on paper in the left side of the blackboard (below the loudspeaker): They say: YES, WHY, BECAUSE …
- Smaller pieces of yellow paper put on the blackboard say: 5 questions will come: How many have birthdays in these different months?
- Rules: The teacher will hide the cards in a while
  You have 5 minutes to make a memo!
  Write date and title of lesson first in your notebook!

  Quite a few students offered an explanation afterwards, and were heard.
- The teacher instructed: *Now write down the reason, if your answer is wrong.*

Grade 4

- 34 students and 11 other teachers (3 women and 8 men) present with notebooks and a few cameras.
- A 3-page lesson plan was distributed to everyone.
- The teacher was talking to the very eager class, and she finally unfolded a line to be put on top of the blackboard as the goal of today: We are going to read very large numbers. The students enthusiastically made suggestions.
- All students were asked for their suggestion, which the teacher seemed to divide into 3 types, each assigned with a number (1-3) and some space on the blackboard. After this everyone was asked to place his/her magnetic nametag at one suggestion out of the 3.
- In turns a few students gave their explanation at the blackboard. Finally all numbers were read aloud in a (very convincing) chorus.

Grade 7

- 42 students, 22 girls and 20 boys sitting in pairs of 2: girl-boy.
- In the class the teacher was already busy writing the homework assignments on the blackboard *before* the lesson start:

  1. (+2) + (+7) = (+4.8) + (-5.2) =
  2. (-2) + (-4) = Etc.

- All boys were asked to go to the blackboard at the same time, and each was asked to write his answer for one of the homework assignments. Then everyone waited for a long time (for the teacher and everyone else to check with their own results). Some answers were written in red or green, I never found out why.
- The teacher asked. What did you learn yesterday? In turns three students were asked – each standing while giving an explanation. Everyone in class turns to the student, who must address the class. After an explanation the whole class will say: We agree (or disagree). According to my colleague, they learned the meaning of (+4) + (-3) as “Toru went 4 east and then 3 west.”
- The teacher wrote the goal of the lesson on top of the blackboard: *Problem posing*.
- Four examples from homework are circled in red, e.g.:

  (+2) + (+7) and (+3.5) + (-2.3).

  The teacher then asked: What do you notice? Again a student is asked to give an answer.
- The task: “*Make 5 problems of each kind in the notebook of your partner*” was given.
- The teacher now walked around, observing and chatting along the way. He then prepared the blackboard for answers like this:

	(+) + (+)	(-) + (+)	(+) + (-)	(-) + (-)
1				
2				
Etc.				

- The teacher instructed: You must be able to solve the problems yourselves! In less than 1 minute! The notebooks were switched back and everyone was busy solving the problems.
- In presentation to the class, the teacher now posed the questions: For the author: *Why did you make this problem?*
  For the solver: *How did you solve it?*

Grade 8

- The attention. At the start and the end of a lesson everyone rises. In grade 8 one student says the salute and everyone bows to the teacher. I was also greeted this way.
- Students who are asked, rise and put forward their suggestion or explanation.
- The mathematics teacher guides and talks a lot, but the students are very active and talking together. They seem very clever and are doing mental addition very

fast. According to my Japanese colleague, this Junior High state school is more popular than another school in the area not so closely connected to Nagasaki University.

- The mathematics teacher underlines or circles the important things in color on the blackboard.
- The teacher summarizes the lesson at the end.

Several of these observations were convincing to an observer. The large number of students in the classes all seemed engaged and mathematically challenged by their teachers, who demonstrated professional classroom management during the lessons by excellent overview and preparation for students' various reactions. The students were attentive and invited to suggest, explain or come to the blackboard to write. The lessons included common discussions (*neriage*) and summing up (*matome*) led by the mathematics teachers. In this way all the lessons seemed carefully planned from start until the end.

The lesson study approach to peer sparring and support is spreading at present through literature, networks and conferences from Asia to Europe and the USA. In Asia the APEC (Asia-Pacific Economic Cooperation) offers inspiration on websites (http://hrd.apec.org/index.php/Lesson_Study). Here lessons from several Asian countries may be studied. "*Classroom Innovations Through Lesson Study*" is an APEC Education Network (EDNET) project that aims to improve the quality of education in mathematics through the use of lesson study.
Lewis takes the tradition to a U.S. context (Lewis, 2002) and some initiatives may be followed on websites (www.lessonresearch.net). This includes research and resources such as multimedia, lesson plans, and workshop handouts. The work focuses on adaptation of lesson study to U.S. settings. Another U.S. initiative is the Chicago Lesson Study Group (http://www.lessonstudygroup.net/), which as part of a lesson study cycle hosts an annual lesson study conference with public research lessons.

11.2.2 Lesson study in Aarhus, Denmark – a pilot

During 2008-09 a group of 11 mathematics teachers at two smaller schools in Aarhus, Denmark entered a pilot lesson-study with me. This school was not among my research schools though. Initially the school management contacted me and asked for "inspiring, professional support" for the mathematics teachers. at two preparation meetings with teacher representatives we agreed on a 22 lesson course with this content for seven meetings:

1. August 12, 2008 Annual plans	AM presents on *Common Goals II* and competencies Annual planning in mathematics teams Teachers mail own preliminary annual plan to AM in 2 weeks
2. August 26, 2008 Annual plans	AM's common remarks to the received annual plans Sparring with AM in smaller groups, e.g. levels 0-3, 4-6, respectively 7-9 Each participant mails final annual plan to AM in 3 weeks
3. September 23, 2008 Student plans + "Lesson study"	AM comments on the final annual plans, possibly as a suggestion (= exercise) to annual teams on the link between annual plan and student plans. Presentations on the Japanese "lesson study" tradition for Danish consideration including the design of lesson plans.
Without AM	Choice (and distribution) of topics for the "lesson study". Each colleague may have a desire for something that is always difficult, or something where you yourself think you have really good ideas. Everyone writes at least one detailed "lesson plan", whether it is used in the common course or not. The volunteer teachers in the next modules distribute their lesson plan (max 3 pages) for all mathematics colleagues + AM one week before.
4. October 28, 2008 Lesson study in two classes	Teaching 1: Class 9.a (all attend the lesson) Teaching 2: Class 7.a (all attend the lesson) "Lunch Break" Joint meeting of all for peer coaching at 11-2 p.m.
5. January 13, 2009 Lesson study in two other classes	Teaching 3: Class: 4.a (all attend the lesson) Teaching 4: Class: 3.a (all attend the lesson) "Lunch Break" Joint meeting of all for peer coaching at. 11-2 p.m.
6. February 3, 2009 Mathematics Enterprising Day	One All Day "contests" with ample "problem solving" in groups across grade levels
7. June 9, 2009 Annual planning	Evaluation and ideas for mathematics teams 2009/10 Annual planning with coaching on levels

At the third meeting four of the teachers agreed to have everyone else watch a mathematics lesson in their classes. The preparation of everybody else then was to read these teachers' lesson plans, which everyone involved received in advance. For each of the two days with lesson study we had a common meeting for one hour in the morning. Here I, as an external consultant, guided conversation and reflection on the academic subject we later were to observe that day. For grade 7 and 9 the topic was common: *scale*. For grade 3 the topic was *probability* and for grade 4 it was *fractions*. It was not customary for these teachers to observe each other's teaching, but all were very positive and curious. An important aspect was the seriousness all displayed by being attentive and taking notes for post peer conversation. Parents and students were briefed beforehand – even that I would probably take pictures (this is normal in Japan). We were simply at work.

For example the program for the fourth meeting October 28th, 2008 was:

8:00	Participants' preparation on the study lesson topic of today: *Scale* I.e. on possible objective, content, organization and assessment at grade 7 and 9.
8:55	Mathematics in class 9.a
9:40	Work on own lesson plan (with coaching from me, copies to be distributed before January 13).
10:10	Mathematics in class 7.a
10:55	Lunch
11:30	Lesson study continued. Time devoted to comments, questions and feedback on each session.
13:30	Discussion of the perspective and possible tips for the future.

As seen in the plan above the preparation of the four study lessons was not done in collaboration. All 11 teachers planned one lesson in their own class in detail and then distributed these plans for peer inspiration. But only the four teachers who volunteered had their mathematics colleagues and me present in their study lessons and afterwards, when all teachers met to discuss what we had seen.
The performing teacher then spoke first. Not everything evolves as planned after all, and it should be possible for the teacher to present considerations and justifications. A round table was then prepared for everyone else to give constructive, positive comments on something observed. And in a second round everyone were expected to pose a question on a critical issue as "it surprised me that ..." or "so you do that ...".

We carried out these days in October 2008 and January 2009, and in a short subsequent round table evaluation (not tape recorded) they were assessed extremely valuable in both a professional and a peer, social context as the atmosphere was characterized by curiosity, but also some restraint. It was e.g. not that easy to persuade teachers to volunteer teaching the study lessons.
These four mathematics teachers were very different in their choice of teacher role. The lesson plans varied from a few lines to two full pages, but none of the four plans directly mentioned points as guiding the mathematics teaching or as intended goals. A few excerpts on the explicit goals from the distributed lesson plan demonstrate the variation and teachers’ emphasis on processes:

Class	Examples on goals from the lesson plans
9.a	By the end of the lesson, the student should using the ruler and calculator be able to: • Explain what scale is with great confidence. • Give examples on the use of scale with great confidence • Explain what 1:25.000 means …
7.a	The tasks of measurement and drawing should give students experience with scale.
4.a	That together we can put words on the fraction concept and use it in relation to everyday life ...
3.a	That students practice the first half of the small table from 1-50. That students strengthen cooperation and assessment competencies when working with the probability concept ...

From the researcher's point of view, the course affirmed that such support and coaching may be implemented and recommended in probably any school, with management's support. The school management did not take part in the actual lesson study, but arranged for sandwiches and coffee on course days. Teachers felt obliged and appreciated, but unfortunately I never thought of asking the participants for a written evaluation.

12 Intervention lesson study with 18 mathematics teachers at one school

In the autumn of 2010, I proposed and implemented a longer lesson study at a large Danish school (671 students) involving all 18 mathematics teachers. At this school one teacher (T12) was a member of the focus group described in chapter 10 (but without any special obligation or role in this course).

This intervention study made three things possible:

1. To investigate the effect of a course on points for all mathematics teachers at one school, then covering grades 1-9. The initial study on grade 8 was decided because grade 8 is a favorite research ground internationally and because mandatory schooling in Denmark is almost coming to an end with one year more to go. Researching lesson study with teachers covering all grades gives a possibility to ask for effects on teaching at primary school grade levels.
2. To investigate the effect for a teacher from the focus group in his peer setting, now teaching new classes. Will this teacher respond differently or especially positively or hesitantly to a peer effort compared with his participation in the initial research and the focus group? He did *not* present any own study lesson to his colleagues at the school.
3. To investigate especially the effect on two grade 8 teachers, not previously involved in the research but now observed and videotaped for one lesson before and one lesson after the course. These lessons are transcribed and analyzed for points and other teacher actions. Especially of course the possible change between the two takes is of interest.

12.1 Invitation and appointment

The invitation and the final program ended up as shown in appendix G. The most important difference when compared to the pilot course was that preparation was done in 3 groups, each formed by 6 teachers during several hours of group work. In turns 4 of the 6 teachers in each group were invited to (and committed to) “perform” one of the two days.
In the end, the “performing” teacher was responsible for finishing the lesson plan and mailing it in advance to everyone in the group and to me.

My role as a participating researcher and instructor varied by giving talks on:

- The overarching focus of the course, which was agreed to be mathematical competencies.
- The development of two selected mathematical topics: Algorithms for +, -, x, : and spatial geometry during grades 1-9.
- The importance of mathematics didactic points in every lesson. In this I referred to my own actual research.

In the mutual planning of study lessons in groups of six teachers, I intervened or supplemented, when asked – but not otherwise. During this I watched the teachers being highly motivated, involving each other in previously private knowledge and experience from teaching. Most teachers were quite new to each other, as the group formation was designed deliberately to assure that teachers, who had traditionally been cooperating on one class or grade, were put in different groups.
I attended all 11 study lessons (see below) and headed the discussion afterwards between the teacher and the group of observing colleagues. Each study lesson was discussed for approximately one hour. The observers took notes, some even pictures or video.

12.2 Eleven study lessons

In this lesson study mathematical competencies were chosen as a common and overarching focus. Also 1-2 mathematical topics were decided beforehand, and it was agreed by teachers to plan for didactic points in every mathematics lesson.
Initially the teachers had talks from me on the importance of points as drivers and indicators to quality mathematics teaching. A common planning of lessons was then done in 3 groups each of 6 teachers. And in turn 4 of the 6 teachers in each group "performed" during one of the two days set aside. Teachers were highly motivated, involving each other.
As no common format to the written lesson plans was decided upon as sometimes seen in lesson study, the plans became very different. Some were rather short and some more detailed, but the teacher's role was carefully described with questions and actions supposed to start and develop student thinking. Some of the teachers planned and executed a controlled ending of the lesson with a summing up of results. This was also strongly recommended by the researcher.

The study lessons took place on six different days with two lessons each day before lunch. After lunch the lessons were discussed the same day. As seen in the scheme below each group of six teachers attended 4 study lessons, an exception was group 2, who had to cancel one study lesson due to sickness.
In the first round of two lessons every observer was free to observe anything. In the second round each teacher colleague was given a specific role by the researcher as e.g.:

- How does the teacher challenge the mathematically weak students?
- How does the teacher challenge the especially fast or gifted students?
- What kind(s) of teaching aids does the teacher use to illustrate or concretize the math?

Date (2010) and group	Grade levels, chosen themes and general competencies	Example of point
September 30 Group 1	4c Similarity, aids and tools + reasoning	As side length in squares, rectangles and triangles are doubled, the area will be four times larger
	9c Similarity, aids and tools + reasoning	As side lengths in polygons are made x times larger, the areal will be x^2 times larger.
October 6 Group 2	4a Multiplication, communication + mathematical thinking	You can multiply in several ways. Which is the best?
	6b Equations, communication + mathematical thinking	We are allowed to add or subtract the same number on either side of the equality sign
October 7 Group 3	6a Negative numbers algorithm and communication	To recognize the difference between - as a sign or - as an operator
	6c Negative numbers and communication	To recognize the difference between - as a sign or - as an operator
November 8 Group 1	1c “Tenner-friends”, aids and tools + reasoning	Some numbers add up to 10 (let’s call them friends). Addition is commutative
	3b Addition algorithms, mathematical thinking + communication	Develop methods for mental and written addition based on your own understanding
November 10 Group 2	2a Subtraction algorithms and reasoning	Finding your best way to subtract
	Cancelled	
November 11 Group 3	1a Subtraction, symbol and formalism	The symbol **-** means to remove something
	8b Algebra, representation + problem tackling (this lesson was videotaped)	The commutative law applies to + and x

Some lessons had more mathematical points stated in the lesson plan than shown above. And most points fitted the definition given to the teachers in the course and in

the present research. Some of the statements indicate rather precise goals or aims for the lesson. But the points were *not* stated as lists of activities or assignments.
The lesson plans still showed large variations. Some were sparse and some very detailed indeed. Two examples from the end of the course (classes 1a and 8b) are attached as Appendix H. The grade 1 teacher here focuses on *symbolism competency* in dealing with subtraction. The grade 8 teacher focuses on *representation and problem handling competencies* (Niss & Jensen, 2002) in dealing with algebraic expressions.

By the end of the peer discussion after each study lesson, I always asked the responsible teacher to react to the questions below from a slide. These questions were also used in the stimulated recall session with a focus group from the 50 teachers in the initial survey (section 10.2):

1. Was it a "standard lesson" – or was it special? 2. What was present that you would call your teacher routine? 3. Was there anything that surprised you – or the students? 4. Was there anything, you would prefer to do differently? 5. How did you choose this structure? In the start – along the way – or at the end? 6. Examples of mathematics teacher actions, you value highly? 7. Examples of conviction, who guided your selection?	1. Blev det en ”standard-time” – eller var den speciel? 2. Hvad var der, som du vil kalde din lærerrutine? 3. Var der noget, der overraskede dig - eller eleverne? 4. Var der noget, du godt ville gøre anderledes? 5. Hvor valgte du denne struktur? I starten – undervejs – eller til sidst? 6. Eks. på matematiklærer-handlinger, du vægter højt? 7. Eksempler på overbevisning, der styrede dine valg?

Often the questions were already dealt with during the rounds by colleagues. Most teachers expressed conscious changes in their teaching routines, and all expressed very positive reactions to the peer support in the refined planning and feedback. This is documented by the questionnaire for all participants, referred to below (section 12.6).

12.3 Video recordings before and after the course

In order to register possible changes in mathematics teachers' use of points in grade 8, I made an appointment with two grade 8 teachers at this school to videotape one lesson before the course and one lesson at the end of the course.
Teachers agreed that the 2 + 2 lessons are fully transcribed and referred to in statistics below.

Before starting the lesson study course

Lessons before AND after lesson study course

Class	Arrival, unpacking	Teacher led correction of homework or return of tasks	Teacher repeats or presents new content to whole class	Seatwork or investigations individual/groups	Common summing up or messages	Total
8a before: 25/8 2010	4:25 0:00-4:25	22:02 4:25-26:27	10:33 26:27-37:00		1:36 37:00-38:36	38:36D
8a after: 10/1 2011			33:30 0:00-33:30	13:40 33:30-47:10	0:33 47:10-47:43	47:43

8b before: 24/8 2010	0:10 0:00-0:10		27:10 0:10-27:20	19:38 27:20-46:58	1:31 46:58-48:29	48:29
8b after: 11/11 2010 *)	1:39 0:00-1:39	16:01 31:09-43.15, 45:30-49:25	29:30 1:39-31:09		2:15 43:15-45.30	49:25

*) The teacher's plan for the late 8b lesson is inserted as Appendix H.

In class 8a no time is allotted for correction of homework in the lesson after the lesson study course, and the teacher now does a lot of talking.

In class 8b there also is a lot of talking from the teacher. As in the first recording he administers an extensive review of recent mathematics in a dialogue with the whole class. No time is allotted to the students' seatwork in this second recording. But 16 minutes are spent on discussing questions on algebra provoked by a recent mathematics test.

A development between lessons in the two classes is marked when it comes to length of lesson phases. But the development seems very different for the two teachers, and a closer look at the classroom communication (below) is of course mandatory before concluding anything.

When looking at the codes for *points* the following pattern emerges:

Class 8a
Male teacher, seniority 5-9 years

POINT CODES (in %) of lesson length (Topic: Polygons and lines)

TAKE 1, August 25, 2010	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class	1.8	4.2		4.4	59.4
Teacher → Student					
Student → Teacher	1.8				

POINT CODES (in %) of lesson length (Topic: Equations)

TAKE 2, January 10, 2011	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class				2.8	58.8
Teacher → Student					
Student → Teacher		1.1			

During most of the take 1 lesson the teacher directed an overview (repetition) a previously taught / learnt topic, partly through correction of a homework assignment, partly through an overview of polygons and lines. The following lesson (not recorded) was announced to be on the Pythagorean Theorem, so the recorded lesson may be regarded as a "warm-up" for that. As seen above the code for elicitation covers more than half the lesson time.

The take 2 lesson – after the lesson study course at this school – was on the solving of first degree equations like 2x – 3 = x + 11. For more than 30 minutes this is a dialogue between the teacher and a not too eager student "invited" to the blackboard. This communication is almost exclusively dependent on the teacher, therefore elicitation codes are (again) quite dominating.

The number of point code references for the two recordings are 4 and 2 respectively, so there is certainly not an increase in the frequency of points being stated.

A few examples of transcript from class 8a:

Take 1 (16:09) coded as elicitation	
T: M, try to put some words on that. S: Put into words, what do you mean? T: What *are* you doing? It looks good. S: I tried to make this slash. Slightly tilted. T: Yes, now one can see, now you have drawn the height. And what's this? (pointing to the long height) S: 6 cm. T: Then write it (T does it however himself). Now you can keep track of how far you have come. We write 6 cm. It is obviously not to scale because it is on the blackboard. But it is fine. What then? S: So I should draw the oblique line. T: Yes, you might as well do that. Try to do it. Now what? Would it be an idea to grab the second ruler again (T means the triangle). The one behind you? [Yes].	L: M, prøv at sætte lidt ord på. E: Sæt ord på, hvad mener du? L: Hvad *laver* du? Det ser fornuftigt ud. E: Jeg har prøvet at lave den der streg. Lidt skrå. L: Ja, nu kan man se, nu har du jo tegnet en højde. Og hvad er den her? (peger på den lange højde) E: 6 cm. L: Så skriv det på (L skriver det dog selv på). Så kan du selv holde lidt styr på, hvor langt du er kommet. Vi skriver 6 cm. Den er selvfølgelig ikke målfast, fordi det er tavlen, vi er på. Men det er fint. Hvad så? E: Så skal jeg tegne den der går skråt. L: Ja, det kunne du da godt gøre. Prøv at gøre det. Hvad så nu? Ville det være en fordel, at tage den anden lineal frem igen (L mener tegnetrekanten). Den, der ligger bag ved dig? [Ja].

S: Because it (inaudible) T: What do you need, as there is on that? S: Perpendicular.	E: Fordi den (uhørligt) L: Hvad skal du bruge, som der er på den der? E: Vinkelret.

The points being elicited by the teacher in this excerpt are the definition of height in parallelograms and a procedure for drawing lines showing it. Later in this lesson the teacher elicits and partly states descriptions of trapezoid, rhombus, equilateral and isosceles triangles, medians and angles and perpendicular bisectors.
The teacher has a hard time in take 1 having the student explain his solution method in front of everybody at the blackboard. The class also does not seem too confident with the assignments. It requires some effort from the teacher, mainly in elicitation suggestions from students, to keep the lesson going. The excerpt above is quite typical.

Take 2 (4:34) coded as elicitation	
S (On the equation 2x – 3 = x + 11): I guess I'll start by taking the x's on one side and numbers on the other side. T: That is where we want to go. That is the country we will travel to in a moment. S: That's where we are going. T: That's good! S: 11 - 3, it is completely wrong? T: No, but you must tell how you will do it. You said you wanted to isolate the x's on one side and the numbers on the other side [Yes] Yes. Well this is what you will, yes. So what could the first step be, F? S: But must the x's be on *that* side - or on *that* side? (pointing) T: That's *not* important [Another student: It does not matter.] It's your choice, you might say. S: Then I would say 2x (writes this) times x? (Looks inquiringly at the teacher, some laugh) T: How does x appear on the right side of the equation? S: Times. T: Where do you see the times sign? S: But if there is nothing, it's times - is it not? T: No. It stands alone, does it not? There is	E (om ligningen 2x – 3 = x + 11): Jeg tror jeg vil starte med at tage x’erne på den ene side og så tallene på den anden side. L: Det er der, vi vil henad. Det er det land, vi vil rejse til lige om lidt. E: Det er, hvor vi vil henad. L: Det er godt! E: 11 - 3, er det helt forkert? L: Næh, men du skal fortælle hvordan du vil gøre. Du sagde, du ville isolere x’erne på den ene side og tallene på den anden side [Ja] Ja. Jamen det er det, du vil, ja. Så hvad kunne de første arbejdstrin være, F? E: Men skal x’erne være på dén side - eller på dén side? (peger) L: Dét er *ikke* vigtigt [Anden elev: Det er lige meget.] Det er dit valg, kan man sige. E: Så vil jeg sige 2x (skriver dette) gange x? (ser spørgende på læreren, nogle ler) L: Hvordan optræder x ovre på højre side af lighedstegnet? E: Gange. L: Hvor ser du det gange henne? E: Men når der ikke er noget, så er det gange - er det ikke? L: Nej. Det står alene, gør det ikke det? Der

an x alone over on the right side of the equation. ...	står et x alene ovre på højre side af lighedstegnet. ...

