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Using History as a 'Goal' in Mathematics Education

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This Ph.D. dissertation is an analytical and empirical study of using history of mathematics in mathematics education.

The analytical part consists in proposing two categorizations based on a literature survey, one for the arguments of using history (history as a tool and history as a goal) and one for the approaches to doing so (the illumination, the modules, and the history-based approaches), and then analyzing the interrelations between these 'whys' and 'hows' of using history in order to see which combinations appear the most favorable if one wishes to realize a certain why.

Based on this, a modules approach is chosen to fulfill the purpose of using history as a goal in the new Danish upper secondary mathematics program. Two historical modules are designed and implemented in a particular upper secondary class, the first module in the students' second year and the second in their third and final year. The purpose of the empirical study is to see whether students at upper secondary level are (1) capable at engaging in meta-issue discussions and reflections of mathematics and its history, (2) if these discussions and reflections in any way are anchored in the taught and learned subject matter (in-issues) of the modules, and (3) if such modules in any way may give rise to changes in students' beliefs about mathematics (as a discipline) or the development of new beliefs.

Based on videos of the implementations, students' essays, mathematical exercises, questionnaires, and followup interviews, the conditions on and ways in which the students are able to carry out and engage in meta-issue discussions and reflections are analyzed and discussed and so are the levels of anchoring of these in the related in-issues. In particular, four different levels of the students' discussions about meta-issues are identified: the non-anchored, anchored comments, anchored arguments, and anchored discussions. It is found that modules like the ones designed in the present study may cause some changes in students' views of mathematics on a content specific level as well as in the way the students hold their beliefs. In particular it is found that the students' beliefs seem to grow in consistency and that the students' desire to justify and exemplify their beliefs increases over the one year period of the study.

Finally, the findings and the performed data analyses are used to pose and answer a number of additional and relevant questions as well as to pose further questions which may not be answered based on the performed investigations.

Uffe Thomas Jankvist, 2009

For Emanuel and Elisabeth

Resumé

Nærværende ph.d.-afhandling er et analytisk og empirisk studie af brugen af matematikhistorie i matematikundervisningen.

Den analytiske del består i fremlæggelsen af to litteraturbaserede kategoriseringer, én for argumenterne for at bruge historie (matematikhistorie som et værktøj og matematikhistorie som mål) og én for tilgangene til at bruge historie (illuminations-, modul-, og historiebaserede tilgange). De indbyrdes forhold mellem kategorierne analyseres dernæst for at afdække hvilke kombinationer der synes mest favorable, hvis man har et bestemt formål med at bruge historie for øje.

Baseret herpå vælges en modultilgang til at indfri den ny gymnasiale bekendtgørelses krav om matematikhistorie som mål. To matematikhistoriske moduler designes og implementeres i en og samme gymnasieklasse, første modul i 2g og andet modul i 3g. Formålet med det empiriske studie er at undersøge om gymnasieelever er (1) i stand til at diskutere og reflektere over meta-perspektiver af matematik og dets historie, (2) om disse diskussioner og refleksioner på nogen måde er forankret i den matematik som eleverne er blevet undervist i og har lært som del af modulerne og (3) om sådanne moduler kan give anledning til ændringer i elevers opfattelser, eller nye opfattelser, af hvad matematik som fag er.

Med baggrund i videooptagelser fra implementeringerne, elevernes essay- og matematikopgaver, spørgeskemaer og opfølgende interviews diskuteres og analyseres betingelserne for samt måderne hvorpå eleverne er i stand til at føre og gøre sig meta-perspektiverende diskussioner og refleksioner, såvel som niveauerne hvorpå disse er forankret i modulernes matematik. I særdeleshed identificeres fire forskellige niveauer af elevernes diskussioner: de ikke-forankrede, forankrede kommentarer, forankrede argumenter og forankrede diskussioner. Det konkluderes at moduler som de her designede og implementerede kan afstedkomme ændringer i elevernes matematikopfattelser og -syn, både på et indholdsmæssigt niveau såvel som i måden hvorpå eleverne besidder deres opfattelser. Specielt synes sammenhængen i elevernes opfattelser at vokse og elevernes behov for at begrunde og eksemplificere deres opfattelser øgedes også i løbet den 1 års periode hvorigennem de blev fulgt.

Resultaterne og de udførte analyser af data bruges dernæst til at stille og besvare en række yderligere spørgsmål af relevans for studiet samt stille et antal spørgsmål som ikke kan besvares på baggrund af de indsamlede data og den udførte analyse.

Preface and Acknowledgements

This Ph.D. dissertation is the final product of government funded studies carried out at Roskilde University within the Danish three-year Ph.D. program, and the defence of it on August 28th, 2009 concludes a study begun on April 15th, 2006.

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1 Setting the Scene

In the last three or four decades there has been a movement towards an inclusion of more humanistic elements in the teaching of mathematics. This has been the case for Denmark, in particular for the Danish upper secondary school, as well as internationally. The various ‘humanistic’ elements embrace, among others, cultural, sociological, philosophical, application oriented, and historical perspectives on mathematics as an educational discipline. The reasons for including such elements in mathematics education concern, for instance, the pedagogical, cognitive, affective and motivational sides of teaching and learning mathematics as well as those of more general education, together with wishes of breaking the isolation of subjects and promoting interdisciplinary and cross-curricular activities. This dissertation concerns the historical element.

In this introductory chapter I shall attempt to place the dissertation in both an international setting and a national one, which is that of the Danish upper secondary school. In doing so, I shall raise questions which appear to be unanswered, i.e. questions concerning the use of history of mathematics in the Danish upper secondary school as well as questions in the field of using history in mathematics education in general. In other words, I shall discuss the present state of the field. Based on the setting of the national and international scene both, I shall outline the contribution of my own empirical research on the use of history of mathematics in the Danish upper secondary school. Towards the end of the chapter, I shall describe my own academic background and my way into this line of research. Lastly, an overview of the dissertation as well as the publications related to this will be provided. But first the setting of the international scene.

1.1 The Field of History in Mathematics Education

When describing the field of using history in mathematics education, including its present state, the sociological setting with conferences, meetings, etc. as well as the field’s object of study must be described. I begin with the sociological setting.

The Academic Fora

History of mathematics in the teaching and learning of mathematics is an area which has attracted an increasing amount of interest within didactics of mathematics. This has given rise to several publications, newsletters, conferences more or less dedicated to this topic, and working groups at more general conferences on mathematics education. *The International Study Group on the Relations between the History and Pedagogy of Mathematics* (HPM) can be traced back to the second *International Congress on Mathematical Education* (ICME) in 1972, when it began as a working group. At the

third ICME of 1976, HPM was set up as an affiliated study group to the *International Commission on Mathematical Instruction* (ICMI) together with the presently more widely known *International Group for the Psychology of Mathematics Education* (PME) (see Fasanelli and Fauvel, 2007). Every fourth year HPM holds a satellite conference to the ICME, and three times a year HPM publishes a newsletter. At the ICME itself, there is a *Topic Study Group* (TSG) on the role of history of mathematics in mathematics education. Another conference dedicated to the history of mathematics and mathematics education is the *European Summer University on the History and Epistemology in Mathematics Education* (ESU), which is a more recent initiative taken by the French Mathematics Education community (IREM) in 1993 (see Barbin et al., 2007). ESU has until now been held every third year. Also the *International Colloquium on the Didactics of Mathematics* (ICDM), which is held in Crete every second year, usually has history in mathematics education as one of its themes. A new initiative, which I co-established, is the working group on *The Role of History of Mathematics in Mathematics Education: Theory and Research*, a group set up for the 6th *Congress of the European Society for Research in Mathematics Education* (CERME) in Lyon 2009. Needless to say, all these conferences, working groups, etc. all produce proceedings in one form or another, adding substantially to the amount of papers on history in mathematics education. The proceedings from the combined conference of HPM2004 and ESU4 held in Uppsala as a satellite meeting to ICME10, for instance, includes a total of 78 papers. But also in more general journals on mathematics education it seems as if the involvement of history is becoming increasingly prominent, especially in journals such as *Educational Studies in Mathematics* (ESM) and *For the Learning of Mathematics* (FLM) of which special issues on history in mathematics education have appeared in 2007 and 1991, respectively. A journal such as *Zentralblatt für Didaktik der Mathematik* (ZDM)¹ has had several papers on history in past times (Gulikers and Blom, 2001), although the number has decreased during the last decade (Jankvist, 2009a). In 2004 the *Mediterranean Journal for Research in Mathematics Education* (MJRME) published a double special issue on the role of history in mathematics education consisting of the papers from the TSG17 at ICME10. But also entire books on the subject are available. Examples are: Fauvel (1990); Swetz et al. (1995); Jahnke et al. (1996); Calinger (1996); Katz (2000);² and most importantly Fauvel and van Maanen (2000), the tenth ICMI Study on history in mathematics education. To the best of my knowledge, this ICMI Study is to date the most comprehensive sample on the topic. I shall return to this book in chapter 2, when I review and discuss the available literature on the use of history in mathematics education.

Present State of the Field

But what then is the present state of the field? As a part of my postgraduate studies, I had the opportunity to go abroad to conferences and meetings and also to visit and spend time with experts in the field. When abroad, I seized my chance to interview some of these scholars and, among other things, ask them about their view on the present

¹ Now, *ZDM Mathematics Education*.

² Calinger (1996) and Katz (2000) are, in fact, proceedings from HPM1992/ICME7 and HPM1996/ICME8, respectively, which have been published by the Mathematical Association of America as separate books in the series of MAA notes.

state of the field. The interviewed researchers were all authors of the ICMI Study and are deeply involved in the HPM society, so their insightful opinions on where and how to direct further research on history in mathematics education draw a perspective picture of the present state of affairs. (All quotations below are brought with their acceptance.)

During June 2007 I attended the summer school in Iceland, arranged by the *Nordic Graduate School of Mathematics* (NoGSME), where I was fortunate enough to interview Abraham Arcavi, the co-author of chapter 7 in the ICMI Study (Tzanakis and Arcavi, 2000). Arcavi revealed:

The community of HPM has been successful in at least two fronts: it called the attention to the potential of history of mathematics in mathematics education and it also provided a lively ‘home’ to learn from each other for all the professions (teachers, mathematics educators, mathematicians, and historians) who work with history. However, HPM still needs much more empirical research on teaching and learning related to history than it is the case now, and there is no lack of research questions to pursue. This avenue is important in order to strengthen HPM both internally and externally. Internally, research, as I envision it, would provide insights which confirm, extend or challenge some of our assumptions and proposals, it may reveal directions not yet pursued and it would certainly sharpen our own views and future plans. Externally, research can be a way to reach out and communicate with other communities within mathematics education like PME, CERME, and others and would open opportunities for its themes to appear more in journals like *ESM*, *JRME* [*Journal for Research in Mathematics Education*], *JMB* [*Journal for Mathematical Behavior*] and many others. Pursuing such opening of the current ‘borders’ will give history a wider stage and will be instrumental in attracting more people. Probably, HPM should aim at working in a similar way than other ‘thematic’ communities already do (such as technology in mathematics education, modeling, and the like) – they nurture inner meetings and discussions, but at the same time they pursue a strong presence in general conferences (plenaries, working sessions, discussion groups) and publish in general journals. In my opinion, research is the main way to pursue a wider and visible presence which would make HPM stronger and ever growing. (Arcavi, 2007)

Arcavi’s call for more empirical research studies in the field of using history in mathematics education may be seen as a consequence of some of the critiques of the available literature. In 2001, Iris van Gulik-Gulikers and Klaske Blom provided a large systematic survey listing “the recent literature on the use and value of history in mathematics education” with a special emphasis on geometry (Gulikers and Blom, 2001). Based on this survey Gulikers and Blom noted:

Most publications are anecdotic and tell the story of one specific teacher, whereas it is unclear whether and how the (generally positive) experiences can be transferred to other teachers, classes and types of schools. [...] The amount of general articles that contribute to the debate outnumbers the practical essays which contain suggestions for resources or lessons. [...] [A] gap exists between historians, writing ‘general’ articles, and teachers, writing ‘practical’ articles. Most of the essays lack a legitimation of the ideas and suggestions. For example, the following questions have hardly

been answered: What makes one think that the use of history deepens the mathematical understanding? Is it really motivating to stress the human aspect of mathematics or is it the enthusiastic teacher who motivates his class? Has any research been done to confirm these previous thoughts? Is there any psychological theory to confirm it? And how do people justify their choice of resources? (Gulikers and Blom, 2001, p. 223, 241-242)

Gulikers and Blom (2001) end by stating that they intend “to design a method for evaluating the *effectiveness* of mathematics teaching which uses classroom material in which history is an integral part” (*italics added*). In the case of using history of geometry, van Gulik-Gulikers’ did this in her own dissertation (van Gulik-Gulikers, 2005) by carrying out a large quantitative, empirical study in upper secondary and pre-university education. She primarily investigated the conceptual and the motivational arguments for using history – some of the “assumptions and proposals” which Arcavi also refer to above. van Gulik-Gulikers’ supervisor was Jan van Maanen, editor of the ICMI Study, former chair of HPM (1996-2000), and since the director of the Freudenthal Institute in the Netherlands. When visiting him in March 2007, I asked him about the results of van Gulik-Gulikers’ research, the present state of the field, and in what direction it should be heading now:

We need more studies like the one Iris van Gulik-Gulikers did. Maybe that should be published in a more international source, because the fundamental article about the teaching of geometry was in *Educational Studies in Mathematics*, but all the observational, quantitative material is not published internationally. It would be important to have that. It would be good to have some studies about the *effectiveness* of integrating historical elements in maths teaching, and to study the influence of the teacher in that, for example. We don’t know about that. We have no information about the teacher conduct and how classes react – there is no clear information about that at the moment. [...] Maybe an important thing is about creating better facilities for teachers, publishing a source book or something like that. A good source book with texts which pupils in school can use to read Euler, to read Cantor, to read Descartes, maybe, and there are other accessible authors, after some editorial work, that is. That would be a good thing and useful for teachers. (van Maanen, 2007, *italics added*)

Man-Keung Siu, author of chapter 8 of the ICMI Study (Siu, 2000b), and Constantinos Tzanakis, co-author of chapter 7 of the ICMI Study (Tzanakis and Arcavi, 2000) and former chair of HPM (2004-2008), concluded in the evaluation of the TSG17 on history at the ICME10 that “it became clear that enough has been said on a ‘propagandistic’ level, that rhetoric has served its purpose”, and hence argue that what is needed now are empirical investigations on the *effectiveness* of using history in the learning and teaching of mathematics (Siu and Tzanakis, 2004, p. 3). In his paper in the revised proceedings from the HPM2004&ESU4, Siu (2007, p. 269) mentions that he, at the time of the conference, was only aware of a total of five such empirical studies evaluating the effectiveness of the use of history in mathematics education within the English literature.³ Now, of course, Siu’s list is not meant to be a comprehensive one and, in

³ McBride and Rollins (1977); Fraser and Koop (1978); Philippou and Christou (1998); Gulikers and Blom (2001); Lit et al. (2001).

fact, more empirical studies on the effectiveness of using history may be found in the selfsame proceedings. Out of the total of 78 papers in these proceedings, about ten percent are either clear-cut or somewhat empirical studies, though not all concerning the effectiveness (Jankvist, 2007b, p. 84).⁴ This relatively high percentage may, perhaps, indicate that the field of using history in mathematics education is beginning to slowly direct itself towards conducting more empirical research studies. When scanning through the last ten years (1998-2007) of *ESM*, *FLM*, *ZDM*, and *JRME*, I was able to identify 27 papers on the use of history in mathematics education (Jankvist, 2009a). Out of these 27, ten, all in the *ESM*, can be regarded as empirical research studies in some sense, four of these in the 2007 special issue.⁵

Based on my knowledge of the available literature on using history of mathematics in mathematics education – a more detailed survey of the literature will be presented in chapter 2 – I’d say it is rich on advocating arguments as to why to include the history, thoughts on how to do it, ideas on what elements of the history to include, and on what levels of education to do this. But when it comes to testing these arguments, thoughts, and ideas on an empirically founded basis, then the ‘richness’ is not as overwhelming, even though “the times they are a-changin’”. The shift in focus, for instance, is indicated by the earlier mentioned CERME6 working group (WG15) “primarily dedicated to theory and *research* on the role of history in mathematics education”⁶ (italics added). In the call for papers for this working group, seven topics were listed, which, except for number 1, should receive special empirical attention:

1. Theoretical and/or conceptual frameworks for including history in mathematics education.
2. The role of history of mathematics at primary and secondary level, both from the cognitive and affective points of view.
3. The role of history of mathematics in pre- and in-service teacher education, both from the cognitive, pedagogical, and affective points of view.
4. Possible parallelism between the historical development and the cognitive development of mathematical ideas.
5. Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments.
6. Surveys of the existing uses of history in curricula, textbooks, and/or classrooms at primary, secondary, and university levels.
7. Design and/or assessment of teaching/learning materials on the history of mathematics.

When I asked Man-Keung Siu about the present state of the field while in Hong Kong in October 2006, he revealed the following with regard to number 7 and his own experiences in holding workshops for teachers:

You have to have something in between, not just the research results in history of mathematics, not just the primary texts, not just the storytelling

⁴ For the exact papers, see table 2.2, page 39.

⁵ For the exact papers, see table 2.2, page 39.

⁶ Quoted from the working group’s ‘call for papers’ to be found at: <http://cerme6.univ-lyon1.fr/group15.php> The group was chaired by Fulvia Furinghetti, former chair of HPM (2000-2004), and co-chaired by Jan van Maanen, Constantinos Tzanakis, Jean-Luc Dorier, and myself. And Abraham Arcavi assisted in writing the ‘call for papers’.

popular accounts. You have to have something in between, and those are materials that would be useful for teachers in the classroom. (Siu, 2006)

In other words, the topics to research empirically within the field of using history in mathematics education are plenty, and there seems to be a general acknowledgement within the community of such empirical research studies being both relevant and highly needed.

1.2 The Danish Upper Secondary Mathematics Program

In 2006 the Danish upper secondary school underwent a reform. This reform also affected the mathematics program for upper secondary level (grades 10-12). One of the effects was that history of mathematics came to assume a more prominent role than earlier. In the new regulation, one of the “academic goals” is that the students must be able to

- demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution (Undervisningsministeriet, 2007, my translation from Danish).⁷

Of course, in order for the students to reach the academic goals they have to be taught the mathematical core curriculum. However, some of the academic goals, such as the one quoted above, may not be reached by teaching the core curriculum alone. For this reason the students must also be taught in some supplemental curriculum. About this, it is said:

The supplemental curriculum in the subject mathematics – including also the interplay with other subjects – must put into perspective and deepen the core curriculum, expand the academic horizon, and provide room for local wishes and take into consideration the specific school.

In order for the students to live up to all the academic goals, the supplemental curriculum, which takes up 1/3 of the teaching, must, among other things, include:

- [...]
- *Teaching modules* on the history of mathematics. (Undervisningsministeriet, 2007, my translation from Danish, italics added)

Worth noticing is that the history of mathematics as part of the supplemental curriculum may take up quite a substantial amount of the teaching. Also, as stated above, the frames within which to include the history of mathematics are, in fact, very broad, allowing for local wishes of teachers and schools to be taken into account. In practice, the above formulation means that a teacher often has quite free hands in choosing what elements, aspects, and perspectives of the history of mathematics to include and focus on in his or her teaching. And since history of mathematics belongs to the supplemental curriculum, as opposed to the core curriculum, individual teachers also have the opportunity to include elements of mathematics which are not normally part of the upper secondary mathematics curriculum.

However, the above description from the Danish regulation for upper secondary school, in my opinion, leaves us with several unanswered questions. For instance, what

⁷ The new regulation of 2006 was somewhat modified in 2007. The word ‘demonstrate’ in Danish has a dual meaning; it may be used both as the word ‘prove’ and as the word ‘display’. Thus, students may only need to display knowledge.

does it actually mean to demonstrate knowledge about the evolution of mathematics and its interaction with history, science, and culture? How can one go about testing whether or not the students actually acquire (and possess) this kind of knowledge and understanding? And in what way they are able to acquire it at all. For instance, in what way do their knowledge, understanding, and reflections of the historical aspects depend on their understanding of the related and taught mathematics? How may teaching modules on elements of the history of mathematics be designed in order to take these issues into account? And when designing such modules, how may it be ensured, on the one hand, that the history is not reduced to anecdotes and, on the other hand, that the mathematics, from an historical viewpoint, is not made ‘anachronical’ or ‘Whig’ (Fried, 2001, p. 395)?

Such questions cannot be answered on the basis of armchair research alone. In order to provide possible answers to questions like these, empirical research on the use of history of mathematics in the setting described above must be carried out.

1.3 The Contribution of the Empirical Research Study

In this section I shall discuss how the research of the dissertation relates to the national and the international scene as described above. In other words, the way it deals with the questions raised by the inclusion of history in the Danish upper secondary mathematics program as mentioned above; the way it deals with some of the mentioned critiques of the available literature; how it relates to the seven CERME6-WG15 topics; and how it contributes to the field in general and hopefully brings it yet a step forward.

In order to study some of the questions raised in the new Danish upper secondary mathematics program (cf. section 1.2), I formulated the following list of hypotheses:

- it is possible to have upper secondary students reflect on aspects of the evolution and development of mathematics;
- it is possible to anchor the students’ reflections on the history in the related mathematical contents of the history;
- it is possible to design teaching modules which favor the above hypotheses, and at the same time ensure that the history is not anecdotal or the mathematics anachronical;
- and such modules may be a means to shape existing or foster new beliefs among the students about what mathematics is, where it is applied in extra-mathematical contexts, and how it interacts with history, science, and culture.

Of course, these hypotheses in some respects reveal a few of my personal viewpoints on the use of history of mathematics in education. For instance, that I consider it to be important to provide students with a ‘picture’ of what mathematics in time and space is, and that history in this sense is not only an implement for learning and teaching mathematics, but that it is something more. Another viewpoint of mine, which underlies the hypotheses, is that in order to understand elements of the history of mathematics one must have some kind of understanding of the involved mathematics also, otherwise the ‘picture’ will be blurred. When having developed conceptual and theoretical frameworks, I shall return to these underlying viewpoints of the above hypotheses and try to explicate them (see section 3.9).

Based on the above, an empirical experiment on the use of history of mathematics was carried out in a Danish upper secondary mathematics class, in order to confirm or reject the above hypotheses. The experiment consisted of two teaching modules on chosen elements from the history of mathematics. The reason for introducing the historical elements in specially constructed teaching modules had, of course, to do with the demand for this in the regulation (cf. section 1.2), but also with yet another hypothesis: namely that time, focus, and concentration were needed in order to provide the students with the conditions for performing the mathematically anchored, historical reflections. Thus, each teaching module had a duration of approximately 15 90-minute lessons. As already accounted for in section 1.2, such time frames are indeed possible within the new regulation for the upper secondary mathematics program in Denmark. The historical elements of the teaching modules were extra-curricular in the sense that they are not normally part of the upper secondary mathematics curriculum. Again, this is also possible within the new regulation, and for this particular class these ‘extra-curricular’ contents of the teaching modules came to count under the 1/3 supplemental curriculum and some of it was recommended for the final third year oral exam. The first teaching module was on the early history of error correcting codes and the second teaching module was on the history of public-key cryptography, more precisely RSA and the history of the mathematics applied in this algorithm. The same class was taught in both modules by the same teacher, and for each module a teaching material in the form of a textbook was prepared (Jankvist, 2008d; Jankvist, 2008h). The first module was implemented during the students’ second year of upper secondary school and the second module during their third and last year. (In chapter 2 a theoretical framework and related terminology will be developed, and in chapter 3 a set of research questions using this terminology will be posed on the basis of the hypotheses above.)

Thus, the study of this dissertation is an empirical research study on the use of history in mathematics education and as such, it aims at contributing to filling out of the gap, of which Gulikers and Blom (2001, p. 242) talk, between historians’ general articles and teachers’ practical articles. The study tests – empirically – some of the “assumptions and proposals”, of which Arcavi (2007) talks, that have been part of the literature for some time now. It may also be seen as an example of an investigation of the “effectiveness” of using history in mathematics education, although in a somewhat different way than that of which Siu and Tzanakis (2004), Gulikers and Blom (2001), and van Maanen (2007) talk (I shall return to this ‘effectiveness’ in chapter 4). (A longer presentation, discussion, and categorization of the “assumptions and proposals”, that is to say the different arguments for using history and the different ways of doing this, will be given in chapter 2.) Furthermore, as a byproduct of the empirical investigation, two teaching materials have been produced, materials including extracts of original sources, but at the same time fulfilling Siu’s characteristic of being something “in between”. So, in terms of the seven topics given by the CERME6 WG15 (see page 5) this study contributes to topics 1, 2, 5, 6, and 7.

1.4 Personal Background

With regard to the study of the dissertation, how did I then personally get involved in the business of history in mathematics education? Well, a couple of different things in

my educational background played a role; things related to the history of mathematics and to mathematics education, respectively.

First, I have studied mathematics at Roskilde University where history of mathematics is, or can be, part of the mathematics program. Actually, my first encounter with the history of mathematics was before entering the university, more precisely in upper secondary school when my mathematics teacher gave me a book on the history of equations (Andersen et al., 1986). Nevertheless, I shall count as the beginning my first student project on the history of mathematics, a third semester project on the two-years natural science basic studies at Roskilde university. This project was a comparison of Euler's introduction of differential calculus and the introduction given in modern textbooks of today (Jankvist et al., 2000). My next encounter with the history of mathematics was in my bachelor project (as part of the mathematics program at Roskilde University). This project was a study of algebraic equation solving from Cardano to Cauchy, more specifically a study of the meaning of combinations, permutations, and the concept of invariance in solving algebraic equations *before* Gauss, Abel, and Galois (Backchi et al., 2002). The study was later published as a two-part article in the Nordic journal of mathematics *Normat* (Jankvist and Sağlanmak, 2005a; Jankvist and Sağlanmak, 2005b). Besides mathematics, I studied computer science for my master's degree. My final computer science student project was also on the history of mathematics, namely the history of higher order logic. The focus of this study was to investigate the history of logic and especially type theory and then see how it was applied in modern proof machines to prove the correctness of algorithms (Gunnarsdóttir et al., 2004). I co-authored my master's thesis with a fellow student of mathematics and computer science. We studied the use of mathematics in the Mars Exploration Rover mission (MER) – at that time very much debated in the media (Jankvist and Toldbod, 2005a; Jankvist and Toldbod, 2005b). MER was a mission that landed two robot vehicles on the surface of Mars and then started exploring the planet. Among other things, this study included a trip to the USA, where we interviewed researchers at Brown University, MIT, and especially at the Jet Propulsion Laboratory (JPL), which had been in charge of building the rovers (Jankvist and Toldbod, 2005c). In our study of the mathematics in MER, we focused on the use of data compression and error correcting codes in the mission, and as a part of that also the history of the two disciplines, that is the history of information theory and coding theory. The master's thesis led to the publication of three articles: one in *Normat* (Toldbod and Jankvist, 2006); one in *The Mathematical Intelligencer* (Jankvist and Toldbod, 2007a); and one in *The Montana Mathematics Enthusiast* (Jankvist and Toldbod, 2007b), the two first being mostly concerned with the hidden mathematics of MER and the latter with the applied researchers of JPL as well as a few elements of the history of the mathematics used in MER.

My experience with mathematics education was of a more general nature. In my natural science basic studies, I followed a course on *Natural Science Educational Theory*, which was an introductory course to the didactics of natural sciences and mathematics. It was in this course I had my first experience with observing mathematics classrooms and conducting interviews. Two other students and I did a small series of observations of and interviews with an upper secondary mathematics teacher, in order to see how her spoken beliefs about her own teaching compared with actual observations of this. When enrolling as a student at the mathematics (and physics) department I suddenly found myself within an entire milieu of didactics and educational research. So even

though my master studies in mathematics were not directly concerned with mathematics education, I still encountered it on several occasions: the weekly seminar would from time to time touch upon educational and didactic matters, as would discussions over lunch, and around the time when I enrolled (year 2000), ‘mathematical competencies’ as part of the so-called KOM-project, which I shall return to in chapter 3, seemed to be a very hot issue of debate in the department. Although I was not involved in the KOM-project in any way, I still found myself wondering about these ‘competencies’, what they were, how they should complement a true curriculum description, to what extent I possessed them myself, etc. So even as a student, I was very much aware of the presence of didactics of mathematics and its existence as a scientific research discipline.

Now, the intersection of the study of history of mathematics and mathematics education was not something which presented itself to me as a possible area of research until much later. In fact, not until I had considered what to write about in my master’s thesis and discussed it with Mogens Niss. He suggested looking into the history of topology and students’ learning difficulties with topological concepts. However, for various reasons I chose the Mars-mission project described above, but the possibility of combining research in the history of mathematics with research in mathematics education stuck with me. When I learned that the department had an open postgraduate position in mathematics education, I decided to apply for it with the project of this dissertation.

1.5 Overview of the Dissertation and Related Publications

The dissertation consists of a total of eight chapters (including this introduction), a bibliography, and a smaller number of appendices. In this section I shall briefly outline the contents of these, as well as mention the publications which the research study has led to so far.

The Chapters

Chapter 2 initially provides a survey of the available literature on history in mathematics education. Next, it proposes two sets of categories for the different arguments of *why* to use history and *how* to do it – what I shall refer to as the ‘whys’ and ‘hows’ – which to some degree make up a theoretical framework for discussing the use of history. In the process of doing so, a terminology is also developed. The framework is then used to categorize the empirical (research) studies on the use of history in mathematics education, which I have come across in my survey.

In chapter 3 the framework and terminology from chapter 2 are used to phrase the three research questions of the empirical study. These questions are then discussed in order to identify the underlying theoretical constructs, which may be used to answer them. In this sense, the chapter offers insights into the justification of mathematics as a taught subject in school, what it means to understand mathematics (according to mathematics education literature), how to look at students’ mathematics related beliefs, discussions on how to conduct the history of mathematics, as well as elements of the sociology and philosophy of mathematics. At the end of the chapter, having discussed the underlying theoretical constructs of the research questions, I explicate my own personal views on the use of history of mathematics in mathematics education.

Chapter 4 includes an initial methodological discussion of the empirical research study

carried out in upper secondary school. It explains the actual conduction of the study, the methods used for data gathering, etc. Also, it discusses the answering of the three research questions on, first, a case-specific level, referring to the two teaching modules, and, second, the addressing of the trustworthiness, generality, and importance of the research study and its findings. Of course, the latter discussion cannot be completed before we know what the actual findings are, and parts of it are therefore postponed to chapter 8.

Chapter 5 is a description of the design, implementation, and evaluation of the first teaching module. The chapter contains an introduction to the early history of error correcting codes, an introduction which serves as a basis for the further discussion of the design of the teaching material as well as the students' work with a set of so-called essay assignments. Thus, this chapter includes the analysis of the students' work with the historical case, that is to say the analysis and discussion for the case-specific answering of a subset of the research questions.

Chapter 6 is a similar description of the (re)design, implementation, and evaluation of the second teaching module. This chapter also contains an historical introduction, this time to the history of public-key cryptography, RSA, and the number theory on which RSA relies. The (re)design of the teaching material is accounted for and the students' work with the essay assignments from this module is analyzed and discussed according to the relevant research questions on a case-specific level. Some comparisons are also made with the findings from the first module.

Chapter 7 concerns the study of the students' beliefs. Questionnaires given out and interviews conducted in the class before the beginning of the first teaching module are analyzed and compared to questionnaires given and interviews conducted in between the two teaching modules and after the second module. Thus the purpose of this chapter is, firstly, to account for the upper secondary students' beliefs about mathematics, especially issues related to its historical evolution and development, but also to its applications in society as well as its interaction with culture. Secondly, the purpose is to see to what extent teaching modules as the ones described in the previous chapters may cause changes in students' beliefs or maybe even give rise to new beliefs.

Chapter 8 is the discussion and conclusion chapter of the dissertation. Here the case-specific answers to the three research questions, discussed in the three previous chapters, are recapitulated. A second methodological discussion is then provided, addressing again the trustworthiness, generality, and importance of the research findings revealed in the previous chapters. Also, additional topics and questions which have arisen during the dissertation are dealt with as part of the discussion.

The Appendices

The first appendix, appendix A, provides a time line of the implementations of the two teaching modules in the upper secondary class.

The second appendix, appendix B, provides an English translation of all four questionnaires given to the upper secondary students.

The third appendix, appendix C, contains a selection of transcripts of video clips of students' conversations while working on essay assignments during the modules. Also these are in English translations.

Publications Related to the Monograph

Firstly, the research study has resulted in two teaching materials for the upper secondary level (all Danish titles in the following have been translated into English):

IMFUFA text 459	Jankvist (2008d)	The Early History of Error Correcting Codes – a Teaching Module for Upper Secondary School
IMFUFA text 460	Jankvist (2008h)	RSA and the History of the Applied Mathematics in the Algorithm – a Teaching Module for Upper Secondary School

Secondly, parts of the research study have been presented in the following journals:

ESM	Jankvist (2009a)	A Categorization of the ‘Whys’ and ‘Hows’ of Using History in Mathematics Education
ReLIME	Jankvist (2009c)	On Empirical Research in the Field of Using History in Mathematics Education
FLM	Jankvist (2009b)	History of Modern Applied Mathematics in Mathematics Education
BSHM <i>Bulletin</i>	Jankvist (2008j)	A Teaching Module on the History of Public-Key Cryptography and RSA
NOMAD	Jankvist (2007b)	Empirical Research in the Field of Using History in Mathematics Education: Review of Empirical Studies in HPM2004 & ESU4
NOMAD	Jankvist (2008e)	Upper Secondary Student’s Beliefs about Mathematics: Focusing on the ‘Three Aspects’
MONA	Jankvist (2007a)	The Dimension of the History of Mathematics in Teaching and Learning – Generally Speaking
MONA	Jankvist (2008a)	The Dimension of the History of Mathematics in Teaching – The Case of Upper Secondary Level

Thirdly, other papers have appeared in conference proceedings, a few of which are early versions of journal papers above, and a few contributions have been made to the HPM *Newsletter*:

CERME6, WG15	Jankvist (2009d)	Students’ Beliefs about the Evolution and Development of Mathematics
HPM2008	Jankvist (2008c)	History of Modern Mathematics and/or Modern Applications of Mathematics in Mathematics Education
ICME11, TSG23	Jankvist (2008f)	On Empirical Research in the Field of Using History in Mathematics Education
ICDM5	Jankvist (2008b)	Evaluating a Teaching Module on the Early History of Error Correcting Codes
Novembertagung18	Jankvist (2009e)	A Teaching Module on the Early History of Error Correcting Codes
ESU5	Jankvist (2008i)	A Teaching Module on the Early History of Error Correcting Codes (Abstract)
HPM <i>Newsletter</i>	Jankvist (2008g)	Proceedings HPM2004&ESU4: Empirical research on using history of mathematics in mathematics education
HPM <i>Newsletter</i>	Furinghetti et al. (2009)	CERME6 Working Group 15 Theory and research on the role of history in mathematics education (A report)

2 The ‘Whys’ and ‘Hows’

Before entering into a discussion of a concrete use of history in mathematics education, that is to say the study of this dissertation, it is a good idea to see what the field has to offer first. In this chapter I shall provide a brief account of my survey of the available literature on history of mathematics in mathematics education (section 2.1). Taking my point of departure in the survey, I shall then address the following fundamental and initial (research) question:

RQ0 What different arguments for using history (the ‘whys’) and what different approaches to using history (the ‘hows’) are given in the literature, and what are the interrelations of these arguments and approaches?

The answering of this question will to a great extent follow that of (Jankvist, 2009a), in which two sets of categories were proposed; one for the whys and one for the hows. Based on these, I shall discuss the interrelations of the different categories of whys and hows as well as some of the criticisms of using history in mathematics education.

As part of the categorization of the whys and hows of using history in mathematics education a, to some degree, theoretical framework and a terminology are developed. This framework is used as a basis for discussing and reviewing the existing empirical research studies of using history, which I have come across in my survey of the literature (section 2.8). But first the general survey.

2.1 A Brief Account of My Literature Survey

As already indicated in the introductory chapter, the publications on history in mathematics education are of various kinds. Generally, the publications are of three fundamentally different types: (1) publications *advocating* in one way or another for history in mathematics education; (2) publications *describing* either concrete uses by teachers or developments of teaching material etc.; (3) actual *research* on history in mathematics education. In this section I shall make some ‘downstrokes’ in the literature in order to exemplify the three different types, and I shall discuss a few of the publications which have inspired me in my own research. But first an overview of the literature samples surveyed.

Overview of the Literature Samples Surveyed

The literature which make up this survey are collective samples on the use of history in mathematics education. These consist of special books or collections of papers, as mentioned in section 1.1, special issues of journals devoted to history in mathematics education and of proceedings from the more recent HPM and ESU conferences. The samples used are displayed in table 2.1.

—	FLM Special Issue on History in Mathematics Education: Vol. 11(1) 1991
Swetz et al. (1995)	Learn from the Masters
Jahnke et al. (1996)	History of Mathematics and Education: Ideas and Experiences
Calinger (1996)	Vita Mathematica: Historical Research and Integration with Teaching (proceedings from HPM1992)
—	FLM Issue addressing History: Vol. 17(1) 1997
Katz (2000)	Using History to Teach Mathematics: An International Perspective (proceedings from HPM1996)
Fauvel and van Maanen (2000)	History in Mathematics Education: The ICMI Study
Hornig and Lin (2000)	Proceedings of the HPM 2000 Conference: History in Mathematics Education: Challenges for a New Millennium
Bekken and Mosvold (2003)	Study the Masters: The Abel-Fauvel Conference (proceedings)
—	MJRME Double Special Issue on The Role of the History of Mathematics in Mathematics Education (proceedings TSG17, ICME10): Vol. 3(1-2) 2004
Furinghetti et al. (2007)	Proceedings HPM2004 & ESU4
—	ESM Special Issue on The History of Mathematics Education: Theory and Practice: Vol. 66 2007
Barbin et al. (2008)	History and Epistemology in Mathematics Education: Proceedings of the 5th European Summer University (ESU5)
—	Papers presented at TSG23 on The Role of History of Mathematics in Mathematics Education at ICME11
Cantoral et al. (2008)	Proceedings HPM2008 (only on CD-ROM)
—	Proceedings CERME6, WG15: The Role of History of Mathematics in Mathematics Education: Theory and Research

Table 2.1 A table of the collective samples used when surveying the literature on history in mathematics education.

Comments on and Examples of the Advocating Samples

The papers of the FLM 1991 special issue on history in mathematics are mostly of an advocating nature, the paper by Russ et al. (1991) on the experience of history in mathematics education being the most ‘propagandistic’ one. The papers do, however, provide various examples from the history of mathematics serving as inspiration for teachers, and some of the authors also describe their experiences from teaching situations (see the examples of descriptive papers below). Something similar may be said about the collection of 23 papers edited by Swetz et al. (1995). The collection is organized according to papers discussing the integration of history at secondary and tertiary level, respectively, and mainly focusses on selected mathematical topics, though a few papers also focusses on design (e.g. Helfgott, 1995). The collection edited by Calinger (1996) is equally concerned with the history of mathematics as with the role of it in mathematics education. Concerning the latter, the papers are not very different in nature from those in Swetz et al. (1995), except, perhaps, for the fact that some papers address specifically the role of history for pre-service and in-service teachers (e.g. Heiede, 1996; Kleiner, 1996). Also, the collection edited by Jahnke et al. (1996) includes mainly advocating studies, but in contrast to the above mentioned samples it uses these to propose some interesting visions for future research in the field. For instance, when the editors write in their introduction:

We need more sound knowledge about what is going on when students of a

certain age are confronted with history of mathematics. We urgently need conceptual ideas about how history could be originally embedded into normal teaching. And, above all, we need a continuous process of exchange between interested mathematics educators, historians of mathematics, and research mathematicians. (Jahnke et al., 1996, pp. viii)

The authors then go on to discuss the relations between historians of mathematics and mathematics educators, arguing that these fields have much more to offer each other than is usually considered the case by their practitioners. The following quote advocating for a use of history has been particularly inspirational for me in my own study:

History of mathematics is considered by many as fundamental research, and integrating history into teaching seems to be a mere application of some more or less trivial by-products of the fundamental historical work. This idea is misleading. The significance of history lies in its contribution to the general culture. Even more than for general history, it is true for history of science that the fundamental relation to culture is bounded by what is termed 'Bildung' in German. If this is accepted, the immediate consequence is that we cannot live any longer with a situation in which mathematics educators have to fumble for subject matter which just might be adequate for teaching uses. (Jahnke et al., 1996, pp. viii-ix)

I shall return to and discuss the idea of 'Bildung' (or 'Allgemeinbildung') in chapter 3.

A couple of the advocating papers which I have found rather inspirational do, in fact, not come from the listed samples, but instead from the *International Journal of Mathematical Education in Science and Technology*. The first of these papers is the one by Siu and Siu (1979). They argue for six different 'profits' of taking history into consideration, three for curriculum planning and three for classroom learning. First, a look at the history may help structure the development of content in a curriculum in a more natural way (here they refer to the so-called recapitulation argument, see page 22). Second, mathematics is a discipline focusing on rigor in the sense that a minimum number of basic essential facts are selected and propositions are derived in a logical way from these. However, from an educational viewpoint, if focusing too much on rigor this may restrict the students' original thinking and provide them with an incorrect image of mathematics. Siu and Siu (1979, p. 563) say: "Ironically, the most important as well as the most difficult task in mathematical education is to make students realize that 'mathematics-as-an-end-product' as presented in textbooks can be very different from 'mathematics-in-the-making'." In this respect, looking to the history may illustrate that even the concept of rigor within mathematics is continually under evolution. Third, the history of mathematics may also serve as a useful guide to pointing out and illustrating interrelations between various branches of mathematics. Fourth, entering the three 'profits' for classroom learning, history is embedded in present mathematics (ideas, notation, etc.) and may therefore also help us understand it. Besides assisting in the understanding of specific concepts, history may also help students understand the global picture of mathematics and make them able to place the more local and fragmentary parts they have to study in a broader context. Fifth, history may show that mathematics is a human endeavor and part of the culture of mankind. Sixth, and last, history may also give the students confidence in the sense that it may show them how mathematics has not come into being exclusively by the hand of geniuses, that it is human to err,

and that co-operation is often a key to success. In section 2.3, when categorizing the whys, I shall deal with these kinds of arguments, or variants of them.

The other advocating paper, from the same journal, which I have found inspirational is the one by Furinghetti (2000).¹ In the context of prospective teachers, she discusses the history of mathematics as a coupling link between secondary and university teaching. Although the paper mainly concerns the use of history as a means for the learning and understanding of mathematics, it also addresses the topics of students’ beliefs and images of mathematics, and it is in this respect that I have found it interesting. In Italy, the author states, students arrive at the university with a good disposition towards mathematics, but after having been exposed to the teaching at university their feeling changes “‘very quickly towards a strictly formalist view of the discipline’ (these are a student’s words). Their conception of mathematics becomes so poor that one student says ‘mathematics exists because it is taught’” (Furinghetti, 2000, p. 44). According to Furinghetti, history has a part to play in remedying this, since “history is a good vehicle for reflecting on cognitive and educational problems, for working on students’ conceptions of mathematics and its teaching, and for promoting flexibility and open-mindedness in mathematics” (Furinghetti, 2000, p. 51). Concerning the idea of promoting flexibility and open-mindedness, Furinghetti has addressed this in other papers, also in relation to her viewpoint that “all kinds of mathematics students (prospective teachers and others) should know the history of mathematics for its own cultural value” (Furinghetti, 2000, p. 51). In a previous FLM-paper she argues that teachers should “approach mathematics as a set of human activities [...] and not as a body of rigidly defined knowledge” (Furinghetti, 1993, p. 38), which is very much in line with the advocating viewpoints of Siu and Siu (1979). Furinghetti states her personal belief that it is possible to provide students with the opportunity to develop what she calls an ‘*ecological image of mathematics*, meaning “an image respectful of the peculiarities of this protean discipline” (Furinghetti, 1993, p. 38). I shall return to students’ images of mathematics and the potential role of history in the making of these in chapters 3 and 7.

The different types of advocating arguments for using history shall be addressed when categorizing these in section 2.3. For now, let us see some examples of descriptive papers.

Comments on and Examples of Descriptive Samples

The collection edited by Katz (2000) is quite similar in nature to Calinger (1996), due to the fact that it also includes selected proceedings from an HPM meeting, but an interesting difference is that there is an increase in descriptive papers (and research papers). The ICMI Study (Fauvel and van Maanen, 2000) in particular reflects this increase in a descriptive approach, although it also provides its share of advocating arguments for the use of history. Part of the descriptive nature of the ICMI Study is, of course, explained by the fact that it is itself a very extensive survey of the literature on using history in mathematics education. Still, it reflects very well the increase in descriptive papers from the earlier samples mentioned in table 2.1. Examples of descriptions of using history in the ICMI Study are those of current practice in teacher

¹ Other examples of papers addressing the topic of history in mathematics education in the same journal are those of Grattan-Guinness (1973) and Grattan-Guinness (1978).

training as described in chapter 4 (Schubring, 2000); the section on ideas and examples of classroom implementation in chapter 7 (Tzanakis and Arcavi, 2000); descriptions of teaching projects inspired by history in chapter 8 (Siu, 2000b); the integration of original sources in pre-service teacher education and classrooms in general described in chapter 9 (Jahnke, 2000); and examples of using other media such as role plays, ancient instruments, computer software, and the Internet for integrating the history in class described in chapter 10 (Nagaoka, 2000). This increase in descriptive studies somehow seems to follow through in the remaining samples displayed in table 2.1 along with an increase in research studies, and I shall discuss the remaining samples in this context later. For now I shall exemplify two different types of descriptive samples from the literature: a teacher's concrete use of history in class and an extensive development of teaching materials relying on original sources.

On several occasions, Jan van Maanen has described his use of the history of mathematics in upper secondary classrooms etc., uses which are interesting since van Maanen at the time was a postgraduate student in the history of mathematics (van Maanen, 2007). One of his first descriptions appears as a rarity of its kind in the FLM 1991 special issue, van Maanen (1991), a frequently cited paper in the literature, in which he outlines a couple of lessons where his upper secondary students were to study a problem from L'Hôpital's 1696-textbook on differential calculus, *Analyse des infiniment petits*, the so-called weight problem. In a paper appearing in the collection edited by Swetz et al. (1995), van Maanen (1995) continues his descriptions based on personal experiences with three historical cases: seventeenth century instruments for drawing conic sections, improper integrals, and one on the division of alluvial deposits in medieval times. The latter, which I shall describe in more detail, is different from the other two in that it describes a project in three first-year classes of the Dutch grammar school, pupils about age 11, and since it was an interdisciplinary project with Latin.

In an medieval setting of a case of three landowners, who all had land on the bank of a river, and fought over an alluvial deposit bordering their land, the students were to investigate the problem by means of a method proposed by the Italian professor Bartolus of Saxoferrato in 1355 (the example with the landowners was that used by Bartolus himself). The ideas of the project were, among others, to demonstrate the importance of mathematics in society, to let pupils 'invent' a number of constructions by ruler and compass, to have them apply these 'inventions' to solve the legal problem of the medieval example, and to have them read extracts from Bartolus' treatise in original Latin language, illustrating "that it is impossible to interpret the sources of Western culture without knowledge of classical languages" (van Maanen, 1997, p. 79). van Maanen's evaluation of the implementations of the project is: "Making contact with Bartolus was only possible via deciphering and translating, but that was simply an extra attraction to most of the pupils. They learned to work with point-sets in plane geometry, and simultaneously their knowledge of general history increased. Last but not least, they were greatly stimulated to learn Latin." The project on Bartolus is just one of several of van Maanen's which all illustrate that "new maths may profit from old methods" (van Maanen, 1997).

The other example of descriptive papers is the most recent in a long line by David Pengelley and collaborators² describing their work on developing materials for classroom

² Other papers in the samples from table 2.1 are: Pengelley (2003a), Pengelley (2003b), Laubenbacher

projects using original sources (Pengelley, 2008). In particular, the group is interested in “learning discrete mathematics and computer science via primary historical sources”, a project which has resulted in 18 sets of material for students projects in classrooms since 2008.³ At HPM2008 in Mexico City, I attended the workshop by Pengelley and Barnett, in which two of their materials were discussed: *Treatise on the Arithmetical Triangle* relying on the original work of Blaise Pascal; and *Early Writings on Graph Theory: Euler Circuits and the Königsberg Bridge Problem* using Euler’s solution to the Königsberg bridge problem⁴ from 1736, today considered the starting point of modern graph theory (Barnett et al., 2008). The idea of these materials, as well as the others displayed on their web site, is to guide students through a reading of the original sources, providing them with clarifying and elaborating questions on the mathematics of the source along the way. Barnett et al. (2008) describe their design and aim as:

The projects are designed to introduce or provide supplementary material for topics in the curriculum, such as induction in a discrete mathematics course, or compilers and computability for a computer science course. Each project provides a discussion of the historical exigency of the piece, a few biographical comments about the author, excerpts from the original work, and a sequence of questions to help the student appreciate the source and learn how to do the relevant mathematics. The main pedagogical idea is to teach and learn certain course topics directly from the primary historical source, thus recovering motivation for studying the material. (Barnett et al., 2008, p. 2-3)

Along side the extracts from original sources, the students are often provided with the terminology of modern mathematics, and in the case of the material on the graph theory, the students were asked to ‘fill in the gaps’ in a modern proof of Euler’s main theorem for solving the bridge problem, thereby also drawing their attention to current standards regarding formal proof.

Comments on and Examples of Research Samples

As should be clear from the above, the older sources mentioned in table 2.1, i.e. those from the 1990s,⁵ offer almost no examples of research papers on the use of history in mathematics education.⁶ However, around the end of the 1990s and at the beginning of the new decade, a minor shift in focus seems to occur. The ICMI Study, although this mainly is of a descriptive (and advocating) nature, also occasionally refers to actual research on the use of history in mathematics education, e.g. in chapter 3 on research perspectives (Barbin, 2000). The HPM2000 proceedings has as one of its themes “the effectiveness of history in teaching mathematics: empirical studies”, and a total of 25 papers are listed under that topic (although some of them are to be considered descriptive

and Pengelley (1996).

³ These projects may be found at: <http://www.cs.nmsu.edu/historical-projects/projects.html>. 16 other projects, developed before 2008, may be found at http://www.math.nmsu.edu/hist_projects.

⁴ The Königsberg bridge problem: is it possible to plan a stroll through the town of Königsberg which crosses each of the town’s seven bridges once and only once? See e.g. Biggs et al. (1976).

⁵ Swetz et al. (1995); Jahnke et al. (1996); Calinger (1996); and Katz (2000) since this includes papers from HPM1996.

⁶ For the exceptions, see table 2.2, page 39.

rather than empirical). As mentioned in section 1.1, the HPM2004&ESU4 proceedings, the MJRME 2004 double special issue, and the ESM 2007 special issue also include a number of empirical research studies. The same goes for the ESU5 proceedings, the HPM2008 proceedings, and, naturally, the WG15-papers in the forthcoming CERME6 proceedings. The survey and discussion of the empirical research studies in these samples, as well as others, shall be postponed to section 2.8, since they may then be related to the framework on whys and hows to be developed in the following sections of this chapter.

Now, research studies on the use of history in mathematics education need not only be empirical. They can also be analytical studies and discussions. One such example is found in the proceedings from the Abel-Fauvel conference (Bekken and Mosvold, 2003). Mosvold (2003) discusses the so-called genetic principle (see page 28) and its history, mainly with reference to Schubring (1978), but also relates it to the, of course much younger, Norwegian tradition.⁷ Another example of such an analytical study is found in the FLM 1997 issue, dealing with history in mathematics education. Here Radford (1997) discusses (and criticizes) the historico-epistemological obstacles in mathematics, as discussed by Brousseau (1997) (see page 21), as a means for understanding and overcoming students' learning difficulties.

Analytical research also includes, for instance, studies in the use of history in existing curricula and textbooks. An example of this is the paper by Smestad (2003), also in the proceedings of the Abel-Fauvel conference. Smestad reviewed the textbooks which came out after the 1997 reform in Norway, in which history of mathematics was included in the curriculum for elementary school (ages 6-16). Smestad concludes that the treatment of history is problematic in this first generation of textbooks. The authors seem to have had problems including the history in a meaningful way and lots of factual errors occur. Smestad also describes his ongoing study at the time, to review the use of history in the textbooks for the Norwegian upper secondary school, in which the history became part of the curriculum in 1994 (inspired by the Danish upper secondary mathematics program of 1987, see chapter 3). The situation for these seem more promising, according to Smestad. Fewer factual errors occur, but still the authors have problems actually integrating the history. Often it is 'pasted on' in either the beginning or the end of chapters.⁸

A special kind of analytical research studies are those which discuss and/or categorize the different arguments for using history and the different approaches to doing so, i.e. the whys and the hows of using history. An early example of such, though not in any way a categorization, is the one by Fauvel (1991) in the FLM special issue. Proposed categorizations, however, are found in the papers by Gulikers and Blom (2001), mentioned in chapter 1, Fried (2001), Furinghetti (1997) and Furinghetti (2004), and as part of the ICMI Study in the chapter by Tzanakis and Arcavi (2000), a categorization closely related to that of Tzanakis and Thomaidis (2000), and later put into a somewhat different perspective by Tang (2007). I shall postpone the presentation and discussion of these classifications till sections 2.3, 2.4, and 2.5, where they will be discussed in relation to that of my own.

⁷ Mosvold's discussion is based on his master's thesis (Mosvold, 2001) and other publications on the same topic (Mosvold, 2002b; Mosvold, 2002a).

⁸ Something similar is the case in many of the newer Danish textbook systems for upper secondary mathematics, where the historical information often is added on in special colored boxes (Jankvist, 2008a).

2.2 Reasons for Proposing a New Categorization

Concerning the above mentioned studies on classifying the whys and hows, one disadvantage of these, as I see it, is that they do not all necessarily separate, strictly, the categorization of the hows from the categorization of the whys. One could argue that this is due to the fact that the whys and hows may be so interconnected that it is not a very straightforward, or maybe even a natural, thing to do. In principle, a ‘how’ often conditions or presupposes a ‘why’. Nevertheless, it is my claim that by trying to *strictly* separate the categorization of the whys from the categorization of the hows and vice versa, new insight can be gained, and the interrelations between the whys and the hows become clearer.

Besides providing intellectual clarity, knowing about the interconnections between the whys and hows can make it easier to analyze teaching material applying history, in order to see if it fulfils certain requirements or goals. Such knowledge would also, I suspect, be useful when making decisions about content, form of presentation, and organization concerning the use of history for teachers, material designers, and curriculum developers. At any rate, the interrelations of the whys and hows of using history is not something which is discussed very often in the literature. It therefore seems to me that the discussion on using history would benefit from yet another – systematic – attempt at creating a new platform from which one may discuss, in depth, the potential role of history of mathematics in mathematics education.

Regarding the categorization to be proposed, it is important to bear in mind that it is not absolute, nor is it meant to be. It serves a certain purpose: to create the requisites for analyzing concrete uses of history within mathematics education, i.e. specific implementations of hows (see, for instance, Jankvist, 2007b and Jankvist, 2008a). Also, it should be mentioned that the categorization do not distinguish between the use of history in classrooms, in textbooks, at different levels of education including teacher training. These are all considered collectively.

Method for Testing the Proposed Categorization

In order to test and justify the proposed categorization of whys and hows, I have drawn on the advocating, descriptive, and research oriented studies of table 2.1, as well as others which I have come across. In particular, screenings of the last ten years (1998-2007)⁹ of four major mathematics education journals (ESM, JRME, FLM, and ZDM) have been conducted in order to identify relevant recent literature since the survey of the ICMI Study. In ESM, 27 papers in some way concerning history and/or the use of it in mathematics education were found within this ten year period, 20 of these being relevant to the issues at hand (cf. the references made in the following sections). In JRME, FLM, and ZDM the numbers of papers found were 1 (0 relevant), 4 (1 relevant), and 0, respectively.

However, none of the identified papers discuss arguments or approaches which are not already dealt with in the ICMI Study. For this reason, the proposed categories of whys and hows will mainly be exemplified through the ICMI Study.

⁹ Because the screenings were made in the beginning of 2008 for the ESM paper (Jankvist, 2009a), the year 2008 is not included in the survey.

2.3 Two Categories of Arguments

In general the arguments for using history are of two different kinds: those that refer to history as a *tool* for assisting the actual learning and teaching of mathematics, and those that refer to history as a *goal* (Jankvist, 2007a, pp. 72-76). Each of these two kinds constitutes its own category of arguments.

Arguments Referring to History as a Tool

The category of history-as-a-tool arguments contains the arguments concerning how students learn mathematics. A typical argument is that history can be a motivating factor for students in their learning and study of mathematics by, for instance, helping to sustain the students' interest and excitement in the subject (e.g. Taimina, 2004, p. 88; Tattersall and McMurrin, 2004, p. 101; Farmaki and Paschos, 2007, p. 84). Or that an historical approach may give mathematics a more human face and therefore make it less frightening (Russ et al., 1991, p. 7). Often pieces of the mathematical development over which past mathematicians have stumbled will also be troublesome for today's students of mathematics (e.g.; Fauvel, 1991, p. 4; Tzanakis and Thomaidis, 2000, p. 49; Bartolini Bussi and Bazzini, 2003, p. 206; Bakker and Gravemeijer, 2006, p. 149), which the students may derive comfort from knowing: the same mathematical concept which they themselves are now having trouble grasping actually took great mathematicians hundreds of years to shape into its final form (e.g. Bakker and Gravemeijer, 2006, p. 165).

Besides having these motivational and more affective effects, history may also play the role of a cognitive tool in supporting the actual learning of mathematics. For instance, one argument states that history can improve learning and teaching by providing a different point of view or mode of presentation (e.g. Kleiner, 2001, p. 143; Helfgott, 2004, p. 161; Jahnke, 2001, p. 195). Other arguments say that an historical phenomenology may prepare the development of a hypothetical learning trajectory, or that history "can help us look through the eyes of the students" (Bakker, 2004, pp. 51, 87).

A special use of history as a cognitive tool occurs in the identification of epistemological obstacles, as introduced by Bachelard (1938). When incorporating the notion of epistemological obstacles into his theory of didactical situations, Brousseau (1997, p. 87) explains: "Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. They can be found in the history of the concepts themselves." Since "certain of the students' difficulties can be grouped around obstacles attested to by history" (Brousseau, 1997, p. 96), history cannot only help in identifying these obstacles, it can also help in overcoming them: "an epistemological reflection on the development of ideas in the history of mathematics can enrich didactical analysis by providing essential clues which may specify the nature of the knowledge to be taught, and explore different ways of access to that knowledge" (Dorier and Rodgers in Fauvel and van Maanen, 2000, p. 169) (see also Schubring, 1988, pp. 141-143; Sierpinska, 1994, pp. 125-137; Radford, 1997, pp. 29-32; Radford, 2000a, pp. 162-163; Radford and Puig, 2007, pp. 147-148; Katz and Barton, 2007, pp. 198-200).¹⁰ An important point of Brousseau's is that history

¹⁰ For studies on epistemological obstacles concerning specific mathematical concepts see, for example: Glaeser (1981); Sierpinska (1987); Gagatsis and Thomaidis (1995); Schubring (2005).

should not be used without modifications. The idea is to draw on historical “arguments in order to choose a genesis of a concept suitable for use in schools and to *construct* or ‘invent’ teaching situations that will provide this genesis” (Brousseau, 1997, p. 96).

The last but very distinctive type of the history-as-a-tool arguments might be referred to as the *evolutionary* arguments, since they claim that there can be no learning of mathematics without history. The most clear-cut evolutionary argument is the so-called *recapitulation* argument saying that ‘ontogenesis recapitulates phylogenesis’.¹¹ The connection of ontogenesis and phylogenesis was, according to Furinghetti (2004, p. 5), formulated by the German biologist and natural philosopher Ernst Haeckel in 1874 and is known as the “fundamental law of biogeny” (Haeckel, 1906, pp. 2-3). Haeckel developed this thought even further to say that “the psychic development of a child is but a brief repetition of the phylogenetic evolution” (Furinghetti and Radford, 2002, p. 634). And it is this argument that translates into the recapitulation argument which may be formulated as: *To really learn and master mathematics one’s mind must go through the same stages that mathematics has gone through during its evolution.* The recapitulation argument not only applies to mathematics as a whole, but also to single mathematical concepts and theories. And it is often in relation to the development of single mathematical concepts that another tool argument related to the evolutionary kind, the so-called historical parallelism, is put to the ‘test’ – historical parallelism concerns the observation of difficulties and obstacles that appeared in history reappearing in the classroom (e.g. Harper, 1987; Sfard, 1995; Zornbala and Tzanakis, 2004; Thomaidis and Tzanakis, 2007; Farmaki and Paschos, 2007; Tzanakis and Kourkoulos, 2007). The idea of parallelism may also be used as a methodology or heuristic to generate hypotheses in mathematics education (e.g. Fauvel and van Maanen, 2000, p. 160; Vasco, 1995, pp. 61-63).

Arguments Referring to History as a Goal

The category of history-as-a-goal arguments contains the arguments claiming that learning aspects of the history of mathematics serves a purpose in itself. Notice that when referring to history as a goal in *itself*, this must not be mistaken for knowledge of the history of mathematics as an independent subject, i.e. history of mathematics for the sake of history of mathematics (history of mathematics *per se*). Instead focus is on the developmental and evolutionary aspects of mathematics as a discipline.

In this sense, it is for instance considered a goal to show students that mathematics exists and evolves in time and space (e.g. Tzanakis and Thomaidis, 2000, p. 46; Barabash and Guberman-Glebov, 2004, p. 75); that it is a discipline which has undergone an evolution and not something which has arisen out of thin air (e.g. Niss and Jensen, 2002, p. 268; Philippou and Christou, 1998, p. 193); that human beings have taken part in this evolution (e.g. Gulikers and Blom, 2001, p. 229; Thomaidis and Tzanakis, 2007, p. 181); that the evolution of mathematics is due to many different cultures throughout history and that these cultures have had an influence on the shaping of mathematics as well as the other way round (e.g. Tzanakis and Thomaidis, 2000, p. 46; Barabash and

¹¹ Ontogenesis is the evolution of a single organism (or individual) from the fertilization of an egg to adulthood and phylogenesis is the evolution of an entire species. The recapitulation argument is also sometimes referred to as the genetic principle, however, the genetic principle is also a principle of education, i.e. an approach, and it will be discussed as such later.

Guberman-Glebov, 2004, p. 85; Høyrup, 2007, p. 260); or that the evolution is driven by internal as well as external forces (e.g. Fried, 2001, p. 392; Charette, 2004, p. 121).

From the history-as-a-goal point of view, knowing about the history of mathematics is *not* a primary tool for learning mathematics better and more thoroughly, even though this may still be a (positive) by-product. In using history as a goal, the learning of developmental and evolutionary aspects of mathematics either serves a goal in itself (cf. above) or serves to illustrate other historical aspects of the discipline.

Meta-Issues and In-Issues of Mathematics

Another way of describing the history-as-a-goal arguments is to say that they are arguments for learning something about the meta-aspects or *meta-issues* of mathematics. By this I mean issues involving looking at the entire discipline of mathematics from a meta-perspective level. Examples of questions suggesting a study of such meta-issues are:

How does mathematics evolve over time? What forces and mechanisms can be present in the evolution? Do society and cultural circumstances play a part in this evolution? If so, how? And does mathematics then depend on culture and society, place and time? Is old mathematics also obsolete mathematics? (Niss, 2001a, p. 10, my own translation from Danish)

As opposed to the history-as-a-goal arguments being concerned with meta-issues of mathematics, the history-as-a-tool arguments are concerned with inner issues, or *in-issues*, of mathematics. By in-issues I am thinking of issues related to mathematical concepts, theories, disciplines, methods, etc. – the internal mathematics. For example, to learn about the number sets (\mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C}), their interrelations, their cardinalities, etc. is to be considered a study of in-issues. On the other hand, learning about the historical development of the different kinds of numbers and the difficulties regarding the acceptance of the irrational numbers, the negative numbers, or the complex numbers concerns aspects of meta-issues of mathematics.

The distinction between in-issues and meta-issues has some resemblance to that of Davis and Hersh (1981) who talk about the ‘inner issues’ and the ‘outer issues’ of mathematics or that of Niss (2001b, p. 163) who talks about ‘knowledge of mathematics from the inside’ and ‘knowledge of mathematics from the outside’. However, one word of caution should be given when associating history-as-a-goal and history-as-a-tool arguments with meta-issues and in-issues, respectively: if the purpose is to teach meta-issues of mathematics, and here I am thinking of meta-issues not concerned with the evolution of the discipline (e.g. mathematics as a pure or an applied science, the aesthetics of mathematical proofs, etc.), then history is but a tool to achieve this. Only when the meta-issues have to do with the evolution, development, etc. of mathematics – which they most often do when discussing the use of history – does history serve as a goal.

But let us now look at the arguments for using history in mathematics education provided in the ICMI Study, in order to see how these may be placed into the two categories of whys.

Categorizing the ICMI Whys

As mentioned in the introduction, the mathematics education literature offers a large number of arguments for the use of history in the teaching and learning of mathematics. However, during my readings of the aforementioned samples I have not encountered any arguments which could not be placed into the categories presented above. As a way to render this claim plausible, I shall discuss ‘the list’ given by Tzanakis and Arcavi (2000, pp. 203-207) – “Some arguments in support for integrating history” – in the ICMI Study. This list contains seventeen different topics placed within the five following “main areas”: (a) the learning of mathematics, (b) the nature of mathematics and mathematical activity, (c) the didactical background of teachers, (d) the affective predisposition towards mathematics, and (e) the appreciation of mathematics as a cultural endeavour. Discussing every argument within every area would require too much space, so instead I shall select arguments from each of the five main areas and show how these may be categorized.

Clearly the heading of the first area, the learning of mathematics, suggests that the focus is on the in-issues of mathematics and hence on history as a tool. For example argument a2 (the second under the first main area, a) talks about history as a resource having a “potential to motivate, interest, and engage the learner” (Tzanakis and Arcavi, 2000, p. 204, a2). The learning of mathematics area offers the only arguments (under topic a1) relating somewhat to the evolutionary tool arguments and this even though the authors seem to argue against recapitulationism, saying that “[h]istory could at best suggest possible ways to present the subject in a natural way” and thus serve as an inspiration for teachers (Tzanakis and Arcavi, 2000, p. 204, a1). Actually, topic a1 deals more with the fact that “mathematics-in-the-making” is different from “mathematics-as-an-end-product” (Siu and Siu, 1979, p. 563). Some of the topics may also deal with history as a goal – depending on interpretation, that is. Argument a3 states that history may act as a bridge between mathematics and other subjects (Tzanakis and Arcavi, 2000, p. 205, a3). If the purpose of this ‘cross-curricular bridge’ is to bring some more popular subjects into the mathematics teaching and through these motivate the students to learn the involved mathematics, then the argument belongs to the history-as-a-tool category. On the other hand, if the point is to show that mathematical creations often are offspring of problems occurring in other fields of science, it belongs to the history-as-a-goal category. However, judging from the discussion of a3 in the ICMI Study, e.g. talking about bringing out connections between apparently unrelated scientific domains, the latter seems to be the case. In any respect, arguments provided for using history are occasionally subject to interpretation. And it is the outcome of this interpretation that decides what category the argument must be placed into.

The area concerning the nature of mathematics and mathematical activity is arranged by the authors under the two topics of “content” and “form”. These topics offer both tool and goal arguments, all according to the interpretation as well. For instance, it is stated that “with the aid of original material, or even extracts from it, both the teacher and the learner may become aware of the advantages and/or disadvantages of modern forms of mathematics” (Tzanakis and Arcavi, 2000, p. 205, b1). Now, if the point of this argument concerns aspects about mathematics in the making, e.g. evolution of mathematical techniques and notation, then it belongs to the category of goal. But if the point is to compare ancient and modern mathematics in order to say: ‘look how

long-winded and troublesome the techniques and notations of the past were, and look how shortly and elegantly we can do the same today', then the argument might be interpreted as a motivational one, i.e. tool, since this may lead to a higher appreciation of, and thus willingness to learn, the mathematics in the textbooks.

The third area deals with the didactical background of teachers. One argument (c3) talks about teachers through the study of history becoming more aware of the 'creative process' of doing mathematics and thus being able to enrich their mathematical literacy and appreciate better the nature of mathematical activity. The literacy part of this argument may be interpreted as a tool argument whereas the appreciation part is more in the category of goal. Another argument (c4) mentions history as a resource for teachers to enrich their didactical repertoire.¹² This argument is linked to what Shulman (1987, p. 8) calls the teachers' "pedagogical content knowledge" (PCK): "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interest and abilities of learners, and presented for instruction." At first glance it may not seem easy to decide where to place such arguments concerning teachers' pedagogical content knowledge. However, what we need to do is to ask questions of the form: What is the purpose of, or motivation for, enriching the teachers' repertoire and for what reason do the teachers need this resource? Do the teachers need it to enhance their students' learning of mathematical in-issues? Or do the teachers need it to discuss certain meta-issues of the evolution of mathematics in class? Depending on the answer, the argument may be classified as dealing either with history as a tool or history as a goal. So not only interpretation may cause us to categorize the same argument as either tool or goal, the underlying motivation for an argument may as well.