The point elicited in this excerpt is on solution procedures to simple first degree equations with an unknown x. During the lesson the teacher guides one student at the blackboard through a very thorough but also cumbersome manipulation of parts that are added, subtracted, multiplied or divided on both sides of the equality sign.
The situation in take 2 seems quite similar to take 1. The teacher does almost all the talking, which also seems necessary. There is a clear focus on methods and reasoning prior to results as also stated in the excerpt above.

Class 8b
Male teacher, seniority 15+ years

CODES (in %) of lesson length (Topic: Primes, composite numbers and communication)

TAKE 1, August 24, 2010	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class					31.8
Teacher → Student					
Student → Teacher		3.4			

CODES (in %) of lesson length (Topic: Algebra, representation + problem tackling)

TAKE 2, Nov.11, 2010	**Conceptual**	**Procedural**	**Result**	**Interpretation**	Elicitation
Teacher → Class			3.3	4.0	58.6
Teacher → Student					
Student → Teacher	6.5	3.4	3.9		

The take 1 lesson is almost without any coded points. Here is one exception:

Take 1 (4.20) coded as student procedural point	
T: What is something like 2/3 of 24? (L asks a certain student) SS: No, I'm just thinking / 16 T: Yes, why? S: I'll just take the 8-table. T: Yes, you can use that –you must take 2/3 – yes? (writes the assignment on the blackboard) S: Divide by 2, no divide by 3 and multiply by 2. T: Was it so? S: Yes of course. T: What is the half of the quarter? S: An eighth.	L: Hvad er sådan noget som 2/3 af 24? (L peger på en elev) EE: Nej, jeg sidder lige og tænker / 16 L: Ja, hvorfor? E: Jeg tager bare 8-tabellen. L: Ja, det kan man godt – du skal tage 2/3 – ja? (skriver opgaven på tavlen) E: Dividere det med 2, nej dividere med 3 og gange med 2. L: Var det også sådan? E: Ja selvfølgelig. L: Hvad er halvdelen af en fjerdedel? E: En ottendedel.

T: Yes, what is a quarter of the third? S: A twelfth.	L: Ja, hvad er en fjerdedel af en tredjedel? E: En tolvtedel.

For a start the teacher involves students in using, e.g., tables in simple, repetitive number training by mental factorization of the number 48. This leads to elicitation of a method by halving and doubling factors, respectively, to find new solutions to that problem.

The elicitation continues to fractions of fractions and definition of primes.

Then the topic is changed to quadrangles with or without any well defined height and their position in a coordinate system, which leads to an exercise on area of such polygons. The lesson ends with students being challenged to calculate areas of various quadrangles.

The teacher does a lot of talking trying to involve students in the communication. No results seem stated as points, and the utmost challenge: to investigate connected midpoints of sides in quadrangles is left as for homework and a following lesson.

The mathematics teacher's lesson plan announces many points to look and listen for in the take 2 lesson (Appendix H):

- *Algebra is calculations with numbers and letters. Number representations.*
- *We have worked with algebra when we use different formulas.*
- *These algebraic expressions consist of parts. Such algebraic expressions are called sums.*
- *The order of parts in a sum is unimportant.*
- *Generally it is a rule that $a + b = b + a$ (the commutative law)*
- *Between coefficients and the characters there are always in such cases a · sign.*
- *The order of factors is unimportant.*
- *The commutative law also applies to multiplication. So $a\cdot b = b\cdot a$.*

These points are actually stated during the lesson. As seen in the statistics, the mathematics teacher also elicits a lot. The students are led to formulate definitions on concepts and the crucial results from the point-list above. One example of a transcript from class 8b:

Take 2 (4.44) coded as elicitation	
T: What if we write 10ab above 16a (writes it on fraction line)? S: Yes, so what? T: Can we do something about it? Yes, do you have a suggestion? S: You say that a divides in a, right? [How can you tell?] Because a is the same as a. Then a divides in a [Ok] a times.	L: Hvad så, hvis vi skriver 10ab over 16a (skriver det på brøkstreg)? E: Ja, hvad så? L: Kan vi gøre noget ved den? Ja, har du et forslag? E: Du siger, at a går op i a, ikke også? [Hvordan kan du vide det?] Fordi a er det samme som a. Så a går op i a [Ok] a gange.

T: If I write like this: 10 + a + b and then write 16 + a (under the fraction line)? S: You can just take away the a's. T: Can we take away the a's now? [Yes]. Really?	L: Hvis jeg skriver sådan her: 10 + a + b og så skriver jeg 16 + a (under brøkstregen)? E: Du kan bare tage a'erne væk. L: Kan vi tage a'erne væk nu? [Ja]. Det kunne vi godt?

The excerpt shows the teacher's way of interacting with his students asking for reasons and constantly increasing the difficulty. The teacher requires great concentration, but maintains the undivided attention of the class.
Especially in this second take, the points also seem very clear to the observer. The transcript is paralleled by the very detailed lesson plan (Appendix H).
The two teachers do not show the same development with respect to point-driven teaching. Even though both teachers seem to dominate the common classroom communication, the answers of students are more hesitant or wrong in 8a than in 8b. This is a condition relating to the respective teachers, and they are both doing their best to involve students in understanding. But the points are most clearly made in 8b.

Often elicitation is done by the teacher repeating what has just been said, but now as a question as in the last line of the excerpt above. Franke, Kazemi and Battey describe *revoicing* as a way to clarify or amplify student ideas. The teachers in question seem to do this with respect for student ideas, but also to give direction and momentum by orchestrating the classroom communication. Researchers have found that revoicing may offer "*mathematical support for ideas and the development of students' identities around the learning of mathematics"* (Franke, Kazemi & Battey, 2007, p. 234).

The high percentage of lessons above coded as teacher elicitation may indeed be a sign of teachers' attention to eliciting and maybe correcting student mathematical points in definitions, methods etc. The mathematics teacher who knows the class well may very well manage a brash, almost teasing dialogue with constantly harder questions. But there is a delicate balance between such invitations for a Socratic dialogue and a dominant teacher imposing his or her own overview.
This research is not designed to follow student outcome and must therefore be restricted to the observations given above and the evaluation of the participants.

12.4 Evaluation by school management

The school pedagogical leader wrote an invitation for a magazine editor to come and visit the school for an interview on the course results. The excerpt of her letter below tells about the role of the school management. At this school, teachers could count on backup for new initiatives like the lesson study – even if there was no special funding for the future.

November 10, 2010 Dear editors … I write because we are completing a mathematics course that we think others might be interested in hearing about. … During this autumn our mathematics teachers have developed the mathematics teaching by attending each other's mathematics lessons and then discussing and reflecting upon precisely what they have observed. Perhaps this is not quite new – what is new is that the lessons observed are carefully planned – even down to the tiny details. … There may be several pages of lesson plan behind every single lesson. We focused on getting the mathematical points presented in mathematics teaching … (and) I can already say that it has been a different and demanding form of coursework, to which the mathematics teachers have been very committed. To prepare an exemplary lesson, implement it with colleagues as observers and then afterwards to study and investigate what happened in the lesson has been exciting and challenging – the pedagogical and didactic discussion is raised to a much higher level – because when is it that you actually discuss an exemplary practice with your colleagues, normally?…	10-11-2010 Kære redaktion … Jeg skriver, fordi vi på skolen er ved at afslutte et matematikkursus, som vi tænker, at andre kunne have interesse i at høre om.…. Vores matematiklærere har i løbet af efteråret udviklet matematikundervisningen ved at overvære hinandens matematikundervisning, for derefter at diskutere og reflektere over netop den lektion, som de har overværet. Det er der måske ikke så meget nyt i – det nye er, at de lektioner, som man tager afsæt i, er nøje planlagte – ja helt ned til mindste detalje … Der kan ligge flere siders lektionsplan til grund for en enkelt lektion. Vi har fokuseret på at få de matematiske pointer frem i matematikundervisningen … (og) jeg kan allerede nu sige, at det har været en anderledes og krævende kursusform, som matematiklærerne har været meget optaget af. Det at forberede en eksemplarisk lektion, gennemføre den med kollegaer som observatører for så bagefter at studerer og undersøge, hvad var det så, der skete i den lektion, har været spændende og udfordrende – den pædagogiske og didaktiske diskussion løftes op på et meget højere plan – for hvornår er det lige, at man får diskuteret en eksemplarisk praksis med sine kollegaer ...

12.5 Evaluation by participants

At the final meeting of the course, a common evaluation was done filling out an anonymous questionnaire and performing an interview with a group consisting of one teacher from each of the three working groups.

The questionnaire had two parts, the first one included tick boxes for various degrees of agreement. The opinion of 14 teachers is indicated below as 4 teachers were absent.

	Please tick one X in each line, 1 = not good, 5 = very good	**1**	**2**	**3**	**4**	**5**
A	How was mathematical competence as the overarching goal?			3	6	5
B	How did it work focusing on only 1-2 topics?			1	5	8
C	How did it work forming groups of 6 teachers?	1	2	3	5	3
D	How did you find the composition of groups in terms of grade level, teacher education and experience??			3	8	3
E	How do you assess the Ames talk in relation to what you had to do yourselves?			2	7	5
F	What do you think of the 4 study lessons as an idea?			1	5	8

The quite uniform answers apart from question C might be read as a pronounced recommendation of such a course. The teachers are all different, but the school culture supports the initiative, and the atmosphere during the three months, when I met these teachers regularly, was enthusiastic and supportive.

The latter part of the questionnaire left room for more elaborate opinions. The teachers gave these answers:

1. Has the course made you more aware of the importance of <u>mathematical points</u>? Give reasons why (or why not).	1. Har forløbet gjort dig mere opmærksom på betydningen af <u>faglige pointer</u>? Begrund hvorfor (eller hvorfor ikke).
• Yes, definitely. I have got better at stopping and presenting points. • Yes, the course has made me aware of mathematical points. It's certainly something I will consider in future planning from annual plan to the individual lesson. • Yes a little. I think it is important to focus on the essential mathematical content – the subject's essence – in teaching. Especially at the end of lessons. Competencies. • You've realized how few academic points you normally include in your teaching. • Yes, a point is different from goals to single lessons, they require more	• Ja, helt klart. Jeg er blevet bedre til at stoppe op og gøre opmærksom på pointer. • Ja, forløbet har gjort mig opmærksom på matematiske pointer. Det er bestemt noget, jeg vil inddrage I fremtidig planlægning fra årsplan til den enkelte lektion. • Ja lidt. Jeg tænker det er vigtigt med fokus på det essentielle matematiksærlige – fagets essens – I undervisningen. Især afslutningen af lektioner. Kompetencerne. • Man har indset, hvor få faglige pointer som normalt indgår i ens undervisning. • Ja, en pointe er andet end mål for den enkelte lektion, kræver mere afdækning af stillede spørgsmål, kommunikationen i lektionen.

consideration of questions asked, the communication during the lesson. • Yes – I can see my class has the advantage of us "highlighting" points during the teaching. • Yes, it has become a focal point. • Yes, I'm more conscious about getting points to stand out clearly. • To be responsible for a lesson on mathematics competencies was really good. It "forces" me to reflect twice as I plan a lesson. Moreover, it "forces" me to articulate myself properly to produce / clarify points! • There are several considerations before a lesson on how the teaching should be done – goals. What is it really I want the students to learn! And how do I get the most possible with me! • Yes! It is important that students see <u>meaning</u> in what they do, otherwise it will be seen as unimportant. • It is important for students to understand that our work will lead to learning – not just solve a task and lead quickly to the next. They should feel that there is focus on what they have learned and what it is useful for. • Yes and no. I've always focused on the importance of academic points and notes for retention of knowledge. • I would rather say that the academic points already in the daily lessons have become a necessary focus of importance.	• Ja – jeg kan se min klasse har udbytte af at vi ”highlighter” pointer i løbet af undervisningen. • Ja, det er blevet et fokuspunkt. • Ja, jeg er mere bevidst om at få pointerne til at fremstå tydeligt. • Det at stå for en lektion i matematiske kompetencer var rigtig godt. Det ”tvinger” mig til at tænke over den en ekstra gang, når jeg planlægger en time. Desuden ”tvinger” det mig til at formulere mig rigtigt for at frembringe / tydeliggøre pointer!! • Der kommer flere overvejelser før en lektion om hvordan undervisningen ønskes udført – mål. Hvad er det egentlig jeg ønsker eleverne lærer! Og hvordan får jeg flest muligt med! • Ja! Det er vigtigt at eleverne oplever <u>mening</u> med det de gør, ellers bliver det oplevet som ligegyldigt. • Det er vigtigt for eleverne at forstå, at vores arbejde skal føre til læring – ikke bare løs opgave og hurtigt videre til næste. De skal opleve, at der er fokus på, hvad de har lært, og hvad det kan bruges til. • Både og. Jeg har altid haft fokus på betydningen af faglige pointer og notater til fastholdelse af viden. • Jeg vil snarere sige, at de i forvejen faglige pointer fra den daglige undervisning har fået et nødvendigt vigtighedsfokus.
2. How have you benefited from participation in <u>joint</u> planning of teaching?	2. Hvordan har du fx haft udbytte af at deltage i <u>fælles</u> planlægning af undervisning?
• Hearing how others are thinking and planning. You'd think it happened often, but it's not my experience. • It has been professionally inspiring and provided a better understanding of colleagues in relation to cooperation.	• Høre hvordan andre tænker og planlægger. Man skulle tro, det foregik ofte, men det er ikke min erfaring. • Det har været inspirerende fagligt og givet et bedre kendskab til kolleger i forhold til samarbejde. Og meget

And very entertaining! • Disturbing in a positive way – made me reflect on my views on the important way of doing things. • Excellent. It was nice to discuss things with one's colleagues. • Suggestions and ideas are discussed and weighed against the points and competencies. • Yes, to learn from the others – professional ideas to be aware of what is happening at other grade levels. • Exchange of ideas – subject oriented and pedagogical sparring. • It was rewarding with sparring in the brainstorm phase and the refinement phase. It seemed pointless to sit together during the planning that lies between the two phases. • It's been great because we have been able to spar with each other professionally. 4-8 eyes and ears and brains produce a good educational planning! • It has been a challenge to plan along the way at other levels than those you teach yourself, fun to try. Requires at times knowledge of the class and the students. • Had good ideas for my own planning. Exchange opinions / attitudes. • Provides insight into the mindset of other grade levels. • It was fine to discuss different professional views and angles. However, I had to plan my own lesson. • This has perhaps moved the most. But especially because there were teachers from all grade levels.	underholdende! • Forstyrrelse på den positive måde – af mit syn på den vigtige måde at gøre tingene på. • Rigtig godt. Det var dejligt at drøfte forskellige ting med ens kolleger. • Forslag og idéer diskuteres og vægtes i forhold til pointer og kompetencer. • Ja, at lære fra de andre – faglige idéer til at blive opmærksom på hvad der rører sig på andre klassetrin. • Udveksling af idéer – faglig og pædagogisk sparring. • Det var udbytterigt med sparring i brainstormfasen og finpudsningsfasen. Det føltes overflødigt at sidde så mange sammen i den planlægning, der ligger mellem de to faser. • Det har været rigtig godt, da vi har kunnet sparre hinanden fagligt. 4-8 øjne og ører samt hjerner frembringer en god undervisningsplanlægning! • Det har været en udfordring at planlægge undervejs på andre trin end dem man selv underviser i, sjovt at prøve. Kræver til tider kendskab til klassen og eleverne. • Fået gode idéer til egen planlægning. Udveksle meninger / holdninger. • Giver indsigt i tankegang på andre klassetrin. • Det var fint nok at få diskuteret forskellige faglige synspunkter og vinkler. Dog måtte jeg selv planlægge mit forløb. • Her har det måske rykket allermest. Men især fordi, der var lærere fra alle trin.
3. If you were responsible for a study lesson, what were the benefits?	3. Hvad har det fx givet selv "at være på", hvis du altså har været det?
• It was somewhat artificial and stressful. Smaller groups and several meetings would have been preferable, but good idea and very instructive.	• Det var lidt kunstigt og stressende. Mindre grupper og flere gange havde været at foretrække, men god idé og meget lærerigt.

• The very "being in focus" was good, but the subsequent evaluation was *very* good and rewarding. • Interesting. Instructive. Good feedback. Developing. Good ideas. • Fun. Specifically to receive “name and fame” for one's teaching method, materials and contacts with students. • I got a lot out of others' feedback on what they had seen in the lesson. • Exciting –some "butterflies" in stomach. • Fabulous experience. It is instructive to be disturbed in one’s self-image. • I have been in the center during the planning. Here you will obviously do your very best! • Become more aware of what you say and do. Try out phrases / sentences. • It's fine. I have no problems with that.	• Selve det ”at være på” var godt, men den efterfølgende evaluering var *meget* god og givende. • Interessant. Lærerigt. Godt med feedback. Udviklende. Gode idéer. • Sjovt. Specielt at modtage ris og ros for ens undervisningsmetode, materialer og ens kontakt til eleverne. • Jeg fik meget ud af de andres tilbagemelding på hvad de havde set i lektionen. • Spændende – lidt ”sommerfugle” i maven. • Fantastisk god oplevelse. Det er lærerigt at blive forstyrret på sit selvbillede. • Jeg har været på i planlægningen. Her vil man selvfølgelig gøre det ypperste! • Bliver mere bevidst om hvad man siger og gør. Prøve formuleringer / sætninger. • Det er helt fint. Det har jeg ingen problemer med.
4. How do you assess <u>the peer coaching</u> after each lesson?	4. Hvordan vurderer du <u>den kollegiale sparring</u> efter hver undervisningstime?
• Instructive and proper. • It has been excellent. It is a new way to collaborate, and with more experience and knowledge of colleagues could become better. • Good + guidance + positive + safe + trusting spirit and mood. Room for diversity. • We must practice even more. It's hard to communicate accurately, as the "on" teacher may be sensitive to comments. • Good and proper. • In the top – fun to watch "stress faults" in combination with overview. • Good – especially because it was tightly structured and managed. • There have been good conversations with good useful feedback. • Good and valuable since you are not always aware of how you influence the students and whether you might do it	• Lærerigt og ordentligt. • Det har været udmærket. Det er en ny samarbejdsform og ville på sigt og med mere erfaring og kendskab til kolleger blive bedre. • God + givende + positiv + tryg + tillidsfuld ånd og stemning. Plads til forskellighed. • Vi skal øve os mere. Det er svært at formulere os præcist, da ”på”-læreren kan være følsom for kommentarer. • God og ordentlig. • I top – skægt at se ”stress-fejl” i kombination med overblik. • God – specielt fordi der var stramt struktureret og styret. • Det har været gode samtaler med gode brugbare tilbagemeldinger. • Godt og værdifuldt fordi man ikke altid selv er opmærksom på hvordan man virker på eleverne og om man evt. kunne

differently. • Rewarding and necessary. • Interesting to see the very different approaches to mathematics teaching. • It was ok. It gave some food for thought. • Rewarding and instructive. Fine with many angles.	gøre det anderledes. • Givende og nødvendig. • Interessant at opleve den meget forskellige tilgang til matematikundervisning. • Det var ok. Det gav lidt grund til eftertanke. • Udbytterigt og lærerigt. Fint med mange vinkler.
5. Where do you see <u>a remaining challenge</u> in this type of academic cooperation?	5. Hvor ser du <u>fortsat en udfordring</u> i denne type fagligt samarbejde?
• More focus on the meetings of mathematics teacher teams instead of larger, more composite teams. • To make assessment interviews even better. • The time. If you work with colleagues who care, it can be very rewarding. • If we can be better to give and receive constructively. • To make time for lesson study. • Yes. • It would be nice in the future to have the option for peer supervision. • It is a great idea to have professional coaching and will be very rewarding. Can we plan lessons together in our grade level teams? • In the mathematics teacher team cooperation or lack of same. Maybe the way we organize ourselves. Create a mathematics professional culture. • To find the time and schemes for that in the everyday routines. • To ensure a better consistency in the grade 1-9 cycle – especially in the transitions primary / middle / lower secondary. • That one evaluates in an open trusting environment. It would probably require an extra person as observer. • E.g. for greater cooperation vertical in our structure (i.e. across the grade	• Mere fokus på møderne i fagteams i stedet for større og mere generelt sammensatte teams. • At gøre evalueringssamtaler endnu bedre. • Tiden. Hvis man arbejder sammen med kolleger der gider, kan det være meget givende. • Hvis vi kan blive bedre til at give og modtage konstruktivt. • At få tid til lesson study. • Ja. • Det vil være godt med en fremtidig mulighed for kollegial supervision. • Det er en rigtig god idé med faglig sparring og vil være meget udbytterigt. Kan vi på vores årgangsteams planlægge lektioner sammen? • I matematikgruppens samarbejde eller mangel på samme. Måske vores måde at organisere os på. Skabe en matematikfaglig kultur. • At finde tiden og rammerne til det i dagligdagen. • At skabe bedre sammenhæng i forløbet 1.-9. klasse – specielt i overgangene indskoling / mellemtrin / udskoling. • At man efterbehandler i et åbent tillidsfuldt miljø. Det vil nok kræve en ekstra person, som iagttager. • Fx ved større samarbejde lodret i vores struktur (altså på tværs af trinene).

levels).	
6. Other comments or <u>ideas</u> you'd like to divulge?	6. Andre kommentarer eller <u>idéer</u>, du gerne vil videregive?
• May also be used as a model in other subjects. • One could profit by planning short sequences with a few colleagues, and attend each others' teaching, and thus gain inspiration. • I imagine an advantage of being with teachers who have the same grade levels and then implementation (contemplation / development) instead. • Exchange of assessment tools at the grade level. • Let the planning of lesson take place in smaller groups. • Professional development at the school can follow the course model. Planning of mathematics lessons at a grade level can be undertaken by grade level mathematics teams. Sparring equally. • Good that we had time to plan lessons together in the three teams.	• Kan også bruges som model i andre fag. • Man kunne med fordel planlægge små forløb med et par kolleger, og overvære hinandens undervisning, og dermed opnå inspiration. • Jeg kunne tænke en fordel ved at være sammen med lærere, der har samme klassetrin og så bruge implementering (fordybelse / udvikling) i stedet. • Gerne udveksling af evalueringsværktøjer på årgangen. • Lad planlægningen af timerne foregå i mindre grupper. • Faglig udvikling på skolen kan foregå efter kursusmodellen! Planlægning af timer i matematik på en årgang kan foregå i årgangsteams!! Sparring ligeså! • Godt at vi havde tid til at planlægge undervisning sammen på de tre hold.

Some main points are summarized:

- The course made participants more aware of the importance of mathematical points in planning. It "forces" teachers to realize what they really want the students to learn and thus to clarify and articulate points themselves. Points also require more consideration of questions asked than usually practiced, i.e. the communication in the classroom.
  Students were seen to profit from teachers "highlighting" points during the teaching by understanding that our work is facilitating learning, not just to solve a task and quickly move to the next.
- The joint planning of teaching was found to be professionally inspiring and provided a better understanding of colleagues in relation to cooperation. The subject oriented and professional exchange of ideas has in some teachers' opinion been the most important affective factor.
- To be responsible for a study lesson was somewhat stressful as it disturbs your self-image as a teacher. The peer coaching after each lesson was rewarding and instructive – especially because it was tightly structured and managed. The room for diversity was also important.

- Teachers wish to create a mathematics professional culture. The experience is a new way to collaborate and would eventually and with more experience and knowledge of colleagues become better. But it is a challenge to find the time and schemes for lesson study and / or peer coaching in everyday routines.

Four teachers were interviewed as a representative group, while the rest were observing the interview. The interview was partly structured by the interview-guide below and also shown on a screen, while followed by me as the interviewer.

Questions for a focus group:	**Spørgsmål til en fokus-gruppe:**
1. *Should* there be a point in every lesson? 2. *Can* we use more communal planning? 3. *The demand* for a written lesson plan? 4. *Could* the course be repeated in another grouping? *What* would that require? 5. *Professional* level? 6. Relationships between *knowledge, attitude and experience*? 7. Would you *recommend* this to others?	1. *Skal* der være en pointe i hver time? 2. *Kan* vi være mere fælles om planlægning? 3. *Kravet* om en skriftlig lektions-plan? 4. *Kunne* forløbet gentages i anden gruppering? *Hvad* ville dét kræve? 5. *Fagligt* niveau? 6. Forhold mellem *viden, holdning og erfaring*? 7. Vil I *anbefale* det til andre?

The transcript below is an excerpt of this group-interview:

1. *Must* there be a point in every lesson?	1. *Skal* der være en pointe i hver time?
T: No, I think there must be a point or several in a sequence. But now I'm teaching the algebra we unfortunately did not quite cover in 8.a, and it is not crucial for the teaching Tuesday, Wednesday and Friday that there is a point each time. T: But one lesson is replaced by another, so there will be points some other times. T: With the considerations we have made on the course, then the points are of course also present, but not necessarily for the students. I: But are they as well? Must points direct what we do as teachers?	L: Nej, jeg synes der skal være en pointe eller flere i et forløb. Men nu er jeg i gang med dét algebra, som vi desværre ikke nåede inde i 8.a, og det er ikke bærende for den undervisning, der har været tirsdag, onsdag og fredag at der er en pointe hver gang. L: Men en time afløses jo også af en time, så der kommer jo pointer på nogle andre tidspunkter. L: Med de overvejelser, vi har gjort os om forløbet, så er pointerne jo også til stede, men ikke nødvendigvis over for eleverne. I: Men er de så også det? Skal pointer styre det, vi laver som lærere?

T: It's probably just what we are looking for: that they get the point. ... if it is a pure training lesson, where such work should continue, perhaps there is no reason to highlight points. Generally it is probably also a point that we carry on with something from last time, just reminders of points from last time, so that we have tracked the brain into what we are actually doing. ... T: ... the subject, which was seen as the most important in school – by many, that was mathematics. But it was also the subject which most thought was boring. And this must also be because many do not see any meaning in mathematics ... I think, that's why it's important, there is a point, that there is a meaning to it ... well not allways, but sometimes ...	L: Det er vel lidt det, vi søger efter: at de fanger pointen. ... hvis det er en ren arbejds-time, hvor de sådan skal arbejde videre, er der måske ingen grund til sådan at ”high-lighte” pointer. Generelt er det vel også en pointe at vi fortsætter noget fra sidste gang, lige får trukket pointer frem fra sidste gang, så vi får sporet hjernen ind på, hvad vi egentlig laver.… L: … det fag, som blev oplevet som det mest vigtige i skolen – af mange, det var matematik. Men samtidigt så var det også det fag som flest synes var kedeligt. Og det må også være fordi mange ikke kan se en eller anden mening med matematik … jeg tænker, det er derfor det er vigtigt, der er en pointe, at der er en mening med det … altså ikke hver gang, men af og til …
2. *Can* we be more common in planning?	2. *Kan* vi være mere fælles om planlægning?
T: We have no time for that in everyday routines. T: I could well imagine that we were more common in teaching ... they don’t understand the third time either, when the explanation comes out the same way! They could benefit from meeting mathematics from more than my mouth ... That we were better able to exploit each other's competencies in these areas ...	L: Det har vi ikke tid til til daglig. L: Jeg kunne godt tænke mig, at vi var mere fælles om undervisning … de forstår det jo heller ikke tredje gang, når forklaringen kommer ud på samme måde! De kunne have gavn af at møde matematik fra andre end min mund … At vi var bedre til at udnytte hinandens kompetencer på de områder ...
3. *The demand* for a written lesson plan?	3. *Kravet* om en skriftlig lektions-plan?
T: You get an opportunity to discuss in precise detail, what you think is important to pass on, when you write it down. T: You could make it shorter. We are running ... with weekly plans, and in bold text below: That’s the point. So parents can see ...	L: Man får en mulighed for at diskutere ud i præcis detalje, hvad man synes er vigtigt at give videre, når man skal til at skrive det ned. L: Man kunne korte det ned. Vi kører … med ugeplaner, og så med fed tekst nedenunder: Det er pointen. Så forældrene kan se …
4. Could the course be repeated in another grouping? *What* would that require?	4. Kunne forløbet gentages i anden gruppering? *Hvad* ville dét kræve?
T: There are (too big) differences in how we	L: Der er (for stor) forskel på hvordan vi

structure our day!	strukturerer vores døgn!
5. *Academic* level?	5. *Fagligt* niveau?
T: The course was very much on structuring of the teaching. And we have probably become better at that, it was certainly on target. T: Academic for whom? For us or....? I mean, we talked very academically in our (group). I: Well, it is in teaching. There are also academics in grade two! T: Yes, yes. So when we had to select which multiplication methods there were, then we talked nothing but academic level. And dismissed those we thought was too incalculable.	L: Forløbet er meget gået på strukturering af undervisningen. Og det er man nok blevet bedre til, det var der i hvert fald fokus på. L: Fagligt for hvem? For os eller …? Altså, vi snakkede meget fagligt i vores (gruppe). I: Altså, det er i undervisningen. Der er jo også faglighed i 2. klasse! L: Ja, ja. Så da vi skulle udvælge hvilke gangemetoder, der var, så snakkede vi da ikke andet end fagligt niveau. Og valgte dem ud, som vi syntes var for uoverskuelige.
6. Relationships between *knowledge, attitude and experience*?	6. Forhold mellem *viden, holdning og erfaring*?
T: I think, we are quick to accept each other's beliefs, because it is often decided by what class you have ... The disagreement we have about how to start a lesson (in parallel classes) arises after all because of the students she has, and which I have. And then a belief that what we have done individually, it works. T: I also think it is healthy in terms of shifting a little your position ... And there has been good progress, because we've had this safe forum ... T: The funny thing is that you are in a workplace with 700 children and 100 colleagues. But when the door is closed, you are so much alone after all. ... It is indeed such like one's intimate sphere will be exceeded. It is healthy enough to get it done – but really it's a lot like being self-employed! Experience, it is somewhat an own harvest. It is not an experience I have made with my mathematics colleagues.	L: Jeg synes, man er hurtig til at acceptere hinandens beliefs, fordi det jo ofte er begrundet i, hvilken klasse man har… Den uenighed, vi har om, hvordan man skal starte en lektion (i parallelklasser), bunder jo i, hvad for nogle elever hun har, og hvad for nogle jeg har. Og så en tro på, at det vi har gjort hver især, det virker. L: Jeg tror også det er sundt mht. at rykke lidt ved ens holdning … Og der har forløbet været godt, for vi har haft det der trygge forum … L: Det sjove er, at man er på en arbejdsplads med 700 børn og 100 kolleger. Men når døren bliver lukket, så er man i dén grad alene jo. … Det er faktisk sådan lidt ens intimsfære, der bliver overskredet. Det er sundt nok, at få det gjort – men egentlig så er det meget et selvstændigt erhverv! Erfaring, det er lidt en egen høst. Det er ikke en erfaring, jeg har gjort mig sammen med mine matematikkolleger.

7. I: Would you *recommend* this to others?	7. I: Vil I *anbefale* det til andre?
T: I think clearly that *lesson study* is something I would recommend ... it's really rewarding!	L: Jeg synes helt klart, at *lesson study* det noget, jeg vil anbefale … det er rigtig givtigt!

The group interview supplemented the anonymous questionnaires with more statements on the role of points and peer coaching:

- You don't need to have a point in every lesson, a teacher believes. But there should be a point or several in a sequence of lessons. What we are looking for, another teacher states, is that they (the students) catch the point. Some of the teachers may use a "point" as the everyday word of "meaning". It does not conflict with the definition in the lesson study course (and this research), but it shows the challenge in using everyday words in other contexts.
- These teachers confirm the impression of lack of time for change. But they are also tempted by the possibility of more common planning and teaching effort. *"…we were better able to exploit each other's competencies*".
- A written lesson plan is seen as an opportunity to discuss in precise detail, what you think is important to pass on, when you write it down. But the teachers were also occupied by the desire to make it short and possibly based on a convenient template.
- On the relationships between knowledge, attitude and experience teachers expressed that "*we are quick to accept each other's beliefs, because it is often decided by what class you have* ... ". Disagreements are not seen as based on eternal beliefs, because beliefs may be changed by seeing what works well in other classrooms. One teacher even finds this "*healthy in terms of shifting a little your position* ... ". Lesson study is recommended without hesitation by these teachers.

As stated by Hiebert and Stigler (Hiebert & Stigler, 2000) lesson study may be looked upon as teacher development. Their consideration applies to American teachers, but also for Danish teachers the collaborative design and testing of lessons may provide a context for any teacher to improve their own mathematical knowledge and teaching skill. The teachers at the school in this research were all different, but they were all convinced and tempted by the professional challenge in researching their own practice with colleagues.

12.6 Preconditions

This school has actively joined in the planning of the course. Before arranging the actual framework of the structure, content and teacher involvement the school management mentioned more or less “en passant” – as I visited the local focus teacher for the last time – that they were seeking a person from outside to stimulate a coming-together among all mathematics teachers combined with an in-depth course on some academic issue. Perhaps this disqualifies the school as being typical of Danish schools.

But an arrangement like this course is possible in other schools as well. At the actual school more teachers stressed the importance of the sparring person being from outside. Also it was stressed that the peer involvement and mutual trust was promoting the purpose.

To imagine every school in Denmark involved in a course like this is really a very tempting idea. At the moment there is a lack of researchers and / or teacher educators with the necessary insight into subjects and research. But this may change by reorienting work at University colleges and institutes towards such an important assignment. Also the economy in schools will demand a reallocation of modest funds. Local arrangements may minimize the use of costly teachers’ time.

In a recent course for mathematics teacher tutors in the municipality of Aarhus where I was one of the instructors, the goal was to develop participants’ skills:

1) *to advise colleagues on methods, teaching materials and learning resources in a best imaginable teaching situation*
2) *to undertake special developmental, advisory and coordinating functions related to discipline at school*
3) *to strengthen the standards of the school and promote professional and educational knowledge in the field.*

During the first full week (30 hours of 39) one of six course modules was on *points* and *lesson study*. After this we met for 3 afternoons each for 3 hours and discussed experiences and challenges. Every teacher had to bring a short report to these meetings. At the final meeting February 2nd 2011, I asked everyone to fill in a questionnaire, and the answers from the eight teachers present to one of the questions were (my highlight):

Q: *In what ways has this course contributed to development and qualification of <u>your own practice</u> and the <u>school practice</u>?*
T1: It **inspires**. T2: I am much more **aware** of what I need to put into the role. There are not that many spoken expectations at the school (yet). T3: I am more **aware** of several areas for action in teaching as well as at the school’s

development of the mathematics subject.
T4: Input for qualification of the teaching (including my own). Contribution to the development of subject teacher teams at the school.
T5: (My offering of) support in mathematics lessons.
T6: I look more **analytical** on my own and other's practice.
T7: **In my own practice it is a focus on points** – the school's practice is still unchanged
T8: It forced me to be much more **conscious** in my own teaching

Evidence like this of teacher *sensitivity* caused by course participation is worth noticing, especially because of the risk also stated by Mason (Mason, 2003) of misinterpreting attention and affect. Questionnaire statements seem to validate the goal of the course.

Furthermore these teachers were already involved in guidance of mathematics teacher colleagues at their own school (In Danish: Matematikvejledere), therefore it is of course of utmost importance, that they demonstrate awareness of the importance of qualities in mathematics teaching e.g. the role of points, themselves. A next stage would be for these teachers to guide colleagues in a similar direction.

13 Research findings

The study consists of several stages:

A. The research on mathematics points was motivated by the actual situation in Danish mathematics teaching and some personal motives (chapters 1-2).
B. No research similar to this present study was found, but several classroom research studies presents research methods and/or traits of teacher and student actions also encountered in this research. The literature survey is not confined to points – as there are almost no studies directly linked to that, but looks more generally at the teachers' role in stating crucial mathematical content in classroom communication. Special interest is devoted to video based research also using coding to follow traits in larger data collections as in the TIMSS and the LPS video studies. And some intervention projects are referred because they identify challenges met on larger scale projects (chapter 3).
C. The point concept is theoretically developed and defined (chapter 4).
D. A study among 50 grade 8 mathematics teachers is done on the extent and ways points are made. The research design from stratification and data collection to a final analysis is illustrated by detailed examples of data coding (chapters 5-8).
E. Reasons / obstacles to the articulation of points (chapter 9) and an intervention study with a focus group of five teachers are described and assessed (chapter 10).
F. Finally another intervention study with all mathematics teachers at one school is described. This part of the research is also supported by use of mixed methods (chapters 11-12).

In this chapter findings are presented in relation to the research questions (RQ1-3) as follows:

13.1 Extent of point articulation

The first research question is on the presence and articulation of points in Danish mathematics teaching:

RQ1: To what extent, how and why do teachers *articulate* mathematical point(s) in Danish mathematics teaching?

Based on etymology and collocations the idea of significant evidence is contained in the developed two definitions (chapter 4):

- ***A mathematical point is a statement presenting a clearly delineated significant mathematical content or climax.***
- ***A didactic point is a mathematical point, teachers have judged particularly important to the student's insight and understanding.***

The latter definition is used in the mathematics teaching context, where points may be articulated both by teachers or students, sometime even in the same dialogue. And

points may be made by the teacher or a student in whole class teaching or in guidance of individual students or groups.
Four different types of points are identified and researched in the study (sections 6.4, 6.5 and 6.6). And examples of dialogue and reasoned coding decisions are presented with each type of point:

Conceptual point	Teacher or a student presents and comments on a mathematical concept using definition, symbol or application
Procedural point	Teacher or a student presents and comments on a rule or a method in an application or example
Result point	Teacher or a student develops or presents and comments on a mathematical result such as a formula, theorem or procedure
Interpretation point	Teacher or a student interprets a model or a result or compares representations

13.1.1 50 teachers are representative

The research describes terms, observations and other types of data obtained during 2008-09 from 50 randomly chosen Danish mathematics lessons and documents the extent and ways in which teachers *articulate* mathematical point(s) in Danish mathematics teaching in grade 8 (chapters 5-8).
Stratification and sampling of municipality owned schools and classes involved correspondence with 106 schools to arrange for 50 acceptances representing geographical variety. Reasons for declination are discussed (section 5.4) and representativeness concluded. The relative weight of student population in the 5 regions of Denmark is maintained and 41 out of the 96 Danish municipalities are represented by both rural schools and urban schools with a variation in school size. The teachers' variation in seniority, educational background and gender – and the lesson topic dealt with in each observed lesson is registered in a casebook (section 8.1).
For each lesson data consist of a video recording, a transcript, a teacher questionnaire and a researcher memo. Also correspondence with schools and teachers before actual visits is kept.

13.1.2 Findings concerning the extent and the "hows"

Danish mathematics teachers do *not* put an emphasis on didactic points.
The points observed are typically quite local to an instance in the lesson, and could seldom be observed as a "driver" for one complete lesson. They often seem to be made in passing and then left. And almost half the mathematics lessons observed were completely without any points.

As to each of the four types of points the specific findings are presented below:

Conceptual points (sections 6.4.1, 6.5.1 and 6.6.1) are made, when e.g. a teacher or a student presents a definition, symbol or application to students and explains the content.
Did this decription work?

7 lessons of the 50 contain conceptual points by the teacher to the whole class.
1 lesson contains conceptual points by the teacher to groups or individual students.
12 lessons contain conceptual points made by students.
These are low numbers considering the importance of concepts to mathematics.

The *conceptual* points chosen as examples in the dissertation were the following:

Lesson	Type	Section	Concepts
35	Teacher → class	6.4.1	Formula y = ax + b illustrated by straight line
46	Teacher → class	6.4.1	Cumulated frequency
26	Teacher → student	6.5.1	1 : 150
3	Student	6.6.1	A square
32	Student	6.6.1	Notation for powers

Concepts are rarely defined as lexical entries.
One such example is lesson 46, where the concept of cumulated frequency is presented and commented upon by the teacher in application to traffic counts. Therefore the quoted excerpt is also coded as a teacher’s interpretation point. But in a monologue the teacher then draws the attention of students to frequency as number of oscillations per second. He ends up by articulating, that frequency to sound is approximately the same as in the lesson traffic count.
From the teacher’s point of view, the analogy is relevant, as he is also the physics teacher to this class. The same word may very well be applied to these examples, but the comparison seems to distort the beginning comprehension of students instead of supporting it as no student reacts to the teacher’s presentation of this point.

The concepts stated by the 50 grade 8 teachers are (in terms by the researcher):

Lesson	References	Coverage	Concepts
7	1	10.9%	Percentage as a relative proportion
46	3	7.2%	Cumulated frequency
13	3	4.1%	Sketches, isometric and perspective drawing
2	3	3.6%	Medians, equilateral, isosceles and legs in right triangle
35	1	1.9%	The formula concept illustrated by a straight line
19	1	0.9%	A vanishing point, parallel lines
50	1	0.7%	Income (in spreadsheet)

For any lesson topic (see the casebook in section 8.1) it is possible for a teacher to state conceptual points, if this was the intention. The emphasis on such statements was clearly different for the majority of the 50 teachers as the table above "thins out" quickly. There may be more reasons for that:

- Didactic points on concepts may be driving the lesson without being articulated. In most of the mathematics teaching researched the teachers used mathematical textbooks. And the progression in textbooks may be expected to be carefully designed with respect to didactic points – even if they are not stated in text or assignments.
- Mathematics teachers may not emphasize or be aware of the importance of didactic points to be stated in class. The notion is new or unknown, whereas goals or aims are stated in teacher guides and curriculum.
- The topic taught may already have been introduced in previous lessons. Therefore concepts were perhaps already presented and thus not being observed in this "snapshot" research. In a teaching schema typical of several lessons on a mathematical topic the concept(s) may be introduced first, procedures, results or interpretations later on. Conceptual points were expected, though, to be more often articulated in a study of 50 lessons.

Procedural points (sections 6.4.2, 6.5.2 and 6.6.2) are made, when e.g. a teacher or a student presents and comments on a rule or a particular method to solve mathematical problems of a certain kind.
Did this decription work?

18 lessons of the 50 contain procedural points made by the teacher to the whole class.
12 lessons contain procedural points by the teacher to groups or individual students.
13 lessons contain procedural points made by students.
In other words this type of point is the one most frequently made by the mathematics teachers.

The *procedural* points chosen as examples in the dissertation were:

Lesson	Type	Section	Procedures
7	Teacher → class	6.4.2	More girls as relative proportion in %
18	Teacher → class	6.4.2	Allocating right angles by use of Pythagoras'
2	Teacher → student	6.5.2	Translation, rotation and reflection
38	Student	6.6.2	Common denominator
20	Student	6.6.2	Area of "pie piece"

Procedural points are articulated by many teachers and students. Procedures did not always call for expressed reasoning and understanding though.

One such example is lesson 38, where a student explains why 2/9 + 4/9 equals 6/9 by saying: *We just move over, right*? Asked by the teacher to explain further, the answer is: *They have a common denominator.* As no one in the class expressed any doubts, this seems sufficient and the lesson immediately continues for other tasks.
From the teacher's point of view, this may seem completely obvious. The task was meant as a simple starter in a subsequent (and daily) 10 minute overview. The lesson swiftly moved on to addition and subtraction of fractions with different denominators – and was indeed planned to. To the observer this underlines to importance of context in understanding teaching practice. It may be risky to look at and evaluate isolated episodes.

The procedures stated by the 50 grade 8 teachers are (in terms by the researcher):

Lesson	References	Coverage	Procedures
20	3	16.6%	Area of a circle sector
7	2	16.2%	Finding relative shares as percentages
35	4	15.0%	Plotting graphs to functions on form y = ax + b
13	4	13.3%	Construction of perspective drawings
38	1	12.1%	Addition of fractions with different denominator
21	2	11.7%	Multiplication of powers, drawing of "soft" curves
29	2	10.9%	Solution of equation, minus x minus
14	1	10.9%	Horizon and vanishing point in perspective drawings
6	2	9.8%	Setting a common factor outside parenthesis
5	2	9.5%	Solution of equations involving fractions
39	1	9.2%	Table with frequency and cumulated frequency
50	3	8.8%	Design of spreadsheet for budget
2	3	8.5%	Rotate triangle in coordinate system, find square roots
18	1	8.1%	Right angle construction using the Pythagorean theorem
10	1	6.3%	Sampling without replacement
46	1	5.8%	Use of frequency table
28	1	5.6%	Multiplication of powers
4	1	1.5%	Multiplication of fractions

Procedural points are the type most often found also during teachers' guidance of groups or individual students. This type is also the one most often stated by students, but the extent is comparable to students' articulation of conceptual and interpretation points.

- This may reflect a relatively high importance attributed to procedures by mathematics teachers.

- It may be a consequence of the research only providing a single “window” for a topic teaching sequence, where a major part of the teaching is on procedures compared to other types of points simply because this needs more time.
- It may be evidence of teachers’ beliefs, that procedures are the core of the subject of mathematics and therefore need to be the driving points behind teaching.

Result points (sections 6.4.3, 6.5.3 and 6.6.3) are made, when e.g. a teacher or a student develops or presents and comments on a mathematical formula, theorem or procedure, which is *not* a definition. Output of assignments as numbers or constructions are not considered results here.
Did this decription work?

3 lessons of the 50 contain result points made by the teacher to the whole class.
No lessons contain result points by the teacher to groups or individual students.
3 lessons contain result points made by students.

The result points chosen as examples in the dissertation were:

Lesson	Type	Section	Results
21	Teacher → class	6.4.3	$a^0 = 1$
28	Teacher → class	6.4.3	$a^0 = 1$
12	Student	6.6.3	Number of A3 sheets in 1 m^2

Lessons 21 and 28 seem comparable, as the same mathematical result is stated by the teacher to the class. The goals stated by the two teachers may be seen in a table further below.
In lesson 21 the result $a^0 = 1$ is suggested by a student and confirmed by the teacher without any explanation but as a rule among others asked for in the quoted excerpt. The teacher’s focus in this lesson seems to be on rules for multiplication by powers to be learned by heart. And students know their rules.
In lesson 28 the teacher’s focus seems to be on reasoning. And the dialogue involves the students in presenting findings as a mathematical result. The excerpt is therefore also coded as a student result point.