The area of affective predisposition towards mathematics is a mixture of both tool and goal arguments. For instance, it is said that history may teach us "[n]ot to get discouraged by failure, mistakes, uncertainties or misunderstandings" which is a history-as-a-tool argument (Tzanakis and Arcavi, 2000, p. 207, d3). An example of a history-as-a-goal argument could be that we may learn from history that "mathematics is an evolving and human subject rather than a system of rigid rights" (Tzanakis and Arcavi, 2000, p. 206, d1).

In the area of appreciation of mathematics as a cultural endeavour, the arguments put forward, not surprisingly, almost all concern history as a goal. For instance, it is said that: "History can provide examples of how the internal development of mathematics, whether driven by utilitarian or 'pure' reasons, has been influenced, or even determined to a large extent, by social and cultural factors" (Tzanakis and Arcavi, 2000, p. 207, e2). A couple of the arguments presented within this area may also concern history as a tool, again depending on interpretation. For example, it is said that through the study of history, students and the teacher may become aware of different approaches to mathematics, which have appeared within other cultures (Tzanakis and Arcavi, 2000, p. 207, e3). Now, if the purpose here is to show the students something about evolution of mathematics within different cultural settings, then the argument is a history-as-a-goal argument.

¹² Examples of other papers discussing in-service and pre-service teachers' possible benefits from knowing and learning about the history of mathematics are Freudentahl (1981), Heiede (1996), and Barabash and Guberman-Glebov (2004). Regarding empirical studies on this matter the following may be mentioned: Philippou and Christou (1998); Waldegg (2004); Su (2007); Horng (2007); Arcavi and Isoda (2007); Furinghetti (2007); Goodwin (2007).

On the other hand, if the purpose is to try and promote the students’ understanding of a given mathematical concept by letting them tackle problems regarding this through different approaches, then the argument is a history-as-a-tool argument.

This ends the categorization of the ICMI list of arguments and as illustrated, these may all be placed into the proposed categories, although occasionally this is done according to an interpretation of the argument and/or the underlying motivation for posing it. Next, I shall address the categorization of the hows.

2.4 Three Categories of Approaches

The manners in which the history of mathematics may be used in the teaching and learning of mathematics can be classified into three major categories of approaches. Within each of these categories history may be included in different grain sizes, i.e. the use may be scaled from very small to very comprehensive. I shall refer to these categories as the *illumination* approaches, the *modules* approaches, and the *history-based* approaches.

The Illumination Approaches

In the illumination approaches the teaching and learning of mathematics, whether it is the actual classroom teaching or the textbooks used, is supplemented by historical information. As indicated above, these supplements may be of varying size and scope. The smaller of these are what Tzanakis and Arcavi (2000, pp. 208, 214) refer to as “isolated factual information” or “historical snippets” which may cover names, dates, famous works and events, time charts, biographies, famous problems and questions (e.g. Swetz, 1995; Swetz, 2000), attribution of priority, facsimiles and so forth. Also the telling of anecdotes and stories belongs to this category (e.g. Siu, 2000a). One way to think of these smaller supplements in the illumination approaches is to think of them as ‘spices’ added to the mathematics education casserole.

At the other end of the scale of the illumination approaches we find what might be referred to as “historical epilogues” (or prologues). This term is inspired by Lindstrøm (1995), who at the end of every chapter in his calculus textbook has an historical epilogue: a section where the history of the mathematics presented in the chapter is outlined along with names, dates, motivating problems, references to original works, anecdotes, discussions of attributions of priority as well as the developmental progress. If original sources are to be used within the illumination approaches, the historical epilogues would be the place to do this, although it would probably only be in the form of small extracts.

The Modules Approaches

The modules approaches are instruction units devoted to history and quite often they are based on cases. The term “modules” is taken from Katz and Michalowicz (2004). Just as in the previous category of hows, the modules approaches may vary in both size and scope. The smallest of these would be what Tzanakis and Arcavi (2000, pp. 217-219) refer to as “historical packages” which are collections of “materials narrowly focused on a small topic, with strong ties to the curriculum, suitable for two or three class periods, ready for use by teachers in their classroom”.

In the middle of the scale we find modules of perhaps ten to twenty class periods. Such modules need not be tied to the mathematical topics in the curriculum, due to which they also provide the opportunity to study branches of mathematics that are not usually part of the curriculum at a given school level. The ways to implement both the historical packages and the larger modules are numerous. They may be introduced through textbook studies, through readings of original sources, or through student projects. Other ways may be through historical plays, the Internet, worksheets, historical problems, mechanical instruments, etc. (Fauvel and van Maanen, 2000, pp. 214-232).

At the upper end of the scale we find the full courses (or books) on the history of mathematics within a mathematics program. These may include an account of historical data, a history of conceptual developments, or something in between (Tzanakis and Arcavi, 2000, p. 208). Such courses can rely on original or secondary sources (or both) depending on the level of history studies intended. Of course, these approaches can also be implemented in many other ways than just through an actual course or reading of a book. One example could be that of extensive student research projects as described in Fauvel and van Maanen (2000, pp. 215-216) and practiced at Roskilde University.

The History-Based Approaches

The last category of approaches covers the ones directly inspired by or based on the development and history of mathematics. Unlike the modules approaches, these approaches do not deal with the study of history of mathematics in a direct manner, but rather in an indirect one. The historical development is not necessarily discussed in the open. On the other hand, it often sets the agenda for the order and way in which the mathematical topics are presented. Turning to the number sets again for a concrete example, the evolution of these would mean that the natural numbers (\mathbb{N}) would be the first taught, then the positive rationals (\mathbb{Q}^+) and some of the positive irrationals (\mathbb{R}^+) before turning to zero and the negatives (\mathbb{Z}), the remaining reals (\mathbb{R}), and finally the complex numbers (\mathbb{C}). In this respect, the history becomes a fully integrated part of the approaches themselves – one might think of them as ‘historical approaches’. A frequently mentioned and debated example of this is the so-called genetic approach (principle), which I shall return to.

This completes the presentation of the three categories of how to use history in mathematics education. In the following I shall turn to the ICMI Study’s characterization of the hows.

Categorizing the ICMI Hows

The three ways in which history may be used in mathematics teaching listed by the ICMI Study are: (a) learning *history*, by the provision of direct historical information, (b) learning *mathematical topics*, by following a teaching and learning approach inspired by history, and (c) developing *deeper awareness*, both of mathematics itself and of the social and cultural contexts in which mathematics has been done (Tzanakis and Arcavi, 2000, p. 208). In the following I shall recapitulate each of the three ways and discuss them according to the previously proposed categories of hows.

The direct historical information way in the ICMI Study consists of two different cases; (1) the isolated factual information and (2) the full courses or books on the history

of mathematics. The first of these is, as indicated earlier, similar to the illumination approaches in the lower end of the scale. The second is the upper end of the scale of the modules approaches. Generally Tzanakis and Arcavi (2000, p. 208) describe the emphasis of the direct historical information way as being “more on resourcing history than on learning mathematics”.

About the second way of using history, Tzanakis and Arcavi (2000, p. 208) state that “this is essentially what may be called the genetic approach to teaching and learning”, which places it in the category of history-based approaches. As mentioned earlier, the genetic approach is a frequently debated way of involving history and for this reason I shall elaborate a bit on the topic. The word genetic comes from the Greek word *genesis*, which in English translates to either creation or development. Actually there are several variations of the genetic approach or the *genetic principle* as it is also called where the word ‘principle’ refers to a principle of education (Mosvold, 2001, p. 31). Schubring (1978), in his extensive study hereof, distinguishes between two genetic principles: (1) the *historical-genetic principle*, which aims at leading students from basic to complex knowledge in the same way that mankind has progressed in the history of mathematics, and (2) the *psychological-genetic principle*, which is based on the idea to let the students rediscover or reinvent mathematics by using their own talent and experiences from the surrounding environment. As examples of each of these I shall briefly present the ideas of Toeplitz’ *genetic method* and Freudenthal’s *guided reinvention*.

Otto Toeplitz named and addressed the genetic method in a lecture dated 1926 (Toeplitz, 1927). Burn (1999, p. 8) explains (or interprets) Toeplitz’ method by saying that “the question which Toeplitz was addressing was the question of how to remain rigorous in one’s mathematical exposition and teaching structure while at the same time unpacking a deductive presentation far enough to let a learner meet the ideas in a developmental sequence and not just a logical sequence.” Tzanakis (2000, p. 112) describes the genetic method as a method in which there is no uniquely specified way of presenting a given subject, which means that the approach is not a method in the strict sense of an algorithm, but rather a general attitude towards the presentation of a scientific subject. In such a presentation the motivation behind introducing new concepts, theories, or key ideas of proofs is based on the evolution of the subject. Problems and questions which were stimulating for the historical evolution are then reconstructed in a modern context and notation, so that they become more accessible to the students. For samples on the genetic method put to practice see e.g. Toeplitz (1949), Toeplitz (1963), Edwards (1977), Stillwell (1989), Tzanakis (2000), Burn (1999), and Burn (2005). Worth mentioning is that Toeplitz in his 1926-lecture distinguished between two different types of the genetic method, the *direct* and the *indirect*:

[A]ll these requisites [...] must at some time have been objects of a thrilling investigation, an agitating act, in fact at the time when they were created. If one were to go back to the roots of the concepts the dust of times [...] would fall from them and they would again appear to us as living creatures. And from then on there would have been offered a double road into practice: Either one could directly present the students with the discovery in all of its drama and in this way let the problems, the questions, and the facts rise in front of their eyes – and this I shall call the *direct genetic method* – or one could by oneself learn such an historical analysis, what the actual meaning and the real core in every concept is, and from there be able to

draw conclusions for the teaching of this concept which as such is no longer related to history – the *indirect genetic method*. (Toeplitz, 1927, pp. 92-93, my own translation from German)

Toeplitz was not the first – nor the last – mathematician to believe that history could be a guide for teaching and learning mathematics. Prominent mathematicians as, for instance, Poincaré (1899), Klein (1908), Polya (1962), and Weil (1978) have expressed their positive beliefs regarding this matter. Also Hans Freudenthal was inspired by the evolution of mathematics, although putting it to a slightly different use than Toeplitz. Freudenthal (1991, p. 48) says: “Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now.” Freudenthal dislikes the term genetic method since “it does not allude to any activity of the learner”, instead he stresses that the students should be guided to “an activity” by their teacher, and that “the learner should reinvent mathematising, not mathematics” (Freudenthal, 1991, pp. 46, 49). ‘Mathematising’ may be described very shortly as “the entire organising activity of the mathematician” (Freudenthal, 1991, p. 31). The question as to what the learner is to mathematise is also answered by Freudenthal, his answer being “reality”, by which he means the “learner’s own reality as laid open to him by his guide” (Freudenthal, 1991, p. 50). (See also e.g. Freudenthal, 1973; Gravemeijer and Doorman, 1999; van Amerom, 2002; Fung, 2004.)

The third way of the ICMI Study, the ‘mathematical awareness’ way, is divided into the study of the ‘intrinsic nature’ and the study of the ‘extrinsic nature’ of mathematical activity. The intrinsic nature involves important aspects of doing mathematics such as the part of general conceptual frameworks and of associated motivations; the evolving nature of mathematics exemplified by notation, terminology, methods, representations as well as notions of proof, rigor, and evidence; and the role of paradoxes, contradictions, intuitions, and the motivation for generalizing, abstracting, and formalizing (Tzanakis and Arcavi, 2000, pp. 211-212). The extrinsic nature of mathematical activity deals with the fact that mathematics is often regarded as a discipline which is largely disconnected from social and cultural concerns and influences. Tzanakis and Arcavi (2000, p. 212) state that the history of mathematics may illustrate the superficiality of such a view in the way that it can: show how a vast number of mathematical aspects are related to other sciences and humanities; show how the social and cultural milieu may influence the development, or the delay of the development of certain mathematical domains; show that mathematics is an integral part of the cultural heritage and practices of different civilizations, nations, or ethnic groups; and show that currents in mathematics education through time have reflected trends and concerns in culture and society.

In the context of the three categories of approaches, the modules approaches seem the most suitable for illustrating the intrinsic and extrinsic natures of mathematics. However, in my view the mathematical awareness way is not as clear an approach as the two other ICMI ways. In fact, it seems more concerned with the whys than with the hows. A point which will be subject to discussion in the following section.

2.5 Remarks on the Categorization of Whys and Hows

Any categorization may be productive in some respects and unproductive in others.¹³ This of course goes for the above proposed categorization as well.

Remarks Relating to the Whys

One type of argument for using history which is not necessarily captured by the tool-goal categorization is that of using history for material design purposes. For instance that history may serve as an inspiration source for a given designer of teaching materials, including textbooks, worksheets, etc. (e.g. van Maanen, 1997; Gravemeijer and Doorman, 1999; Radford, 2000b; Kleiner, 2001; Streefland, 2003; van Amerom, 2003; Fung, 2004; Bakker and Gravemeijer, 2006; Barbin, 2007). Such designers may not be interested in restricting themselves to using history as either a tool or a goal; possibly they would want to do both. But even though the categorization may not encompass every teaching material design argument for using history, it may still have something to offer the teaching material designer. For instance, knowing about the different interrelations of the whys and hows may help teaching material designers in choosing an approach for using history which suits their purpose. In any case, the categories of whys and hows (and the following discussion of their interconnections) presented in this article may still be used to analyze the material produced by a given designer, e.g. as a way of revealing the designer’s most favored purpose of using history. But more importantly, it may also be a way of telling to what extent the designer in choosing a certain approach rules out some of the whys of using history.

The ICMI list by Tzanakis and Arcavi does not contain every possible argument for why to use history in mathematics education even though it provides the most comprehensive selection I have seen. For instance, the evolutionary tool arguments are not given too much attention. This may be due to the fact that many people find the evolutionary arguments, and especially the strict recapitulation argument, somewhat old fashioned. And yes, it is probably true that not many people today hold this point of view in its pure form. Nevertheless, it is still a certain type of argument, and a quite frequently debated one as well. Furthermore, (empirical) research related to the argument has been and is still being conducted as the number of references listed earlier indicate. Another argument which does not appear in the ICMI list is the one claiming that history of mathematics should be taught since the history of a subject is part of the subject (e.g. Heiede, 1992, p. 152). However, the explanation for not including such an argument may be that it is, in this respect, inane since it says nothing about why to *use* history. At any rate, the argument seems to be more concerned with history as a goal than history as a tool, since the underlying motivation for presenting it appears to be that ‘history may provide a more authentic or more complete picture of what mathematics is’. Yet another argument is proposed by Fried (2007, p. 203). He argues that history of mathematics may play a role in mathematics education by contributing to “students’ growing into whole human beings”, which is an argument touching upon the discussion of *general education* or *Allgemeinbildung*. In any respect, history in this line of reasoning becomes a tool to achieve this ‘growth’ of the students.

¹³ A general discussion of gains and losses of classification may be found in Bowker and Star (1999).

A less systematic list of whys is given by Fauvel (1991, p. 4). The article by Siu and Siu (1979) is, in fact, also an elaboration on six different whys, largely similar to those of the ICMI list and Fauvel's list. Neither Fauvel nor Siu and Siu explicitly discuss a categorization of the whys in these papers. As mentioned earlier this is, however, done by Gulikers and Blom (2001, pp. 225-231), who divide the arguments into conceptual, (multi-)cultural, and motivational arguments, distinguishing for each category between the relevance for teachers and the relevance for pupils. This categorization cuts across the one of this article. The category of motivational arguments includes mainly history-as-a-tool arguments, whereas the category of (multi-)cultural arguments contains both tool and goal arguments. The category of conceptual arguments embraces a variety of tool arguments and, in contrast to the classification of the ICMI Study, deals explicitly with the evolutionary tool arguments. For example, Gulikers and Blom (2001, p. 226) talk about the "desirability or even the necessity of teaching and learning mathematics along the line of its historical development". As already mentioned, any categorization serves a certain purpose, Gulikers' and Blom's being that of clarifying "the discussion on the role of the history of geometry in education" (Gulikers and Blom, 2001, p. 223). Fried (2001) has a different purpose for providing his classification of the whys (and hows), namely to discuss how mathematics education and the history of mathematics may coexist without resulting in anachronism or 'Whig' history. For this purpose Fried (2001, p. 392) distinguishes between three themes: "(1) that the history of mathematics humanizes mathematics, (2) that it makes mathematics more interesting, more understandable, and more approachable, (3) that it gives insight into concepts, problems, and problem-solving". Furinghetti (2004, p. 3) distinguishes between two themes: "(A) History for reflecting on the nature of mathematics as a socio-cultural process" and "(B) History for constructing mathematical objects" where "(A) includes the idea of history as a means to promote mathematics in the classroom in order to humanize mathematics" and "(B) concerns the core of the problems related to teaching/learning mathematics" (see also Furinghetti, 1997) – a classification which in some respects also parallels the idea of history as a tool and history as a goal.

I would like to draw the attention to one additional argument, one from Fauvel's list and one which is explicitly treated by Gulikers and Blom (2001, p. 229). This argument states that history "[h]elps to explain the role of mathematics in society" (Fauvel, 1991, p. 4). The argument concerns meta-issues but not necessarily those of the evolution of mathematics because of which it may argue more for history as a (motivational) tool than for history as a goal. More importantly, this argument of Fauvel's touches on a different aspect of the discussion of the whys, namely that of the 'indispensability' of the arguments.

Indispensability of the Whys

By the *indispensability* of the whys I am thinking of what arguments can be put forward only on the basis of history and not, for instance, philosophy or mathematical modeling. For example, Fauvel's argument about history helping us to explain the role of mathematics in society is not a very indispensable argument. Applied mathematics, technology, or modeling are often more obvious candidates to illustrate the role of mathematics in society than history is. Of course this does not make Fauvel's argument into a non-argument for using history, but it does suggest that some of the whys

for using history in mathematics education may be more indispensable than others. Actually, the discussion of indispensability spans a scale of different versions or variants of indispensability, also depending on the preconditions invoked; from the fundamentally unavoidable to the for all practical purposes unavoidable; from being a better choice than anything else to merely being a good idea. In this sense Fauvel's argument merely becomes a good idea.

Within the two categories the evolutionary tool arguments may be the strongest ones regarding indispensability – that is, of course, only if one is a firm believer in this type of argument. If for instance the 'old fashioned' view of the recapitulation argument is taken for granted, then there is no other way to learn mathematics than through the use of history. Whether any of the other history-as-a-tool arguments may rise to such a fundamentally unavoidable status as the evolutionary arguments is more questionable. However, the motivational tool arguments, e.g. showing that professional mathematicians also make mistakes, may in a given situation turn out to be a better choice than anything else. If within a given school system it is considered important to teach certain meta-issues concerning the evolution of mathematics, the history-as-a-goal arguments may be considered indispensable in terms of being for all practical purposes unavoidable.

Remarks Relating to the Hows

As with the categorization of the whys, the categorization of the hows is just one of many imaginable ones, each being made for some specific purpose. Different categorizations with different purposes may to a larger extent stress, for instance, the type of history in use (factual, conceptual, cultural, etc.); the amount of history being used; the ways the students work with the history (worksheets, student-projects, etc.); or the materials used (original sources, secondary sources, or a mixture). However, even though these aspects may not be part of the definition of the three categories of hows presented in this paper, it should be clear that they are, in some way or another, still embraced by them. Concerning the use of original sources, as mentioned above, a few comments are in order.

A distinction between the use of primary (original) sources and the use of secondary sources when integrating history is essential. When using secondary sources, the students are exposed to a given historian's presentation and possibly interpretation of history, and they must make their choices based on this (Furinghetti, 2007, p. 136). When reading original sources the students must, on the other hand, perform their own interpretation of what actually took place, why a certain mathematician developed a theory in one way or another, whether or not the written is true, what internal and/or external forces drove the development of the work, etc. In short: they are to perform their own inquiry, a type of inquiry to which secondary sources essentially are closed (Fried, 2001, pp. 402-403). Studying the original sources, however, requires a lot of work on the students' behalf. Jahnke (2000, p. 291) describes it as "the most ambitious of ways in which history might be integrated into the teaching of mathematics, but also one of the most rewarding for students both at school and at teacher training institutions." The extent to which original sources are being used does, of course, have an impact on what the students learn: what students may gain from just 'sniffing' a few picks from an original source and what they might learn from being exposed to systematic readings of original

sources are immensely different things. Obviously, original sources have to be chosen with great care, depending on the educational level in question, in order to make sure that the students have a realistic chance of actually working with them.

One objection I do have towards the ICMI authors' classification of arguments for using history and approaches to doing so, is that they sometimes mix the whys and hows. That is to say they from time to time use the hows to describe the whys and vice versa. One example of this is argument b1, dealing with the aid of original material, which is more of an approach than an actual argument for using history. The third category of the hows, the mathematical awareness way, is however the most conspicuous example of the mixing of whys and hows. The hows of this category are defined by their concern with either the intrinsic or the extrinsic nature of mathematical activity. As seen in the previous section the descriptions of these natures are, in fact, a recital of whys rather than a description of how they are meant to be put to use. This, of course, reveals some of the interconnections between the whys and the hows. For instance, that the mathematical awareness way is concerned with the meta-issues rather than the in-issues of mathematics. But it also makes the distinction between the whys and the hows less sharp and thereby, perhaps, blur other possible connections between them. My objection, of course, is primarily one of classification. In practical teaching situations, there may be reasons to use history if it is used in a certain way.

A similar situation may be found in the categorization of hows by Gulikers and Blom (2001, pp. 231-235). The very definitions of different hows are, from time to time, given in terms of related whys. An example of this is the following: "According to conceptual, and also motivational arguments, [i.e. whys] *a teaching and learning approach inspired by history* [i.e. a how] *to introduce mathematical concepts or methods can be followed*" (Gulikers and Blom, 2001, p. 231). Tzanakis and Thomaidis (2000, pp. 46-49) are more careful in not mixing whys into their actual description of the hows. Also they focus on which whys the different approaches may stimulate. The description itself is, however, quite similar to that of the ICMI Study, except that the deeper awareness way is described in terms of using original and/or secondary sources. Another categorization which is also quite strict in describing different hows is the one provided by Fried (2001, pp. 392-393), who distinguishes between a "strategy of addition" and one of "accommodation". In the first, history is merely added to the existing curriculum, whereas in the second, history changes the presentation of the mathematics curriculum without enlarging it. Fried's 'accommodation' clearly covers the history-based approaches. To some extent the 'addition' may correspond to the illumination approaches, but whether or not it also covers the modules approaches is less clear. Fried does not, however, have it as his aim to describe the interrelations between the categories of whys and hows.

2.6 Interrelations Between the Whys and Hows

In the previous sections I have tried to provide a set of lenses through which the different approaches of using history in mathematics education can be viewed. In this section, these lenses are used in an attempt to make the possible interrelations between the whys and hows stand out; the purpose being to analyze these relations critically. The following discussion will be structured around figure 2.1, which shows the six possible

interconnections between the two proposed sets of categories.

The illumination approaches may be used to illuminate meta-issues regarding the evolution of mathematics, i.e. history as a goal. However, due to the nature of the illumination approaches, the meta-issues may only be 'illuminated', not discussed in depth, in such a presentation. Of course, in the upper end of the scale, in the historical epilogues, one stands a better chance of dealing in depth with the meta-issues. The illumination approaches, however, are much more suitable when it comes to using history as a tool. Especially the motivational and affective history-as-a-tool arguments are well off, since this approach may 'spice' up the presentation, which is to say that history may become the sugar that makes the student swallow the 'bitter pill' of mathematics (Heiede, 1992, p. 153), i.e. the in-issues. Concerning the cognitive tool arguments the illuminations approaches seem ill-suited and the evolutionary tool arguments are downright impossible to realize through this type of approach.

Since the cognitive history-as-a-tool arguments often are of a more demanding nature than the motivational ones, at least where implementation is concerned, a use of history as a cognitive tool for teaching in-issues may require additional time and space. For this reason the modules approaches seem a better place to achieve success with these arguments. The modules approaches are also a much more suitable place to try and bring out the different meta-issues of the development of mathematics, no matter if it be in the lower end of the scale, the middle section, or the top end. Historical packages in the lower end provide a means for putting the meta-issues of the curriculum-tied in-issues on the agenda for a short period of time. In the more extensive modules of the middle section as well as the full courses and books of the top end, an opportunity opens up for introducing new in-issues of mathematics, thus fertilizing the soil for a discussion of related meta-issues. Such modules further provide the opportunity to study branches of mathematics which are not normally part of the curriculum at a given school level. This again opens up for a larger variety of meta-issues to be discussed. As mentioned earlier, original sources may play a central role in the modules approaches, both as a way of realizing the cognitive history-as-a-tool arguments and in particular the history-as-a-goal arguments. This is evident from Jahnke (2000, pp. 291-292), who lists three general ideas which he thinks best suited for describing the special effects of such use: (1) replacement, (2) reorientation, and (3) cultural understanding. The first idea deals with

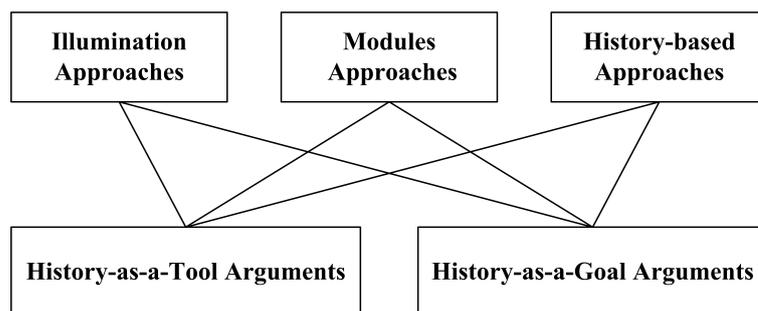


Figure 2.1 The six possible connections between the categories of whys and hows.

the replacement of the usual with something different. For instance, that history allows mathematics to be seen as more than just a corpus of knowledge and techniques. The second idea challenges one's perception by making the familiar unfamiliar. For instance, when the study of history causes a reorientation of our views (e.g. Furinghetti, 2007). The third idea allows us to place the development of mathematics (and mathematics education) in a scientific, technological, or societal context of a given time and place. Regarding the modules approaches and the evolutionary kind of tool arguments, this connection may best be realized in the middle section or top end of the scale. However, as we shall see, the history-based approaches provide a much more feasible way of realizing these arguments.

The main concern of the history-based approaches is on the in-issues of mathematics, i.e. the focus of these approaches is on the learning of mathematics, and therefore the history merely acts as a tool in this context. Meta-issues, and hence history-as-a-goal arguments, only become part of the agenda if they, in some way, can assist or motivate the learning of the in-issues in question. Within the history-based approaches the genetic principle is tightly connected to the evolutionary kind of history-as-a-tool arguments. Schubring (1978, p. 192, my own translation from German) says that the historical-genetic and the psychological-genetic principles: “[b]oth assume that there exists a parallel between the evolution of knowledge of the individual attempting to obtain knowledge and the historical societal evolution of knowledge. The conceptual basis for this is – explicitly or implicitly – the biogenetic law.” Hence, the genetic principle, in the historical or the psychological version, is in fact a realization of the recapitulation argument put into practice, whether it be in its strict form or in terms of historical parallelism.

So far I have only discussed the arguments for using history, the whys, and the approaches to doing so, the hows, as well as their interrelations. Not surprisingly, it must be expected that when arguments *for* the use of history exist, arguments *against* it exist as well.

2.7 Selected Criticisms of Using History

Because the whys and hows of using history fall into the two sets of categories, the criticisms of using history relate to these categories as well. Some criticisms, however, appear to be of a more overarching nature and may be viewed as concerning the very interrelations of the whys and hows as opposed to just the individual whys and hows. Again I shall take my departure point in the ICMI Study which lists ten typical criticisms of or objections to using history (labeled O1-O10), some due to Siu (2007, pp. 268-269). I shall begin with the overarching criticisms.

Overarching Criticisms

The overarching criticisms are of two kinds: those that simply reject history as something important, and those that may recognize the use of history as an option, but reject it due to other causes. Examples of the first kind are: “History is not mathematics. If you must teach history, then you need to teach mathematics itself first: teach the subject first, then its history”; “Progress in mathematics is to make the tackling of difficult problems a routine, so why bother by looking back?” (Tzanakis and Arcavi, 2000, p.

203, O1, O5). Examples of the second kind are: "History may be liable to breed cultural chauvinism and parochial nationalism"; "Students may have an erratic sense of the past which makes historical contextualisation of mathematics impossible without their having had a broader education in general history"; "Lack of time" – already there is not enough time to teach mathematics and the integration of history will require even more time (for a deeper discussion of this objection see Fried, 2001, p. 394); "Lack of resources" – there is not enough available material which integrates history; "Lack of expertise" – the teachers do not know enough about the history of mathematics; and "Lack of assessment" – it is not clear how to integrate historical matters in the student's assessments (Tzanakis and Arcavi, 2000, p. 203, O3, O4, O7-O10). Regarding the lack of expertise, Burn (1998, pp. 11-12, my own translation from Norwegian) says: "Even more than in mathematics the pupil in the history of mathematics is dependent on the teacher's presentation. The pupil is at the mercy of the teacher. If the teacher makes mistakes in mathematics, it can be discovered and corrected by an accurate thought process. A teacher's mistake in history is, on the contrary, not easily accessible for correction." One mistake which might be liable to occur if care is not taken, is that of anachronism.

Criticisms Related to the Whys

Examples of criticisms of history as a goal seem to be somewhat rare. However, since we know from Jahnke (2000, p. 291) that the main reasons for using original sources are concerned with history as a goal, the following criticism from Siu's list may be interpreted as one: "Does it really help to read original texts, which is a very difficult task?" (Siu, 2007, p. 269). It is the word 'help' that allows us to relate this criticism with the whys instead of the hows, to which the use of original sources otherwise would belong.

The criticisms of history as a tool include that "[h]istory may be tortuous and confusing rather than enlightening" and that "[m]any students dislike history and by implication will dislike history of mathematics, or find it no less boring than mathematics" (Tzanakis and Arcavi, 2000, p. 203, O2, O4). The latter criticism is roughly the opposite of the motivational argument for using history. Of course, any means of motivation can only be used on a subjective and personal level. This is in fact the weakness of this argument; that it assumes an historical approach to really be of interest to a given student. According to Schubring (1988, p. 138) it then becomes a prerequisite condition that historical questions do in fact have value within our culture, which perhaps is not the case for today's students in the same way it was for students in earlier times. On the cognitive side of using history as a tool, Schubring (2007, p. 339) states that a major flaw in relying on historical parallelism is that it "presuppose[s] history of mathematics as a definitely established corpus of knowledge which is beyond controversy", a presumption which is "far from being true." One criticism of using history as a tool which is not mentioned in the ICMI Study is that history may be an obstacle in itself, like for instance van Amerom (2002, p. 297) concludes on the basis of an empirical research project: "On the one hand history is welcomed as a change in routine exercises, on the other the low attainers mostly find it confusing and difficult, rather than interesting." Bakker (2004, p. 266) reports a similar finding.

As already implied, the evolutionary argument and in particular the recapitulation

argument are subject to a variety of criticisms. Dorier and Rogers, for instance, refer to the very strict recapitulationist paradigm as mere “nonsense”, due to the fact that the past cannot be reconstructed with any certainty, since we are missing essential data and an historical fact or event never is pristine, it is always seen through interpretative lenses, thus being only partial and subjective (Fauvel and van Maanen, 2000, pp. 168-169).

Criticisms Related to the Hows

The small biographies and anecdotes of the illumination approaches that are often seen in textbooks are not always entirely in accordance with historical truth (Freudentahl, 1981, p. 30). This may be due to such supplements often serving a certain purpose – e.g. the purpose of providing entertaining reading – and that this may be realized better if the truth is twisted a bit. Regarding the use of anecdotes Siu (2000a, p. 4) says: “When we make use of anecdotes we usually brush aside the problem of authenticity. It may be strange to watch mathematicians, who usually pride themselves upon their insistence on preciseness, repeat without hesitation apocryphal anecdotes without bothering one bit about their authenticity.” Siu does not see this as a problem, though, as long as we only take anecdotes for what they are, and do not attribute more significance to them than they are entitled to. Another aspect of the illumination approaches concerns the use of pictures, facsimiles, and so forth. Regarding the use of portraits of earlier mathematicians, Burn (1998, p. 11, my own translation from Norwegian) says: “Portraits of earlier mathematicians placed in textbooks are meant to make the mathematics more humane. But many of these people seem strange and the portraits show a distance. Actually they are not the faces belonging to anybody whom I wish to meet, nor do they give me the impression of a community which I wish to belong to”. Besides being humorous, this quote also illustrates something else. Namely, that Burn seems to have an unspoken idea of what the purpose of showing portraits in textbooks really is, i.e. that they must have a positive (perhaps motivating) effect on the reader rather than being realistic.

As to the middle and upper end of the scale of modules approaches, the study of original sources and full courses on the history of mathematics are subject to another criticism. Tzanakis (2000, p. 111) says that “a rather naive attitude is to follow the historical development of a mathematical discipline as closely as possible, presumably using original books, papers, and so on”, and he further states that such “a strictly historical approach is not didactically appropriate, since, contrary to what is sometimes naively assumed, the historical evolution of a scientific domain is almost never straightforward and cumulative. It involves periods of stagnation and confusion, and new concepts or proofs are not introduced in the simplest and most transparent way.” For this reason Tzanakis (2000, pp. 111-112), and also Edwards (1977, pp. vi-viii), argue that one should choose an approach between the naïve historical approach and the strictly deductive one (presentation of concepts, theorems, and proofs in a clear-cut manner and from there derive both understanding and applications). Tzanakis and Edwards are in fact spokesmen for the genetic method and therefore they also suggest this as a useful compromise to avoid the extremes of the other two approaches.

As mentioned before, the history-based approaches to some extent crystallize into the genetic approach, which also is subject to criticisms. According to Schubring (1978), especially the historical-genetic principle has been criticized over the years. One criticism

given by Karl.-H. Schwager in 1956 points out that history is always made by adults and that it therefore may be difficult for a child to fully understand the problems related to history (quoted in Schubring, 1978, p. 189). Also Lietzmann, Klafki, and Freudenthal have, according to Schubring (1978, pp. 189-197), criticized the historical-genetic principle. They were of the opinion that history should not make up the base for the education, but that the students themselves should be the starting point. Both the pedagogic and the didactic perspective should be the students' own experiences from the environment surrounding them every day. Recent criticism of Toeplitz' genetic method does not seem easy to find (Mosvold, 2001, p. 32), however one discussion of the indirect genetic method by Kronfellner may be found in Fauvel and van Maanen (2000, pp. 71-73).

The sixteenth and last objection in Siu's list is in fact more of a critical question than an objection to the use of history, since it asks: "Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?" (Siu, 2007, p. 269). With this 'objection' in mind, let us take a look at the list of empirical research studies which I have come across in my survey of the literature.

2.8 A Survey of Empirical Studies on Using History

The survey of empirical research studies were done based on the samples listed in table 2.1, the screening of ESM, FLM, JRME, and ZDM (1998-2007), and other (old and recent) journal papers, postgraduate dissertations, etc. which I have come across during the time of my study. First, I shall provide an overview of the found empirical studies. Afterwards I shall categorize these according to their use of history as a tool or as a goal.

Overview of Empirical Studies on Using History

Empirical studies may be many different things: from large scale quantitative studies to small scale qualitative studies, from experimental investigations to a teacher testing out a course using methods of questionnaires and interviews. When scanning the literature and deciding which studies to regard as empirical, I have mainly considered whether or not a paper had any display of empirical data to support findings and conclusions, and if there seemed to be some kind of (implicit or explicit) underlying discussions or descriptions of research design and methodology. These criteria have resulted in the list of 81 somewhat or clear-cut empirical studies depicted in table 2.2.

Surely, the list of table 2.2 is not a complete list of every empirical study ever made on the use of history in mathematics education. Nevertheless, it is a fairly comprehensive one, and as such it may provide us with a sound indication of where the emphasis of the empirical studies within the field lies.

Categorizing the Empirical Research Studies

Just like some of the arguments against the use of history were of an overarching nature, so are some of the empirical studies. In a sense, they also relate to the interrelations between the whys and hows, like the overarching criticisms did. As an example take the studies which address in-service teachers' actual use of history in their classrooms or their general views and attitudes towards why and how to use history.

Jahnke et al. (1996)	Bartolini Bussi and Pergola (1996)
Calinger (1996)	Bero (1996)
Katz (2000)	Bruckheimer and Arcavi (2000); Dorier (2000); Isaacs et al. (2000); Winicki (2000)
Hornig and Lin (2000)	Hsiao and Chang (2000); Hsieh (2000); Hsieh and Hsieh (2000); Lakoma (2000); Lin (2000); Liu (2000); Ming (2000); Ohara (2000); Prabhu and Czarnocha (2000); Su (2000); Troy (2000); Tsukahara (2000); Winicki-Landman (2000)
MJRME Special Issue	Barabash and Guberman-Glebov (2004); Fung (2004); Waldegg (2004); Zorbala and Tzanakis (2004)
Furinghetti et al. (2007)	Demattè (2007); Hornig (2007); Isoda (2007); Liu (2007); Smestad (2007); Su (2007); Tzanakis and Kourkoulos (2007); Winicki-Landman (2007)
ESM (1998-2007)	Philippou and Christou (1998); Radford (2000b); van Amerom (2003); Durand-Guerrier and Arzac (2005); Bakker and Gravemeijer (2006); Farmaki and Paschos (2007); Arcavi and Isoda (2007); Furinghetti (2007); Radford and Puig (2007); Thomaidis and Tzanakis (2007)
Barbin et al. (2008)	Dimitriadou (2008); Glaubitz (2008); González-Martín and Correia de Sá (2008); Liu (2008); Morey (2008); Paschos and Farmaki (2008); Thomaidis and Tzanakis (2008)
TSG23, ICME11	Lawrence (2008)
Cantoral et al. (2008)	Ceylan Alibeyoglu (2008); Gonulates (2008); Jankvist (2008c); Kourkoulos and Tzanakis (2008); Nataraj and Thomas (2008); Peard (2008); Reed (2008); Smestad (2008)
WG15, CERME6	Blanco and Giovart (2009); Jankvist (2009d); Kjeldsen (2009b); Lawrence (2009); Tardy and Durand-Guerrier (2009); Thomaidis and Tzanakis (2009)
Other samples	McBride and Rollins (1977); Fraser and Koop (1978); Arcavi et al. (1982); Arcavi et al. (1987); Harper (1987); Sfard (1995); Demattè and Furinghetti (1999); Lit et al. (2001); Jankvist (2008j); Jankvist (2008b); Charalambous et al. (2009); Jankvist (2009b); Jankvist (2009c); Kjeldsen and Blomhøj (2009)
Ph.D. dissertations	van Amerom (2002); Bakker (2004); van Gulik-Gulikers (2005); Goodwin (2007)

Table 2.2 A list of the 81 empirical studies found in the samples in table 2.1 as well as others, which I have come across in my survey. Remarks: Samples which was not found to include empirical studies are not listed in the table. The surveys of proceedings edited by Hornig and Lin (2000), Barbin et al. (2008), and Cantoral et al. (2008) do not include the papers written in Taiwanese, French, and Spanish. Nor does the survey of contributions in Barbin et al. (2008) take into account abstracts, since these do not display any empirical data.

Overarching Studies

Fraser and Koop (1978); Smestad (2007); Smestad (2008); Gonulates (2008)

The studies concerning mostly the use of history as an affective and motivational tool are:

Studies Using History as an Affective and Motivational Tool

Charalambous et al. (2009); Hsiao and Chang (2000); Hsieh (2000); Hsieh and Hsieh (2000); Lawrence (2008); Lin (2000); Liu (2007); McBride and Rollins (1977); Philippou and Christou (1998); Troy (2000)

Studies focusing on history as a cognitive tool (e.g. epistemological obstacles) make up the largest group of empirical studies:

Studies Using History as a Cognitive Tool

Arcavi et al. (1982); Arcavi et al. (1987); Arcavi and Isoda (2007); Bakker (2004); Bakker and Grave-meijer (2006); Barabash and Guberman-Glebov (2004); Blanco and Giovart (2009); Bruckheimer and Arcavi (2000); Bartolini Bussi and Pergola (1996); Ceylan Alibeyoglu (2008); Dimitriadou (2008); Dorier (2000); Fung (2004); Furinghetti (2007); Glaubitz (2008); González-Martín and Correia de Sá (2008); Goodwin (2007); van Gulik-Gulikers (2005); Horng (2007); Isoda (2007); Kjeldsen (2009b); Kourkoulos and Tzanakis (2008); Lakoma (2000); Lawrence (2009); Lit et al. (2001); Liu (2000); Liu (2008); Ming (2000); Morey (2008); Nataraj and Thomas (2008); Ohara (2000); Paschos and Farmaki (2008); Peard (2008); Prabhu and Czarnocha (2000); Radford (2000b); Reed (2008); Su (2000); Su (2007); Tardy and Durand-Guerrier (2009); Thomaidis and Tzanakis (2008); Thomaidis and Tzanakis (2009); Tsukahara (2000); van Amerom (2002); van Amerom (2003); Winicki (2000); Winicki-Landman (2000); Winicki-Landman (2007)

Of studies somehow referring to the evolutionary arguments for using history, historical parallelism, etc., we find:

Studies on Historical Parallelism Etc.

Bero (1996); Durand-Guerrier and Arsac (2005); Farmaki and Paschos (2007); Harper (1987); Radford and Puig (2007); Sfard (1995); Thomaidis and Tzanakis (2007); Tzanakis and Kourkoulos (2007); Waldegg (2004); Zorbala and Tzanakis (2004)

Of course, some studies recognize history as a cognitive tool and at the same time as a motivational or affective tool (or some other combination of the roles), but I have tried to classify them according to what appears to be their main objective.

It should be mentioned that some of the studies concerning history as a tool also acknowledge the role of history as a goal, although this is not their main objective (e.g. Hsieh and Hsieh, 2000; Liu, 2000; Su, 2007). Other studies focus on history as a goal, but mention the possible side effects in terms of history as a tool (e.g. Kjeldsen and Blomhøj, 2009; Jankvist, 2009b). Of studies mainly concerned with using history as a goal we find:

Studies Using History as a Goal

Demattè (2007); Demattè and Furinghetti (1999); Isaacs et al. (2000); Kjeldsen and Blomhøj (2009); Jankvist (2008b); Jankvist (2008j); Jankvist (2008c); Jankvist (2009b); Jankvist (2009c); Jankvist (2009d)

Thus, only four studies in the above list, not counting my own, focus on the use of history as a goal.

Categorizing the above empirical studies was not always an easy task. Some of the classifications are results of interpreting what seemed to be the authors' underlying motivation. In particular, this was the case when having to separate some of the tool studies into those using history as a motivational tool and those using it as a cognitive tool. This being said, it was fairly easy to separate the studies focusing mainly on history as a tool from those focusing mainly on history as a goal.

I shall not go into individual descriptions of the 81 studies in table 2.2, arguing for each of their classifications above, since this would be a rather lengthy process and more importantly, it would not necessarily add much to the research study of this dissertation, the reason being that it first and foremost is concerned with the use of history as a goal. For the same reason I shall, however, consider the few empirical studies on history as a goal listed above.

The Empirical Studies on History as a Goal

Demattè and Furinghetti (1999) discuss a questionnaire of 39 questions used to study 13 year old students' beliefs about mathematics as a socio-cultural process. The questionnaire was given to a total of 288 students. Several of the 39 questions relate directly to matters of history as a goal as discussed in the sections above. An example of a finding related to these questions is that students on the one hand have doubts about the existence of mathematics independent of the Western culture, but on the other hand believe students all over the world to study the same mathematics (Demattè and Furinghetti, 1999). Based on the (history as a goal) questions, the authors conclude that the typical student has only primitive ideas on the birth and development of mathematics. The line of thought from this study is continued in Demattè (2007), a paper displaying 35 different questionnaire questions relating to students' images of mathematics and discussing briefly some of the students' answers to them.

Isaacs et al. (2000, p. 123) describe a "cultural origins of mathematics" unit introduced to first year pre-service primary mathematics teachers with the goal of modifying their "world view of mathematics". Through historical examples of geometry the authors tried to show the students that geometry is a practical science used to solve problems; a constructive and aesthetic medium where patterns, transformations, and geometrical relationships predominate; the interplay between geometry and religion in history and culture; geometrical measurements as an introduction to numbers which are not rational; and matters of logical justifications in geometry (Isaacs et al., 2000, pp. 124-126). 57% of the students found that the unit had helped them change their attitude towards mathematics, 25% were undecided and 18% disagreed. In particular, the authors mention the deepening answers of one student, who pointed to the fact that the unit had helped dispel some of the myths associated with mathematics: that it is based on genius mentality, it has nothing to do with reality (life), that mathematics must be done mentally and fast (if you cannot, you are not mathematical), and that mathematics is purely a Western phenomenon.

In their paper, Kjeldsen and Blomhøj (2009) discuss how history (and philosophy) of mathematics have been integrated at university level mathematics education through problem-oriented project work (based on examples from Roskilde University natural science basic studies). For two out of three examples of students' projects, the authors illustrate how the students gained insights into elements of the history of mathematics. The first example is a project on a mathematical model as an explanation of cell division proposed by Rashevsky in 1934. Kjeldsen and Blomhøj (2009, p. 94) describe some of the (history as a goal) outcomes of this project as: "the students learned about some of the problems in the history of the development of a new scientific discipline of mathematical-biology in the 1930s. Through this project work, the students managed to put themselves in a position where they could understand the discussion that took place in the 1930s, and they were able to uncover (some) of the causes behind the conflict. This gave the students a consciousness about history of mathematics, tolerance and insights into problems of communication that can be observed today between scientists from different sciences about 'proper' methods to produce scientific knowledge." The other example concerned whether linear programming, as developed by Dantzig and Kantorovich, respectively, was in fact a multiple discovery. The student's (history as a goal) profits from the project were: "how mathematics has developed historically in

space and time, in culture and society. This case especially demonstrated that despite the universal character of mathematics, its historical development is not unaffected by the surrounding society” (Kjeldsen and Blomhøj, 2009, p. 100). The framework used by Kjeldsen and Blomhøj in discussing these student outcomes is the so-called KOM-report (Niss and Jensen, 2002), which I shall rely on myself and therefore discuss in some detail in chapter 3.

2.9 Recapitulation

So, what are some of the main points of this chapter? First and foremost, the analysis of the different arguments for and approaches to using history promoted in the literature has shown how these whys and hows may be placed into two proposed sets of categories: for the whys the set of history-as-a-tool arguments concerned with students’ learning of the inner issues of mathematics, and history-as-a-goal arguments concerned with the use of history as a self-contained goal; for the hows the set of illumination approaches, modules approaches, and history-based approaches.

Second, it is illustrated that a distinction between the use of history being either primarily concerned with the historical meta-issues of mathematics (development, driving forces, societal and cultural circumstances, time and place, etc.) or with the in-issues of mathematics (concepts, theories, disciplines, methods, etc.) provides a means for exposing some of the interrelations of the whys and hows. It is, however, argued that when uncovering these interrelations one should not, in the categorizations, define the hows on basis of the whys, or vice versa, since this may blur the picture of the interrelations. Instead, the categorizations of whys and hows should be strictly separated. Something the two sets of categories presented here aim at fulfilling. As for the interrelations between the whys and hows, it is clear that not all hows are equally suitable for fulfilling the different whys. Some relations simply appear ill-suited, while others appear non-realizable or contradictory. Knowing about these relations would benefit mathematics teachers, curriculum developers, and material designers, e.g. since choosing an ill-suited ‘how’ may obstruct the fulfilling of an intended ‘why’. The material designer who, for instance, wants to use history both as a goal and a tool, and possibly also as an affective as well as a cognitive tool, can use the knowledge about the interrelations to navigate between the different hows in order to fulfill the intended whys.

Third, it is shown that also the criticisms of using history may be ordered according to the categories of whys and hows as well as their interrelations.

Fourth, the categorization, including subcategories, of the whys also proved efficient in classifying the literature on the empirical research studies (bringing in a category of overarching studies).

Finally, we have seen that the literature on history in mathematics education have gone from being of a mostly advocating nature over a descriptive one to now comprising some amount of actual research (empirical and analytical both). As for the empirical research studies surveyed, these indicate that the focus among researchers within the field lies almost entirely with history as a tool, and in particular history as a cognitive tool. Out of the 81 empirical studies found, only four focussed on history as a goal (not counting the ones being results of the present study). Throughout the rest of this dissertation, I shall address empirically and experimentally the use of history as a goal.

3 Research Questions and Theoretical Constructs

In this chapter the theoretical considerations supporting the research of the dissertation will be presented and related to the framework developed in the previous chapter.

Before presenting the theoretical considerations, the use of theoretical constructs will be discussed briefly on more general premisses. The choice of theoretical constructs will be based upon a discussion of the dissertation's three research questions, which shall be presented in their final form in section 3.2. After having presented the theoretical constructs, which in the rest of the dissertation shall be used to answer the three questions, I shall reveal my own personal position towards the research questions in section 3.9, as well as towards the use of history in mathematics education in general.

3.1 Of What Use are Theoretical Constructs?

Before trying to provide an answer to the question in the heading of this section, another question to address briefly might be: what is the role of theory in mathematics education research? According to Niss (2007, p. 1308-1309), theory may serve six different purposes: (1) it may provide *explanation* of some observed phenomenon; (2) it may provide *predictions* of certain phenomena; (3) it may provide *guidance for action and behavior*, e.g. in specific implementations in order to achieve certain goals; (4) it may provide a *safeguard against unscientific approaches*, enabling us to avoid “haphazard and inconsistent choices with regard to terminology, research methodology, and interpretation of results”; (5) it may provide *protection against attacks from the outside*, that is attacks from researchers in other scientific disciplines questioning the foundations and results of our research; but most importantly, (6) it may provide a *structured set of lenses* through which phenomena may be “approached, observed, studied, analysed or interpreted”. Concerning this latter purpose, Niss continues:

This takes place by selection of the elements to be considered important in the context, by focusing on certain features, issues, or problems; by adopting and utilising particular perspectives, and providing a *methodology* for answering questions concerning the domain(s) considered. (Niss, 2007, p. 1309)

Lester, Jr. (2005) addresses the role of theory in mathematics education research through a discussion of the nature of research frameworks. Lester, Jr. (2005, p. 458) uses the metaphor of a *scaffold* for a framework: “A scaffold encloses the building and enables workers to reach otherwise inaccessible portions of it. Thus, a research framework is a basic structure of the ideas (i.e., abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated”. Such a research framework “helps

us develop deep understanding by providing a structure for designing research studies, interpreting data resulting from those studies, and drawing conclusions.” Inspired by the anthropologist Magret Eisenhart, Lester distinguishes between three kinds of frameworks: *theoretical*, *practical*, and *conceptual*. Within theoretical frameworks the researcher uses accepted conventions of argumentation and experimentation associated with the theory, and research questions are rephrased in terms of this and to some extent determined by it. The goal of the research and the gathered data is to support, extend, or modify the theory in question. Although approaching research in mathematics education through theoretical frameworks has the advantages of “facilitating communication, encouraging systematic research programs, and demonstrating progress”, it also has the serious disadvantages of forcing researchers to explain their results by ‘decree’ rather than evidence, making data ‘travel’ to serve the theory, not setting a discourse helpful in everyday practice, and not offering enough opportunities for validation by triangulation (Lester, Jr., 2005, p. 459). A practical framework is based on accumulated practice knowledge and ‘what works’. One advantage of practical frameworks over theoretical frameworks is that the problems addressed are those of the people directly involved. However, a drawback is that the practical frameworks suffer from being, at best, only locally generalizable, e.g. because of narrow insider perspectives. Conceptual frameworks focus more on justification than on explanation and base themselves on previous research and theory both. However, instead of relying on only one overarching theory, as with theoretical frameworks, they build on a variety of sources and can be “based on different theories and various aspects of practitioner knowledge, depending on what the researcher can argue will be relevant and important to address about a research problem” (Lester, Jr., 2005, p. 460). With reference to Gravemeijer and Claude Levi-Strauss, Cobb (2007, p. 29) refers to such a scenario as a *bricolage*, a term relating to the French *bricoleur*, a handyman skilful of using whatever tools available in a given situation. Thus, in conceptual frameworks, as well as in a bricolage, it is the problems which drive the research and helps identify the theoretical constructs used to build the scaffold. One problem of taking such a bricolage or conceptual framework approach is, as also indicated by Cobb (2007, p. 31), that of co-existence and conflict. That is to say the applied theoretical constructs must be able to co-exists in such a manner that they do not conflict with or contradict one another.

Schoenfeld (1992b, p. 180) distinguishes between three different lines of research within the learning sciences in general: *method-driven* research, *theory-driven* research, and *problem-driven* research. (Possibly one could also speak of a fourth type; data-driven research. This would be research which takes its departure point within already accumulated data, e.g. the PISA or the TIMSS.) Theory-driven research corresponds to Lester’s theoretical frameworks, whereas method-driven research seems to fit better within the practical frameworks. Problem-driven research may fit under either Lester’s conceptual frameworks or the practical frameworks. Also Arcavi (2000, p. 145) distinguishes between theory-driven and problem-driven research, claiming that such a distinction “help[s] to pinpoint the main initial impetus” of what researchers in mathematics education do (though most may use a “dialectic integration of the two”).

The research study of this dissertation is clearly a problem-driven one, which the research questions in the following section will illustrate. These research questions have been rephrased in terms of the, to some degree, theoretical framework of tool and goal developed in chapter 2. As briefly touched upon in the introduction of this chapter, the

idea is to identify the – implicit or explicit – underlying theoretical elements, assumptions, underpinnings, or points of view on which the problem driven research rests, in order to build a conceptual framework to help answer them. The analysis of the research questions will take place in section 3.3, and the rest of the chapter will be used to build the scaffold. It is my hope that the framework of tool and goal along with the scaffold built in this chapter together may provide explanation and justification: explanation of the research findings, and justification of the interpretation and explanation of these findings. Another kind of justification which the theoretical constructs on some level also may contribute to is the discussion of importance of the research questions themselves and even the methods applied in providing answers for these, e.g. methods for data gathering and designing experiments. The close relationship between theory and method is addressed by Silver and Herbst (2007). In addition to some of the points made by Niss and Lester above, they also mention the use of theory as “a tool that helps give closure to the corpus of data to study a problem, complementing the sources identified by the problem” (Silver and Herbst, 2007, p. 50).

3.2 Research Questions (RQs)

By means of the framework developed as part of the categorization of the whys and hows of using history in mathematics education (chapter 2), and prompted by the hypotheses in chapter 1, the research questions of the present empirical research study may now be formulated as:

- RQ1 In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of ‘history as a goal’?
- RQ2 In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and ‘ensured’ through a ‘modules approach’?
- RQ3 In what way may teaching modules focusing on the use of ‘history as a goal’ give rise to changes in students’ beliefs about (the discipline of) mathematics, or the development of new beliefs?

3.3 Identifying Theoretical Elements of the Research Questions

Behind all three research questions stands the so-called relevance or justification discussion for teaching mathematics, which is to say the discussion of reasons for letting mathematics be a subject taught to pupils at all educational levels. Besides having to do with *why* mathematics should be taught, this discussion also embraces questions of *what* mathematics should be taught (and once justification is settled, *how* to teach it). The latter concerns which mathematical in-issues should be part of a given curriculum, but also which meta-issues a given mathematics program considers important to include in the teaching and learning of mathematics. Questions concerning what meta-issues should be part of our mathematics general education are relevant in this sense, but also the discussion of whether or not mathematics itself should be part of our general education is relevant.

Research question 1 concerns students' capabilities to engage in meta-issue discussions and reflections about history as a goal. To deal with this question, it must be addressed more deeply what meta-issues of the evolution and development of mathematics actually are. Such an addressing must rely on a discussion of the nature of mathematics, what mathematics itself is in time and space, and to what extent mathematics is a product of time and space. In discussing these matters, I shall also identify and address some more general themes from the history of mathematics, themes which later shall play an important role in the empirical design of the teaching modules (due to the selection of historical cases in them). This also includes some discussion on how to actually conduct history of mathematics.

In research question 2, *anchoring* of the meta-issue discussions and reflections in the related in-issues of the modules, that is the subject matter substance of these, is referring to things said about the meta-issues which could not have been said without actually knowing and understanding the related in-issues. So in order to address this research question, one must discuss what it means to acquire and learn mathematical in-issues. That is to say, what it means to *understand* mathematics. Of course, there are many theories within didactics of mathematics which seek to explain this, or somehow provide insight into this very intriguing question. By surveying the available literature a bit, I shall try to locate theoretical constructs which may be applicable in the answering of research question 2, and then discuss these in more depth.

For research question 3, when talking about possible changes in students' beliefs and the development of new beliefs, one must naturally first address the question of what beliefs are. That is to say present and discuss some of the available and more widely used definitions and categorizations of beliefs, in order to make up one's mind about which to apply in the research study. In doing so, I shall also focus on students' beliefs about *mathematics as a discipline* as well as the changing of beliefs.

The beliefs literature, as well as a definition of what is to be understood by students' beliefs about mathematics as a discipline will be presented and discussed in section 3.8. The survey of literature on understanding mathematics and more in-depth discussions of selected theoretical constructs within this will take place in section 3.7. In section 3.6 I shall address some of the more general topics and issues from the history and historiography of mathematics. But first, I shall engage in the underlying discussion of the relevance and justification of mathematics education, first within a general context (section 3.4) and next for the Danish context in particular (section 3.5).

3.4 Justification and Relevance of Mathematics Education

Blomhøj (2001, p. 223-224) points to the fact that the problem of justification and relevance of mathematics education has both an objective and a subjective side. The objective side concerns the reasons for mathematics education placed in a societal context. The subjective side, on the other hand, concerns the single individual's (e.g. teacher or student) sense making of participating in mathematics education. Of course, such a distinction between the objective and subjective may be made concerning the justification of any subject. Nevertheless, the situation for mathematics is a special one in terms of objectivity and subjectivity, a situation sometimes referred to as the *relevance paradox* (Niss, 1994, p. 371).

The Relevance Paradox

The objective side of the paradox has to do with “the unreasonable effectiveness of mathematics” (Wigner, 1960; Hamming, 1980). That is to say mathematics applies to – and *is* applied in – a wide range of extra-mathematical subjects and practice areas. (The nature of mathematics both as a pure and an applied science will be addressed in section 3.6.) This use of mathematics permeates society at large, in its past and present functioning and evolution as well as in its future development. The use of mathematics in technology constitutes one example of this, both the material technology (physical objects and systems, e.g. computers and other microelectronic devices) and the immaterial technology (computer software, codes, geographical coordinates, calendars, money transactions, graphical representations, measurements of time, space, weight, currency, etc.) (Niss, 1994, p. 369-370). Other examples, also of an immaterial nature, are the various decision-making and controlling processes which take place as part of the infrastructure of a society and various forms of descriptions and predictions, e.g. about the weather and climate. The application of mathematics to these extra-mathematical areas is brought about by mathematical modeling, including the building, usage, and validation of such models. However, the embeddedness of mathematics into mathematical models and (other) immaterial technologies, as well as the further potential embedding of these into material technologies, brings about the subjective side of the paradox – namely that the mathematics in society becomes *invisible*, hence irrelevant, to us. Or as Philip J. Davis, co-author of *The Mathematical Experience*, sees it:

... it’s invisible to people because it’s in programs, it’s in chips, it’s in laws...
So you don’t see it, and if you don’t see it, you don’t think it’s there. (Davis, 2005 in Jankvist and Toldbod, 2005c, p. 17)

And not only is the mathematics hidden to the layman, it may also be hidden to the specialists, e.g. engineers and physicists. For example, when interviewed, several of the applied scientists working with the Mars Exploration Rover mission at JPL would, on first hand, reply that they did not use much mathematics in their work. But of course they did, only the mathematics was embedded in commercial software packages – the mathematics had to some extent been outsourced and therefore also hidden (Jankvist and Toldbod, 2007a, p. 14). Scenarios like these are exactly what constitutes the relevance paradox: “the simultaneous objective relevance and subjective irrelevance of mathematics” (Niss, 1994, p. 371). The invisibility of mathematics in society does of course not make it easier to justify mathematics education. We can continue Davis’ line of thought by saying: If you don’t think it is there then why should you care to know about it and study it?

But if the role of mathematics in society is not recognized by the public, what role does mathematics then serve, and why is mathematics then still a major subject in school? The reason is that mathematics education serves several purposes, and that various different arguments for having it there will be put forward by politicians, mathematics educators, mathematicians, people in the humanistic sciences, teachers, parents, and the students themselves. When dealing with some of these, it may be a good idea to distinguish between two different categories of issues; the descriptive/analytic and the normative (Niss, 1996). The analytic category deals with ‘what actually *is*’ (the case) and ‘*why*’, whereas the normative deals with ‘what *ought to be*’ (the case) and ‘*why*’. I shall address a few of the analytic issues first.

A Few Analytic Issues of Justification

One purpose which mathematics education has served in the educational systems throughout time is that of a ‘critical filter’. That is to say mathematics, e.g. in upper secondary education, serves as a “controlling access to many areas of advanced study and better paid and more fulfilling occupations” (Ernest, 1998, p. 42). Students – and parents – are aware of this ‘sorting mechanism’ of mathematics as a subject, and will therefore often pursue studies in mathematics despite the fact that the use of mathematics in society (mathematical modeling etc.) is hidden to them.

Another argument for the dominant role of mathematics education in the educational system is that the study of mathematics promotes students’ logical thinking. Blomhøj (2001, p. 227) refers to this issue as building on the idea of a “thinking muscle”, which may be trained through mathematics and applied in various other areas as well. However, according to Blomhøj, modern research shows that this is not the case: mastering of mathematical reasoning in one area of mathematics cannot even be transferred to other areas of mathematics, despite the fact that there may be significant structural similarities between the areas.

The last of analytical arguments which I shall mention is the so-called utilitarian argument. Niss (1994, p. 374) describes this as follows: “The dominant interest of society at large in relation to mathematics education is to provide for the *utilization, maintainance and development* of mathematics as an *applied science* and as an *instrument for practice* as means for technological and socioeconomic development, with the ultimate purpose of increasing the material wealth in society.” To accomplish this, mathematics as a pure science must be kept alive and well, and in order to do that society must provide a fairly advanced mathematics education, at least to a selected few. For the majority of the developed countries the utilitarian argument is probably the general answer to the justification problem of mathematics education. But it may, of course, also have to do with tradition within an educational system, from primary and all the way up through tertiary levels. The utilitarian argument is subject to critique, at least in its more traditional form. Ernest (1998, pp. 38-39) classifies the argument as a ‘myth’, and rates it as “greatly overestimated” and as only providing “poor justification” since the view that “academic mathematics drives its more commercial, practical or popular applications” ignores “the fact that a two way formative dialectic relationship exists between mathematics as practised within and without the academy.” Furthermore, regarding mathematics’ role in society as an (utilitarian) argument for justifying mathematics education, Ernest (1998, p. 40) claims that the overt role of academic mathematics in many of the more basic material and immaterial technologies is minimal, at least when it comes to operating them. Therefore “mathematics skills beyond the basic are not needed by most of the general populace in industrial societies” to ‘cope’ with potential changes as a result of the present information revolution, “if to ‘cope’ means to serve, as here, rather than to critically master”. Thus, the discussion of the role of mathematics in society indeed also touches upon *what* mathematics should be part of the mathematics education (curricula etc.) provided by the society as well as whom should receive it (the general populace or the selected few).

A Few Normative Issues of Justification

This brings me to the more normative issues, because what does it mean to “critically master”? Ernest is here referring to empowerment and social justice concerns, i.e. the empowerment of a given student as a highly numerate critical citizen in society. That is a citizen who is, as an example, capable of seeing through misuses of mathematics, e.g. in graphical representations, statistics, etc. and critically questioning the mathematical models which he or she is subjected to as a citizen. Such intelligent and concerned citizenship is important if society is to be developed in a democratic way (Niss, 1994, p. 376). Skovsmose (1990) talks about a *democratic competence* as “a socially developed characteristic [...] which people to be ruled must possess so they can be able to judge the acts of the people in charge”.¹ As indicated, mathematics education has an important role to play here, due to the use of mathematics throughout society. But how may mathematics education equip people with a democratic competence and the capabilities to exercise this kind of citizenship? Surely not every citizen need become an expert in mathematics. Instead it is suggested that the citizens should be provided with an insight into the experts’ expertise, and that this be done through an element of ‘general education’ or *Allgemeinbildung* (Niss, 1994, pp. 376-377; Blomhøj, 2001, pp. 241-242). Niss elaborates this view of “mathematics education for democracy”:

Mathematics education should be provided to everyone in order to give *insight into ‘the general’*, by which I mean: the constitutive features of and the essential driving forces behind the development of nature, society, and the lives of human beings. Insights into the general does not consists in facts and skills alone and for their own sake, but serves the acquisition of overview, knowledge and judgement of main patterns, connections and mechanisms in the world; the ultimate end being to create prerequisites for taking positions on and acting towards processes of significance to society and the individual. (Niss, 1994, p. 377)

Thus, in the terminology of the framework developed in chapter 2, such an “overview, knowledge and judgement” surely requires more than just an insight into the mathematical in-issues, it requires an insight into the meta-issues of mathematics as well.

Ernest provides a list which may be seen as an exemplification of the “insight into the general”. Ernest (1998, p. 49) adapts a distinction between (1) “developing technological capabilities” and (2) promoting “appreciation and awareness”. The first refers to teaching and having the students learn the mathematical in-issues (of a given curriculum) and developing their skills in relation to these. The second, however, has more to do with the meta-issues of mathematics. Ernest mentions the following seven elements of awareness to illustrate what he believes should be understood by mathematical appreciation:

1. Having a qualitative understanding [of] some of the big ideas of mathematics such as infinity, symmetry, structure, proof, chaos, randomness, etc.;
2. Being able to understand the main branches and concepts of mathematics and having a sense of their interconnections, interdependencies, and the overall unity of mathematics;
3. Understanding that there are multiple views of the nature of mathematics and that there is controversy over its philosophical foundations;
4. Being aware of how and the extent to which mathematical thinking

¹ Other sources addressing this are: Skovsmose (1994a); Skovsmose (1994b); and Ensor (2008).

permeates every day and shopfloor life and current affairs, even if it is not called mathematics;

5. Critically understanding the uses of mathematics in society: to identify, interpret, evaluate and critique the mathematics embedded in social and political systems and claims, from advertisement to government and interest-group pronouncements;

6. Being aware of the historical development of mathematics, the social contexts of the origins of mathematical concepts, symbolism, theories and problems.

7. Having a sense of mathematics as a central element of culture, art and life, present and past, which permeates and underpins science, technology and all aspects of human culture. (Ernest, 1998, p. 50)

Worth noticing in Ernest's list is that several of the mathematical awareness elements, or meta-issues, which he mentions are presented in such a way that dealing with them actually requires a certain amount of anchoring in the related mathematical in-issues. Niss (1996, p. 43) elaborates a bit further on his views of 'insights into the general' in a 'list' of goals for mathematics education at higher levels, involving aspects of mathematical modeling as well as "the scientific and philosophical nature and status of mathematics and of its position in society and culture, and into the history and development of mathematics as a subject which is a result of human activity". However, for the purpose in this dissertation, these views are better accounted for through a description of the mathematics program of Danish upper secondary school and the Danish report on competencies and learning of mathematics, the so-called KOM-report (Niss and Jensen, 2002).

3.5 The Danish Regulations and the KOM-Report

Before engaging full scale into the types of mathematical awareness dealt with in the KOM-report and in the Danish regulations, I shall report a little on the use of history of mathematics in the history of the Danish mathematics program for upper secondary school – also since the use of history is the main concern of this dissertation.

The History of Using History in Danish Regulations

History of mathematics in the Danish upper secondary mathematics program is not an altogether new initiative. The inclusion of history of mathematics can, actually, be traced all the way back to the regulation of 1953:

To understand the context of culture, it is relevant that elements, of a universally human interest, from the history of mathematics are included, and this along with illustrative examples from epochs within the history of mathematical thinking, serving the purpose of illustrating how fundamental problems have arisen and been solved. (Undervisningsministeriet, 1953, my translation from Danish)

Even though history of mathematics was mentioned in the 1953-regulation it was not an explicit demand that it be included in the teaching. To what extent it actually was included or if teachers merely thought of the above passage as a kind of 'good advice' is difficult to say. In 1961 the history of mathematics disappeared from the

regulations and remained gone for a period of 26 years (Jankvist, 2008a, p. 25). When it reentered in 1987, it was as one of the so-called ‘three aspects’ (these will be explained and discussed below). The demands for including elements of the history of mathematics were definitely more explicit in 1987 than in 1953, which on some occasions also caused that the history of mathematics found its way into the textbook systems,² although not to the same extent that this happened with the new regulation of 2007 (Jankvist, 2008a). In the 2007-regulation the historical evolution and development of mathematics has a more dominant role, since it is now one of the nine “academic goals” of teaching mathematics at upper secondary level. Of course, it is still too early to say anything about the overall effect of these more dominating demands for including history in comparison with, for instance, those of the 1987-regulation.