The results stated by the 50 grade 8 teachers are (in terms by the researcher):

Lesson	References	Coverage	Results
20	2	6.0%	Area formula to rectangle
21	3	5.1%	Calculation rules to powers
28	1	4.7%	Division between powers to the same root, 10^0.

- The sparse findings may be due to high level thinking not offered or accessible to all students in a class. No result points were articulated by teachers to individual students or groups and only four by students. The latter may be explained by an overload of routine exercises requiring practice in the classroom.

Interpretation points (sections 6.4.4, 6.5.4 and 6.6.4) are made, when e.g. a teacher or a student interprets a mathematical model, result or compares representations.
Did this decription work?

13 lessons of the 50 contain interpretation points made by the teacher to the whole class.
7 lessons contain interpretation points by the teacher to groups or individual students.
11 lessons contain interpretation points made by students.

Interpretation points described as examples in the dissertation were:

Lesson	Type	Section	Interpretations
41	Teacher → class	6.4.4	Area and unity for 2 dimensions
13	Teacher → class	6.4.4	Eye height in perspective drawings
14	Teacher → student	6.5.4	Location of vanishing point and horizon line
22	Student	6.6.4	"… for maximum 2 days"?
24	Student	6.6.4	33 minutes = 0.55 hour

In lesson 24 the mathematics teacher is somewhat caught by surprise because a substitute teacher has made the class do a test, but did not correct and hand it back to the class. The lesson was intended to be repetitive on conversions of volume and time and the excerpt quoted shows a student explaining conversion from time in minutes to decimal notation in hours.
None of the fellow students seem able to follow the comparison between time unit representations by this student, and they are commenting loudly. The teacher then has a hard job deciding whether to intervene with supplementary explanation (if he is able to), or to dismiss the eager student. Instead he decides to stress the importance of being able to do this conversion by asking: *When is it smart?* A student's suggestion: *In mathematics homework,* is accepted as the one and only relevance.
This was not exactly what I would generally hope for. But in the isolated episode it is quite understandable as all students were thinking of homework and test results.

The interpretations stated by the 50 grade 8 teachers are:

Lesson	References	Coverage	Interpretation
7	4	14.7%	Meaning of percentages
14	2	10.7%	Horizon in perspective drawings
35	4	8.6%	Slope and intersection points to lines
41	1	7.7%	Area and unity with 2 dimensions
20	2	5.9%	Decimal points and units by multiplication

21	1	4.3%	Hyperbola graph representation
13	1	4.2%	Eye height in perspective drawings
23	1	3.3%	The unknown to equations as a *joker to card games*
38	2	3.3%	Common denominator and shortening
46	1	2.3%	Frequency in *physics*
2	1	1.6%	Pythagoras' theorem to right triangles
40	1	1.6%	Decoding text for mathematical information
29	1	0.9%	Combinatorics and *a distribution of marbles*

- These points are the second most registered type by the research. As seen in the table above this is partly due to a few teachers (23, 46, 29) offering interpretation in the context of other subjects or everyday life to help students to understand.
- De-mathematization is the point in many of the examples (7, 14, 21, 13, 2).

Aggregated conclusions are:

- **56% of the lessons have no common teacher to class points at all.**
- **Only 22% of the lessons have more than 10% of the lesson time characterized by points being made by the teacher in communication to the whole class.**

Do Danish mathematics teachers then "miss the point"?
Not quite. Many mathematics lessons *do* include a statement in the form of a mathematical point, but some types are articulated in the same lesson, why percentages below total more than 58%.

- **58% of the mathematics lessons included points in one form or another.**
- **44% included teacher to class points.**
- **30% included teacher to individual student points**
- **52% included student points in communication with the teacher in either whole class communication or during teachers' guidance of individuals or groups.**

The four types of points do *not* appear equally often. Procedural points are by far the type most often present. And many points are only stated briefly or by chance at a sudden opportunity. Most points do *not* seem planned or driving the lesson.

So far the *extent* and the "*how*" on articulation of points is documented – not only as stated by teachers, but also as stated by students.

13.1.3 Findings on the "whys"

Also important is that most mathematics teachers (74%) elicit student statements in their communication with the class or individual students or groups. Elicitation is teacher communication inviting students to answer or contribute (sections 5.9.1 and 6.7.1), and does not include teachers' instructions, repetitions (uptake) or response. Elicitation is found and coded in the transcripts as teacher actions may be expected to lead to more students' own articulation of points.
The 50 teachers were deliberately *not* informed on the research focus on points when asked to fill out a questionnaire after the lesson. Instead a question was on the most important thing, the teacher wanted the students to learn in that lesson. To search for an answer to the "*why*", all questionnaires were analyzed for teachers' answers to this question (Appendix B):

What was the most important thing, you wanted the students to learn:
In the lesson of today?
In previous lesson?
In next lesson?

It was quite common to note a correspondence in topics between the lesson observed and the neighboring ones. But among the 45 questionnaires returned, only a few answers could be said to fully answer the question posed on goals. The majority indicated the "most important thing" simply by stating a topic by a headline, as it is quite often done in mathematical textbook chapter headings. The table below shows the answers to "the lesson of today question" exactly as they were stated by the teachers.

Lesson	**Content**	**Lesson goal as stated by teacher**
1	Semester test	What mock exam consists of
2	Coordinate system	Parallel displacement, rotation and mirroring
3	Algebra, area	Repeat square, rectangle, calculate perimeter and area by generalization, higher level of abstraction
4	Fractions, skills	Train fraction, perspective, finish mathematics test
5	Equations	Equations
6	Negative numbers	Not received
7	Fractions	Fractions, decimal numbers, percent, per mille and relative
8	Currency	Convert foreign currency to Danish kroner
9	Equations	Using undergone theory to calculate / solve equations
10	Probability	Routine in probability
11	Reduction	Repeating and developing reduction

12	Polar coordinates	Read and understand long text, construct a coordinate system different from the usual one, cooperation
13	Perspective drawing	To draw from self chosen models (cubes, wooden blocks, 3 principles)
14	Perspective drawing	Introduction to 3 drawing principles
15	Arithmetic	Mathematical skills, as they must be able to calculate such tasks by the end of next school year
16	Use of formulas	Hands on + use of formulas
17	Arithmetic	Evaluate their own skills in various mathematical topics
18	Pythagoras	Pythagoras, work with tasks
19	Arithmetic	Analysis of mock exam, oral mathematics by tasks at blackboard, individual skill training
20	Area	Area of geometrical forms, conversion m → cm
21	Powers	Power calculation rules + graph hyperbola
22	Statistics	Training statistics
23	Equations	What an equation is, how it eventually is solved
24	Area and time	Repetition as $cm^3 \rightarrow m^3$, time, equation systems
25	Quadrangles	(left blank)
26	Problem solving	Solve mathematical problems
27	Calculation hierarchy	To create understanding and “rythm” in use of calculation hierarchy
28	Powers	Rules to power calculations in relation to the task
29	Equations	Upstart of equations and solution of these
30	IT Competence	Use of the program “Jing”
31	Repetition	To give me insight into student achievement of last two weeks of teaching. Students should be able to individually ask about issues and areas where they feel they need special attention.
32	Powers	Not received
33	Percentages	Understand the different kinds of % calculations
34	Individual practice	Different for different students
35	Equation of line	That b in the expression f(x) = ax + b indicates the intersection point of graph and the y-axis
36	Speed	Not received
37	Pythagoras	To detect a correspondence among side lengths in right triangles
38	Reduction	Reduction + fraction + number understanding
39	Statistics	Get to grips with subject concepts
40	Decode information	Decode text for mathematical information
41	Algebra, area	Not received
42	Volume – project	Comprehensive plan on volume, density, sheer liters: Students work with their liter task and tasks on the volume and density.
43	Angles, compass	Compass divided into 360°, fractions, use of other units

44	Probability	Routine in sample space and the probability-ratio
45	Equation of line	Practice drawing of linear functions
46	Statistics	Not received
47	Decode information	Mastering percentage calculation in contexts. Being able to ”read” what is significant, and what is not.
48	Perspective drawing	Perspective drawing: Front and x-perspective
49	Equation of line	Understand linear functions
50	IT competencies	Use spreadsheet as a tool to get overview of many numbers

There are several possible explanations for these ways to express lesson goals:

- Lesson goals are not sufficiently advocated by teacher guides – or perhaps not adapted.
- Textbook headlines use to be sufficient announcements to students. And they are most often a declaration of the mathematical topics, not the goals.
- A superficial indication of goals seems to be sufficient for colleagues, parents and in student plans.
- Teachers are not supported in precise lesson planning, but expected to find their own way.
- The questionnaire may mislead, as space was not sufficient for deeper descriptions.
- Teachers wanted to spend as little time as possible on filling out the questionnaire.

Not being conscious of points ahead makes it is difficult to keep track and plan for progression when teaching the whole class or guiding individual students or groups. As argued above (section 4.4) didactic points are connected to the intentions of mathematical teaching by planning explanations, communication or activities for students to experience one or more mathematical points to develop insight and competence. Points may appear as either characteristics of crucial *moments* in mathematics lessons or “guiding” *the way* mathematics is being taught.

It is difficult to provide a deeper explanation of the “whys” in RQ1 than the reasons so closely connected to topic headlines and activity lists as indicated above by the teachers in the list of teachers’ lesson goals above.
But any articulation of a mathematical point made deliberately has a purpose. Based on the description of the four types of points in this research (section 6.4), the articulation may be assumed for these reasons even when not stated by the teacher:

- When a conceptual point is stated, the purpose is to define, present or clarify concepts e.g. by application.
- When a procedural point is stated, the purpose is to demonstrate or develop a method.
- When a result point is presented, the purpose is to present or develop a mathematical result.

- When an interpretation point is stated, the purpose is to interpret a concept, a model or a result or to compare representations.

The 50 randomly chosen mathematics teachers were not specifically asked more about “why”, but 7 teachers out of the 50, all in the same region, later joined a focus group for closer contribution and collaboration involving such reflections. The outcome of this is described earlier (section 10.2) and also referred to below (section 13.2.1).

13.1.4 Lesson structure

In 1980 a group of Danish mathematics teachers (N = 71) representative of the capital region was *interviewed* on patterns in their mathematics lessons (section 3.2.1). This research indicated three typical phases of varying length (Hansen, 1980):

Correction of homework	Presentation of new content	Individual seatwork

Three types of lessons were found to dominate:

- Type 1, e.g. 15 % of the teachers preferred two phases: One very long phase of teacher presentation (2) and one very short individual seatwork phase (3). Correction of homework was absent or very short in the beginning of lessons.
- Type 2, e.g. 56 % of the teachers preferred the same two phases as type 1, but more equal in time. These lessons could also include a shorter phase for correction.
- Type 3, e.g. 10 % of the teachers also preferred two phases: One very short introduction to new content (2) and one very long individual seatwork phase (3).

In the present research the 50 lessons were actually *observed*. The time allocated for five separate types of lesson activity is registered, and lessons sorted accordingly:

Arrival	Homework	Presentation of new content	Seatwork	Summing up

This overview of the different patterns (graphically shown in section 8.4), although not stipulated in the research questions, may be considered as a side benefit:

- *Arrival*, unpacking and messages took less than 3 minutes in the beginning of 66% of the researched lessons. A few mathematics lessons are later interrupted by students arriving late or other teachers leaving messages or looking for students. Often this is done without disturbing the teaching, sometimes so discretely that most students don’t even notice.
- *Homework* was dealt with in 66% of the 50 lessons, in 46% for more than 5 minutes for a common and often very careful correction of assignments found hard by the students.

This means that 34% of the lessons were without any correction of homework. Such lessons usually had a longer phase of individual seatwork. 5 of the 12 lessons containing the *most teacher to class points* did not include any correction of homework. Therefore it may be concluded, that homework correction does not seem to be used consistently for the stating of points. Of the 54% of the lessons having 5 minutes or *less* allotted for correction or presentation of homework almost all have a very long phase for *seatwork*.

- *New content* is presented briefly by most teachers, in 40% of the 50 lessons this takes less than 5 minutes and for 66% less than 10 minutes. All 12 lessons containing *most teacher to class points* in lesson time were among the top 28 lessons according to lesson time for teachers' presentation of new content.
- *Seatwork* was arranged in 90% of the 50 lessons, in 60% for more than 20 minutes. This means that only 10% of the lessons were without any individual seatwork. Such lessons usually had a longer phase of common homework correction.
- A *common summing* up of results at the end of a lesson was absent in 54% of the lessons. Approximately half the 50 lessons simply ended with the bell ringing and everyone suddenly realizing during seatwork, that time was up.

There are good reasons for seatwork. Insights are constructed by the learners at their pace, by different means and support. It's therefore part of a teachers' toolbox for differentiation to organize individual or group work for some of the time.
If seatwork takes place without much common presentation or summing up it may be problematic. Teachers easily get involved in assistance to individual students and thereby risk losing their overview of common progress or needs by other weak or able students. This was seen in the research.
It is a professional teacher challenge to lead quality communication in the classroom. Not all teachers seemed comfortable with this role. Elicitation techniques here become a means to invite students to state the points. This is part of the mathematical knowledge for teaching (section 9.6.1).

A few more patterns may be seen in the research data:

Relatively more points were made by teachers with mathematics as one of their *major subjects*. And relatively more points were made by teachers teaching at *large schools* (section 8.7).
Teachers' *lesson plans* in writing are sparse or nonexistent. The rather few plans shown to and received by the researcher are mostly activity lists and self-designed assignments.
Lesson goals are typically presented to students also as activity lists, e.g. a number of tasks to be solved. Mathematical goals are seldom stated in communication to students and thus invisible to observers in typical grade 8 lessons.

Lesson content is mostly based on texts and assignments from a textbook. In the textbook a series of lessons may be suggested to strive for the same goal, and are often explicitly stated in the teacher manual.

The teacher questionnaires ask for connection between lessons. This is answered but almost exclusively by mentioning a common mathematical topic. And mathematics teachers seldom referred to any connections between lessons while actually teaching.

In 78% of the lessons the teacher reviewed or presented new content in whole class teaching. Reviews were often a general overview of rules.

13.1.5 Solidity and scope of quantitative findings

The care taken to ensure accountability through a large respondent sample and handling of contacts and responses is described above. It was decided to study 50 lessons to ensure sufficient strength in the data, sufficient to be manageable by one person.

Triangulation by including and comparing transcripts, questionnaires and memos contributes to trustworthiness of the findings.

Lessons observed were also intended to be as authentic as possible. This concern was highlighted and justified in the initial written communication to school and teachers (Appendix A). This meant that mathematics lessons on project work or cross-curricular work were excluded by the teachers. The significant number of mathematics lessons that are invested in such work are not represented among the 50 lessons observed.

Both student and teachers were almost always prepared for the visit, as parents in almost all schools were previously asked for permission to record video. But it’s my impression that the lessons most often developed as normal.

Most mathematics teachers in this grade 8 survey are assumed to have mathematics as one of their major subjects in teacher education (an average of 46% did, 24% did not, 30% did not indicate in the questionnaire). According to the latest Danish statistics 77% of the teachers teaching mathematics to grade 8 do have mathematics as one of their major subjects at this moment. (*Undersøgelse af linjefagsdækningen i folkeskolen. Gennemgang af resultater,* 2009, p.11). Most teachers also are very experienced, as 54% have 10+ year seniority and 46% a 15+ year. This may be supposed to ensure that actions registered by research also were established routines.

13.1.6 Limitations

The many parameters in this research may imply a risk to overlook one or more important patterns. Keeping as open minded as possible and registering four types of points, each in three types of classroom dialogue *and* the lesson topic context, school size, teachers’ educational background, etc. may imply a risk of losing the focus on points. E.g. I did not realize that time in the school year might be an important

parameter before the take 3 sessions in the focus group. And elicitation did not occur to me as a possible indicator of student involvement until the intervention studies. Finding point free lessons with obvious teaching qualities like lesson 44 also did make me a bit more hesitant about "missing the point" proclamations. In all, the increasing amount of elicitation was a big surprise in the intervention studies.
I consider the methodology as well suited for the RQ's though, taking into account the amount of time and workload affordable. Even so it is often quite discussable what code to apply to dialogues. My solution to this has been to be frank and openly discuss my premises and choices.

Another question is of course this: Does one see more clearly now what it is, point-driven mathematics teaching? Teachers involved in focus group and the lesson study clearly confirmed this. But a short or superficial explanation to stress the importance and let teachers take action won't do it. The research showed, that great care must be taken to explain the difference between point as a mathematical and didactical concept on one side – and then point as an everyday word, or even as a synonym to teaching goal.

The observations in this research are snapshots of teaching as each class is visited only once. Therefore arbitrariness must be accepted in the choice of content, organization, working methods, assessment forms etc. met in the individual lesson. Only because of a relatively large number like 50 can some patterns may be deduced. A few lessons e.g. included a test with students silently working individually for a longer period (lesson 1, 15 and 17). These lessons all were one half of a "double-lesson".
In total 36% of the lessons observed were one half of a "double-lesson", most often the first part (marked with a "D" in the table in section 8.2). Some teachers announced rather different content and organization for the two lessons, sometimes decided by the need for a test, sometimes motivated by a desire for variation. This was mentioned by the teacher and noted in researcher memos, e.g.:

Lesson 1: *I was told that all mornings students start with a 20-minute "reading band" irrespective of the subject. It was mostly fiction after student elections – but could also be in a textbook. The idea is perhaps later to leave this to subject reading. As I observed it in the mathematics lesson the mathematics teacher sat completely silent reading herself for 20 minutes. The video was first started when there remained 2 minutes of reading.*

Lesson 15: *The first lesson was launched with set number 13 in a collection of skill tasks. The textbook was not in use until the second lesson (which was not filmed).*

Lesson 17*: I arrived in the second part of a double lesson where the first 15 minutes went with the last part of a 1-hour skill test. Everyone was whispering the few times the teacher was asked about something, and it was impossible to come close without disturbing.*

Approximately after 26 minutes recording the lesson changes completely as students fetch worksheets from a sorted collection in a box for individual training of their weak skills.

Lesson 19: *During this first part of the lesson the class was correcting a mock exam in problem solving with a circus theme. After this the work is continued with the textbook working on a powers and large numbers. I left after 49 minutes of this double lesson.*

Lesson 33: *The session was organized so that T could review percentages and allow students to start any exercises ("so I could see how he taught"). The following lesson the students are going to solve tasks with teacher support. T does not walk around as he prefers that students come to his desk. In the recorded lesson, however, T walks around quite a lot – although he occasionally sits down at his desk if no students come to him. 3 students went to the bathroom during this lesson and a teacher colleague walked into the classroom with three books for some of the students*.

The lesson observed was most often the lesson with more teacher actions as decided by the teacher, the lesson not observed was sometimes reserved for student seatwork.

In practice it is sometimes difficult to determine whether a dialogue contains a point. All recorded communication has been listened to at least twice and some of the transcripts also discussed with others. But definitions for the four types of points registered by this research leave quite a lot to be considered in context. In the analysis I offer several examples to illustrate the options and my reasoned choice.
In a few cases the articulation, noise or need for discretion made it hard to follow a communication between the teacher and individual students. This is marked in transcripts. The teacher questionnaire did not offer any possibility to inform on previous or anticipated progress either.

13.2 Strengthening the occurrence and role of points

The first part of the research indicates room for improvement in Danish mathematics teaching at the grade 8 level. Teachers' possibilities to change are challenged by several obstacles and requirements. I have presented and analyzed some conditions for the current research in the form of textbooks, organization and the different agendas in working with a mathematical topic (sections 9.2-3).
Also the school culture including management, teacher communities, teacher beliefs and professionalism, routines and student behavior present conditions for development as registered by the research (section 9.4).
But teachers are *the* key persons to a stronger role of points in mathematics teaching. Several researchers emphasize teachers' subject knowledge and teaching experience (Shulman, 1986; Stigler & Hiebert, 1999; Ma, 1999; Niss & Jensen, 2002; Hill et al, 2008; Krauss et al, 2008).

The second research question was this:

RQ2:	**In what way can the occurrence and role of mathematical points be *strengthened* in mathematics teaching practice?**

13.2.1 Stimulated recall and teacher attitudes

In 2009-10 the research involved a focus group of seven of the 50 randomly selected mathematics teachers in a seminar. "Stimulated recall" interviews with each of the seven focus teachers confirmed that the take 1 clips shown were representative and typical of each of these teachers' mathematics teaching (section 10.2). Conclusions are:

- **The stimulated recall interviews indicate quite varying mathematics teacher routines, mathematics teacher beliefs and feelings about teacher identity. Only three out of seven see themselves as mathematics teachers. The remaining four identify themselves more with other subjects or as class teacher.**

The small number of teachers does not justify too many generalizations, though, but all teachers express interest in research and debate on qualities in mathematics teaching. The stated beliefs and attitudes of teachers impose different teacher roles and varying possibilities for their students to make conceptual, procedural, result or interpretation points.
One teacher lets students benefit from lesson organization according to cooperative learning. Other teachers expect and demand a quiet classroom associated with their own need for calmness and concentration.
Points are *not* a distinct issue with regard to these teachers in my interview questions and their answers. Among teacher actions highly valued are a precise language, students' taking part in classroom communication and teachers' responsibility for lesson organization and flow when meeting different students' prior knowledge and needs.
Being made aware of the research emphasis on points I offered these teachers peer assistance in preparation of some upcoming mathematics lessons to evaluate the potential in such support (sections 10.3 and 13.4).

13.2.2 Lesson study

Experience and recommendations from lesson study are reviewed in literature and reports.

- **Lesson study supports the idea of mathematical points as each lesson is designed around to achieve a single goal in a topic (section 11.2). And the corresponding lesson plan works in several ways:**

It is the teacher's screenplay for the lesson's activities, also telling peers about the thinking of the performing teacher and identifying the points to look for and comment on as an observer.

During my study in Nagasaki in 2008 the teachers always showed excellent overview and seemed prepared for different student reactions (section 11.2.1).

- **Lessons seemed carefully planned from start until the end.**
  The Japanese teachers systematically included common discussion and summing up to highlight the mathematical point of the lesson.

I tried out *some* lesson study routines with mathematics teachers at two small Danish schools in a pilot study during 2008-09 (section 11.2.2).

- **These Danish teachers were found to be curious, positive and serious in attending the study lessons and taking notes for later discussion with peers. Observation of lessons and subsequent discussions were assessed by me as valuable in a professional, peer and social context.**

These observations are made by me as a participating researcher and partly based on an evaluating round-table. The pilot study did not include any quantitative assessment of teacher strategies or a written evaluation as e.g. by a teacher questionnaire. This is done in the subsequent lesson study at one large school referred below (section 13.5).

13.3 Intervention studies

The third research question was on that:

RQ3:	**To what extent and by which means can mathematics teachers be *supported* in *point-driven* mathematics teaching?**

This was researched by two intervention studies.

13.4 Focus group intervention

In 2009-10 I arranged individual sparring by e-mail between me as a proficient mathematics teacher colleague and each individual of the seven focus teachers aiming at developing and refining their lesson plans (chapter 10).

Teachers in this group now teaching grade 9 are very different according to whether they taught in schools in cities or small towns, seniority, gender, educational background (section 10.1). They also expressed different beliefs and attitudes (section 10.2).

The seven teachers were visited immediately after the sparring and a lesson videotaped (take 2) and five of them again one month later (take 3).