In any case, the inclusion of history in the Danish upper secondary mathematics program today must be seen and understood in the light of the 1987-regulation, which is to say the so-called ‘three aspects’. In the remaining part of this section, I shall account for mathematical awareness and, in particular, the inclusion of history at upper secondary level by first describing the ‘three aspects’ as well as their origin. Secondly, I shall relate these aspects to the KOM-report and then the KOM-report to the 2007-regulation.

The ‘Three Aspects’ and their Origin

The ‘three aspects’ of the 1987-regulation were the *historical aspect*, the *mathematical modeling aspect* and the aspect concerning the *inner structures of mathematics*. The aspects were described in the following way:

i. The historical aspect.

The students must acquire a knowledge of elements of the history of mathematics and mathematics in a cultural and societal context.

ii. The modeling aspect.

The teaching must provide the students with a knowledge of the construction of mathematical models as representations of reality and an idea of both the possible applications and limitations of mathematical models as well as making them capable of completing a modeling process in simple situations.

iii. The inner structures of mathematics.

The students must acquire an understanding of the characteristic thought processes and methods in mathematics as well as an insight into the role of these in the development and structuring of mathematical subject areas.

(Undervisningsministeriet, 1987, §20, my translation from Danish)

The treatment of the aspects was to take place as part of the teaching on the core curriculum through specially planned teaching modules on one or several of the aspects, varying according to the level of the mathematical education (mandatory level, intermediate level, or advanced level). In these teaching modules the teacher also had the opportunity to include supplemental curriculum in order to get the aspects into play. On mandatory level the modules should have a length of at least 20 lessons (each lesson being 45 minutes) and on advanced level approximately 25 lessons.

The ‘three aspects’ can be traced back to Niss (1980) where they appear as three out of ‘four subjects’ which were recommended to be included in the upper secondary

² For example the series of books called *Ind i matematikken* (Into the Mathematics): (Clausen et al., 1988); (Clausen et al., 1989b); (Clausen et al., 1989a); (Clausen et al., 1993).

mathematics program (Jankvist, 2008e, pp. 10-14):

Subject no. 1. The presence of the teaching of mathematics in school, its structure, setting, and background.

Subject no. 2. Mathematical models and model building for concrete extra-mathematical problems. This includes the interplay between characteristics of these problems and the characteristics and possibilities of the model, as well as the model's limitations, in order to say something about reality from looking at the mathematical model.

Subject no. 3. The concepts, methods, and building of mathematics, investigated according to the coherence of the discipline.

Subject no. 4. The development and societal position of mathematics seen from an historical perspective. (Niss, 1980, pp. 55-56, my translation from Danish)

The historical aspect thus corresponds to subject no. 4. The motivation for introducing this subject is, according to Niss (1980, p. 56), that the traditional teaching of mathematics often communicates mathematics as being a finished discipline without a history of development, a discipline detached from time and space and society, and consisting of a set of concepts, results, and methods, perhaps accompanied by examples of application. This equips the students with a picture of mathematics as something having no volume in the sense of connections to cultural, societal, and historical surroundings, the mathematics becomes 'flat' and static (Niss, 1980, p. 56):

[M]athematics has gone through and still goes through different kinds of development. Are these developments not made the object of teaching, a distorted picture of mathematics will be induced. Not that I believe there exists a finished and correct picture of mathematics which should be served to the students. But a cutting away of these aspects reduces mathematics to a one- or two-dimensional projection of the multi-dimensional organism it actually is. (Niss, 1980, p. 56, my translation from Danish)

In illustrating this multi-dimensional organism to the students it is important, according to Niss, to treat the inner as well as the outer driving forces of the development and evolution of mathematics – something I shall return to in section 3.6.

The modeling aspects corresponds to subject no. 2. The purpose of this subject is for the students to acquire a knowledge and an understanding of applications of mathematics and the background for it in extra-mathematical connections. Also, the students must be able to apply mathematics themselves as a means for treating such extra-mathematical problems, and critically analyze and judge already existing applications (models) (Niss, 1980, pp. 54-55). Subject no. 3 is the aspect concerning the inner structures of mathematics. The argumentation for this subject, like the former, has to do with the power mathematics has in extra-mathematical problems, and especially the deductive nature of mathematics is decisive and fundamental in this connection (Niss, 1980, p. 55). Niss points to the fact, though, that the theoretical coherence of the subject cannot be illuminated in its whole. For that reason such illuminations should be done 'regionally' instead of 'globally', but still in such a fashion that the treatment of the chosen 'regions' is exemplary (students should get an insight into the general – *Allgemeinbildung*). Subject no. 1 never found its way into the regulations for the Danish upper secondary mathematics program. Nevertheless, this subject is still as relevant today as it was three decades ago. Students of today still have a right to know why they must be educated in

certain subjects, certain topics and not others within the subjects, why the regulations look like they do, etc. Niss even states that this contributes to the students' beliefs: "The more layers and nuances which are part of the students' beliefs about mathematics, the more reflected and useful for them they become" (Niss, 1980, p. 55, my translation from Danish).

The 'three aspects' have remained intact as such in the regulations from 1987 up till the new regulation, and to some extent also in this. The new regulation was in some ways inspired, at least on a rhetoric level, by the so-called KOM-project, which was financed by the Natural Science Educational Council and the Department of Education.

The KOM-Report

The KOM-report, Danish abbreviation for *Competencies and Learning of Mathematics* (my translation of the title to English), the final report of the KOM-project, was published in 2002 (Niss and Jensen, 2002). The 'three aspects', or the three out of the four subjects, is to be found in this report.

The KOM-report lists a total of eight essentially different, yet closely connected, *mathematical competencies* together with three types of *overview and judgement* concerning mathematics as an area of practice. (These have also been referred to as the eight competencies of 1st order and the three competencies of 2nd order, see Niss, 1999.) The three types of overview and judgement are almost identical with 'three aspects' of the 1987-regulation, something which is also mentioned by the authors in the section on the upper secondary level (Niss and Jensen, 2002, pp. 267-269). About the type of overview and judgement concerning the "historical evolvement of mathematics, internally as well as in a societal context" it says:

The object for this type of overview and judgement is the fact that mathematics has evolved in time and space, in culture and society. [...]

The type of overview and judgement should not be confused with knowledge of 'the history of mathematics' viewed as an independent topic. The focus is on the actual fact that mathematics has evolved in culturally and socially determined environments, and on the driving forces and mechanisms which are responsible for this evolution. On the other hand, it is obvious that if overview and judgement regarding this evolution is to have *solidness* they must rest on concrete examples from the history of mathematics. (Niss and Jensen, 2002, p. 68, my translation from Danish, italics added)

The talk of "solidness" in this quote suggests that the historical meta-issues of the concrete examples somehow must rest on, and be anchored in, the mathematical in-issues of the chosen historical cases. For the upper secondary level the overview and judgement concerning history is deepened even further:

In the teaching of mathematics at the upper secondary level the students must acquire knowledge about the historical evolution within selected areas of the mathematics part of the level in question. The central forces in the historical evolution must be discussed including the influence from different areas of application.

Through this, the students must develop a knowledge and an understanding of mathematics as being created by human beings and, in fact, having undergone an historical evolution – and not just being something which has

always been or suddenly arisen out of thin air. (Niss and Jensen, 2002, p. 268, my translation from Danish)

So this type of overview and judgement is, indeed, very similar to the historical aspect. The counterpart of the modeling aspect in the KOM-report is called “the actual application of mathematics in other subject and practice areas”. This type of overview and judgement, like the modeling aspect, has to do with the application of mathematics in extra-mathematical areas, e.g. everyday life, society, or science, and the fact that such applications come into play through the building and use of mathematical models (Niss and Jensen, 2002, p. 67). The actual building of such models is in the KOM-report also part of an independent competency which I shall return to below. First, the counterpart to the aspect concerning the inner structures of mathematics. This is called “the nature of mathematics as a subject” (Niss and Jensen, 2002, p. 69). The argumentation for this type of overview and judgement is almost identical to the ones given in the 1987-regulation and that of subject no. 3 (Niss, 1980, pp. 55-56).

As already indicated, the three types of overview and judgement are not themselves mathematical competencies (not of 1st order, anyway). For that purpose they are too general and too directed towards mathematics as a subject area instead of mathematical situations. Niss and Jensen talk about them as a “type of ‘active insights’ concerning the character of mathematics and its role in society” and state that “these insights equip those who possess them with a set of viewpoints, which provide overview and judgement about the connection of mathematics to circumstances and allotting in nature, society, and culture” (Niss and Jensen, 2002, p. 66, my translations from Danish). Mathematical competencies, on the other hand, “consist in having knowledge about, understanding, being able to practice, apply, and commit oneself to mathematics and mathematical activities in a profusion of connections in which mathematics plays a part or can play a part”, or in other words a kind of “insightful readiness to act expedient in situations which hold a certain kind of mathematical challenges” (Niss and Jensen, 2002, p. 43, my translation from Danish). The idea of mathematical competencies have some similarities with Ernest’s idea of developing technological capabilities, and the three types of overview and judgement share many viewpoints with Ernest’s idea of mathematical appreciation and awareness (cf. section 3.4). The type of overview and judgement concerning the evolution and development of mathematics, especially overlaps with Ernest’s points number 6 and 7 (see page 50).

The KOM-report divides the eight competencies into two groups or two capacities which students should come to possess: (1) “to be able to ask and answer questions in and by means of mathematics” and (2) “to be able to handle language and tools in mathematics”. Group 1 contains the following four competencies:

(a) the ability to pose such questions and be aware of the kinds of answers available (*mathematical thinking competency*), (b) be able to answer such questions in and by means of mathematics (*problem solving competency* and *modeling competency*, respectively), and (c) the ability to understand, assess, and produce arguments for solving mathematical questions (*reasoning competency*). (Niss and Jensen, 2002, p. 45, my own translation from Danish, italics added)

Group 2 consists of the four remaining competencies which are about:

(a) being capable of handling different representations of mathematical entities, phenomena, and situations (*representation competency*), (b) being

able to deal with the special representations of mathematical symbols and formalisms (*symbols and formalisms competency*), (c) being able to communicate in, by means of, and about mathematics (*communication competency*), and (d) being able to make use of and relate to various technical aids for mathematical activity (*aids and tools competency*). (Niss and Jensen, 2002, p. 46, my own translation from Danish, italics added)³

It is important to understand that even though there are eight different and relatively bounded competencies, this does not mean that the competencies are without connections to one another. Often the situation is the exact opposite, that a competency cannot be obtained and possessed in isolation from the other competencies (Niss and Jensen, 2002, p. 43). For example, all of the four competencies in group 1 are likely to be present in a given situation of building a mathematical model.

I shall not describe the eight competencies in detail at this time – this will be done in chapter 8, when dealing with additional questions related to the research study. I shall, however, use a little effort on describing how the eight competencies and the three types of overview and judgement interrelate. The nature of mathematics as a subject is the type of overview and judgement which most closely follows the competencies, however the point is that “only if the special nature of mathematics as a subject is made the object of elucidation and consideration itself, is a conscious and articulated form of overview and judgement created” (Niss and Jensen, 2002, p. 69, my translation from Danish). Niss and Jensen (2002) continue by pointing to the mathematical thinking competency, the reasoning competency, and the symbols and formalisms competency as being the competencies which especially contribute to the creation of a foundation for such an articulated form of overview and judgement. Concerning the actual applications of mathematics, this type of overview and judgement goes hand in hand with the modeling competency, and to some extent the other competencies of group 1 as well. However, the competency and the type of overview and judgement must not be mistaken for one another: In the competency it is a matter of being able to act mathematically in an extra-mathematical situation. In the type of overview and judgement many concerns must be taken in order to be able to relate to a given application, both sociologically and science-philosophically. Concerning the type of overview and judgement about the historical evolution of mathematics it is not as clear what the corresponding competencies are, especially since the KOM-report does not operate with an ‘history of mathematics competency’ (Niss and Jensen, 2002, p. 68). On the other hand, it can be claimed that examples from the history of mathematics in reality may draw on any of the eight competencies as well as any combination of these.

The 2007-regulation

The 2007-regulation in some sense bases itself on the KOM-report, at least rhetorically, and therefore also implicitly on some of the ideas in Niss (1980). Under “purpose” of the teaching of mathematics in the 2007-regulation it says:

Through the teaching the students must obtain a knowledge about the important sides of the interaction of mathematics and culture, science, and technology. Furthermore they must develop an insight into the ways

³ The English names of the eight competencies are consistent with names used in the in-progress translation of the KOM-report.

in which mathematics can contribute to the understanding, formulation, and treatment of problems in various subjects, as well as an insight in mathematical reasoning. By means of this, the students must become capable of relating themselves to other peoples' use of mathematics and also obtain sufficient *competencies* to be able to complete a tertiary education involving mathematics. (Undervisningsministeriet, 2007, my translation from Danish, italics added)

The choice of the word “competencies” in the the last sentence of the purpose description is due to the influence of the KOM-report. The description of purpose in the new regulation is common for all three levels of mathematics education (mandatory, intermediate, advanced). Under the topics of “academic goals” and “supplemental curriculum”⁴ variations do exist from one level to the next. These variations, however, mostly concern the level of difficulty in the sense that it increases from level to level. The most comprehensive is therefore found in the description for the advanced level. Under academic goals we find the following three, out of a total of nine goals, which are very close to the three first subjects in (Niss, 1980) and therefore also the ‘three aspects’:

- account for mathematical reasonings and proofs as well as deductive sides of the building of mathematical theory
- demonstrate knowledge about applications of mathematics within chosen areas, including knowledge about application in the treatment of a more complex problem
- demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution (Undervisningsministeriet, 2007, my translation from Danish)

Another place where the views of the KOM-report seem to have entered into the regulation is in the description of the “identity” of mathematics:

Mathematics builds upon abstraction and logical thinking and embraces a long line of methods for modeling and problem treatment. Mathematics is indispensable in many professions, in natural science and technology, in medicine and ecology, in economics and social sciences, and as a platform for political decision making. At the same time mathematics is vital in everyday life. The expanded use of mathematics is the result of the abstract nature of the subject and reflects the knowledge that various very different phenomena behave uniformly. When hypotheses and theories are formulated in the language of mathematics new insight is often gained hereby. Mathematics has accompanied the evolution of cultures since the earliest civilizations and human beings' first considerations about number and form. Mathematics as a scientific discipline has evolved in a continual interrelationship between application and construction of theory. (Undervisningsministeriet, 2007, my translation from Danish)

When the students are to “demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution”, as stated in the academic goals of the regulation, one must assume that it is within the

⁴ Remember that since not every one of the academic goals may be reached through the core curriculum, 1/3 of the curriculum should be supplemental (cf. section 1.2).

frame of view of the above “identity” that they are expected to do so. Another way of phrasing this is to say that one purpose of the teaching of mathematics at the Danish upper secondary level is to shape the students’ beliefs about mathematics according to the above description of identity.

Concerning the historical evolution and development of mathematics, the views in both the KOM-report and the 2007-regulation – and certainly also those in (Niss, 1980) – are clearly concerned with the use of history as a goal in mathematics education rather than the use of history as a tool. In these descriptions, the purpose of including elements of the history of mathematics has to do with showing the students that mathematics is dependant on time and space, culture and society, that mathematics is not ‘God given’ but developed by humans, etc. – nothing about using history as a cognitive, affective, motivational, or pedagogical tool in the teaching and learning of mathematics! However, this does not mean that the use of history at the Danish upper secondary level should be detached from cognition. As seen in the previous quotes from the KOM-report, a “solidness” – or anchoring – of the historical elements in the mathematical in-issues of these is considered quite important.

3.6 Meta-Issues of Mathematics in Time and Space

Now that we know what kind of role history of mathematics should play in the Danish mathematics program and the KOM-report, we may turn our attention to the history of mathematics itself, and to the nature of mathematics itself as well as the nature of its evolution and development, which is the meta-issues.

From the previous sections in this chapter, we already know that mathematics is more than just a *pure science*, it is also an *applied science*, and it is furthermore an *educational subject* (taught and studied both). On top of this, mathematics is a *system of tools for societal practice* (cf. the role of mathematics in society), and it is a certain kind of platform for gaining *aesthetic experiences*. Thus, the nature of mathematics may be looked upon as five-fold, that is as having five different faces (Niss, 1994, p. 367-368; Siu, 1977, pp. 18-19).⁵ Niss (2001a, p. 15) states that mathematics is certainly not the only science possessing these five faces, but that it is unlikely that any other science possesses all five faces to the same extent and simultaneously, in the way that mathematics does.

All of the five faces, as well as the interplay between them, are reflected in the evolution and development of mathematics through time and space. In this section, I shall address some of the more general topics and issues related to the history and historiography of mathematics – also in the sense of *Allgemeinbildung* – dealing with aspects mentioned in the KOM-report as well as the interplay between some of mathematics’ five faces. The general topics and issues are:

- Inner and outer driving forces of mathematics, including matters of internalism and externalism, Whig and anti-Whig.
- Pure and applied mathematics.

⁵ Other authors argue that mathematics has a ‘different-number-than-five’-folded nature. Hansen (2008), for instance, discusses the dual nature of mathematics (pure and applied mathematics), and Semadeni (2008) discusses the the triple nature of mathematics (deep ideas, surface representations, and formal models).

- Epistemic objects and epistemic techniques.
- Discovery versus invention of mathematics.
- Multiple developments (discoveries/inventions) of mathematics.

All of these general topics and issues shall play a significant part later on in the dissertation (especially in chapters 5, 6, and 7), since they were used as frames for having the students discuss meta-issues of chosen historical cases, as well as a way of trying to anchor these discussions in the related in-issues of the cases. In order to understand these general topics and issues, and as a background knowledge for evaluating the students' later discussions of them, we must have a look at some of the underlying theoretical constructs and discussions. Let us consider each of them in turn.

Inner and Outer Driving Forces, Internalism-Externalism, Whig and Anti-Whig

As pointed out in section 3.5, one element of mathematics in time and space which is considered to be particularly important for Danish upper secondary level is that of illustrating both the inner as well as the outer driving forces and mechanisms in the evolution and development of mathematics. In a more science-theoretically embedded context, inner and outer driving forces translate into internal and external factors of the development of science. In fact, the debate of *internalism* and *externalism* has permeated the history of science for decades.

The debate concentrates itself around the question of what and how much meaning the external factors of, for instance, a technological, societal, or cultural nature should be taken into account in the dynamics of developing scientific theories, and likewise what role the more internal factors, e.g. those of a more sociological nature, play in the development. According to the *internalistic* historiography, science is all in all an autonomous activity and its evolution and development can only be understood from within a scientific discipline itself, without including interrelations with extra-scientific factors. The *externalistic* historiography, however, attributes exactly such factors a crucial role in the developmental history. Externalism does not disclaim the fact that science may have a certain inner logic and autonomy, but it does state that any deeper understanding of the scientific evolution and development must be based on the external influences (technological, political, or cultural) which the scientific community is exposed to (Kragh and Pedersen, 1991, p. 188). The internalism-externalism debate dominated and shaped the academic history and sociology of science from the beginning of the Second World War to the ending of the Cold War (Shapin, 1992, p. 333).⁶ Today many professional historians of science consider the debate to be a pseudoproblem, since a consensus has developed that these two aspects neither shall nor can be separated (Kragh and Pedersen, 1991, p. 188; Høytrup, 1982, p. 29). Kragh and Pedersen state that “history of science must be ‘contextualized’ in such a way that it reflects an integration of both external and internal aspects in a given historical setting” but continues to say that “in spite of this consensus (which is more an ideal than a practice) it is still relevant to distinguish between factors which are external and internal, respectively, in the scientific development” (Kragh and Pedersen, 1991, p. 188, my translation from

⁶ For an historical account of the beginning, development, and ‘end’ of this debate, see Shapin (1992).

Danish). The note about this consensus being more of an ideal than a practice seems to be an observation which especially applies to the history of mathematics.

According to Richards (1995, p. 123), when the historians of science, in general, were writing obituaries for the internalism-externalism debate, the historians of mathematics were digging the ditch between the two camps even deeper. At the time, Richards concluded “that the division between these two camps is not only *a* but *the* critical problem in the history of mathematics” (Richards, 1995, pp. 123-124). Kjeldsen et al. (2004) ascribe this tension to the composition of people working in the field:

The ideal historian of mathematics should be a mathematician as well as a full-fledged historian. Some of the practitioners identify more with one of the fields and the two groups often have very different ways of working with the historical texts. Mathematicians often tend to stress internal factors to the point of neglecting all external influences, whereas historians tend to embed their histories in a broader cultural, institutional, philosophical, and/or political context but tend to care less about the strictly logical conceptual development of the mathematical techniques. (Kjeldsen et al., 2004, p. 12)

A parallel to this distinction between historians and mathematicians is, according to Kjeldsen et al. (2004, p. 12), the “distinction between whig and anti-whig historical writing”. The notion of *Whig* history, or *Whiggish*, is due to the British historian Herbert Butterfield who in 1931 defined this as a way of measuring the past in terms of the present (Butterfield, 1973).⁷ Another way of saying this is that “what one considers significant in history is precisely what leads to something deemed significant today” (Fried, 2001, p. 395). Kragh (1987, p. 89) calls this type of history *anachronical* and explains that within such an historiography it is “considered legitimate, if not necessary, that the historian should ‘intervene’ in the past with the knowledge that he possesses by virtue of his placement later in time”. Kjeldsen et al. (2004, p. 12) state that a mathematician studying the history of his or her subject often is inclined to take such an approach and, hence, judge the contents of earlier mathematics on the standards of modern mathematics. Within the field of history of mathematics, Rowe (1996, p. 3) refers to such historians as *mathematical historians* (as opposed to the *cultural historians*). Extreme examples of mathematical historians are those of Bourbaki, in particular Jean Dieudonné and André Weil. In a discussion of why and how to do the history of mathematics, Weil (1978, p. 232), for instance, claims that “it is impossible for us to analyze properly the contents of Book V and VII of Euclid’s without the concept of group and even that of groups with operators, since the ratios of magnitudes are treated as a multiplicative group operating on the additive group of the magnitudes themselves.” A (cultural) historian, on the other hand, will approach the history from an *anti-Whig* point of view and will, thus, “typically look for differences in the mathematics of different times and in different locations, and explore historical changes in mathematics, without using modern ideas as a yardstick” (Kjeldsen et al., 2004, p. 12). Kragh (1987, p. 90) similarly talks about the *diachronical* ideal which “is to study the science of the past in the light of the situation and the views that actually existed in the past; in other words to disregard all later occurrences that could not have had any influence on the period in question.”

⁷ The term takes its name from the British Whigs, advocates of the power of Parliament, who opposed the Tories, advocates of the power of the King and the aristocracy.

Even though the question of internalism versus externalism is still alive within the history of mathematics, some historians in the field have chosen a path which reduces the importance of such a distinction. Epple (1998, p. 307), for instance, in his study of the history of topology discusses a method of causally coherent historical narratives. According to Kjeldsen et al. (2004) a consequence of this narrative is that the internalism-externalism dichotomy disappears since the historians must insist “on *both* the objective character of mathematical knowledge *and* the fact that this knowledge was constructed in a fabric of social and communicative action” (Epple, 1998, p. 307). As noted by Kjeldsen et al. (2004) (pure) external approaches to the history of mathematics are rare. However, new insights may be revealed if taking the external factors into account, especially in the more modern history of mathematics. For example, Kjeldsen et al. (2004, p. 13) state that “various aspects of the technological developments during the Second World War may have had a deeper influence on the development of 20th century mathematics than it is usually assumed.” One aspect of this, of course, has to do with the many new and modern applications of mathematics in various technological devices, in particular computers, in the twentieth century. As we shall see in chapters 5 and 6, such external matters had an important impact on the development of error correcting codes and public-key cryptography both, the two historical cases of the teaching modules of this dissertation’s empirical research study.

Illustrating inner *as well as* outer driving forces in the evolution and development of mathematics when using history of mathematics in mathematics education is one way of dealing with the internalism-externalism debate on the educational level, and at the same time illustrating some of the approaches to the actual conduction of the history of mathematics to the students. As indicated earlier, the problem of making the history of mathematics Whig is also one which is very present when using history in mathematics education (Fried, 2001; Fried, 2007). I shall return to this in chapter 8, when the actual uses of history of the present research study has been described in detail.

Pure Mathematics versus Applied Mathematics

Another topic which is rather general in the history of mathematics, and quite relevant for the historical cases of the two teaching modules, is that of pure mathematics versus applied mathematics.

In earlier times a sharp distinction between pure and applied mathematics was not an issue in the same sense it is today. For example, mathematicians like Euler and Gauss excelled in pure and applied mathematics both, as well as other disciplines. During the twentieth century, however, pure and applied mathematics – or mathematicians – seemed to drift apart. Pure mathematicians would pride themselves of studying abstractions which had no practical applications whatsoever (e.g. G. H. Hardy whom I shall return to later). But also applied mathematicians seemed to develop a more hostile attitude towards pure mathematics and pure mathematicians (Davis, 1994, p. 130). Philip J. Davis who worked as an applied mathematician during World War 2, holds a Ph.D. in pure mathematics from Harvard University, and is now a Professor Emeritus in applied mathematics at Brown University, describes the situation:

On the sociological and psychological level there seem to be a fair amount of hostility between the pure and applied mathematicians. The pure mathematicians like to think that they are at the top of the heap, and ‘the applied

mathematicians are just people down there that are just taking the stuff that we, the pure people, develop and...'. Then the hostility has another aspect, they say that the applied mathematicians are getting all their funding; the pure mathematicians are getting some funding, but it is not like the funding of the applied mathematicians. I think that in terms of the progress of the science of mathematics it goes both ways: The applied problems suggest pure mathematics, and pure mathematics is useful in the real world. (Davis, 2005 in Jankvist and Toldbod, 2005c, p. 20)

According to Chandler Davis, Editor-in-Chief of *The Mathematical Intelligencer*, the hostility of twentieth century pure mathematicians toward the applied mathematicians may be phrased as: "I am too noble to get my hands dirty on mechanical problems like you mere engineers", as well as more defensively: "You are destroying my true science if you entangle me with your reality" (Davis, 1994, p. 132). In fact, he claims that "Most 20th-century mathematicians talk as if they had a subject-matter outside of time and space" (Davis, 1994, p. 132). But as indicated by Philip J. Davis the hostility is directed the other way as well:

There was a very good applied mathematician that used to work years ago for the Bell Telephone Laboratories: Hamming. He was the one with Hamming codes. I knew him. He was a very clever fellow and he had a considerable disgust for some parts of pure mathematics. He said: 'If I knew that the safety of an airplane depended upon the Lebesgue integral, I would never fly in it'. What he meant by that was that the theories of measure and the Lebesgue integration and so on were perfectly irrelevant to the sort of problems he was interested in. (Davis, 2005 in Jankvist and Toldbod, 2005c, p. 21)

Chandler Davis suggests some different, but interconnected, explanations for the twentieth century 'hostility' between pure and applied mathematics and mathematicians. His first explanation is more or less that of (pure) mathematics serving as a 'critical filter' in society, as discussed in section 3.4. Davis phrases it like: "Society tells the student, even the working-class student, you may be a dentist if you pass the test, you may be a military officer if you pass the test... and the decisive test is in math" (Davis, 1994, p. 137). In this way, Davis says, (pure) mathematics performs the most decisive winnowing of students in formal education, and it helps in maintaining social stratification in the advanced industrial capitalist society. Twentieth century mathematics is more important than Latin grammar was as a 'sieve' in the nineteenth century, and it "far surpasses the authentically applied mathematics which maintains its interfluence with other sciences and engineering" (Davis, 1994, p. 137). In such a system, the (pure) mathematicians are given prestige and, unwillingly or not, come to act as the authority to determine young people's lives. According to Davis this only makes up a partial explanation, since mathematics did not begin to perform as a selection gate until around 1940. Thus, a second explanation is needed to include the earlier decades of the century. Davis considers the twentieth century's general turn toward abstraction, a phenomenon which occurred throughout the Western world: "Indeed there is a clear affinity between the 'modern' mathematics of the category, the scheme, and the topos, the 'modern' music of the row and the cluster, and the 'modern' painting and 'modern' poetry of multiple isms; there is the same defiant pride in incomprehensibility" (Davis, 1994, pp. 137-138). But also this explanation seems insufficient to Davis, and he therefore

extends the explanations in terms of the earlier mentioned ‘unreasonable effectiveness of mathematics in the natural sciences’. This ‘unreasonable’ success provided mathematics with an exceptional claim to be left alone. As opposed to engineers who had (and still have) to direct their research attention toward the economic power, mathematicians needed not lay off subjects of past concern (e.g. Fermat’s last theorem or the Riemann Hypothesis) just because new needs in technology arose. A situation which opened up for the possibility of pure mathematicians secluding themselves from any applications of their subject.

Another reason for this situation may have to do with the world wars of the twentieth century. During these, new applications of mathematics saw the light of day (or the dusk, some might say). To mention a few of the more obvious, mathematics has played an important role in ballistics and weapon development, design of airplanes, ships, tanks, etc., the Manhattan-project (the nuclear bomb), the Los Alamos project (the hydrogen bomb), the breaking of the German Enigma cipher, and the development of the computer (e.g the British Colossus or the American ENIAC).

Not all pure mathematicians were equally fond of these new applications of mathematics. One who was indeed very much in opposition to the use of mathematics in war, and therefore also to the applied mathematics, was the English number theorist G. H. Hardy. In an address to the British Association for the Advancement of Science in 1915, Hardy expressed his views on science and war, views which are particularly interesting in terms of the later use of number theory in the development of public-key cryptography (the historical case of the second teaching module): “A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life” adding that the study of prime numbers did neither, while “the greatest mathematicians of all ages have found in it a mysterious attraction impossible to resist” (Wells, 2005, p. 120). In his book *A Mathematician’s Apology* from 1940 (Hardy, 1992), Hardy develops this view further to say that there are two kinds of mathematics (and mathematicians); real mathematics of the real mathematicians and ‘trivial’ mathematics. It is clear, says Hardy, that the trivial mathematics is ‘useful’ and ‘does good’ in a certain sense, and that the real mathematics does not, but the question is whether any of them do *harm*, which, according to Hardy, equals the effect they have on war:

Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years. It is true that there are branches of applied mathematics, such as ballistics and aerodynamics, which have been developed deliberately for war and demand a quite elaborate technique: it is perhaps hard to call them ‘trivial’, but none of them has any claim to rank as ‘real’. They are indeed repulsively ugly and intolerably dull; even Littlewood^[8] could not make ballistics respectable, and if he could not who can? So a real mathematician has his conscience clear; there is nothing to be set against any value his work may have; mathematics is, as I said at Oxford, a ‘harmless and innocent’ occupation.

The trivial mathematics, on the other hand, has many applications in war.

⁸ From around 1911 and on the mathematician John Edensor Littlewood was the collaborator and close friend of G. H. Hardy.

The gunnery experts and aeroplane designers, for example, could not do their work without it. And the general effect of these applications is plain: mathematics facilitates (if not so obviously as physics or chemistry) modern, scientific, 'total' war. (Hardy, 1992, p. 140-141)

One aspect which Hardy leaves out of his discussion of war and mathematics is that of funding. As pointed out by Chandler Davis, the effectiveness of mathematics proper led to freedom of research among pure mathematicians whereas the applied mathematicians were more constrained by societal and technological needs. However, as pointed to by Philip J. Davis, when the applied mathematics directs itself towards these needs the funding will follow more easily than it will for pure mathematics. And during wartime there is no question where the money will go, they will go to the research that may advance warfare the most. This was the situation during World War 2 and to a large extent during the Cold War which followed – a situation which in return of the use of mathematics in war produced new mathematical disciplines (e.g. mathematical programming, operations research, and some new branches of mathematical statistics) and results, primarily in applied mathematics but also in mathematics proper.

Despite the in-fight between pure and applied mathematicians, pure mathematics and applied mathematics are still two sides of the same coin: mathematics – and they still live off and feed one another. Clearly applied mathematics draws on pure mathematics, though often with some delay in time, but as already suggested there is also a feedback. Aspects of applied mathematics may provide new insights to problems in pure mathematics too. One example of this are the error correcting Golay-codes (these will be discussed further in chapter 5) about which the coding theoretician Ralph Kötter said in a personal interview:

For example the Golay code. The classification of simple groups. That is a real contribution to mathematics, it has nothing to do with engineering, nobody in engineering cared about that. [...] So there is stuff going back and forth between pure and applied mathematics. (Kötter, 2005 in Jankvist and Tolddod, 2005c, p. 27)

Epistemic Objects and Epistemic Techniques

One thing is (some) pure mathematics becoming applied mathematics over time, another thing is the use and 'application' of established mathematical concepts, methods, etc. in the development of new (pure) mathematics itself. One way of looking at this development is through the approach of epistemic *objects* and epistemic *techniques*, as introduced into the history of mathematics by Epple (2000) on the basis of Rheinberger (1997). This approach seems a very appropriate configuration if one wishes to study the evolution and development of mathematics in time and space, and also the effect of inner and outer driving forces on this development since these too, as we shall see is the case for the epistemic objects and techniques, are likely to change over time and in different spaces. Furthermore, from an educational point of view, the notions of objects and techniques seem a promising way of anchoring students' meta-issue discussions in the related in-issues. This I shall return to in chapters 4 and 5, but for now, let us get an idea of the notions.

Within a context of history of modern microbiology and experimental systems, the smallest units of research, Rheinberger (1997) distinguishes between what he calls

epistemic things and technical objects. The epistemic things or objects, research objects or scientific objects as Rheinberger also occasionally calls them, are the “material entities or processes – physical structures, chemical reactions, biological functions – that constitute the objects of inquiry” (Rheinberger, 1997, p. 28). Epistemic objects are themselves characterized by their vagueness, paradoxically, since they are the objects under investigation and therefore embody what one does not yet know. However, Rheinberger states, they are not simply hidden objects to be uncovered, they are absent in their experimental presence. Rheinberger refers to Latour (1987, pp. 87-88) who within the context of ‘science in action’ claims: “The new object, at the time of its interception, is still undefined. [...] At the time of its emergence, you cannot do better than explain what the new object is by repeating the list of its constitutive actions; ‘with *A* it does this, with *C* it does that. It has *no other shape than this list*. The proof is that if you add an item to the list you *redefine the object*, that is, you give it a new shape.” In order to enter such a stage of operational redefinition, the researcher needs an agreement referred to as the experimental conditions, which is the technical objects. It is through these technical objects that the epistemic objects become ‘materialized’ in a wider field of epistemic practices and cultures. The technical objects, in contrast to the epistemic objects, “tend to be characteristically determined within the given standards of purity and precision”, says Rheinberger (1997, p. 29) and continues:

But the point to be made is that within a particular experimental system both types of elements are engaged in a nontrivial interplay, intercalation, and interconversation, both in time and space. The technical conditions determine the realm of possible representations of an epistemic thing; and sufficiently stabilized epistemic things turn into the technical repertoire of the experimental arrangement. (Rheinberger, 1997, p. 29)

Epple (2000) has adapted and applied Rheinberger’s concepts of epistemic objects and technical objects to research in the history of mathematics. Within the context of mathematics research the concept of technical objects is a bit off since mechanical devices seldom play the most important role in mathematical research. Instead, mathematicians rather use intellectual courses of action in their work. Epple therefore talks about epistemic techniques instead technical objects, so that we in the context of mathematics talk about *epistemic objects* and *epistemic techniques* – in short just *objects* and *techniques*. Epple uses the metaphor of a ‘cabinet maker workshop’ where the objects are being worked on by the cabinet maker using more or less worked out and developed techniques. Epple explains:

The objects do certainly not concern finished existing things, but rather partially understood, partially not understood mathematical, intellectual constructs to which the time-dependent studied questions refer. The techniques, to some extent, make up reliable functional procedures which may at least provide partial answers to the questions. (Epple, 2000, p. 149-150, my translation from German)

Together the epistemic objects and techniques constitute what Epple calls the *epistemic configuration*. This is where the research mathematicians (pure or applied), within a given space and time, perform their work – it is their ‘mathematical workshops’. Kjeldsen (2009a) has used the concepts of objects and techniques to study the early history of the modern theory of convexity. She notices that the “concepts of epistemic objects and techniques seem to be promising working tools for micro-historical approaches to

history of mathematics precisely because they are constructed to distinguish between how problem-generating and answer-generating elements of particular research episodes function, interact, and change in the course of the work of a specific mathematician or group of mathematicians” (Kjeldsen, 2009a, p. 88). In this respect the historiographical approach of the epistemic configuration may seem closer related to internalism. However, this relation to internalism is not decontextualized since the epistemic configurations are located in time and space. And, at any rate, the ‘problem-generating elements’ do open up for discussions of the effect of outer driving forces and influences on the ‘mathematical workshop’.

When using the approach of epistemic objects and techniques it is, however, important to be aware of the fact that objects and techniques change in time and space, from one ‘workshop’ to another. As implied in the quote above by Rheinberger, an assumption is that what in one instance of time and space is an epistemic object under investigation may in another instance be a technique used to study another object. Such a propagation of objects to techniques is common in the evolution and development of mathematics. As an example, once Fréchet had developed the mathematical concept of a metric in 1906 this could be used by Hausdorff to axiomatize the notion of a topological space in 1914 (Katz, 1998, p. 818-819). Epple (2004, p. 151) also talks about “the dynamics of epistemic configurations in research processes” and points to the fact that the configuration elements may change places: “The epistemic objects of one research episode may turn into tools for another or they may vanish from a mathematical laboratory altogether. Techniques may themselves move in the focus of research interest and become modified for new tasks.” So objects may not only become techniques, techniques may also become objects. One example of an epistemic technique in mathematics – although not in the sense of a well-defined, well-understood, and previously studied object – which later became the object under investigation itself is that of complex numbers. Cardano and his contemporaries used complex numbers as a technique to solve algebraic equations, just as long as the complex numbers canceled out in the end and led to solutions in the (positive) reals. But with Caspar Wessel in 1799 the complex numbers themselves became the epistemic object of study. Another example is that of negative numbers. In medieval Italy, negative numbers were introduced as a technique, by introducing plus and minus signs, to handle overweight and underweight bales of goods. And also negative numbers served as a technique, like the complex numbers, in algebraic equation solving. However, it was not until negatives were accepted as roots to these equations that they were finally accepted as numbers and as mathematical objects in themselves. (Sfard (1991) discusses this situation in the context of learning and understanding mathematics, which I shall return to in section 3.7.)

A specific configuration of the dynamic relationship between objects and techniques, which neither Rheinberger nor Epple seem to notice, is that object and technique in one context may *exactly* shift places in another. That is to say, in context 1, A is the object studied by means of B , the technique, but in context 2, B is the object studied by means of A , now serving as the technique. Such examples may be rare in the history of mathematics, nevertheless we shall see one from the early history of error correcting codes in chapter 5.

As the above examples show, and as implied in the quotes already given, the difference between epistemic objects and techniques is *functional* rather than structural. Therefore we may not once and for all draw a distinction between different elements of a given

mathematical workshop. One might then ask, why the distinction should be held at all: Why construct a division whose only effect is that it permanently has to be undone? One answer to this is the quote by Kjeldsen (2009a) on page 64, another is provided by Rheinberger (1997, p. 31) himself: “because it helps to assess the game of innovation, to understand the occurrence of unprecedented events and with that, the essence of research.” Both answers suggest the potential of the notions of objects and techniques in illustrating aspects of ‘history as a goal’ in a teaching and learning situation.

Discovery of Mathematics versus Invention of Mathematics

The fourth of the general topics and issues is of a more philosophical (or epistemological) nature, since it asks whether mathematics (mathematical objects, concepts, etc.) is discovered or invented. This topic does not play a role in the actual design of the two modules or in the answering of research questions 1 and 2, but it plays a crucial role in the answering of research question 3 on students’ beliefs. This will be explained in detail in chapter 7, but since the topic also relates to the next of the general topics and issues, I shall sketch a picture of it here.

In its most absolute form the discussion of discovery versus invention breaks down to that of Platonism and anti-Platonism. *Platonism*, taking its name from the Greek philosopher Plato (429-347 B.C.), regards mathematical objects as real objects which exist quite independent by our knowledge of them (cf. Plato’s *Parable of the Cave* as given in *The Republic*). G. H. Hardy, for example, was a Platonist. When discussing idealism versus realism⁹ in his *Apology*, he says: “Pure mathematics [...] seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way” (Hardy, 1992, p. 130).

The most extreme of the anti-Platonist views is that of *formalism*. From a formalist point of view there are no mathematical objects. Mathematics consists of axioms, definitions, theorems, etc. These are not *about* anything. They are just strings of symbols which mathematicians may manipulate according to a set of rules; in a sense mathematics is a ‘game’ we play. Of course, formalists know that mathematical formulas may find an application to, for instance, physical problems, and that they within such a physical interpretation may acquire a meaning, i.e. may be true or false. But according to the formalist this is only within the physical world, within mathematics the formula has no meaning and no truth value.

Despite the fact that Platonists and formalists quarrel about existence and reality of mathematical objects, they have no objections to the principles of reasoning applied within mathematics. Mathematical constructivists however do. They only regard what can be obtained by a finite construction as genuine mathematics. Therefore they do not consider non-constructive existence proofs to prove anything. For instance, if one assumes that an object does not exist and from there derives a contradiction, one

⁹ Mathematical *realism*, like realism in general, holds that mathematical entities exist independently of the human mind. *Idealism* is any system or theory which maintains that the ‘real’ is of the nature of thought, or that the object of external perception consists of ideas. Though Plato is often referred as an ‘idealist’ he does, in fact, not describe ‘Ideas’ as being in the mind. Instead he describes them as having their own independent existence. It was not until the Christian philosopher Augustine reinterpreted Plato’s philosophy that ‘Ideas’ ended up *in* the mind (Radford, 2004).

has according to the constructivists still not proved the existence of the object; the proof of existence is tied to the possibility of construction. The only thing which truly exists in the constructivist view are the natural numbers, and from these all meaningful mathematics must be constructed (Hersh, 1997, p. 63).

Constructivism and formalism were hot issues around the first half of the twentieth century. Today some mathematicians may still call themselves constructivists or formalists, but in philosophy one speaks more commonly of Platonists versus fictionalists (Hersh, 1997, p. 139). Fictionalists are anti-Platonists of any kind; constructivists, formalists, etc. But what are the majority of mathematicians today then: Fictionalists or Platonists? Hansen (2008, p. 8) believes that no mathematician nowadays holds the Platonist views: “rather, they would stop with Aristotle, who dismissed the existence of an independent world of ideas but discriminated between concrete and abstract aspects in the world of sensations.” In contrast, Davis (1994, p. 133) claims that (pure) mathematics of the twentieth century has been heavily influenced by a “naïve Platonism”, i.e. “the doctrine that every properly defined mathematical object exists.” According to Davis, the bizarre thing is that mathematicians insist on living by the doctrine, but at the same time do not seem to really believe in it.

But why is all this important? Is it not only a question of philosophy, one could ask, what is its relevance in terms of ‘history as a goal’? Well, take for example the question of what we may and may not patent, an important question within science and applied science in particular. One would think that only inventions could be patented for commercial use, not discoveries. However, DNA strings, which are surely not invented, are being patented continuously. And within mathematics it is possible to patent mathematical formulas and algorithms, something which is in strong contrast to the (naïve) Platonist view held by many mathematicians. As we shall see in chapters 5, 6 and 7, the question of patenting and thus also that of discovery versus invention is a quite relevant question for the chosen historical cases of the two teaching modules.

Multiple Discoveries/Inventions

The last of the general topics and issues is also one which is highly relevant for both of the historical cases in the modules (to be explained in chapters 5 and 6), namely that of multiple developments in science, in the literature referred to as *multiple discoveries*. One of the more famous and well known examples of this in the history of mathematics is that of Newton’s and Leibniz’ independent developments of infinitesimal calculus. As a topic of study among historians and sociologists, the phenomenon of multiple discoveries is not a new one. Ogburn and Thomas (1922) mention 148 examples of what they find to be multiple discoveries. Multiple discoveries need not be simultaneous, though they can be, the main criterion is that they have occurred in time and space independent of each other.

One discussion on multiple discoveries concerns how common or rare these are. Ogburn and Thomas (1922), with their 148 examples, advocate that multiple discoveries are a very general aspect of the evolution of natural sciences. Merton (1973, p. 356), one of the most important researchers within natural science sociology, proposes the hypothesis that multiple discoveries are, in fact, the very dominant pattern of development; that “all scientific discoveries are in principal multiples, including those that on the surface

appear to be singletons.” Merton provides ten kinds of evidence for his hypothesis. The first of these concerns the class of discoveries long regarded as singletons, which later turn out to be rediscoveries of unpublished work. The following six kinds of evidence all concern types of forestalled multiples: “discoveries that are historically identified as singletons only because the public report of the discovery forestalled others from making it independently” (Merton, 1973, p. 358). Should the researcher choose to publish anyway, a remark concerning the awareness of previous publications is often provided. Merton’s last three kinds of evidence concern the way researchers act. Merton suggests that the behavior of researchers testifies to their belief that all scientific discoveries are potential multiple discoveries. A ‘race’ is therefore taking place, and researchers will try to protect themselves from being overtaken. Researchers will keep diaries, letters, dated notes, and other records of their discoveries to cover their back, or they will leak information to others in such a manner that they do not reveal too much. An example of researchers’ behavior is that of Andrew Wiles (which will become relevant in the chapters to follow). For seven years Wiles had worked in secrecy on the proof of Fermat’s last theorem since he found that it created too much interest when he mentioned it to colleagues. When a flaw was found in his first proof, given in June 1993, he ‘locked himself up’ for about a year until he in September 1994 could provide the complete proof. The reason for a behavior like this has to do with the sociology of scientific research. Merton (1973, p. 361) explains: “the culture of science puts a premium not only on originality but on chronological firsts in discovery, this awareness of multiples understandably activates a rush to ensure priority.” An example which Merton does not seem to acknowledge is that of the time not being ‘ripe’ for certain scientific developments. The history of the proof of the impossibility to solve the quintic in radicals may make up such an example (Jankvist and Sağlanmak, 2005b). Ruffini had been convinced of the insolubility of the quintic for years and been trying to convince other mathematicians, e.g. Lagrange, about it, but nobody seemed to believe him. Of course Ruffini’s proof, though continuously corrected between 1799 and 1813, was incomplete, but this does not seem to be the main reason. Some people were still trying to solve the quintic, thus the time may have been ‘unripe’. Not until twenty-five years later with Abel in 1824 did people begin to pay attention.

Other researchers of course disagree with Merton. One who is in opposition to Merton’s hypothesis is Patinkin (1983) who claims that many of the asserted multiple discoveries, in fact, are singletons if they become subject to a more thorough analysis. Such an analysis, says Patinkin, must rely on two things. First, an evaluation of the ‘central message’ in the publication containing the discovery, i.e. is the discovery central in the publication, e.g. in the title or abstract, or does it only occur as a subordinate result. And second, the degree to which the discovery stimulates further research among colleagues and in the field of inquiry in general. Patinkin’s second criteria to some extent acknowledges the role played by third parties, that is to say that sometimes the establishment of a discovery as a multiple or a singleton must come from the outside (Cozzens, 1989). In the history of mathematics, the historians often play the role of this third party, trying to establish past or present priority quarrels of mathematical results. However, the sociological approaches to multiple discoveries may not always be fully adequate to the study of history of mathematics. In her study of the Kuhn-Tucker theorem, Kjeldsen (Kjeldsen, 1999; Kjeldsen, 2000) finds that the sociological approaches, like those of Merton and Patinkin, may not be sufficient to describe certain aspects

of the history. Because these approaches are mostly concerned with the settlement of priority, they do not take the contexts in which the results have been developed and presented into consideration. And exactly this is an important aspect, one which also has to do with the maturity, or the ‘ripeness’, of the communities: “results do not shift contexts just like that. Often it takes a ‘new discovery’, and often it takes time, in which the result and the field of research have to mature, before one may assess what other scientific connections the result may enter into, and before one may see the general and wider context” (Kjeldsen, 1999, p. 218, my translation from Danish). Also the underlying implications of science being accumulative in nature, the inevitability of discoveries, and hence the de-emphasizing of geniuses in the development of science are aspects which many historians of mathematics undoubtedly will hold against the sociological theories of multiple discoveries.

In the light of the previous topic of discovery versus invention, the term ‘multiple discovery’, although I have used it above in consistence with the literature, may perhaps not be the most appropriate term. The reason being that it makes certain assumptions about the nature of scientific, in this case mathematical, development. It appears to me that a better term would be that of *multiple developments*, which I shall prefer to use throughout the rest of the dissertation.

This concludes the discussion of meta-issues regarding general topics and issues related to the history and historiography of mathematics, and as such the discussions of underlying theoretical constructs for research question 1. Next I turn to the theoretical constructs underlying the discussion of research question 2.

3.7 Understanding Mathematical In-Issues

An answering of research question 2, on the anchoring of the students’ meta-issue discussions and reflections in the related in-issues, must rely on an understanding of what is usually meant by understanding mathematics. To obtain this, I shall survey the literature a bit (without making it into an actual review).

When discussing understanding within mathematics education, this may either be related to (1) the understanding of certain topics, for example continuity, or (2) it may cut across topics or ‘leave’ the topics. As for 1, roughly speaking, such discussions usually concern: first, the understanding of the mathematical concepts in question; second, understanding of the related theorems, what these cover, their scope and limitations; and third, an understanding of why the used techniques actually work, and how, when, and under which circumstances they work. In the case of 2 one may, for instance, consider understanding of the interrelations and connections between different topics, e.g. between algebra, topology, and geometry. Or one may try to understand different ways of thinking, e.g. abstraction, generalizing, etc., thus ‘leaving’ the topics. To some degree, the eight mathematical competencies of the KOM-report, as discussed in section 3.5, are an example of such an approach to understanding. In this case, one leaves the specific topics of mathematics and discusses understanding on more general terms. Or as another example, one may try to understand the historical roots and development of a mathematical concept, theory, method, etc., which may provide a deeper understanding although not necessarily a better one in terms of 1 above. Or, as yet an example of leaving the topics, one can try to understand why mathematics is so

usable in extra-mathematical contexts. When scanning the literature on understanding within mathematics education, I found that most publications concern the understanding of certain topics (1), in particular mathematical concepts, though some also focus on the different ways of understanding (2), while others mix the two.

In the following subsection I shall present examples of theoretical constructs dealing with the understanding of concepts, ways of thinking, as well as combinations of these. This subsection, which constitutes the survey, serves the purpose of a 'stepping stone' for explaining the theory mainly to be used in the answering of research question 2, namely Anna Sfard's recent theory of commognition, since this theory builds on or relates to various of the other theoretical constructs. (The reasons for choosing exactly Sfard's theory of commognition shall be made explicit in chapter 5, when we are more familiar with the actual data of the research study.)

Cognitive Networks and the Processes or Acts of Building Them

One of the earliest categorizations of understanding within mathematics education is that of Skemp (1976). With reference to the Norwegian mathematics educator Mellin-Olsen, Skemp distinguished between *instrumental understanding*, which means to know how, and *relational understanding*, which means to know both how and why. In this sense, understanding becomes a style of knowing. Over the years Skemp expanded his model of understanding. First with a *logical understanding* (Skemp, 1979) and next with a *symbolic understanding* (Skemp, 1982), and this in such a way that all four kinds of understanding could occur in two different modes: the *intuitive* and the *reflective*. Thus, Skemp's model makes up "a 2×4 matrix yielding eight different elements to contend with in the analysis of understanding" (Herscovics and Bergeron, 1983, p. 76).

Another way to look at Skemp's early model is to look at the 'products' of the two kinds of understanding. Hiebert and Lefevre (1986, p. 23) discuss the relationship between skills and understanding, saying that "skills and understandings are important because they signal two kinds of knowledge that play crucial, interactive roles in the development of mathematical competence". These two kinds of knowledge are referred to as *procedural knowledge* relating to skills and *conceptual knowledge* relating to understanding. Procedural knowledge includes, for instance, knowledge about symbolic language, procedures, algorithms, and non-symbolic operating strategies, i.e. knowledge about the how. Conceptual knowledge is knowledge about the why, e.g. why a certain proof is correct. The point is not, however, to favor one of these types of knowledge over the other, the point is to focus on the *relationship* between them.

Focusing on such relationships is a common approach and often they are considered as connections in *cognitive networks* (e.g. Herscovics and Bergeron, 1983; Hiebert and Carpenter, 1992). In mathematical terms we may think of such networks as undirected graphs consisting of unspecified sets of vertices and edges, being non-constant over time. The vertices make up different pieces of information, knowledge, representations of ideas or actions, and understanding of these may be defined as a connection through an edge to a network (Hiebert and Carpenter, 1992, p. 75). The number of edges in a network, or various networks, may be taken as a measure for the understanding of the involved pieces of information: the more connected one's networks are, the better one's understanding. Such networks are internal representations inside the mind of an individual and as such cannot be directly accessed by others. They may, however, be

accessed through the external representations, i.e. spoken language, written symbols, pictures, or physical objects – that is to say, different forms of communication. Hiebert and Carpenter (1992, p. 66) assume that there is a relationship, the nature of which is undefined, between the internal and the external representations. Besides providing an access to the internal networks, this relationship also provides means for influencing the networks, changing them, and causing them to develop. Growth of understanding takes place “gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected information” as well as old connections being terminated (Hiebert and Carpenter, 1992, p. 69). A much more general way of phrasing this is to say that understanding is a process.

Dreyfus (1991) talks about the process of developing understanding in terms of *representation*. “To represent a concept”, he says, “means to generate an instance, specimen, example, image of it” (Dreyfus, 1991, p. 31). A representation of such an instance can be either *symbolic*, which refers to externally spoken or written usually with communication as the aim, or *mental*, which refers to the internal schemata or frames of reference with which a person interacts with the external world. Part of understanding is then the process of developing a single or several, possibly competing, mental representations of a mathematical concept or object. In the networks metaphor we might think of such representations as subgraphs of the cognitive networks. The richer one’s mental representations are, i.e. the more subgraphs one has and the larger they are, the more likely one is to be successful in mathematics. Poor mental representations, i.e. subgraphs which contain only few vertices and edges, do not allow for flexibility in mathematical situations, e.g. problem solving. Rich mental representations are not enough alone, though, one also needs to be able to switch from one representation to another and to be able to *translate*. Translation here refers to the ability to, for instance, translate a formulation of a problem in an extra-mathematical context to one in a mathematical context (e.g. mathematical modeling).

Another couple of researchers who talk about understanding as processes are Tall and Vinner (1981, p. 151) who say that to “understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived.” This is done by distinguishing between *concept image*, which is “the total cognitive structure that is associated with the concept, including all the mental pictures and associated properties and processes”, and *concept definition*, which is “a form of words” used to specify or define a concept (Tall and Vinner, 1981, p. 152). In this sense, a concept image has many resemblances with Dreyfus’ mental representations: A person’s concept image is build up over years and through experiences of all kinds, and it may change as the person acquires new knowledge, new beliefs, meets new stimuli, and matures academically. Concept definition to some degree resembles Dreyfus’ symbolic representations, except that Tall and Vinner distinguish between two kinds of concept definitions: the personal and the formal. The formal concept definition is the (more objective) definition of a concept (or mathematical object) which is accepted by the mathematics community at large. The personal concept definition, on the other hand, is a more subjective one constructed by an individual. It may be a personal reconstruction by a student of a formal definition and it is then “the form of words that the student uses for his own explanation of his (evoked) concept image” (Tall and Vinner, 1981, p. 152). A concept definition (personal or formal) generates its *own* concept image, the *concept*

definition image, which is part of an individual's larger concept image. If we refer to the networks metaphor we may imagine the larger concept image as networks made up of several subgraphs, each representing different concept definition images (notice that in Dreyfus' model, the mental representations were themselves subgraphs in the cognitive networks. Thus, if comparing concept images to mental representations, the concept definition images become 'subgraphs of subgraphs'). Just as Dreyfus' mental representations could be in competition with each other, part of a concept image or a personal concept definition can be in conflict with other parts of the image or definition. Such a part is called a *potential conflict factor*. However, a person's concept image or definition may possess such factors 'forever' without it ever leading to actual conflicts, only if the potential conflict factors are evoked simultaneously will they result in a cognitive conflict. Such conflicts may then be looked upon as part of a student's process of learning and understanding mathematics. Dreyfus and Vinner (1989, p. 356) state that students most often will use their concept image and not their concept definition to decide whether a mathematical object is an example of a given concept or not. One hypothesis is that a student's concept image only is as elaborated as the set of examples of that concept with which he or she is familiar. A very serious type of conflict is when a student's concept image is in conflict with the formal concept definition. According to Tall and Vinner (1981, p. 154) "such factors can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors unless the formal concept definition develops a concept image which can then yield a cognitive conflict."

Sfard (1991) acknowledges these twofold descriptions of mathematical understanding (Skemp, 1976; Hiebert and Lefevre, 1986; etc.) when building her theoretical construct of "the dual nature of mathematical conceptions", in that she distinguishes between an *operational conception* and a *structural conception* of abstract mathematical entities. By a *conception* Sfard (1991, p. 3) thinks of the internal, subjective, "whole cluster of internal representations and associations evoked" by a *concept*, the latter being a mathematical idea in its 'official', objective form. In this way, Sfard's perception of concept and conception closely resembles Tall's and Vinner's (formal) concept definition and concept image, respectively. By operational conception Sfard means the ability to conceive a concept as a dynamic operator, a process which takes an input and delivers an output. Structural conception is more abstract and is conceived when one is able to think of a concept as an object, i.e. a static structure existing somewhere in space and time. As different and incompatible as these two kinds of conception may seem, they are in fact complementary, opposite sides of the same coin. In this sense, the two kinds of conception are intertwined and inseparable in the learning and understanding of a mathematical concept. From an historical analysis of the evolution and development of numbers and functions, Sfard concludes that most often, and for most people, the first step towards an understanding of a new mathematical concept is operational conception. For the case of numbers, Sfard summarizes:

The development of the number concept has just been presented as a chain of transitions from operational to structural conceptions. As seen more than once, however, even before the processes which engendered the new kinds of numbers were reified, namely turned into full-blown objects, mathematicians were able to perform them and even to combine them with other operations to obtain more complex computations. I shall say, therefore, that the processes were interiorized and even condensed: They could be easily performed (so

were interiorized) and they could be referred to as procedures executed inside a ‘black box’ – something that no longer had to be described in full detail when considered as a part of a composite process (so they were condensed). This three-component pattern, interiorization/condensation/reification, seems to repeat itself at almost every turning point in the history of mathematical ideas – and in the process of learning. (Sfard, 1995, p. 38)

The three stages of interiorization, condensation, and reification forms a hierarchy which students must pass through in their process of coming to understand a mathematical object (say A). The first stage of *interiorization* is where the learner gets acquainted with the processes which will eventually give rise to a new concept, e.g. counting leads to natural numbers, subtracting may yield negatives, etc. These processes are operations on lower-level objects and with reference to Piaget, Sfard defines a process to be interiorized if it can be carried out through acquired mental representations (Sfard, 1991, p. 18). The next stage up, *condensation*, is where the learner becomes capable of thinking about a given process as a whole, without the need to go into details: “It is like turning a recurrent part of a computer program into an autonomous procedure: from now on the learner would refer to the process in terms of input-output relations rather than by indicating any operations.” (Sfard, 1991, p. 19). Thinking about this ‘autonomous procedure’ as a whole is when a new concept is ‘officially’ born. According to Sfard (1991, p. 19), a progress in condensation will show by a “growing easiness to alternate between different representations of the concept”, which is similar to Dreyfus’ idea of success in mathematics as the ability to switch between several mental representations of a concept. The stage of *reification* is entered when a concept is detached from its related processes, and the learner is capable of grasping it as a full-fledged object in itself – i.e. the concept has been reified. Due to what Sfard (1991, pp. 3, 20) calls the “deep ontological gap between operational and structural conception”, she conceives “reification [as] an instantaneous quantum leap: a process solidifies into object, into a static structure”, whereas interiorization and condensation occur more gradually as quantitative changes rather than qualitative ones. Once a learner has reified a concept into an object, this learner possesses a structural conception of that object and it may be used as input in other learning processes of new higher-level concepts (those that originate in processes performed on the object A) to provide understanding of new objects (say e.g. B). In this way mathematical understanding becomes an iterative process. Concerning the difficulty, “the ontological gap”, of reification, Sfard summarizes:

One final remark concerning the difficulty of reification: As an ontological shift it is an inherently complicated process. At least two serious reasons for its being very difficult to attain may be mentioned. One of them was called the *vicious circle* of reification. The use of this name refers to the fact that the reification of primary processes (those which underlie the given concept) seems to be the precondition for the ability to deal with secondary processes (those which are applied to the given concept), whereas the latter seem, in turn, to be a precondition for the former. For example, $3 - 5$ must be treated as a legitimate mathematical object before it can be manipulated and combined – through secondary processes – with other numbers. On the other hand, to speak about such operations as $3 - 5$ and $1 - 3.5$ as numbers, one must be able in advance to use them as inputs to the secondary processes. After all, it is the only way in which one may realize that $3 - 5$

and $1 - 3.5$ obey the same rules as 2, 5, 12 and the like, thus behaving like genuine numbers; such a realization is indispensable to justify and to motivate reification.

The second type of obstacle arises when some semantic concessions must be made before the new abstract object is fully accepted. For example, to talk about a square root of a negative quantity as a number, people must free themselves from their deeply rooted conviction that number is something which expresses quantity – a result of a measuring procedure. It is thus the very process that engendered the concept of number that must now be given up. (Sfard, 1995, p. 39, italics added)

Not all researchers think of understanding as a process,¹⁰ even though they may still refer to the network metaphor. Sierpinska (1990, p. 24), for instance, defines “understanding as an act (of grasping the meaning) and not as a process or way of knowing.” She generates the following categorization of acts of understanding a mathematical concept:

Identification of objects that belong to the denotation of the (or: a) concept (related to the concept in question), or: identification of a term as having a scientific status; this act consists in a sudden perception of something being like the ‘figure’ in the Gestaltist^[11] experiments.

Discrimination between two objects, properties, ideas, that were confused before.

Generalization consists in becoming aware of the non-essentiality of some assumption, or of the possibility of extending the range of applications.

Synthesis is grasping relations between two or more properties, facts, objects, and organizing them into a consistent whole. (Sierpinska, 1990, p. 29)

Sierpinska points out that “the necessary condition for all these acts to occur is experiencing, using, and applying”, which is to say that if one behaves passively towards mathematical objects, they will remain hidden and no understanding will take place – one has to act. In Sierpinska’s model, Skemp’s two kinds of understanding, as explained earlier, may be thought of as acts of understanding according to the styles of knowing they produce: instrumental or relational.

This ends the small survey of understanding in mathematics, and thus the presentation of the foundations for the following, more detailed, presentation of Sfard’s theory of commognition (Sfard, 2008b). This theory views mathematical understanding as a form of interpersonal communication and focusses on mathematical discourse rather than talking about mathematical concepts. Since the understanding of in-issues in research question 2 of this dissertation concerns the ways in which students’ meta-issue *discussions* are anchored in the in-issues, an approach concerned with interpersonal communications seems promising. And furthermore, as we shall see in chapter 5, the idea of focusing on discourses may not only be applied to the in-issues, but also to the meta-issues.

¹⁰ In addition to the above discussions of understanding as a process, there are also more *dynamic* descriptions of this. One of the wider know is that by Pirie and Kieren (1994), who provide a dynamic model for growth in mathematical understanding as a process.

¹¹ This refers to the perception of understanding in Gestaltpsychologie.

Understanding Mathematics as Change in Discourse

Under the assumption that understanding and learning is the occurrence of some form of change, Sfard asks what it is that changes when a person learns mathematics. After reviewing the viewpoints of several educational, psychological and/or philosophical directions (behaviorism, cognitivism, acquisitionism, and participationism), a task which I shall not pursue here, Sfard concludes that *thinking* is what changes when one understands or learns. And to the question of what thinking is, her answer is that the collective activity which is the best candidate for the precursor of thinking is *communication* (Sfard, 2008a). That is to say she defines thinking as “an individualized version of (interpersonal) communication” (Sfard, 2008b, p. 81), thereby also referring to the fact that communication does not have to be verbal or outspoken, and that it can take place inside the mind of an individual (e.g. in the process of understanding) as well as between two or more persons. On this basis, Sfard introduces her term *commognition* as a contraction of communication and cognition (commognition=communication+cognition) stressing the fact that these two processes are different interpersonal and intrapersonal manifestations of the same phenomenon. The next step is the definition of a *discourse* as “the different types of communication, and thus of commognition, that draw some individuals together while excluding some others” (Sfard, 2008b, p. 91). Sfard then roughly reasons that mathematics is a way of thinking, thinking is a form of communication, communication is a discourse, and therefore mathematics is itself a discourse. Thus, if thinking is what changes when one understands or learns, this means that understanding or learning mathematics is the same as changing the discourse (Sfard, 2008a).

Sfard (2008b, p. 161) describes mathematics as a “multilayered recursive structure of discourses-about-discourse”, and that its objects therefore, in themselves, are “discursive constructs”. In this sense, mathematics is an autopoietic system, i.e. a system that produces the things it talks about, a feature which produces a new version of Sfard’s old ‘vicious circle’: “Some familiarity with the objects of the discourse seems a precondition for participation, but at the same time participation in the discourse is a precondition for gaining this familiarity” (Sfard, 2008b, p. 161). Mathematical discourse is distinct from other discourses such as zoology, chemistry, or history in terms of its autopoieticity and therefore also by the ‘things’ the system produces. Sfard characterizes these ‘things’ as either *word use* (e.g. three, function, derivative, triangle, etc.), *visual mediators* (symbolic artifacts, geometrical figures, graphs, etc.), *routines* (how to look, how to convince, how to inscribe, how to prove, etc.), or *endorsed narratives* (e.g. $2 + 2 = 4$; $e^{i\pi} + 1 = 0$; $ax^2 + bx + c = 0$; ‘In a triangle, equal sides face equal angles’). An individual’s changing of the discourse, i.e. process of understanding, involves the building or *realization* of discursive constructs or objects. ‘Number’, for example, is a discursive object created by people to communicate about the world. A child who is to learn about ‘number’ for the first time, however, may not find ‘number’ in the world as he or she would with, for instance, a fish, ‘number’ is to be found in the discourse. A fish is an example of what Sfard calls a *primary object*, or *p-object*, which exists independently of human discourses, whereas the *discursive objects*, *d-objects*, arise in the process of assigning a name (noun or pronoun) or symbolic artifact to a specific primary object, then known as a *simple d-object*, or to a set of p-objects or previously created d-objects. Thus, a d-object is a pair: ⟨noun or pronoun, specific (set of) object(s)⟩. For example, if we name a specific fish (simple p-object) ‘Joe’ we

have ⟨Joe, fish⟩, and as an example of a set of previously created d-objects we may take ⟨basic quadratic functions, $\{(1, 1); (2, 4); (3, 9); \dots\}$ ⟩ (pairs (x, y) with $y = x^2$). So-called *compound discursive objects* arise by naming extant objects, either discursive or primary, in one of the following three ways:

By *saming*, that is, by assigning one signifier (giving one name) to a number of things that, so far, have not been considered in any way ‘the same’ but are mutually replaceable in a certain closed set of narratives [a set of narratives is called closed if it contains all the narratives that can be logically derived from those already in the set]

By *encapsulating*, that is, by assigning a signifier to a set of objects and using this signifier in singular when talking about a property of all of the set members taken together

By *reifying*, that is, by introducing a noun or pronoun with the help of which narratives about processes on some object can now be told ‘as timeless’ stories about relations between objects. (Sfard, 2008b, p. 170)

An example of saming p-objects into a d-object is that of saming fish, bird, mammal, etc. into animal. An example of encapsulation could be the above of basic quadratic functions, where the set of pairs of ordered numbers are encapsulated. Reusing this example, Sfard (2008b, p. 171) says: “reifying the operation of squaring 2 leads to the ordered pair (2,4), which again can also be realized as a point in the Cartesian plane. Combined with encapsulation of all such pairs created by letting the left element of the pair range over all possible numerical values, the reification leads to the discursive object called basic quadratic functions.” All those p-objects and d-objects that arise through saming and encapsulating familiar p-objects are termed *concrete objects*. *Abstract objects*, on the other hand, are defined as those d-objects that originate when reifying discursive processes. Based hereon, Sfard (2008b, p. 172) now defines *mathematical objects* as “abstract discursive objects with distinctly mathematical signifiers”, which means signifiers that are regarded as mathematical, whereupon she provides the following claim: “Mathematical objects are not any less material than the primary objects, except that rather than being a single tangible entity that predates the discourse, they are complex hierarchical systems of partially exchangeable symbolic artifacts.” At the same time, though, it should be noticed that these (mathematical) “objects are personal constructions, and different mathematicists^[12] may associate different objects with the same signifier”, which may cause communication difficulties (Sfard, 2008b, p. 193).