Registered changes in this intervention study were (section 10.5):

- **A decrease in the mathematics teachers' articulation of points (from an average of 15.2 % of lesson time in take 1 to 4.3% in take 2) towards an increase in the students' articulation of points (from an average 4.9% in take 1 to 21.7% in take 2).**
- **An almost uniform increase in the teachers' elicitation (from an average 7.2% of lesson time in take 1 to 15.0% in take 2 and 31.6% in take 3).**

I have discussed several components influencing the occurrence of mathematical points. But for the handling of research questions 2 and 3, I have been researching one of them more comprehensively: e.g.: school culture and peer support.
The occurrence of points may be strengthened by connection to different concept *representations*, different *interpretations* of a result, different *uses* of the same method, different *methods* to solve the same problem and consistency in the understanding students have of a point through dialogue and their questions and answers. Examples of this actually happening were found in the transcripts.

13.4.1 Peer coaching

Mathematics teachers appreciate peer support. This was documented in my focus teacher interviews and peer coaching by e-mail correspondence.
The coaching by e-mail shows (section 10.3) that most of these teachers are willing and able to change their routines and enhance the mathematical content in the topics already chosen. Also more teachers express satisfaction with this input.
The focus teachers behave differently, though, to the peer coaching. For some teachers this work as a welcome support for change. Some teachers make major changes to plans while others are more modest. To a few the offer of peer coaching is registered, but teaching continues unaffected.
Most teachers do indeed adapt some ideas suggested in peer coaching, and express more awareness of the importance of teaching leading towards mathematical points. This is documented both by interviews before take 2 and before take 3 (section 10.4)

The effect of the peer coaching is not visible in the *point* coding statistics, e.g. the durability is not documented either. With some teachers there is an increase in student points from take 1 to take 2. The average amount of student points is considerately higher in take 2 than in take 1.
But most interesting is a significant increase in the amount of elicitation by these teachers (section 10.4). Even considering the small sample this almost uniform increase in teacher elicitation along with statements from the interview sessions indicates a clearly more conscious mathematics teacher classroom management.

13.4.2 Solidity and scope of the qualitative findings

For practical reasons the focus group was invited from a group of 11 teachers in one region (section 10.1). The seven teachers who accepted were spread across

geography, education, gender and seniority and I consider them representative of the group of 50. But certainly they were positive to the project, now being involved in research a second time. Similar positive results may not be expected or guaranteed in any other group of seven teachers.
The analysis of teacher attitudes and beliefs are based on three interviews with each teacher, all except one were audio taped:

- The first interview was at the seminar with focus colleagues present. As described above the stimulated recall was supported by two excerpts of a video recording (take 1) and an open interview-guide. The atmosphere was very friendly allowing laughs and confidentiality. Teachers expressed different view without seemingly being affected by the presence of others.
- The second interview with the seven focus teachers was at their school on the take 2 lesson as a result of peer coaching by e-mail.
- The third interview with five of the focus teachers was at their school on the take 3 lesson recorded after several months of no contact between focus teacher and the peer researcher.

There was no increase in the articulation of points in % of lesson time from take 1 to take 2 and take 3. Rather the opposite, i.e. a decrease! But the increase in elicitation time was significant. The scope of this may be that teachers develop their role as head of the common class communication. Teachers are not “simply” stating what to do, but involve students more actively in dialogues.
Also to be noted is that this type of peer sparring is inexpensive and flexible. You don’t have to ask anyone but a proficient colleague, and you may decide when. By using the internet this can take place anytime and anywhere, thus not demanding desks or quiet rooms at a school.

13.4.3 Limitations

Statistics makes little sense with only 7 (and later 5) informants. Seven or five focus teachers are too few for a quantitative reliable conclusion. On the qualitative part, the teacher interviews from take 3 also indicate considerable difference in respect to change.
It is not possible to do more than attach an existence proof to peer sparring. My peer intervention is not one of a colleague either. And there is a difference from individual teachers doing it to a mathematics teacher culture of collaboration.
Peer support on an individual basis will not support a common development of quality mathematics teaching. It may be scaled in this direction by eager and proficient colleagues, but unless supported by teacher teams and school management it will stay an “island in the sea”. Whether or when teachers will state more points by peer sparring require a more prolonged longitudinal study.

The lesson agenda as one specific in a sequence of many lessons to some extent also formats the *type* of points possible. Often an introduction of new mathematical topics involves new concepts. And conceptual points may be expected to dominate. But later in a lesson sequence on the same topic – or when the topic is reviewed at the end of term – student assignments may call on procedures, results and interpretations to be the typical points.

The registered change over a few months among focus teachers for fewer points in respect of lesson time to more teacher elicitation may be due to more active classroom management or a change in lesson agendas because of grade 9 examinations coming up.

13.5 Lesson study with all 18 mathematics teachers on one school

The lesson study course in 2010-11 with me as a participant researcher was with *all* 18 mathematics teachers at one relatively larger school (chapter 12).

Mathematics lessons observed with two grade 8 teachers before and after the lesson study showed a different development with respect to point-driven teaching:

8a: Teacher to class points decreased from 10.4% of lesson time to 2.8%
Teacher elicitation was stable in extent from 59.4% to 58.8%
Student points decreased from 1.8% to 1.1%

8b: Teacher to class points increased from 0.0% of lesson time to 7.3%
Teacher elicitation increased from 31.8% of lesson time to 58.6%
Student points increased from 3.4% to 13.8%

Both teachers dominated the common classroom communication by strongly eliciting student ideas before and after the course. But their teaching progressed rather differently because students were involved in two different ways:
In class 8a one student was at the blackboard for a longer period, in class 8b the teacher communication was with the whole class. This may well illustrate both the different didactical contracts and the different teacher attitudes.

The qualitative study of only these two teachers certainly does not qualify to suggest any scaling up of such an initiative. Both the teachers in question were active in contributing to common discussions at the school and they both have mathematics as a major subject. But there was a great difference in the way points were articulated by these teachers and their students.

An evaluation of this lesson study course was made by the participating teachers in anonymous questionnaires and indicated (section 12.5):

- Lesson study makes participants more aware of the importance of mathematical points in planning and in communication in the classroom.
- The joint planning of teaching is professionally inspiring.

- The subject oriented and professional exchange of ideas is in some teachers' opinion the most important affective factor. Coaching after each lesson is found rewarding and instructive, and the room for diversity considered very important.
- Teachers' wish to create a professional mathematics teaching culture. Lesson study is recognized as a new way to collaborate and would eventually and with more experience and knowledge of colleagues become even better.

A subsequent group interview with four representatives supplied the anonymous questionnaires with more statements on the role of points and peer coaching (section 12.5):

- What we are looking for, a teacher states, is that *they* (the students) get the point.
- Teachers are tempted by the possibility in a more common planning and teaching effort. *"…we were better able to exploit each other's competencies*".
- A written lesson plan is seen as an opportunity to discuss in precise detail, what you think is important to pass on, when you write it down. Teachers also indicate lack of time and suggest lesson plans to be short and possibly based on a convenient template.
- On the relationships between knowledge, attitude and experience teachers expressed that "*we are quick to accept each other's beliefs, because it is often decided by what class you have ...* ". Disagreements are not seen as based on eternal beliefs, because beliefs may be changed by seeing what works well in other classrooms. One teacher even finds this "*healthy in terms of shifting a little your position ...* ". Lesson study is recommended without hesitation by these teachers.

13.5.1 Solidity and scope of lesson study findings

It is shown, that mathematics teachers at one school with a limited use of resources can be supported through a change to a more manifestly point-driven mathematics teaching. Many teachers claimed to see the point in mathematical points. And to some extent this is also documented by the research.

Lessons in two grade 8 classes not previously videotaped were observed before and after the lesson study to document and analyze change. The two teachers did *not* develop an especially point-driven teaching (section 12.4) but a very high percentage of the lessons were coded for teacher elicitation. Both teachers may be characterized by elicitation for student ideas and direction to the classroom communication. I assume this as a sign of teachers' intention to have *students* articulating the mathematical points in definitions, methods etc.

The lesson study was evaluated using a questionnaire sent to the teachers and a group interview. This evaluation was very positive. In the questionnaire the teachers affirmed the importance of overarching goals and few different topics for the study lessons. They found the researchers' talk on points very relevant for their own work, and were happy to be mixed with mathematics teacher colleagues not well known from daily collaboration. Teachers definitively felt aware of the importance of mathematical points and benefited from participation in joint planning of teaching. They assessed the peer coaching after each lesson as good and valuable, some stressed the room for diversity and some the well structured conversation. Without hesitation, such professional coaching in an open trustful environment was recommended by the participant teachers.

13.5.2 Limitations

This research is not designed to follow student outcome and must therefore be restricted to the observations given above and the evaluation of the participants. Peer cooperation and support in their own school is difficult to arrange for mathematics teachers without external support.

The school management and the 18 mathematics teachers at the research school were all pre-positive. The school management acted in an economical and supportive manner and one person, me, was put in charge. Whether this necessarily has to be a visiting consultant or a researcher I don't know. The teachers at this school thought so. It is my impression, and also supported by interviews with teachers, that an internal choice of such a person at least must respect the demand for mathematical proficiency and an updated view on mathematics didactics, i.e. professional competencies as those referred to at the very beginning of this thesis (section 1.1).

There is a conceptual challenge as mathematics teachers do not seem to be trained to use mathematical *points* as drivers and goals of mathematics teaching. Most points in the study lessons fitted the definition in the present research which was given to the teachers before planning. Some teacher statements more indicate points as precise goals or aims for the lesson. Points also meant meaningfulness to some participants. But the points were *never* stated as lists of activities or assignments by these teachers. In a group interview with four of the lesson study teachers, some were hesitant about the time for such common planning but still valued the collaboration as teacher development.
Long term robustness of teacher change over time is *not* documented either.

14 Discussion and recommendations

How new is all this? And do the results suggest urgent initiatives?

14.1 The survey study of 50 teachers

None of the 50 teachers in the first part of my research were told beforehand about my special interest in didactic points. I wanted to find out *how* mathematics in Danish grade 8 classrooms is taught today. Therefore teachers were kindly asked to act as they would normally do, and the data collected by observation were supplemented with questionnaires and memos to maximize trustworthiness.

Many observed mathematics grade 8 lessons seem without a guiding goal. In questionnaires, lesson goals were described by many teachers through referral to some mathematical concepts, a mathematical topic or simply by activity terms. During actual lessons these goals were also stated this way to students. Some possible explanations to this are listed above (section 13.1.3).
There was seldom a specific lesson plan. I did receive a few lesson plans, though, some even describing a sequence of several lessons on the same topic.

Many lessons seem to be "stand alone" lessons without any stated goal reference to surrounding ones. The connection between lessons mentioned was normally described in terms of a common or familiar topic, not in terms of a common goal. The 18 lessons being one of two in a double lesson were not necessarily on the same content. To some teachers variation seems to overrule coherence.

Teachers did not review or present any new content in 22% of the lessons observed (section 8.2). And in two thirds of the lessons the teachers were doing this for less than 10 minutes. Some mathematics teachers mainly seem to distribute and organize student work with assignments. The organization quickly becomes individual or group based seatwork, where the teachers now walk around the classroom, occasionally stopping for a chat, by a question to confirm student assumptions – or as often was seen – being dragged in a tedious helping a few really weak students to keep pace with the rest.
Such "walk between desks" (Clarke, 2004) is of course a deliberate action aimed at getting students to think and articulate themselves. It is also a way to assess student progress in planning continued teaching. The teachers' choice of subsequent elicitation or summing up then becomes decisive to the success or failure of students' efforts on assignments.
Assessment seems mainly done via regularly homework or small tests. But this is also to be expected in a snapshot of teaching. A few teachers did arrange self evaluation by schemes or portfolios in the lessons observed, though.

These results are quite disturbing. Some of them are a confirmation of the Danish 2006 review of mathematics at the middle school level (*Matematik på grundskolens mellemtrin*, 2006) which stated that teachers in schools seldom use the ministerial curriculum guidelines as a planning tool (section 1.1). This is still so, but is expected by the Ministry of Education to improve by changed curriculum guidelines to primary and lower secondary mathematics in 2009. Among the teacher actions considered important to mathematics teaching are distinct invitations on dialogue in the classroom (section 9.4). But the present research seems to indicate that this is difficult for many teachers.

Above I discussed some possible criteria for validity and credibility (sections 8.9 and 8.10). Not all fit equally well though to any kind of research. Especially I argued that my interpretation was enhanced by triangulation and respondent validation. Schoenfeld includes the following aspects of trustworthiness (Schoenfeld, 2007, p. 82; Schoenfeld, 2001, p. 230) and a description of quality of empirical and theoretical research:

- *Descriptive and explanatory power*
- *Prediction and falsification*
- *Rigor and specificity*
- *Replicability*
- *Triangulation.*

My survey of 50 teachers' articulation of points is certainly not an *explanation*. Even findings by cross-tabulation of the co-occurrence between e.g. teachers' points and teachers' educational background or their employment in larger schools (section 8.7) are only evidence of some possible factors that may be further researched.
I find it reasonable to assume, that there are many more factors influencing the articulation (or absence) of points. An explanation of the findings with respect to minimal lesson planning could be ignorance or lack of time and other resources, or it might be explained by the survey as being non-representative. Quite different causes, connected to the "lack of time" issue from the questionnaires, may be found in (the quick shift in) different foci presented and demanded by school authorities, and parents, e.g.:

- "The including school", expectation of a possible *integrating* teaching of the weak (and very weak) students as well as the especially gifted ones in whole class teaching. Most teachers struggle with this demanding task.
- The mathematics teacher like any other teacher makes a regular and huge effort to establish a constructive climate for work and study, occasionally meets with students outside the norm as well as the noisy and unbalanced.
- *Parents* are increasingly asking teachers or principals for more or better individual help for their special child. A few parents even lack respect for the role and conditions of the professional teacher.

- The mathematics teaching sometimes seems to be inferior to *cross-curricular work* or even themes in common to more subjects and grade levels in school. During the school year the week-scheme is regularly changed or abandoned.
- The peer coaching, which could be subject oriented by regular meetings among all mathematics teachers, is presently almost entirely done among teachers teaching different subjects in the *same* class.

I also wish to remind of some formal conditions and uncertainties listed in chapter 1. In such a combination of demands and expectations – that will probably remain at schools and within education – several needs for reforms to ensure quality teaching and learning are stated from outside the teaching communities.

Boaler has closely followed the development in the U.S. and reminds (Boaler, 2008) that in "*September 1989, the nation's govenors gathered in Charlottesville, Virginia, and set a challenge for the new millennium: American children should top the world in mathematics and science by the year 2000*" (p. 4). According to PISA 2009 (Highlights From PISA 2009, 2010), the U.S. still ranks below the middle in this mathematics testing. The U.S. average score in mathematics literacy (487) was lower than the OECD average score (496) in 2009, as it was in 2003 and 2006.
Also in Denmark, international comparisons play a role in the political debate. A government paper from 2010 stated (section 1.1) that Danish students in 2020 should be among the top five students in the world (DANMARK 2020, 2010). Another Danish government paper on "Professionalism and Freedom" (*Faglighed og frihed*, 2010) stated these key goals for the primary and lower secondary school to be reached in 2020 (my highlight):

1. All children must be able to read at the end of grade 2. **2. What students can now do in grade 9, in the future they must be able to in grade 8.** 3. Fewer students in special classes and special schools. 4. Teachers of the future must be recruited among the best students. 5. Education must be based on knowledge, not on habit. 6. Students, parents and school boards must to a greater extent be involved in developing the school. 7. Clear goals and transparency of results must reduce the need to manage in detail.	1. Alle børn skal kunne læse ved udgangen af 2. klasse. **2. Det, eleverne nu kan i 9. klasse, skal de fremover kunne i 8. klasse.** 3. Færre elever skal i specialklasser og på specialskoler. 4. Fremtidens lærere skal rekrutteres blandt de bedste studerende. 5. Undervisningen skal baseres på viden, ikke på vane. 6. Elever, forældre og skolebestyrelser skal i højere grad være med til at udvikle folkeskolen. 7. Klare mål og åbenhed om resultater skal mindske behovet for at styre i detaljen.

Statement 2 suggests quite a knowledge boost. To the government an extra year of learning increment is supposed to correspond to the level that would be necessary to lift the Danish students from the current average ranking in the PISA studies (in effect it is slightly, but still significantly above middle) to a level in the top five. The goals must also, according to this paper, be fewer and clearer, making it easier for teachers, students and parents to understand and relate to these goals. There must at each grade in all subjects be clear and precise goals for what all students should learn (the baseline) and what in addition shall be pursued so that all students are challenged to some extent.

Such ambition is interesting and understandable, as read by this researcher and mathematics teacher. If funds could be allocated, I feel sure most mathematics teachers would welcome such a challenge. But they are not asked and funds seem non-existent. Policy papers are not always results of hearings or negotiations in a pedagogical or scientific community. The effort suggested seems to depend on reallocating resources already in the educational system. To my best knowledge and experience, teachers would then need to rearrange their preparation and use of resources to an unprecedented extent – and by agreements on municipality level. Is this at all possible?

14.2 The intervention studies

The interventions I've made *predict* that change can be achieved under certain conditions. The outcomes are measured by questionnaires and interviews at the very end of efforts. This is no proof of a longer lasting effect of such interventions. And it's my strong conviction that the researcher played a decisive role as mediator in the peer processes in the focus group coaching and the lesson study sessions at one large school.

But we did gain some new knowledge. The research has shown two ways of intervention perfectly possible without changing frames a lot, and with means at hand.
Both studies required an introduction to the idea and importance of point:

1) In the first intervention study the invited group of seven teachers engaged in detailed planning by peer sparring. The teachers differed in personal background, school, attitude and experience. This study showed that different teachers appreciate a targeted individual peer coaching, and effects may be significant on these teachers' elicitation in classroom dialogues. We do not know anything of a lasting effect though.
2) In the second intervention study at one school, where the school management took the initiative to contact me, all 18 mathematics teachers became involved in a lesson study with great success. These teachers were also very different in personal background, attitude and experience – but they were at the *same* school. This study shows that teachers' curiosity towards subject colleagues and their

willingness to act with professionalism at the same school can build bridges across any local “teaching gap”.

There still are some reservations connected to the *point* concept, though. Even in the intervention studies, some teachers used the word point as a synonym to goal or meaning (section 12.5). It is not. It may be a goal to mathematics teaching to let students drill number processing or do number crunching with spreadsheets – or increase cooperative skills. But points are connected to the mathematical content. Procedural points are e.g. to explain a procedure for multiplication of powers or to find an average.

14.3 A way ahead

In the end it’s all about increasing students’ mathematical competencies. The intended goals and levels for these are described in the National curriculum guidelines (*Fælles Mål 2009*, 2009). But we really don’t know if there is a strong correlation between students’ mathematical competencies and a point-driven mathematics teaching. This was an axiom to the present research (section 2.1) and may still be challenged.
The two intervention studies show that peer sparring should be arranged as soon as possible. And the lesson study model is so promising that I shall suggest this as one of the first actions in a series progressively becoming more demanding and comprehensive:

A mathematical supervisor
Be sure that all schools have at least one mathematics teacher sufficiently proficient and respected by teachers to guide them. This mathematics supervisor must have special responsibility to take initiatives that encourage mathematics teachers' professional discussion, mentoring of colleagues, contacting resource centers and researchers.
Formally this teacher must be supported by time, status and support from school management.

A mathematical team
All mathematics teachers form a professional team in every school. Arrange for the current mathematics supervisor to offer peer mathematics support in three ways:

- Prepare common meetings supported by ideas in existing publications (Mogensen, 2008).
- Offer peer support for individual teachers as described in the intervention study above (chapter 10) (Andersen, 2010).
- Subscribe to 1-2 Nordic journals on mathematics teaching and exhibit these in the staff room to ensure a flow of new ideas.

A lesson study
Schools are encouraged to arrange a mathematics lesson study following the model for lesson study described in the intervention study above (chapter 12).
If it is adapted to local priorities it's almost certain to be rewarding!

It's tempting to increase the scale of the study by taking the lesson study design from chapter 12. But before this I suggest a more systematic research and development phase on a school basis, partly influenced by the recommendation of several "drivers" for reforms as mentioned by Tate and Rousseau (Tate & Rousseau, 2007, p. 1231):

1. The curriculum guidelines and recommended assessment tools may be studied and considered for mathematical points by teacher teams.
2. Mathematics teachers and school management should recapture the initiative for school development and a high proficiency with a point-driven mathematics teaching and explain why to politicians. Teachers and management know most about this and teaching won't develop without them.
3. Combine resources for development of mathematics teaching in fewer – or even better: one single program of upgrades and improvement on mathematical points.
4. Involve mathematics teacher communities on more levels: in journals, other institutions, among parents, press, politicians and public on the importance of mathematical points to trigger enthusiasm and economic awareness. Go public with alternative evidence on student efforts on mathematical points from portfolios or projects.
5. Argue and demonstrate how both the more weak *and* the more gifted students may profit by a point-driven teaching.

Some suggestions require persuasion and influence on higher levels:

Mathematics teacher-training should at the next adjustment be brought up-to-date with new knowledge from this project. There must be clear references to educational goals and the statements above (perhaps in combination) for two recommendations:

- Teaching practice preparation, practice teaching and evaluation is recommended to involve lesson study with clear demonstration of various forms of mathematical points.
- Discussion and exemplifications of "Subject Matter Knowledge", "Pedagogical Content Knowledge (PCK)" and "Mathematical Knowledge for Teaching (MKT)" etc. should enjoy (continued) promotion. For strategic reasons these concepts should also have a Danish wording!

Finally I estimate that research as above may be equally relevant in upper secondary education as it was for the grade 8 classrooms in public schools. It is therefore also an idea to propose collaborations between one or more universities and nearby teacher training accordingly.

And one should remember: *Not everything that counts, can be counted*. A. Einstein.

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Appendices

A: Letter to schools

Teacher education in Aarhus, October 2008

Dear Headmaster
I hope you will let the school participate in a research program. Participation will require a very small amount of one grade 8 mathematics teacher’s time. You can find more details about my research below.

How is mathematics taught today?
You may be aware of the small number of reports on *outcome measurement* available for Danish mathematics teaching. E.g.: PISA and the lower secondary examinations. *The purpose of my research is slightly different.* The existing investigations of the *teaching* of mathematics are in fact all very small or very old. So we actually (and unfortunately) do *not* know much about *how* mathematics is taught today. The goal of this project is to investigate the current situation.

Who is asking?
The study is conducted by senior lecturer Arne Mogensen from the Teacher Education in Aarhus. And it is part of a Ph.D. research project supported by the Danish Research Board and University of Roskilde. As a former teacher at Aarhus Municipal Schools between 1976-87 and during my subsequent involvement in teacher education in Aarhus, I have worked for many years with the development of high quality professional primary and lower secondary school mathematics teaching.

E.g. with portfolios as a learning strategy and evaluation method, and with gifted students. I was also a member of the government's mathematics committee for a national plan of action on primary and lower secondary schools and I chaired the curriculum committee in mathematics for the new teacher education.

I hope this introduction gives you confidence that your school would be participating in an unbiased investigation. I am aware that many opinions on teaching are expressed by those with limited understanding and experience, and that is why I feel it is important that those who are most close to the situation have an input.

Sincerely,
Arne Mogensen
VIA University College, Teacher education in Aarhus
Trøjborgvej 82, 8200 Århus N

Please return agreement to participation.