Besides Sfard’s basic assumption that learning mathematics is a change of discourse, another basic assumption of her’s concerning learning of mathematics is that it is an *individualization* of historically established discourse. By individualization she is thinking of an individual proceeding from being able to participate in a collective implementation of an activity to performing it competently of his or her own accord. Concerning the development of a learner’s *word use*, Sfard (2008b, p. 181) identifies four stages which a learner passes through in his or her process of individualization of the use of mathematical nouns: (1) a *passive* use of such signifiers, e.g. when a child does not use the word ‘number’ when speaking, but starts counting upon hearing the word; (2) a *routine-driven* use where the child actively uses the word, although only in a restricted

¹² A *mathematist* is a participant in mathematical discourse including students, teachers, and mathematicians.

number of specific routines as part of certain discourses; (3) a stage of *phrase-driven* use enters when words becomes linked with phrases rather than whole routines, e.g. the word ‘number’ gets its own life as a noun and may be used in propositions etc. where it fits grammatically; (4) the final stage of *object-driven* use slowly begins as the learner’s use of the word is being guided by the signified object, and he or she is aware of the availability and contextual appropriateness of his or her different personal realizations of the word. Sfard summarizes the relationship between these four stages of word use and the development of object construction:

As one advances through these stages, the use of the word becomes broader and more flexible. In this process, the increasingly skillful ‘peripheral participant’ overcomes multiple hurdles inherent in the processes of saming, reifying, and encapsulating. First, creation of subsuming discourses involves loss of some of the previously endorsed generalizing narratives. Second, saming processes in mathematical discourses may often appear counterintuitive, as they do not match our everyday experience. Two properties make them quite different from the identity-preserving transformations of concrete discourses: the discreteness of the symbolic operations that transform one realization into an equivalent one and the fact that they leave behind them a trace of visible intermediary forms. With relation to reification, the learner may suffer from an anxiety of unreleased signifiers and be baffled by the counterintuitiveness of process-object duality.^[13] The action of encapsulation faces learners with yet another type of difficulty, one that finds its expression in their frequently observed inability to translate the properties of elements into properties of the set, and vice versa. (Sfard, 2008b, pp. 193-194)

We may thus think of the four stages of word use in the process of individualization as a top-level description of a learner’s development, whereas the three ways of object construction (saming, encapsulation, and reification) are more general descriptions, i.e. not only present in word use, appearing in the historical development as well as being interwoven into the individualization. In addition to the above, it may be mentioned that another way of thinking about passive versus active word use is in terms of the word being objectified or unobjectified. For example, when a child instead of speaking about number only speaks of ‘marbles in a box’ the use is unobjectified whereas a statement like ‘the marbles in the boxes are the same number’ is objectified. In general, Sfard speaks about *objectification of discourse* as the discursive process that makes us use mathematical words and symbols as if they signified discourse-independent objects (Sfard, 2008a). *Objectification* is a double process consisting of both *reification* and *alienation* – “of turning statements about processes into impersonal statements about objects” (Sfard, 2008b, p. 63). Sfard describes objectification as a “double-edged sword” since, for instance, reification enhances the effectiveness of communication (replacing verbs with nouns), while alienation removes the human subject and effectively disguises the fact that the reified objects are discursive constructs and, as such, are created by humans rather than given (Sfard, 2008b, p. 50).

Besides the above concerning ‘word use’ some operational remarks can be made about ‘routines’ in mathematical discourse as well. Sfard defines routines as sets of

¹³ E.g. “that composite symbolic expressions, such as $4 + 5$, $134/29$, or $2x + 1$, may be used both as prescriptions for processes and as the products of these processes” (Sfard, 2008b, p. 182).

meta-discursive rules that describe recurrent patterns, e.g. the way we take turns communicating in conversations or as interviewer and interviewee, the act and re-act of greeting one another, etc. Such patterns can be described as resulting from rule-governed processes – human communication is itself a rule-governed activity, or a ‘game’, as Wittgenstien observed (Sfard, 2008b, p. 200). In opposition to meta-discursive or meta-level rules are *object-level rules*, e.g. the law of gravitation or mathematical narratives on geometrical shapes as ‘the sum of the angles in a triangle is 180° ’. Object-level rules speak about the behavior of mathematical objects, not about the actions of the discursants (the participants in a discourse), which is what the meta-level rules do. More generally, Sfard (2008b, p. 201) provides the definition that “object-level rules are narratives about regularities in the behavior of objects of the discourse, whereas metarules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives.” Important things about meta-discursive rules are that they are dynamic structures, constantly created and recreated in the course of interactions. Most often they are tacit and they may be quite value laden as well, both on a personal level as well as within a community of discourse (in the form of norms). Not only does Sfard distinguish between the object-level and the meta-level rules, she also does so among the meta-level rules themselves:

The set of metarules that defines routines can be divided into three subsets that specify, respectively, the applicability conditions, the course of action (procedure), and the closing conditions of the routine. The first and the last of these sets constitute the *when* of the routine and the middle defines its *how*. Two people whose particular performance may seem identical may, in fact, be implementing different routines, set apart by their applicability and closing conditions. Whereas learning a routine’s *how* is often a fairly straightforward task, learning its *when* may be a lifelong endeavor. (Sfard, 2008b, p. 221)

Meta-discursive rules constrain the ‘how’ and the ‘when’ of discursive procedures – mathematical proving is a good example – both in the development of mathematics itself as well as in the process of understanding mathematics. But how do routines develop? Sfard divides discursive routines into three types which differ from one another by what tasks they accomplish (their closure): explorations, deeds, and rituals. In terms of historical development and individualization the claim is that deeds and rituals are developmental predecessors of explorations. A routine is an *exploration* “if its implementation contributes to a mathematical theory”, or the other way round, an “exploration is a routine whose performance counts as completed when an endorsable narrative is produced or substantiated” (Sfard, 2008b, p. 224). The word ‘endorsable’ indicates that the narrative may be either endorsed or rejected according to the rules of a given mathematical discourse. Equation solving and routines of mathematical defining and proving are examples of mathematical explorations. A *deed* is defined as “a set of rules for a patterned sequence of actions that, unlike explorations, produce or change objects, not just narratives” (Sfard, 2008b, p. 237). And *rituals* are defined as “sequences of discursive actions whose primary goal (closing conditions) is neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people” (Sfard, 2008b, p. 241). Concerning the development of routines in the process of individualization, Sfard explains:

All three types of routines play a role in the development of discourses. In

the case of *metalevel* learning, when the routine to be learned involves new metarules or new mathematical objects, its reinvention by the learner is highly unlikely. In this case, the learning would typically occur through scaffolded individualization, that is, through interaction with mathematicians who are already insiders in the target discourse. When the process of individualization begins and the child makes her first steps in collective implementations of new routines, her performance is ritualized. The *how* of the routine is usually individualized well before the *when*. Thus, far from being the unintended result of an ineffective learning-teaching process, ritual is a natural, mostly inevitable, stage in routine development. [...] Becoming a participant of a new discourse is made possible by our propensity for thoughtful imitation. It is the child's readiness to follow in other people's footsteps that powers the process of individualization, at least in its initial stages. Later through gradual rationalization, the ritual will ideally be transformed into a full-fledged exploration. (Sfard, 2008b, p. 259)

Meta-level learning, as discussed in this quote, is one of two types of learning that may be identified if learning mathematics is regarded as a change in discourse (one of Sfard's initial claims). The other type is *object-level learning*, which manifests itself in the expansion of the existing discourse by extending a vocabulary, constructing new routines, and producing new endorsed narratives, i.e. an endogenous expansion of the discourse. *Meta-level learning*, on the other hand, is defined as involving "changes in metarules of the discourse and is usually related to exogenous change in discourse", which means "that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008b, p. 256). One 'trigger' of meta-level learning is that of a *commognitive conflict* – a situation where different discourses are acting according to different meta-rules, most likely because they endorse contradicting narratives. A commognitive conflict should not be confused with a cognitive conflict, as discussed by Tall and Vinner, for at least three reasons: a cognitive conflict stands between the interlocutors and the world, whereas the commognitive conflict stands between incommensurable discourses; a cognitive conflict is an optional way for removing misconceptions whereas a commognitive conflict is practically indispensable for meta-level learning; and a cognitive conflict is resolved by the student's rational effort, whereas a commognitive one is resolved by the student's acceptance and rationalization (individualization) of the discursive ways of the expert interlocutor (Sfard, 2008b, p. 258). A common feature is, however, the observation of a conflict only being potential: "a person whose mathematical discourse is a collection of rituals but who is never exposed to situations in which this discourse would prove insufficient has no reason to complain about the lack of understanding" (Sfard, 2008b, p. 274). As a final remark, it may be mentioned that meta-level learning and development introduce yet a circularity or 'vicious circle', one which concerns meta-rules change: To modify a meta-rule you need to understand the usefulness of such change, but the only way to become aware of the usefulness of a meta-rule is to enact it (Sfard, 2008a).

This ends the discussion of understanding in mathematics education, and thus that of the theoretical constructs underlying research question 2 and the later answering of it.

3.8 Students' Beliefs about Mathematics

Now I shall address the theoretical constructs on which an answering of research question 3, students' beliefs about mathematics, may draw.

According to Philipp (2007, p. 259), the "term *belief* is so popular in the education literature today that many who write about beliefs do so without defining the term", or as Thompson (1992, p. 129) concluded after reviewing the literature: "For the most part, researchers have assumed that readers know what beliefs are." Readers may very well know what beliefs are, or have an understanding thereof, but as an investigation by Furinghetti and Pehkonen (2002) show, the understandings may vary largely from one mathematics education specialist (or reader) to another.

In this section I shall attempt to shed some light on some of the characterizations of beliefs to which I shall confine myself in my own research. This also means describing the relationship between beliefs and related factors such as knowledge, values, affect, etc. as well as addressing matters of belief systems, belief structures, and belief categorizations. Also, I shall address the matter of changing beliefs, especially students's beliefs.

A Definition of Beliefs

Philipp (2007, p. 258) describes beliefs as "lenses through which one looks when interpreting the world":

Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (Philipp, 2007, p. 259)

From this definition, which I shall adhere to in this dissertation, it is clear that beliefs is an 'entity' which stands in relation to other 'entities' such as *attitude*, *emotion*, and *knowledge* (and possibly a few others also).

Both attitude and emotions are part of *affect*, which may be thought of as a disposition, tendency, or feeling, attached to an idea or object, which one takes toward some aspect of his or her world (Philipp, 2007, pp. 258-259). Emotions comprise "feelings or states of consciousness, distinguished from cognition" and emotions "change more rapidly and are felt more intensely than attitudes and beliefs" (Philipp, 2007, p. 259). Emotions may be positive or negative, and they are less cognitive than attitudes which, on the other hand, may be described as "manners of acting, feeling, or thinking that show one's disposition or opinion" (Philipp, 2007, p. 259). Also attitudes may be positive or negative, but they are felt less intensely than emotions. They are more cognitive than emotions but less cognitive than beliefs, and they change more slowly than emotions but more quickly than beliefs. Some researchers dealing with affect consider beliefs as a third component of affect, and we thus sense the hierarchy – belief, attitude, emotion – listed according to the most cognitive component. Among beliefs researchers though, beliefs are most commonly considered as an independent 'entity' next to that of affect, however acknowledging that beliefs and affect together "affect the way one interacts with his or her world" (Philipp, 2007, p. 258).

The relationship between knowledge and beliefs makes up a long-standing discussion within beliefs research (Pehkonen, 2008). Thompson (1992, pp. 129-130) refers to one type of distinction between belief and knowledge as *conviction*: beliefs are held with varying degrees of conviction whereas knowledge is not. As another dimension of distinction she mentions *consensuality*: beliefs are not consensual, whereas knowledge is. That is to say what is knowledge for one individual may be belief for another and that people generally are aware of the fact that others may believe differently, and even that their stances may be disproved. Concerning knowledge, on the other hand, people find “general agreement about procedures for evaluating and judging its validity” (Thompson, 1992, p. 130). Furinghetti and Pehkonen (2002, p. 43) propose to distinguish between two different kinds of knowledge: “objective (official) knowledge that is accepted by a community and subjective (personal) knowledge that is not necessarily subject to an outsider’s evaluation”. Based on this, they argue that beliefs should be considered to belong to the subjective knowledge, and that they as such have different degrees of stability and therefore are open to change. In this sense, it is normal to talk about *central* beliefs, those which are not very likely to change, and more *peripheral* beliefs as those which may be changed more easily (I shall return to this below).

A couple of other terms which are often associated with beliefs are those of conception, value, identity, and belief system. Conception may be defined as “a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences” (Philipp, 2007, p. 259). A value is the worth of something, a deeply held and often cherished belief which one acts upon – ‘a rule of life’. In contrast to beliefs, which are often considered to be held with a true/false dichotomy, values are associated with a desirable/undesirable dichotomy, and they are less context-specific than beliefs. Identity is a more comprehensive ‘entity’, which may be seen as “the embodiment of an individual’s knowledge, beliefs, values, commitments, intentions, and affect as they relate to one’s participation within a particular community of practice; the ways one has learned to think, act, and interact” (Philipp, 2007, p. 259). The term belief system is used as a metaphor to represent how an individual’s beliefs are structured (Pehkonen, 2008, p. 3). This term will need further explanation.

Belief Systems, Structures, and Categorizations

It is widely believed that an individual’s beliefs never appear altogether independently (Pehkonen, 2008, p. 2), and that when one adopts a new belief, this will automatically form a part of that individual’s subjective personal perception, that is one’s *belief system*. Thus, beliefs are structured in an individual’s belief system which may be seen as a compound of one’s conscious and subconscious beliefs, hypotheses, expectations, and possible interrelations between these. Just as existing beliefs are subject to change, so are belief systems. Thompson (1992, p. 130) says: “belief systems are dynamic in nature, undergoing change and restructuring as individuals evaluate their beliefs against their experiences.” Green (1971) has identified three dimensions of belief systems having to do with the interrelations of individuals’ beliefs: (1) Though beliefs are never held in total independence of other beliefs, “beliefs are held in *clusters*, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs” (Green, 1971, p. 48, italics added). According to Thompson (1992, p. 130) such “clustering prevents crossfertilization among clusters of belief or confrontation

between them, and makes it possible to hold conflicting sets of beliefs.” Conflicting beliefs are something very often found in research studies (we shall see examples hereof in chapter 7), and Green’s clustering property provides an explanation model for such inconsistencies within individuals’ belief systems. (2) The next of Green’s dimensions is that of *psychological centrality*, which is the already mentioned observation that some beliefs appear to be *central*, while others appear to be *peripheral*. (3) The last of Green’s dimensions is the *quasi-logical* structure of belief systems. This refers to the observation that some beliefs are *primary* and some are *derivatives* of these. Op’t Eynde et al. (2002) explain the relationship between this dimension and the two others:

As constituting elements of a person’s entire belief system, belief clusters are framed in a person’s subjective rationality (i.e. *quasi-logicalness*). People always strive for a coherent belief system; only then are they able to function in an intelligible way. Beliefs that are perceived as incompatible will be changed, and consequently also the clusters of which they are a part. Depending on the *centrality* of these changing beliefs in a person’s entire belief system, this will also affect other clusters. (Op’t Eynde et al., 2002, p. 25)

Thus, changing central beliefs may have far-reaching consequences in a person’s belief system, and for that reason some researchers compare changes in central beliefs with ‘paradigm shifts’ (Carter and Yackel, 1989).

Various attempts have been made to try and organize people’s beliefs about mathematics. One of the more recent categorizations of students’ mathematics-related beliefs is that of Op’t Eynde, de Corte, and Verschaffel (Op’t Eynde et al., 2002). They review four earlier categorizations of students beliefs (due to Underhill, 1988; McLeod, 1992; Pehkonen, 1995 (also to be found in Pehkonen and Törner, 1996) and Kloosterman, 1996), and provide a new more comprehensive framework of their own, structured under three different topics:

- 1. Beliefs about mathematics education:** **a.** beliefs about mathematics as a subject; **b.** beliefs about mathematical learning and problem solving; **c.** beliefs about mathematics teaching in general
- 2. Beliefs about the self:** **a.** self-efficacy beliefs; **b.** control beliefs; **c.** task-value beliefs; **d.** goal-orientation beliefs
- 3. Beliefs about the social context:** **a.** beliefs about the norms in their own class (**a1.** the role and the functioning of the teacher; **a2.** the role and the functioning of the students); **b.** beliefs about the socio-mathematical norms in their own class (Op’t Eynde et al., 2002, p. 28)

I shall not go into a detailed discussion of the components of these three categories (clusters) of beliefs, only mention that in the context of this framework of students’ mathematics-related beliefs it is point 1a, students’ beliefs about mathematics as a subject, which is most closely related to the investigation of this dissertation. However, due to the embeddedness of this point in an educational context, it is fair to argue that there is a dimension missing in the categorization of Op’t Eynde et al. (2002) – a dimension we may call *beliefs about mathematics as a discipline* (a more detailed definition will be provided below). In fact, this dimension is dealt with more independently by Underhill (1988) and Pehkonen (1995), but in Op’t Eynde et al. (2002) it has been played down significantly as illustrated in the quote above and in figure 3.1 (left). Nevertheless, this shortage may be remedied by adding the extra dimension – like in figure 3.1 (right).

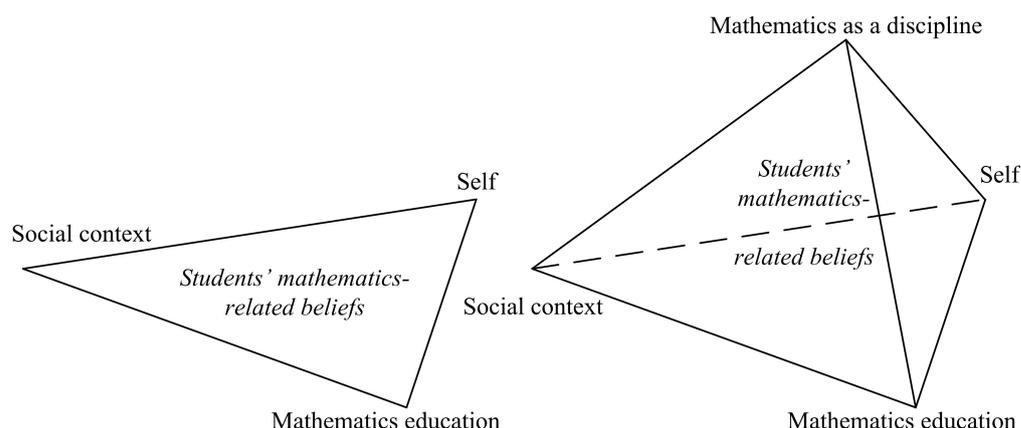


Figure 3.1 Left: “Constitutive dimensions of students’ mathematics-related belief systems” illustrated by a triangle with the corners: *mathematics education*, *social context* (the class), and the *self* (Op’t Eynde et al., 2002, p. 27, figure 2). Right: An expansion of the left hand side triangle to a tetrahedron, the fourth dimension above the triangle illustrating *mathematics as a discipline*.

The idea of placing this fourth dimension outside the triangle in figure 3.1, i.e. not just turning this into a square, has to do with the fact that mathematics as a discipline is rather different than mathematics as a subject, which is included in the triangle under mathematics education. However, if students are to obtain an image of and develop beliefs about mathematics as a discipline through their teaching and learning of mathematics, then this can only happen in the interplay between their social context (class), their mathematics education, and their self, which is to say the triangle making up the base of the tetrahedron.

Students’ Beliefs about Mathematics as a Discipline

The beliefs literature often talks about students’ beliefs about mathematics, students’ beliefs about the nature of mathematics, and occasionally also students’ beliefs about mathematics as a discipline. But what do these terms cover? In the following I shall attempt to provide my own definition of the dimension of students’ beliefs about mathematics as a discipline.

‘Beliefs about mathematics’ as an independent category of beliefs, as for example defined by McLeod (1992), Pehkonen (1995), and Kloosterman (1996), comprises a wide spectrum of beliefs, e.g. “beliefs on the nature of mathematics as such, and beliefs on the subject of mathematics” (cf. number a1 in the categorization by Op’t Eynde et al., 2002) and “beliefs on the nature of mathematical tasks, beliefs on the origins of mathematical knowledge, and beliefs concerning the relationships between mathematics and the empirical world (applications of mathematics)” (Pehkonen and Törner, 1996, p. 102). Underhill (1988), in his categorization, operates with a category called ‘mathematics as a discipline’. According to Op’t Eynde et al. (2002, p. 17), Underhill’s

category “refer to the beliefs students have about the nature of mathematics; for example, mathematics is about addition, subtraction, multiplication, and division, plus a collection of routine problems.” Thompson (1992, p. 132) defines ‘conception of the nature of mathematics’ in terms of mathematics as discipline: “conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics”. Schoenfeld (1992a) provides a list of typical student beliefs about the nature of mathematics based on a review of the literature:

1. Mathematics problems have one and only one right answer.
2. There is only one correct way to solve any mathematics problem – usually the rule the teacher has most recently demonstrated to the class.
3. Ordinary students cannot be expected to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically and without understanding.
4. Mathematics is a solitary activity, done by individuals in isolation.
5. Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
6. The mathematics learned in school has little or nothing to do with the real world.
7. Formal proof is irrelevant to processes of discovery or invention. (Schoenfeld, 1992a, p. 359, numbering added)

Under the assumption that when one has been exposed to mathematics education for years, one should have an idea about the nature of mathematics, Kloosterman (2002) asked American high school students what they would answer if an out-of-space alien turned up and asked them what mathematics is. To his surprise students generally were incapable of providing answers and would in general begin to talk about whether they liked mathematics or not, why they needed to learn it, etc. Kloosterman (2002, pp. 258, 264) concluded: “The nature of mathematics as a discipline is not an issue that United States high school students think about” and that “students have a hard time separating the activities in a mathematics class from the nature of the discipline itself”. When students are pressed to talk about the nature of mathematics, “they mention that mathematics can be used to solve a variety of problems and that it involves numbers” and they “often mention the procedural nature of mathematics but almost never mention deduction or proof” (Kloosterman, 2002, p. 258).

As indicated by the above, the meaning of the frequently used terms ‘beliefs about mathematics’, ‘beliefs about the nature of mathematics’, and ‘beliefs about mathematics as a discipline’ are overlapping and mixed up, to some degree they seem to mean the same and then again not. For that reason I shall provide my own definition of *mathematics as a discipline*, the fourth dimension in figure 3.1. Where the first category of Op’t Eynde et al. (2002), including the subcategory of mathematics as a subject, has more to do with Schoenfeld’s typical student beliefs number 1 through 5, the category of mathematics as a discipline shall relate more to beliefs 6 and 7. Of course, such a distinction depends on the interpretation of the typical beliefs, an interpretation which may be difficult not knowing the original questions asked to the students. For instance, does belief 4 have to do with problem solving in school or with mathematicians’ creation of new mathematics? Thus, instead of defining the category of mathematics as a discipline through ‘answers’, I shall do it through questions. I shall define students’ beliefs about mathematics as a discipline – in contrast to their beliefs about mathematics as a (taught) subject – to

include, for example, their beliefs about:¹⁴

How, when, and why they imagine that the mathematics in their textbooks came into being?

If mathematics is discovered or invented?

Where mathematics is applied in their everyday life or elsewhere in society?

Whether mathematics has a greater or lesser impact on society today than one hundred years ago?

Whether parts of mathematics can become obsolete?

If mathematics is a science (scientific discipline)?

What they think a researcher in mathematics does? And in what the research consists?

In terms of the previous discussions in sections 3.4, 3.5, and 3.6, the category of mathematics as a discipline includes students' view about mathematics as a pure science, an applied science, a system of tools for societal practice, and a platform for aesthetic experiences (four of the five faces of mathematics of which Niss (1994) talks). It includes elements of Ernest's (1998) examples of mathematical awareness, the 'three aspects' of the Danish 1987-regulation for the upper secondary mathematics program, and the three types of overview and judgement from the KOM-report (Niss and Jensen, 2002).

Of most interest to the study of this dissertation are students' beliefs about mathematics as a discipline concerning the evolution and development of mathematics (including the history of mathematics). As seen from the review of the empirical studies on history in mathematics education in section 2.8, often the conducted research has been on teachers' beliefs about the history of mathematics. Except for the studies of Christensen and Rasmussen (1980), Demattè and Furinghetti (1999), and Demattè (2007), I have not come across any empirical research studies dealing with students' beliefs about the evolution and development of mathematics. Furthermore, none of these three studies deal – experimentally – with possible changes in students' beliefs about these matters. Thus, the research to be presented in this dissertation (in chapter 7) may be one of the first steps aiming at filling out of this 'gap'.

Beliefs as a Tool or Beliefs as a Goal

As may already be clear to the reader, most often when beliefs are dealt with in the literature, it is with the purpose of improving mathematical thinking and learning.¹⁵ Beliefs, both cognitive and affective ones, are investigated in order to identify the 'ingredients' which do or do not make students capable of solving mathematical tasks, or teachers capable of teaching differently and/or more effectively. Certain beliefs are identified as advantageous in the learning of certain mathematical contents, the solving of related tasks, etc., and studies are then conducted on how to change already existing beliefs into these more favorable ones. Even studies which talk about students' or teachers' beliefs about mathematics, beliefs about the nature of mathematics, or beliefs about mathematics as a discipline seem to do so always as a (necessary) step on the way to improving mathematical thinking, learning, understanding, instruction, etc. In this sense, beliefs are regarded as means – or *tools* – to achieve understanding in the individuals' constructive learning process. Only rarely is the act of providing students

¹⁴ Several of these questions are inspired by Christensen and Rasmussen (1980) and Niss (2001a).

¹⁵ A few examples are Schoenfeld, 1985; Schoenfeld, 1992a; Törner, 2002; Leder and Fortaxa, 2002.

or teachers with certain beliefs, e.g. by changing existing ones, about mathematics or mathematics as a discipline considered a *goal* in itself. And when this is done (see e.g. Furinghetti, 1993; Niss, 1994; Ernest, 1998) the term ‘beliefs’ are usually not mentioned. Instead, mathematical appreciation, mathematical awareness or providing students with a more coherent *image* of what mathematics is are the words or phrases used.

In the light of sections 3.4 and 3.5 of this chapter, one could argue (Jankvist, 2009d) that the beliefs discussion lacks a more goal-oriented (or normative) dimension. A dimension which addresses students’ mathematical world view and proposes and evaluates some desirable beliefs in order to turn students into critical citizens by providing them with intelligent and concerned citizenship, a democratic competence, and with *Allgemeinbildung* in general. Looking to the KOM-report, this would include providing the students with a more balanced and multifaceted image of mathematics as a discipline, the influence of mathematics in society and culture, the impact of society and culture on mathematics, and the historical evolution and development of mathematics as a product of time and space. As pointed out by Niss (1980) and Niss and Jensen (2002), students would have to develop such an image through the study of exemplary cases, e.g. when concerning history, this could be in terms of discussing some of the general topics and issues mentioned in section 3.6. The topics of invention vs. discovery and pure vs. applied mathematics in section 3.6 also illustrate that such an image of mathematics to a large degree would have to be made up of beliefs, since there are no certain answers to many of the questions concerning these meta-issues. Students will have to *reflect* on the meta-issues in question, based on the exemplary cases and from there develop their individual, subjective image, or set of beliefs, about mathematics as a discipline and its relationship with the extra-mathematical world. And as we shall see in the following, reflection is considered a key to changing already existing beliefs, and perhaps also to foster new ones.

Changing Beliefs

The last aspect of the beliefs discussion I shall touch upon is that of changing beliefs. Whether beliefs are looked upon as tools to enhance mathematical thinking and learning or as goals in order to provide students with a more profound and balanced image of mathematics, the focus is often on changing already existing beliefs or developing new beliefs which are considered more beneficial or more desirable. This is due to the widely held notion that in order to change students’ behavior, teachers’ practices, and their ‘mathematical world view’ one must first change their beliefs, since beliefs act as filters which affect what one sees, and people have difficulty seeing what they do not already believe (Philipp, 2007, p. 276).

As discussed, some beliefs are central and some are peripheral. The stronger a belief is held, i.e. the more central it is, the harder it is to change it. This has to do with the fact that central beliefs may have derivative beliefs and in this way form a very connected structure. This means that if an individual is to change a highly central and connected belief, this person will have to change a long line of derivative beliefs as well, in order to obtain a coherent belief system. Thus, not only *which* beliefs are held but also *how* they are held are important features in challenging and changing existing beliefs. According to Green (1971, p. 48), evidence plays a crucial role, since non-evidentially held beliefs, i.e. beliefs which are not supported by any evidence in

the form of examples etc., are unlikely to change through rational criticism, reason, or evidence. In contrast, evidentially held (or evidence-based) beliefs may change through further evidence, better reasons, or reflection. Cooney (1999) has identified four different types of (teacher) student standpoints toward changes in beliefs, which he refers to as the *isolationist*, the *naive idealist*, the *naive connectionist*, and the *reflective connectionist*. The beliefs of the isolationist are not very connected, but separated and clustered, and he or she is not capable of accommodation and will reject any beliefs that conflict his or her own. The naive idealist is a received knower who, unlike the isolationist, absorbs what others believe, though often without analysis of his or her own beliefs in respect hereto. The connectionists hold a more reflective stance toward the world and are able to accommodate and reformulate core beliefs when faced with disturbances while holding on to their values, only the “naive connectionist fails to resolve conflict or differences in beliefs, whereas the reflective connectionist resolves conflict through reflective thinking” (Cooney, 1999, p. 172).

But even if changes do occur they may not necessarily last. Especially studies concerning teachers' beliefs have shown that even though changes in beliefs are spotted, these changes need not influence the teachers' practice, and if they do, it may only be for a limited period of time (Philipp, 2007). The same may be assumed for changes in students' beliefs. However, this suggest that the higher the level of reflection associated with changes in beliefs, the larger the probability that the changes may last.

Thus, reflection seems to be a core element in changing beliefs. The question, of course, becomes how to provide the circumstances which may foster such reflection in connection to changes in beliefs. And another question is how to ‘measure’ the changes in beliefs. In order to do this one must, of course, first establish what the ‘original’ beliefs are, before the activities to change these are set in motion. But doing this may not necessarily be a straightforward task since beliefs may be hidden. Or to put it differently, students may not always be conscious about their own beliefs. According to Lester, Jr. (2002, p. 352), Kath Hart at a PME once asked: “Do I know what I believe? Do I believe what I know?” Lester’s version of this question is: “Do students know what they believe?” Furinghetti and Pehkonen (2002) argue that one should take into consideration the beliefs that students hold consciously as well as unconsciously. But how to do this? Lester, Jr. (2002, pp. 352-353) expresses doubt about some of the more usual methods for doing this: “I am simply not sure that core beliefs can be accessed via interviews [...] or written self-reports [...] because interview and self-report data are notoriously unreliable. Furthermore, I do not think most students really think much about what they believe about mathematics and as a result are not very aware of their beliefs.” One remedy of this, taken in the research of this dissertation, is to provide classroom situations in which students are expected to work actively with issues related to their beliefs about certain matters of mathematics. For now, however, I shall pass over such more methodological concerns and leave them to later chapters.

To end this chapter, I shall, as promised, state my own position on the use of history, taking into account elements from the chapter as well as the previous one.

3.9 My Personal Position on the Use of History

Even though the focus of the research study of this dissertation is history as a goal, factors of history as a tool may still occur as byproducts or side effects of this approach. For that reason, I shall briefly account for my own view on history as an assisting tool in the teaching and learning of mathematics. I do believe that history may have a very motivating effect on some students. As accounted for in Jankvist (2008c) and Jankvist (2009b), particularly a history of modern mathematics or one of modern applications of mathematics seem to intrigue some students (I shall return to this in chapter 8). However, as will be seen in chapter 7, other students are not at all taken in or fascinated by an historical dimension; some just appear indifferent while others are hostile. Nevertheless, I do believe, in accordance with other research results (cf. chapter 2), that history do have a role to play as a motivational, affective, and pedagogical (e.g. in terms of PCK) tool in mathematics education. Concerning the role of history as a cognitive tool I am more cautious. Not that I am a die-hard skeptic, but I think that more empirical evidence will have to be provided to support the claim of history as a cognitive tool, whether it be in the case of identifying and overcoming epistemological obstacles or surveying possible historical parallelisms, the latter in the more soft-core inspirational versions as well as in the hardcore ‘ontogenesis-recapitulates-phylogenesis’ version of the evolutionary arguments. However, if we distinguish between the learning of mathematical topics and the developing of mathematical competencies, as discussed by the KOM-report, then there may be something ‘new’ to say about the use of history as a cognitive tool.¹⁶ I shall return to this in chapter 8.

When considering history as a goal, I am in favor of providing students with a more coherent, profound, balanced, and multifaceted image or picture of what mathematics as a discipline is, as discussed by Furinghetti (1993), Niss (1994), and Ernest (1998). Especially, I find the normative descriptions by Niss and Jensen (2002) to be quite clear concerning the part to be played by history in the provision of such an image. Further elaboration on the part of history as a goal, in terms of *Allgemeinbildung*, is given by Radford (2008a, p. 166), who claims that an historical dimension is a necessity of a 21st century citizen’s mathematics education, since a concrete understanding of reality cannot be attained without an historical approach to it, and therefore: “Mathematics, with its tremendously sophisticated conceptual equipment, should be a window towards understanding other voices and subjectivities, and understanding ourselves as historically and culturally constituted creatures.” However, this is not the focus of the types of mathematics education provided today in general. In fact, the one for Danish upper secondary level may be one of the few exceptions. And not even here it is very clear how to actually go about ‘using’ the history in order to achieve the somewhat loosely defined educational goals of the regulation. This brings me to my next point.

In agreement with the KOM-report, I believe that it is essential to have the historical dimension anchored in the mathematical dimension (meta-issues anchored in in-issues). If not, there is a fear that the historical dimension will become anecdotal and be served in the form of biographical Christmas-tales with no mathematical contents whatsoever. If the use of such (illumination) approaches gets out of control, the students will not necessarily be provided with a very coherent, profound, or balanced image of what mathematics as a discipline in time and space is. Many aspects of the history of

¹⁶ This is also discussed in Kjeldsen (2009b).

mathematics is, naturally, best explained to the students by means of the involved mathematics. Illuminations approaches bear the risk of ‘illuminating’ the externalistic sides of the historical elements in question and leaving untouched the internalistic ones. And as it has already been argued above, this is not a favorable approach to conducting the history of mathematics, and therefore neither to the use of it in mathematics education. It is my belief, and working hypothesis, that many meta-issue elements of a certain historical case will be better understood by the students if they simultaneously develop some kind of understanding of the related in-issues – the empirical data to be revealed later will be a test of this.

As argued in chapter 2 and in Jankvist (2009a), a purpose of ‘history as a goal’ seems best manifested through some kind of modules approach. The historical cases chosen for such an approach should, however, be exemplary, e.g. in such a way that they embrace as many general topics and issues related to the history and historiography of mathematics as possible. Or differently put, the history of mathematics in general, or the ‘global’ history, should be illustrated through specifically chosen exemplary ‘regional’ cases. As illustrated and discussed in sections 3.6 and 3.8, some of these general topics and issues have no final, correct ‘answers’, e.g. that of invention versus discovery. In some sense, it becomes a matter of personal opinion and therefore also one of personal beliefs. In this respect, the important thing is not always whether the students lean toward one viewpoint or another, but whether they are brought to *reflect* upon their own personal opinions, views, and beliefs, and then brought to discuss and defend these evidentially among each other. In this way they are not only creating an image of mathematics, they are also ‘living’ it and testing it out. Or in the words of Radford (2008a, p. 165), “any process towards knowledge (in other words, all processes of *objectification*) is also a process of *subjectification* (or of the constitution of the ‘I’)”. Radford (2008a, p. 165) recapitulates this view in the sentence: “*Knowing something* should be at the same time *being someone*.” We may rephrase this sentence into: *believing something is the same as being someone*. And it is in this way that students’ beliefs about ‘mathematics as a discipline’ become a goal (rather than a tool to enhance in-issue learning), a goal of *Allgemeinbildung*.

Besides relying on theoretical constructs the research questions, or an answering of them that is, must also rely on some ‘methodological constructs’. The methodology used and developed to answer the three research questions of this dissertation shall be explained in the following chapter.

4 Methodology

The three research questions posed and analyzed in the previous chapter will be addressed on two levels. First, they will be answered in terms of the implementation of the two specific teaching modules. Second, the answers concerning the specific cases will be used as a basis for trying to address the three research questions on a more general level. In this chapter I shall describe the methods used to address the research questions on both these levels. In order to do so, some more general considerations concerning methodology will also be presented and discussed. However, an answering of the research questions requires, first and foremost, a clarification of the terms on which these are researchable and to what extent. I shall address this matter now and then return to it later in the chapter as well.

4.1 On What Terms are the Research Questions Researchable?

As should be clear from the introductory chapter (1), my approach to researching the use of history in mathematics education is an empirical approach. Due to the setting of this empirical research within the frame of Danish upper secondary school, the focus of the study has been on ‘history as a goal’ rather than ‘history as a tool’ (cf. section 3.5). Thus, with reference to the discussion of the ‘effectiveness’ of using history in mathematics education, as touched upon in chapter 1, this study concerns the effectiveness of using history as a goal.

From the survey of empirical studies on history in mathematics education in section 2.8 we saw that very few empirical studies have focussed specifically on the use – or effectiveness – of history as a goal. This means that the present study must also, at an initial stage, deal with the development of a way for doing that. In this light, it seems reasonable to approach the three research questions (RQ1-3) through a methodology of qualitative research rather than one of quantitative research. Qualitative research may be defined as studies which are concerned with causal relationships on a conceptual level, whereas quantitative research concerns the production of quantitative measurements for practically anything. Mason (2002, pp. 2-3) says, within a social science context, that qualitative research “aims to produce rounded understandings on the basis of rich, contextual, and detailed data”, data gathered by methods which are “flexible and sensitive to the social context in which data are produced (rather than rigidly standardized or structured, or removed from ‘real life’ or ‘natural’ social context, as in some forms of experimental method)”.

Especially research questions 1 and 2 (asking about in what sense, to what extent, on which conditions and levels) seem to call for ‘rounded understandings’ based on rich, contextual, and detailed data rather than data removed from the ‘natural’ social context in which they were generated. Research question 3 is more suitable for quantitative

methods, since it asks to the possible changes in students' beliefs. But it also asks how these changes may be brought about, which again calls for more 'rounded understandings'. Some authors argue for a combination of quantitative and qualitative methodologies (Brown, 1992, p. 156), e.g. by combining a concentration on large scale data bases with in-depth microgenetic analyses of a few students or a group of students – an approach which to some extent may be applied for research question 3.

In summary, to provide answers for research questions 1 and 2, in-depth studies of single implementations of designed teaching modules is a first step in addressing the 'effectiveness' of history as a goal. Furthermore, this together with the actual formulation of the two questions seem to suggest a line of qualitative research rather than one of quantitative research, not least due to the lack of previous empirical, experimental studies on history as a goal and hence of any pre-made methodologies for handling the research questions. For research question 3, on the other hand, it can be argued that a combined strategy of both quantitative and qualitative methods may be suitable (I shall explain the details later). Of course, the quantitative measurements for research question 3 shall not rely on "large scale data bases". The basis for answering the question will be restricted to the same population used to answer research questions 1 and 2, since question 3 asks after some of the effects of carrying out the research on questions 1 and 2.

4.2 Research Design

Once having decided on the empirical method to be mainly one of qualitative research, the next step is to decide on the design of the research study. As already mentioned, the purpose of including history in the Danish upper secondary mathematics program is history as a goal rather than history as a tool. And as argued in chapter 2, the modules approach is a very suitable approach for using history as a goal. The modules approach also fits nicely the frame for including elements of history at the Danish upper secondary level since modules here may have a quite long duration. Hence a modules approach was chosen.

Although deciding to use a qualitative research method and a modules approach are important initial steps in laying down the research design, they are only the first decisions out of many to be made. In this section, I shall discuss some of the other decisions made, and the pros, cons, and possible consequences of them, regarding the answering of the research questions.

Why Two Teaching Modules?

Since elements of the history of mathematics may enter in any and/or several of the three years of Danish upper secondary school, it made sense to try doing this in two different years, in order to be able to detect differences in students' engagement in meta-issue discussions and reflections. Due to the study of students' beliefs (RQ3), the possibility of seeing students' developments from one year to the next, and to some extent also due to practical concerns, it was decided to conduct the two teaching modules in the same class with the same mathematics teacher. For this reason, the two teaching modules had to be different, i.e. concern different historical cases. Another possibility would of course have been to implement the same teaching module in two different classes, at

the same or on different levels, but for the reasons just given such an approach was not chosen. Thus, the study carried out was not a traditional design research study in the sense of a cyclic process of doing thought experiments and instruction experiments (e.g. Gravemeijer and Cobb, 2006), or put another way: designing, implementing, evaluating, redesigning, re-implementing with another population, and so on. This study consisted of a design with one case, an implementation, an evaluation, a design with another case, an implementation with the same population, and an evaluation. Still, of course, some redesign was possible from case to case (just not in the cyclic design research sense).

This redesign mainly consisted of the varying of parameters from the first module to the second, especially in the topics of the meta-issue discussions of the modules. In order to observe and access the students' discussions and reflections on meta-issues, a scene had to be set in which such discussions were to be carried out. A set of essay assignments for each module made up such a 'scene', since the students were to work on these in groups and hand in written answers (also in groups). Particularly the general topics and issues related to the history of mathematics, as presented and discussed in chapter 3, made up the parameters in the essay assignments, and therefore also the students' meta-issue discussions, which could be adjusted from one module to the next. Some of the topics and issues had a larger degree of built-in insurance of getting the meta-issue discussions and reflections anchored (cf. RQ2) in the related mathematical in-issues of the module, by which I still mean that some of the things said about the meta-issues could not have been said without knowing the related in-issues, or because the meta-issues themselves emanate from the in-issues. For instance, a discussion of epistemic objects and techniques (section 3.6) may ensure anchoring to a larger degree than a discussion of multiple developments would (section 3.6), since you need to be able to identify actual mathematical objects and techniques in order to discuss them, whereas parts of a discussion on multiple developments may be carried out in a non-anchored way. Thus, by varying the topics of the essays from module to module one might get an idea of how much built-in insurance is needed to obtain an anchoring of the meta-issues in the in-issues. (This will be exemplified in chapters 5 and 6).

Beside being able to vary the parameters, there were two other reasons for doing two modules with the same population. The first, as mentioned above, concerned the possible developments in the students' abilities to engage in meta-issue discussions and reflections, also due to their development in maturity. The second, of course, was to see if, and if so then to what extent, such teaching modules on the history of mathematics may give rise to changes in students' beliefs about mathematics (RQ3). Since beliefs and maturity are not likely to change and develop from one day to the next, some substantial calendar time was needed to 'measure' this. (A time line for the study will be provided later.)

Why Produce New Teaching Materials?

Due to the fact that the regulations were new, not much teaching material focusing on elements of the history of mathematics was available. The new textbook systems did include some elements of the history, but as shown in Jankvist (2008a) the approaches used here were mostly those of illumination, and even when smaller modules were suggested ('historical packages'), these did not live up to the intentions of the KOM-report. Looking to the English literature for suitable material did not solve the problem

either since, as Siu (2006) implies (cf. section 1.1), there are not too many materials “in between” the original sources, historians’ research results, and popular accounts. Thus, it seemed that suitable teaching materials would have to be produced.

Another equally important argument for producing my own teaching materials had to do with the fact that I for several reasons wanted to use elements from the history of modern applied mathematics and modern applications of mathematics, something done only rarely within the field of history in mathematics education (Jankvist, 2009b). These reasons were the following: (1) A history of modern applied mathematics addresses the topic of inner and outer driving forces in the development of mathematics, as pointed to by the KOM-report (cf. section 3.5), in a somewhat natural way, since outer driving forces often play in when (old and new) mathematics finds its applications. (2) Besides the KOM-report’s overview and judgement concerning the historical development of mathematics, a history of modern applied mathematics could also address the other two kinds of overview and judgement (actual applications and the nature of mathematics, respectively), as well as elements of mathematical awareness as described by Ernest (1998), if choosing appropriate historical cases. By this I mean cases for which the issues could be built up in front of the eyes of the students in parallel with the explaining of the related meta-issues. (3) Touching upon aspects of all three types of overview and judgement might also provide the students with a richer picture of mathematics in time and space (cf. section 3.6), and therefore also potential changes in the students’ beliefs about mathematics as a discipline (cf. section 3.8). (4) I was myself already familiar with some areas of the history of modern applied mathematics. More precisely, the history of error correcting codes (part of my master’s thesis) and some history of elementary number theory and RSA (from a course on discrete mathematics and its applications I had taught at Roskilde University). But more importantly, I knew that these two cases could provide the basis for discussing elements of the general topics and issues related to the history of mathematics as discussed in chapter 3, and I knew that the mathematics involved stood a good chance of being explained to upper secondary students. (5) Furthermore, there may in fact be an implicit point in dealing with a history of mathematics which is not ancient, namely that the students might relate better to these histories since their settings in time and space will appear more familiar to them. The applied aspect of a newer history may further provide the students with a reason (should they feel the need for one) for actually having to learn this mathematics. (6) And finally, a history of applied mathematics may assist in revealing some of the hidden mathematics in the students’ everyday life and in society in general (section 3.4), thus also providing evidence to the fact that mathematics is actually used and that it is still being developed.

How to Design the Teaching Materials?

In general, not much literature is available on how to write and design teaching material and textbooks. As mentioned in chapter 2, the literature on history in mathematics education is rich in ideas on how to include history in the teaching, how to use original sources, etc. However, when it comes to providing actual guidelines for designing teaching material, there is not much help to get. One of the few exceptions, though, is the Ph.D. dissertation by van Gulik-Gulikers (2005) in which she provides the following guidelines:

Educational designers, course book writers and curriculum developers, when having to decide on a choice from historically based subjects, will have to find a balance between the level of the pupils, the degree of [difficulty of] the chosen subject and the available instruction time. It is important to consider to what extent the original source material can be used in its original form or whether it should be adapted and translated [...] The availability of classroom resources and of a teacher's manual is of vital importance, since many teachers do have insufficient historical expertise. (van Gulik-Gulikers, 2005, p. 223)

The balance between subject matter, the students' capabilities, and the available time was indeed an important element in the design of the teaching materials for the two modules. Concerning the use of extracts from original sources, I decided to translate all these into Danish which will be illustrated in chapters 5 and 6. No teacher's manual was written since there was only one teacher involved and coaching her was a possibility.

A design principle I adapted from the literature on the genetic method (cf. section 2.4) was that of using modern notation in the presentation of the mathematical in-issues. Thus, the selected theory of error correcting codes and public-key cryptography was presented to the students in modern notation along with the history of these two cases and the already established concepts and theories on which they were first founded. However, due to the disciplines of error correcting codes and public-key cryptography being so relatively new, the notation used in the original sources only differed slightly from the modern one. And when quoting original sources, the original notation was kept intact. Applications of error correcting codes and public-key cryptography, both the ones in question and others, were also discussed along the way. All this was done in order to try and meet the intentions of the new regulation and the KOM-report (section 3.5), e.g. showing the students that the evolution and development of mathematics draw on internal as well as external driving forces.

Another design principle was picked up from a textbook on set theory (Niss, 1985) I had used for an assignment once, while being a student.¹ This book distinguished in its presentation between the mathematics and comments given to this by having one on the even numbered pages and the other on the uneven numbered pages. I adopted a similar approach; setting the text of the teaching material with two different fonts; one for in-issues, that is mathematics, and one for meta-issues: historical, applicational, and others. The idea of this was to make it easier for the students to read and use the text, thus providing a service for them in this sense.

The last feature I shall mention concerning the design of the teaching material are the aforementioned essay assignments. While working in their groups, the students were asked to discuss various aspects concerning the development and evolution of error correcting codes and public-key cryptography as well as aspects of their applications. Following their group discussions, the groups were to write essays on the topics in question and hand these in. In particular, the students were to do one very large final essay assignment at the end of the implementation of each module. However, in order to get them accustomed to doing and writing these essays, the first module also included a number of smaller introductory essay assignments. During the implementation of the

¹ Surely other textbook writers use similar design principles, but the 'template' I used was the one of (Niss, 1985).

module the students also were to work on a number of more traditional mathematical exercises/problems, some being historical in the sense that they used extracts from original sources. These were posted at the end of each chapter and dealt with the mathematical subject matter (in-issues) of the particular chapter.

Who should Teach the Modules?

The next question to take into consideration was one of implementation: should I teach the modules myself or should the class' mathematics teacher do it? Both approaches have pros and cons.

A major downside of conducting the teaching myself would be that I was more likely to make my studies a target for the criticism of Gulikers and Blom (2001), who says that most publications on the use of history in mathematics education tell the story of one specific teacher (and often this teacher is also the author of the paper), and that it is unclear how the (generally positive) experiences can be transferred to other teachers, classes, and types of schools (cf. section 1.1). This critique might to some degree be avoided by video filming my own teaching, but then the mathematics teacher would have to undertake this task. I was not sure she would do that, and even if she wanted to, how would she then feel about leaving her students at my mercy for two long teaching modules. Another problem of this approach concerned the students' exams. Parts of the teaching modules could end up in the students' oral exams. And if so, would I then have to examine the students on these parts? And if the teacher was to do it, would she then not be much better off by having taught the modules herself in the first place? On the other hand, if I was the one to be video filmed while teaching, I would not have to worry about criticizing the teaching since I would only be criticizing myself. Having to criticize a teacher for his or her teaching may indeed be a delicate matter, since you come to know the teacher very well after working together for such a long time. Another positive aspect of teaching the modules myself was that I could be sure that the historical elements which I wanted to promote would be dealt with in the teaching.

By having the mathematics teacher teach I would still be subject to Gulikers and Blom's critique of having only one specific teacher. However, this teacher would be a typical upper secondary mathematics teacher. She would not be the designer of the materials and therefore not have any personal interests riding on them. This would definitely make any existence proof which might come out of the study more reliable. Also, if the two designed teaching modules were to be useable in any sense, they had to be suitable for teaching by a typical, ordinary upper secondary mathematics teacher. On the downside though, such a teacher would necessarily be ethically committed to the students' profit from the modules, rather than to the needs of my empirical study which in worst case could compromise the research. Furthermore, with the mathematics teacher teaching I could not be entirely sure of the teacher doing exactly what I wanted her to. At any rate, a lot of coaching would have to be done in order to get the teacher to conduct the teaching of two teaching modules, the contents of which she was likely to be only remotely familiar with. However, if the teacher was willing to engage in such a coaching process of both learning the contents of the modules and modifying her teaching practice towards the goals of these, then the benefits of having her teach would certainly overshadow the downsides. Fortunately the teacher I was cooperating with was willing to do all this, hence this approach was chosen. It gave me the opportunity to

observe the classroom teaching and the students' work on both mathematical exercises and essay assignments in the sense of being a 'fly on the wall'.

The 'fly on the wall' approach is sometimes also referred to as a non-participant (or observer) method. Of course, it is not possible to completely be a 'fly on the wall' when observing in a classroom, especially not if you bring in video camera, tripod, microphone, all wired together, and sit there writing in your notebook next to the students. For this reason, some researchers also talk about "data generation" rather than "data collection" (Mason, 2002, p. 52). As opposed to a non-participant study you find, of course, a participant study. Even though I suppose I could have been considered some kind of a participant had I done the teaching of the module, this is not what is usually meant by a participant study. In the usual sense of the method, a participant is one who 'infiltrates' the classroom as a student and participates in all the classroom activities without the other students, and possibly the teacher, knowing about this (Cohen and Manion, 1994, p. 107). This research method of course has some severe limitations concerning the data gathering; video filming, audio recording, and taking field notes are difficult to do without 'blowing your cover'. On the other hand, you are probably as close to being 'a fly on the wall' as possible. However, for my research project, a participant study in this sense was really not an option, firstly because I was the author of the teaching material, and secondly because I at the age of 31 would have stood little chance of blending in with a bunch of 17 year old students.

What Data Sources to Gather?

Having decided to be a non-participant/observer, the next question concerned what kind of data to gather from the two teaching modules. The first decision made was to give the students a questionnaire before beginning the implementation of the first module. This questionnaire contained questions on their view of mathematics as a discipline, e.g. the evolution and development of mathematics, the applications of mathematics, the role of mathematics in their everyday life and in society in general. The majority of questions in the questionnaire were open questions in the sense that they were not multiple-choice questions, and that the students were expected to write down answers consisting of several sentences. For the precise formulation of the questions see appendix B.1. Several of the questions in this questionnaire were inspired by an older research project by Christensen and Rasmussen (1980) in which the purpose was to investigate students' beliefs in regard to what seven years later would become known as the 'three aspects' (cf. section 3.5). My questionnaire had two purposes, the most important one of course being that of uncovering students' beliefs (RQ3). The other purpose had to do with the selection of students for followup interviews as well as the selection of five students to a focus group which I was to observe during classes. The precise criteria for selecting students will be discussed in chapters 5 and 6. The important thing to notice now, as mentioned earlier, is that the students were to work in groups. Of course, in order for them to have *discussions* of meta-issues there needs to be more than just one student, but previous studies also show that students tend to be more relaxed when working together with other students than alone (Schoenfeld, 1985, pp. 281-282).

The idea of giving questionnaires and conducting followup interviews was continued throughout the implementations of the two modules. The reasons for this were several: The first was to see if the questionnaires would indicate any changes in the students'

beliefs. For this reason some of the questions from the first questionnaire were repeated in the questionnaires after the two modules, for example in the context of the two historical cases (see appendices B.2 and B.3). Another reason for continuing giving questionnaires was to be able to ask the students more specific questions concerning their understanding of the meta-issues and the in-issues of the modules, sometimes in an almost test-like manner. Finally, a fourth questionnaire, largely identical to the first (see appendix B.4), and a set of followup interviews were given to the students almost five months after the completion of the second module and the third questionnaire, just before they were to graduate from upper secondary school, which was also one year after the first questionnaire. The reason for this last questionnaire was, as mentioned in section 3.8, that beliefs are not something which are likely to change over a short period of time. So in short, during the period of a year in which the two modules were implemented, the class was given four questionnaires and 10-12 students were interviewed, based on their questionnaire answers, four times during this year as well.

The interview strategy was what may be referred to as semi-structured interviews. In semi-structured interviews a set of questions is prepared for the interviewee, but the interview itself is structured more along the line of a conversation than an interview, and with the possibility of providing regulatory feedback. The questions are not necessarily posed in the order they are put down on the paper and other questions may be added to the interview in case the interviewee touches on something which seems important. Small-talk is allowed as well, in order to make the interviewee more comfortable with the situation. Thus, semi-structured interviews allow “the researcher to get fairly comparable answers from the group of interviewees to the same set of questions, but at the same time reduces unintelligible answers and misinterpretations” (Ottesen, 2009, p. 71). In the followup interviews the students were interviewed individually. The main reason for this was that I expected it to be easier to discuss the students’ answering of the questionnaires if they were interviewed one by one. Kvale (1997, p. 108) discusses the positive and negative effects of group interviews. On the positive side, when interviewing people in groups, the interaction among the interviewed may lead to spontaneous and emotional statements about the topic of discussion. However, a negative effect is that such group interviews may be difficult to control, and that the gathered data as a result of this may become difficult to analyze due to the chaos of the interview itself – a situation which may be suspected to increase in likelihood the more loosely the interviews are structured. Since I wanted to video film the focus group during their work on assignments, I was already sure to get plenty of spontaneous and emotional statements. So for that reason too, I chose to interview the students individually as well as for the reason to be able to follow single students through the two implementations.

Video taping groups of students while they work as opposed to just doing audio recordings has certain advantages. Of course, video captures the ‘picture’ of the situation, e.g. facial expressions, gesticulations, etc., but it has certain other methodological advantages as well (Brown, 1992, p. 174). With video tapes as opposed to just field notes, which are difficult to get as objective as the tapes, you always have the opportunity to reanalyze your data, and using different pairs of ‘lenses’, by watching the tapes as many times as required. On top of this, having a video may make it a lot easier to do transcriptions of students’ conversations, especially if several students are talking at the same time. The downside of using video as opposed to for instance just an audio recording device is, of course, all the gear you have to carry round, set up, operate,

and take down again. Something which makes the illusion of being ‘a fly on the wall’ harder to maintain. My main reason for video filming was the essay assignments. As said, these were to be handed in to the teacher, and I would get copies.² However, what I wanted to capture was the discussion process: what went ahead of their finalized and hand-in essays, and how they arrived at their conclusions. I suspected – and hoped – that such group discussions would provide insights into the meta-issue reflections the students were engaging in, both in terms of research question 1 and 2.

When video filming, one question is how much to video film. I decided to go with the methodological philosophy of ‘better having too much than too little’, or ‘it is better to have data and not use them than to need data and not have them’. However, as straightforward as this ‘philosophy’ may sound, it actually makes another methodological question more pressing: How do you select exemplary situations from your data material? The larger your amount of data (I have 40 hours of video film), the more there is to choose from and the more difficult the choices become. One method I used to keep track of what was going on in the films was to take supportive field notes while filming the focus group and the classroom teaching. These field notes were organized in three columns (see figure 4.1): the first containing the recording time on the video camera; the second a characterization of the activities going on as being concerned with either in-issues or meta-issues; and the third comments on the specific activities, e.g. a certain theorem

21/8-2007		TAPE (20) + (21)	
tid	karak.	komm.	
3.15	i	perfekt kvote	
5.40	i	i perfekt kvote gode kvote	
ca. 11	i	familie af Hæmning-kvoter	
14.40	med	hvad skal vi med mange kvoter	
17	om	offentliggjort Hæmning-kvoter	
18.40	om	Gstlay	
19	om	Shannon	
ca. 20	om	Gstlay-kvoter	
25.15	i/med	foldningskvoter	
26	—	turbo-kvoter	
28		oplej til essay-opp. 54	
ca. 32		op. osv. besvarelser	

Figure 4.1 The above field notes are from the 21st of August 2007, video tape 20. The left hand column *tid* is time, and the right hand one *komm.* is comments. The middle column *karak.* is the characterization of activities in *i*, *om*, or *med* referring to ‘in-mathematics’, ‘about-mathematics’, and ‘by means of mathematics’.

² To be accurate, the students would turn in their assignments electronically and I would simply download them from the school’s intranet.

the teacher was proving, an exercise the students were working on, which student went to the blackboard, etc. Concerning the characterization of activities a subdivision of the meta-issues as being mostly concerned with matters of application or matters of history, philosophy, etc. was also made. More precisely, the idea was somewhat based on the classical IMFUFA trichotomy of ‘in-mathematics’ (in-issues), ‘about-mathematics’ (meta-issues on history, philosophy, etc.), and ‘by means of mathematics’ (applications, modeling), in Danish: *i-matematik*, *om-matematik*, and *med-matematik*. In the analyzing process, these field notes proved to be an effective tool in identifying first candidates for exemplary situations concerning the answering of research questions 1 and 2.

Besides collecting the students’ essay assignments, several mathematical exercises/problems which they had to hand in were collected as well (some of these being historical ones using original sources). This was done in the hope of gaining some insight into the individual students’ understanding of the mathematical in-issues of the teaching module.

The teacher was also exposed to four rounds of interviews. The first interview in particular had to do with her expectations to the pending implementation of the first teaching module. But she was also given some of the same questions as students had been given in the first questionnaire in order to uncover her views and beliefs about mathematics. The second interview, the one in between implementations, concerned both an evaluation of the first module and expectations to the second. The third interview after the second module was again mostly an evaluation. The fourth and last interview mainly concerned the teacher’s reflections on the two modules in hindsight, now perhaps having a better idea about the students’ outcomes of the modules, as well as her ideas for using the materials in future classes.

Overview of Data Sources

To recapitulate: Before, during, and after the two teaching modules, data from the following sources were gathered:

- First round of student questionnaires and followup interviews – before implementations
- First teacher interview – before implementations
- First module videos of the teaching, supporting field notes
- First module videos of the focus group, supporting field notes
- First module hand-in of mathematical exercises/problems
- First module hand-in of introductory essay assignments
- First module hand-in of final essay assignments
- Second round of student questionnaires and followup interviews – in between implementations
- Second teacher interview – in between implementations
- Second module videos of the teaching, supporting field notes
- Second module videos of the focus group, supporting field notes
- Second module hand-in of mathematical exercises/problems
- Second module hand-in of mathematical historical exercises/problems
- Second module hand-in of final essay assignments
- Third round of student questionnaires and followup interviews – right after implementations

- Third teacher interview – right after implementations
- Fourth round of student questionnaires and followup interviews – ‘long after’ implementations
- Fourth teacher interview – ‘long after’ implementations

Frame and Time Line for the Complete Study

As already explained, the study was carried out in one Danish upper secondary class, the first module in the students’ second year and the second module in their third and final year of upper secondary school. Students in their second year are usually of age 17-18, and in their third year of age 18-19. At the beginning of the first module the class consisted of 26 students, a number which dropped to 23 by the completion of the module and stayed the same during the implementation of the second module.

The upper secondary school – *Ørestad Gymnasium* – where the study was carried out was a new school, and the class I followed was among the first generation of students from this school. The school has a profile of focusing on media, communication, and culture (<http://www.oerestadgym.dk/>) – aspects which the historical cases of the two modules also touched upon. The school also focusses on the use of technological devices (e.g. intranet and pod casts) in teaching, however, this was not part of the two modules. Due to the school being new, it was in a process of running-in, something which for instance caused the breaking of the first module in two (see chapter 5). On the upside, though, the school had a rather loose way of laying their teaching schedule, which sometimes was an advantage in planning the modules.

Apropos planning, table 4.1 provides an overview of the time line of the complete study. (A more detailed and full-fledged plan and time line is found in appendix A.)

Year	Months	Part of study
2007	March-April	First round of questionnaires and interviews
2007	April	First teacher interview
2007	May and August-September	Implementation of first teaching module
2007	September	Second round of questionnaires and interviews
2007	October	Second teacher interview
2007	October-November	Implementation of second teaching module
2007	November-December	Third round of questionnaires and interviews
2007	December	Third teacher interview
2008	April	Fourth round of questionnaires and interviews (students only)
2009	February	Fourth teacher interview

Table 4.1 Time line for the complete empirical study.

Having accounted now for the methodology and the research design of the empirical study, the next issues to be addressed are the terms on and extent to which the research questions may be answered. In order to discuss this, some more general methodological considerations are appropriate.

4.3 Three Dimensions of Educational Research

Schoenfeld (2007), in what seems to be the most comprehensive of recent discussions on ‘method’ in mathematics education,³ argues that research must be examined along three somewhat independent dimensions:

- trustworthiness,
- generality, and
- importance.

These three dimensions may be depicted in a coordinate system in space, assigning an axis to each of them (Schoenfeld, 2007, p. 82). A research study may thus be ‘measured’ according to how high it scores on each of the axes. In order to obtain an idea of where the study in this dissertation may be placed in such a three dimensional system, I shall discuss it according to Schoenfeld’s three dimensions (sections 4.4 and 4.5). But first a description of the dimensions and their components.

Trustworthiness

Concerning the first dimension, *trustworthiness*, Schoenfeld (2007, p. 93) is referring to “the degree of believability of the claims made in a piece of research” and states that the “core issue is: if claims are made, do the warrants for them ring true?” As criteria for judging the trustworthiness of a piece of empirical research, Schoenfeld mentions:

- a study’s *descriptive* power;
- its *explanatory* power;
- whether the claims made are in some sense *falsifiable*;
- whether the study makes *predictions* and, if so, how well those predictions fare;
- how *rigorous* and *detailed* the work is;
- whether the work has been described in ways that allow for attempts at *replication* and, if so, whether the findings are duplicated or extended;
- and whether the study offers *multiple lenses* on the same phenomena and *multiple lines of evidence* that converge on the same conclusions.
(Schoenfeld, 2007, p. 93, bullets and italics are added)

The *descriptive power* refers to the capability of a theory or model to identify ‘what counts’ in relation to the phenomena under investigation. For instance, a mathematics educator setting up an empirical study will have to focus on the factors which are essential for answering the posed questions and conducting the related analysis. Once having identified such factors, the interplay between them and the phenomenon under investigation must be examined, i.e. it must be explained “how and why the phenomenon functions the way it does” (Schoenfeld, 2007, p. 83).

The degree to which a theory or model may do so has to do with its *explanatory power*. An analogy from mathematics may be that of an existence theorem of some kind, and the situation that this theorem may be proven in both a non-constructive way and a constructive way. Now, the constructive proof for the theorem certainly has a much larger explanatory power than the non-constructive one, since the latter only shows that

³ Schoenfeld’s chapter in the *Handbook* is special in the sense that it has as its end purpose to address so-called ‘clinical trails’ in educational research. Nevertheless, the considerations made in the first parts of the chapter are quite useful for addressing and evaluating empirical research.

something exists (e.g. by contradiction) while the former provides information on both why and how. In the multifaceted field of mathematics education there are, of course, various ways to try and explain why and how certain phenomena occur and function. But as Schoenfeld (2007, p. 83) expresses it: “What is important is that the attempt be made and that claims must be held accountable to data.”

Also, a theory or model must be *falsifiable* (according to some ‘margin’). This means that it should be challengeable by means of gathered data and empirical evidence.

To some extent a theory or model should be able to make *predictions* about the outcome of certain situations addressed by that theory or model. If a piece of research possesses descriptive and explanatory power, it stands a fair chance of being able to ‘predict’ as well. Schoenfeld (2007, p. 86) points to the circumstance that mathematics education research operates with a wide range of prediction. For example, the kind of predictions made as a consequence of qualitative studies may be very different in nature from those based on quantitative (and e.g. very statistically oriented) studies. However, Schoenfeld (2007, p. 85-86) also states that: “a great deal of descriptive work contains implicit claims of generality [see below], and thus of prediction. As soon as there is the implication that ‘in similar circumstances, similar things will happen,’ one is, at least tacitly, making a prediction”.

Such studies involving prediction should, due to their explicit or implicit claims of generality, possess potential *replicability* as well (Schoenfeld, 2007, p. 87). The reason for this is that in such studies there are some key aspects which are assumed to be replicable. The task for the researcher, of course, consists in characterizing those key aspects in a manner detailed enough, so that other researchers may profit from them in the right ways and perhaps refine the ideas further.

This of course has to do with *rigor*. By being rigorous in one’s descriptions of theoretical notions and empirical actions (including methods and data) others will be more likely to understand, benefit from, and use them in their own research or practice, again in right ways. By ‘right ways’ I am referring to ways consistent with one’s intentions.

If notions, constructs, etc. are too loosely defined and not well enough explained, i.e. the descriptions of these are not *detailed* enough (lack in *specificity*), problems can occur since they may be used in ways conflicting with the original ideas behind them. In worst case, such ‘misuses’ can lead to an undermining of the notions and constructs themselves.

The last element of the dimension of trustworthiness is that of providing *multiple lines of evidence* (multiple sources), or providing multiple lenses. Often multiple sources of evidence also means multiple methods for data gathering. For example observations, questionnaires, students’ worksheets, and interviews may be different methods all used to challenge, confirm, or expand the information gathered from each source (Schoenfeld, 2007, p. 87). When this approach is taken in order to attack and capture a complex phenomenon from different angles, it is often referred to as methodological *triangulation* (Cohen and Manion, 1994, p. 233-235). Thus, triangulation may serve both as a method in itself (multiple lenses) and as a way to validate conclusions (multiple lines of evidence).

Generality

Concerning the second dimension, that of *generality*, Schoenfeld (2007) distinguishes between four different types of generality of a body of research as ways to think about the scope of that research:

- the *claimed* generality,
- the *implied* generality,
- the *potential* generality, and
- the *warranted* generality.

Often authors will imply the generality of a phenomenon, either tacitly or explicitly, by suggesting the typicality of circumstances of their study. “Implying generality is one thing”, Schoenfeld (2007, p. 93) says, “providing solid evidence for it is another.”

From this point of view, many studies in the field of mathematics education may be of only a limited, warranted generality. This does not, however, imply that they should not be taken seriously or that they are not interesting. In fact, many such studies may be of high interest, e.g. because of their potential generality (or generalizability), because they suggest a new line of research or because they make theoretical contributions. Examples of such studies may be those of ‘existence proofs’:

[M]any studies in mathematics education can be seen as existence proofs. They may demonstrate the existence of a phenomenon worthy of investigation [...] or of instructional possibilities [...]. The findings of studies with existence proofs are not yet general – but there may be the potential for them to be. (Schoenfeld, 2007, p. 88)

Importance

A study’s potential may not only concern the generality of the study, but also the importance of it. *Importance* is the third dimension which Schoenfeld addresses. By importance of a research study he refers to a value judgement of the topic of study. Regardless of a given study having a warranted generality or not, one must question its importance:

- What difference does the study make?
- What consequences might it be expected to have?
- What is its contribution to theory and practice?