How are schools and classes sampled?
50 (random) schools are sampled, and "your" school is one of them. The selection of schools has followed the same procedure as that used for large studies (such as PISA or tryouts of national electronic testing), i.e. schools are well distributed in size and across regions.

Role of the school
My project requires that I be allowed to record video in one lesson in one grade 8 class.
The focus will be on the teacher who is constantly followed with one camera. The choice of class (if more than one grade 8 class exists) will be done according to statistically random selection. In this instance it will always be the 8.a. or the grade 8 class that is first in an alphabetical arrangement.
It is requested that the lesson to be recorded should be typical of a mathematics lesson for this class, in this school. To this end, the date for the visit can be chosen at random.

I hope very much that you will agree to participate in this project. If this is possible, I would be grateful if you could provide me with a list of suitable dates for my visit, or indicate your flexibility regarding this matter.

Once I have received the schools' proposals, I will contact the schools individually to inform them of my preferred date and time for carrying out the recording.

After the recording, the teacher in question will be asked to complete a short questionnaire (about education, teaching experience and headlines for the planned content of the "surrounding" lessons) as well as providing me with copy of pages or other teaching materials that might be included in the lesson to be videoed. There will be no expense to the school.
On the appointed date I, or a colleague, will show up at the school office at least half an hour before the lesson to be shown to the room and prepare the video equipment for recording from the lesson start to finish. If the lesson is the first of several coherent mathematics lessons only the first max. 50 minutes will be recorded.

Publishing
Over the next year I will analyze the video recordings, and based on selected analytical categories, I will endeavor to characterize grade 8 Danish mathematics teaching. The recorded material will be anonymised in the analysis, and no images will be reproduced or other identifying information released without the school's written consent.

The results from this Danish analysis and possible international recommendations will be published as a PhD thesis in about two years - and will at this point become a focus for discussion among mathematics teams, in teacher education and in-service training. A revised written report of the complete material will be sent to all participating schools as a small thank you for your help.
I hope for a favorable reception to this query.

If you need more information please feel free to call us on tel. 8743 0360 (Arne Mogensen, private), 8739 2802 (work) or e-mail: armo@viauc.dk.

Parents' understanding and acceptance can be obtained by using the form proposed on the back of this pamphlet.

Please also complete and attach the calendar form (below). Here you can suggest dates for my visit or note your flexibility, which would be much appreciated.

December	January	February	March	April	May
49		6	10	14	19
50			11		20
51	3	8	12	16	
		9	13	17	
				18	

Please return your **acceptance of participation in this research program** before November 7, 2008 at the latest, and mark it for the attention of Arne Mogensen, VIA University College, Teacher education in Aarhus, Trøjborgvej 82, 8200 Aarhus N (stamped addressed envelope enclosed) or via e-mail to armo@viauc.dk stating the name of the school headmaster, the school's eighth class and their mathematics teacher.

Response (acknowledgment) is sent to schools in week 47
On receipt of a school's acceptance to be part of the project, I will contact the school and confirm an agreement with both the class teacher (e-mail) and the school principal.

Teacher education in Aarhus, October 2008

For parents of grade 8

Regarding permission to shoot video

In the context of a research project on good
MATHEMATICS EDUCATION I want to record one mathematics lesson on video in your child's class.

The video is exclusively intended for research, and is part of a research project supported by the Danish Research Board and Roskilde University. This includes one grade 8 class in 50 randomly selected schools in Denmark.

The head teacher has already given its permission, but I shall hereby also ask for yours. Although the idea is to film the teacher, one cannot avoid recording students on the video. But no images or names will be published without written permission. Recording at the school will take place in one mathematics lesson later this year.

If you do not want your child to be present in this lesson, please indicate this on the tear off slip below and ask your child to return the completed slip to her/his mathematics teacher as soon as possible. Removing students from the video recorded lesson will mean that the school is no longer eligible to participate in the project. Thanking you in advance for you consideration.

Sincerely,
Senior lecturer Arne Mogensen,
VIA University College, Teacher education in Aarhus

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I do not want my child _________________________ to participate in the mathematics lesson, where video is being recorded.

____

Signature

Appendix A: Letter to schools (DANISH VERSION)

Læreruddannelsen i Århus, oktober 2008

Kære skoleleder

Jeg håber, du vil lade skolen deltage i et forskningsprogram, der kun kræver en ganske *lille* smule af én matematiklærer i 8. klasse. Pjecen her beskriver det.

Hvordan undervises der i matematik i dag?

Der er mange forsøg for tiden på *udbytte-måling* af dansk matematikundervisning. Fx i PISA og i folkeskolens afgangsprøver.

Ærindet her er et helt andet! De undersøgelser, der findes af *undervisningen* i matematik, er nemlig alle meget små eller meget gamle. Derfor ved vi reelt (og desværre) *ikke* så meget om, *hvordan* der undervises i matematik i dag.

Projektets formål er at undersøge dét. Og om der også kan overvejes andre måder.

Hvem spørger?

Undersøgelsen udføres af lektor Arne Mogensen ved Læreruddannelsen i Århus. Og den indgår i et ph.d. forskningsprojekt støttet af Forskningsstyrelsen og Roskilde Universitet. Som lærer ved Århus Skolevæsen 1976-87 og siden ved læreruddannelsen i Århus har jeg arbejdet i mange år med udvikling af faglig kvalitet i folkeskolens matematikundervisning.

Fx med porteføljer som læringsstrategi og evalueringsmetode samt med dygtige elever.

Jeg var også med i regeringens matematikudvalg for en national handlingsplan til folkeskolen og var formand for læseplansudvalget i matematik for den nye læreruddannelse.

Jeg håber på, at det kan underbygge tilliden til en saglig undersøgelse. Der er mange, der udtaler sig på ret spinkelt grundlag om skolen. Men vi, der arbejder i den, må være interesseret i at udsagn også er godt underbyggede.

Med venlig hilsen

Arne Mogensen

VIA University College, Læreruddannelsen i Århus

Trøjborgvej 82, 8200 Århus N

Tilsagn om deltagelse bedes returneret.

Hvordan udvælges skoler og klasser?
Der er udvalgt 50 (tilfældige) skoler, og ”din” skole er altså én af dem.
Det er foregået på samme måde, som når der udvælges skoler til andre store undersøgelser (som fx PISA eller afprøvning af elektroniske test), dvs. at skolerne også er godt fordelt på størrelser og regioner.

Skolens rolle
Jeg vil meget gerne have lov til at optage video i blot én lektion i én 8. klasse.
Fokus vil være på læreren, der hele tiden følges med ét kamera.
Desværre kan skolen ikke selv pege på en bestemt 8. klasse, hvis der er flere. Der må nemlig ikke være tvivl om det statistisk tilfældige valg. Derfor vælges altid 8.a., eller den 8. klasse, der kommer først i en alfabetisk ordning.
Det er heller ikke meningen, der skal være nogen anderledes forberedelse eller undervisning end klassens sædvanlige, så det skal virkelig helst være en tilfældig lektion (dvs. dato), der videooptages. Det er netop idéen at indfange den typiske matematikundervisning.

Jeg håber meget på dit/jeres tilsagn, som jeg jo også er helt afhængig af.
Og at I enten vil anføre mange tidspunkter på blanketten i brevet her, hvor skolen gerne ser, det finder sted. Eller vil anføre, at I er helt fleksible.
Blandt skolens forslag til uge(r) og/eller evt. ugedage håber jeg at kunne vælge en lektion, der også tidsmæssigt findes bekvem for skolen til videooptagelse. Det meddeler jeg selvfølgelig skolen i god tid.

Den pågældende lærer vil blive bedt om at udfylde et meget *kort* spørgeskema (om uddannelse, undervisningserfaring og overskrifter for det planlagte indhold i de ”omgivende” lektioner) samt om at udlevere kopier af sider eller andet undervisningsmateriale, der evt. indgår i video-lektionen.
Men ellers er det <u>ikke</u> til ulejlighed eller udgift for skolen.
På den valgte dato vil jeg eller en kollega dukke op på skolens kontor mindst en halv time før lektionen for at få udpeget lokalet og kunne klargøre video, der så kan optages fra timens start til slut. Hvis timen er den første af flere sammenhængende matematiklektioner, optages <u>kun</u> de første max. 50 minutter.

Publicering
Videooptagelserne vil næste år blive analyseret af mig. Og baseret på udvalgte analysekategorier vil jeg søge at give en karakteristik af dansk matematik-undervisning i 8. klasse. I det arbejde vil videooptagelse og -analyse selvfølgelig blive anonymiseret, og der vil ikke uden skolens skriftlige tilladelse blive gengivet billeder eller andre identificerbare oplysninger efterfølgende.

Resultatet af den danske analyse og evt. internationale anbefalinger vil om to år blive sammenskrevet i en ph.d. afhandling – og dermed til drøftelse i matematik fagteam, i læreruddannelse og i efteruddannelse. En bearbejdet skriftlig rapportering af det samlede materiale vil blive sendt til alle deltagende skoler som en lille tak for hjælpen.
Jeg håber på velvillig modtagelse af denne forespørgsel.

Ønskes flere oplysninger er man velkommen til at ringe på tlf. 8743 0360 (Arne Mogensen, privat), 8739 2802 (arbejde) eller e-maiT: armo@viauc.dk

Forældrenes forståelse og accept kan fx indhentes ved at bruge blanketforslaget på bagsiden af denne folder.

Derefter beder jeg dig/jer venligst returnere vedlagte blanket, hvor der er plads til at ønske særlige ugenumre i et skema som herunder. Men er det lige meget for skolen, er det selvfølgelig nemmere for mig.

December	Januar	Februar	Marts	April	Maj
49		**6**	**10**	**14**	**19**
50			**11**		**20**
51	**3**	**8**	**12**	**16**	
		9	**13**	**17**	
				18	

Accept af deltagelse i forskningsprogrammet bedes inden 7. november 2008 sendt til Arne Mogensen, VIA University College, Læreruddannelsen i Århus, Trøjborgvej 82, 8200 Århus N (frankeret svarkuvert vedlagt) eller via e-mail til armo@viauc.dk med oplysning om navn på skole, skoleleder, skolens 8. klasse og deres matematiklærer.

Svar (bekræftelse) sendes til skolerne i uge 47.
Når jeg modtaget skolernes tilsagn, kontakter jeg skolen og bekræfter en aftale. Det vil ske direkte til både lærer (e-mail) og skoleleder.

Læreruddannelsen i Århus, efterår 2008

Til forældre i 8. klasse

Vedrørende tilladelse til at optage video

I forbindelse med et forskningsprojekt om god
MATEMATIK-UNDERVISNING vil jeg gerne have mulighed for at optage én matematiktime på video i jeres barns 8. klasse.

Videoen er kun beregnet til forskning og led i et forskningsprojekt, der støttes af Forskningsstyrelsen og Roskilde Universitet. Der optages én matematiktime i en 8. klasse på 50 tilfældigt valgte skoler i Danmark.

Skolelederen har allerede givet sin tilladelse, men jeg skal hermed også sikre mig jeres. For selv om tanken er at filme det, som læreren gør, kan man ikke undgå, at elever kommer med i billedet. Men der bliver ikke offentliggjort billeder eller navne uden skriftlig tilladelse.
Optagelse på skolen vil ske i én matematiktime senere i dette skoleår.

Hvis I ikke ønsker, jeres barn er med i denne time, beder jeg om besked snarest via blanketten nederst til matematiklæreren. Skolen vil så ikke kunne deltage.
Hvis det er OK, skal I ikke foretage jer noget. På forhånd tak for velvilje.

Venlig hilsen
Lektor Arne Mogensen, VIA University College, Læreruddannelsen i Århus

- - - - - - - - ✂ -

Jeg ønsker IKKE, at mit barn ________________________ deltager i den matematiktime, hvor der optages video.

____

Underskrift

B: Questionnaire for the mathematics teachers

Dear colleague teaching grade 8.
Please help me by answering the following.
Mark with ✖ where appropriate. Thanks in advance, Arne Mogensen

Date:	School:
Name:	Class: Other:
Mathematics as a “line-subject”: And/or another background to mathematics teaching:	Seniority as a teacher (number of years):
Did you want to take grade 8 mathematics lessons this year?	
Textbook system (if used) in grade 8:	
Precise page number(s) referred to during <u>this</u> lesson (if so):	

Goal

The purpose of the lesson is described in the teaching guide of the system.	
YES:	NO:

I have a separate or special goal with my teaching in this lesson.	
YES:	NO:

If you had a written plan for this lesson, was it:		
Rather general?	Fairly detailed?	Very specific?

If it is possible, I would very much like to have a copy (you may use the addressed envelope).

What was the most important thing, you wanted the students to learn:	Or X here:
In the lesson of today?	No special goal:
In previous lesson?	I don’t recall:
In next lesson?	To be decided:

Content

Other material for <u>all students</u>? (if possible also enclosed in copy)	Publisher: Own:
Material for only some students? (if possible also enclosed in copy)	Publisher: Own:

Assessment

Was assessment included in this lesson?	If YES, what kind?	NO
When was the last time, you used assessment in a mathematics lesson?		

Would you be willing to fill out an extended questionnaire at a later date?	YES:	NO:

B: Questionnaire to mathematics teachers (DANISH VERSION)

Kære kollega i 8. klasse
Jeg håber, du vil hjælpe mig med svar på følgende.
Sæt ✖, hvor det passer bedst. På forhånd tak, Arne Mogensen

Skole:	Dato:	
Navn:	Klasse: 8.a	Andet:
Matematik som linjefag: ELLER baggrund for matematik:	Anciennitet som lærer (antal år):	
Har du selv ønsket matematiktimer i 8. klasse?		
Evt. bogsystem i din 8. klasse:		
Evt. sidetal i dette bogsystem i denne lektion:		

Mål

Målet med lektionen passer med lærervejledning i et bog-system.	
JA:	NEJ:

Jeg har selv formuleret et andet eller et særligt mål med lektionen.	
JA:	NEJ:

Hvis du havde en skriftlig plan for denne lektion, var den så:		
Overordnet	nogenlunde detaljeret	meget detaljeret

Er det muligt, vil jeg meget gerne have en kopi (brug evt. svarkuvert).

	Hvad var det vigtigste, Serne skulle lære? **Formulér målet (kort):**	**Eller kryds af her:**
Dagens lektion		Ikke et særligt mål
Forrige lektion		Husker den ikke
Næste lektion		Ikke bestemt endnu

Indhold

Andet materiale for alle: (om muligt vedlagt i kopi)	Forlagsmateriale: Eget:
Materiale til blot nogle elever: (om muligt vedlagt i kopi)	Forlagsmateriale: Eget:

Evaluering

Indgik der evaluering i lektionen?	Hvis JA, hvilken slags:	NEJ:
Hvordan har du sidst evalueret i en matematiklektion?		

Vil du evt. udfylde et udvidet spørgeskema for mig ved en senere lejlighed?	JA:	NEJ:

C: Codes in % of lesson length

		Teacher → class				**Adjusted**	Teacher → student				**Adjusted**
No.	Content	*Conceptual*	*Proce-dural*	*Result*	*Inter-pretation*	**total**	*Conceptual*	*Proce-dural*	*Result*	*Inter-pretation*	**total**
1	Semester test										
2	Coord. system	3.6	8.5		1.6	**12.1**		3.8		3.8	**3.8**
3	Algebra, area										
4	Fractions, skills		1.5			**1.5**		4.9			**4.9**
5	Equations		9.5			**9.5**					
6	Neg. numbers		9.8			**9.8**					
7	Fractions	10.9	16.2		14.7	**30.9**					
8	Currency										
9	Equations							9.2			**9.2**
10	Probability		6.3			**6.3**				3.1	**3.1**
11	Reduction									3.3	**3.3**
12	Pol. coordinates							2.0			**2.0**
13	Perspect.drawing	4.1	13.3		4.2	**17.4**		7.6		1.4	**9.0**
14	Perspect.drawing		10.9		10.7	**13.9**		15.3		2.0	**17.3**
15	Arithmetic										
16	Use of formulas										
17	Arithmetic										
18	Pythagoras		8.1			**8.1**		8.5			**8.5**
19	Arithmetic	0.9				**0.9**					
20	Area		16.6	6.0	5.9	**28.5**		3.2			**3.2**
21	Powers		11.7	5.1	4.3	**19.2**					
22	Statistics							4.4		7.6	**10.1**
23	Equations				3.3	**3.3**					
24	Area and time										
25	Quadrangles										
26	Problem solving						10.5			10.5	**10.5**
27	Calcul. hierarchy										
28	Powers		5.6	4.7		**10.3**					
29	Equations		10.9		0.9	**11.8**					
30	IT Competence										
31	Repetition										
32	Powers										
33	Percentages										
34	Indiv. practice										
35	Equation of line	1.9	15.0		8.6	**20.4**					
36	Speed										
37	Pythagoras										
38	Reduction		12.1		3.3	**15.4**					
39	Statistics		9.2			**9.2**					
40	Decode info.				1.6	**1.6**					
41	Algebra, area				7.7	**7.7**					
42	Volume-project							3.8			**3.8**
43	Angles, compass										
44	Probability										
45	Equation of line										
46	Statistics	7.2	5.8		2.3	**13.0**					
47	Decode info.										
48	Perspect.drawing										
49	Equation of line							2.8			**2.8**
50	IT competencies	0.7	8.8			**8.8**		5.7			**5.7**

Lessons without any common teacher points	28
Lessons with > 10% common teacher points	11
Lessons with few common teacher points	11

Lessons without any teacher points in seatwork	35
Lessons with > 10% teacher points in seatwork	3
Lessons with few teacher points in seatwork	12

C: Codes in % of lesson length

		Student → teacher				**Adjusted**	Episodes or instants				
No.	Content	*Conceptual*	*Proce-dural*	*Result*	*Inter-pretation*	**total**	*Elicitation*	*Hint*	*Missed point*	*Overview*	*Summing Up*
1	Semester test	5.9				**5.9**	15.6				
2	Coord. system	3.6				**3.6**	1.6		1.5	1.2	
3	Algebra, area	2.4		3.9		**6.3**	27.5			7.9	
4	Fractions, skills		3.2			**3.2**	20.4		9.6		
5	Equations						4.6				
6	Neg. numbers								6.9		
7	Fractions				3.6	**3.6**	24.3			5.1	0.9
8	Currency						12.6				
9	Equations						21.5		3.8		
10	Probability		3.1			**3.1**	6.3	21.9	31.5		
11	Reduction						22.3	**5.8**			
12	Pol. coordinates	14.8		2.0	14.8	**16.8**		2.0	0.4		
13	Perspect.drawing						7.7	9.8	1.9	2.9	0.7
14	Perspect.drawing				3.3	**3.3**	21.7			3.2	
15	Arithmetic								6.8		
16	Use of formulas										
17	Arithmetic										
18	Pythagoras						21.1	45.1		5.9	
19	Arithmetic	1.9				**1.9**					
20	Area		11.4	9.7		**11.4**	26.3	1.1	2.5	28.1	
21	Powers						6.9	6.1	6.3		
22	Statistics				1.7	**1.7**	25.1		1.7	12.9	
23	Equations	0.6				**0.6**	29.6				
24	Area and time				11.9	**11.9**	30.1		3.0		
25	Quadrangles	2.6				**2.6**	17.2				
26	Problem solving						40.0	0.9			
27	Calcul. hierarchy		3.6			**3.6**	3.6			3.6	
28	Powers	2.8	6.7	3.9		**9.5**					
29	Equations						10.0				
30	IT Competence						23.6	16.9			
31	Repetition										
32	Powers	6.1	7.1			**13.2**			13.2	13.2	
33	Percentages		3.3			**3.3**	14.0			26.0	
34	Indiv. practice									5.5	
35	Equation of line	0.4	2.5		0.4	**2.8**	8.1		3.0		
36	Speed						37.4				
37	Pythagoras						30.0				1.8
38	Reduction		2.3			**2.3**	12.1			3.0	
39	Statistics		20.8		20.8	**20.8**	11.6	3.7	13.1		
40	Decode info.						2.7				
41	Algebra, area	11.7			7.7	**11.7**					
42	Volume – project		8.4		11.4	**11.4**	19.2	18.3			
43	Angles, compass						12.8				
44	Probability	10.5	4.8			**15.3**	40.9				3.8
45	Equation of line								3.9		
46	Statistics		30.1			**30.1**	20.2				
47	Decode info.						9.8				
48	Perspect.drawing						6.0				
49	Equation of line		3.1			**3.1**	26.1				
50	IT competencies										

Lessons without any student points	24
Lessons with > 10% student points	9
Lessons with few student points	17

D: Invitation for seminar to headmaster and focus teachers

Teacher education in Aarhus, June 2009

Dear central region **headmaster**

Earlier this school year you were kind enough to let me visit a grade 8 class at your school, where I recorded video of the mathematics teacher. I've now been to all of the 50 randomly selected schools in Denmark.

It has been very exciting to see both the similarities in the mathematics teachers' approaches – regardless of experience, education, school and students. But also *differen*ces, according to different circumstances and conditions.

The mathematics teacher at your school kindly agreed to complete an expanded questionnaire. I would like to follow up this meeting by inviting the teacher to join with a small group of colleagues who also participated in the research with their grade 8 classes, for a **Saturday seminar at the college in Aarhus on November 7th 2009 between 9-14.** Refreshments will be provided, but other expenses will be the responsibility of the participant. I am aware that I am making this request just before the summer recess. But I hope nevertheless that you are able to encourage my mathematics colleague to take up this opportunity for contact to a research environment - with the associated possibility for extension of competence.

The benefit of participation for the teachers is that they will have the opportunity to meet with selected mathematics colleagues from other municipalities, and at the same time will be contributing to and being briefed on preliminary results of research for continued development of their and others mathematics teaching.

There will also be time in the seminar to look at selected issues in more depth. So, the participating teachers will have an opportunity to discuss their own teaching.

Following the seminar, I would like to leave open the opportunity to return to the participating teachers' classes – which are then likely to be grade 9.

- If you think the teacher would like the opportunity to join the seminar and subsequent activities, please forward the attached letter to the teacher concerned and note that it would be helpful if I could have their answer before the holidays. After the holidays I will return with more detailed information regarding the seminar
- I would be grateful if you could inform me if for some reason you feel the teacher cannot, will not or may not participate.
- I enclose an envelope.

Thank you and good summer holiday!

Sincerely,

Arne Mogensen
VIA University College, Teacher education in Aarhus
Trøjborgvej 82, 8200 Århus N
For more information please call: 8743 0360 (private) or mail: armo@viauc.dk

Teacher education in Aarhus, June 2009

Dear **math-colleague** in Central region

Earlier this school year you were kind enough to let me visit your grade 8 classroom and make a video recording. I've now been to all the 50 randomly selected schools in Denmark. It has been very exciting to see both the similarities in the mathematics teachers' approaches – regardless of experience, education, school and students. But also *differen*ces according to different circumstances and conditions.

You were kind enough to tick the option that you were happy for me to return with an expanded questionnaire. I would like to use this opportunity to invite you to join a small group of colleagues, who also participated with their grade 8 classes, for a **Saturday seminar at the college in Aarhus November 7th 2009 at 9-14.**
Refreshments will be provided, but other expenses will be the responsibility of the participant. I am aware that I am making this request just before the summer recess. But I hope nevertheless that you are able to take up this opportunity to make contact with research environment - with the associated possibility for extension of competence.