And such value judgements of the importance of a study must be made reflectively (Schoenfeld, 2007, p. 93).

4.4 Answering the Research Questions on a Case-Specific Level

How may the collected data be used to answer the research questions on a case-specific level? To answer this, Schoenfeld’s dimension of trustworthiness must be discussed. However, not all elements can be discussed at this point in the dissertation, since some require a completed data analysis. This particularly goes for the elements of descriptive and explanatory power, but to some extent also falsification, rigor, and specificity. Thus, in this section I shall discuss the method for answering research questions 1 and 2 (students’ abilities to engage in meta-issue discussions and the anchoring of these, respectively) on the basis of *triangulation*, *prediction*, and *replicability* for each of the

two teaching modules, and afterwards the answering of research question 3 (possible changes and developments in beliefs due to the modules) on a similar basis.

Before I begin, it is worth mentioning that in the data analysis not all of the eighteen data sources as depicted above may be of equal significance in answering the different research questions. For instance, some data may be more relevant for answering question 1 than question 2, and vice versa. And some will mostly be of relevance to question 3, e.g. the first round of questionnaires and followup interviews. Furthermore, the significance of a given data source may depend on the possible answers to the research questions. For instance, some sources may become more relevant if the answer is negative since they may then suggest possible explanations for the negative answers. I shall return to this scenario below.

Concerning the First Teaching Module

I begin by addressing the possible and meaningful triangulations on data sources in order to provide evidence for the answering of research questions 1 and 2. To answer these questions, the idea is to begin by analyzing the first module hand-in essay assignments (introductory and final ones). Such an analysis may provide a 'broad' answer to the research questions, based on the essays from all student groups. For question 1, it is possible to compare data from the essay assignments, the second questionnaire, and the second round of interviews to perform a triangulation in order to support conclusions. For question 2, an indication of an anchoring of the meta-issue discussions in the related in-issues may be supported by looking into the mathematical exercises/problems of the students in a particular group, i.e. the same triangulation as before, only involving yet another data source. A deepening of the answers to question 1 and 2 may be performed for the focus group in particular, by involving the video material in the triangulations also. In the second round of followup interviews, the students were asked questions about their answering of the essay assignments. Thus, a meaningful triangulation to perform for the focus group could be that of hand-in essays, videos, and followup interviews. Again, if required, for question 2 it could become relevant to include the hand-in mathematical exercises in the triangulation as well.

Thus, the triangulations to be performed serve as methodological triangulations in the form of addressing and uncovering the phenomena at hand, but also as a way to validate interpretations and conclusions. The idea of first providing a broad answer and then later to narrow it down in terms of studying individual students is also suggested by Brown (1992, p. 156), who claims that such an approach "enables us to see the magnitude of the effect in terms of outcome measures and to get a feel for the phenomenon itself by looking at a particular child or group in depth."

Now, as for the video films of the classroom teaching, these may be relevant for different scenarios. One example could be if students appear to do better at engaging in some of the essay assignments compared to others. In this case, the success could be due to the topics of these assignments having been dealt with very thoroughly as part of the teaching. Something the video films of the teaching could give an idea about. In a somewhat similar manner, the videos of the classroom teaching could be used as an extra dimension in triangulations involving the mathematical exercises to see if students' problems in understanding certain concepts could be traced back to a teaching situation. Another scenario in which the classroom videos might become relevant are in the case of

negative answers to the research questions. Say, for instance, that the answer to research question 1 is negative, i.e. that the students when doing their essay assignments are not capable of engaging in certain meta-issue discussions and reflections. A watching of the classroom teaching videos might reveal situations where a topic had not been dealt with properly or explained correctly, either due to a true misconception on the teacher's behalf or due to poor coaching on my behalf.

Schoenfeld's element of prediction may be interpreted in two different ways:

1. prediction may refer to the actual capability of a study to predict outcomes of certain experiments, or
2. it may refer to the potential of a given study, its design and methods, to perform predictions.

In the first sense, not much prediction is possible in the first teaching module. This mainly has to do with the fact that no previous empirical, experimental studies have focussed clearly on the use of history as a goal. And that no pre-made methodologies exist for that reason, as well as it not being possible to pose any literature-based hypotheses about the outcome of the teaching module. In the second sense, however, the first teaching module bears a higher degree of prediction. For instance, some prediction may take place as the implementation proceeds, e.g. of the ways in which the students cope with the designed classroom activities. Such predictions can not be stated before formulating the research questions and designing the experiment, as would be expected of the first interpretation. But the design and method used have the potential to make predictions from lesson to lesson, and, more importantly, from implementation to implementation, should the investigation be carried out again.

A similar situation is seen for the element of replicability, since this may be viewed in the two same ways as the element of prediction. In the first sense, it is not possible to replicate anything, because nothing has been done previously. In the second sense, some kind of replicability of results from, for instance, essay assignment to essay assignment could be the case during the implementation. On the overall, the first teaching module bears the potential of replicability of results, from lesson to lesson and from implementation to implementation.

Concerning the Second Teaching Module

The triangulations to be performed on the data material from the implementation of the second teaching module are not very different from those of the first, except that the data are different. Again the idea is first to use the (second module) essay assignments in combination with the third questionnaire and round of followup interviews to provide a broad answering of the research questions. And then deepen the answers by looking at the videos of the focus group, and possibly in the answering of research question 2 also including some of the collected mathematical exercises (including the historical ones). Moreover the video of the classroom teaching may provide answers in situations like the ones discussed above for the first teaching module.

Another way to obtain insight and to deepen the answers of the research questions could be to perform triangulations on data sources from the first module and sources from the second as a way to reveal new information and as a way to further validate conclusions and interpretations. When dealing with the focus group in particular, this might prove to be beneficial. And also if case-studies of single (focus group) students

are to be conducted, such a ‘cutting across the data’ approach might turn out to be what is needed to obtain new knowledge about students’ meta-issue reflections etc.

Since the second teaching module is similar to the first, the second of course also bears the potential of being able to both predict and replicate results. However, concerning the actual capability of doing these things, the second teaching module is much better off than the first. Based on the analysis of the first module, predictions can be made regarding the nature of the students’ meta-issue discussions and reflections, the anchoring of these in the related in-issues, the way they work with the essay assignments in the groups (particularly the focus group), how the design of these as well as the teaching material in general functions, how the teacher and the teaching influence the work done on the essay assignments and the reflections done in connection with these, etc. Of course, there are substantial differences in the two teaching materials and the topics of the essay assignments (the variation of ‘parameters’), but since some common key-factors may be identified in both, e.g. the kind of topics and issues addressed in the essay assignments, the basis for prediction should be present. Concerning replicability, the entire implementation of the second teaching module may be seen as an attempt to replicate the findings of the implementation of the first module. Again there are differences in the material, some redesign will have occurred and other aspects of the implementations may vary, but the students and the teacher are the same.

Concerning Students’ Beliefs

The most relevant data sources for answering research question 3 are the questionnaires, especially the first and fourth, and the followup interviews for these. As explained earlier, the first questionnaire served the purpose of providing a before-picture of the students’ general ideas of what mathematics is, how it has evolved, where it is being used, and so forth. The fourth questionnaire, which was more or less identical to the first, served the purpose of providing an after-picture, and thus, making it possible to see if any changes in students’ beliefs had come about or if new beliefs had arisen. Of course, there is the possibility that some students in the fourth round may recall having been asked the same questions before. However, this is not necessarily a bad thing since it might indicate that the students have become aware of their beliefs (cf. section 3.8), which may be seen not as a change in their actual beliefs, but as a change in the way they hold these. In many cases, whether the students recall the questions or not, it is likely that they do not recall what they answered a year earlier.

Though the population of the class was rather small, as mentioned 23 students, some quantitative measurements of the students’ beliefs and potential changes in these over the one-year period, may be provided by means of the questionnaires and interviews. Again, as is the idea with the student groups’ essay assignments, the questionnaires may be used to provide a broad answering of the research question, an answer which may be deepened and to some degree verified by (a qualitative) looking into changes for individual students (interviewees). Some of the questionnaire questions were repeated in one form or another throughout all four rounds of questionnaires and interviews. For these questions in particular, it may be possible to follow changes in beliefs and developments of new beliefs, for the class in general as well as for individual students. Students’ beliefs in relation to questions which were not present in all four questionnaires may on some level be investigated through the hand-in essay assignments (from both

modules), and for the focus group students through the videos (from both modules). Again, the answering of this research question may possibly be deepened by involving the videos of the classroom teaching, like the students' understanding of the in-issues through the mathematical exercises of the modules.

Concerning the possibility of attributing any identified changes in beliefs to the teaching modules, the only way to go about this seems to be to look for elements in the students' questionnaire answers, interview statements, and possible utterances on video which may be directly related to the two teaching modules.

This completes the treatment of the dimension of trustworthiness for now. Next are those of importance and generality.

4.5 Addressing the Research Questions on a General Level

Importance is, in my opinion, closely connected to relevance: Is it relevant to carry out a certain study or not? And relevance is, of course, again connected to context. A study might be relevant in one context and not in another. The relevance, and thus also the importance, of the research study of this dissertation has already been addressed on various occasions throughout the previous chapters; in an international as well as in a national context. As for the expected consequences of the study, the differences it may make, and its possible contributions to theory and practice, these are matters which cannot be addressed before the findings of the study have been presented and the research questions have been answered. Thus, I shall have to return to these.

As for generality, due to the fact that there are no previous clear-cut, empirical, experimental studies on the use of history as a goal in mathematics education, the kind of scientific generality of which Schoenfeld (2007) speaks must be considered to be out of reach of the present study. However, the study may have other kinds of 'general' characteristics.

The first kind of these have to do with the 'typical' circumstances under which it was carried out. The students subject to the experiment were not specially chosen in any way. The school was a new school which perhaps made the administration and the personnel more willing to engage in such experiments, but aside from that the experiment might as well have been carried out at any upper secondary school in Denmark. Nor was the teacher specially chosen in any way. Of course, it may always be argued that the enthusiasm of the teacher, as well as the goals the teacher has with his or her teaching, have an important effect. Seen from an international perspective, the Danish circumstances do of course make it easier to implement teaching modules on historical cases like the ones described here. In other countries with different types of curricula, the possibilities for doing this may be somewhat limited. But seen from a scientific point of view, this only justifies the setting of the experiment, since the new regulation for the Danish upper secondary mathematics program perhaps provides the most beneficial setting for testing out what can and cannot be obtained by using history (as a goal) in mathematics education.

Another point has to do with the fact that students are being filmed and are the center of attention during such two modules. These circumstances may influence the students to act differently, e.g. that they will put more effort into their work because they know that someone cares about what they are doing, or that they will put less

effort into it because they find the entire thing to be ‘stupid’ and rather would have gone without having to participate. However, nor this is a special circumstance for the particular class involved, it is something typical for any class of students participating in an experimental study of this kind.

Concerning the terms on and extent to which the three research questions may be answered, what can be hoped for in relation to the empirical study of the dissertation is that it will be able to give an existence proof in the sense described by Schoenfeld: that it may demonstrate the existence of phenomena worthy of investigating further. That is to say that the findings are not yet general, but that they may have a degree of potential generality or generalizability. In this sense, the present empirical study may be seen as a ‘pilot’ for larger research projects. Further work would then include finding out what such projects should involve, how they could rely on the performed qualitative study, but also how they should differ from this, and how the qualitative results could be made more quantitative, and thus more general.

At any rate, the methodology discussion cannot be completed at this point in the dissertation, since certain questions cannot be answered yet. For example, if I in one way or another have had a ‘lucky hand’ in my research design; or, what new territory was, and perhaps further can be, gained with the design; and what the limitations and potentials were of the applied ‘empirical probe’ in the investigation? These and other matters concerning the methodology (e.g. the four not yet discussed elements of trustworthiness) must be revisited after the data analysis has been completed and the findings have been exposed. Hence I shall return once again to Schoenfeld’s dimensions of trustworthiness, generality, and importance in the final chapter of the dissertation, chapter 8 – ultimately in order to ‘place’ the present study in the coordinate system of these three dimensions.

5 First Teaching Module

The purpose of this chapter is to provide evidence for the answering of research questions 1 and 2. A further description of the design and implementation, than that of chapter 4, and to some extent also an evaluation of the first teaching module shall be provided here. This is done on the basis of the data sources described in chapter 4. Also, an analysis, e.g. in form of triangulations, shall be made and related to elements of chapter 3. All shown extracts from student papers, questionnaires, and interviews in this chapter, as well as the following chapters, have been translated from Danish. The mathematics of error correcting codes is explained along the way when dealing with examples involving this. But first, an introduction to the early history of error correcting codes is required.

5.1 The Early History of Error Correcting Codes

In 1948 Claude Shannon, at the Bell Laboratories, published his *Mathematical Theory for Communication* (Shannon, 1948) – the starting point of information theory. As part of this theory, Shannon also considered channel coding, i.e. *error correcting codes*, with the purpose of adjusting for any errors occurring during transmission as a result of noise (see figure 5.1). Shannon proved that ‘good’ error correcting codes exist (Shannon, 1948, p. 14). However, his proof only showed the existence of such codes, it did not say how to construct them. But he did provide one example of a good (efficient) code (Shannon, 1948, p. 16). This example was due to the mathematician Richard Hamming.

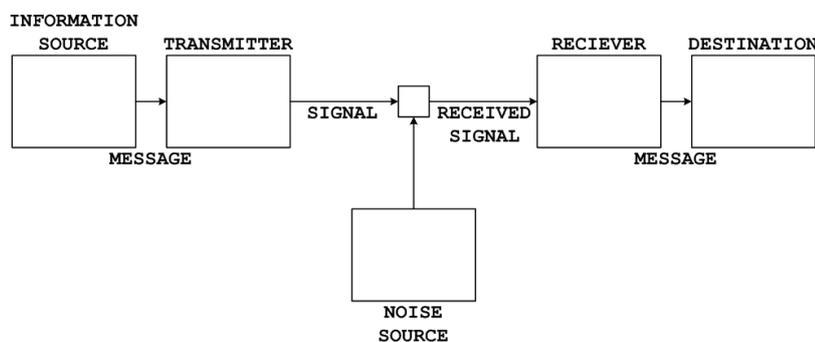


Figure 5.1 Shannon’s illustration of a system of communication (Shannon, 1948, p. 5). Encoding with error correcting codes takes place before the message reaches the transmitter. Errors can occur as a result of the noise source. Error decoding takes place after the message is received.

He had become an employee of the Bell Labs a few years earlier coming from the Los Alamos project, where he had been working with computer calculations. At Bell Labs Hamming did not have first priority at the computers, so many of his calculations were run over the weekends. At this point in time the computers were only using error detecting codes, meaning that whenever the computer detected an error due to noise, they would drop the current calculation and move on to the next in line. After finding his calculations dropped two weekends in a row, Hamming said to himself: “Damn it, if the machine can detect an error why can’t it locate the position of the error and correct it?” (Thompson, 1983, p. 17). Continuing this line of thought, Hamming began to develop his error correcting codes and, supposedly, by some time in 1947 he was able to provide Shannon with the example of one of his codes, the (today) so-called Hamming (7,4)-code (each codeword in the code consists of seven symbols, four of these being information symbols). When the Bell Labs heard about Hamming’s codes they wanted to patent them. Hamming himself was quite skeptical about this: “I didn’t believe that you could patent a bunch of mathematical formulas. I said they couldn’t. They said, ‘Watch us.’ They were right. And since then I have known that I have a very weak understanding of patent laws because, regularly, things you shouldn’t be able to patent – it’s outrageous – you can patent” (Thompson, 1983, p. 27). The patenting of Hamming’s codes led to a long delay of their publication. Not until 1950 was Hamming able to publish his family of codes (Hamming, 1950).

Hamming, being a mathematician, used mathematics in the creation of his error correcting codes. That is to say Hamming used already established mathematical *techniques* in order to describe the *objects* under investigation, i.e. the codes (cf. epistemic objects and techniques in section 3.6). For instance, he used the generalized concept of distance known as a metric to define what today is known as the Hamming distance. Also, he used elements of vector space theory and linear algebra – he thought of possible binary n -tuples as the coordinates of corners of an n -dimensional cube and his

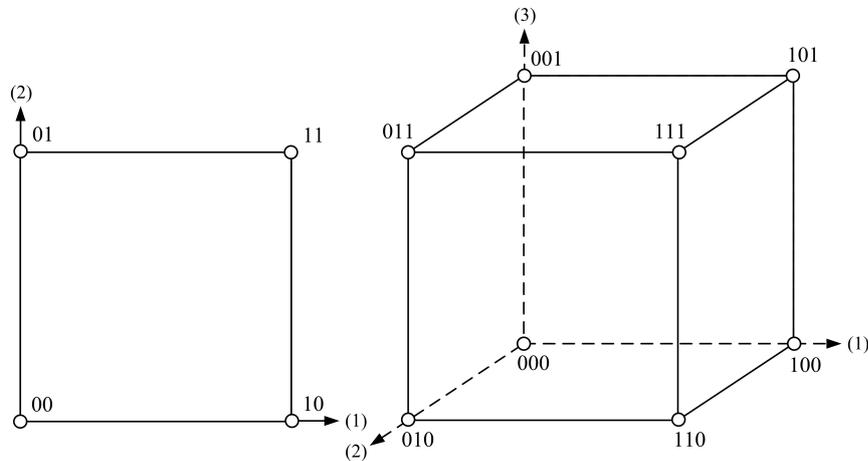


Figure 5.2 Binary tuples of length 2 pictured as points in the plane, and binary tuples of length 3 pictured as points in space (Jankvist, 2008d, p. 39).

codes as subsets of the sets of corners. An idea of this may be obtained from figure 5.2. From this figure the *Hamming distance* between two codewords is also easily illustrated. For instance, the Hamming distance between the two codewords 111 and 101, written as $d(111, 101)$, is 1 since these two corners of the three dimensional cube are one apart, or the codewords differ in exactly one place.

In order to determine the error correcting capabilities of a given code, Hamming introduced spheres into his discrete metric spaces. A *sphere*¹ is centered in a codeword and all the n -tuples inside or on the boundary of this sphere are the ones that may be corrected into the codeword in the center, i.e. they are at most the ‘distance’ of the radius of the sphere away from the codeword. Now, a code for which all the possible n -tuples are included in spheres around the codewords is called a *perfect code*. Hamming-codes are such perfect codes. Figure 5.3 shows the situation for the Hamming (7, 4)-code.

By the time Hamming was able to publish his family of codes in 1950, another mathematician, Marcel Golay, had already generalized the (7, 4)-code presented in Shannon’s article to all other codes in the family (Golay, 1949). This fact has led to an ongoing dispute about who actually can be called the creator of the family of Hamming codes (Thompson, 1983, pp. 56-59). Golay also invented a few additional codes, four to be precise, of which two are also perfect. Especially one of these, a tertiary (23, 12)-code called \mathcal{G}_{23} , is interesting, since coding theoreticians in 1973 proved that the only (non-trivial) perfect code which may correct three or more errors is this Golay

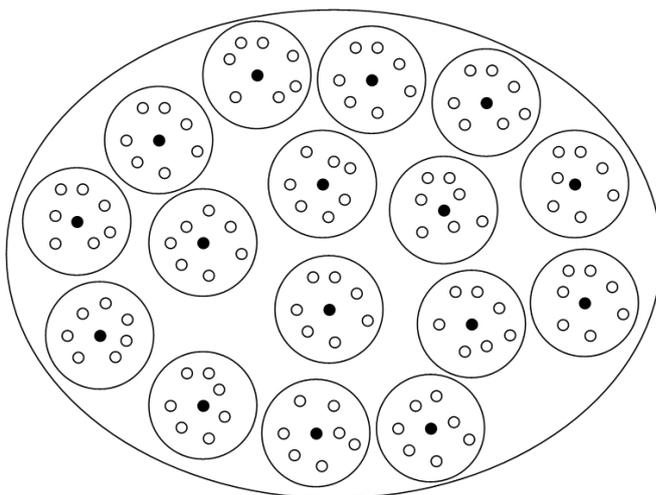


Figure 5.3 An illustration of the perfect binary Hamming (7, 4)-code (Jankvist and Toldbod, 2005b, p. 72). A \bullet denotes a codeword and a \circ another binary 7-tuple. The number of codewords for the code is $2^4 = 16$, and the number of binary 7-tuples is $2^7 = 128$. All these tuples are contained in a packing sphere, illustrating that the code is perfect. Every packing sphere contains $128/16 = 8$ tuples, the center one being a codeword.

¹ Actually, these are what in metric spaces normally are referred to as closed *balls*, since they contain both the inside and the boundary, but I shall follow the convention in coding theory and call them spheres.

(23, 12)-code. From a coding theoretical viewpoint, this result is important, since it tells us that there is no need to keep searching for perfect codes which may correct three or more errors. From the viewpoint of mathematics proper, this result indeed tells us something too: since error correcting capabilities may be viewed in terms of spheres with certain radii, the result tells us something about what sphere packings are possible in which metric spaces (Thompson, 1983, chapter 2).

5.2 Design of the Teaching Material

In the teaching material (Jankvist, 2008d), the students were given an introductory example of two people sending text messages to each other over their mobile phones, in order to set the scene for error correcting codes. Also, they were introduced to binary numbers and binary representation in general. The early history of error correcting codes (perfect codes, sphere packings, etc.) as presented in the previous section was made available to the students. This was mainly done in mathematical terms though sometimes also in the form of stories, as above, depending on the level of difficulty of the in-issues. Coding theory today relies heavily on linear algebra and hence on the use of vectors, matrices, etc. However, the very early coding theory may be, and was to a large degree in history, explained without lining up the large apparatus of linear algebra. Thus, in my presentation of the early coding theory, I was able to rely only on a very limited amount of linear algebra, introducing the codewords as n -tuples instead of vectors, and the construction of these by means of simple linear combinations instead of using the short hand notion of matrices.

Because Danish upper secondary students are not accustomed to writing essay assignments as part of their mathematical training, a number of smaller introductory essay assignments was to be done during the module. This was a way of making the students familiar with the idea of writing essays and this different way of studying, before taking on the large final essay assignment. In the introductory essay assignments the students were, for example, asked to search the Internet for information on the Bell Labs, in order to answer some more specific questions about this institution. In the final essay assignment the students had the opportunity to draw on all the introductory essays, as well as their acquired knowledge from the teaching material (concerning in-issues as well as meta-issues). No additional literature or Internet searches were required to complete this task.

An overview of the essay assignments to be explained, described, discussed, and analyzed throughout the rest of this chapter may be given in the following way:

- **Introductory essay assignments:**
 - **First introductory essay** on binary numbers and binary representation.
 - **Second introductory essay** on Bell Laboratories.
 - **Third introductory essay** on Hamming's use of the concept of metric.
- **Final essay assignment:**
 - **Main essay** in which the students are to provide two accounts for the early history of error correcting codes: a when-and-who account and a why-and-how account.
 - **First supportive essay** on Shannon's (mathematical) presentation of the Hamming (7,4)-code.

- **Second supportive essay** on the use of epistemic objects and techniques in Hamming's development of his codes.
- **Third supportive essay** on the story of the Golay \mathcal{G}_{23} code.
- **Fourth supportive essay** on multiple developments and settling matters of credit.

5.3 Implementation of the First Teaching Module

The first teaching module was carried out over a period of eight weeks in total. It began in the end of April 2007 and was supposed to have ended in the beginning of June, but, as mentioned, exams and summer vacation interrupted the finishing of it. So the rest of the module was put off until the beginning of the students' third year in September. Before the summer vacation the students had eight double lessons, each being 90 minutes, during which the mathematical content of the teaching material was covered. At the end of their second year the students had a mathematics test, constructed by their teacher. Besides dealing with the normal curriculum mathematics for upper secondary level, the test also included a collection of small coding theoretical problems that I had designed (see section 5.6). The first double lesson after summer vacation was spent on evaluating this test in class. This was thought to function as a small repetition of the teaching module, before the students were to do the final essay assignment. However, when the students had worked a bit on their essays, it was soon realized that more repetition was needed: two months of summer vacation had left its traces. For that reason, about three double lessons were spent on repetition before continuing on the final essay assignment. The students spent four double lessons in class, and possibly also additional time outside class, on their essays before handing them in. During these lessons, the teacher would circle the classroom and assist the groups in their work, should they have any questions. Overall, the teaching was a combination of blackboard teaching, individual assignments, and group work – quite normal for Danish upper secondary school.

Before beginning the module, the teacher and I had several meetings, in person and on the phone, as well as email correspondences about the teaching material, especially its content and the presentation of it. During the implementation of the module, the teacher and I would also be in regular contact. Both for the teacher to be able to ask any questions she might have, e.g. concerning the material, what exercises to give out etc., and in order for me to comment on what I had observed while video filming in the classroom. That is to say I would from time to time coach the teacher if, for instance, there were certain things I wanted her to pay special attention to, something I wanted her to say to the class or stress in her presentation, etc. Such coaching would typically take place immediately before or after lessons.

The students were divided into six groups, each group having to turn in one paper, answering the final essay assignment. As mentioned in chapter 4, the selection of interviewees was based on the students' answering of the first questionnaire. Twelve students were selected, and this in such a manner that they represented the class best possible, both concerning their mathematical capabilities (according to themselves as well as to the teacher) and their beliefs about mathematics as a discipline. Out of these twelve students, five were chosen to be in the focus group (group 1). These students were also chosen in such a way that they would be representative of the class as was

best possible. However, other criteria played into this selection process as well: the students should not be too different from each other with regard to their mathematical capabilities; they should be able to work together in a constructive and civilized manner; they had to be somewhat stable in attendance, so that I did not end up having nobody to film (on these issues, I had to rely on the teacher's experience); the group should be composed of students of both sexes; and the group should include students who were positive towards the inclusion of history in mathematics, as well as students who were skeptic. To increase the likelihood of two or more focus group students being present at every class, the focus group was made up of five students, whereas the other groups typically were made up of four. The five remaining groups were put together by the teacher with the sole constraint that I should have at least one interviewee in each group.

During the implementation, the students of the focus group would come sit at a table in front of the video camera, whenever they were to work on essay assignments, and sometimes also if they were to work on mathematical exercises for a longer period of time. The rest of the time the video camera was recording the teacher teaching or the general activities in the classroom.

5.4 Students' Introductory Essay Assignments

As seen, the students were given three smaller introductory essay assignments. Two of these required searches on the Internet and one included an interpretation of an extract from an original source.

Binary Numbers and Binary Representation

The students' first encounter with the idea of writing essay assignments in their mathematics class was with an assignment on binary numbers and binary representation. They had been introduced to these ideas in the teaching material and in the teaching, and they were now asked to use the Internet to do searches on 'history of binary numbers', 'history of binary representation', etc. to see what they could find. Based on that, they were to write a short essay addressing the following questions:

- a. From when, historically speaking, do the ideas of binary numbers and binary representation stem?
- b. From where, or from whom, do the ideas of binary numbers and binary representation stem?
- c. Why did the ideas of binary representation originate?
- d. Why are binary numbers and binary representations applied, today as well as in previous times?

The students were asked to provide the Internet addresses from where they had retrieved information, so there were some possibilities of checking if the students had been cribbing. Of course this did happen, though actually not too often. And if students were in fact copying they would still have to translate from English to Danish. The answers which follow do not appear to be subject to cribbing.

As an example of one of the more elaborate answers concerning the history of binary numbers, one group gave the following description of Leibniz' contributions:

In the end of the 17th century the German philosopher and mathematician Gottfried Leibniz constructed the first machine capable of performing addi-

tion, subtraction, multiplication, and division. He is primarily known as a philosopher and for his contribution to the differential calculus, but he did in fact also discuss several ideas which later were to become central within computer science. In his famous *Explication de l'Arithmétique Binaire* from 1703 he shows that instead of using the ten-base number system, one might just as well perform all calculations in a two-base number system where the only symbols needed are 0 and 1. (Group 3)

Another group gave the following concise description of the application of binary numbers and binary representation:

The binary number system, or the two-base number system, consists of the digits 0 and 1. It is applied in computeres as machine code and in punch cards. In punch cards a 1 is represented by a hole and a 0 by no hole. In the digital (logical) circuits of electronics, and thereby also computers, the two values can be represented by the presence or the absence of a current (voltage or 0 volt). (Group 4)

The binary representation in punch cards was noticed by several groups. One group, however, was also able to make connections between binary representation and the Western history in general:

During the period of industrialization, the binary numbers turned out to be a useful invention. Silk was brought from the East to Europe, and large factories for weaving the silk were built. Normally, weaving silk is a very difficult and manual job demanding both craftsmanship and precision. But during the industrialization binary numbers were used in designing the weaving machines, so that these could weave the silk themselves. As a consequence, not much manual work was needed. The method was simpel. Big pieces of cardboard were used, and holes were made in these. The holes were placed on the cardboard exactly where you wanted the machine to react in one way or the other. For instance, if it should move the color to somewhere else, or stop weaving in one place, etc. It worked a little like one of these mechanical pianos where small holes are placed exactly where you want the piano to play a note. (Group 6)

The examples above show the students' first encounter with meta-issues relating to the history and application of mathematics, and their essay writing about them – an encounter which all in all seemed to go rather smoothly. On average, the hand-in essays were around three quarters of a page long, and they were written as interconnected sentences addressing elements of the four questions, not just as separated answers to these. The students had approximately 45 minutes in class to work on their essay, and could finish it outside class if necessary. The essay was to be turned in during next class, two days later (see appendix A.2). In terms of anchoring their first essays in the in-issues, I also believe that the examples show such an anchoring to some extent, e.g. when the students discuss matters of a ten-base versus a two-base number system, or when they address how to represent a two-base system either mechanically or digitally. In particular I enjoyed the following example where one of the groups had found the old 'binary joke' on the Internet:

There are 10 kinds of people: those who understand binary, and those who don't. (Group 2)

Needless to say, if one does not 'understand binary numbers', this joke is just not funny.

Besides the purpose of this introductory essay assignment being to get the students accustomed to working in this manner, it also had the purpose of introducing them to some of the questions being asked, when studying the history of mathematics, e.g. questions of when, who, why, and how – questions they would encounter again in the final essay assignment.

The Bell Laboratories

In the second of the introductory assignments, the students were to search the Internet again (and provide the Internet addresses used). This time to see if they could find information linking Bell Labs to Shannon and Hamming, e.g. by searching on ‘history of Bell Labs’, ‘Shannon and Bell Labs’, or ‘Hamming and Bell Labs’. Based on this information, as well as that in the teaching material, they were to discuss:

- a. What can be said about Bell Labs when it is known that their research department made up 12 percent of the institution’s entire technical staff? Are 12 percent to be considered a lot or a little?
- b. Is it possible to say anything about whether Bell Labs conducted basic research or if they were only interested in practical applications?

Both of these questions were somewhat related to matters of World War 2, with the post war military funding of scientific research programs at universities and at institutions like Bell Labs, which the students had been told about in the teaching material.

One group wrote the following about Hamming:

After the Manhattan project ended he got a position in 1946 with Bell Telephone Laboratories in New Jersey. Here he worked within a group (‘Young Turks’) together with Claude E. Shannon, Donald P. Ling, and Brockway McMillian. Here he developed error-correcting codes (Hamming codes). Within the area of digital filters his name is associated with a function called a Hamming window. (Group 4)

Most groups thought that 12 percent seemed a quite high percentage of the personnel to be involved with conducting research. However, some of the students had recently visited the Danish medical company Lundbeck, which led one group to answer:

Since only 12% of their department personnel consists of researchers you could say that this is no large number, because Bell Labs in the beginning was a research center. On the other hand, if the ones selling the products and the ones leading the company and all others doing all sorts of things in the company are included in this number, then it is a relatively large number! This scenario can be found also with a company like Lundbeck, where 21% of their department personnel are researchers compared to Bell Labs’ pitiful 12%. (Group 6)

Concerning Bell Labs’ interest in basic research versus applications, group 6 continues:

Bell Labs might have started out as a company which values research highest [...] Since researchers are being directed in their research, the choice of free research is taken away from the researchers. [...]

Companies have been created within natural science. It is no longer done out of free will, and people do not research freely. Therefore research is no longer conducted as when it was carried out by Aristotle, Pythagoras, and Plato in the dawn of time. (Group 6)

Another group (group 3) states that Bell Labs are involved in both research and development, and that you therefore cannot say that Bell Labs are involved in basic research, but that a bunch of enthusiastic researchers develop their research further. Group 5, on the other hand, says that Bell Labs do both basic research and practical applications, and support their claim by saying that Bell Labs “on top of this even won a Nobel prize in physics last year”. Group 6 is really the only group that enters into the discussion of post war military funding saying that:

During the years of and after World War 2 research all over the world received a lot of financial support. Bell Labs' research received a lot of money during the Cold War, because you needed to be able to communicate with each other on the battle field and in other respects. For this reason they were told to develop their idea about the telephone net further. This financial support had a large influence on the way Bell Labs conducted their research. They no longer needed to think about how much money they spent, only how to get the needed material. For that reason they were in need of nothing and their materials were of high standard. (Group 6)

From the above, the students once more appear capable to enter into meta-issue discussions, on some levels at least, in this case, meta-issues concerning elements of the history of applied mathematics. Again the students were given time in class to work on the essay, but they had to finish it at home before handing it in during the next class (see appendix A.2). Generally, the essays on Bell Labs were a little shorter than the ones on binary numbers. Concerning an anchoring of the addressed meta-issues in mathematical in-issues, this particular essay assignment appears less qualified than the previous one on binary numbers. The mathematical focus of the assignment was somewhat weak, there was no built-in insurance that the students would meet with in-issues in the same way that there was for the previous assignment. And the students did not seem capable of bringing in this dimension themselves at this point in time in the implementation of the module, at least no one did.

Hamming's Use of Metric

The third of the introductory assignments was much more mathematically founded, and thereby forced the students to work with the in-issues of the historical case. First, the students were presented with the following extract from Hamming's 1950 paper:

Into this space of 2^n points we introduce a distance, or, as it is usually called, a *metric*, $D(x, y)$. The definition of the metric is based on the observation that a single error in a code point changes one coordinate, two errors, two coordinates, and in general d errors produce a difference in d coordinates. Thus we define the distance $D(x, y)$ between two points x and y as the number of coordinates for which x and y are different. [...] This distance function satisfies the usual three conditions for a metric, namely

$$\begin{aligned} D(x, y) &= 0 \text{ if and only if } x = y \\ D(x, y) = D(y, x) &> 0 \text{ if } x \neq y \\ D(x, y) + D(y, z) &\geq D(x, z) \text{ (triangle inequality)}. \end{aligned}$$

(Hamming, 1950, p. 11)²

Based on this the students were asked to do some comparisons with what they had been exposed to in the teaching material:

- a. Account for Hamming's definition above of a metric being the same as the definition of Hamming distance in the teaching material.
- b. Account for the conditions which Hamming's metric above fulfill being the same as those which the Hamming distance is shown to fulfill in the teaching material.

For question a, the teaching material definition of Hamming distance read: "The Hamming distance $d(\mathbf{x}, \mathbf{y})$ between two binary tuples \mathbf{x} and \mathbf{y} of length n equals the number of places where \mathbf{x} and \mathbf{y} differ" (Jankvist, 2008d, p. 18). For question b, the second condition above which Hamming's distance function satisfies was split in two (i and iii) in the teaching material (Jankvist, 2008d, p. 19):

- i. $d(\mathbf{x}, \mathbf{y}) > 0$ if $\mathbf{x} \neq \mathbf{y}$.
- ii. $d(\mathbf{x}, \mathbf{x}) = 0$.
- iii. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
- iv. $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{z})$.

Next, the students were given a small example introducing the idea of a 'technique' (in terms of epistemic objects and techniques as described in section 3.6) and given the following question:

- c. In what way does Hamming's use of the generalized concept of distance, 'a metric', serve as a technique in the quote above? What problem does this use of a metric help solve?

Most groups appear to have seen that the two definitions of Hamming distance are the same, though they do not all explicitly state this. One exception is the following:

Looking at Hamming's definition of a metric it can be seen that this basically concerns the same as the Hamming distance [in the material]. He just uses D instead of d . As he writes, the number of errors is the same as changes in coordinates. This is the same as if we say: This code[word] has one error 000011 in comparison to the next 000111. And we can also see that only one of the coordinates is changed, the fourth coordinate. (Group 6)

None of the groups provided an entirely correct and clear-cut answer to question b. Most groups observed that the triangle inequalities, Hamming's third condition and the fourth (iv) in the material, are the same, some spending more effort than others discussing the fact that the x , y , and z are rearranged within the two. Many groups were also clear about Hamming's first condition being identical to the second (ii) in the material. However, seeing that Hamming's second condition was a contraction of the first (i) and the third (iii) in the material was a bit more problematic. Some groups would leave out one of the four conditions of the material in order to make a one-to-one correspondence (typically condition iii), and one group even talked about the material having "an extra 'part' saying that $d(x, y) = d(y, x)$ " (Group 3). Only one group (the focus group) gave a correct answer, though it could have been more clearly stated:

Definition 2.11 [in the material] fits with the one above [Hamming's]. If two binary tuples are the same, the difference is 0 (ii), and if they are different the difference is $d(x, y) > 0$ (i). The distance between two different tuples

² In the students' version of the teaching material there was a misprint: the triangle inequality read $D(z, y) + D(y, z) \geq D(x, z)$. The teacher did, however, draw the students' attention to this misprint during the work on the assignment.

$d(x, y)$ and $d(y, x)$ is also said to be equal to each other in this [Hamming's] definition (iii). (Group 1)

This group, however, had some problems with the triangle inequality, mainly due to a misprint in the material (see footnote 2, page 120), something which had been clarified by the teacher but apparently had still gone unnoticed by the group members.

The last question (c) concerning Hamming's use of the generalized concept of distance as a technique, seemed to be rather difficult for the students to grasp. Some groups would answer with very general statements like for example: "The concept of a Hamming distance also plays a role in the rest of his mathematics." (Group 1); "The two ways [Hamming's and the one in the material] of treating the concept of distance are very alike." (Group 4). One group stated that they were not able to answer the problem (Group 6). The closest to an answer was provided by group 2 who said:

A metric is defined by four conditions. Hamming used these conditions due to other mathematicians to define the actual Hamming distance. (Group 2)

The answer I had hoped for, was an observation that the generalized concept of distance served as a technique for Hamming, in his work with the error correcting codes (the objects under investigation). By defining his own discrete distance as a special case of the general concept of distance by fulfilling the conditions of a metric, Hamming had created a tool which he could use in studying and developing his error correcting codes. An example of this, which the students were familiar with when working on the essay assignment, were the use of Hamming distance in decoding transmitted codewords with the so-called nearest neighbor decoding procedure, i.e. decoding a transmitted word to the codeword which is the least Hamming distance away.

The overall purpose of this introductory essay assignment was twofold. First, to have the students work with the mathematical subject matter (in-issues) in a bit more historical context with the use of an original source, and second, to slowly introduce the idea and notions of objects and techniques before this was to be used in full scale in the final essay assignment. The first part seemed to go rather well, despite not all groups providing correct answers. The students were generally willing to and, largely, capable of engaging in and discussing the extract from the original source. And the fact that there were some notation differences did not seem to throw them too much off track. For the second part, however, much is left to wish for. Only one group (group 2) gave something resembling an answer to question c. One conclusion to this result might be that the question was phrased and presented in a way not accessible enough – I shall return to this in the recapitulation in section 5.7. Another conclusion could be that epistemic objects and techniques make up a general issue too difficult for upper secondary students to engage in. However, the students did engage in it somewhat successfully as part of the final essay, as I shall discuss next.

5.5 Students' Final Essay Assignments

In this section, I shall paint an overall picture of the students' answers to the final essay assignment by displaying a variety of extracts from these, and by providing my own comments. The final essay assignment consisted of one main essay assignment and four supportive essay assignments, which were to be made before the answering of the main assignment. The final hand-ins were generally between 2 and 4 computer written

A4-pages long. I shall begin by outlining the main essay.

Outline of the Main Essay Assignment

The object of the main essay assignment was the writing of history of mathematics. The students were asked to provide the following:

- a. A presentation [of the early history of error correcting codes] basing itself solely on *when* and *who*. Elements from your discussions of the first and fourth supportive essay assignments should be part of this presentation.
- b. Another presentation basing itself on *why* and *how*. Amongst other you should describe what has dictated the emergence and genesis of coding theory, or the early coding theory. Elements of your discussions of objects and techniques in the second and third supportive essay assignments should be part of this presentation. And also your answers to the second and third introductory essays should be taken into consideration.
- c. Which of the two presentations do you prefer? Why?
- d. What do you think may be learned from studying the history of mathematics?

As mentioned earlier, the idea of the four supportive essay assignments was to create a basis for the students to answer the main essay assignment on as well as to fulfill the intentions of the 2007-regulation and the KOM-report along the way (cf. section 3.5).

In the following, each supportive essay assignment will be presented and extracts from the students' answers will be shown and discussed. The students' answers of the main essay assignment are treated after looking at the supportive assignments.

First Supportive Essay Assignment

The first of the supportive assignments was a mathematical exercise where the students were given an extract from Shannon's 1948-paper in which he introduces the Hamming (7, 4)-code:

The following example [...] is a case in which exact matching to a noisy channel is possible. There are two channel symbols, 0 and 1, and the noise affects them in blocks of seven symbols. [...] An efficient code, allowing complete correction of errors [...], is the following (found by a method due to R. Hamming):

Let a block of seven symbols be X_1, X_2, \dots, X_7 . Of these X_3, X_5, X_6 and X_7 are message symbols [...]. The other three are redundant and calculated as follows:

$$X_4 \text{ is chosen to make } \alpha = X_4 + X_5 + X_6 + X_7 \text{ even}$$

$$X_2 \text{ is chosen to make } \beta = X_2 + X_3 + X_6 + X_7 \text{ even}$$

$$X_1 \text{ is chosen to make } \gamma = X_1 + X_3 + X_5 + X_7 \text{ even}$$

When a block of seven is received α, β and γ are calculated and if even called zero, if odd called one. The binary number $\alpha\beta\gamma$ then gives the subscript of X_i that is incorrect (if 0 there was no error). (Shannon, 1948, p. 16)

The students were asked to justify that this code was, despite the difference in presentation, the same as the (7, 4)-code they had been introduced to, in modern notation, in the teaching material. Also, they were asked to show by example, that the error correcting procedures of the two formulations of the code were, in fact, equivalent.

The purpose of this assignment was twofold. One purpose was to check the students' skills on the involved in-issues. This was, however, done in an historical context by having them work with the original presentation of the (7, 4)-code as given in Shannon's 1948-paper – this being the second purpose, showing them an extract from an original and historical source. Thus, in this sense, the assignment was similar to the third introductory essay assignment on Hamming's use of metric.

Five out of the six groups were capable of completing this exercise in a satisfactory manner, i.e. justifying that the two presentations of the Hamming (7, 4)-code were identical. Due to the technicalities of this assignment, I shall refrain from giving a detailed description of all the students' answers. Instead I concentrate on the focus group's answer in section 5.6, where I also shall provide the exact phrasing of the first supportive essay questions.

Second Supportive Essay Assignment

In the second supportive essay assignment the students were to look at a series of selected quotations from Hamming's 1950-paper and then try to identify some of the already established mathematics, that is to say the mathematical techniques, and then, in terms of these techniques, discuss which ones he used to develop the codes, i.e. the objects under investigation.

The quotes concern the generalized concept of distance and the elements of linear algebra which Hamming uses, including the use of his "geometrical model":

When examining various problems connected with error detecting and correcting codes it is often convenient to introduce a geometric model. The model used here consists in identifying the various sequences of 0's and 1's which are the symbols of a code with vertices of a unit n -dimensional cube. (Hamming, 1950, p. 11)

The n -dimensional cube of which Hamming talks, is the cube depicted earlier in figure 5.2, i.e. a cube or a space made up of binary n -tuples. Hamming then introduces (packing) spheres:

To continue the geometric language, a sphere of radius r about a point x is defined as all points which are at a distance r from the point x . (Hamming, 1950, p. 11)

By referring to a previous example, when he investigates the "code points" (001), (010), (100) and (111) in the 3-dimensional cube, Hamming says:

Thus, in the above example, the first three code points are on a sphere of radius 2 about the point (1, 1, 1). In fact, in this example any code point may be chosen as the center and the other three will lie on the surface of a sphere of radius 2. (Hamming, 1950, p. 11)

When Hamming at bit later studies 1-error detecting codes (codes which can detect one error), he begins his description of this study by saying:

The problem studied in this section is that of packing the maximum number of points in a unit n -dimensional cube such that no two points are closer than 2 units from each other. (Hamming, 1950, p. 11)

The exact tasks which the students were given based on these quotes were:

- a. Based on the quotes above and the rest of the information throughout the teaching material, you are to identify and provide a list of the techniques which Hamming uses to

study his codes (the objects). You are to include elements from the third introductory essay assignment.

- b. For each of the identified techniques in the list describe which aspects of the objects (the codes) exactly these techniques are to illuminate.

As with the first supportive essay assignment, these questions too rely on a great deal of technicalities concerning the theory of error correcting codes. For that reason, some of the details will not be fully dealt with here, but will be left to section 5.6 when discussing the focus group's answer of the assignment.

The answers to the second supportive essay assignment may be divided into three kinds. The first kind consists of answers looking at techniques used by Hamming in general, that is to say no distinction was made between techniques already available to Hamming from the very beginning, and techniques that Hamming had to create himself in the process of developing and describing his codes, e.g. the Hamming distance and nearest neighbor decoding. One group of students presented the following list:

The techniques Hamming uses to study codes: Hamming-distance, decoding with nearest neighbor, weight of the words, t -detecting codes, t -error correcting codes, the syndrome,... (Group 6)³

The second kind of answers were those taking into consideration only the already developed and available techniques, which was the original intent with the assignment. As an example of this, one group gave the following clear-cut answer:

Hamming uses generalized concept of distance; elements of linear algebra; geometrical models; and unity n -dimensional squares. (Group 5)

The last kind of answers consisted of a mix of the two first, that is both the already available techniques and techniques created in the process of development. The majority of the answers to this essay assignment were of the third kind and only one group gave a clear-cut answer of the second kind (the answer shown above). Even though the example of answers of the first kind shown above does not concern the already established techniques it does, in fact, still concern techniques. For instance, the group mentions Hamming-distance which is a metric and did serve as a technique for Hamming in order for him to describe the error correcting capabilities of his codes.

More interestingly for the second research question is the fact that all groups, to some extent, mention either metric or Hamming-distance as a technique used by Hamming. This does suggest some kind of anchoring of the students' treatment of meta-issues concerning the birth and early development of error correcting codes in the acquired mathematical content (in-issues) of the module. Had the students not been taught the mathematics of error correcting codes, they would not have been able to address the history of these codes in terms of objects and techniques. Not in the manner they actually did, anyway, by identifying used (old or new) techniques. Of course, the objection may be made that the students were repeating names or words from the teaching material, but question b had as its goal to make sure that the purposes of the techniques mentioned were also accounted for. As mentioned above, I shall provide an example of this in section 5.6.

³ The terms used in this quote will be explained in section 5.6.

Third Supportive Essay Assignment

Based on the second supportive assignment on objects and techniques and the story of the \mathcal{G}_{23} code (cf. section 5.1), the students were asked:

- a. From Hamming's use of packings in n -dimensional cubes, as described in the quote above, and the earlier told story of the binary Golay code \mathcal{G}_{23} what may then be said about the relationship between objects and techniques? Provide arguments for your viewpoints.

The students' answers of the third supportive essay assignment varied quite a bit. From the almost rapid statements:

You must crawl before you can walk [...] Both Golay and Hamming had to work hard using techniques before they could establish objects. (Group 1)

to the more substantial ones:

Objects and techniques, one might say, are different ways of placing symbols so they obtain different meanings. For example the minus sign: where minus symbolizes the negative number in -4 , while it symbolizes an arithmetical problem in $7 - 3$. Due to this we may conclude that *techniques may become objects and vice versa*. For example the arithmetical problem $7 - 3$ becomes -4 . (Group 6, italics added)

Probably the last part of this quote should have read $3 - 7$ becomes -4 . In any case, calling a symbol like the minus sign an object is probably not entirely correct, at least not in Rheinberger's meaning of the word (see section 3.6). On the other hand, a negative number like -4 is a mathematical object, but then the example is a little off. Most important, though, is the group's remark saying that "techniques may become objects and vice versa". In fact, this was mentioned briefly, though in a different wording, in the teaching material, so the students were expected to know about this relationship. However, it was not exemplified, which is what the students attempts to do in the above quote.

In the story of Golay's \mathcal{G}_{23} code the relationship between objects and techniques is a bit more peculiar, because objects and techniques actually took each other's places in two different historical contexts: \mathcal{G}_{23} is a perfect code which means that the space related to this code may be filled with packing spheres, leaving no tuples outside a sphere. Hence, for Golay, packing spheres would have been a technique to describe the error correcting capabilities of the \mathcal{G}_{23} code, the latter being the object. Now, as mentioned in section 5.1, the \mathcal{G}_{23} code later came to play a role in the context of packings of metric spaces, since the code theoretical result about \mathcal{G}_{23} served as a way of saying something about what packings are possible in which spaces. Thus, in this context the \mathcal{G}_{23} code served as a technique to say something about the objects under investigation, the sphere packings (cf. the discussion of objects and techniques shifting places in section 3.6, page 65).

No groups were able to give the answer described above, which could indicate that the question was either too hard or should have been made more clear to the students. In any case, the third supportive essay was a somewhat difficult one.

Fourth Supportive Essay Assignment

In the fourth supportive essay assignment the students were introduced to the still ongoing discussion about whether Hamming or Golay should be considered the founding

father of the family of Hamming codes. Based on some selected quotations from this discussion, the students were asked the following questions:

- a. Who do you think should be ascribed the honor for the family of Hamming codes? Provide arguments for you viewpoint.
- b. Why do you think it may be of such importance for people to establish the correct originator to a given mathematical result? Which for the mathematicians personal driving forces may be at stake in the research in mathematics (and the research in science in general)? Is it possible to say anything about the 'sociology' in scientific communities?

The majority of the groups agreed on Hamming being the creator of the family of Hamming codes, arguing very much in accordance with the viewpoints in the quotations they had been exposed to. One group took to the point of view that the first to publish should get the credit.

More interestingly, however, are the students' answers to question b. Here, answers like the following were given:

A lot of work behind – bummer if someone else gets the credit. (Group 1)

... if you don't know who the creator was, nobody knows who has the right for patent. (Group 6)

One might imagine that what actually drive mathematicians and scientists to wander toward new paths would be the thought of recognition, fame, and money along side the thought of bringing mankind to a higher level. (Group 4)

The latter clearly being the more reflective – and solemn – of the three.

Main Essay Assignment

In their when-and-who presentation (see question a, page 122) of the early history of error correcting codes, some groups, based on the teaching material, gave extensive lists with years and persons, while other groups merely gave short accounts, highlighting the major events in terms of years and people.

The why-and-how presentations (question b) varied a lot in scope as well as in angle of approach. Some groups concentrated on Hamming's personal annoyance with his calculations being dropped:

Hamming needed to have some information coded, but since the machine wasn't capable of handling errors it stopped, and during the weekends there were nobody to restart it. That's why he made a system capable of correction... (Group 3)

Others focused more on the overall need (in society) for error detecting and correcting codes:

The problem in communication was that people had trouble reproducing either exactly or approximately the original message as sent by the source. Often it was transmitted correctly, but sometimes noise would arise and the message was almost unusable. People wanted to make sure to avoid these errors by creating some new ways of transmitting messages... this is the background for people developing the coding system further. (Group 4)

Regarding the students' own opinion on the two accounts and if they preferred one over the other (question c), the majority seemed to prefer the why-and-how account. One

group (group 6) stated that this account provided a certain depth to the area as opposed to the other one. Another group (group 1) stated that only the why-and-how account was of any use, since this approach provided an insight into the related mathematics of the history. Also, one group (group 3) thought it to be pointless to discuss the when-and-who approach without telling the how as well – the first account could not go without the second and vice versa, they said. There was also a group in favor of the when-and-who account:

We believe it to be more tangible getting the hard facts than getting some long elaborated explanation on why and how. After all, there is a reason for us choosing a natural science education ;-) (Group 2)

Perhaps this statement may be seen as an indication of the fact that not all students were equally fond of essay writing being brought into the mathematics class. (The second round of questionnaires and interviews confirmed this.)

In regard to the last question (d), what one might learn from studying the history of mathematics, we may again refer to the notions of history as a tool and history as a goal. Some of the students seemed to have an initial idea of history being an aid to assist the learning of the mathematics in question, which is interesting since this was not something the teacher had discussed with them. Three of the groups said:

By working with mathematics from scratch, i.e. its history, one obtains a better understanding of the given mathematical subject. (Group 1)

In our opinion you can obtain knowledge about the methods behind mathematical results by studying the history of mathematics. You can get an insight into the thought process and approach behind a theory and use this, for example, in the development of new theories. (Group 4)

It ought to give a better understanding of the subject... (Group 6)

The latter answer is a bit more vague than the two first and seems to be indicating that perhaps history did not provide this better understanding after all. Nevertheless, all of these answers deal with the use of history as a tool rather than the use of history as a goal. In fact, none of the groups' answers had a clear focus on history as a goal, though one group did mention the application aspects and the enormous use of error correcting codes in our everyday life. One explanation might be that the students do not distinguish between mathematics and the history of mathematics. Another, of course, may be that they find it hard to believe that they should spend time on something in mathematics class which does not have as its main purpose to increase their mathematical skills.

As a final remark about the students' essays, it should be mentioned that after the implementation of the module, and after the second questionnaire and round of followup interviews, I went through the final essay assignment with the class. I discussed their hand-in essays with them, provided them with additional answers, and gave them the opportunity to ask questions which I would then answer (though I would not discuss any questionnaire questions which I knew would be repeated later). Normally something like this would have been done by the teacher, but she had asked me to do it, and I accepted as a gesture of gratitude to her participation in my research, and because it gave me the opportunity to thank the students for their participation also.

5.6 Observing the Focus Group (Group 1)

In this section I shall first account for the selection of the focus group students and then introduce them. Next, I shall discuss the focus group's final essay assignment and the making of it, the latter also by means of a video clip. Finally, I shall discuss the focus group students' understanding of the mathematical contents of the teaching module, in order to be able to evaluate the anchoring of their meta-issue discussions and reflections in the related in-issues.

Selection and Introduction of the Focus Group Students

The focus group consisted of five students whom were, as well as the other interviewees, mainly chosen based on their answering of the first questionnaire (see page 115). In total there were twenty questions in the first questionnaire (see appendix B.1), examples of which are:

1. On a scale from 1 to 5, where 5 is the subject you like the most and where 1 is the subject you like the least, where would you place mathematics?
2. Are you good at mathematics?
3. Regardless of what you answered to the previous questions, mention the things you like the most about the subject mathematics.
4. Regardless of what you answered to the previous questions, mention the things you like the least about the subject mathematics.
5. Do you believe it to be important for you to learn mathematics? If yes, why? If no, then why not?
6. Do you believe it to be important for people in general to learn mathematics? If yes, to whom is it then most important and why? If no, why not?
13. Have you at any point in time during your schooling come into contact with the history of mathematics? If yes, when and in what way?
14. Is the history of mathematics something which interests you or which you may think could be of interest to you? If yes, *what* about the history of mathematics? If no, why not?

Along with these questions the students were asked questions concerning their beliefs about or images of mathematics; a combination of more historical and developmental questions, sociologically oriented questions, and some philosophical questions (these questions shall be the subject of attention in chapter 7).⁴ The five students selected for the focus group were: Gloria, Sean, Andrew, Jonathan, and Lucy. The following descriptions of the students are based on the first questionnaire questions presented above (1-6, 13, and 14) and the first round of followup interviews.

In question 1, **Gloria** rates her interest as close to 4 (5 being the highest on a scale from 1-5), although it depends on the mathematical topic in question, she says. She considers herself good at mathematics when it concerns more 'abstract' aspects, not details as rules for calculating fractions etc. Of the things she likes about mathematics, she mentions working with mathematics together with other students: "It really provides you with some energy having sat there and worked for two long hours, for instance, where you haven't thought of anything else than the mathematics (which you hopefully succeed with) and your collaborator. That I love." Also, she likes the many different

⁴ For yet an analysis of the students' answers to some of these questions, see Jankvist, 2009d.

ways of doing things in mathematics and the fact that they can be proven, something which is not the case for biology, she mentions, where you merely state that this is the way it is. The downside of mathematics, as she sees it, is that it is not a subject that takes into account your personality and feelings. In the interview she expressed this view in the following manner:

It doesn't matter whether you like fractions or not. You follow me. It's kind of like... Well, in Danish and English and social studies and history, there it's your interpretation of people's behavior and actions in texts and history, which you kind of interpret, and say that he is an evil person and he is greedy of power, and this and that. And here it's kind of like: this gives 5.

Gloria is not sure if it is important for her to learn mathematics, but she wants to, she says. She does not think that all mathematical topics are equally important for all people. Her mother, for instance, has no need to know about vectors in her job as a producer (Gloria does not say what her mother is a producer of). Gloria has previously been introduced to elements of the history of mathematics – “Aristotle and a lot of other old people”. She thinks that the history of mathematics could interest her, though she does not know why: “I think it is very exciting that some people have dedicated their lives to exploring it. It's just difficult knowing whether it will interest you or not when you know nothing about it.”

Sean rates his interest in mathematics to 5, and he believes himself good at mathematics (in fact, the teacher thinks him very capable). He likes the logic of mathematics and the challenges in solving problems, but dislikes too long and out of control calculations. Sean personally considers mathematics important because it satisfies his curiosity about matters which are explained easiest by means of mathematics, he says. In the interview he mentions “formulas for a theoretical increase in temperature for overheating” and “statistics for overpopulation of, for example, both people and animal, and what to do about it”. He considers it “a way to put a structure on the world”, to explain “why it is the way it is”. Sean also thinks it is important for people in general to learn about mathematics, e.g. so they can scale cooking recipes, get the best bargain when shopping, etc. He has previously been exposed to some history of mathematics, Pythagoras' theorem and Fermat's last theorem. He says that the history of mathematics does interest him, but he cannot mention any specific elements, and that it depends very much on what he finds interesting at a given point in time.

Jonathan rates his interest in mathematics as a 3 and believes himself to be “relatively good” at mathematics. He likes numbers, he says, and is good at doing calculations in his head. On the other hand, he is fed up with geometry and coordinate systems(!). Jonathan thinks that he wants to study physics and chemistry at the university, and since mathematics is a tool in these subjects he thinks it quite important to learn mathematics. In question 6 on whether it is important for people in general to learn mathematics, he answers that it is definitely important to know something about mathematics: “Mathematics is everywhere, and knowing something about it only makes everything easier.” Jonathan has no prior experiences with the history of mathematics and he considers himself “rather indifferent towards when different elements of mathematics have been invented”, to him the mathematics itself is more important, he says. He does, however, have an idea about what history of mathematics is, for example that it is not only concerned with years etc. but also thoughts from one period of time and other people expanding on these thoughts in different periods and so. In particular, he

mentions the story of Andrew Wiles whom the class has seen a movie about:

It was interesting to see how much, well, he was really, really into this, this guy. I don't think I've ever seen anyone before who was so much into math. He was weeping of happiness when he finally solved it. In some way I understand why, because it's been... people have been wondering about this, and it took so long, and then that it had to be him who finally saw how it all was connected. That must be big, right, in some way. For people who aren't interested in mathematics it might seem kind of 'whatever'.

Andrew is not too interested in mathematics but answers 3 to the first question. He considers himself an average student in mathematics, he is more into biology, he says. In fact, he wants to study biology and in order to do so, he reckons he needs mathematics at advanced level. He likes that mathematics can be used for something, e.g. statistics etc., which may also be used in biology. In particular, he dislikes "everything that has to do with sine and cosine". He thinks it important for people in general to learn the most basic things of mathematics: "Kind of like learning to read. It is very useful for all. Not that everybody need to learn all that sine and cosine..." Andrew has previously been introduced to elements of the history of mathematics both in upper secondary school and before. He remembers having "read about different 'legendary figures' of mathematics and their considerations as well as their time". To the question whether he believes the history of mathematics may interest him he answers: "No. I'm not that hooked on mathematics that I want to spend more time on it than absolutely necessary – unfortunately." In the interviews it is revealed that some of the places he has encountered the history of mathematics is in the short (illumination) paragraphs or pages in the textbooks. Andrew admits to reading those as well as finding them somewhat interesting. He is very interested in history in general, he says. But it is not "how they found out about one thing or the other" that interests him, "it's more the persons who interests me a little", for instance in the form of biographies. When asked what it is that does not interest him in the history of mathematics he replies: "All the numbers."

The last student, **Lucy**, rates her interest in mathematics as a 5 and considers herself good at most mathematics. What she likes about mathematics is that it is a challenging subject, that there always are new things to learn, new ways to solve problems, and the satisfaction that comes after having worked hard on a problem and then finally arriving at a solution. What she does not like is that sometimes there are a lot of things to keep track of, and "when there is a topic you just don't get". She thinks that mathematics is good to have as a foundation later in life, and that the study of it also "develops one's brain in some way". She believes that people in general should know about mathematics in order to make society function: "If people didn't know how to calculate, how would the world economy then look?" Her prior experiences with the history of mathematics involves seeing a movie (probably the one with Andrew Wiles, which Jonathan mentioned), something with Pythagoras, and something she does not remember. She believes, however, that "you get a better understanding if you know something about the history", and also she thinks that she may find it exciting to see how mathematics has come into being: "I mean, you havn't just found out about it from one day to the next, it has kind of like arisen."

The Focus Group's Final Essay Assignment and the Making of It

Before engaging in a presentation and analysis of the focus group video clip, I shall provide a short overview of the group's hand-in final essay. This overview will be supplemented by some comments on their actual making of the essay, based on my observation of their work.

The focus group provided a very good answering of the first supportive essay assignment, a two page essay with rearrangements of formulas, small examples, and calculations. The remaining three supportive essay assignments as well as the main essay assignment were also answered on two pages altogether (the final hand-in was four pages). This imbalance may be due to different things. First of all, since the students took on the first essay assignment first, they may have spent most of the available time in class on this before moving on to the other assignments. Secondly, since the first supportive assignment concerned in-issues, it may have been intuitively clearer to the students, or at least some of them, how to proceed with the assignment. And thirdly, the student typing up the essay on the computer, Sean, definitely seemed more interested in the mathematics than in the history – something which is evident from the videos where he participated much more actively in the work on the first essay assignment than in the work on the following ones. (A more thorough description of the students' answer of this assignment will be given later.)

The focus group turned the second supportive essay assignment into a 'tour de force' through the teaching material, indiscriminately identifying various definitions, notions, and concepts they stumbled upon as a technique (Hamming distance, Hamming weight, t -error detection, t -error correction, decoding to nearest neighbor, Hamming spheres, packing radius, syndrome decoding, n -dimensional cube). Applying such a 'scattergun method', it is no wonder that the students got a few of the epistemic techniques right (n -dimensional cube and Hamming distance). From a mathematical in-issue point of view, the students manage to explain the purpose of the techniques in a satisfactory manner (since I shall not display them, the reader will have to trust me on this). However, they do so in a way completely detached from the historical circumstances, e.g. which of the techniques were already available to Hamming, which did he create himself, what was *Hamming's* purpose with the technique, etc. The hand-in paper and the videos both support this observation.

The students of the focus group found the third supportive assignment too difficult, and the teacher had problems helping them without revealing the answer. After working on the assignment for about twenty minutes they lost interest and focus, started goofing off and ended up writing a couple of vapid statements like the one exposed in section 5.5 (see pages 125).

They then moved on to the fourth supportive essay assignment. Although what is turned in as an answer of this assignment is not necessarily more informative than those of the two preceding assignments, the discussions the students had while working on it are. A transcript of these discussions may be found in appendix C.1 and will be discussed in the following subsection.

The main essay assignment is not very elaborating either. It is half a page providing mainly a summary of the preceding supportive essay assignments, a few historical errors (e.g. that binary numbers were invented for the sake of error correcting codes) and a few statements concerning the last couple of questions (see section 5.5). Due to lack of time

on the focus group's behalf, most of the main essay was completed outside the classroom, and the few videos of the group working on the assignment in class do not provide more elaborated discussions than the hand-in itself. But let us turn to a video clip that actually does this, the clip from the group's work on the fourth essay assignment.

A Video Clip from the Fourth Supportive Essay Assignment

The focus group's actual hand-in answer of the fourth supportive essay assignment may appear a bit silly:

- a) We believe that the honor for the Hamming codes should be ascribed to the sweet Richard Wesley Hamming (1915-1998 (poor guy (We LOVE U big time))). We think that the dear Hamming did know of those 'potentials of further development' which his (7,4)-code possessed when he passed it on to Shannon, who later published generalizations of this code exactly.
- b) So we know whom to hate/love – according to temper. A lot of work behind – bummer if someone else gets the credit. (Group 1)

However, some of the discussions the students had during their work on the assignment are not silly at all. In the following, I shall describe the video clip containing the students' work on the assignment, occasionally displaying interesting quotes referencing the line numbers in appendix C.1.

The discussion begins with Andrew pointing to the fact that the codes are named after Hamming, so maybe he should be given credit. Sean seems to be in favor of this view, saying that others rely on Hamming's theories. Gloria joins the discussion, shortly after stating:

Gloria: Yeah, but they are called Hamming spheres, right. What is there to ask about? (line 26)

At this point in time Jonathan plays the role of the critic, pointing to the fact that Hamming did not publish first, that Golay developed the codes further,⁵ that maybe Golay received credit back then, and telling the others "you need to provide reasons" (line 25). Andrew then enters a new argument onto the scene:

Andrew: Hamming knew that his (7,4)-code could be generalized when he gave it to Shannon, but [he] kind of didn't really have time to publish it. (lines 31-33)

As seen in section 5.1, this argument is not in line with the reality of Hamming's paper being delayed until 1950 by the application for patent, not due to lack of time. What might have confused Andrew is the fact that Hamming was annoyed with the 'waste of time' in computer calculations due to insufficient coding, this being what Hamming did not think his time suited for. Andrew, however, manages to persuade the others for a while, wrongly quoting the teaching material in his support, and finally making the argument that Hamming "put the theory higher than his own name by passing it on to one who had time [Shannon]" (lines 36-37). At this point in time the discussion only takes place between Andrew, Jonathan, and Gloria, the latter mainly asking clarifying questions in order to take notes. Jonathan, even though on some level agreeing with Andrew, still plays the critic:

⁵ This correct observation does not go into the hand-in answer, here it mistakenly says that Shannon was the one to generalize the (7,4)-code.

Jonathan: The debate concerns whether or not he knew of the generalization of his (7,4)-code when he passed it on. Or if Golay was the first to discover it. (lines 61-62)

At this time, Sean is not participating in the discussion anymore, and Lucy has not yet entered it.

The discussion takes a turn when Andrew tries to clarify his viewpoint to Gloria by saying:

Andrew: Kind of like that we want to help you do this assignment, Gloria, but we are too busy, so we pass it on to you and you can take credit for it once you have made it. (lines 67-68)

This remark somehow triggers Gloria to venture an outburst:

Gloria: This I don't know, because I want to answer 'yes, he knew!', but this doesn't fit with what I want to answer in b. Because look at this. In b there is a question which says: Why do you think people care about finding out who is responsible for a mathematical result? Which personal driving forces of the mathematicians play a part in mathematical research? Can you say anything about the sociology? People who study mathematics, they are so nerdy. They enter completely into their own shell... [...] They spend their lives on it. There's nobody who spends his life on something and then says: 'Ohh well, it doesn't matter, you can take the credit. (lines 72-77, 79-80)

This observation brings Gloria to question Andrew's statement that Hamming did not have time by asking why not and what Hamming then did at Bell Labs. Andrew's short reply to this is that Hamming "worked" (line 92), an argument which Gloria does not accept. At this point, Jonathan, who has been flipping through the teaching material, finally points to the patent issue:

Jonathan: It says here that he had to wait with publishing the codes until the thing with the patent had been resolved. (lines 94-95)

Andrew acknowledges this as a better argument than his own. And they move on to the second question of the essay assignment: Why it may be important to establish who was the creator of something. Jonathan tries to frame the discussion of the second question by saying:

Jonathan: It says: Many educated people are of the opinion that the honor should be ascribed to the first person to publish something. That is the dilemma with Hamming, right. Because both Shannon and Golay have published their stuff before Hamming himself publishes it. (lines 112-115)

Andrew and Gloria, however, keep on discussing the issue of honor and credit, i.e. indicating that it is important to establish ownership in order to credit the correct person. Within the next few moments the discussion is resolved, Sean reenters the discussion providing an alternative answer which does not really catch on, Jonathan draws parallels to Wiles' proof of Fermat's last theorem, something he has already done once earlier (line 78), and Lucy participates constructively in the discussion for the first time:

Jonathan: They can't get the credit for it before it has been published.

Gloria: No, it is the thing with it being a huge work load and it being a huge effort they make to make it happen.

Jonathan: That is why it must be a fucking downer to see someone else get...

Sean: And then you know who to go totally berserk at if you actually find

an error in it.

Gloria: You don't do that.

Sean: Yeah, because if there are further things...

Jonathan: He also got slaughtered that guy with Fermat's... the guy who made Fermat's...

Lucy: There *was* an error.

Jonathan: Yes, there was an error. Then he was just slaughtered. Then he went totally down. And then he corrected it again.

Lucy: Imagine actually wanting to sit and go through that long proof and then finding an error in it.

Andrew: We'll just write that there is a lot of work behind and its a bummer if someone else gets the credit, right. (lines 125-140)

And that is what they do, wrap their more interesting discussions into a few not nearly as interesting one-liners.

Nevertheless, the discussions of the focus group students while working on the assignment do support the hypothesis that upper secondary students are capable of carrying out meta-issue discussions, also in a quite reflective manner (whether they then manage to get it into their hand-in or not). As for research question 2, concerning anchoring, there is also something to be said based on the students' discussions. Namely that the students begin using some of the mathematical terms and names that they have acquired through the teaching module. It is not surprising that the students use the term 'family of Hamming codes' since this is used in the assignment question. However, Andrew will on several occasions refer to the (7,4)-code as the one to be generalized into the others (cf. appendix C.1) – something which is historically correct, since this was the code Golay saw in Shannon's paper, and further suggests a familiarity with the (7,4)-code itself. The presence of such familiarity is not a far fetched claim at all, since the (7,4)-code was the example used throughout the teaching material as well as in the first supportive essay assignment. Of course, the claim may be checked by looking into Andrew's hand-in mathematical exercises, his work on these in class, his participation in the first supportive essay assignment, as well as in the second questionnaire, and the second round of interviews. In a similar manner it may be checked to what extent Gloria understands the term 'Hamming sphere' she is using in the beginning of the discussion. These things, among others, will be done in the following subsection.

Focus Group Students' Work with the Mathematical In-Issues

I shall begin by discussing the coding theoretical problems which the students were given in an exercise as part of their test at the end of their second year. In this exercise a certain code,

$$\mathcal{C} = \{00000000; 00110011; 11001100; 11111111\},$$

was given, and the students were told that out of the 8 binary symbols in each codeword, 2 were information symbols. The students were asked to answer nine questions, a through i, about the code \mathcal{C} . (a) First they were to find the parameter m which is defined as the number of codewords in the code, i.e. for \mathcal{C} $m = 4$, and they were to find the information rate of the code which in this case is $2/8 = 1/4$. (b) Next they were to investigate if \mathcal{C} is a linear code. A linear code is one for which two arbitrary codewords $\mathbf{x}, \mathbf{y} \in \mathcal{C}$, and it is so that

- i. $\mathbf{x} + \mathbf{y} \in \mathcal{C}$, and
- ii. $a\mathbf{x} \in \mathcal{C}$, where a is either 0 or 1.

For binary codes condition *ii* is trivial, something the students had been made aware of, and \mathcal{C} is a binary code so all we have to do is to verify condition *i*. This is easily done by inspection, thus \mathcal{C} is a linear code. (c) For a linear code, a theorem says that the minimum Hamming distance, d_{\min} , between two codewords in the code equals the minimum weight, w_{\min} , of the code, which is the least number of 1's in the codewords except in the zero-codeword, $\mathbf{0}$. For the code \mathcal{C} , the minimum weight is 4, thus $d_{\min} = w_{\min} = 4$. (d) According to a couple of theorems in coding theory, theorems proved in the teaching material, a code \mathcal{C} can *detect* t errors if and only if $d_{\min} \geq t + 1$, and it can *correct* t errors if and only if $d_{\min} \geq 2t + 1$. Since \mathcal{C} has $d_{\min} = 4$, it can detect 3 errors but only correct 1. (e) A Hamming sphere S is defined by its center in a codeword \mathbf{b} and a radius r . The students were asked to write up the words contained in the Hamming sphere $S(\mathbf{b}, r) = S(11001100, 1)$, which are the eight words obtained by altering one binary symbol in the codeword at the time:

$$S(11001100, 1) = \{01001100, 10001100, \dots, 11001101\}.$$

(f) Next, the students were to assume they had received the word $\mathbf{v} = (00001100)$ after a transmission. This word was to be decoded by nearest neighbor decoding, which means to find the codeword closest, in terms of Hamming distance, to the received word. There are two such words in \mathcal{C} ; 00000000 and 11001100, which is because there are two corrupted symbols in the received word, and as found before we can only correct for one error. Thus, nearest neighbor decoding is of no use in this case. (g) A code \mathcal{C} has a packing radius, $p(\mathcal{C})$, which is the largest positive integer for which the set of balls (or Hamming spheres) with radius $p(\mathcal{C})$ centered in \mathcal{C} 's codewords do not overlap. A theorem, proven in the teaching material, says that the packing radius for a code with minimum distance d_{\min} is

$$p(\mathcal{C}) = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor.$$

Thus, our code has $p(\mathcal{C}) = \lfloor 1.5 \rfloor = 1$. (h) In order to determine if a code is a perfect code, we can inspect the Hamming spheres of the code and see if all words are contained in a sphere. For large codes, even the (7,4)-code which exists in a space of 128 7-tuples, this is a rather tedious task. Fortunately, there is a small theorem, also proven in the teaching material, which says that the minimum distance of a perfect code always is an uneven natural number. Our \mathcal{C} has $d_{\min} = 4$, thus it is not a perfect code. (i) In the last question, the students were asked to expand \mathcal{C} into a code with ten symbols in each codeword, two still being information symbols, and to do this in such a way that the code remained linear. This can be done in the following way:

$$\mathcal{C}^* = \{0000000000; 0011001100; 1100110011; 1111111111\}.$$

Only three students in the entire class were able to answer question i in a correct manner, and one of them was Lucy (see figure 5.4). Lucy also provided correct answers to questions a, b, c, d, and h, the last one by using the theorem about perfect codes. In question f, she missed the zero-codeword as a solution, but got the others right. Jonathan provided correct answers to questions a, b, c, d, and f. Sean provided correct

f) $v = 00001100 = 11001100$

g)

h) nej, minimumsafstanden er lige.

i) $\mathcal{C} = \{0000000000; 0011001100; 1100110011; 1111111111\}$

Figure 5.4 Lucy's answers to questions f, g, h, and i in the test. Answer h reads: "No, the minimum distance is even".

answers to the first part of a (in the second part, he wrote the information rate as $8/2$) and b, c, d, and f. Andrew provided correct answers to the first part of a (he did not write down an information rate), the minimum distance in c (he wrote the minimum weight as 0), and f. For some reason Gloria did not take the test.

As explained, when starting up the rest of the module after the summer vacation some lessons were spent on repetition. This provided the students with the opportunity to refresh their memories of error correcting codes, but it also gave some of them the chance to work once more with things they had missed the first time round – something Andrew and Sean seemed to take advantage of.⁶ As part of the repetition the students were to turn in a number of mathematical exercises from the teaching material. In this set of exercises they were to rework a figure similar to Shannon's (figure 5.1) to include compression and decompression, the question being whether this happens before or after error coding. They were to carry out some binary calculations involving multiplication. There were some proof questions, e.g. to show the three first of the four conditions for Hamming's distance function being a metric (see page 120). They were to do some more calculations of t -error detecting and t -error correction capabilities of given codes. And finally they were to do an exercise on Hamming spheres, which is the one that I shall concentrate on now. The students were given the code

$$\mathcal{C} = \{0000, 1111\},$$

and they were then to decide which of the 16 possible 4-tuples would be correctly decoded by use of nearest neighbor for this code. Furthermore, they were to draw the two Hamming spheres for the code (in the plane) and place the 4-tuples according to these. Obviously, this code is a linear code and it has $d_{\min} = 4$ and can therefore correct 1 error. All focus group students answered this exercise in a correct fashion (see figure 5.5), including Gloria who thus can be said to have some understanding of Hamming spheres. This time round all focus group students were capable of doing the exercises

⁶ Sean mentions this himself in the second round of interviews.

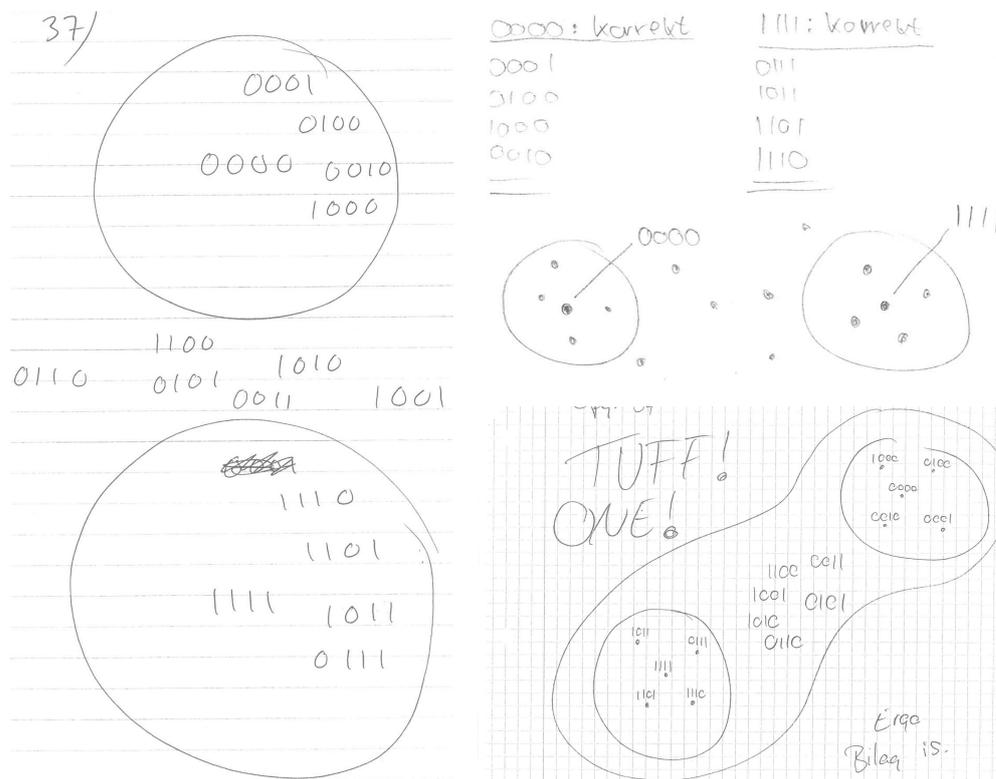


Figure 5.5 Gloria's (left), Jonathan's (right, top), and Andrew's (right, bottom) drawings of the Hamming spheres for the code $C = \{0000, 1111\}$. The text in Jonathan's answer means "correct". The comment "TUFF ONE!" in Andrew's answer appears to refer to the entire set of exercises rather than just the one on Hamming spheres.

on t -error detection and correction correctly, even Andrew who had missed this in the test. The proof problems, however, were more varying in quality, which may have been due to the fact that the students had not been working explicitly with proof exercises as part of this module. Often, concrete examples were provided instead of proofs.

Next, I shall discuss the students' participation in the work on the first supportive essay assignment. As already mentioned, the focus group provided a very good answering of this essay. There was a cooperative effort among the students, Lucy perhaps excepted, in working out the calculations and reasonings for this assignment. The first problem of the assignment was not one which necessarily presupposed much understanding of the (7,4)-code, but on the other hand one which was thought to develop some understanding. The students were asked to look at the so-called *syndrome* s :

- Write up the equations for s_1 , s_2 and s_3 in the presentation of the (7,4)-code provided in the teaching material. Write up the equations for α , β and γ in Shannon's presentation of the (7,4)-code. Decide which variables in this presentation correspond to which variables in the other presentation.

The syndrome is a linear combination similar to that of Hamming's $\alpha\beta\gamma$ (cf. page 122), and is defined as $(s_1s_2s_3)$ with

$$\begin{aligned}s_1 &= x_5 \oplus 0x_1 \oplus 1x_2 \oplus 1x_3 \oplus 1x_4 \\s_2 &= x_6 \oplus 1x_1 \oplus 0x_2 \oplus 1x_3 \oplus 1x_4 \\s_3 &= x_7 \oplus 1x_1 \oplus 1x_2 \oplus 0x_3 \oplus 1x_4,\end{aligned}$$

where \oplus means addition modulo 2 ($0 \oplus 0 = 0$; $0 \oplus 1 = 1 \oplus 0 = 1$; $1 \oplus 1 = 0$). By deleting the non-significant terms ($0x_1$, etc.) and by renaming the index numbers of the variables, it is seen that this is exactly the same as Hamming's

$$\begin{aligned}\alpha &= X_4 + X_5 + X_6 + X_7 \\ \beta &= X_2 + X_3 + X_6 + X_7 \\ \gamma &= X_1 + X_3 + X_5 + X_7,\end{aligned}$$

e.g. x_4 corresponds to X_7 , x_3 to X_6 , x_2 to X_5 , etc. The biggest hurdle for the focus group in this assignment turned out to be what was actually going on in the original extract from Shannon's paper. After half of a double lesson the group decided to ask the teacher and had her explain the assignment to them. While Gloria, Sean, and Lucy took a break, Jonathan and Andrew stayed behind and worked on the problem individually. When the others came back from the break, Andrew was able to present a solution. Sean, Gloria, and Jonathan reworked Andrew's solution while typing it in and corrected a few mistakes in the process. They called the teacher over again, Andrew explained their solution to her, and she confirmed the correctness of it.

After having worked the first problem of the first essay assignment, the students' work was interrupted by the three repetition lessons. When returning to the essay assignment, the students spent half a double lesson on recapitulating before engaging fully in the next question. Here, the students were asked to look at the received word $\mathbf{v} = (0011110)$ and decode it by the procedure in Shannon's presentation, i.e. calculate the index $\alpha\beta\gamma$ of the error. The exact questions given were:

- b. Using the example of the teaching material and the received word $\mathbf{v} = (0011110)$ calculate the binary number $\alpha\beta\gamma$ for \mathbf{v} in Shannon's presentation. What correction does $\alpha\beta\gamma$ result in? What happens when we in the example from the teaching material add the word (0001000) to \mathbf{v} ?

The problem is solved by first writing up:

$$\begin{aligned}\alpha &= X_4 + X_5 + X_6 + X_7 = 1 + 1 + 1 + 0 = 1 \\ \beta &= X_2 + X_3 + X_6 + X_7 = 0 + 1 + 1 + 0 = 0 \\ \gamma &= X_1 + X_3 + X_5 + X_7 = 0 + 1 + 1 + 0 = 0.\end{aligned}$$

Thus, $\alpha\beta\gamma = 100$ and the error occurs in the fourth place of \mathbf{v} , which must then be corrected to the codeword (0010110) . In terms of this example, the students were to compare Hamming's decoding procedure with the more modern one of syndrome decoding. Syndrome decoding is an elaborated form of nearest neighbor decoding in which the syndrome $(s_1s_2s_3)$ is first calculated. The syndrome is related to a so-called 'class leader' \mathbf{f} , which is the codeword with least weight and the same syndrome as the received word. The students had been exposed to the long and general derivation of this

procedure in the teaching material, but here it shall suffice to say that in our example the ‘class leader’ corresponding to the received word \mathbf{v} is $\mathbf{f} = (0001000)$, and that \mathbf{f} and \mathbf{v} are linked by the syndrome (111). The decoding is carried out very easily by adding \mathbf{f} to the received word which alters and hence corrects the error in the fourth position. After having recapitulated (rereading parts of the material and going through examples with the syndrome), the students in the next lesson began working seriously on this second problem. Sean had studied the syndrome at home and tried to explain his understanding to the others – except Lucy who was not present. This, however, resulted in Sean studying the material on his own once more, and some twenty minutes later he was able to explain the steps of the example in the material, calculate the syndrome of \mathbf{v} to (111), and interpret the corresponding \mathbf{f} to mean that “there is an error in the middle” of \mathbf{v} . Andrew seemed to understand Sean’s explanation this time. Gloria also accepted it, but then raised the question of $\alpha\beta\gamma$. At this point in time, the students do understand how to calculate $\alpha\beta\gamma$: they have just written down in their paper examples saying that 000 means no errors, 010 means an error in second position, and 111 an error in seventh position. However, Sean had either miscalculated $\alpha\beta\gamma$ to 001, or, according to Jonathan’s explanation afterwards, he had turned his paper upsidedown at some point. In any case, the students were rather puzzled about syndrome decoding revealing an error in the fourth position, and $\alpha\beta\gamma$ revealing one in the first position:

Gloria: What really happens when we add this word $[\mathbf{f}]$ to \mathbf{v} ?

Andrew: Then we correct the fourth position.

Gloria: Yeah, but that doesn’t make any sense when the error is in the first. That is stupid!

Sean: The error is in the first in Shannon’s presentation [Hamming’s decoding procedure in Shannon’s paper] while the error is in the fourth in Hamming’s [modern syndrome decoding].

This is followed by a small discussion of syndrome decoding, a discussion which reveals that Gloria may have understood more than some of the others since she keeps pointing out that \mathbf{f} is chosen because it has the same syndrome as \mathbf{v} . Jonathan and Andrew disagrees at first, mainly because they are more concerned with $\alpha\beta\gamma$, but Sean starts calculating and ends up agreeing with Gloria. At any rate, the discussion of $\alpha\beta\gamma$ continues and it is not resolved until the teacher intervenes. Her interpretation of what is going on is that they may have switched α and γ round since this would give them the right result, 100, and she asks them what that would mean in the binary system:

Andrew: That means 4.

Teacher: Super. That is 4. What does that mean?

Andrew: That is where we have to correct. The fourth...

Teacher: The fourth what?

Andrew: The fourth position must be corrected.

Teacher: Super!

Jonathan: Then everything fits!

Gloria: Then everything *totally* fits!

At this point the students are quite taken in by the assignment. For example, a couple of minutes later while typing in their answer to the problem, Gloria spontaneously exclaims: “Right now, I think this is really cool. Isn’t it?”, and when nobody seem to react: “Okay, *I* think it’s cool.” To the first question in the second questionnaire, about what had been most interesting in the module, Gloria also said: “I definitely thought it

was most fun when you, by yourself, had to find the connection between the syndrome and $\alpha\beta\gamma$." This, together with the above presentation, I think, supports the notion that some learning of the subject matter (and in-issues) is taking place, and that the students are developing an understanding of the (7,4)-code, its structure, and its decoding. The students themselves also seem to feel this. For example, at one point in time, while working on the first supportive essay assignment, Andrew says: "I actually think you learn more from this... when you get assignments where you get an 'aha'-experience all of a sudden."⁷

The claim above is also supported by the second questionnaire (see appendix B.2), from which I shall now display some of the focus group students' answers. All five students provided correct answers to question 8, in which they were to describe in their own words what is to be understood by Hamming distance:

Jonathan: It is a distance between to codes [words], i.e. how much they look alike. It is not a normal euclidian distance, but more a generalized concept of distance, e.g. $d(x, y)$.

Andrew: Hamming distance is the smallest distance between two codes [words] \rightarrow smallest number of differences (\div zero code [word]). E.g. 00000; 00110; 11000; 11001; 11111, Hamming distance = $d_{\min} = 2$.

Gloria: Hamming distance is a metric between two codes. Not euclidian distance, but the *difference* between them. E.g. $d(10\underline{1}, 10\underline{0}) = 1$, the last place is different.

Lucy: The distance between two binary number sets. E.g. 110011 100010. Hamming distance = 2.

Sean: The number of differences in two codewords. $C(01001001) C(01010011)$, $D_{\min} = 3$.

In question 9, where they were to explain what a Hamming sphere is and what perfect codes are. Gloria, Andrew, Jonathan, and Sean all provided acceptable answers to either both, or one or the other:

Jonathan: A Hamming sphere is a way of explaining it visually – minimum distance.

Andrew: Center=code[word]. Other 'words' in the sphere can be decoded to the center by nearest neighbor decoding = perfect code (-?).

Gloria: A Hamming sphere is a sphere with a codeword in the middle. Within the sphere lies those codewords [words] with a distance from the code [codeword] which isn't large enough to make it into another code [into other codewords] in the 'alfabet' [code]. For perfect codes all the spheres fill out the 'entire set' [space].

Lucy: Really haven't understood what a Hamming sphere is...

Sean: A sphere with the codeword in the middle and all the other possible words which you can decode within the periphery of the sphere.

The above questions are especially interesting for Gloria due to her 'word use' of Hamming sphere. Surely her terminology is not free of errors, but some understanding

⁷ The first supportive essay also included a third question: c. From looking at Shannon's presentation of the (7,4)-code, which other concept may then be ascribed to Hamming (though perhaps indirectly)? The answer to this question was *syndrome decoding*. The focus group did provide the correct answer, but mainly due to Sean consulting a friend from another group.

of what Hamming spheres and perfect codes are is definitely present. Lucy was the only one not capable of answering this question, and in fact she did not answer the next question either.

Question 10 asked which perfect codes are known (the answer being that Hamming and Golay codes are the only non-trivial ones):

Jonathan: Hamming's (7,4)-code and Golay's \mathcal{G}_{23} -code.

Andrew: The (7,4)-code and co.

Gloria: Definitely the (7,4)-code since that's the one we've been working with the most.

Lucy: [No answer.]

Sean: (7,4)-code and family.

Of the four students who answer, the thing to notice is that they all mention the (7,4)-code, which shows that the students are able to provide an example of a perfect code. The answers from Jonathan, Andrew, and Sean also indicate that they are aware that the (7,4)-code is not the only one.

In question 7, Gloria was able to give quite a good description of what is understood by binary representation, and so were Andrew, Sean, and Jonathan (again Lucy did not answer). Concerning Lucy and her lack of answers, she may be seen as a 'victim' of the module being split into two. In the test before the summer vacation she did very well, but after the summer vacation it seems as if she did not quite manage to get back in the game.

5.7 Case-Specific Analysis, Discussion, and Recapitulation

What we have above is a kind of existence proof that it is possible to have students engage in meta-issue discussions of mathematics in time and space (the general topics and issues of epistemic objects and techniques and multiple developments; cf. section 3.6), to have them perform reflections in relation to history as a goal, and do it in such a way that these discussions and reflections to some degree are anchored in the related and taught subject matter (mathematical in-issues). However, in order to answer research questions 1 and 2 (e.g. in what sense, to what extent, and on what levels this is possible) and, thus, begin to transform the existence proof into a more constructive one, further analysis is needed. For such analysis, I shall rely on some of the theoretical constructs presented in chapter 3, namely the theory of Sfard (2008b). Towards the end of the section I shall also discuss the class' answers to the second questionnaire. And finally, I shall evaluate the representativity of the focus group students and the composition of the focus group.

A Discursive Approach of Commognition to Research Questions 1 and 2

As mentioned in chapter 3, Sfard's theory of commognition seems a promising tool to apply in the empirical investigation of this dissertation. One reason for this is that the idea of commognition not only applies to the learning and understanding of mathematics and mathematical concepts, as many of the other theoretical constructs of understanding discussed in section 3.7, but applies to understanding and learning in general. This means that it also may be used for analyzing the students' discussions and reflections of the meta-issues, not only their understanding of the mathematical in-issues. That is to

say the theory may be applied to the answering of research question 1, and not only research question 2.

Also, the discursive approach of Sfard may offer appropriate ‘lenses’ for looking at the data since we may view the discussions and reflections of the students as following different discourses. In general, we may talk about two different – yet related – discourses being present in the investigation: a *meta-issue discourse* on the one hand, and an *in-issue discourse* on the other. As explained in chapter 2, the meta-issues of history as a goal mainly concern those of the evolution and development of mathematics, something I shall refer to as the *historical discourse* in the following. In the context of the essay assignments, however, there are also other meta-issue discourses present, for example a philosophical discourse, a sociological discourse, a psychological discourse, and possibly others as well, depending on the discussions taking place among the students and the questions of the assignments. The in-issues, also explained in chapter 2, are the issues related to mathematical concepts, theories, disciplines, methods, algorithms, proof techniques, etc., which is to say the internal mathematics. In the present investigation the in-issues are all related to the mathematical content or subject matter of the teaching module. Using Sfard’s notion, the students engaging with these in-issues as part of their work then become mathematicists, somebody who are participating in a *mathematical discourse*.

As stated above, the meta-issue discourses and the in-issue discourse, though being different discourses, are related. Exploring this relationship is in some respect the core of research question 2: the students’ meta-issue discussions and reflections being anchored in the mathematical in-issues of the teaching module. Or we may look at it as the students’ understanding of the meta-issues relating to and building on their understanding of the related mathematical subject matter. In terms of discourses, we may imagine the meta-issue discourses and the mathematical discourse as running along somewhat separate tracks, and the anchoring being potentially present in situations when these two discourses ‘intersect’. I shall refer to these ‘intersections’ as possible or *potential anchoring points*.

Students’ Meta-Issue Discussions and Their In-Issue Anchoring

Let us take a look at the video clip from the fourth supportive essay assignment (section 5.6) once again, more precisely the discussion between Andrew and Gloria who each argue based on different narratives. Andrew’s narrative is that Hamming did not publish due to lack of time, a practical problem which Andrew later rephrases as Hamming caring more about getting the codes out than about credit and honor. Gloria’s picture of a real mathematician is a person who devotes his/her life to some mathematical problem, and more or less seals himself/herself off from the outside world in the pursuit of a solution (a view which could be due to the movie about Andrew Wiles).

From a data gathering perspective, the wonderful thing about Gloria is that she often thinks aloud, something which may provide us with a glimpse of her intrapersonal communication, based in the interpersonal setting of the group work situation, of course. In Gloria’s small monologue (lines 72-80, appendix C.1) she is able to change her narrative from that of Andrew’s to a new of her own. Gloria’s new narrative contradicts that of Andrew, which eventually leads her to put her foot down:

No, it is god damn because everybody know that such research mathemati-

cians... that they spend their lives on this. You can't come up with some brilliant mathematical thing now – because so much have already been created – you can't do it without really spending a lot of energy on it. And anybody who really make an effort, damn it, wants to get some credit for it.
(lines 117-121)

From a discursive point of view, Gloria changes from Andrew's plain (and practical) historical discourse to a more psychologically oriented discourse: It makes no sense that a person who spends years on developing something should just pass the results and subsequent credit on to someone else. According to Sfard (cf. section 3.7), thinking is communicating, and both thinking and communicating is definitely what Gloria does in her small monologues (lines 72-80 and lines 117-121). Sfard's basic assumption is that learning and understanding are changes in discourse, and changing discourse is exactly what Gloria does.

The two narratives of Andrew and Gloria result in a kind of discursive conflict, we could refer to it as a *meta-issue discursive conflict*. As explained earlier, the conflict is resolved by Jonathan consulting the teaching material and pointing out the aspect of patenting the codes, something which forces the group to reorient the basis of their meta-issue discussion, a reorientation which fits Gloria's discourse. The important thing to notice, however, is that it is the change in discourse Gloria brings about that actually leads to the resolution of Andrew's wrong assertion about lack of time and brings the other focus group students to further change their discourse, thus learning and understanding something about the meta-issues of mathematics-in-the-making (rather than only the in-issues of mathematics-as-an-end-product). The discursive conflict bears some resemblances with Sfard's notion of a commognitive conflict: it is a conflict which stands between incommensurable discourses; and it is resolved by the students' acceptance of the discursive ways of the expert interlocutor, in this case the teaching material. The reason, however, for not referring to it as a commognitive conflict and instead calling it a meta-issue discursive conflict is that Sfard links her notion to a mathematical discourse, i.e. an in-issue discourse. Another difference is that Sfard states that a commognitive conflict is practically indispensable for meta-level learning (meta-level learning not having anything to do with meta-issues in this case, cf. section 3.7), something which can not be claimed for the discursive conflict described above.

One element of Sfard's theory that does apply, within a mathematical discourse, to the video clip from the fourth supportive essay assignment is that of a learner's development of 'word use' in his or her individualization of the use of mathematical nouns. That is to say, the four stages of passive use, routine-driven, phrase-driven, and object-driven use, the three latter ones all being active uses. As seen in the previous section, the video clip offers two interesting, active 'word uses': Gloria's use of the mathematical noun 'Hamming sphere' and Andrew's use of the noun '(7,4)-code'. I shall argue that both Gloria and Andrew in the course of the module reach beyond the third stage of 'word use' concerning these two mathematical nouns.

Now, Gloria did not take the test at the end of the second year, so we do not know what her understanding of Hamming spheres were at this time (otherwise we could have used question e of the test). In the set of exercises after the summer vacation Gloria has a very nice and correct answering of the exercise of drawing the two Hamming spheres for the code $\mathcal{C} = \{0000, 1111\}$ (see figure 5.5). And in questionnaire 2, after the module, Gloria definitely showed understanding of both Hamming spheres and perfect codes.

Gloria's use of the word Hamming sphere in the hand-in exercise may very well be somewhat routine-driven, since it may have occurred as part of "specific routines" in "constant discursive sequences" (Sfard, 2008b, p. 181). However, when Gloria uses the word in the video clip, the use is neither routine-driven nor phrase-driven, the latter being when "entire phrases rather than the word as such constitute the basic building blocks" (Sfard, 2008b, p. 181). Instead, the word has a 'life of its own' as a noun, which, according to Sfard, indicates that the process of individualization is completed: "One can now insert this word in any proposition in which there is a slot for this particular grammatical category." This is exactly what Gloria does, she points to the fact that the packing spheres are named after Hamming – Hamming spheres – so what is there to discuss, he has already been given credit. The word's 'life of its own' is further supported by the fact that the essay assignment asks about Hamming codes, not about Hamming spheres. Thus, 'Hamming sphere' is a word in Gloria's vocabulary she can use whenever grammatically – and mathematically – appropriate.

Something similar may be said about Andrew's use of the word '(7,4)-code'. Three times during the video clip (lines 32, 48, 89), Andrew mentions the (7,4)-code specifically even though the assignment addresses the Hamming codes in general. Judging from Andrew's previous work on the first supportive essay assignment, he definitely has some kind of understanding of the (7,4)-code – remember that he solved the first part of the assignment himself by paring the x_i s and the X_j s ($1 \leq i, j \leq 7$) in the two presentations. At the time of solving the assignment, Andrew's use of the word may have been routine-driven, but when he uses it in the context of the fourth supportive essay assignment, it appears more phrase-driven. Andrew uses the word '(7,4)-code' in connection with the generalization of the specific code to the family of Hamming codes – something which is correct in the sense that Golay made the generalization based on the (7,4)-code in Shannon's paper. Of course, the issue of the assignment is Golay's 'generalization' of the Hamming code, but still the use of the two words together is striking:

- "... Hamming knew that his (7,4)-code could be generalized when he gave it to Shannon..." (lines 31-32)
- "... he passed his (7,4)-code on to Shannon. And it was Shannon who made the generalizing..." (lines 48-49)
- "... or that he did know that there was some generalization of his (7,4)-code." (lines 88-89)

In questionnaire 2, however, Andrew may be closer to an object-driven use of the word when he writes "The (7,4)-code and co." under question 10 on which perfect codes are known, 'perfect codes' being a term which he also appears to have an understanding of.

The two examples of 'word use' above are examples of potential anchoring points between the mathematical and historical discourses, and at some points the psychological discourse. The question then is if, and if so then to what extent, an anchoring of these meta-issue discourses in the mathematical in-issue discourse is present in these points. Or phrased differently, to what extent do the meta-issue discourses rely on the in-issue discourse? Concerning Gloria's use of 'Hamming sphere' it may make more sense to talk about an anchored comment rather than an anchored discussion, since the topic of Hamming spheres is never picked up by the other group members. Nevertheless, Gloria cross references the historical discourse with the mathematical discourse, something she could not have done had she not learned some of the subject matter of the teaching

module. Concerning Andrew's use of the '(7,4)-code' more may be said. First of all, Andrew is more persistent in referring to the (7,4)-code throughout the discussion, and the specific code clearly plays a role in his own understanding of what is afoot within the meta-issue discourse. Thus, from an interpersonal communicative point of view, a relation between the two discourses appears present. A relation somehow relying on an anchoring being present as well, since Andrew does have a mathematical understanding of the (7,4)-code as seen from his work on the first supportive essay, and since he constantly seems to want to refer the meta-issues of the fourth supportive essay discussion to this code. Furthermore, Andrew's narrative, and thus the relation between the two discourses, sets the agenda in long stretches of the discussion (from lines 31 through 71). And Jonathan (lines 61-62) also picks up on Andrew's narrative when he tries to rephrase the basis for the discussion:

The debate concerns whether or not he [Hamming] knew of the generalization of his (7,4)-code when he passed it on. Or if Golay was the first to discover it.

Students' Answers to the Second Questionnaire

Having spent quite some time on the focus group, it seems reasonable to broaden our view again and take a look at the other students of the class. I shall do so by looking at the students' answers to the second questionnaire. In terms of the students' understanding of the in-issues, I shall consider questions 7-10, as I did for each of the focus group students. And in terms of the students understanding of the meta-issues, I shall consider questions 6 and 11-14 (see appendix B.2). The remaining questions of the second questionnaire will be addressed in chapter 7 or 8.⁸ Due to the readability of the questionnaire results and the relatively small population of the class, the students' answers have been indexed in the following manner:

one < few < some < many < the majority < the *vast* majority,

a partition which in percentage intervals roughly corresponds to 0-5%, 6-15%, 16-35%, 36-50%, 51-80%, and 81-100%, respectively.

I begin with the in-issue questions. As was the case for the focus group students, the vast majority of the students in the class provide correct/acceptable answers to question 8 on the meaning of Hamming distance; describing the meaning in their own words, providing examples, or both. Only one student does not answer the question. For question 9, the meaning of Hamming spheres and perfect codes, some provide correct answers to the question of Hamming spheres and to the question of perfect codes as well, although many only provide correct answers to one or the other. Out of these, some students focus on the geometrical aspects of the Hamming spheres, and their ability to provide visualizations of a code. Some do not answer the question or state they do not remember or have not understood. One student just answers that a perfect code is "really good". To question 10, regarding what perfect codes are known, many answer both the Hamming and the Golay codes, the (7,4)-code and the Golay codes, or various other combinations of specific Hamming and Golay codes, a few also mentioning the family of

⁸ Actually questions 22, 23, and 24 will not be addressed at all in this dissertation. For a discussion of the students' answers to these questions, see Jankvist (2008e).

Hamming codes. Some students only answer the (7,4)-code. One student simply answers “Golay codes”. A few do not answer or provide incorrect answers. In question 7 on what is to be understood by binary representation (referring to the first introductory essay) the majority provides answers which are fully or somewhat acceptable, some by referring to factual situations with, for instance, 1s and 0s or high and deep tones. The latter example shows that some students are aware that it is not the 1s and 0s which define a representation as binary, but the fact that there are two and only two different ‘symbols’. Some students provide non-acceptable, though not completely incorrect answers. A few students do not answer or provide incorrect answers.

The first of the meta-issue questions (question 6) also refer to the first introductory essay, since it asks from where the binary numbers originate. In the material the students had been told that binary representation could be traced back to African tribes who communicate by a deep and a high tone, and that Australian aborigines and some tribe people from New Guinea count in twos. They were also told that the Chinese work *I Ching* from around 1100 uses some binary representation of numbers, though not a real number system with an associated arithmetic. One such may be found in the Indian mathematician Pingala’s work from around 300-200 B.C. Many students referred to one or more of the above in their answers to question 6. Gloria, for instance, writes: “The binary number system may be traced back to some ethnic groups in the past, e.g. African tribes and ancient China.” More surprisingly, however, is that some students believe the binary number system to originate from Bell Labs (including Andrew, Sean, and Lucy), neither the teaching material nor the teacher made any such claims. A few students do not answer or provide other incorrect answers.

The next of the meta-issue questions (11) asked what motivated Hamming to develop his error correcting codes. Some students do not answer this question or provide incorrect answers (Sean included). However, the majority of the students actually provide answers which show that they are familiar with this part of the early history of error correcting codes (the remaining focus group students included). A few state things like: “He wanted to make the use of Bell Labs’ computers more efficient, the calculations which were performed on these, and possibly the wasted time involved in the procedure up till then.” Some state that Hamming was annoyed with the machines coming to a halt and dropping his calculations. A few explicitly mention that the computers were only able to detect errors, not correct them: “Before, when there were only error-detecting codes, Hamming found himself annoyed by the fact that the computers would come to a stop every time it found an error. It couldn’t correct it by itself.”

Question 13 asked which influence Hamming, being a mathematician as opposed to for instance an engineer, reasonably could be assumed to have on his work with the codes. A few do not answer this question. Some provide answers like: “He did not build bridges.”; “It may have had an influence that he worked with numbers and formulas, and not constructions or something.” However, the majority of the students (including the five focus group students) provide acceptable answers or qualified guesses, pointing to matters such as Hamming having a more mathematical or theoretical perspective on the codes than an engineer might have had. One student says: “Had he been an engineer he would probably have invented a faster computer.” A few say that he presented the codes in a logical way. Others that he was able to boil it down to formulas, and prove things about them. One says that “coding theory is mathematics”.

Question 14 was a different matter. The question asked what other area the theory about perfect codes has contributed to, the answer being packing of metric spaces (cf. the third essay assignment, page 125). The vast majority of the students did not even provide answers to the question or said that they did not know, had forgotten, etc. Only a few students provided guesses and these were all incorrect. This picture is, however, in accordance with the students' lack of correct answers to the third supportive essay assignment.

The last question I shall touch upon is question 12, which asked what is to be understood by objects and techniques. A few students do not provide an answer to this question, but the rest do. Some students provide answers, relating to an example of the teacher's, saying that numbers, e.g. 3 or -4 , are objects, and that arithmetical operations performed on these, e.g. $7 - 3$, are techniques (cf. the answer of group 6 to the third supportive essay, page 125). The answer of Sean is an example of this:

Sean: Object: a thing, e.g. -3 . Technique: a relation, e.g. $4 - 3$.

A few students mistakenly consider the (arithmetical) operations to be the objects, but do have an idea about techniques being some form of action. Jonathan and Andrew belong to this group:

Jonathan: Objects can be operations, such as $+$, $-$, \oplus , and \cdot . Techniques are when you use them, i.e. you perform an action.

Andrew: An object can be a sign, while a technique is an arithmetical problem. Object: no action. Technique: action.

Finally, some students provide more sound answers, e.g.: "Objects are at a given time in history subject to investigation. Techniques are used to investigate the objects." A few of the students in this last group provide answers relating to the historical case of error-correcting codes. Gloria and Lucy are among them:

Gloria: Objects are those treated by techniques: for example, a code is an object for the technique $\alpha\beta\gamma$, if it is placed in the formula.

Lucy: Objects: e.g. error-correcting codes. Techniques: e.g. concept of distance.

Representativity of the Focus Group Students

As explained earlier, the focus group students were chosen to represent the class in general as best was possible. That is to say the group consisted of students whose beliefs about and attitudes towards mathematics more or less reflected the viewpoints of the class, e.g. the group consisted of both students who were for and against the inclusion of history. As a way to evaluate this representativity of the focus group students, we may compare their hand-in essays to those of the other groups as well as their answers of the second questionnaire questions discussed above to those of the other students.

As for the length of the the final essays, the focus group's was among the longest; four pages. However, two of the pages were on the first supportive essay exclusively. This leaves two pages for the other three supportive essays and the main essay, something which averages the length of the other groups essays fairly well. As for the quality of the hand-in essays, the focus group's final essay appears to be a little bit below average. The group does not do too good a job on the essays concerned with meta-issues, only their first supportive essay on the in-issues is of good quality. For the introductory essays, the first is rather poor as compared to those of the other groups, concerning quality and

length, and the focus group never handed in their second introductory essay. However, the first part of the focus group's answer to the third introductory essay is quite good (cf. page 120), but again this only concerned in-issues rather than meta-issues. Thus the focus group's essays cannot be claimed to be of a higher quality than those of the other groups. If anything, perhaps a bit poorer.

The focus group students' answers to the second questionnaire questions appear to be more on average. Like the rest of the class, the focus group students were able to account for the meaning of Hamming distance. Most of them were able to provide examples of perfect codes, account more or less for the meaning of these as well as Hamming spheres, and explain the meaning of binary representation – fairly much in accordance with the other students' answers to the in-issue questions. As for the questions concerned with meta-issues, the focus group students' answers to the origin of binary numbers is a bit off, since three of the students state that they stem from the Bell Labs. On the question on Hamming's motivation for developing the codes most focus group students show familiarity with this aspect of the history, which the majority of other students also do. To the question of the influence of Hamming being a mathematician rather than an engineer, all five focus group students provided acceptable answers, also in accordance with the majority of the class. And like the rest of the students, none of the focus group students were able to answer the question concerning the use of coding theory in the packing of metric spaces. Concerning the question on the meaning of objects and techniques, the above presentation clearly shows that the five focus group students represent the different types of answers provided to this question by the class in general.

So compared to their classmates, the five focus group students certainly appear to be representative. Whether they are also representative compared to students in other classes, other schools, etc. is a completely different question – a question I shall address in chapter 8.

Recapitulation

Let us recapitulate what information the first module has revealed, and what may be gained further by a second module.

Concerning research question 1, definitely the students are capable of saying something about the development of early coding theory (the general topics and issues of objects and techniques and ascribing credit in case of multiple developments), and at least some of them (the focus group students) are able to have meta-issue discussions on these matters within their group. For the focus group we may also conclude that some of the students carry out meta-issue reflections concerning the issues posed to them, both of the intrapersonal as well as the interpersonal communicative kind; Gloria's reflections and following meta-issue conflict with Andrew about Hamming's 'lack of time' are examples hereof. Concerning the conditions on which this is possible, the essay assignments seem a promising means for this, something the introductory essay assignments, as described in section 5.4, also bear witness to. Of course, not all of the students' essays are equally good and reflected, some of them are rather vapid. However, as the description of the video clip of the focus group illustrated, the answer to an essay assignment is not necessarily fully representative for the discussions and reflections which took place as part of the working process. This may, of course, go both ways: a good answering may be based on lousy discussions as well, though I find

this to be less likely than the other way round. One question to ask is whether a sound discussion with a poor hand-in answer means that the students are then aware of the discussions they have had at all, and if it is fair to talk about a learning profit in such a case. However, as illustrated by the second round of questionnaires, some learning have taken place regarding the general topics and issues of the essay assignments, also in the case of some of the focus group's more vapid hand-in essay answers. Beside being either 'good' or 'bad', the students' essays also varies on other levels: some are more anecdotic than others; some simply seem based on common sense rather than referring to the historical information in the teaching material; some posses a degree of anchoring of the meta-issues in the mathematical subject matter, while others certainly do not appear to do so. Concerning anchoring, this of course also depends on the design of the essay assignment itself, and in terms of research question 2, the degree of the 'insurance' of anchoring in the assignment.

As an illustration of the above, the focus group's answering of the second supportive essay assignment serves as an example of how unexpected things may occur in empirical investigations. The essay, which was designed around the use of epistemic objects and techniques in a certain historical context, was thought to have a high built-in insurance of anchoring since it is not possible to identify mathematical techniques and account for their purposes without knowing something about the mathematics itself. Thus, the assignment was thought to be well balanced between the historical discourse and the mathematical discourse as compared to some of the other essay assignments. The focus group, however, managed to cut down the historical discourse to a level of non-existence with their scattergun approach to identifying the techniques as random definitions and theorems in the teaching material (cf. page 131). Fortunately, the focus group appears to be the only group having attacked the second supportive essay assignment using this method. The answers from the other groups appear more reflected and balanced between the two discourses. On the other hand, some of the assignments designed with a lower degree of anchoring insurance actually resulted in anchored discussions and reflections taking place, the video clip of the focus group's work on the fourth supportive essay assignment being an example of this. Worth noticing is that this only happened at the end of the module, when the students were familiar with the in-issues. In the second introductory essay, the one on Bell Labs, which did not have any built-in insurance of anchoring, none of the students' answers touched upon in-issues whatsoever. Thus, some anchoring may grow on its own with even little fertilization (low degree of built-in insurance), but only if the seeds have been sown. Concerning the part of research question 2, asking about the different levels on which an anchoring may occur, the analysis of the video clip shows at least two such different levels based on Sfard's notion of 'word use': Gloria's *anchored comment* on Hamming spheres, and the *anchored discussion* of the (7,4)-code between Andrew and Jonathan.

The answers to the second questionnaire provide us with some insight into the students' understanding of the in-issues in general, and hence the degree of (potentiell) anchoring. The vast majority of the students had a good conception of what Hamming-distance is, many had an idea of what Hamming spheres or perfect codes are (some of both), and the vast majority was able to provide examples of perfect codes, either naming both Hamming and Golay codes or just by mentioning the (7,4)-code. This certainly shows a high degree of potential anchoring for the related questions of the final essay assignment. Also, the majority of students appeared to have a sound idea of what

is to be understood by binary representation, suggesting some degree of anchoring of the first introductory essay.

In terms of any future implementations of module 1, a few matters of redesign may be considered. The students' introduction to the notion of techniques (and objects) used by Hamming in his development of the codes in the third introductory essay did not go as smoothly as hoped for. The main reason for this was probably that the students were not prepared well enough. The example used to introduce the idea of a technique talked about the general concept of distance *a topology*, something the students had been told a story about in the teaching material, being a technique for Hausdorff to study classical results of functions in one variable (Jankvist, 2008d, pp. 21-22, 24). Though this story was somehow related to the use of general distance measures, and the students were familiar with functions in one variable, the example seems ill-suited since none of the students were able to extract the relevant information from it and answer question c. The fact that the students were largely capable of understanding the notions of objects and techniques (as seen also from the second questionnaire), and applying this in the the final essay assignment (the second supportive essay), shows that it was not the general issue of objects and techniques itself which was the problem. It suggests that in a future implementation a more suitable example should be chosen to present the notion of techniques. The third supportive essay on the story of Golay did not go over too well either. No actual answers to the question asked were provided in any of the students' essays, and in the second questionnaire, no students were able to answer question 14 on coding theory having contributed to packing of metric spaces. Firstly, this essay may have been too difficult for them and perhaps it was not significantly clear to the students what they were expected to do. Secondly, the essay question may have had an element of 'guess the answer the material designer is looking for' in it. These two things are of course related, because it may be difficult to clearly explain the answer which you are looking for without revealing it. The essays which worked out best were the ones with more open questions that could be answered with somewhat open answers as well – not an open question with a 'closed' or fixed answer like in the third supportive essay.

The answering of research question 2 in particular may profit from the analysis of a second teaching module. As described in chapter 4, one of the ideas of the second teaching module was to change the 'parameters' of the essay assignments, especially the degree of the built-in insurance of anchoring. More precisely, the conditions were 'loosened' somewhat qua the selection of general topics and issues for the essay assignments. Since we now know that elements of anchoring may occur in assignments not designed with insurance in mind, the data from the second module may help identifying some more specific conditions needed to provoke anchoring, thus turning the existence proof of the first module into a more constructive one. In terms of research question 1, a second module may help identify what goes wrong when the essay assignments turn out 'bad', and why others turn out 'good'. Potentially speaking, there is a learning profit on the students' behalf from having engaged in the module on the early history of error-correcting codes. The answers to the second questionnaire bear witness to this, for the in-issues as discussed above, but also for the meta-issues, despite one of these perhaps being out of reach (question 14). The question is what it would take for the students to become more capable of displaying and articulating this learning profit in their final essay assignments. Perhaps more practice could be one way, something a second module could contribute to answering.

Before ending this chapter and engaging in the second teaching module, yet another matter needs to be addressed and accounted for, namely the recomposition of the focus group.

Recomposition of the Focus Group

After implementing the first module and reviewing some of the data, I found that the focus group had not always been as ‘focused’ as one could wish for. I therefore decided to alter it slightly by exchanging two students – Lucy and Sean – with two others. In the following, I shall spend a little time accounting for this change in setup halfway through my empirical investigation.

From a research point of view, video filming Lucy and her (lack of) interaction with the other focus group students seemed a waste of time since she rarely uttered a word or participated in the discussions. As is evident from the presented data, Lucy’s lack of interaction in the focus group can not be ascribed to mathematical incapability. The reason is probably that Lucy was a ‘quiet girl’, and that she often would feel ‘run over’ by the other focus group students. In the second interview she revealed: “I just think that it was a difficult group to be in, because all the others are very talkative, and I’m really not. [...] If you are kind of quiet then it is difficult to get through, if there is something you don’t understand or something you want to say, but the others keep on talking.” So Lucy was taken out of the focus group because, firstly, the video films were wasted on her, and, secondly, she needed to be in another group where she would have more room to interact with the other members.

Sean accompanied his second questionnaire with a written remark: “I don’t want to be interviewed”. Besides expressing this dislike in being interviewed, Sean had on several occasions seemed negative towards the group work sessions. According to the teacher, Sean was usually a very capable student, but the topic of error correcting codes seemed to cause him problems of understanding. The frustration that he was experiencing because of that, might have been what caused him to show a negative attitude every time the group had to assemble itself in front of the video camera. From time to time this negativity would spread to the other group members and valuable time would pass with no productivity.

Concerning the remaining three focus group students, a few remarks should be provided also. Gloria entered in the first teaching module with a positive attitude towards the integration of history of mathematics, and when interviewed after the module she was still positive. She thought that the historical aspects of the module had functioned well together with the applicational ones. However, if she had to choose between these two aspects, which the students had to in question 4 of questionnaire 2 (see appendix B.2), she would choose application. Jonathan, who had started out as somewhat skeptical, seemed to have changed his opinion in the second interview, now being positive towards the historical elements. Jonathan was also quite happy about the applicational elements, something he seemed to have missed in his mathematics education – he mentioned not seeing the point of differential calculus before actually applying it in his physics class. When asked if he thought that the history of mathematics could also be interesting without the application side, he answered “Yes”, and once again brought up the story of Wiles and Fermat’s last theorem. Andrew, on the other hand, stayed skeptic. He found the historical accounts in the teaching material to be somewhat

protracted. They could have been worse, he said, but still he would have preferred if it had been cut to the bone: "... mention their names, how they came to work with this, and then how it evolved. Focus more on the mathematics instead of the persons, I think." Interestingly enough, this last remark is quite the opposite of what Andrew uttered in the first interview, where he claimed to be more into the persons (biographies) and disliked "All the numbers". What brought about this change in opinion is not easy to say. Maybe his statements in the first interview were less reflected, because he had little experience with elements of history in mathematics and had not yet been given the chance to properly make up his mind about the question. Or, maybe he was just fed up with history due to the amount of it in the final essay assignment. At any rate, when selecting the new students for the focus group, the composition of skeptics and non-skeptics within the group should be taken into account and balanced.

Another thing to be taken into consideration was the group dynamics. In this respect, Gloria seemed to be the student in the focus group who had most clout. Not that she always acted as the leader or was the most capable student, mathematically or historically, but very often she would be the one to set the agenda: when to 'cut up' and 'goof off'; when to work seriously and concentrated; and when working, often deciding the issues of discussion and the angles of approach to the essay questions. Andrew, though not too interested in mathematics according to himself, actually made an honest effort during the module and not least in the essay assignments. Jonathan, from my observer point of view, seemed a very capable student and participant in the meta-issue discussions. However, for some reason his inputs to the discussions often seemed to drown or not get picked up on by the others, and not make it into the hand-in paper, e.g. his remark that Golay was the one to generalize Hamming's (7,4)-code, and not Shannon. Jonathan was not 'quiet' like Lucy, he participated actively in the discussions, only the focus group's way of working did not seem to be his cup of tea. In the followup interviews, he asked if the composition of the group was based on the questionnaire answers, implying that it was not based on who worked well together, and to the direct question of how the focus group had functioned, he replied: "Not always too well. Some of them 'fool around' with each other a little too much, or what you want to call it, but I suppose it'll have to be like that.". However, when making changes in the focus group anyway, it did not necessarily have to be "like that".

So part of the criteria for choosing the two new focus group students were that they had to be on approximately the same mathematical level, they needed similar 'profiles' judging from the questionnaires and interviews (i.e. they should be representative for the class in general), they should not be 'quiet' like Lucy had been, at least one of them should be a skeptic towards the integration of history, and they should be able to function together with the three remaining focus group students. Furthermore, if the latter could be done in such a way that the focus group would become a bit more 'focused', it could improve Jonathan's chances of being heard within the group. This might mean bringing in students who were not already among the interviewees. In order to get the best impression of such students they would have to be interviewed after the second questionnaire. Thus, the selection could not be postponed until the second module. On the positive side, however, the second questionnaire could serve as an extra dimension for choosing the two new students. I shall return to this in section 6.6 of the next chapter.

6 Second Teaching Module

This chapter is similar in structure to the previous one, only it concerns the second teaching module on the history of public-key cryptography, RSA, and the related number theory. Again, a description of the design and implementation is given (sections 6.2 and 6.3). The idea, as in the previous chapter, is to first provide an overall description of the students' hand-in assignments (sections 6.4 and 6.5), and subsequently give an in-depth description of the focus group students' work (section 6.6). For the focus group, this includes triangulations between the various data sources along the way. The chapter is ended by a case-specific analysis, discussion and recapitulation, all given in order to provide evidence for the answering of research questions 1 and 2. However, first an introduction to the part of history of mathematics treated in the module.

6.1 History of Public-Key Cryptography, RSA, and Number Theory

One of the oldest problems in cryptography is that of distributing the private encryption and decryption key between two parties. After centuries of having to deal with this, a solution was finally found in 1975 by a small group of cryptographers at Stanford University. Actually they solved the problem in two different ways, first by coming up with a safe way in which to generate a common integer, i.e. the key, between two parties, and second by proposing an entirely new system of cryptography, public-key cryptography. It is the latter which is of interest to us presently.

Whitfield Diffie, a former mathematics student of MIT, was the one to get the revolutionary idea for the new scheme. Early on in the 1970s Diffie had realized the potential of the so-called ARPANet, which later was to develop into the Internet, as well as the need for keeping information secret, for example during money transactions, on such a net. Diffie had learned of Martin Hellman, another key-distribution 'fanatic' at Stanford. Together with Ralph Merkle, a computer scientist involved in the first way of solving the key distribution problem, the three of them formed a team.

But what was Diffie's idea? Well, Diffie thought, what if we had such a thing as a one-way function: An injective function f for which it for every x in its domain is easy to calculate $f(x)$, but where it for every $y = f(x)$ in its range for all practical purposes is impossible to calculate $f^{-1}(y) = x$. Of course, the term 'for all practical purposes' is not a well defined mathematical term, but the idea is that it may take seconds to calculate $f(x)$ while it may take millions of years to calculate $f^{-1}(y)$. With this idea Diffie and Hellman were able to describe a new system of cryptography. The idea was that a person – Bob – by means of a one way function generated a public encryption key, one to which only he knew the decryption key, that is the inverse function. Another person – Alice – who wanted to send a secret message to Bob could then use his public key, which would be posted somewhere public, to encrypt the message and send it. Bob would then

be the only one capable of decrypting this message. Due to the nature of the one-way function, a cryptanalyst – Eve – eavesdropping on the line would stand no chance of breaking the code, even though she knew both the encrypted message, dependent on $f(x)$, and the public key. The situation is illustrated in figure 6.1. However, Diffie and Hellman could not find a one-way function to fit the description. They spent almost a year looking for one before finally giving up and publishing the idea of the system in 1976 (Diffie and Hellman, 1976).

Two computer scientists at MIT, Ronald Rivest and Adi Shamir, got hold of this paper and began a quest in search for the one-way function. Rivest and Shamir would come up with ideas and pass them on to their friend and colleague, the mathematician Leonard Adleman, who would then put the ideas to the test. After Adleman had shot down 42 ideas Rivest came up with one that finally paid off (Bass, 1995). Rivest had turned his attention to number theory and especially to the problem of prime factoring large numbers. Now, generating a very large number n , for example 200 digits long, by means of multiplying two, also, large primes p and q is a straightforward operation. Going the opposite way, however, prime factoring n , is ‘for all practical purposes’ impossible. Using number theoretic results, Rivest then devised a method for generating both public and private keys relying on this one-way function. The public encryption key consisted of two numbers; n , the product of two large (secret) primes p and q , and a number e which was determined in such a way that $\gcd(e, (p-1)(q-1)) = 1$, \gcd being the greatest common divisor. The encryption procedure on the message M , a natural number, revealing the cipher text C was defined as $C \equiv M^e \pmod{n}$, meaning that the integer C is *congruent* to the integer M^e modulo the integer n , i.e. that n divides $(C - M^e)$ with remainder 0, $n \mid (C - M^e)$. The private decryption key, besides also consisting of n , consisted of a number d which was an inverse of e modulo $(p-1)(q-1)$, that is to say $ed \equiv 1 \pmod{(p-1)(q-1)}$ (d can be calculated using the Euclidian algorithm and Bezout’s identity, see Jankvist, 2008h, p. 87). The decrypting procedure was defined as $C^d \equiv M \pmod{n}$. The encryption and decryption procedures due to

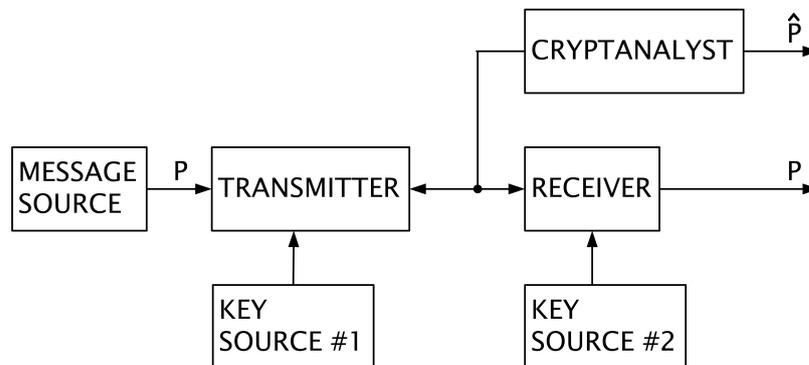


Figure 6.1 Diffie’s and Hellman’s illustration of public-key cryptography (Diffie and Hellman, 1976, p. 647). One public key (#1) is used to encrypt before transmission. Another private key (#2) is used to decrypt after reception.

Rivest, Shamir, and Adleman later became known as RSA (their initials). Also worth mentioning is that also Rivest, Shamir, and Adleman seem to have been quite aware of the potential of the early Internet. In the introduction to their paper they say: “The era of ‘electronic mail’ may soon be upon us; we must ensure that two important properties of the current ‘paper mail’ system are preserved: (a) messages are private, and (b) messages can be signed. We demonstrate in this paper how to build these capabilities into an electronic mail system” (Rivest et al., 1978, p. 120). The three of them patented RSA and in 1982 they started their own company offering RSA-solutions – a company which they sold in 1996 for the price of \$200,000,000.

Of course, it needed to be proved that the decryption procedure of RSA actually led to the original message M . Rivest, Shamir, and Adleman did this using well established – old and even antique – number theory (Rivest et al., 1978). To be exact, they proved the result using Euler’s theorem (1735-36), and the one hundred year older special case of this known as Fermat’s little theorem (1640), as well as the Chinese remainder theorem (around 400) which is found in *Sunzi suanjing*. About the history of these three theorems, not much is known of the latter or its creator, Sun Zi. More is known about the other two. Fermat’s little theorem is found in a correspondence he had with another French (‘hobby’) mathematician, Frenicle, dated October 10th 1640 (Struik, 1969, p. 28). Prior to this, Fermat had mentioned a few special cases of the theorem in correspondences with the French monk Mersenne, dated June that same year (Weil, 1984, p. 55). Fermat was greatly inspired by his readings of Diophantus’ *Arithmetica*. However, he never published much of his work on number theory which was partly due to the fact that only very few of his contemporaries were interested in the topic. A century later though, the line of interest was picked up by another prominent mathematician, namely Euler. Euler’s interest in number theory arose through his correspondences with Goldbach in 1729-30. When formulating and proving the generalization of Fermat’s little theorem, Euler was not aware of Fermat’s work. In fact, it is believed that Euler did not get access to Fermat’s work until several years later. For instance, Euler did not have access to the 1670 edition of Diophantus’ *Aritmetica* containing Fermat’s notes and comments until 1748. Actually, Euler was somewhat alone in his pursuit of number theoretical results, although a few others shared his interests. Among these were Lagrange who also assisted Euler in recovering Fermat’s lost manuscripts.

As an interesting twist to the history of public-key cryptography, it was made public in 1997 that the British Government Communication Headquarters (GCHQ) already knew of the system back in 1969 (see Singh, 1999, pp. 279-292). During the 1960s the British military had been playing with the idea of equipping soldiers with radios in order for them to constantly be in contact with their superiors. However, it was soon realized that the distribution of keys eventually would impose a serious problem upon such a system. One of the top cryptographers of GCHQ, James Ellis, was asked to look into this. In 1969 Ellis had arrived at essentially the same idea as Diffie was to arrive at six years later. Just as Diffie and Hellman later would be unable to identify a suitable one-way function, so was Ellis. For four years Ellis and the rest of GCHQ knew that their problem had a solution in theory, but that they could not apply it without the crucial missing one-way function. In 1973 a young mathematician and number theoretician, Clifford Cocks, was employed by GCHQ. After six weeks in the department he learned of this ‘crazy’ idea of Ellis’ and decided to look into it. Being so familiar with number theory, Cocks was able to find a solution in no more than half an hour – a solution which

four years later would prove to be identical to that of Rivest, Shamir, and Adleman. It took Cocks quite some time to realize the importance of the work he had done, but once he finally did, he recalls thinking that his discovery would not have pleased the English mathematician G. H. Hardy much (Singh, 1999, p. 286) (see also page 62). In 1974 one of Cocks' old friends and colleges from Cambridge University, a mathematician by the name of Malcolm Williamson, was employed by the GCHQ as well. Cocks told him about Ellis' idea and his own solution to the problem. Williamson's first reaction was that this was not possible, that Cocks must have made an error. Williamson decided to prove this, but of course could not. In the process, however, he found the method for secretly exchanging integers in public which Hellman also found at approximately the same time. Now, GCHQ being a secret organization was of course not interested in making any of their discoveries public and therefore neither in patenting them. For that reason, Ellis, Cocks, and Williamson had to silently watch from the sideline as the academics from Stanford and MIT got both the honor and money out of this cryptographic innovation. In the beginning of the 1980s Diffie learned, probably via the NSA, about Ellis' work. In 1982 Diffie went to Cheltenham to see Ellis and set the record straight, but all he got out of Ellis was the remark: "Well, I don't know how much I should say. Let me just say that you people did much more with it than we did" (Singh, 1999, p. 290).

6.2 Design and Redesign of the Teaching Material

In the teaching material (Jankvist, 2008h) the students were introduced to the idea of private-key cryptography through the example of Caesar cryptography (substituting each letter in a message with the letter three places ahead in the alphabet), which led to a discussion of the key-distribution problem, a presentation of Diffie's and Hellman's idea for public-key cryptography, and the use of a mathematical one-way function. Furthermore, the students were introduced to a selection of elementary number theory (prime numbers, Euclid's algorithm, the fundamental theorem of arithmetic, etc.), and from there it moved on to congruences, Gauss' work on these, and a presentation of the historical theorems used to prove the correctness of the RSA algorithm. As in the first module, the mathematics was presented to the students in modern notation and the history was unfolded simultaneously. Depending on the level of difficulty this was again done either in the form of 'stories' or by showing the students translated extracts from the original sources. For example, the works of Hardy (and Riemann) were introduced in the form of stories whereas translated extracts were shown from, for instance, Euclid's *Elements* and *Sunzi suanjing*. The idea of setting the text with two different fonts; one for in-issues and one for meta-issues was used in this teaching material as well. Due to the nature of the mathematical case, RSA, and the use of number theory in this algorithm, the second teaching material included far more theorems and proofs than the first on error correcting codes. Normally, students of upper secondary level are not used to being exposed to such a load of mathematical proofs. Therefore, those proofs that could be omitted, without the reader losing too much understanding, were marked with an asterisk (Proof*) in the teaching material. Of course, it was up to the teacher to make the final decision on which proofs to include or exclude, but a 'guideline' was offered while at the same time all proofs were presented to the interested reader.

Also, the idea of deepening the in-issues through mathematical exercises and the

meta-issues through essay assignments was continued in the second teaching module. However, in terms of redesign the number of essay assignments in the first module had seemed to be too high – something which could be deduced from the videotapes as well as the students' responses in the second round of questionnaires and interviews. Due to this and due to the fact the students now were familiar with doing essay assignments, the only essay assignment in the second module was the final one. As in the first module this consisted of a main essay and a number of supportive essays:

- **Final Essay Assignment:**
 - **Main essay** consisting of two different accounts of the historical case: when-and-who and why-and-how.
 - **First supportive essay** dealing with G. H. Hardy's *Apology*, and his views on pure and applied mathematics relating this to RSA.
 - **Second supportive essay** dealing with inner and outer driving forces concerning both the development of public-key cryptography and number theory.
 - **Third supportive essay** having to do with multiple developments and public versus non-public research.

(The essays will be explained in detail in section 6.5.) Instead of the introductory essay assignments, so-called historical exercises were introduced. Here the students had to look at extracts from original sources, answer questions about them and try to grasp the connection to the more modern presentation of the results that they had been exposed to in the teaching material. The essay assignments about Hamming's use of metric and Shannon's presentation of the $(7, 4)$ -code in the first module had been of such a kind, and the students had expressed their liking of these. To differentiate this kind of assignments from the essay-assignments they were given the name 'historical exercises', since the original sources used in these were indeed historical:

- **Historical exercises:**
 - **First historical exercise** on the Chinese remainder theorem in *Sunzi suanjing*.
 - **Second historical exercise** on Fermat's little theorem from the 1640 correspondence with Frenicle.
 - **Third historical exercise** on Goldbach's conjecture from the 1742 correspondence with Euler.

(The precise exercises will be shown and discussed in section 6.4.) Another thing which the students had seemed to respond well to in the first module was those introductory essay assignments which required Internet searches. For this reason, Internet searches were made part of the final essay assignment of the second module.

Something differing from the first module to the second were the parameters of the final essay assignment (see also section 4.2). By that I am thinking of the general topics and issues used to frame the supportive essay assignments, e.g. the discussion of objects and techniques and the discussion of multiple developments. In particular the idea of objects and techniques was used in the first module as a way of ensuring a certain amount of mathematical anchoring of the meta-issue reflections and discussions in the related in-issues. In the second module, this constraint was 'loosened' in the choice of general topics and issues, in order to study to what extent the meta-issue discussions and reflections would still be anchored in the subject matter (in-issues).

6.3 Implementation of the Second Teaching Module

The teaching module was carried out during a period of six weeks in the same class' third and last year of upper secondary school. The implementation ran over fourteen double lessons, each being 90 minutes. The same teacher taught the module. This time also, the students were divided into six groups in which they were to do the final essay assignment. The composition of the groups, however, differed from that of the first teaching module (for this reason the numbers of the second module groups are accompanied by an asterisk indicating that group x of the first module is not the same as group x^* in the second). The main reason for this was that the teacher wanted to change some of the groups, since she had found the working relationships among some of the students to be rather infertile. The composition of the focus group was left up to me. As explained in section 5.7, Sean and Lucy were taken out of the focus group and replaced by two other students of my choice – a choice I shall account for in section 6.6. Sean and Lucy were placed in other groups by the teacher. My only constraint of the teacher's reorganization of the groups was that there should be at least one interviewee in each of the groups.

Yet another change in the implementation of this teaching module was brought about by the teacher. In the first module she had been giving a somewhat traditional blackboard presentation of the mathematical contents of the teaching material followed by the students doing mathematical exercises. Inspired by the number of proofs in the second teaching module, she changed her focus a little, now having the students carry out one proof from the teaching material each on the blackboard. Of course, the difficulty level of the proofs in the material vary, e.g. from proving small theorems about integers to proving Euclid's algorithm, the fundamental theorem of arithmetic, the Chinese remainder theorem, or the proof of the correctness of the RSA algorithm. The teacher would delegate these proofs to the students according to her judgement of their mathematical skills, even assigning to one student an induction proof, something the class had never been exposed to before. All in all, this initiative functioned well in practice, though some students would of course not be as enthusiastic about the idea as others. But they all seemed to make an effort. One consequence, however, of all the time spent on proofs was that less time was spent on mathematical exercises, which several students commented on in the interviews after the completion of the module.

In the implementation of the second teaching module the teacher was more at ease with dealing with the history of the presented mathematics than she had been in the former module. This showed in the conduction of the final essay assignment, since the students now on occasion would quote the teacher for having said this or that concerning the meta-issues under discussion. Seen from an overall perspective, the activity of having the students do the final essay assignment in the second module went well. This, of course, had to do with the fact that they now knew what the task was about and how to proceed with it, thanks to the first teaching module.