The benefit of participation is that you will have the opportunity to meet with selected mathematics colleagues from other municipalities, and at the same time will be contributing to and being briefed on preliminary results of research for continued development of your and others mathematics teaching.
There will also be time in the seminar to look at selected issues in more depth. So, you will have an opportunity to discuss your own teaching.
Following the seminar, I would like to leave open the opportunity to return to your class – which I presume will then be grade 9.

- If you are tempted and would like to attend the seminar, please can you contact me by mail, telephone or letter, I enclose an envelope for your convenience. Please make sure you clear your decision with your headmaster first. **After the holidays I will provide a concrete proposal and program for the day.**
- I would be grateful if you could inform me if for some reason you feel that you cannot, will not or may not participate.
- I also enclose an envelope.

Thank you and good summer holiday!
Sincerely,
Arne Mogensen
VIA University College, Teacher education in Aarhus
Trøjborgvej 82, 8200 Århus N

For more information please call telephone: 8743 0360 (private) or mail: armo@viauc.dk

D: Invitation til seminar for skoleleder og fokuslærere (DANISH VERSION)

Læreruddannelsen i Århus, juni 2009

Kære **skoleleder** i region Midtjylland
Tidligere på skoleåret var du venlig at lade mig se indenfor i en 8. klasse på din skole, hvor jeg optog video med matematiklæreren. Jeg har nu været på alle de 50 tilfældigt valgte skoler i Danmark.
Det har været meget spændende at se, både hvor meget matematiklærere gør på *samme* måde – uanset erfaring, uddannelse, skole og elever. Men også hvor meget, man gør *forskelligt*, sådan som omstændigheder og vilkår jo også er forskellige.

Matematiklæreren var venlig at afkrydse muligheden for evt. at lade mig vende tilbage med et udvidet spørgeskema. Det vil jeg gerne benytte mig af ved at invitere ham/hende og en mindre gruppe kolleger, der også deltog med *deres* 8. klasser, til et **lørdagsseminar på seminariet i Århus 7. november 2009, kl. 9-14.**
Her vil jeg selvfølgelig stå for forplejning og program, men ikke kunne dække anden udgift.
Jeg er helt klar over, at det er i sidste øjeblik før sommerferien. Men jeg håber alligevel med brevet her, at du kan lade min matematikkollega benytte denne mulighed for kontakt med et forskningsmiljø – med den tilhørende mulighed for kompetenceudvidelse.

Guleroden er (måske), at man her mødes med udvalgte matematikkolleger fra andre kommuner, selv bidrager til og bliver orienteret om foreløbige resultater af en særdeles relevant forskning for fortsat udvikling af (også andres) matematikundervisning.
Men seminaret skal også gøre det muligt at gå i dybden med udvalgte spørgsmål.
Man vil fx den dag blive spurgt nærmere ud om sin matematikundervisning ;-)
Det er muligt, jeg så derefter gerne vil vende tilbage til endnu et besøg i den pågældende matematik-klasse, som jeg formoder de fleste skal undervise i 9. klasse.

- Hvis du mener, læreren bør have mulighed for at medvirke, beder jeg dig videregive det vedlagte andet brev, som jeg så meget håber på at få et svar på inden ferien. **Så vender jeg tilbage efter ferien med et konkret forslag til program for dagen.**
- Hvis man af en eller anden grund ikke kan, vil eller må deltage – vil jeg selvfølgelig også meget gerne have det at vide.
- Jeg vedlægger en kuvert.

På forhånd TAK og god sommerferie!
Med venlig hilsen
Arne Mogensen
VIA University College, Læreruddannelsen i Århus
Trøjborgvej 82, 8200 Århus N
Ønskes flere oplysninger kan man ringe tlf. 8743 0360 (privat) eller mail: armo@viauc.dk

Læreruddannelsen i Århus, juni 2009

Kære **matematik-kollega** i region Midtjylland

Tidligere på skoleåret var du venlig at lade mig se indenfor i din 8. klasse og optage video. Og jeg har nu været på alle 50 tilfældigt valgte skoler i Danmark.
Det har været meget spændende at se, både hvor meget matematiklærere gør på *samme* måde – uanset erfaring, uddannelse, skole og elever. Men også hvor meget, vi gør *forskelligt*, sådan som omstændigheder og vilkår jo også er forskellige.

Du var venlig at afkrydse muligheden for evt. at lade mig vende tilbage med et udvidet spørgeskema. Det vil jeg gerne benytte mig af ved at invitere dig og en mindre gruppe kolleger, der også deltog med *deres* 8. klasser, til et **lørdagsseminar på seminariet i Århus 7. november 2009, kl. 9-14.**
Her vil jeg selvfølgelig stå for forplejning og program, men ikke kunne dække anden udgift.
Jeg er helt klar over, at det er i sidste øjeblik før sommerferien. Men jeg håber alligevel med brevet her, at du er fristet af denne mulighed for kontakt med et forskningsmiljø – med den tilhørende mulighed for kompetenceudvidelse.

Guleroden er jo, at du her mødes med udvalgte matematikkolleger fra andre kommuner, selv bidrager til og bliver orienteret om foreløbige resultater af en særdeles relevant forskning for fortsat udvikling af (også andres) matematikundervisning.
Men seminaret skal også gøre det muligt at gå i dybden med udvalgte spørgsmål.
Du/I vil fx blive spurgt nærmere ud om din matematikundervisning ;-)
Det er muligt, jeg så derefter gerne vil vende tilbage til endnu et besøg i din matematik-klasse, som jeg formoder de fleste af jer skal undervise i 9. klasse.

- Hvis du er fristet, og gerne vil medvirke, beder jeg dig give mig et praj på mail, telefon eller brev. Jeg vedlægger en kuvert. Jeg regner med, at det så også er afklaret med din leder.
  Så vender jeg tilbage efter ferien med et konkret forslag til program for dagen.
- Hvis du af en eller anden grund ikke kan, vil eller må deltage – vil jeg selvfølgelig også meget gerne have det at vide.
- Jeg vedlægger også en kuvert.

På forhånd TAK og god sommerferie!
Med venlig hilsen

Arne Mogensen
VIA University College, Læreruddannelsen i Århus
Trøjborgvej 82, 8200 Århus N

Ønskes flere oplysninger kan man ringe tlf. 8743 0360 (privat) eller mail: armo@viauc.dk

Please X and return soon (i.e., preferably before the holidays).

Happy to participate in the research seminar on mathematics teaching November 7th 2009 ☐

Can not participate in the research seminar on mathematics teaching November 7th 2009 ☐

Reasons if so:

School: ____

Headmaster signature: ____________________________________

Mathematics teacher signature:
____

E-mail: ____

- -

Sæt venligst X og returner venligst snarest (dvs. helst inden ferien).

Deltager gerne i forskningsseminar om matematikundervisning 7/11 2009 ☐

Kan ikke deltage i forskningsseminar om matematikundervisning 7/11 2009 ☐

Evt. begrundelse:

Skole: ____

Skoleleders underskrift: ____________________________________

Matematiklærers underskrift: __________________________________

Evt. e-mail: ____

E: Invitation for peer sparring and video recordings 2 and 3

Teacher education in Aarhus, November 2009

Dear central region **headmaster**
Re. research on mathematics teaching

Last school year you were kind enough to let me visit a grade 8 class at your school and record video of the mathematics teacher. I have since visited 50 randomly selected schools in Denmark, and recently (November 7, 2009) held a Saturday seminar in Aarhus, with seven of the teachers from Jutland, including the mathematics teacher from your school.
I believe the teachers have appreciated this opportunity for contact with a research environment. It brought together randomly selected mathematics colleagues from 7 municipalities, all made valuable contributions and were briefed on preliminary the results of the research which seems highly relevant for their continued development as mathematics teachers. Everyone also received a DVD with their own video recording. I also had the opportunity to visit the teachers again, now teaching grade 9.

Following the success of the seminar, and indeed the project so far, I would like to invite some of them to pursue "cooperation" a little longer. I have already mentioned this possibility to them, but your support as a leader is obviously important.
The experience will be useful for the mathematics teacher team, and we may even develop a template for future systematic peer sparring. I would be pleased if your school will contribute.
The goal is to investigate (and possibly prove) that it is possible to support mathematics teachers in the development of mathematics teaching using academic points through peer coaching.
Specifically, I invite you to participate in:

- **Mathematics coaching via e-mails or meetings for three weeks, with a particular focus on three mathematics lessons spread across weeks 2 to 5, 2010.**
- **A third video recording on April 2010 of which the teacher will get a copy on DVD.**

- If you think the teacher would be able to help, I would be very grateful if you could pass on the attached letter to which I would appreciate an answer
- If for some reason your school cannot, will not or may not participate - I would also appreciate notification.
- I enclose an envelope for your convenience.

Thank you!
Sincerely,
Arne Mogensen
VIA University College, Teacher education in Aarhus
Trøjborgvej 82, 8200 Århus N
For more information please call telephone: 8743 0360 (private) or mail: armo@viauc.dk

UNIVERSITY COLLEGE

Roskilde Universitetscenter

Teacher education in Aarhus, November 2009

Dear Central region **math-colleague**

Thank you for your participation in the research on point-driven mathematics teaching.
I appreciated the first visit to your grade 8 class last school year, your participation in our seminar in Aarhus November 7 - and this year's visit to video record you again, now with grade 9.

I sense that you also consider it appropriate to examine how to possibly develop your (and others) mathematics teaching through a kind of peer network. The professional focus of you and your colleagues is so promising that I would like to invite some of you to continue "cooperation" a little longer.

The goal is still to consider whether it is possible to support mathematics teachers in mathematics teaching using academic points through a peer network.

There is not (yet) in Denmark any widespread tradition of peer cooperation on mathematics teaching objectives and content, including the academic points.
This is probably because it takes time and structure for you and your colleagues to develop mathematics professional peer communication. The experiences and opportunities we can identify with our work, perhaps offers a way forward because it can be implemented locally.

Specifically, I invite you to

- **Mathematics sparring through e-mails or meetings for 3 weeks, with a particular focus on 3 mathematics lessons during separate weeks between 2 to 5, 2010.**
- **A third video recording on April 2010, which you will get a copy of on DVD.**

- If you are tempted, and would like to help, I would be very grateful if you could give me an early hint by mail, telephone or letter. The easiest is to use the attached form. Please ensure that you have cleared your decision with your supervisor.
- I would also be grateful if you could inform me if for some reason you cannot, will or may participate.
- I enclose an envelope.

Thanks in advance!
Sincerely,

Arne Mogensen
VIA University College, Teacher education in Aarhus
Trøjborgvej 82, 8200 Århus N

For more information please call telephone: 8743 0360 (private) or mail: armo@viauc.dk

Teacher education in Aarhus, November 2009

For parents of grade 9

Regarding permission to shoot video

In the context of a research project on good
MATHEMATICS TEACHING I want to record one mathematics lesson on video in your child's class.
The video is part of a research project supported by the Danish Research Board and Roskilde University. The class is one of 50 randomly selected from schools throughout Denmark. You may remember that I made a similar request last year.

Although, like last time, my aim is to film the teacher, one cannot avoid the students coming into the frames. I therefore again ask for your permission. Names of students and teachers will not be published, but I would like to have the possibility to use the video in a teaching context, e.g. in teacher training and perhaps single images from the video in a book.

The recording will take place in one mathematics lesson later this school year. <u>If you do not want</u> your child to be present in this lesson, I ask that you pass on your concerns to the mathematics teacher as soon as possible using the tear off slip below.
<u>If you agree to the filming, there is no need to reply.</u>
<u>Once again thank you for your consideration.</u>

Sincerely,
Associate Professor Arne Mogensen,
VIA University College, Teacher education in Aarhus

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I do NOT want my child ________________________ to participate in the mathematics lesson, when video is being recorded.

____

Signature

E: Invitation til kollegial sparring (9. klasse) samt video-optagelse 2 og 3

VIA UNIVERSITY COLLEGE

Roskilde Universitetscenter

Læreruddannelsen i Århus, november 2009

Kære **skoleleder** i region Midtjylland
Vedr. forskningsprojekt om matematikundervisning

Sidste skoleår var du venlig at lade mig se indenfor i en 8. klasse på din skole, hvor jeg optog video med matematiklæreren. Jeg har siden været på 50 tilfældigt valgte skoler i Danmark, og for nylig (7. november 2009) også afholdt et lørdagsseminar i Århus med 7 af lærerne fra Midtjylland, bl.a. matematiklæreren fra din skole.

Jeg tror, lærerne har sat pris på denne mulighed for kontakt med et forskningsmiljø.
Her mødtes tilfældigt udvalgte matematikkolleger fra 7 kommuner, der alle bidrog til og blev orienteret om foreløbige resultater af en særdeles relevant forskning for fortsat udvikling af (også andres) matematikundervisning. Alle fik også en DVD med egen video-optagelse. Samtidigt fik jeg mulighed for igen at besøge lærerne, nu med undervisning i 9. klasse.
Det er netop sket og deres faglige fokus så lovende, at jeg gerne vil invitere nogle af dem til at fortsætte "samarbejdet" lidt endnu. Jeg har før luftet muligheden for dem, men din opbakning som leder er naturligvis vigtig.
Erfaringen være nyttig i det lokale matematik fagteam, og vi kan måske også her udvikle en skabelon for en fremtidig systematisk kollegial sparring. Det er flot, hvis skolen vil medvirke til det.
Sigtet er altså at undersøge (og evt. dokumentere), at det faktisk kan lade sig gøre at støtte matematiklærere i udvikling af matematikundervisning med faglige pointer gennem kollegial sparring.

Helt konkret inviterer jeg til

- **Matematik-faglig sparring via e-mails eller møder i 3 uger, men med særligt fokus på 3 matematiktimer placeret i hver sin uge blandt uge 2 – 5, 2010.**
- **En 3. video-optagelse i april 2010, hvor læreren selv får en kopi på DVD.**

- Hvis du mener, læreren bør have mulighed for at medvirke, beder jeg dig videregive det vedlagte andet brev, som jeg så meget håber på at få et svar på ved lejlighed.
- Hvis man af en eller anden grund ikke kan, vil eller må deltage – vil jeg selvfølgelig også meget gerne have det at vide.
- Jeg vedlægger en kuvert.

På forhånd TAK!
Med venlig hilsen
Arne Mogensen
VIA University College, Læreruddannelsen i Århus
Trøjborgvej 82, 8200 Århus N
Ønskes flere oplysninger kan man ringe tlf. 8743 0360 (privat) eller mail: armo@viauc.dk

VIA UNIVERSITY COLLEGE

Roskilde Universitetscenter

Læreruddannelsen i Århus, november 2009

Kære **matematik-kollega** i region Midtjylland

Tak for din deltagelse i forskningsprojektet om pointe-styret matematikundervisning.
Jeg har sat stor pris på det første besøg på din skole i 8. klasse sidste skoleår, vores seminar i Århus 7. november – og at jeg i år kunne videofilme dig igen, nu med 9. klasse.

Jeg fornemmer, at du også finder det relevant at undersøge, hvordan man evt. kan udvikle sin (og andres) matematikundervisning gennem en form for kollegialt netværk. Dit og kollegernes faglige fokus er så lovende, at jeg gerne vil invitere nogle af jer til at fortsætte "samarbejdet" lidt endnu.

Sigtet er stadig at undersøge, om det faktisk kan lade sig gøre at støtte matematiklærere i matematikundervisning med faglige pointer gennem et kollegialt netværk.

Der er (endnu) ikke i Danmark nogen udbredt tradition for kollegialt samarbejde om matematikundervisningens mål og indhold, herunder de vigtige faglige pointer.
Det skyldes formentlig både, at det kræver tid og struktur, hvis man kollegialt på skolen skal udvikle den matematik-faglige samtale. De erfaringer og muligheder, vi kan pege på med vores samarbejde, er måske netop den vej, der virker, fordi den kan sættes i værk lokalt.

Helt konkret inviterer jeg til

- **Matematik-faglig sparring via e-mails eller møder i 3 uger, men med særligt fokus på 3 matematiktimer placeret i hver sin uge blandt uge 2 - 5, 2010.**
- **En 3. video-optagelse i april 2010, hvor du igen selv får en kopi på DVD.**

- Hvis du er fristet, og gerne vil medvirke, beder jeg dig give mig snarligt et praj på mail, telefon eller brev. Det letteste er at bruge den vedlagte blanket.
- Jeg regner med, at det så også er afklaret med din leder.
- Hvis du af en eller anden grund ikke kan, vil eller må deltage – vil jeg selvfølgelig også meget gerne have det at vide.
- Jeg vedlægger en kuvert.

På forhånd TAK!
Med venlig hilsen

Arne Mogensen
VIA University College, Læreruddannelsen i Århus
Trøjborgvej 82, 8200 Århus N

Ønskes flere oplysninger kan man ringe tlf. 8743 0360 (privat) eller mail: armo@viauc.dk

INVITATION til

- **Matematik-faglig sparring via e-mails eller møder i 3 uger, men med særligt fokus på 3 matematiktimer placeret i hver sin uge blandt uge 2 - 5, 2010.**
- **En 3. video-optagelse i april 2010, hvor du igen selv får en kopi på DVD.**

Hvis du vil medvirke (TAK!) sender jeg en bekræftende mail med mere om det konkrete.

Arne Mogensen

Sæt venligst X og returner venligst snarest.

Deltager gerne i et fortsat samarbejde om matematikundervisning 2010 ☐
Sæt så også x ved 3 af ugerne herunder
Uge 2 .... ☐
Uge 3 .... ☐
Uge 4 .... ☐
Uge 5 .... ☐

Video-optagelse i april 2010
Vælg én af ugerne herunder, tilføj gerne dato og tid – ellers blot senere
Uge 14 .... ☐
Uge 15 .... ☐
Uge 16 .... ☐

Kan ikke deltage det fortsatte samarbejde om matematikundervisning 2010 ☐
Evt. begrundelse:

Skole: ____

Skoleleders underskrift: ______________________________

Matematiklærers underskrift: __________________________

Evt. e-mail: ______________________________________

Læreruddannelsen i Århus, november 2009

Til forældre i 9. klasse

Vedrørende tilladelse til at optage video

I forbindelse med et forskningsprojekt om god **MATEMATIK-UNDERVISNING** vil jeg gerne have mulighed for at optage én matematiktime på video i jeres barns klasse.
Videoen er led i et forskningsprojekt, der støttes af Forskningsstyrelsen og Roskilde Universitet. Matematiklæreren og klassen er én af 50 tilfældigt valgte på skoler i hele Danmark, der sidste år blev filmet på helt samme måde i én time.

Selv om tanken igen er at filme det, som læreren gør, kan man ikke undgå, at elever kommer med i billedet. Jeg skal derfor igen bede om jeres tilladelse.
Der bliver ikke offentliggjort navne på elever eller lærer, men denne gang vil jeg gerne have mulighed for at bruge videoen i undervisnings-sammenhæng, fx på læreruddannelsen og evt. enkeltbilleder fra videoen i en bogudgivelse.

Optagelse på skolen vil ske i én matematiktime senere i dette skoleår.
Hvis I ikke ønsker, jeres barn er med i denne time, beder jeg om besked snarest via blanketten nederst til matematiklæreren.
Hvis det er OK, skal I ikke foretage jer noget. På forhånd tak for velvilje.

Venlig hilsen
Lektor Arne Mogensen, VIA University College, Læreruddannelsen i Århus

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Jeg ønsker IKKE, at mit barn ________________________ deltager i den matematiktime, hvor der optages video.

____

Underskrift

F: Questionnaire to focus teachers

Dear colleague teaching grade 9
I hope that you AGAIN will help me answer a few questions.
Thanks in advance, Arne Mogensen

School:	Date:
Name:	Class: 9.a Other:
Did you select to take mathematics lessons in grade 9?	
Textbook system (if used) in grade 9:	
Page number(s) referred to this lesson (if so):	

Goal

The purpose of the lesson is described in the teaching guide of the system.	
YES:	NO:

I have a separate or special goal with my teaching in this lesson.	
YES:	NO:

If you had a written plan for this lesson, was it:		
Rather general?	Fairly detailed?	Very specific?

If possible, I would very much like to have a copy (you may use the addressed envelope).

	What were the most important things the students should learn? Express 1-2 POINTS (shortly)	**Or X here:**
In the lesson of today		No special goal
In previous lesson		I don't recall
In next lesson		To be decided

Content

Other material for all students? (if possible also enclosed in copy)	Publisher: Own:
Material to only some students? (if possible also enclosed in copy)	Publisher: Own:

Peer sparring

Did the invitation to brief Arne before on lesson content result in some new thoughts?	If YES, what kind:	NO:
Could you imagine peer sparring with colleagues at the school be more widespread?		

May I contact you once more, if I need more feedback?	YES:	NO:

F: Spørgeskema til fokus lærere (DANISH VERSION)

Kære kollega i 9. klasse
Jeg håber, du IGEN vil hjælpe mig med svar på et par spørgsmål.
På forhånd mange tak, Arne Mogensen

Skole:	Dato:	
Navn:	Klasse: 9.a	Andet:
Har du selv ønsket matematiktimer i 9. klasse?		
Evt. bogsystem i din 9. klasse:		
Evt. sidetal i dette bogsystem i denne lektion:		

Mål

Målet med lektionen passer med lærervejledning i et bog-system.	
JA:	NO:

Jeg har selv formuleret et andet eller et særligt mål med lektionen.	
JA:	NO:

Hvis du havde en skriftlig plan for denne lektion, var den så:		
Overordnet	nogenlunde detaljeret	meget detaljeret

Er det muligt, vil jeg meget gerne have en kopi (brug evt. svarkuvert).

	Hvad var det vigtigste, eleverne skulle lære? **Formulér 1-2 POINTER (kort):**	**Eller kryds af her:**
Dagens lektion		Ikke særlige pointer
Forrige lektion		Husker den ikke
Næste lektion		Ikke bestemt endnu

Indhold

Andet materiale for alle: (om muligt vedlagt i kopi)	Forlagsmateriale: Eget:
Materiale til blot nogle elever: (om muligt vedlagt i kopi)	Forlagsmateriale: Eget:

Kollegial sparring

Satte invitationen til at orientere Arne om lektionens indhold forinden nye tanker i gang?	Hvis JA, hvilken slags:	NO:
Kunne du forestille dig kollegial sparring med kolleger på skolen være mere udbredt?		

Må jeg kontakte dig nok en gang, hvis jeg får behov for mere feedback?	JA:	NO:

G: Introduction and agenda for lesson study at a Danish school

Lesson study in mathematics – now what's that?

Originally a Japanese tradition, peer sparring through *lesson study* is spreading via literature, networks and conferences from Asia to Europe and the USA. *Lesson study* is the name of a professional development process in which Japanese teachers systematically examine their own practices. The goal is to make their teaching more effective, and the core of the *lesson study* is a group of teachers collaborating on a small number of *study lessons*. Work on a *study lesson* will take place over several stages:

1. Investigation and preparation, where teachers develop a detailed plan for the study lesson together.
2. Implementation, where one teacher teaches a class in the study lesson while others observe.
3. Reflection and improvement, where the group meets to discuss their observations from the lesson.
4. Repeated implementation and reflection, when another teacher teaches another class based on the study lesson, while group members observe. Then the group will meet again and discuss their observations.