6.4 Students' Historical Exercises

As explained above, the teaching material included three historical exercises in which the students were exposed to translated extracts from original sources. The students had to turn in the two first of these individually, the last one was optional as a part

of the main essay in the final essay assignment. In the following, I shall explain the exercises and their solutions and then discuss the students' answers.

The Chinese Remainder Theorem

In the first of the historical exercises, the students were exposed to Sun Zi's original formulation of the Chinese remainder theorem:

Now there are an unknown number of things. If we count by threes, there is a remainder of 2; if we count by fives there is a remainder 3; if we count by sevens, there is a remainder 2. Find the number of things.

Answer: 23.

Method: If we count by threes and there is a remainder 2, put down 140. If we count by fives and there is a remainder 3, put down 63. If we count by sevens and there is a remainder 2, put down 30. Add them to obtain 233 and subtract 210 to get the answer. If we count by threes and there is a remainder 1, put down 70. If we count by fives and there is a remainder 1, put down 21. If we count by sevens and there is a remainder 1, put down 15. When [a number] exceeds 106, the result is obtained by subtracting 105. (Yong and Se, 1992, p. 178-179 (26))

Based on the quote the students were given the following questions:

- Carry out the calculations of Sun Zi's problem by means of the method given by Sun Zi.
- Next, carry out the same calculation by means of the modern formulation of the Chinese remainder theorem in the teaching material.
- Do you think Sun Zi provides a proof for the Chinese remainder theorem? Give arguments for your answer.

Now, what Sun Zi tells us in the first paragraph is that we are interested in finding an unknown, say x , about which we know that

$$\begin{aligned}x \bmod 3 &= 2 \\x \bmod 5 &= 3 \\x \bmod 7 &= 2.\end{aligned}$$

Sun Zi tells that $x = 23$, and then provides the method for calculating x :

$$\begin{aligned}x \bmod 3 &= 2 \rightarrow 140 \\x \bmod 5 &= 3 \rightarrow 63 \\x \bmod 7 &= 2 \rightarrow 30.\end{aligned}$$

The numbers on the righthand side are numbers which fit in x 's place on the lefthand side, respectively. Next we are told to perform the addition $140 + 63 + 30 = 233$, and then subtract 210 in order to obtain the result 23. However, why we must do so may still not be entirely clear and for that reason – we must assume – Sun Zi continues:

$$\begin{aligned}x \bmod 3 &= 1 \rightarrow 70 \\x \bmod 5 &= 1 \rightarrow 21 \\x \bmod 7 &= 1 \rightarrow 15.\end{aligned}$$

If we perform the addition now, we get the result $70 + 21 + 15 = 106$. Sun Zi tells us that when a number, i.e. a solution, exceeds 106 we obtain x by subtracting 105. Of course, for modern day standards quite a bit is left for the reader to see without explanation, e.g. that $105 = 3 \cdot 5 \cdot 7$, and that the reason we must subtract 210 from 233 is because 210 is a multiple of 105. The reason that Sun Zi explains his method by means of an example must be viewed as a form of style and tradition in China at the time (Yong and Se, 1992), the method is, as we shall see, quite general (see also Jankvist, 2008h, pp. 49-53).

In modern day textbooks the Chinese remainder theorem is phrased in terms of a system of congruences, the phrasing in the teaching material was the following: Let m_1, m_2, \dots, m_n be positive integers, relatively prime by pairs. The system

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

then has a unique solution x modulo $m = m_1 m_2 \cdots m_n$, meaning there is a solution x for $0 \leq x < m$ and all other solutions are congruent to this solution modulo m . By relatively prime we understand that two integers have greatest common divisor (gcd) 1. Now, when formulated like this, the Chinese remainder theorem is an existence theorem telling us there is a unique solution, but the theorem in itself does not tell us how to construct it. The proof, however, does. I shall not provide the full proof from the teaching material since it builds on several lemmas. Instead, I shall illustrate the idea of constructing a solution by means of Sun Zi's problem, which was what the students were asked to do in the exercise. First we write up the system using congruences:

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 2 \pmod{7}. \end{aligned}$$

Since 3, 5, and 7 are all primes we know they are relatively prime, and we know from the Chinese remainder theorem that a unique solution x modulo $m = m_1 m_2 m_3 = 3 \cdot 5 \cdot 7 = 105$ exists. The first step of the procedure of constructing a solution is to calculate the expressions

$$M_k = \frac{m}{m_k},$$

for $k = 1, 2, \dots, n$. That is to say M_k is the product of all m_i s except m_k . Since m_i and m_k ($1 \leq i, k \leq n$) are relatively prime they have no common factors other than 1 when $i \neq k$ and therefore we must have that $\text{gcd}(m_k, M_k) = 1$. In our example we get

$$M_1 = \frac{105}{3} = 35, \quad M_2 = \frac{105}{5} = 21, \quad M_3 = \frac{105}{7} = 15.$$

From a lemma we know there exists integers y_k which are inverses of M_k modulo m_k , i.e. that $M_k y_k \equiv 1 \pmod{m_k}$. We can construct the sum

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \dots + a_n M_n y_n,$$

and show that this sum is, in fact, a solution to the system of congruences. What we notice first is that since $M_j \equiv 0 \pmod{m_k}$ when $j \neq k$, then all the terms in the sum for x except that of number k must be congruent to 0 modulo m_k . Because $M_k y_k \equiv 1 \pmod{m_k}$ we see that

$$x \equiv a_k M_k y_k \equiv a_k \pmod{m_k},$$

for $k = 1, 2, \dots, n$. That is to say we have reasoned that x is indeed a solution to the system of n congruences (only we have not argued that it is unique as well). In the problem of Sun Zi we find, by inspection, that $y_1 = 2$ since $2 \cdot 35 = 70 \equiv 1 \pmod{3}$; $y_2 = 1$, since $1 \cdot 21 \equiv 1 \pmod{5}$; $y_3 = 1$ since $1 \cdot 15 \equiv 1 \pmod{7}$. (If the numbers are larger, the solutions may be obtained using the Euclidian algorithm, see Jankvist, 2008h, pp. 48-49.) The solution therefore is

$$\begin{aligned} x &= a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3 \\ &= 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \\ &= 233 \\ &\equiv 23 \pmod{105}. \end{aligned}$$

Thus, the solutions are of the form $23 + 105k$, where k is an integer. From this solution we may deduce that Sun Zi's integers (70, 21, 15) in the second part of his method are carefully chosen: chosen to fit calculations somehow similar to those above.

The second part of the exercise (b), i.e. performing the calculations above apparently imposed no problems on the students. In all the handed in answers of the exercise, the students were capable of solving the problem by means of the modern Chinese remainder theorem. Of course, the teaching material contained an example from which the students to some extent could see how to proceed, so this may not be very surprising. What is interesting, however, is the fact that they all were able to read and understand the extract from the original historical source. This is also evident from the students' answers to the first part of the exercise (a). One issue here though, is that many seem to have problems writing up Sun Zi's calculations without sneaking in elements from the modern formulation, e.g. use of congruences etc. A few students even skipped question a, the reason appearing to be that they saw no difference between questions a and b.

Concerning the students' comments in c, many did not believe Sun Zi to have proved anything. Comments like "he only provides an example", or "there is no explanation or argumentation" were quite common. Only a couple of students seem to be able to abstract away from the old Chinese style of presentation. One says:

At first I couldn't see what the statement did, but now [after having performed the modern calculations as well] the connection seems quite well established. He could have a part – a great part – in it [the theorem].

Another seems more reserved concerning the actual 'proof' by Sun Zi, but takes into account the date of it:

Sun Zi shows us how you find an unknown number of objects once you know the remainder, when 3, 5, and 7 are divided into the number. However, he only shows a concrete example. He does not generalize the remainder theorem as you would today when you prove it within mathematics. For that reason his presentation probably would not be sufficient today. For example,

you don't know if this is something that goes for all numbers or not. And if so, then what numbers. Also, he does not prove uniqueness. But this text was written many years ago, and maybe this was sufficient back then. But today it probably would not pass as a proof, more likely as an example.

Many of the students take the provided text at face value, while only a few, like the one above, tries to interpret it. Maybe for that reason, no students seem to notice the choice of Sun Zi's numbers to be exactly those they obtained themselves in the process of finding the inverses. My original idea was to have the teacher comment on this as well as on different styles in proof in history and culture, but unfortunately this never happened since she did not go over the hand-in exercises in class. They were only commented and returned to the students.

Fermat's Little Theorem

The second of the historical exercises was on Fermat's original formulation of his little theorem in his correspondence with Frenicle:

Every prime number is always a factor [*mesure infailliblement*] of one of the powers of any progression minus 1, and the exponent [*exposant*] of this power is a divisor of the prime number minus 1. After one has found the first power that satisfies the proposition, all those powers of which the exponents are multiples of the exponent of the first power also satisfy the proposition.

Example: Let the given progression be

1	2	3	4	5	6	
3	9	27	81	243	729	etc.

with its exponents written on top.

Now take, for instance, the prime number 13. It is a factor of the third power minus 1, of which 3 is the exponent and a divisor of 12, which is one less than the number 13, and because the exponent of 729, which is 6, is a multiple of the first exponent, which is 3, it follows that 13 is also a factor of this power 729 minus 1.

And this progression is generally true for all progressions and for all prime numbers, of which I would send you the proof if I were not afraid to be too long. (Struik, 1969, p. 28)

Based on the above the students were asked to do four things:

- a. Explain in your own words what Fermat says in the quote above, including what he understands by the words 'exponent', 'power', and 'progression'.
- b. Write up the three next columns in the progression given as example by Fermat.
- c. Translate Fermat's example with the prime number 13 to modern notation using congruences.
- d. What happens if you insert other primes than 13 in the given progression? Does the proposition (Fermat's little theorem) still hold then?
 - (a) In Fermat's example, the top line contains the 'exponents' which the number 3 is taken to the power of, and the bottom line the 'powers', i.e. the results of taking 3 to the power of the exponents: $3^1 = 3$; $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; $3^5 = 243$; $3^6 = 729$. By 'progression', Fermat understands the line of 'exponents' together with the line of 'powers'. As an example of the students' answers to question b, figure 6.2 shows former focus group student Sean's correct answer to the problem. Concerning c, the modern

b)	1	2	3	4	5	6	7	8	9
	3	9	27	81	243	729	2187	6561	19683

Figure 6.2 Sean's answer to question b in the historical exercise on Fermat's little theorem.

formulation of Fermat's little theorem often reads: If p is a prime number and n is an arbitrary integer for which $p \nmid n$ (p does not divide n with remainder 0), then

$$n^{p-1} \equiv 1 \pmod{p}.$$

(A proof may be found in Jankvist, 2008h, pp. 56-57.) Another formulation of the theorem is: Let p be a prime, then for every integer n we have

$$n^p \equiv n \pmod{p}.$$

Seeing that this formulation follows from the above is actually quite easy. What we need to do is look at the two cases: $p \mid n$ and $p \nmid n$. If $p \nmid n$ we just multiply with n on both sides of the congruence sign in the first formulation of Fermat's little theorem to obtain the second. If $p \mid n$ we notice that both sides of $n^p \equiv n \pmod{p}$ will be 0 modulo p , and the congruence will therefore still be true. Now, for the example given by Fermat we have either $3^{12} \equiv 1 \pmod{13}$ or $3^{13} \equiv 3 \pmod{13}$ using modern notation. As for d, the modern formulation of Fermat's little theorem tells us that not all numbers will do: p must not be a divisor of n . For example, if we take p to be 3 instead of 13 we have $p \mid n$, and therefore 3^2 is not congruent with 1 modulo 3.

With only a few exceptions (around 15%) all students were capable of answering question a in a satisfactory manner. All students answered question b correctly. For an example of an answer to question a, and one of b as well, see figure 6.3. No students entered Fermat's example into the general context of the modern formulation of the little theorem ($3^{12} \equiv 1 \pmod{13}$). What they did instead, and which was okay given the formulation of question c, was to try and translate the third paragraph of the quote of Fermat into modern notation. In doing so, they typically wrote answers like:

$$13 \mid (27 - 1) \text{ and } 3 \mid (13 - 1) \text{ since } 3 \mid 6 \text{ so } 13 \mid (729 - 1).$$

Or if they used congruences:

$$27 \equiv 1 \pmod{13} \text{ and } 13 \equiv 1 \pmod{3} \text{ since } 6 \equiv 0 \pmod{3} \text{ so } 729 \equiv 1 \pmod{13}.$$

A couple of students, however, wrote: " $13 \mid (27 - 1)$, $3^3 - 1 \equiv 0 \pmod{13}$ ", which is closer to the modern formulation of the little theorem. As for d, the answers differed more. Some did not answer the question. Out of the rest, about half said that it still holds and the other half that it does not. Often their conclusions depend on the concrete examples they have chosen, e.g. the student who gave the previous answer to c wrote: " $3^x - 1 \equiv 0 \pmod{y}$, $3^4 - 1 \equiv 0 \pmod{5}$. It's working." Some say no without providing any arguments. However, few students answer in a much more reflected manner, for instance:

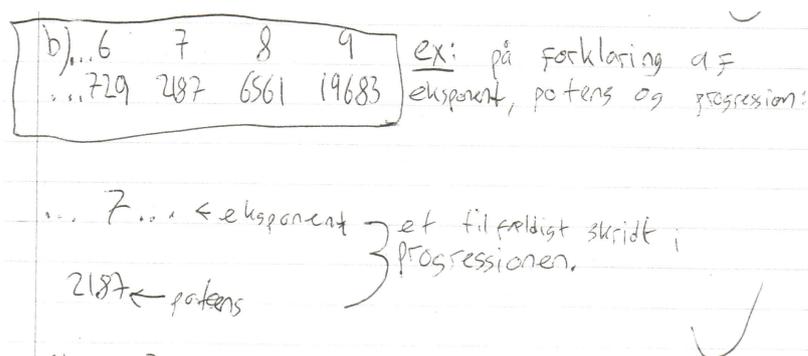


Figure 6.3 A student's answer to questions a and b. In the top it reads: "ex: [example] of explanation of exponent, power, and progression". At the bottom it says "an arbitrary step in the progression", and "←exponent" points to 7 while "←power" points to 2187.

$$7 \nmid (27 - 1), 3 \nmid (27 - 1), 11 \nmid (27 - 1)$$

So no, the progression does not work for other prime numbers than 13. Every prime must have its own progression. However, there are times when other primes do actually apply, but those must be seen as coincidences:

$$7 \mid (729 - 1), 7 \mid 531440 \text{ (which is } 3^{12} - 1).$$

Goldbach's Conjecture

The third and last of the historical exercises concerned Goldbach's conjecture as first formulated in a letter to Euler:

I also shall hazard a conjecture: that every number which is composed of two prime numbers is an aggregate of as many numbers as we like (including unity), till the combination of all unities [is reached]. [Goldbach adds in the margin:] After rereading this I find that the conjecture can be demonstrated in full rigor for the case $n + 1$, if it succeeds in the case for n and if $n + 1$ can be divided into two prime numbers. The demonstration is very easy. It seems at any rate that every number greater than 2 is an aggregate of three prime numbers. (Struik, 1969, p. 47)

The last sentence saying that "every number greater than 2 is an aggregate of three prime numbers" is what today is known as 'Goldbach's original conjecture'. After receiving Goldbach's letter, Euler soon realized that a stronger conjecture could be made, namely that every equal number (greater than 2) can be written as the sum of two primes (not counting unity, 1, as a prime). And it is this conjecture which is known as Goldbach's conjecture, despite the fact that Euler was the first to formulate it this way. The students were familiar with this version of Goldbach's conjecture from the teaching material, and after having been provided with the quote above they were asked to do the following:

- Explain what Goldbach means by 'composed'.
- What does Goldbach implicitly assume when he talks about "the combinations of all unities"?

- c. For the numbers 1 through 12 write up all the combinations which Goldbach considers.
 d. In the combinations, identify all those 'aggregates' which fulfill 'Goldbach's original conjecture' and Euler's stronger conjecture, respectively.

Questions a, b, and c are best explained by looking at the examples provided by Goldbach in his letter, examples the students had to do for themselves:

$$\begin{aligned} 4 &= 1 + 1 + 1 + 1 \\ &= 1 + 1 + 2 \\ &= 1 + 3 \end{aligned}$$

$$\begin{aligned} 5 &= 1 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 2 \\ &= 1 + 1 + 3 \\ &= 2 + 3 \end{aligned}$$

$$\begin{aligned} 6 &= 1 + 1 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 2 \\ &= 1 + 1 + 1 + 3 \\ &= 1 + 2 + 3 \\ &= 1 + 5 \end{aligned}$$

From these examples it is clear that Goldbach means 'sum' when he said composed. Concerning b, he implicitly means that every number n which is a sum of two primes is a sum of as many primes as one wishes up to n , taking into account that for Euler and Goldbach the number 1 (unity) was a prime. Concerning c, it may not be entirely clear from the first part of the quote alone (i.e. without the three examples) if Goldbach only considers sums of primes (including unity) or natural numbers in general. However, it is quite clear that the number 11 should not be written up since it is not a sum of two primes counting unity (for the reason of which an aggregate of two (prime) numbers cannot be written). Question d first concerns 'Goldbach's original conjecture', i.e. the students should identify all those aggregates consisting of three primes ($3 = 1 + 1 + 1$; $4 = 1 + 1 + 2$; $5 = 1 + 1 + 3$; $6 = 1 + 2 + 3$; $7 = 2 + 2 + 3$; $8 = 2 + 3 + 3$; $9 = 3 + 3 + 3$; $10 = 2 + 3 + 5$; $12 = 2 + 3 + 7 = 2 + 5 + 5$; and possibly also that for $11 = 1 + 3 + 7 = 1 + 5 + 5$), and secondly locate the aggregates adhering to Euler's stronger conjecture, namely: $4 = 2 + 2$; $6 = 3 + 3$; $8 = 3 + 5$; $10 = 5 + 5 = 7 + 3$; $12 = 5 + 7$.

This exercise being optional, only one group of students answered it (group 6*). This group answered correctly to question b. In question a they answered wrongly by suggesting: "When Goldbach says 'composed' he means the product of 2 primes (composite numbers). This then means that he says that a product of two primes is a sum of an unknown number of numbers." In c they listed *every* possible combination of the numbers 1 through 12, including aggregates with composite numbers (e.g. for 5, $1 + 2 + 2$ and $1 + 4$), and also all those for the number 11. In d they pointed to their answer of c and said only: "The combinations fulfilling these requirements are those only consisting of primes." Of course this is correct, but for 'Goldbach's original

conjecture' they should have pointed to the fact that it was only the aggregates consisting of three primes, and for Euler's they should have pointed to the equal numbers and their corresponding aggregates consisting of only two primes, neither of which they did.

Thus, judging from group 6*'s answer alone, this historical exercise seems to have been difficult for the students. However, judging from the way in which the class performed on the two previous historical exercises, the students should have been able to do the historical exercise on Goldbach as well. At any rate, it would have been interesting to have had answers to the third historical exercise from all of the students.

6.5 Students' Final Essay Assignments

In this section, I shall give an overall impression of the students' answers to the final essay assignment by displaying a variety of extracts from these, and by providing my own comments. The final essay consisted of one main essay assignment and three supportive essay assignments, which were to be made before the answering of the main assignment. The hand-ins were generally between 4 and 7 computer written A4-pages long. I shall begin by outlining the main essay.

The Main Essay Assignment

The main essay assignment of the second teaching module was very similar to the one of the first module, both in its questions and in its relying on the students to have answered the supportive essay assignments first. The students were asked to provide the following:

- a. A presentation basing itself solely upon *when* and *who*. Elements of your discussion in the third supportive essay [see below] should be part of this presentation.
- b. Another presentation basing itself solely upon *why* and *how*. Among other things you should try to describe what dictated the genesis and development of public-key cryptography and RSA. Elements of your discussions in the first and second supportive essay assignments should be part of this presentation, and so should elements of the previously answered historical exercises.
- c. If you previously have been through the teaching module on *The Early History of Error Correcting Codes* you are expected to include, in the why-and-how presentation, a discussion of *objects* and *techniques* in the development of RSA.
- d. What insights in the history of mathematics do you believe may be obtained by approaching the history through inner and outer driving forces?
- e. If you previously have been through the teaching module on *The Early History of Error Correcting Codes* you are also expected to discuss what insights you believe may be obtained by approaching the history in terms of objects and techniques.^[1]
- f. What do you believe one may learn from the when-and-who presentation and the why-and-how presentation, respectively?

¹ The phrasing of the questions in c and e is a bit misleading since the students, of course, had been through the teaching module on *The Early History of Error Correcting Codes*. In more straightforward language it means that *all* students in class were expected to answer these questions along with the other questions. The reason for the ambiguous formulation was that the teaching materials were published afterwards, and other teachers might decide to use the material for the second module and not that for the first.

In their answering of these questions, just as in the first module, the students had the opportunity to draw on the teaching material as much as they wanted to.

In the following, each supportive essay assignment will be presented and discussed, and translated extracts from the students' answers will be displayed. The students' answers of the main essay assignment above is postponed until after the treatment of the supportive essays.

First Supportive Essay Assignment

In the first of the supportive assignments the students were to discuss some of G. H. Hardy's views on mathematics, based on a selective reading of his *A Mathematicians Apology* (the students read about two thirds of the English edition of the book as part of their preparation for this assignment). The students were asked to do the following:

- a. Account for Hardy's characteristic of pure mathematics and applied mathematics, respectively.
- b. What is Hardy's view on pure mathematics and applied mathematics, respectively? Which examples does he use in his discussions of this?
- c. Discuss Hardy's statements about number theory^[2] on the basis of your knowledge about the development of RSA. For instance, what does the example with RSA tell us about the importance of basic research? How is this connected to Hardy's statements about number theory?
- d. What do you generally think about Hardy's *Apology* and the view of mathematics and mathematicians which he exhibits in this book? To what extent do you believe that mathematicians of today share Hardy's opinions concerning the different aspects discussed in his book?

None of the groups seem to have any problems answering the first question (except group 6* which did not answer the first supportive essay at all). A somewhat typical example of an answer is:

G. H. Hardy characterizes pure mathematics as being number theory, for instance. Amongst other he believes geometry and mathematics of the Greeks and Pythagoras to belong to this category. Pure mathematics is not directly useable in everyday life, but it is still a necessity.

He sees applied mathematics as more dull mathematics, but at the same time he uses it every day. As belonging to applied mathematics he mentions chess as an example. In regard to this he says that the mathematics behind chess, despite being difficult and original mathematics, is boring and not very surprising. For that reason he does not believe applied mathematics to be most important. On the contrary he describes pure mathematics as being beautiful, something which he, for example, illustrates with proofs by Pythagoras and Euclid. (Group 2*)

Another group (group 3*) brings in Hardy's discussion of mathematics and art³ to say that Hardy believes applied mathematics to be ugly, trivial, and dull, also since it is created for specific purposes, e.g. in war.

² See page 62.

³ "The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics." (Hardy, 1992, p. 85)

Concerning Hardy's view on applied and pure mathematics (b), especially number theory, one group (the focus group) answers:

Hardy says that pure mathematics is the most important since the one who masters pure mathematics also masters applied mathematics, and not the other way round. Those who master applied mathematics do not necessarily master the background for it. So they can't do the pure mathematics which is the basis of what they do. He provides the example that we live in a world where people specialize in different things. He compares the specialists to the applied mathematicians since they can only do one thing, while the pure mathematicians master all things more or less. Besides this he also believes that it is only the applied mathematicians who are to blame for all evil invented in the world to carry out warfare, because they are the ones to specialize in applying the pure mathematics for 'evil' purposes. (Group 1*)

The discussion of mathematics as a means to do 'evil' was something several of the groups touched upon. In fact, several of the students were somewhat surprised to learn about the extended use of mathematics in war and as a consequence of this seemed quite interested in discussing the theme.

As an example of an answer to question c, the following is fairly typical:

He [Hardy] likes abstract mathematics, the kind of mathematics you cannot conceive immediately, mathematical rules where human aspects cannot be applied to anything. He sees it [mathematics] more as a science, and not as something which has concrete applications. Hardy believes prime numbers and number theory to be one of the purest forms of mathematics. It has no practical applications. Therefore it cannot be used [applied] nor misused. If Hardy had lived longer he would probably have become disappointed in seeing how number theory was applied in RSA.

"A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life." This is a quote by Hardy where he obviously is being ironic since he doesn't believe the study of primes to be neither. RSA may tell us that basic research in mathematics is quite important since RSA couldn't have been realized had it not been for the basic research from 200 years earlier. This doesn't fit too well with Hardy's statements about number theory and primes. (Group 3*)

Of course, Hardy may have been overemphasizing some of his views in the *Apology*, and indeed it may be discussed to what extent he actually held the views he put forward in the book. A few groups touch upon this, e.g. when the group in the quote above state that he is being ironic, and other groups explain the colorful viewpoints and value-laden presentation of his book with the fact that he experienced 'misuse' of mathematics in two world wars.

Regarding the students' own opinions about the *Apology* (d), and what they believed to be the views of mathematicians of today, one group said:

It [the book] is somewhat skeptical regarding applied mathematics and pure mathematics. We believe that you can look upon mathematics as a science [which is] more than just a tool, and therefore it is okay if you do research in mathematics and just develop it, because maybe you can use the mathematics later on, as was the case with the British security agency. We are for applied

mathematics. All mathematics should be applied to something and lead somewhere, [if not] all the mathematics above primary and secondary school level would be useless to anyone. Were it not for applied mathematics we probably wouldn't have all the technology we have today. Mathematicians applying mathematics aren't 'the bad guys'. No, it is the governments who use the applied mathematics in wrong doings that are ['the bad guys']. Anyway, we believe mathematicians today to be somewhat indifferent since they probably don't believe that they are the ones responsible, but on the contrary the people who use the knowledge to do evil. (Group 4*)

The above quote actually illustrates the views of the students quite well. Many of them said that they found the book interesting, but that they did not agree with all the viewpoints of Hardy, in particular they pointed to mathematics as something being used for 'good' purposes as well. Now, concerning the viewpoints of modern day mathematicians the groups did not always agree. Some believed mathematicians today to be much more concerned with applied mathematics, e.g. due to the advancement of technology etc., while others thought they might very well share the views of Hardy, or, like the group above, be indifferent. It should be mentioned that probably the only basis the students had for their views on this matter were the more contemporary mathematicians, to whom they were introduced as part of the modules. A couple of groups also reflected upon the 'flow' from pure mathematics to applied mathematics, for example group 1* who concluded their answering of the first essay by writing:

History has shown that pure mathematics in time might as well become applied mathematics, even though people couldn't see any future meaning [application] in it from the beginning. (Group 1*)

Second Supportive Essay Assignment

In the second supportive essay assignment the students were introduced to the idea of discussing the history of mathematics in terms of *inner* and *outer* driving forces. They were then to employ this for the development of number theory and public-key cryptography:

- a. For each of the mathematicians of older date introduced in the teaching material (Euklid, Sun Zi, Fermat, Euler, Gauss, Riemann, Hardy) you are to discuss their personal motivation for working with the parts of number theory which they did. How does the personal motivation for each of them relate to the discussion of inner and outer driving forces?
- b. For the mathematicians of newer date introduced in the teaching material (Diffie, Hellman and Merkle; Rivest, Shamir and Adleman; Ellis, Cocks and Williamson) you are to discuss their personal motivation for working with the parts of cryptography and/or number theory which they did. How does the personal motivation for each of them relate to the discussion of inner and outer driving forces?
- c. Discuss your answers to the two above questions on inner and outer driving forces in relation to each other.

The typical answer to questions a and b is a list of mathematicians describing their personal motivation in relation to inner and outer driving forces. These lists may, however, vary substantially in both length and 'depth'. Let us take Diffie as an example:

Diffie studied mathematics in the US, and afterwards worked different jobs relating to computer technology. He was very fascinated by the Internet,

which must have been an inner driving force for him. The outer driving force was that he saw it could turn into a world wide computer network. He considered the problems with trading via such a net, e.g. a person who wanted to buy things with a credit card might get into security problems with hacking etc. (Group 3*)

Diffie: He worked for a security agency where he made the key systems. (Group 4*)

Now, apart being incorrect, the latter quote is also quite short compared to the first. Fortunately, only a couple of the groups write answers like the latter. Most groups give longer descriptions and discussions in line with the first. Something else which the first quote above shows is that the students mix up the inner and outer driving forces within mathematics as an academic area with the personal motivations of the individual researchers involved with a given development. This actually showed to be a fairly general tendency, as following quotes will illustrate, and as such I shall return to and discuss it later.

Of course, it was not always possible to give ‘in-depth’ descriptions of the motivations for the mathematicians, especially the older ones as Sun Zi and Euclid. However, some of the students would still perform somewhat reflected ‘speculations’:

Euclid: mathematician, lived 300 B.C., wrote books about geometry, and three books about number theory. Euclid’s algorithm: if you cannot find a gcd which divides 2 numbers, Euclid’s algorithm produces a solution.

As a function of time we may conclude that he basically worked out of an inner driving force since applied number theory didn’t really exist at that time the way it does today. However, you may talk about knowledge as being power in this time [of history], and if he was striving for more power this may be considered an outer driving force. (Group 6*)

Concerning Euclid in particular, it seems that some of the students are capable of drawing on knowledge they already had about this period. One group, for example, after attributing the definitions of primes, composite numbers, etc. to Euclid, continued:

[...] It is not to say from where Euclid obtained his driving force to do mathematics. However, it is known that Euclid lived in a time where mathematics was seen as a religion, and where the striving after the ideal world was a driving force for many, here amongst Euclid. (Group 5*)

The teaching material said nothing about knowledge being power, mathematics being a religion, or a striving towards the ideal world. Interestingly, such drawing on their own historical knowledge is not something which happens in the descriptions of the 17th, 18th, and 19th century mathematicians. Whether this is because the students did not know very much about these periods, or if it is because they found the information in the teaching material sufficient in describing these mathematicians, is not to say. Of course, it might also be a matter of transfer, i.e. not being able to transfer and use knowledge from one subject to other subjects, because the subjects may be placed in non-overlapping and non-connected ‘clusters’ (somewhat similar to the beliefs clusters described in section 3.8).

Concerning question c, I shall show only one quotation to illustrate the students’ answers of this assignment, the reason being that I shall discuss this question in detail

for the focus group in section 6.6. However, the quote is quite long and to a large extent encompasses the more fragmented and shorter answers of some of the other groups.

From the two previous questions, it is quite clear that the period under which mathematicians live has great influence on the mathematical research being conducted. In older times mathematicians did mathematics out of a desire to do so; often mathematics was a free-time activity or even a hobby. Mathematical research was driven from within, mathematicians sought to solve problems for the sake of mathematical research itself. In comparison, more contemporary mathematicians were influenced by external driving forces. As Hardy also points out, war is an outer driving force for mathematics. Often wars have raised new questions that afterwards have been solved by scientists in terms of developing new areas within their respective fields, including mathematics. Another clear outer driving force in this sense is money. This [war and money being outer driving forces] has, more or less, been the case for mathematicians like Diffie, Hellman, Ellis, Cocks, Rivest, Shamir, Adleman, etc. while the situation was different for Euclid, Fermat, Euler, Gauss, Riemann, and Hardy. (Group 2*)

Third Supportive Essay Assignment

The last of the supportive essay assignments concerned the topic of multiple developments in the history of mathematics, like the story of Hamming and Golay in the first teaching module, although this time with the 'twist' of classified research being involved:

- a. Search the Internet and see which informations you can find on NSA and GCHQ, respectively (e.g. use the search words 'NSA' and 'GCHQ'). Which types of organizations are NSA and GCHQ? Is it possible to say anything about how much research is being conducted within these organizations?
- b. What is the motivation for researchers at the universities and researchers in the security agencies, respectively? Is there, and if so how, any difference in the 'sociology' in these two scientific communities?
- c. Whom do you think should be credited for public-key cryptography and for RSA, respectively? Provide arguments for your viewpoints.

Concerning the first question (a), the students were somewhat surprised to learn that NSA is the world's largest employer of mathematicians and that they have tens of thousands of employees. GCHQ, according to their own web site, probably holds the highest concentration of mathematicians in Britain. However, compared to the size of NSA, GCHQ is a considerably smaller organization. Also, the students were a bit shocked to learn that GCHQ are trying to recruit new 'spies' by running advertisements in popular computer games (MediaGuardian, October 18th, 2007).⁴ Unfortunately, not all groups list the web pages from which they have retrieved their information, but the few that do mention www.wikipedia.org among others. A few groups seem to just quote the information from the web pages, but more commonly the groups have used the information to write up small essays mentioning and discussing the information retrieved. For NSA one group wrote:

⁴ [urlhttp://www.guardian.co.uk/media/2007/oct/18/digitalmedia.advertising](http://www.guardian.co.uk/media/2007/oct/18/digitalmedia.advertising), (Web site located on March 3rd, 2008.)

We found: NSA, which means National Security Agency, is one of USA's security agencies whose job is to listen in on, gather, and analyze all sorts of communication. NSA is the world's largest employer for mathematics. They mainly hire mathematicians. Since they also cooperate with NATO and have the most worldwide network, you must assume that the activity is large. (Group 5*)

About the motivation of the agency researchers versus that of the university researchers (question b), the answers of the students are largely captured by the following quotes:

The motivation of the researchers at the universities seems to a large degree to be connected to a desire of wanting to do this. They do research in order to go further and further within the world of mathematics. At the same time, they are not trying to conceal any information from the surrounding world since they want everybody to know what they are doing research in, and how far they have gotten. The motivation with the security agencies, on the other hand, is probably closer connected to a necessity than to a desire. They are interested in developing the 'technology' as much as possible, so they can use it in praxis. Therefore they are not interested in anyone knowing anything about their research, and how far they have gotten, since this might reveal their ideas. (Group 2*)

When you do research for a security agency you know that it is a work you do and not your 'hobby'. You are paid money but you do not receive credit in public since the security agencies will classify your work, and because the work is done in order to advance the organization, not the public. With researchers at the universities it is the other way round since here you receive honor in the public space if you are the first to publish on a new breakthrough within mathematics or something else. (Group 6*)

A funny thing about the latter quote is that the students implicitly are saying that researchers at the universities are not paid money to do their research. As an example of an answer which is less reflected and somewhat 'off', we have the following:

The motivation for people at the universities is an interest in developing the mathematics. For the security agencies the purpose is to use the mathematics in a practical application. There are differences between the two communities, one is pure mathematics, the other is applied mathematics. (Group 5*)

In some way, the students of group 5* may have mixed up their answers of the two previous supportive essay assignments (the discussion of Hardy and pure and applied mathematics, and the discussion of the motivation of mathematicians of older and newer date) when they were to answer this question.

As could be expected, the students were divided by their opinion on who should be credited for the creation of public-key cryptography and RSA (question c). Some would favor the researchers at the universities:

Since Diffie and Hellman were the ones to find the method they should be given the credit for it. GCHQ may have found it earlier, but they didn't publish it – for that reason it is more fair that Diffie and Hellman get the credit. (Group 3*)

And other groups would favor the cryptographers of GCHQ:

We believe that Cocks is the only right person to credit for both RSA and public-key cryptography. The reason for this is that he found the one-way function, and this function is the basis for RSA. (Group 6*)

Another group of students (group 2*) stated that there is no definitive answer to the question of attribution, and that it depends on how you look at it.

Main Essay Assignment

In their answering of the when-and-who account (question a) some groups, based on the teaching material, merely answered with short lists while other groups provided more extensive accounts on people and years as well as what contributions had been made. An example of the first kind is:

Euclid, 300 B.C.: Euclid's algorithm which finds greatest common divisor.
 Étienne Bézout: *générale des équations algébriques*, published at Paris in 1779, used to go the reverse way of Euclid's algorithm.
 Carl Friedrich Gauss, 1801: founder of Congruence.
 Sun Zi: year 280-473.
 Pierre de Fermat, 1640: can be used to find which numbers are not primes.
 Wilson 1760 [...]
 Leonhard Euler, 1730s: Euler's theorem. (Group 4*)

The omitted part about Wilson concerns what is known as Wilson's theorem, and was an extract from wikipedia, with reference though. An example of the second and more extensive kind of account is:

Euclid – 325 B.C. to approximately 265 B.C. The first one we know in this connection [number theory] must be Euclid. It was he who began number theory. He wrote 13 books, which have been used in thousands of years for teaching in, amongst other, geometry and number theory. In his books you may find prime numbers, composite numbers, numbers being relatively prime, and greatest common divisor.

Sun Zi – approximately between 280 and 473. The Chinese remainder theorem provides us with an insight into the mathematics of the old China, as well as how Sun Zi handled modulo and congruences in his own way. This was between 280 and 473, the uncertainty of these numbers are due to the writing which was used.

Fermat – 1601 to 1665. He made ground-breaking work within number theory (Fermat's little theorem). He developed probability theory and differential calculus.

Euler – 1707 to 1783. He made proofs by means of 'phi' [his ϕ -function]. He treated congruence and linear congruence. In Euler's function you may find how many numbers are relatively prime to an arbitrary number N .

Gauss – 1777 to 1855. He wrote Gauss' *Arithmeticea* where he used the concepts congruence, modulo, remainders – more modern versions of what others had worked with. (Group 3*)

Needless to say, none of these answers are flawless. But what is interesting is that even in the when-and-who account, students include elements of the taught and learned mathematical in-issues, even though they have not been asked to do so and the assignments were not designed to 'ensure' this. Of course, one might think the why-and-how

account (question b, page 166) to be a more natural place to treat in-issues, but most often this account was used to discuss meta-issues, e.g. matters of personal motivation, inner and outer driving forces, and matters of mathematics and war inspired by Hardy's viewpoints. I shall not show any examples of this since a lot of the accounts were, naturally, based on the supportive essay assignments which have already been presented.

Question c was included to 'ensure' a degree of anchoring, should this not occur in the other assignments, by having the students look at the history of public-key cryptography through the 'lenses' of objects and techniques. This approach to history had not been discussed as part of the second module at all, so in a certain way this question was also a test of what they had gained from the first module. The idea, of course, was to look at public-key cryptography and the RSA algorithm as being the objects under investigation, and then discuss the techniques used to create and develop these. Not all groups provided equally good answers to question c: a few did not answer at all, some talked about the notions of objects and techniques, instead of answering the question, but some also gave quite elaborated answers. A few examples showing the 'existence' of some understanding of the notions of objects and techniques are:

The RSA algorithm rests upon three mathematical theorems: the Chinese remainder theorem, Fermat's little theorem, and Euler's theorem. Congruences are a fantastic tool [technique] when dealing with integers of several hundred digits as you are in RSA. Carl Friedrich Gauss (1777-1855) defined congruences, moduli, and remainders in his work *Arithmeticae*. [A definition of congruence is provided.] The security in the RSA system builds upon prime factoring being a difficult calculation problem. If $p \mid ab \rightarrow p \mid a \vee p \mid b$. The fundamental theorem of arithmetic [the theorem is written up]. Euler's φ -function [a definition is provided]. Inverse elements: In the RSA system the decryption algorithm is an inverse element of the encryption system [a definition of inverse elements is provided]. (Group 2*)

In cryptography, as in coding theory, there are many objects which become techniques and the other way round. RSA is in itself an algorithm, which is a technique, but may equally well be seen as an object within public-key cryptography. In a similar way, public-key cryptography may be seen as an object within safe communication. In that way you may move further and further out, but also *in* by objects becoming techniques. (Group 1*)

Judging from the lack of answers to questions d and e (see page 166) it seems that they may have been too difficult for the students. Only two of the six groups provided answers to these questions, counting answers saying that these matters were incorporated implicitly in previous answers. In this sense these two questions may tell us something about the limitations of the students' capabilities to engage in meta-issue discussions, something I shall return to in section 6.7. For now, I shall turn to the last question (f) on the students' discussion of and comparison between the when-and-who and the why-and-how accounts. More groups answered this question, for example the two following:

The when-and-who account gives you a quick historical overview of what has happened, whereas in the why-and-how account you may gain a thorough insight into why things have been developed. (Group 4*)

The when-and-who account provides cryptography with an historical perspective so that you, in terms of time, may view it from above. We learn that societal factors may influence on the development within mathematics. The why-and-how account shows us how you go from a thought, or a desire, of something to inventing it. It gives us an insight into the driving forces which play in, something which the first account does not provide. (Group 1*)

Besides these two groups being quite right about the why-and-how account providing a deeper and more thorough insight into the evolution and development of mathematics than the when-and-who account, their statements, in fact, also show something else. They show us that the students may not only be able to discuss matters of the history of mathematics, but they may also, on some level, even though the majority did not answer questions d and e, be able to discuss the presentation of the history itself.

6.6 Observing the Focus Group (Group 1*)

In this section, a more thorough analysis of the focus group's answering of the final essay assignment will be given. The approach is to hold their hand-in paper up against some of the discussions they had while working in class, the purpose, of course, being to provide means for answering research questions 1 and 2. In doing so I shall also discuss the focus group students' understanding of the mathematical subject matter (in-issues) of the teaching module. But first an introduction of the two students replacing Sean and Lucy in the focus group.

Introducing the Two New Focus Group Students

The two new focus group students were Lola and Harry. Lola was from the old group 2 and was among the interviewees. Harry, on the other hand, had not been interviewed before, but he was chosen because of his effort in group 6 during the first module. This meant that Harry was interviewed about his answers to both questionnaire 1 and 2 in the same interview of round two. But let us take a closer look at these two new focus group students' answers to questionnaire 1 and 2 and the followup interviews.

Lola rates her interest in mathematics as a 4.5 (on a scale from 1 to 5) and considers herself good at mathematics (questionnaire 1). The things she likes about the subject mathematics are "logic, cracking your head, 'the brain is functioning', seeing finished results and understanding connections". Things she does not like are way too difficult problems, and things which cannot be placed in wider perspectives. To the question whether she considers it important to learn mathematics, she answers "yes" and continues: "I have played with the thought of becoming an architect where mathematics plays a major role. + I think that the natural science will have a larger and larger impact on our world." She does not, however, find it equally important for everybody to know mathematics at a higher level. That is only relevant for people like engineers etc., but other people need to understand the basic mathematics of their everyday life. Lola has not been introduced to the history of mathematics besides something about Pythagoras and also something about Galileo Galilei in her physics class. To the question of whether she believes history to be something which could interest her, she answers: "No, I do

not find the history interesting. I'm only interested in calculating & using my brain." This questionnaire answer was supplied by the following comment in the interview:

I just don't find it exciting to consider why Pythagoras figured out that his theorem had to be the way it did. To me it's just more exciting being told what the theorem is, and then use it for something which you can relate to yourself.

Lola's outspoken skepticism towards the inclusion of history was an important reason for choosing her since this view would replace that of Sean's. Another important reason was that she was a much more talkative girl than Lucy had been, something needed to function in a group with both Andrew and Gloria. Thirdly, Lola seemed to be on approximately the same mathematical level as the other focus group students, judging from the teacher's idea of her and her participation in the first teaching module (hand-in exercises and answers in questionnaire 2).

Harry puts down his interest in mathematics as a 4 and thinks himself good as well. What he likes about mathematics is that "You learn many new mathematical models, which can be used later in life. It challenges your logical sense." He does not like it if it gets too difficult and hard to follow. It is important for him to learn mathematics because of his further education (in the interview he says that he is going to study nano-technology at the technical university), but also due to household economics. To the question whether he considers it important for everybody to learn mathematics he answers: "Yes. Everyone. So we can have a society which functions well, smart people. The survival of human beings depends on our brains." Harry has had a little bit of history of mathematics in upper secondary school, he says, something with famous people etc. To the question of whether he thinks he might find history of mathematics interesting he answers: "The part of the history of mathematics which is being used today is interesting. Everything which is not used is only interesting when we find ourselves in a situation where we need to use it. Otherwise it becomes too history-like." Despite this statement about things becoming "too history-like", Harry seemed to have a much more positive attitude towards the teaching modules, the essay assignments, and the meta-issues of mathematics in general than Sean had (something which was later confirmed in the second round of interviews). Another reason for choosing Harry was my impression of him as a quite determined and goal-oriented student from observing the class in general. It was my hope that he could bring a bit more 'focus' into the focus group. When the old focus group was 'focusing' they had some very interesting discussions. However, a lot of (recording) time was spent on discussing matters not related to the essays or exercises, such as the previous and the coming weekends' parties, other teachers, friends, grades, last night's tv-shows, etc. As mentioned, Gloria, though a very capable student, was often the one to set the agenda of the group, including when to cut-up and when not to. The fact that Harry, like Gloria, was a 'strong' student and seemed a mathematically capable one as well (according to the teacher, but also judging from his exercises in coding theory), made me believe that he could act as a counterpoise to Gloria and thereby help keep the discussions on track for longer stretches of time. Furthermore, Harry and Jonathan were friends, so if Harry was able to take charge of the group from time to time, this might also result in Jonathan not being overheard as often as he was in the first module.

The Focus Group's Final Essay Assignment and the Making of It

The focus group provided a quite good answering of the final essay assignment, the main essay as well as the three supportive ones. The total length of the hand-in paper was four pages.

Unlike the situation in the first module, the modified focus group's work on and preparation of the final essay offer a variety of good meta-issue discussions, concerning the main essay as well as all three supportive essays. This made the selection of which clips to present more difficult. In my selection I have tried to choose clips including elements which were not emphasized during the analysis of the first module final essay, as well as clips including students whom were not the most discussed during the analysis in chapter 5. Also, I have tried to identify clips which possess potential anchoring points of the meta-issue discussions in the related subject matter (in-issues).

Actually, the focus group is one of the two groups who answered all questions of the main essay. The length of the main essay is a little more than one and a half page, which is a bit above average for this assignment. However, the focus group only considers the mathematicians of newer date in the main essay (i.e. in questions a and b). It is not entirely clear from the video why they only consider the newer mathematicians, but it may be because they began the main essay assignment while finishing up the third supportive essay, which concerned only the cryptographers. Judging from the presentation of the main essays in section 6.5, the discussion of the older mathematicians unfortunately seems to be one of the natural places to include elements of the mathematical subject matter. For this reason, the only places in the focus group's discussions of the main essay which offer possible anchoring points are those places where they discuss objects and techniques in relation to public-key cryptography and RSA. But since objects and techniques were not among the main general topics and issues of the final essay for the second module, I am not going to use these clips to discuss anchoring.

Instead, I shall use a clip from the group's work on the first supportive essay, the one on pure and applied mathematics and Hardy's *Apology*. The students worked and discussed in relation to this essay assignment for approximately an hour. The chosen clip is from their work on question c – Hardy's statements on number theory versus the creation of RSA. In their discussion, the students drew on related mathematical concepts and notions from both RSA as well as number theory. Originally, the clip was 14 minutes long, but some minutes have been edited out of the transcript since they concern matters not related to the essay.

In addition to this clip, I shall also present a clip from the second supportive essay assignment (question b), in which the students discuss inner and outer driving forces in the development of public-key cryptography and personal motivations of the researchers involved. This video clip has been chosen for three reasons: (1) it concerns meta-issues which were not touched upon in the first module, i.e. inner and outer driving forces; (2) it involves a very interesting discussion of these meta-issues, in which all five students engage actively and in which a couple of the students definitely do not agree with each other; and (3) it is a clip of 12 minutes, no omissions, where the students stay focussed on the task – in fact the longest of such consecutive clips in my entire data collection. Now concerning the actual answering of the second supportive essay, something should be addressed. As mentioned, a somewhat general tendency for the groups of students was that of mixing up the inner and outer driving forces in the development of a

mathematical area with the personal motivations of the researchers involved in the development. That is to say the students would look at one particular researcher and ask what his *personal* inner and outer driving forces were. The way the assignment was intended, was to have the students do the one first and then the other. Nevertheless, in spite of this misunderstanding, interesting discussions and written answers still emerged from the students' work.

The third supportive essay offers interesting discussions, especially in relation to the students' Internet searches on GCHQ and NSA. However, these discussions offer no potential anchoring points to be explored, they are all concerned with meta-issues and in this sense they do not provide any new information in addition to the other clips.

I shall now turn to the presentation of the video clips. Since the students worked the second supportive essay assignment before the first, I shall begin by looking at the video clip from that one.

A Video Clip from the Second Supportive Essay Assignment

In this first clip the students are working on question b of the second supportive essay (inner and outer driving forces of the development of public-key cryptography and personal motivations of the people involved therein). Their full answer to question b was:

b) Diffie, Hellman, and Merkle: These people, who invented public-key cryptography, were mostly driven by an inner driving force since they were interested in the area themselves, and developed it out of own interests. Diffie is especially fascinated by the Internet, and both Diffie and Hellman believe that cryptography should be for the benefit of all. This may be seen as an outer driving force.

Rivest, Shamir, and Adleman: Rivest and Shamir are very clearly driven by their interest in the area. But at the same time they are also driven by their interest in Diffie and Hellman's new theory, which develop their interest in finding a code [one-way function] for this theory. That is to say it is an outer driving force which influences them in a certain direction. But Adleman is solely driven by an outer driving force since he is not interested in the area personally, but is being pressed by Rivest to find numbers which fit the theory.

Ellis, Cocks, and Williamson: They are all driven by the outer driving force because they work for GCHQ, which is a military organization. They are therefore strongly influenced by the organization concerning their work. (Group 1*)

In the very beginning of the discussion (see appendix C.2), Harry states that he finds it obvious that Diffie and Hellman were driven by an inner driving force whereas Rivest, Shamir, and Adleman were driven by an outer. This immediately results in a series of comments from Andrew who believes that there is also an outer driving force present in the form of the need for security. Gloria brings up Diffie's work related to the Internet, and she and Jonathan argue that all mathematicians somehow need to possess an inner driving force – arguments which Gloria brings up again and again in the rest of the discussion. Jonathan begins flipping through the teaching material to check out Harry's argument, while Andrew and Gloria seem to reach some common ground concerning

Diffie and outer driving forces (lines 21-29 and 54-56). Jonathan then finds what he is looking for and reads aloud: “It says right here: Diffie was especially fascinated and interested in the beginning Internet, the so-called ARPANet, when it at first belonged to the US military. And he was one of the first to realize the possibilities, blah blah ...” (lines 35-37). Unfortunately, Jonathan also throws in a comment about Hellman and Diffie wanting to compete with the NSA and this being an inner driving force for them. What he is referring to is the reaction of Hellman’s university colleagues, who said that when doing cryptography you would be competing with the NSA and their million dollar budget. Jonathan himself soon realizes that this is not a real competition but more a figure of speech from the colleagues. Harry, however, catches on since this supports his view of Diffie and Hellman being driven by an inner driving force and quite some time is spent on Jonathan and Harry discussing this competition aspect. Then Lola enters the discussion reading aloud from the text, something which kickstarts a discussion between Gloria and Harry:

Lola: Check this out: Diffie recalls the day in 1975 when he first got the idea for public-key cryptography like this: I went downstairs for a Coke and almost forgot the idea. I remembered that I had been thinking of something interesting, but I could not recall what it was. Then it came back to me in an adrenaline rush of excitement. For the first time while I had been working with cryptography I was aware that I had discovered something really valuable. He is...

Harry: He is really interested in it himself, right?

Gloria: Yes yes, but all of them are. There is actually an idea behind him producing this mathematics: it is for developing the Internet.

Harry: Yes, but nobody told him that he should do it to develop the Internet. It’s like in wars, there are many...

Gloria: No, nobody... but I also think... No, but the outer circumstances, it also has to do with the situation of the world and its state, and how far you are and what it can be used for.

Andrew: And again, the thing with you getting the credit for it.

Harry: But I also think that these three persons...

Gloria: But he also wants to do this for the sake of the world.

Harry: It’s like that with all of them. I mean, it’s for the benefit of the world, all the things these guys are doing.

Jonathan: All the previous ones haven’t said that they wanted to do this for something. Here they say that they want to do it so that everyone may benefit from it.

Harry: Yes, but that still isn’t an outer driving force. It is just his own thought as to what his project may amount to. Isn’t it? (lines 77-99)

Gloria is rather strong in her belief about Diffie being visionary and having higher goals (e.g. the Internet) as opposed to just doing cryptography out of pure interest. Harry is not convinced yet, but he does not seem as opinionated as he did in the beginning either.

Next, the students begin to discuss the GCHQ guys and whether they were driven by inner or outer driving forces. Harry, Jonathan, and Lola reach an agreement rather fast: for these guys it must be outer driving forces that counts. They were hired by an organization, they were told what to look at, they were paid money, etc. (lines

110-120). Jonathan decides to check with the teacher for a definition of outer driving forces (lines 121-126). Gloria seizes this opportunity of the teacher's presence to reenter the discussion from earlier on Diffie and Hellman:

Gloria: But can't the state of the world also be an outer driving force: Okay, now we are like this, we need some Internet which works even better?

Teacher: Yes, I think you can say that that can be an outer driving force.

Lola: So you can say that the task they are being given is an outer driving force?

Harry: Yes, but the task they are being given must be given by an outer driving force before you can say that they are driven by an outer driving force. But they are not being given this task.

Gloria: Listen. All mathematicians are also driven by an inner driving force, otherwise they wouldn't do it. Otherwise they wouldn't care to spend their life on it.

Harry: But those three persons were not told to make a better system.

Gloria: No, no... (lines 127-137)

Notice here that Gloria presents the argument which she and Jonathan gave in the beginning of the discussion (lines 12-14), only this time in a more elaborated version. What she seems to be doing is reusing an argumentation which she gave in the fourth supportive essay of the first module, where she said that mathematicians (Hamming and Golay) would not be willing to spend their lives on something without receiving credit for it (an element which Andrew also brought up in the present discussion, lines 54-55, 91). The difference in Gloria's argument is that she now is able to provide her argumentation using the notion of personal inner driving forces. While Gloria pursues her quest, Harry and Lola are elaborating on the discussion about what an outer driving force is. The teacher asks a clarifying question and thereby actually make Harry and Gloria reenter their discussion from before:

Teacher: Oh, you mean outer driving force as someone else posing the task.

Gloria: It doesn't have to be. It doesn't have to be somebody else who gives the task.

(lines 140-141 omitted)

Gloria: Try and listen to this: Outer driving forces are understood as those forces which affect the research of mathematics from the outside.

Harry: They began this not knowing if it was going to lead somewhere or not. What he [Hellman] writes is that it's just idiots who keep on trying, isn't that right? They continue because they are interested in it themselves.

Gloria: And because they want to succeed, so the world can evolve a little and the Internet can come to work.

Harry: But then you can say that about everybody, because everybody wants the world to evolve.

Gloria: That's not right. This guy who says that I want to play with prime numbers, and I don't care if it can be used for anything.

Harry: There is no doubt that they all make the world evolve, and all have both an inner and an outer driving force. What I'm saying is just that the three [Diffie, Hellman, and Merkle] have more of an inner because they are interested in the stuff and have started looking into it without anything having influenced them to do it. And when they start, they don't even

know if they will achieve anything which means that they don't think about money.

Gloria: Can't we just say that you can discuss what has driven them the most because we, in the group, do not entirely agree, and it isn't something which we can... (lines 138-159)

For Gloria, the outer driving forces, or at least the ones she finds interesting, are 'the evolution of the world' and 'the greater benefit of mankind', whereas Harry apparently believes the major driving force to be money. The teacher tries to smoothen out the discussion a bit by saying that she sees no conflict in saying that for some of them it is a combination of both inner and outer driving forces, after which she draws their attention to Adleman. Her argument is that there is no (personal) inner driving force for Adleman since he was asked to do the checking by Rivest and Shamir. The discussion ends by Harry changing his original view of Diffie and Hellman being driven by inner and the RSA guys by outer to say: "So it is a mix, right? Because Rivest, he is really interested in coding, and he is the one to take initiative to it all, right?" (lines 193-194).

Jonathan and Harry now write up the answer to question b. The discussion continues, but in a less focused way than before and often more in relation to what actually to write in the essay. Some ten minutes later Gloria and Lola get the computer and begin writing down the answer to question c, which eventually becomes:

c) It is obvious that practically every researcher in mathematics has an inner driving force. Lack of this kind of motivation can stop the creative process in getting new ideas which are to be tested, and new connections which are to be examined.

Through time [history] the mathematicians' outer driving force may be increased since the importance and usability of mathematical research stands clearer and clearer, and the situation of the world to a higher degree demands it. (Group 1*)

As pointed out earlier, the discussion of inner and outer driving forces concentrates itself mainly around the individual mathematicians and their personal motivation for engaging in the field, i.e. not necessarily around driving forces in the field as such. And when the students ask the teacher for advice in settling a matter related to this mixing up, she does not object to the students' confusing and mixing up of the two different perspectives. In the teaching material the students were introduced to the notions of inner and outer driving forces by statements saying that the inner are those forces which drive research in mathematics from the inside and that the outer are those forces which affect the development from the outside (Jankvist, 2008h, p. 105). As examples of inner driving forces, unsolved problems such as Fermat's last theorem, the Goldbach conjecture, and the Riemann hypothesis were mentioned. As an example of outer forces war was mentioned, with reference to Hardy and funding, e.g. the increases in funding due to World War 2 and the Cold War, which led to developments in pure and applied mathematics both. Thus, the examples by which to introduce the notions were mainly chosen from the historical cases of the modules. Discussing inner and outer driving forces for scientific disciplines is not always easy and despite the use of 'familiar' examples above, the notions of inner and outer driving forces, could very well have been made more accessible to the students – something I shall return to in section 6.7 when discussing possible redesign of the module. At any rate, despite the strong personification of inner and outer driving forces the above discussion of the focus

group still shows the students' capabilities to discuss meta-issues, e.g. elements of the development of public-key cryptography and RSA, outer driving forces such as the Internet, and the personal motivation of the developers having to do with applied or pure mathematics (Gloria's reference to primes above). The discussion between Harry and Gloria is particularly interesting, since they seem to have different ideas of what is to be understood by outer driving forces, both of them being right in their own way. I shall discuss this further in section 6.7.

A Video Clip from the First Supportive Essay Assignment

The focus group's hand-in answer to question c of the first essay assignment reads:

c) Applied mathematics builds upon pure mathematics. For example, RSA could not have been developed before they received help from the number theoretician Adleman, who was a pure mathematician. Hardy points to that pure mathematics may only be conceived in one way. That is to say prime numbers only can be prime numbers no matter who is looking at them. However, applied mathematics is or can be relative. Hardy believes that number theory is pure mathematics because it has no application. He says that it is so pure that it cannot be used for anything. But here he is wrong, because it is used repeatedly today, e.g. in RSA. (Group 1*)

The discussion (see appendix C.2) begins with Harry saying: "Isn't it very logical that he [Hardy] says that number theory is only connected to mathematics, pure mathematics that is? And if we have to compare this to RSA then what we have to say is that they had to have this third guy [Adleman] in order to be able to encode, right? If they didn't have him they wouldn't know how to encode. And he was a pure mathematician, right." Upon Harry's remark, Andrew and Jonathan clarify among themselves which of the researchers Adleman is, and once Andrew is clear on this he says: "But we agree that Hardy he says that number theory is pure mathematics, right?" To which Jonathan answers positively. Lola enters the discussion and they clarify the assignment (question c) among each other. This leads to the following communication between Andrew and Jonathan:

Andrew: Well, at least you can say that if number theory is pure mathematics then it is very basic [basic research], right? And then we have all this RSA which is being developed using this number theory, and as Jonathan says, it is Adleman who's kind of the crucial link in this...

Jonathan: Yeah, if they hadn't had this number theorist guy then they wouldn't have gotten anywhere. The others kind of want to develop it and are very interested in it, and they have this problem. But they kind of have to work together with a number theoretician.

Andrew: But that underpins that the applied mathematics must build on the pure mathematics, right?

Jonathan: Yes. (lines 24-34)

These statements actually refer to the discussion about Adleman, which the group had with the teacher in the previous video clip. Lola, who is taking notes on the computer, does not recall all of the story of RSA which brings first Andrew and later Jonathan to try and refresh her memory:

Andrew: Well, we've got these two RS – I can't remember what it stands for,

but that doesn't matter – they've got this idea that they want to develop this odd thing, right. But they can't really get moving until they get this guy Adleman, who is a pure mathematician who kind of... well, he...

(lines 56-68 omitted)

Jonathan: Yeah. They've got this problem that they need this one-way function. So they come up with some suggestions for some solutions. Then they pass them on to Adleman and say 'try and see if these ones check out'. And then Adleman sits and tests them...

Andrew: But Adleman, he was the the guy who wasn't interested in this thing at all, right?

Jonathan: Yeah, he wasn't really interested, but he helps them out. (lines 72-75)

Some time now passes with the students evaluating what they have for question c so far and what they need to do next. Jonathan points to the fact that they are to discuss Hardy's statements and that they therefore need to locate the relevant statements in the *Apology*, maybe even quote them. He and Andrew now ask Harry to join them in this, which after a few minutes of reading on the part of Harry, results in the following discussion:

Harry: It says [in the teaching material] that one of the problems in number theory is to decide if a number is a prime or not. And in this book [the *Apology*], he [Hardy] says that finding primes is pure mathematics, because when you are a mathematician then you already have the frame for the area you are working within, you know what a prime number is, in contrast to a physicist or a chemist who works with some applied mathematics, they have to work with things relatively, I would say: Here is a table [points to the table], but to you... for some other person this might not be a table, or for some other thing in a different universe, this might not be a table. But a prime number will always be a prime number.

Andrew: So the pure mathematics cannot be discussed, you might as well say.

Lola: That depends on how you conceive that the history of mathematics has been developed, because if someone else had been sitting and thinking over a number and had found some other connections, well then a prime might suddenly not have the same meaning.

Harry: But now they have this frame...

Andrew: You could say that, what we are kind of working with... it is our frame, it is the numbers we have, it is our frame, and then within these there is a lot of pure mathematics, for example prime numbers.

Lola: Yes, primes can never be different, if you look at how we look at mathematics.

Andrew: No, if you stick within our frame.

Lola: But if we imagine that the numbers had some different values or whatever you'd say, then...

Andrew: Yes, but then you are changing to a new frame, and then there is a connection within this frame.

Lola: Well... but then you could also say...

Harry: In that way it might be kind of relative, I can see that. But what

he says is that you can't discuss it. He says that in this world we live in here there can be two different physicists who tell you what this is [points to the table] while two different mathematicians [equally] can tell you what a prime number is.

Lola: Yes, and that is what Hardy he says, right? Do you want me to write that?

Andrew: But then you could also say that the pure mathematics is as objective as anything can ever become, right, because it isn't colored by anything.

Harry: But we have to see it in connection to RSA. (lines 105-137, the end of the clip)

Some of the viewpoints of Hardy which the group refers to are those mentioned in section 3.6.⁵ I shall not enter a discussion of whether the students' are right or not, concerning their own viewpoints and philosophical remarks. However, I do believe that they interpret Hardy's viewpoints more or less correctly, once more showing their ability to engage in discussions of meta-issues of mathematics, its development, and possible application. Even more interesting is the entering of mathematics into the discussions. This happens twice: when Harry brings up primes, in the last part of the clip and they all enter into this discussion; and when Jonathan earlier on mentions the need for a one-way function. Studying prime numbers was an essential part of the second module and the students did not know much about number theory (if anything at all) and primes beforehand, except the definition of a prime number. In the above quote the students do, however, seem at ease with discussing matters relating to prime numbers. No questions are asked pertaining to the notion of primes, nor the beginning statement by Harry that it is not easy deciding if a number is prime or not. The same can be said about Jonathan's mentioning of the one-way function. None of the students ask any questions about this, which may indicate some kind of conception of the notion. Of course, one could argue that the students might not want to pose 'stupid' questions to each other and thereby reveal what they might not know. However, I do not believe this to be the case since the students generally do not seem to hesitate to ask 'stupid' questions in front of each other, or in front of the camera. One example of this might be when Lola asks: "Okay, but it [RSA] is developed now, right?" (line 68).

It should be noted that Gloria actually is present the entire time during the above discussion, but that she almost does not intervene at all (only briefly in lines 40 and 58). The reason for this appears to be that she is very preoccupied calculating her grade point average and speculating on future exams, something she talks widely about during the omitted parts of the transcript (omissions in lines 35 and 79). However, we do learn one thing from Gloria's 'non focused' interventions, namely that she would prefer not to be examined in mathematics (the oral exam that is)⁶ – something I shall

⁵ "Pure mathematics [...] seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way" (Hardy, 1992, p. 130).

⁶ In the Danish system, the ministry decides which schools and which classes in these will be examined in which subjects. The students will have a fixed number of exams but the subjects will differ. And since the mathematics exam consists of both a written national and mandatory exam as well as an oral exam in the locally decided curriculum, there is actually a chance that the oral mathematics exam will not take place.

return to in section 6.7. But first, let us look at the students understanding of the mathematical subject matter (in-issues). Both what is touched upon above (primes and one-way function) as well as other matters from the available data sources.

Focus Group Student's Work with Mathematical In-Issues

As explained earlier, during the second teaching module the teacher assigned to each student a proof of one of the theorems in the teaching material, which was to be presented on the blackboard in front of the class. These presentations delivered by the focus group students may tell us something about their understanding of the mathematics of the module.

The first of the focus group students to present a proof was Harry. He was given the proof of the Chinese remainder theorem and as we have seen when discussing the first historical exercise (section 6.4), this is not an easy proof. Harry, however, was able to give a clear and thorough presentation and as part of this even point out a misprint in the teaching material. More precisely, when earlier illustrating the proof of the theorem we considered the term $M_j \equiv 0 \pmod{m_k}$ when $j \neq k$ in showing that the constructed sum was indeed the solution. In the students' version of the teaching material the first expression here mistakenly read $M_j \equiv 0 \pmod{m_j}$ (m_j instead of m_k). When reaching this point in the proof, Harry said:

And then we also see... I don't know if there is a misprint in the book, because it says capital M_j ... [writing what it said in the book]. As far as I have understood, little m_j is not a divisor in (capital) M_j with remainder 0, since M_j is exactly without this [points to m_k in the fraction $M_k = m/m_k$], so its probably a misprint, I would say. It should be k here [pointing to the j in M_j].

Upon this, the teacher commented that it should be m_k and not M_k , and she and Harry entered a small discussion of whether it could also be M_k . Now, despite the fact that Harry did not suggest the exact right correction, he still located the error and reasoned that something was wrong. In addition to having a clear picture of the entire proof (existence and construction of a solution, and showing the uniqueness of it), Harry also showed himself to have a feeling for the details and mathematical contents of the proof (in this case the construction of the fraction $M_k = m/m_k$) – all in all a sound understanding of the in-issues in question.

Next to present a proof was Jonathan, who gave the proof of Fermat's little theorem (for the exact formulation see the discussion of the historical exercise, page 163). This proof is not easily explained without introducing other preliminary lemmas and theorems. Therefore I shall not provide an in-depth description of the proof and Jonathan's presentation of it, but only some general observations which the reader will have to take my word for. What can be said from watching Jonathan's presentation of the proof is that he seemed to have a very good perception of the general structure of the proof, which fell in three parts, the first and third part drawing on previously proven theorems (Jankvist, 2008h, pp. 56-57). Some of the details of the proof, such as smaller calculations etc., might not have been communicated too well by Jonathan in his presentation since many of the other students asked questions about them afterwards. However, what Jonathan seemed to lack in communicative skills he possessed in understanding: he was able to answer the other students' questions as well as the clarifying questions of the teacher,

remembering the two other theorems used in the first and third part of the proof.

Andrew's proof to deliver was that of Euler's theorem, which states that if n and m are relatively prime, then

$$n^{\phi(m)} \equiv 1 \pmod{m},$$

where $\phi(m)$ refers to Euler's ϕ -function: $\phi(m)$ gives the number of integers r with $1 \leq r < m$ such that r is relatively prime to m , i.e. $\gcd(r, m) = 1$. As mentioned earlier Euler's theorem is a generalization of Fermat's little theorem: If m equals a prime p , then of course n and p are relatively prime. Furthermore, for any prime p we have $\phi(p) = p - 1$ giving us

$$n^{\phi(p)} \equiv 1 \pmod{p} \Leftrightarrow n^{p-1} \equiv 1 \pmod{p},$$

which is exactly Fermat's little theorem. I shall not go into details of the exact proof (the theorem and the proof may be found in Jankvist, 2008h, p. 61), only say that the proof builds on the proof of Fermat's little theorem. Thus, Andrew not only needed to understand the proof of Euler's theorem, he needed to understand details in the proof of Fermat's little theorem as well. In his presentation of the proof, Andrew got off to a good start writing up the theorem and beginning the proof by making correct references to the proof of Fermat's little theorem. Andrew structured the proof fairly well, following the textbook of course, and arrived at the correct result. However, some of the intermediate steps were not clearly accounted for, and Andrew erroneously put down commas between integers in a sequence instead of multiplication signs ($nr_1, nr_2, nr_3, \dots, nr_{\phi(m)}$ instead of $nr_1 \cdot nr_2 \cdot nr_3 \cdots nr_{\phi(m)}$), thus not having a product as he was supposed to. This can be explained by the fact that the numbers previously had been put down as a sequence, but when asked by one of the other students if these were commas Andrew confirmed. Not until Jonathan pointed out that they should in fact be multiplication signs did Andrew notice the mistake. Upon the completion of the proof, the teacher asked Andrew to account for some of the intermediate steps, which he was not entirely capable of. Once more Jonathan entered the discussion and was able to account for the missing intermediate reasonings of Andrew's presentation. Hence Andrew's presentation not only tell us something about Andrew's understanding of this proof, it also elaborated our view of Jonathan's understanding of Euler's as well as Fermat's theorems and the proofs for them.