A (to Danish teachers very) detailed lesson plan supports the *lesson study* process for all colleagues in the group in several ways. It is a **teaching tool** because it is the screenplay for the lesson's activities. But it is also a **communication tool** because it tells others about the thinking of those teachers who have planned the lesson. And finally, it is an **observational tool** because it will identify the points that should be seen after the lesson and a place for observers to record and share what they have seen.

Study lessons are generally planned by groups of 4-6 teachers who primarily teach the same grade levels. Each group will usually perform 2-3 annual *lesson study* cycles, located appropriately in relation to important school events such as festivals and tests. Groups that work with a study lesson have a weekly meeting time, usually after lessons. This provides time for teachers to share their work across groups. Besides the teachers who have worked with the study lesson, other teachers at the school are keen to come and observe and discuss the study lessons.

The lesson-study format invites a peer, professional dialogue that can be directly translated into clear points and a perhaps better class discussion on the teaching. But it requires mutual respect and responsiveness to give room for not only the teacher but also students' own explanations on many levels. And it requires knowledge about different approaches to the same mathematical topic.

In a course with lesson study, the key phrase is collegiate, professional communication.
During the fall we will combine this work with my presentations on 1-2 mathematical topics, common to several grade levels. You will all be expected to study these presentations and exchange ideas. This way, I hope to meet the different expectations during this 24-hour course.

I look forward to the cooperation envisaged as described below.
Yours Arne Mogensen

Mathematics with points

In meetings with management, mathematics teachers and Arne Mogensen ("instructor") it has been decided to choose *the 8 mathematics competencies in Common Goals 2009* as the single overarching goal that will guide the work of the 4 + 4 study lessons, where each participant will see all 4 lessons. The management will divide the 18 participants into three groups covering a broad range of class levels, education and experience.
The project focuses on two mathematics topics followed throughout the school years. Among the ideas identified as important by the mathematics teachers are **arithmetic algorithms** and **spatial geometry**.

	Date & time	**Content**
1	Thursday 26/8, 13-16 (Week 34)	1. Presentation of selected *competencies* from grade 8 *Common Goals*, such as communication, representational, aid and tools competence (including IT) that can be developed through work with the academic subjects **arithmetic algorithms** and **spatial geometry** for grades 0-9. 2. Talk 1 lesson-study with Japanese film AND lesson plan!
2	Thursday 9/9, 13-16 (Week 36)	1. The three groups (of 6 participants) start joint planning. For each *study lesson* first a content sub goal is selected.

		E.g.: *Students should learn to multiply (or divide) 2-digit numbers by 1-digit.* 2. Each group connects the sub-goal with the overarching goal. E.g.: *Students should develop aids and tools competence and communication competence in working with multiplying 2-digit numbers by 1-digit.*
3	Thursday 23/9, 13-16 (Week 38)	1. Selecting classes for the first two ”study lessons". 2. All in group contribute to one detailed plan for the two selected study lessons. Bring your own ideas. 3. A lesson plan is produced that suits actual class and can be distributed to everyone on the weekend before the class by the two teachers who "are on" in week 40.
4	Group 1: (Week 39) Thursday 30/9, 10-15 Group 2: (Week 40) Wednesday 6/10, 10-15 Group 3: (Week 40) Thursday 7/10, 10-15	Implementation No. 1, where two teachers from each group teach each class in the study lesson, and the other group members observe. Subsequent, joint reflection and ideas for improvement.
5	Thursday 28/10, 13-16 (Week 43)	1. Talk 2 on lesson study and experiences. 2. Selecting classes for the next two ”study lessons". 3. All in group contribute to one improved AND detailed plan for the now selected study lessons. 4. A lesson plan is produced that suits actual class and can be distributed to everyone the weekend before class by the two teachers who "are on" in week 43.
6	Group 1: (W eek 45) Monday 8/11, 10-15 Group 2: (Week 45) Wednesday 10/11, 10-15 Group 3: (Week 45) Thursday 11/11, 10-15	Implementation No. 2, where two other teachers from each group teach each class in the study lesson, and the other group members observe. Subsequent, joint reflection and ideas for improvement.
7	Thursday 18/11, 13-16 (Week 46)	Evaluation of the progress & Can we keep the conversation running?

Please have the following available:

- Mathematics curriculum: *Common Goals 2009*. There will be (limited) texts for preparation.
- Possible mathematics books for own class.
- It is an assumption that there is access to internet and "smart board" in the seminar room.

Mathematics with points

The school management divides the participants into three groups mixed in terms of grade, education and experience. **In the course each group focuses on one (or at most two) mathematical topics**, which are followed throughout their school grade levels. In meetings with management, mathematics teachers and Arne Mogensen ("instructor") it is decided to choose **the 8 mathematics competencies in Common Goals 2009** as the overarching goal that will guide the work of each group:

Representation	Symbols and formulas	Communication	Aids and tools
Reasoning	Modeling	Problem handling	Thinking

Ideas for the 1-2 common ***topics*** are gathered NOW from among the mathematics teachers using this "wishlist ". There can be a max of 3 crosses - but preferably at least one!
Return to D. by Friday June 4th, 2010.

Mathematical topic	See e.g. **Level goals** p. 14-17 or **Curriculum** p. 20-28	Max 3 crosses
Work with numbers and algebra	Arithmetic methods (algorithms)	X
	Equations and functions	
	The coordinate system	
	Practical calculations (Applied mathematics)	(X)
Work with geometry	Area	
	Patterns	
	Constructions in the plane	
	Spatial geometry	X
Work with statistics And probability	Data collection and statistics	
	Games and simulation (chance)	

(The preferences were relayed to Arne 8/6, and the result marked by crosses above)

Names or initials: ___________________________ Class in 2010-11: ______

G: Introduktion og program for lektionsstudium på en dansk skole

Lektions-studier i matematik – hvad er nu dét?

En oprindeligt japansk tradition for kollegial sparring gennem lektions-studier (*lesson-study)* breder sig for tiden gennem litteratur, netværk og konferencer fra Asien til Europa og USA. *Lesson-study* er navnet på en professionel udviklingsproces, hvor japanske lærere systematisk undersøger egen praksis. Formålet er at gøre deres undervisning endnu mere effektiv, og det centrale i *lesson-study* er en gruppe læreres samarbejde om et lille antal *studie-lektioner*. Arbejdet med en *studie-lektion* sker så i nogle faser:

1. Undersøgelse og forberedelse, hvor lærere sammen udarbejder en detaljeret plan for studie-lektionen.
2. Implementering, hvor én lærer underviser en klasse i studie-lektionen, mens andre observerer.
3. Refleksion og forbedring, hvor gruppen mødes for at diskutere deres observationer fra lektionen.
4. Gentagen implementering og refleksion, hvor en anden underviser en anden klasse i studie-lektionen, mens gruppe-medlemmer observerer. Herefter mødes gruppen igen og diskuterer deres observationer.


En (for danske lærere meget) detaljeret lektionsplan støtter *lesson-study* processen for alle kolleger i gruppen på flere måder. Den er et ***undervisnings-redskab***, fordi den er drejebog for lektionens aktiviteter. Men den er også et ***kommunikations-redskab***, fordi den fortæller andre om tænkningen hos de lærere, der har planlagt lektionen. Og endelig er den et ***observations-redskab***, fordi den udpeger de pointer, der skal ses efter i lektionen, og et sted for observatørerne at registrere og dele det sete.

Studie-lektioner planlægges normalt af grupper på 4-6 lærere, som fortrinsvis underviser på samme (klasse)trin. Hver gruppe vil normalt udføre 2-3 årlige *lesson-study* forløb, placeret hensigtsmæssigt i forhold til vigtige skole-arrangementer som fester og prøver. Grupper, der arbejder med en studie-lektion har en ugentlig mødetid, normalt efter skoletid. Der er afsat tid for lærerne til at dele deres arbejde på tværs af grupper. Udover de lærere, der har arbejdet med studie-lektionen, vil andre lærere på skolen gøre alt for at komme og observere og at diskutere studie-lektionerne.

Lesson-study formatet inviterer til en kollegial, faglig dialog, der umiddelbart kan omsættes i tydelige pointer og en måske bedre klassesamtale i undervisning. Men det kræver gensidig respekt og lydhørhed at give plads til ikke blot lærerens, men også elevernes egne forklaringer på mange niveauer. Og det kræver viden om forskellige tilgange til det samme matematiske emne.

I et kursusforløb med lektions-studier er det centrale altså den kollegiale, faglige samtale. På skolen vil vi i efteråret kombinere arbejdet med mine oplæg om 1-2 faglige emner, der skal være fælles på flere klassetrin, og som alle så må studere og udveksle idéer til. På den måde håber jeg, vi kan imødekomme de forskellige ønsker, der er til et kursusforløb på 24 timer.

Jeg glæder mig til samarbejdet, der er planlagt som beskrevet på næste side.
Med venlig hilsen Arne Mogensen

Matematik med pointer

På møder med ledelse, matematiklærere og Arne Mogensen (”instruktør”) er det besluttet at vælge *de 8 matematiske kompetencer i Fælles Mål 2009* som ét fælles overordnet mål, der skal lede arbejdet i 4 + 4 studie-lektioner, hvor hver deltager kommer til at se de 4. Ledelsen fordeler de 18 deltagere i tre grupper, der dækker bredt mht. klassetrin, uddannelse og erfaring.
I forløbet fokuseres på højest to matematik-faglige emner, der følges gennem hele skoleforløbet.
Blandt idéerne hertil har matematiklærerne peget på **regnemetoder** (algoritmer**) og rumgeometri**.

	Dato & tid	**Indhold**
1	Torsdag 26/8, 13-16 (Uge 34)	1. Præsentation af udvalgte *kompetencer* blandt de 8 i *Fælles Mål*, fx kommunikations-, repræsentations- og hjælpemiddel-kompetence (bl.a. IT), der kan udvikles gennem arbejdet med de *faglige emner* **regnemetoder** (algoritmer) og **rumgeometri** i 0.-9. klasse. 2. Oplæg 1 om lesson-study med japansk film OG lektionsplan!

2	Torsdag 9/9, 13-16 (Uge 36)	1. De tre grupper (à 6 deltagere) starter en fælles planlægning. Til hver *studie-lektion* vælges først et indholdsmæssigt delmål. Fx: *Eleverne skal lære at gange (eller dele) 2-cifrede tal med 1-cifrede.* 2. Delmålet forbindes af hver gruppe med det overordnede. Fx: *"Eleverne skal udvikle hjælpemiddel-kompetence og kommunikations-kompetence i arbejdet med at gange 2-cifrede tal med 1-cifrede.*
3	Torsdag 23/9, 13-16 (Uge 38)	1. Valg af klasser til de to første "studie-lektioner". 2. Alle bidrager i gruppen til én <u>detaljeret</u> plan til de to valgte studie-lektioner. <u>Medbring</u> egne idéer her. 3. En lektionsplan sammenskrives, så den er tilpasset aktuel klasse og kan udsendes til alle senest weekenden før af de to lærere, der "er på" i uge 40.
4	Gruppe 1: (Uge 39) Torsdag 30/9, 10-15 Gruppe 2: (Uge 40) Onsdag 6/10, 10-15 Gruppe 3: (Uge 40) Torsdag 7/10, 10-15	Implementering nr. 1, hvor to lærere fra hver gruppe underviser hver sin klasse i studie-lektionen, mens de andre gruppe-medlemmer observerer. Efterfølgende fælles refleksion og idéer til forbedring.
5	Torsdag 28/10, 13-16 (Uge 43)	1. Oplæg 2 om lesson-study og erfaringer. 2. Valg af klasser til de to næste "studie-lektioner". 3. Alle bidrager i gruppen til én forbedret OG <u>detaljeret</u> plan til de to nu valgte studie-lektioner. 4. En lektions-plan sammenskrives, så den er tilpasset aktuel klasse og kan udsendes til alle senest weekenden før af de to lærere, der "er på" i uge 43.
6	Gruppe 1: (Uge 45) Mandag 8/11, 10-15 Gruppe 2: (Uge 45) Onsdag 10/11, 10-15 Gruppe 3: (Uge 45) Torsdag 11/11, 10-15	Implementering nr. 2, hvor to andre lærere fra hver gruppe underviser hver sin klasse i studie-lektionen, mens de andre gruppe-medlemmer observerer. Efterfølgende fælles refleksion og idéer til forbedring.
7	Torsdag 18/11, 13-16 (Uge 46)	Evaluering af forløbet & Kan vi holde liv i samtalen?

Til hver gang medbringes i det mindste:

- Faghæfte 12 for matematik: *Fælles Mål 2009*. Der vil være (begrænsede) tekster til forberedelse.
- Evt. <u>matematikbøger</u> til egen klasse
- Det er en forudsætning, at der er adgang til net og "smartboard" i et kursuslokale.

Matematik med pointer

Ledelsen fordeler deltagerne i tre grupper, der dækker bredt mht. klassetrin, uddannelse og erfaring. ***I forløbet fokuserer hver gruppe på helst ét (eller højest to) matematik-faglige emner***, der til gengæld følges gennem hele skoleforløbet.

På møder med ledelse, matematiklærere og Arne Mogensen ("instruktøren") er det besluttet at vælge ***de 8 matematiske kompetencer i Fælles Mål 2009*** som dét fælles overordnede mål, der skal lede arbejdet i hver gruppe:

Repræsentation	Symboler og formler	Kommunikation	Hjælpemidler
Ræsonnement	Modellering	Problembehandling	Tankegang

Idéer til de 1-2 fælles ***emner*** indsamles NU blandt matematiklærerne på denne "ønskeseddel". Der kan max sættes 3 krydser – men helst mindst ét!
Afleveres til Dorte senest fredag 4. juni 2010.

Matematisk emne	Se fx **Trinmål** s. 14-17 eller **Læseplan** s. 20-28	Max 3 krydser
Arbejdet med tal og algebra	Regnemetoder (algoritmer)	X
	Ligninger og funktioner	
	Koordinatsystemet	
	Praktisk regning (Matematik i anvendelse)	(X)
Arbejdet med geometri	Areal	
	Mønstre	
	Konstruktioner i planen	
	Rumgeometri	X
Arbejdet med statistik og sandsynlighed	Dataindsamling og statistik	
	Spil og simulering (chance)	

(Ønskerne blev formidlet til Arne 8/6, og resultatet vist med krydser herover)

Navn eller initialer: ___________________________ Klasse i 2010-11: ______

H: Two lesson plans from *lesson study*

Time Schedule for Mathematics in 1.b, 11 November 2010
Introduction to minus
Symbolism is the mathematical competence we prioritize.
Goal: Students will be acquainted with the symbol minus and get some practical experience with its meaning and use.

Time Start: (Max. 15 min.)	Teacher writes "-" on the blackboard Entrance Question: "What is it?" If the children do not suggest that it is a minus, T writes numbers on both sides of the sign. "What is it now?" **Point: "It's a minus!"** "How do you know that?" "What do we use it for?" **Point: "This means that you subtract or remove something!"** "If you're buying something in a shop - you may then need to subtract?" "Do you know any game where you must subtract?"
Practical exercises: (2 times 12 min.)	Half of the children are provided with a bag of 10 plastic shapes. Now they must find a mate without a bag. The partner takes a handful of shapes from the bag, and displays them. The child with the bag must now find out how many are left. (You are allowed to feel, but not to look inside). The partner gets the bag and now finds a new mate. The children stand in pairs at a number line at the tiles. One stands at 10. The other throws a cube, and moves his partner the number shown on the dice down the number line. The children change places and try again.
Time of completion: (Max. 6 min.)	"Today you learned what minus means. The next week you will learn more ways to subtract!"

Lesson Plan 8b
Mathematics (A historic day) Thursday 11-11 (unfortunately not at.11:00)

- Domain Algebra:
- Goal: Students must be able to solve simple and basic algebraic problems with +, - and *
- Preconditions 8b is a class, which is arguably more humanistic-oriented than the natural sciences. Their preparation and work ethic leaves much to be desired.

Point(s)

- Algebraic calculations with numbers and letters, applications. Short on number representations and formulas.
- Parts in a sequence and their order. The communicative law: a + b = b + a.
- Product, factors and order of factors. The communicative law: a * b = b * a.

Competencies
Representation competence

- Ability to understand and avail themselves of various algebraic symbols and objects.
- 2a as 2 * a.

- A rectangle area by terms such as length 2a, width 3a. Area 6ab.

Problem handling competence

- Students must propose different ways of solving and formulating the mathematical problems using different algebraic expressions.

T writes: Algebra? on blackboard or Smartboard.
Q: What is meant by the word concept algebra?
Point: Algebra is calculations with numbers and letters. Number representations.

Conversation about the concept of algebra and hopefully known applications.
Point: We have worked with algebra when we used different formulas.

T: How do you think our everyday life would have been if you did not know about algebra?
-- -- --------
T writes: 1) 8 + 3 2) 8 + 3 + 6 3) 8 + 8 + 8 +8 (treated first)
4) 2a + 6b 5) 2a + 6b + 3a (discussed later)
Q: What do you call this?
Q: What do the terms consist of?
1) 3 + 8 2) 3 + 6 + 8 3) 6b + 2a
Q: Does this make a difference?
Point: These algebraic expressions consist of parts. Such algebraic expressions are called sums.

Q: Can we say anything about the nodes sequence?
Point: The order of parts in a sum is unimportant.
Point: Generally it is a rule that **a + b = b + a (the commutative law)**

T: If we write a, b, n or x, what are these letters representing?
T: Is there anything in front of these letters?
T writes example: 2*a = 2a. Is this true or false?
T: What do you call the number 2 in relation to the letter a?
Point: The * sign is always between coefficients and characters in such cases.

T writes: 1) 2*3 2) 2a*3b
T: What do you call these expressions?
T: What are they composed of?
T: What do you call numbers that represent a multiplication problem?
T: Can we do something about them?
T writes: 3) 3*2 4) 3b*2a
L: Has this changed anything?
L: Can we conclude that the same rule applies as for +?
L: How should the rule be formulated in terms of multiplication?
Point: The order of factors is unimportant.
Point: The commutative law also applies to multiplication. So a*b = b*a

T: Does this rule perhaps also apply to the last 2 arithmetical calculation types: minus and division?
Conclusion: Repetition and emphasis of the points.

Thus written but no way to guarantee the success

H: To lektionsplaner fra *lesson study* (DANISH VERSION)

Timeplan for matematik i 1.b, 11. november 2010
Introduktion til minus
Symbolisme er den matematiske kompetence som vi vægter.
Mål: Børnene skal gøres bekendt med symbolet minus, og de skal have nogle praktiske oplevelser med hvad det betyder, og hvordan det bruges.

Timestart: (max. 15 min.)	Læreren skriver "-" på tavlen Indgangsspørgsmål: "Hvad er det?" Hvis børnene ikke foreslår at det er et minus, skriver L tal på begge sider af tegnet. "Hvad er det nu?" **Pointe: "Det er et minus!"** "Hvor kender I det fra?" "Hvad bruger man det til?" **Pointe: "Det betyder at man trækker fra eller fjerner noget!"** "Hvis du skal købe noget i en butik – kan du så have brug for at trække fra?" "Kender I nogen spil, hvor man skal trække fra?"
Praktiske øvelser: (2 gange 12 min.)	Halvdelen af børnene udstyres med en pose med 10 plasticfigurer. Nu skal de finde en makker uden en pose. Makkeren tager en håndfuld af figurerne op af posen, og viser dem. Barnet med posen skal nu finde ud af, hvor mange der er tilbage. (Man må gerne mærke, men ikke kigge). Makkeren får posen og finder derefter en ny makker. Børnene stiller sig to og to ved en tallinje på fliserne. Den ene stiller sig ved 10. Den anden slår med en terning. Og flytter makkeren ligeså langt ned af ned ad tallinjen, som terningen viser. Børnene bytter plads og prøver igen.
Timeafslutning: (max. 6 min.)	"I dag har I lært hvad minus betyder. De næste uger skal I lære flere forskellige måder, man kan trække fra!"

Lektionsplan 8b
Matematik (hin historiske dag) torsdag d. 11-11 (desværre ikke kl. 11.00)

- ❖ Overordnet emne Algebra:
- ❖ Delmål: Eleverne skal kunne løse enkle og basale algebraiske opgaver med +, - og *
- ❖ Forudsætninger 8b er en klasse som nok er mere humanistisk orienteret end naturfagligt. Forberedelsen og arbejdsmoralen lader meget tilbage at ønske.

Pointe(r)

- Algebra regning med tal og bogstaver, anvendelsesområder. Kort Talrepræsentationer og formler.
- Led og leddenes orden. Den kommunikative lov: a + b = b + a.
- Produkt, faktorer og faktorernes orden. Den kommunikative lov: a * b = b * a.

Kompetencer
Repræsentationskompetence

- Færdighed i at kunne forstå og betjene sig af forskellige algebraiske symboler og objekter.
- 2a som 2*a.

- Et rektangels areal udtrykt som f. eks. Længde 2a, bredde 3a. Areal 6ab.

Problembehandlingskompetence

- Eleverne skal ud fra forskellige algebraiske udtryk formulere og foreslå behandlingen af de matematiske problemstillinger.

Læreren skriver: Algebra? på tavle eller smart board.
Spørgsmål: Hvad forstår i ved ordet begrebet algebra?

Pointe: Algebra er regning med tal og bogstaver. Talrepræsentationer

Samtale om begrebet algebra og forhåbentlig kendte anvendelsesområder.

Pointe: Vi har arbejdet med algebra når vi anvender forskellige former for formler

L: Hvordan tror I vores hverdag havde set ud, hvis man ikke kendte eller kender til algebra?

L skriver: 1) 8 + 3 2) 8 + 3 + 6 3) 8 + 8+ 8+8 (behandles først)
4) 2a + 6b 5) 2a + 6b + 3a (behandles efterfølgende)
Sp.: Hvad kalder man dette?
Sp.: Hvad består udtrykkene af?
1) 3 + 8 2) 3 + 6 + 8 3) 6b+2a
Sp.: Gør dette en forskel?

Pointe: Disse algebraiske udtryk består af led.
Sådanne algebraiske udtryk kaldes for sum

Sp. Kan vi sige noget om leddenes rækkefølge?

Pointe: Leddenes orden i en sum er ligegyldig,

Pointe: Der gælder åbent en regel om, at **a + b = b + a (den kommutative lov)**

L: Hvis vi skriver a, b, n eller x, hvad repræsenterer disse bogstaver?
L: Står der noget foran disse bogstaver?
L skriver: 2*a = 2a. Er dette sandt eller falsk?
L: Hvad kalder man tallet 2 i forhold til bogstavet a?

Pointe: Mellem koefficienterne og bogstaverne er der altid i sådanne tilfælde et *tegn.

L skriver: 1) 2*3 2) 2a*3b
L: Hvad kalder man disse udtryk?
L: Hvad består de af?
L: Hvad kalder man tal, der repræsenterer et multiplikationsstykke?
L: Kan man gøre noget ved dem?
L skriver: 3) 3*2 4) 3b*2a
L: Har dette ændret noget?
L: Kan vi heraf slutte at der gælder den samme regel som for +?
L: Hvordan skal reglen så formuleres for multiplikation?

Pointe: Faktorernes orden er ligegyldig.

Pointe: Den kommutative lov gælder også for multiplikation. Altså **a*b =b*a**

L. Gælder denne regel mon også for de 2 sidste regningsarter minus og division?
Afslutning: Repetition og understregning af pointerne.

Således skrevet men ingenlunde garanti for gennemførelsen.