Lola and Gloria were supposed to have presented proofs as well, but for some reasons did not. Most likely they were not present on their days of presentation, or they may have been unprepared. In any case, the teacher presented their proofs. Gloria did, however, present her proof to the teacher outside class:

At one point in time I ran into Susan [another student] and Gloria, who didn't present their proofs in class. They begin talking about grades, and I offer that they can present their proofs to me right here and now. And they were up for it [...]. Susan did her proof, it wasn't very difficult, but it was a good presentation. Gloria hadn't looked at hers at all, but she sat down on the spot and swotted it [while Susan did hers]. And she was actually capable of doing it just as good as I had been, when doing it for her in class. (Teacher interview, round 3)

Gloria handed in the set of exercises including the two historical ones on Sun Zi and Fermat, and Lola handed in another set including, among others, exercises on prime

numbers. Thus, we shall look at these in order to get a better idea of their understanding of the mathematical subject matter, as well as that of the other focus group students.

The first set of exercises to be handed in was that on primes. The students were to use the following theorem and its corollary to decide if a list of numbers were primes or composites:

- If n is a composite number, then n has a prime factor which is less than or equal to \sqrt{n} .
- A number n is prime if it cannot be divided by any prime less than or equal to its square root.

(For a proof of these results, see Jankvist, 2008h, p. 35.) The concrete numbers to be examined were the following: 103; 117; 313; 1013; 1729; 4013; 5013; 7717. Of these $117 = 3^2 \cdot 13$, $1729 = 7 \cdot 13 \cdot 19$, and $5013 = 3^2 \cdot 557$ are composite numbers, the rest are prime numbers. Lola gave only correct answers to this exercise, the two last of which are provided in figure 6.4. Of course, strictly speaking, Lola needed not have written up 71 in the list of primes in the first answer. However, this is the only time she makes this mistake, for all the other numbers in the lists consist of primes less than the square root, as in the second answer on the figure. Furthermore, using the sum of digits to solve the case for 5013 actually shows some ingenuity since this ‘trick’ was not mentioned in the teaching material at all. (Taking the sum of digits is equivalent to calculating modulo 9, so if the result of sum of digits is divisible by 3, then surely the original number is also divisible by 3.) The set also included two other small exercises, both of which was on mathematical induction. Lola only did one of these, and she did not get it right, partly because she copied the text of the exercise incorrectly from the material. Andrew and Harry handed in this set of exercises as well. (Jonathan and Gloria might also have

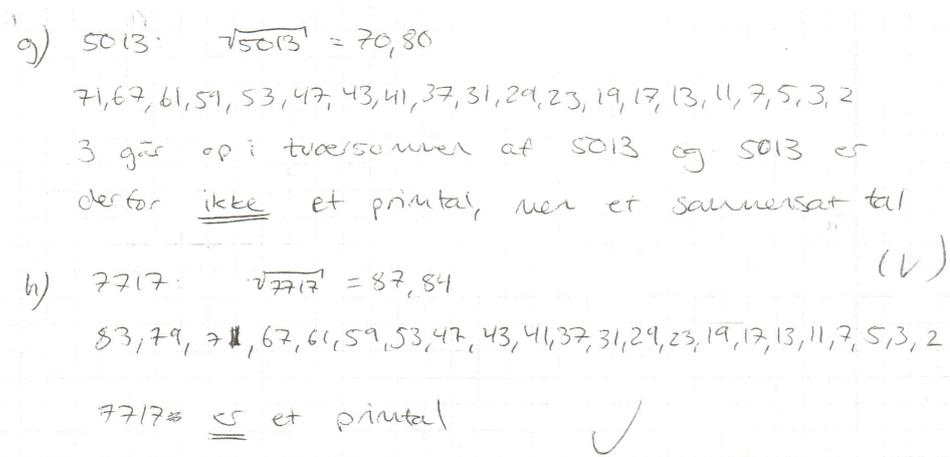


Figure 6.4 Lola’s answers to the last two questions in the exercise on primes. The text for 5013 reads: “3 divides the sum of digits of 5013, and 5013 is therefore *not* a prime number, but a composite number.” (The reason the teacher has marked the correctness of this in parentesens is because she would have liked to have the prime factoring of the number as well.) The text for 7717 reads: “7717 *is* a prime number.”.

handed theirs in, but if they did not do it on time, I might not have received copies from the teacher.) Andrew's answers are similar to those of Lola: he only did two of the exercises (the same two), he got the induction proof wrong (for the same reasons as Lola) and he answered correctly to all the questions of prime numbers. In fact, Andrew also used the idea of digits of sums, and he used it for both 117 and 5013. Harry's hand-in set consisted of all three exercises and only very few flaws appeared. For the exercise on primes, Harry also provided the prime factorizations of the composite numbers.

From the set including the historical exercises, I have copies of the hand-ins by Gloria, Jonathan, and Harry. The set consisted of four exercises, two of which were the historical ones on Sun Zi and Fermat. The two others were on finding an inverse and solving a linear congruence, respectively. The students were to find an inverse of 2 modulo 17. If a and m are integers with $m > 1$, then an integer \bar{a} which satisfies

$$\bar{a}a \equiv 1 \pmod{m},$$

is called the *inverse* of a modulo m . If a and m are relatively prime a theorem says that \bar{a} exists and is unique modulo m (for a proof see Jankvist, 2008h, p. 48). Now, 2 and 17 are clearly relatively prime, so the inverse of 2 modulo 17 exists. Thus we need to find the \bar{a} which satisfies

$$\bar{a}2 \equiv 1 \pmod{17}.$$

This can be done in various ways, trial and error being one. One way of proceeding is to say that the above congruence is the same as $17 \mid 2\bar{a} - 1$, and then solve the equation $17 = 2\bar{a} - 1$ for \bar{a} , i.e.

$$17 = 2\bar{a} - 1 \quad \Leftrightarrow \quad \frac{17+1}{2} = \bar{a} \quad \Leftrightarrow \quad \bar{a} = 9.$$

This is the method Harry uses. Gloria's approach is to consider possible numbers with remainder 1 modulo 17, she writes:

$$\begin{aligned} 1 + 17 &= 18 \\ 1 + 2 \cdot 17 &= 35 \\ 1 + 3 \cdot 17 &= 52 \\ &\vdots \end{aligned}$$

and then from this deduces that $18/2 = 9$ is a solution. Jonathan's solution builds upon the fact that when two numbers a and b are relatively prime they may be written on the form $1 = sa + tb$, which is a special result of Bezout's identity, a theorem the students had been introduced to in the teaching material. Finding the integers s and t may in general be done by use of the Euclidian algorithm (for an example, see Jankvist, 2008h, pp. 48-49), but for small integers as we have here it may be done by inspection: $1 = 1 \cdot 17 + (-8) \cdot 2$. This appears to be what Jonathan has done, see figure 6.5. From this expression it is easy writing up

$$(-8)2 \equiv 1 \pmod{17},$$

thus giving us -8 as the solution. Jonathan does this, but then continues to 'add' 17 to -8 to obtain 9, which he puts as the solution. The phrasing of the exercise asked the

4) Find den inverse
 b: 2 modulo 17
 sfd(2, 17) Den inverse \bar{a} til $a=2$
 $17 = 8 \cdot 2 + 1$ $= -8 \pmod{17}$
 $1 = 1 \cdot 17 - 8 \cdot 2$ $17 - 8 = 9$ ✓

Figure 6.5 Jonathan's answer to the exercise on finding the inverse of 2 modulo 17. "41 b:" refers to the number of the exercise. In the top it says: "Find the inverse [of] 2 modulo 17". The rest of the assignment is then split up in two columns, where 'sfd' in the first one is the Danish abbreviation for greatest common divisor (gcd). The second column begins with: "The inverse \bar{a} of $a = 2$...".

students to find one inverse, so the answer 9 is a fully correct answer. However, 9 is only one solution in the set of solutions:

$$\dots, -42, -25, -8, 9, 26, 43, \dots$$

As seen from figure 6.5, Jonathan is aware of -8 being a solution as well. Harry also mentions that -8 is a solution because $-8 \cdot 2 = -16$ which is congruent to 1 modulo 17. Gloria writes 9 as the solution, but the writing of $1 + 3 \cdot 17 = 52$, which would lead to $\bar{a} = 26$, suggests that she as well is aware of there being several solutions.

If a , b , and m are integers with $m \geq 1$, then a linear congruence is an expression of the form

$$ax \equiv b \pmod{m},$$

where x is a variable. The students were asked to solve the linear congruence

$$2x \equiv 7 \pmod{17}.$$

From the previous exercise we know that 9 is an inverse of 2 modulo 17. If we multiply by 9 on both sides in the equation we obtain:

$$18x \equiv 63 \pmod{17}.$$

We may split this up, saying that since $18 \equiv 1 \pmod{17}$ and since $63 \equiv 12 \pmod{17}$, then if x is to be a solution we must have that $x \equiv 63 \equiv 12 \pmod{17}$. That is to say that $x = 12$ is a solution, but of course this is not the only solution. Again we have a set of solutions, this time:

$$\dots, -39, -22, -5, 12, 29, 46, \dots,$$

or in short $x = 12 + k \cdot 17$ for $k \in \mathbb{Z}$. This approach is the approach applied by both Jonathan and Harry. None of them supplies much explanation in writing, only

calculations. Gloria's approach to solving the problem is a bit different since she appears to sort of 'translate' the given expression before beginning to solve it, see figure 6.6. Gloria puts down 12 as the answer. Jonathan puts $x \equiv 12 \pmod{17}$. Harry puts down the same expression, though he for some reason writes $x \equiv 12 \pmod{17} \Rightarrow x = 29$. Thus, Harry is the only one who (implicitly) suggests two different solutions, but he does not generalize them to encompass all possible solutions, and neither do Jonathan and Gloria.

42 B) $2x \equiv 7 \pmod{17}$

Vi skal finde et tal, der gænge med 2 har resten 7 (mod 17)

$7 + 17 = 24$ $\frac{24}{2} = \underline{12}$

den inverse til 2 (mod 17) er 9 (se 41, B)

$\Rightarrow 9 \cdot 2x \equiv 9 \cdot 7 \pmod{17}$

$x \equiv 63 \pmod{17}$

$63 = 3 \cdot 17 + \underline{12} = ?$ ✓

Figure 6.6 Gloria's answer to the exercise on linear congruence. "42 B)" refers to the number of the exercise. The first text bit reads: "We must find a number which multiplied by 2 has the remainder 7 (mod 17)". And the second text bit reads: "the inverse of 2 (mod 17) is 9 (see 41,B)", 41,B referring to the previous exercise. The "=" and the correction mark are the teacher's.

Concerning these three students' answers to the two historical exercises, a few things may also be mentioned. Gloria is one of the students who have trouble not mixing Sun Zi's method for solving the given problem with the modern one of the teaching material. She does, however, provide more or less correct calculations and answers to both historical exercises, giving an answer to question c the second historical exercise much in line with that shown in figure 6.3. And she also answers "No, no, no, it's not going to work!" to question d, though no explanation is provided. Jonathan provides an entirely correct solution to the exercise on Sun Zi, differentiating very well between the old and new method. The same is the case for the exercise on Fermat's little theorem where everything is correct as well, except perhaps a few unclear explanations in question a. Not surprisingly, all of Harry's calculations in both historical exercises are correct as

well. More interesting for Harry, though, is that he, unlike many of the other students including Jonathan and Gloria, actually provides rather long and elaborated comments in question c of the Sun Zi exercise – as a matter of fact, Harry’s answer was the one shown earlier on page 161.

The above presentation of the focus group students’ answers and solutions to mathematical exercises and their classroom presentations of proofs show some understanding of the subject matter, more precisely aspects of the elementary number theory of the module (and therefore also prime numbers). However, the picture may appear somewhat fragmented due to the inconsistency of data sources for the individual students. I shall attempt to remedy this and provide a more coherent picture by looking at the third questionnaire, which all five focus group students answered. Thus, in the following, I shall discuss the five students’ answers to questions 6-10, 13, 14, and 16 in the third questionnaire (see appendix B.3). Some of these questions concern number theoretic issues and some concern matters of cryptography.

In question 6 the students were asked what is to be understood by private-key and public-key cryptography. All of the focus students seem to have a good idea of this. As an example, Gloria wrote: “With private-key cryptography both the sender and the receiver know the same en- and decryption key (it must be shared first!). With public-key cryptography the ‘future receiver’ begins by sending: n . This is public! I.e. not secret. The sender of the message may now encrypt, and only the receiver of the message, who knows p and q , can decrypt the message.”

Question 7 was on the key distribution problem, i.e. the problem of distributing the keys in a private-key cryptography system. The students were asked to explain what the key distribution problem is. The only one who did not answer this question was Lola, all the others provided correct answers in which everyone of them referred to private-key cryptography. In question 8 the students were asked where private-key and public-key cryptography are used, respectively. The only one who did not answer this was Andrew:

Jonathan: “Private: the wars, Enigma for instance, DES/Incifer, etc. Public: RSA, the Internet.”

Andrew: “I don’t know.”

Gloria: “Caesar cryptography is a private-key. The Enigma machine from WW2 is also private. The Internet uses public-key cryptography when secret information must be transmitted (home banking, etc.)”

Lola: “Private: digital signature?; home banking; debit card. (Private=decryption. Public=encryption) Public: E-mail? The Internet – nothing happens when the keys are exchanged.”

Harry: “Private: E.g. during the second world war with the Omega [Enigma] machines. And the early Caesar cryptography. Public: E.g. the bank and in military applications.”

Next the students were asked what is to be understood by an ‘algorithm’. All five students appeared to have an idea of this. Lola referred to it as a kind of recipe, Jonathan and Harry talked about a set of rules, and both Harry and Gloria compared it to a manual, Gloria using the words “step by step”. Andrew on the other hand cited the definition from the teaching material: “An algorithm is a finite set of precise instructions for the execution of a calculation or the solving of a problem.”

In terms of the previous discussion in this section, question 10 was an interesting one since it asked what the notion of ‘number theory’ covers:

Jonathan: “It covers concepts such as prime numbers, composite numbers, relatively prime, gcd, and least common multiple.”

Andrew: “Composite numbers, prime numbers.”

Gloria: “Number theory covers the pure mathematics? Nope... I’m not really sure.”

Lola: “Covers numbers with specific properties, for instance prime numbers. Usually you work with integers.”

Harry: “All numbers I suppose, both binary and those from base 10. You could invent you own number system, if there was a need for that.”

Question 13 related to the use of a mathematical one-way function, asking which property of division of (large) integers RSA builds upon:

Jonathan: “Prime factoring.”

Andrew: “I don’t know.”

Gloria: “gcd()...? Greatest common divisor seems to play a role in almost all the formulas, theorems, proofs, and algorithms we have been through.”

Lola: “Something with modulo, perhaps? – a kind of ‘remainder’ by division.”

Harry: “RSA is a way of making a one-way code [function]. This one-way code has to do with division of large integers exactly, since it is difficult to prime factor large integers, and also finding precisely those two primes which the RSA-code [key] in question is build of.”

In the followup interview Jonathan explained his answer ‘prime factoring’: “Well, in order to break the RSA algorithm you have to be able to prime factor these very, very, long numbers, right. So if you can’t prime factor, you can’t break it... or why it is so difficult to break, at least.”

In question 14 the students were asked to mention the theorems from number theory on which the correctness of the RSA algorithm in particular builds upon. The answer to this question is, of course, the Chinese remainder theorem, Fermat’s little theorem, and Euler’s theorem. To this question both Harry and Lola answer the theorems of Fermat and Euler. Andrew says he does not know and so does Gloria. Only Jonathan mentions all three theorems. In the interviews Andrew was given the opportunity to answer question 14 anew, and this time round he was able to mention Euler’s theorem, the proof of which he had presented himself. Also Lola was given the opportunity to review her answer in the interview and when doing so she added the Chinese remainder theorem to the list.

Question 16 concerned the relationship between the theorems of Fermat and Euler, the answer being that Euler’s is a generalization of Fermat’s little theorem. Jonathan and Andrew answered correctly to this question, which was expected since they were the ones to present the proofs of these two theorems in front of the class. Harry also answered correctly. Gloria did not answer. And Lola said: “Euler’s was about integers, Fermat’s was about primes – that is all I know.”

This completes the presentation of the data relating to the focus group of the second module. Next I shall use the focus group data as well as that of the other students in the class, as presented in the previous sections of the chapter, to discuss answers to research questions 1 and 2, in the process of which some of the data also shall be more thoroughly analyzed.

6.7 Case-Specific Analysis, Discussion, and Recapitulation

This section may be seen as the sequel to section 5.7, the purpose being a further transformation of the debated ‘existence proof’ into a more constructive one, i.e. the existence proof of it being possible to have upper secondary students discuss and reflect upon meta-issues of mathematics in relation to history as a goal, and this in such a way that their discussions to some degree are anchored in the related and taught subject matter (in-issues). Again I shall base my analysis of the focus group’s discussions on elements from the theory by Sfard (2008b).

Students’ Meta-Issue Discussions and Their In-Issue Anchoring

In the analyzed clip from the first module we witnessed a so-called *meta-issue discursive conflict* between the viewpoints of Andrew and Gloria (cf. section 5.7). In some sense, the focus group’s discussion on (personal) inner and outer driving forces, which takes place in the clip from the second supportive essay, may be seen as one long discursive conflict, exemplified by the ‘clashes’ between Harry and Gloria. Harry’s approach to the question is mostly one of an historical discourse arguing that Diffie and Hellman were driven by inner driving forces to “invent a theory” resolving the key-distribution problem, whereas Rivest, Shamir, and Adleman were driven by outer circumstances in the form of Diffie’s and Hellman’s public-key system (lines 1-4, 15-17). Andrew and Gloria oppose this viewpoint immediately, doing so by means of both an historical and a psychological discourse. They argue the presence of outer driving forces in the work of Diffie and Hellman, e.g. the beginning of the Internet and the need for security in such a net (the historical discourse). And they also argue that inner driving forces are present too, but only in the form of the motivation a mathematician must have to spend his or her life on a certain task. Andrew uses this to say that it is a combination of both inner and outer (lines 10-11): “It’s both inner and outer. You of course have to have the inner before you even want to do it. And the outer, there is a need for security.” Gloria uses this as a launchpad for a new and more social discourse, arguing that Diffie was much more of an idealist than any of the others, and that he in being so also was driven and influenced by outer forces:

- “... There is actually an idea behind him producing this mathematics: it is for developing the Internet.” (lines 84-85)
- “... it also has to do with the situation of the world and its state, and how far you are and what it can be used for.” (lines 88-90)
- “But he also wants to do this for the sake of the world.” (line 93)
- “But can’t the state of the world also be an outer driving force: Okay, now we are like this, we need some Internet which works even better?” (lines 127-128)
- “And because they want to succeed, so the world can evolve a little and the Internet can come to work.” (lines 147-148)

The last of these quotes is the one to which Harry replies that you can say that about everybody. In order to respond to that Gloria changes her social discourse into one of mathematical in-issues, saying (lines 151-152): “That’s not right. This guy [Hardy] who says that I want to play with prime numbers, and I don’t care if it can be used for anything.” As was the case with the mentioning of Hamming spheres and the (7,4)-code in the discussions during the first module, Gloria’s mentioning of prime numbers here is a potential anchoring point for the meta-issue discourse in the related in-issues (i.e. the

mathematical and the historical ‘tracks’ intersect). However, Harry and the others do not really catch on, so the discussion of inner and outer driving forces does not take on a mathematical discourse. Nevertheless, during the discussion Harry seems to change his original viewpoint and acknowledge that it may in fact be a combination of both inner and outer driving forces. And one important reason for Harry changing his mind seems to be Gloria’s mathematically anchored ‘punch line’ from above, since Harry’s answer to this is (lines 153-154): “There is no doubt that they all make the world evolve, and all have both an inner and an outer driving force. ...” Harry’s change of mind and his acknowledgement of Gloria’s and the other focus group students’ viewpoints may also be witnessed from the written answer in the final essay (see page 178), which Harry wrote on the computer together with Jonathan.

The video clip of the first supportive assignment on Hardy’s *Apology* also holds a couple of potential anchoring points. The first is Jonathan’s mentioning of a mathematical one-way function (line 69). Like Gloria’s comment on primes, the reference to the one-way function does not enter into the discussion in general. Harry’s later reference to primes, however, does (lines 105-113). The departure point for this discussion is Harry’s summary of Hardy’s views: “... a prime number will always be a prime number” (line 113). This leads to a discussion involving Harry, Andrew, and, in particular, Lola. In order to discuss Harry’s reading of Hardy, Lola enters into a much more philosophical, in some places almost ontological, discourse, saying things like: “... a prime might suddenly not have the same meaning” and “... if we imagine that the numbers had some different values or whatever you’d say...” (lines 117-118, 125-126). Andrew’s approach to the discussion seems to be one of a philosophy of science or theory of science nature, discussing different perspectives (‘frames’) for looking at pure mathematics and ending by saying: “But then you could also say that the pure mathematics is as objective as anything can ever become, right, because it isn’t colored by anything.” As opposed to the situation in the previous clip, the anchoring points (‘prime numbers’ or ‘primes’) occur several times during this discussion, not just once or twice, and the underlying mathematical discourse is picked up by all three students participating. In fact, the notion of prime numbers acts as a reference point throughout the latter part of the discussion of Hardy’s view on pure and applied mathematics and his statements on number theory. To confirm if this ‘reference point’ is in fact also an anchored reference point, in terms of the meta-issue discussion being anchored in the related in-issues, we should look into the students’ understanding of the mathematics in question, i.e. the notion of primes.

Harry’s proof of the Chinese remainder theorem as well as his hand-in mathematical exercises show that he has a sound understanding of pretty much all of the elementary number theory presented in the teaching material, including primes of course. For Lola, the data sources are a bit more sparse, but fortunately the written data we have from Lola concern primes. In the mathematical exercise on deciding if numbers are primes or composites, Lola does quite a good job using also the ‘trick’ of sums of digits. For Andrew, we have the same exercise on primes and he also uses sums of digits a couple of times to rule out a number as prime. Also, we have Andrew’s proof of Euler’s theorem. Despite the fact that Andrew seemed to have missed some of the intermediate steps of the proof, his presentation still shows an understanding of concepts such as primes, relatively prime, and Euler’s ϕ -function. Thus, it seems fair to say that the reference point of the discussion is in fact an anchored one.

Of course the use of the word ‘prime numbers’ (and ‘primes’) in the discussion above may also be seen and analyzed as a form of ‘word use’, as described by Sfard (2008b). Judging from the above, it seems clear that all three students are on the object-driven stage concerning the (active) use of the mathematical noun ‘prime numbers’. They are all capable of inserting the noun into grammatically appropriate places, the use of the word is not linked to a specific routine, nor is it linked to certain phrases. In fact, this provides further justification that some kind of reification of this mathematical concept/object must have taken place for each of these three students.

The same seems to be the case for Gloria’s use of the word ‘prime numbers’ in the clip from the second supportive essay. Having seen and discussed Gloria’s understanding of the mathematics in her hand-in set with the historical exercises, it seems fair to say that she does have an understanding of various elementary number theoretic concepts including prime numbers. Thus, the mentioning of primes is in fact an anchoring point of Gloria’s more social meta-issue discourse in the mathematical subject matter. Of course, since she only uses the word once, it is difficult to see if the use is only phrase-driven. However, my guess is that it is more object-driven, since she uses the word in a context where she cleverly makes her meta-issue argument by referring to a situation involving mathematical in-issues. Also Jonathan’s use of the word ‘one-way function’ may be looked at as a kind of word use, in order to evaluate the appearance of actual anchoring in this potential anchoring point. Jonathan’s understanding of the concept of a one-way function must be based on his answers in the third questionnaire. As shown earlier, Jonathan seems to have a quite good understanding of the RSA one-way function since he answers “prime factoring” to question 13, and since he is able to deepen this answer in the followup interview by discussing the difficulty of breaking RSA. Thus, his remarks appear to be object-driven or at least phrase driven, but in any case anchored.

Evaluating the (Focus) Group Dynamics

One question that should be addressed is the effect of altering the composition of the focus group. As a point of departure for such an evaluation, I shall take the students’ own comments on the recomposition of the group, especially those of the three ‘old’ focus group students. When asked about the new focus group and how it had functioned, Andrew replied:

I think it has functioned fine. We’ve had some longer discussions this time, I think, compared to the old focus group. That was actually okay. Also I’d say that we’ve had some new eyes on the things, right, it was some different types sitting there compared to those in the old group, if you can say it like that. And of course that results in some new discussions and approaches. I think it’s been good.

Also Jonathan was quite positive in his evaluation of the new focus group:

Yeah, the groups were rearranged. I think that was good in some way or another, because there are some, at least I think we had some the previous time, there were many... some of the others who were goofing off with each other all the time, right. This time we had Harry who is a bit more serious, or what you’d say, and then there wasn’t as much fooling round. And also I work better with him than I do with some of the others.

The full clip of 12 minutes from the second supportive essay actually shows a somewhat typical situation, where Andrew and Gloria begin to goof off (appendix C.2, lines 39-45), but where the situation does not escalate as it usually would have during the first module. The reason for this is that Harry and Jonathan do not pick up on it, but instead continue to work on the essay, thereby making Andrew and Gloria concentrate on working a while longer.

Gloria admitted that the modified focus group had been more focused and she provided a reason for this as well: “Maybe it was because we didn’t have Sean. We were kind of silly when we were together; me, Andrew, and Sean.” Gloria seemed quite pleased with having ‘little Lola’, as she called Lola, join the group. However, she was not at all thrilled with being in the same group as Harry and referred to him as “so irritating”, explaining further:

Harry, he is a real nuisance to have in a group, because you can’t discuss with him. You just can’t. He gets kind of pissed when you are discussing, and such. And that I just can’t. I can’t comprehend it. It’s simply too heavy a load, because then it can’t amount to anything if you can’t kind of like debate different topics and reach an agreement at some point in time. He just gets really mad if you disagree with him, and then I get completely furious, and it’s just so *arrrrrrg Harry* and so... so it is actually not too good to be in a group with him, I think. But it did get better in the end, so in a way you could say that it was a social challenge, I don’t know.

Gloria then asked me what I had thought of the new group. To this I replied that I had found it to be somewhat more focussed than the previous one, to which Gloria then said the following about Harry:

That is true, he is good at keeping focus, he really is.

Well, I don’t know, but there seemed to be something with the two of you, and sometimes it actually also generated some discussions. And I’m glad to have that on tape, so I’m actually okay satisfied.

That’s nice. I just can’t discuss with Harry. Sometimes I get all quiet when he begins to say something. I just get like ‘well, it doesn’t matter’. I don’t feel like talking anymore. Or I just leave the group, that’s what I usually do in biology.

So concerning the modification of the focus group we may conclude that it did in fact make it more focussed. Besides the students own comments, the existence of a 12 minutes long focussed clip to some extent also proves this. Bringing in Lola seemed to pose no problems at all. She functioned well together with all of the other members of the group and she did prove to be more talkative than Lucy had been. Also, Lola herself seemed pleased with being in the group. When answering a question on what she thought about having to do the essays, she compared the focus group to her old group:

First time I really thought it to be a drag, but I think that was due to my group. We were two who had to do everything, which made it too big a task. But this time I thought it to be really good, and it was amazingly educative. We managed to treat the things in the book in a totally different way. So this time I liked it, but last time I couldn’t stand it.

(I shall return to the students’ experiences with the essays in chapter 8).

Harry seemed pleased to have joined the group also, when asked in the followup interviews: “It has been very exciting, and even though I sometimes wanted the work to

be a little more focussed, I still think that the way of working in the focus group was more focussed than before [in the old group].” Harry’s presence in the group, however, had both positive and negative effects. It seems possible to ascribe much of the stronger focus in the group to Harry’s presence, the videos suggests this and so do the three ‘old’ focus group students in the quotes above. Especially Jonathan seemed happy with Harry and the hypothesis that having Harry in the group would result in Jonathan being ‘heard’ more often in the discussions seem to be proven correct. Also the hunch that Harry and Gloria might be opposite poles and therefore would be able to generate interesting discussions seemed to prove true. The negative effect, on the other hand, is that this hunch possibly turned out to be a little too true. According to Gloria’s own statements above, the presence of Harry might have made her more quiet than usual. Of course, while being filmed in the focus group, she probably would not get up and leave as she would have in biology. Having observed the group for some time, I would say that Gloria might be exaggerating a bit about Harry in order to make her point, but of course they may have a prehistory of working together in biology class that we do not know. Still, I see what she might mean by referring to Harry as hardheaded. Harry was a very capable mathematics student, and his self confidence and self assuredness might very well stem from the fact that he was aware of that. Probably Harry is what math educators would refer to as *gifted*, something I did not fully realize when selecting him for the group. My selection was mainly based on the fact that he seemed to have a more serious attitude towards the meta-issues. Nevertheless, as an outside observer, I feel confident that if any of the students should be able to act against Harry’s ‘hardheadedness’, then it would be Gloria with her persistence and her ‘arrrrrgh Harry’ attitude. The video clip from the second supportive essay assignment shows an example of this: Gloria clings to her point of view concerning Diffie, which in the ends forces Harry to compromise. The video clip from the first supportive essay assignment shows a very different situation though. Here Gloria hardly ever intervenes in the essay discussion. One reason, as suggested earlier, might be that she is too absorbed in calculating her grade point average. A second reason may be what she refers to herself, that Harry’s presence sometimes makes her quiet. But a third reason might also be that Gloria did not read Hardy’s *Apology*, or at least did not read it well enough to discuss specific matters of it. Furthermore, the discussion in this clip, as described in the above analysis, takes a more mathematical and philosophical direction, while Gloria appears to be stronger in the sociological and psychological discourses of the meta-issue discussions in module 2, this being a possible fourth reason. Gloria said in the third questionnaire and followup interview that she had found the mathematics of the second module rather difficult, this also being the reason that she dreaded an oral exam in it. Thus, not feeling at ease with the elementary number theory might suggest a fifth reason for Gloria not engaging in this particular discussion, especially knowing that Harry probably mastered it quite well and that he therefore would have the higher ground, should the discourse become too mathematical. However, the hand-in mathematical exercises by Gloria suggests that she does have some understanding, even though she may have found it difficult. Thus, I lean more towards the other four reasons, though it is impossible to say which of these in fact is the dominant one.

Students' Answers to the Third Questionnaire

As in the previous chapter, I shall now go from considering the focus group in particular to consider the class in general, thus broadening the view. I shall do so by providing an overall picture of the students' answers to the third questionnaire. In terms of the students' understanding of the in-issues, I shall consider questions 6-10, 13, 14, and 16, as I did for each of the focus group students. And in terms of the students' comprehension of the meta-issues, I shall consider questions 15, and 17-21 (see appendix B.3). The remaining questions of the third questionnaire will be addressed in chapter 7 or 8. In presenting the questionnaire results I shall use the same index as previously:

one < few < some < many < the majority < the *vast* majority,

corresponding roughly to 0-5%, 6-15%, 16-35%, 36-50%, 51-80%, and 81-100% of the population, respectively.

I shall begin with the questions concerning the in-issues. As was the case for the focus group students, the vast majority of the class seem to have an understanding of the notion of private-key and public-key cryptography in question 6. In question 7, on the key-distribution problem, the vast majority of the students are also able to provide correct answers. For question 8, where private-key and public-key cryptography are used, the situation is not as clear-cut. Some students provide examples for both private-key and public-key cryptography. Some students provide only public-key examples (typically the Internet or home banking). Some students answer to why or how instead of where. And a few do not answer or answer that they do not know. The situation for question 9, what is to be understood by an algorithm, is just about as diffuse. Many provide acceptable answers, some describing an algorithm in terms of cryptography and others by referring to it as recipes, manuals, etc. Some, like Andrew, provide the definition from the book. A few provide wrong answers and another few do not answer. In question 10, what the notion of number theory covers, the majority provides answers like those of the focus group students. That is to say they list concepts like primes, composites, modulo, etc., but actually some also write \mathbb{N} or \mathbb{Z} . A few answer incorrectly and another few do not answer. For question 13, the property of division of integers which RSA builds upon, many do not answer at all. Some provide incorrect answers/guesses, but some also provide good or acceptable answers similar to those of Jonathan and Harry. A few say that they do not know or do not remember. One says that she cannot understand the question. Question 14 (the theorems on which the correctness of RSA builds) has a similar story. Here as well many do not answer at all and a few provide incorrect answers/guesses. Some students are, however, able to provide partially correct answers, mentioning one or two of the wanted theorems. And a few, like Jonathan, are able to provide the full list of Euler's theorem, Fermat's little theorem, and the Chinese remainder theorem. In the followup interviews a few students are able to mention one or two theorems in addition to their questionnaire answers. As for the last of the in-issue questions, question 16 on the relationship between the theorems of Euler and Fermat, there are again many who do not answer or say that they do not know. Some, however, do provide entirely correct answers, while some others provide incorrect answers. A few provide partially correct answers, e.g. by writing up the theorems and trying to compare them.

Let us now have a look at the six questions concerning some of the meta-issues of the historical case. Question 15 asked for the students' thoughts on why it might have taken

so long for the key-distribution problem to be solved (the solution being public-key cryptography), when the mathematics for the RSA had been available since the 18th century. Many students gave answers which ascribed this to the Internet, computers, or technology in general, saying that the need for it had not been as urgent in the previous centuries. Some provided other answers, for example saying that the people who could do it were not around yet, or suggesting that number theory was pure mathematics and back then not at all related to applications. Some did not answer or say they did not know.

In question 17 the students were asked what was to be understood by inner and outer driving forces. In the answers to this question, all students, except one who does not answer the question, seem to have an understanding of outer driving forces being something outside the mathematics itself, but which influences the development in mathematics. The examples they typically provide are wars, military, money, or the Internet. The understanding of what is meant by inner driving forces is more tricky. The majority of the students (including all five focus group students) link the term 'inner' to something inside the researchers themselves, i.e. their personal motivation, interest, etc., and not to something inside mathematics as an area or discipline. This confirms the prediction based on the essay assignments of the students' understanding of inner driving forces as being *personal* inner driving forces. Some students, however, do describe the notion of inner driving forces as it was meant to be understood, e.g. referring to unsolved problems within mathematics. One example is:

Inner: forces which drive the research in mathematics from the inside, e.g. the questions which mathematics seek to solve solely for the purpose of its own research.

Outer: the forces which affect research in mathematics from the outside, war for example is an outer driving force for mathematics.

In question 18 the students were asked what is to be understood by pure and applied mathematics, respectively. Some students' answers are very closely related to the definition given by Hardy in the *Apology*, e.g. by saying that pure mathematics cannot be used for anything whereas applied can, or by providing some of Hardy's examples of pure and applied mathematics. Some other students, however, seem to provide definitions which are more their own. Interestingly, all five focus group students belong to this category. A few examples of their answers are:

Gloria: Pure mathematics is algorithms, relations, etc. which are invented *without* any specific intention. As the name says, *pure* mathematics. The applied mathematics is (most often) the pure mathematics being placed in a situation where it suddenly is useful, and where new possibilities open up.

Harry: Pure mathematics is the foundation within mathematics, the stuff that lies behind all mathematics, e.g. Pythagoras. Applied mathematics is a more specialized form of mathematics which you can use in everyday life, or at least for something 'solid', e.g. coding theory.

A few link their answer to the previous question: one saying that pure mathematics only concerns the development of mathematics whereas applied affects the research in society; and another saying that pure mathematics is carried out due to a researcher's curiosity about something or due to his or her pleasure, and that applied is when the math is used for something, e.g. nuclear weapons or RSA. Some do not answer the question or say they do not know.

Question 19 concerned Hardy's *Apology*. The students were asked what Hardy's view on pure and applied mathematics was, and how he viewed number theory in connection to this. The majority of the students have reasonable answers to this question, showing that they have some idea of Hardy's views. Many of them state that Hardy thought pure mathematics to be the best, that it was superior to applied mathematics, that it was more important, that it cannot be misused in wars, etc. One example of the latter is: "Hardy ♥ pure mathematics. He didn't want that his mathematics could hurt others." Out of these students some also provide remarks on Hardy's views on number theory, saying that he thought number theory to be pure mathematics, the purest mathematics, or that it could neither be used nor misused. Some do not answer the question.

In question 20 the students were asked why they thought that two parallel and independent developments of public-key cryptography and RSA took place at the same time (Stanford & MIT vs. GCHQ). Though the answers to this question are many and varied, some groupings may be made. Actually, many students seem to think that it mainly is due to a growing need around this period. One student answers: "Because the time was ripe for it, because of the beginning of the Internet." Another says: "Outer driving forces demanded a breakthrough within cryptography." Some focus on the element of secrecy, e.g. saying that since the research of GCHQ was classified, the university researchers knew nothing about it, thereby implying that if the GCHQ research had been public, the others of course would not have repeated it. Some answers are a bit more off, saying that it was a race between the two, e.g. for patent or money. A few students do not answer.

And finally in question 21 the students were asked what can be said about the importance of basic research in pure mathematics, based on the story of RSA. Some students say that pure mathematics (often) make up the building blocks for future applications or applied mathematics. One of the more elaborated answers is that from Andrew: "The pure mathematics is the foundation for the development of the applied mathematics, even though this would question, e.g. in Hardy, how pure the pure mathematics really is." Some students focus on pure mathematics possibly becoming applied mathematics; that it may be applied, only we do not know for what yet; that it is relative to the context in which it is seen; or that you need (basic) research for development. Jonathan is among these and he answers: "You need to have some basic research in place in order to make ground-breaking discoveries/inventions, you might say." A few students provide more vapid answers, saying that it is always good to have some basic research. Some do not answer the question.

Recapitulation

Let us recapitulate shortly what information the second module has provided us with, also in addition to the first.

Concerning the first research question, on whether the students are capable of engaging in meta-issue discussions and reflections in terms of 'history as a goal', the above presentation and analysis of data confirms the results of the first module, i.e. that the upper secondary students are indeed capable of this, but at the same time extends the results by pointing to possible limitations of these capabilities. From the evaluation of the focus group's work on the first and second supportive essays, we see that the students are capable of carrying out their meta-issue discussions shifting between various

discourses: between philosophical (sometimes ontological) and mathematical discourses when discussing pure and applied mathematics, and between historical, social, and mathematical discourses when discussing (personal) inner and outer driving forces. In the process the students are also sometimes able to resolve meta-issue discursive conflicts which arise. As already pointed out, the students' discussions and answers to some of the essay questions become a bit home-spun and 'fluffy', especially that on inner and outer driving forces. This means there are some limits to the quality of the students' meta-issue discussions. And also for the degree of reflections they perform as part of their work on the essays; it is not necessarily as high as it could have been, had the questions not allowed for fluffiness as much as they did (I shall elaborate on this below).

The students generally seem to have a better hold of the problems at stake in the discussion of pure and applied mathematics (though some of the group's answers to this essay are subject to fluffiness too). One reason for this may be that the students could rely on Hardy's *Apology* in their discussions and answers of this essay. An example of fluffiness in the third supportive essay is when the students turn the discussion of whom should be credited for public-key cryptography and RSA into a matter of fairness, without suggesting and applying any criteria to base their conclusions on. Though the students had touched upon a similar discussion of multiple developments and credit (of the family of Hamming codes) in the first module, they may not have been adequately prepared for this question. At any rate, the core problems of the general topics and issues of the second module (pure versus applied mathematics, inner and outer driving forces, and multiple developments), though relatively easy explained, are not necessarily easy carried out or resolved. Even for historians and philosophers of mathematics and science a lot of untangling of the available conceptual frameworks and applied notions would be required, so in that sense it may not be so strange that the students' discussions are a bit fluffy. But despite the fluffiness and with knowledge of only one or two historical cases, the students still engage fully in the discussions and do quite a fair job in debating the issues.

Of course, for the non-focus groups we do not know as much about how they actually carried out their meta-issue discussions, but the hand-in essay assignments and not least the students' answers to the third questionnaire show evidence of meta-issue discussions and reflections having taken place. Concerning limitations of the students' capabilities in meta-issue discussions and reflections, the lacking answers to questions d and e (see page 6.5) of the main essay may tell us something. The questions asked what insights in the history of mathematics they believed could be obtained by approaching the history through inner and outer driving forces and objects and techniques, respectively. These questions are not easy, and not even the purpose of them may be easy to understand. In a sense they are on another cognitive level, since the students in order to answer them have to reflect on their own learning and relate this to the conduction of history of mathematics. In both modules some students were in fact able to say something about the carrying out of history of mathematics (their discussions of the when-and-who versus the why-and-how accounts), and in the following chapter on students' beliefs, we shall see examples of students who are able to reflect on their own learning (in terms of changes or developments in beliefs), but combining these capabilities seems to be a more difficult task for them.

Also in the second module the essay assignments in general appear to be a suitable setting for having the students engage in meta-issues of the evolution and development of

mathematics. The clip from the first supportive essay, in which the discussion takes on a quite philosophical discourse, even suggests a new element of these essay assignments. During this clip, Harry begins playing the role of Hardy, when responding to Lola's and Andrew's more inquiring remarks. This actually seems to function well for the group and it provides the students with a deeper insight into Hardy's views and his philosophical stance. From a design perspective, this implies a way of framing the students' meta-issue discussions in a sort of role play setting, where the students might take on the identity of different past mathematicians, philosophical schools, or historical methodologies as a way of adding an extra dimension to their discussions and reflections. And as seen from the analysis of the clip, such a role play seems to function quite well as part of an underlying mathematical discourse.

This brings us to the second research question, the anchoring of the students' meta-issue discussions and reflections in the related mathematical in-issues of the teaching modules. In the first module we saw that an anchoring to some degree can be ensured by the selection of the general topics and issues for the essays (the example of objects and techniques). However, we also witnessed that essays with very little or no degree of such insurance may still lead to anchored discussions and answers. One of the ideas of having a second teaching module was indeed to loosen the parameters of the modules to see to what extent an anchoring would still occur, i.e. by choosing general topics and issues not necessarily ensuring this. None of the general topics and issues of the second module essays had a built in insurance in the same way the objects and techniques essay of the first module had. Still, the groups' hand-in essays show several examples of potential anchoring points, especially in the essays involving the older mathematicians, where the students from time to time seem to have a need for mentioning, for example, the purpose of Euclid's algorithm or other number theoretic concepts in his books, Gauss' definition of congruence, and Euler's ϕ -function. Naturally the why-and-how account of the main essay allowed for the possibility of including in-issues, but unlike the situation in the first module, the when-and-who account in the second module also gave rise to several groups mentioning mathematical concepts and notions in relation to the various mathematicians, thus making these accounts less disconnected and less anecdotic. Concerning the question of whether these potential anchoring points in the written essays are actually anchored, the students' answers to the in-issue questions of the questionnaire suggest that at least some of them are. Though not all students see the deeper connection between the old number theory and RSA (the three theorems used to prove the correctness of RSA, question 14), the majority of them have some idea about what (elementary) number theory covers (question 10).

Also, the focus group discussions show clear signs of anchoring, despite the fact that the parameters of the essays were loosened. Particularly the clip from the first supportive essay with the discussion between Harry, Lola, and Andrew bear witness of that. This very discussion is a quite clear example of what was named an *anchored discussion* in the recapitulation of chapter 5 (in the first module that of the (7,4)-code). Jonathan's remark on the one-way function earlier in the same clip, however, is an example of an *anchored comment* (comparable to that of Hamming spheres in the first module). In addition to these two examples of the two kinds of anchoring found earlier, the data of the second module suggest yet a kind. In the clip from the second supportive essay, where Gloria changes to a mathematical discourse, in order to make the point of her meta-issue social discourse, she actually uses anchoring as a way to underpin and

substantiate her meta-issue claim – as an *anchored argument*. Concerning the question of the conditions on which such anchoring of the discussed meta-issues is possible, surely none of the above would have been possible had the teaching material (and the teaching) not focussed as much on the mathematics of the historical case(s) as it did. Furthermore, it seems that once the teaching material and the teaching is mathematically anchored, then this anchoring, in one form or another, will transfer to the students' meta-issue discussions.

Although the loosening of the parameters in the second module provided insights into what is needed for an anchoring of the meta-issues in the in-issues to be present, it also, as mentioned before, allowed for a higher degree of home-spun philosophies and fluffy arguments to flourish during the discussions. In a way the loose parameters made this legal. Thus, in terms of any future redesign and implementations of the second module, one idea would be to try to ensure a higher degree of anchoring, especially in the essay assignment on inner and outer driving forces. And at the same time, doing this in such a way that a confusion and mixing up of inner and outer driving forces with personal motivations would become less likely to occur. Perhaps asking the students questions on what kind of mathematics was already available on 'the shelves' when public-key cryptography and RSA was developed, and what (new) mathematics had to be created in the process, might do the trick. A similar question could be applied in the first module, where some of the students had trouble differentiating between techniques already available to Hamming (available on the shelves) and techniques he had to create in the process (not yet on the shelves). Also the notions of pure and applied mathematics might benefit in clarity from a definition given in terms of 'shelves': applied mathematics is the pure mathematics which is taken off the shelves to be applied. Of course, there is a very intricate balance between the notions of pure and applied mathematics, because they may change over time and because of the feedback mechanisms between developments in the two. Still, posing the questions for these general topics and issues in terms of availability on the 'shelves' would to a higher extent force the connections between the meta-issues and the in-issues to be dealt with. Concerning the general topic of multiple discoveries, an increase in quality of the third supportive essays on credit is certainly within reach. This probably only requires preparing the students a little better by providing some criteria for them to judge upon, e.g. the degree of a development's impact on the scientific community, instead of having them propose such criteria themselves.

Identifying factors which make the students' essays turn out either 'good' or 'bad', as suggested in section 5.7, is difficult, especially since so many factors are present during the students' work on the essays. However, the altering of the focus group from the first to the second module shows us that the composition of the groups may be one quite important factor. Of course, it is not easy to say how much the hand-in essays from the focus group would have differed had the group not been altered, but from a research perspective, the recomposition of the group paid off in terms of longer and more focussed discussions. And since such 'good' discussions may be taken as a prerequisite for the hand-in essays being 'good' as well, it seems reasonable to ascribe some importance to the group compositions.

One last question which you, as a reader, might be left with is what happened to Sean and Lucy now that they were out of the focus group. Well, they were placed in other groups, of course, Sean in group 6* and Lucy in group 2*. Sean managed to

'escape' the third round of interviews, but from his answering of the third questionnaire it appears that he may have had a better understanding of the mathematics in the second module than in the first (cf. also Sean's answer in figure 6.2). In the fourth and final round of interviews I asked him if things had been better in the second module compared to the first: "Well, in the beginning, I think so, yes. And I also think that towards the end I was more on top of cryptography than I was on coding." Lucy was placed in a group with two other girls, and according to herself, she actually had to be the talkative one in this group. On how the group functioned, she replied:

They are not the ones whom I work best with, but it was okay. They had to be kickstarted, then they did something.

But did you get the opportunity to say more? You said that you found it difficult to intervene in the conversation in the focus group.

Yes, but here I was the only one who said anything. I sat and had conversations with myself all the time, because the others didn't talk at all.

As seen in section 6.5, Lucy's group (group 2*) was responsible for several of the better and mathematically anchored answers to the questions in the final essay, so maybe Lucy did profit from the reorganization of the groups.

An important element of the study presented and discussed in this and the previous chapter concerns the generality and generalizability of the findings. A 'devil's advocate' – or just any skeptic reader – might say: Okay, I do believe your results for this particular population, but what about other students? How normal or how average are the students in this particular class really? Is the school some kind of an elite school? And what about the teacher, surely she must be rather extraordinary to engage in such teaching activities? For now, my short answers to these four questions shall be: approximately the same; rather normal; no; and, not really. In chapter 8 I shall deepen these answers as I reenter the discussion of the research questions in order to address the generality of the case-specific answers. With the present chapter, our story of the focus group(s) and the focus group students' work in the two modules ends for now. However, we are not finished with these students just yet. Since we know them so well now, all seven of them shall play an important part in answering research question 3 on changes in beliefs, from an individual student perspective. This will take part in the next chapter (chapter 7).

7 Students' Beliefs

This chapter concerns research question 3 of the dissertation: In what way teaching modules like the ones just described and discussed can give rise to changes in students' already existing beliefs about 'mathematics as a discipline', or the development of new beliefs. In this respect the term 'teaching module' (or module) covers everything from the teaching material and the intentions behind the design of it to the actual implementation, the setting and context, the involved activities, learning situations, etc. The way I shall go about dealing with the question is to provide overall descriptions of the students' answers to the first and fourth questionnaire and rounds of followup interviews – before and after pictures – and then compare the answers in them. In order to investigate if changes or new beliefs may be ascribed to the modules, I shall follow two different approaches: First, I shall follow the students' answers to those of the questions which were asked on several occasions in the students' questionnaires and followup interviews (not just in the first and fourth round). Second, I shall follow the focus group students individually through the implementations of the modules and all of the questionnaires and interviews in order to discuss any changes or new developments which might have occurred in their beliefs, as well as why these changes might have occurred.

As with the two previous chapters, this chapter will be concluded with an analysis, discussion, and recapitulation. There I shall discuss changes in the students' beliefs, also according to the beliefs literature discussed in chapter 3: first the more question and content specific ones, changes in the way the students hold their beliefs; second, the nature of these changes exemplified by individual students; third and last, the possible attribution of changes in beliefs or the development of new beliefs to the teaching modules. But before we reach the discussion section, there are lots of data to be presented and accounted for. I begin with an introduction of the questionnaire questions.

7.1 About the Questions Asked

As mentioned in chapter 4, some of the questions in the first questionnaire were taken from the study by Christensen and Rasmussen (1980) – a study on what was to become known as the 'three aspects' in the Danish upper secondary mathematics program of 1987 (cf. section 3.5).¹ The questions of the first questionnaire concerned the historical aspect as well as a few matters of the aspect of inner structures of mathematics and the aspect of modeling, the latter in relation to applications of mathematics.

¹ For a comparison of the students' answers today with those from 1980, see Jankvist (2008e).

Another inspiration source for the questions was the 2007-regulation's description of the 'identity' of mathematics: that mathematics is indispensable in many professions; that it is a platform for political decision making; that it is vital in everyday life; that it has accompanied the evolution of cultures since the earliest civilizations and human beings' first considerations about number and form; and that it as a scientific discipline has evolved in a continual interrelationship between application and construction of theory (cf. page 56). That is to say the questions were tightly connected to the idea of using 'history as a goal' (cf. section 2.3) as well as the dimension of beliefs about 'mathematics as a discipline' (since some of the questions were used to define this, cf. section 3.8). In this sense, the questions to be discussed in the following may be grouped under three topics:

- the sociologically oriented questions,
- the historical and developmentally oriented questions, and
- the philosophically oriented questions.

In the first questionnaire there were four sociologically oriented questions. The first was on the importance for people in society to learn mathematics (question 6, appendix B.1) and whether it can be considered more important for some people than for others. The second was on the application of mathematics (question 7), where the students were asked to mention places where mathematics is used in their everyday life or in society in general. In the third question the students were asked if they believe mathematics to have a greater or lesser influence in society today than a hundred years ago (question 8). And in the fourth they were asked if they believe mathematics to be a science, if so about what, and if not then why not (question 17). Concerning question 17 it is important to understand that the Scandinavian meaning of 'science' – in Danish called *videnskab* – is different in nature from the English word *science*. The Danish word *videnskab* is, as the German word *Wissenschaft*, much more inclusive and refers to practically all forms of systematic knowledge about nature (the natural sciences), culture (the Humanities), and society (the social sciences), whereas the English word *science* is more exclusive and generally only covers the sciences which use the scientific method to study nature (as opposed to, for instance, the social sciences which may use the scientific method to study human behavior). However, since there is no comprehensive English term synonymous with the Scandinavian word *videnskab*, I have used the English word 'science' in the translation of question 17, and the students' answers must thus be interpreted by the reader in this way. In Danish, the translation of *science* would be *naturvidenskab* ('nature science'), which covers the classical natural scientific disciplines such as astronomy, physics, chemistry, biology, and geology as well as hybrids like physical chemistry, biophysics, and geochemistry. In the translation of interviews and questionnaire answers I have used the word 'natural science' in the normal meaning of the word *science*, in order to distinguish *naturvidenskab* from *videnskab*. Thus, the reader should bear in mind that when the word 'natural science' is used in this chapter, it is usually not in the English meaning of the word. For instance, in a Scandinavian context, it is an open question whether or not *naturvidenskab* covers mathematics, whereas the English word *natural science* does not cover mathematics, since mathematics and logic are considered as belonging to the formal sciences. The exact formulations of the sociologically questions shall be presented along with the students' answers in the following sections. Questions 7, 8, and 17 were all inspired by Christensen and Rasmussen (1980).

There were four questions on historical and developmental matters of mathematics. Three of these took as their point of reference the students' mathematics textbooks. The first asked *how* the students believe the mathematics in their textbooks to have come into being (question 9). The second *when* they think it came into being (question 10). And the third *why* they think it came into being (question 11). In their study, Christensen and Rasmussen (1980) used similar questions asking *how* the students imagine the mathematics in the textbooks to have been created, and *when* they think it was created (no question on *why*). The fourth question was also inspired by Christensen and Rasmussen (1980) asking what the students believe a researcher in mathematics does, e.g. at a university, and what the research consists in (question 18).

Finally there were three questions of a more philosophical nature. The first of these concerned whether or not parts of mathematics can become obsolete and if so, then in what way (question 12). This question was inspired by Niss (2001a, p. 10). The next question was a more general one, an old and much debated one as such, since it asked whether mathematics is discovered or invented (question 16). This question was also asked by Christensen and Rasmussen (1980). Because the question was quite abstract, the previous one (question 15) was a more concrete version of it, asking if the negative numbers were discovered or invented.

Due to the rather abstract nature of some of the above questions, in particular questions 16 and 17, some of the students' answers are rather vague, indefinite, or ambiguous. For that reason it is sometimes difficult to interpret what the students are actually saying, something the reader may bear in mind. Nevertheless, often it is still possible to make meaningful observations concerning the possible changes in the students' beliefs. Another remark which should be made concerns the distinction between beliefs and knowledge, as discussed in section 3.8. To decide whether a given answer or statement from a student is a belief of his or hers, or if it is in fact a knowledge that (s)he has, is quite difficult, sometimes probably impossible. For that reason, and because research question 3 concerns the possible *changes*, I shall not make much effort to distinguish between knowledge and beliefs in my presentations of data in sections 7.2, 7.3, and 7.4. In the analysis in section 7.6 I shall, however, distinguish between beliefs and what I shall term 'views' (the explanation will be given there).

In the fourth questionnaire a few additional questions were provided in order to expand or deepen the students' answers to some of the above questions. Some of these new questions had come up during the followup interviews and were for that reason found relevant to be posed separately. I shall account for this in section 7.3.

7.2 First Questionnaire and Round of Interviews

In the following I shall focus on presenting the students' answers to the first questionnaire, and only comment on them occasionally (comments and comparison will be made in section 7.3). The numbers of the questions in the presentation refer to the numbers in the original questionnaire (appendix B.1). Concerning the readability of the questionnaire results and the relatively small population of the class, the students' answers will again be indexed according to:

one < few < some < many < the majority < the *vast* majority,

roughly corresponding to 0-5%, 6-15%, 16-35%, 36-50%, 51-80%, and 81-100%, respectively (cf. the students's answers to the second and third questionnaire in sections 5.7 and 6.7). Due to the ordering of the questions under the three topics discussed above, the numbers of the questions do not appear in the same order as in the questionnaire. I begin with the sociological questions.

The Sociologically Oriented Questions

6. *Do you believe it to be important for people in general to learn mathematics? If 'yes', for whom is it then most important and why? If 'no', why not?*

The vast majority believes that mathematics is useful/important for people in general to learn, and that it may be used in various places throughout the society and the everyday life – at least to a certain level, though not necessarily the level of upper secondary mathematics. For example one student answers: “For sure. Mathematics is everywhere, and knowing something about it only makes everything all the more easy.” Many believe that it is important to learn mathematics in order to function as a member of society: “Yes. Everybody. So we can get a well functioning society with wise people. The human survival depends on our brains, you know.” Some think it depends on the job you have and provide examples like architects, accountants, economists, researchers (statistics), business people, and people working with opinion polls. A few mention “logical thinking”. Less common answers are: “Everything is math, people say, and [I] can't see what can be accomplished by language [subjects]... It's thanks to math, physics, chemistry, etc. that the world has evolved to what we know today.”; “No, it depends on what occupation you want to have. For many jobs in the world it is not at all necessary to know mathematics on the level which we are learning it [in upper secondary school] – [I] think it ought to be more basic.” The followup interviews reveal no further information on the question.

7. *From time to time you hear that mathematics is used in many different contexts. Can you mention any places from your everyday life or elsewhere in society where mathematics is being applied, either directly or indirectly?*

The vast majority of the class mentions shopping, for example in connection with supermarkets and other places where you buy things. The majority of these also mentions construction of houses, bridges, and architecture in general as well as more economically related contexts such as banks, interest rates, stocks, salaries, taxes, and insurance companies. Some mention the role of mathematics as a tool in other subjects such as physics, chemistry, astronomy, science in general or in relation to school. Many mention the application of mathematics in relation to computers. One of the more interesting answers points to the steering system of the Copenhagen Metro. The student believes that mathematics must be applied somewhere since there are no drivers in the train. The same student lists several of the above given answers as well and then she ends by writing: “but I'm having difficulties seeing where mathematics on a higher level can be applied in everyday life.”

In the followup interview some of the students are asked how their answers to question 6 would have differed if the more ordinary uses of mathematics (the four basic operations of addition, subtraction, multiplication, and division as well as taking percentages, etc.) were not to be included. To this question a couple of students again mention

mathematics in computers. One student mentions the school as the place where such mathematics would be used, in mathematics classes as well as in formulas in physics and chemistry. A few students believe it to depend on the occupation you might have, for example that engineers, statisticians, and pilots will need to know more than just the basic arithmetic operations.

8. *Do you think mathematics has a greater or lesser influence in society today than 100 years ago?*

The vast majority of the students believe the influence is greater. This answer is in general based on the increased amount of technology in our everyday life. Answers as “definitely, more computer=more mathematics” and “everything develops and everything has to be high-technology” are often given. A few of those who believe that mathematics has a greater influence today also point to economic affairs as a reason, or that “the use of mathematics has become more advanced in our time”. Some think that mathematics has the same influence today as it had a hundred years ago, and only very few believe that the influence today is lesser. One of the more exceptional answers of the latter kind is: “No, I don’t believe that, because even though we use mathematics a lot more in space etc. we have modern machines to do it.”

The followup interviews largely confirm the beliefs described above. To the deepening question of why a student found the influence today to be greater she answered:

Because today you can, for instance, get an education at... or study mathematics at the university and things like that, and you couldn’t do that a hundred years ago. [...]

So it is something relatively new that you can study mathematics at the university?

No not new, but I do believe at a higher level. That is, you didn’t know as many things back then as you do today.

And you couldn’t get an education as a mathematician in the same way, you think?

No.

The student who argued for lesser influence due to the use of modern machines is also given the opportunity to expand on her view in the interviews. She finds, amongst other things, that mathematics appears less present because we rely on technical aids to a great extent, and because the use of mathematics is mostly about “pushing some buttons”.

17. *Is mathematics a science? If yes, about what? If no, what is it then?*

The vast majority of the students answer “yes” to this question. The arguments provided, however, are very different. For example: “Yes, there are researchers in mathematics.”; “Yes, it belongs to the natural science faculty.”; “Yes, it is scientific. I must admit that I don’t know why.” To the question of what mathematics is a science about, the students also provide various suggestions. Some believe it to be a science about numbers and/or figures. One thinks that mathematics deals with reality and facts. Another thinks it is a science about truth. A third that it is a science about logical thinking. And a fourth that it is a science about knowledge, maybe a natural science. Of those (some students) who answer “no”, several think that mathematics to a higher degree is a tool for other sciences. Some believe that it is not a science because it is a language. One student

answers: “No, I don’t think so. It is not something which you conduct research in, you know, you invent something.”

In the followup interviews the science versus tool problematique is explored further. Here one student says that he sees science and tool as the same thing, because you can create new things with the knowledge you already have. Another says: “I just think that science... I think it’s more with doctors and such, well medicine and stuff in outer space and that. But I think that mathematics is a good tool to calculate and figuring things out, finding the exact position of a star or something. Then you’ve got that knowledge.” The question of whether mathematics is a natural science or not also arises in the interviews. One student argues that “it is because they [mathematicians] use the same methods as the people in other natural scientific subjects. [...] They examine, demonstrate, and prove.” Another student is convinced of the opposite:

Mathematics is, I suppose, the tool to explore nature, and physics is very much about laws and so, biology is about, well yeah, biological things, and mathematics is then the tool you use to do research within the other natural sciences. I probably wouldn’t call it a natural science, but rather the tool you use in the other subjects.

This student, however, does not doubt that mathematics in itself *is* a science, in fact he believes mathematic to be a science as well as a tool.

The Historical and Developmentally Oriented Questions

9. How do you think that the mathematics in your textbooks came into being?

The majority of students believe that mathematics is due to people in history who have been wondering or been curious about something and therefore attempted to explain what they observed. Many of the students who think so, believe the people responsible for mathematics to be some special wise persons and great “minds of ideas”, a few mention Pythagoras. One student suggests that the ones responsible are “some very patient, half autistic people who have been wondering about the connections, rules, etc. between things. For example the angles of triangles, the lengths of the sides, and so on.” Some believe mathematics to be an accumulation of experiences, observations, and experiments, possibly anchored in nature. A few of these emphasize the cumulative nature of mathematics. Others believe that mathematics was created because of a need, for instance in connection to trade or “in order to make things more manageable”. The followup interviews do not add to the above description.

10. When do you think it came into being?

The majority believes that the mathematics in their textbooks came into being “sometime long ago”. The suggestions concerning exactly when are plentiful: “From even before da Vinci’s time!”; “When the numbers were invented”; “When we began using Arabic numerals”; “Way before it says in the books”. Some point to antiquity and provide as argument that, for example, “the construction of, for instance, the pyramids must have required at least some mathematics.” Out of this majority believing ‘long ago’, some share the perception that mathematics has always existed, or at least as long as human beings have been around. One says: “Mathematics in general has existed since the dawn of time, but highly developed [mathematics] has only emerged within the last 200-100

years.” Only one student believes the mathematics in the textbooks to be of a more recent date and he is not afraid to pinpoint this to “40 years ago”.

In the followup interviews, events in the history of mathematics were occasionally fixed within not too unreasonable orders of magnitude, for instance, the beginning of mathematics to 4000-5000 years ago; Pythagoras to the first couple of centuries; and Fermat’s last theorem to “the Middle Ages or something”. But only few students were able to do this. Whether this is due to lack of knowledge about history of mathematics or lack of knowledge about history in general, or maybe both, is difficult to say. Finally, one of the students seemed very strong in her belief that it was impossible to practice mathematics without the Arabic numerals. When asked why, she answered: “the mathematics you do today, you wouldn’t have been able to do that... [without the Arabic numerals]”.

11. *Why do you think it came into being?*

The majority of students believe there was a need to have mathematics at one’s disposal. A few even talk about a necessity: “For example with constructions, it has been important to be able to predict/calculate if, for instance, the walls can support the roof etc. Better to find errors on the drawing board than when the final construction collapses.” Many students mention the development of society and related aspects as the main driving force. Again, some mention that people have been wondering about something and then followed their ideas and impulses. One student ascribes the cause to “The will of God - or Big Bang, if you like.” In the followup interviews one student said: “Because people had too much time on their hands, for example, so they were given jobs as mathematicians.”

18. *What do you think a researcher in mathematics (at universities and the like) does? What does the research consist in?*

To this question, many of the students decide to pass. One answers: “I haven’t got a clue! But I have actually wondered about it and I’m quite puzzled.” A couple of other examples of ‘don’t knows’, though including suggestions, are: “I don’t know, but something with proofs.”; “Don’t really know! Examining if theories and formulas are correct. The use of mathematics to the world or something.” The idea of controlling the correctness of already existing formulas and theories actually characterizes a group of answers (guesses) to what the research consists in, namely that it is a kind of ‘clean-up job’. One student phrases it as: “To go through proofs and the like, old theorems, etc. Try to find ‘errors’.” Some do, however, believe that mathematicians might also propose new theories: “Find new and better formulas. Faster ways of calculating, or, for instance, ways of calculating something which we haven’t been able to calculate before.”; “Solve previously unsolved riddles (the case with Fermat’s last theorem). Propose new theories, etc.”

Many of those who in question 17 believed that mathematics was a science now have no idea as to what a researcher in mathematics actually does. Nor do a few of those who believed that mathematics is a tool. The few who believed that mathematics is a language are most inclined to say that research in mathematics is a kind of ‘clean-up job’. This belief that mathematicians spend most their time checking old theorems and formulas is one that also appears in the followup interviews. Here a student also says that if anybody conducts research in mathematics, “then it must be some very

complicated mathematics, I think.” A couple of others are more clear about what they think a professional mathematician does: “Well, then he chooses something [...] Let’s say he chooses the quintic equation, sees if there are any easy ways to... to find out how to solve it.” To the deepening question of whether it is meaningful to talk about ‘new’ mathematics, many thought that it is not. Often beliefs about this seem connected to the belief that mathematics is something which has always been, and that we are only exploring the already existing: “I think, well, that the mathematics is there, so in a way I don’t think you ever could create new mathematics, if the mathematics is there. You can discover new mathematics.” Many share the belief that mathematics comes into being, e.g. by way of discovery, because there is an application oriented need for it: “I believe that the mathematics we have now, we have that because, through time, there has been some needs for finding methods for calculating certain things, and therefore they have come into being.” Another answers: “I can’t really imagine that you found some mathematics and then thought that it couldn’t be used for anything...” And then there is one single student who short and concisely answers: “I just can’t imagine what their working days are like. Then they just sit there and gawp at their calculator. I think, it is just weird.”

The Philosophically Oriented Questions

12. *Do you think that parts of mathematics can become obsolete? If yes, in what way?*

To this question the vast majority answers a clear “no” or that it appears unlikely, for instance: “a proof is a proof” or “the basic things we build our mathematical development on are used and tested so often that it won’t become obsolete”. Some provide a “no” with modifications: “Don’t think that it can become obsolete, but that theorems/theories can be disproved and thereby provide a foundation for ‘new mathematics’.” Or more remarkable: “No, but there are probably some things which will not be used so much in the future. Such as vectors.” Only a few answer “yes” or “maybe”. The followup interviews provide no additional information.

15. *Are the negative numbers discovered or invented? Why?*

In the answers to this question the class is divided in two approximately equal groups, one in favor of discovery and the other in favor of invention. The argumentations provided are quite differentiated though. A few students believe the negatives to have been discovered in connection with or immediately after the positive numbers. Others believe that they always have been there, but that it might have taken some time to “learn to express them” or that people were “able to see it, but might have had difficulties explaining it.” Among the arguments for invention we find: “They are invented, I think, because you would get something wrong if they weren’t there”; “On the face of it, invented because you can’t have something which isn’t there”; “They are invented because you needed values smaller than 0”; “Think they are invented since it appears strange that a number all of a sudden should fall from the sky or something”. From time to time the same arguments are used for both discovery and invention: “Discovered. If we imagine a man who has bought a cow, but doesn’t have enough money, so that he owes money away, i.e. a negative number”; “Discovered. If you were in debt to someone, maybe”. One student plays it safer: “I’d think they were invented because almost all

mathematics is invented, but at the same time also discovered.” No extra information is provided by the followup interviews.

16. *Do you believe that mathematics in general is something you discover or invent?*

The majority of the students believe that mathematics in general is something you discover. Only a few believe that it is something you invent. Some students, though, believe that it might be a combination of the two. Many of those who believed that negative numbers were something discovered stick to this point of view for mathematics in general. A few of the answers are: “Discover. I don’t believe you can invent mathematics. It is something ‘abstract’ you find in already existing things.”; “Discover. Because mathematics is already invented. What happens today is only that you discover new elements in it.” A lot of those who believed that negative numbers were discovered and a few of those who believed them to be invented now seem to think both: “Many things might begin as an invention, but afterwards they are explored and people discover new elements in the ‘invention’ in question”; “Both, [I] think that you discover a problem and then solve it by inventing a solution or applying already known rules of calculation”; “You invent formulas after having discovered relationships”. Some of those who believed negative numbers to be invented now believe mathematics in general to be discovered: “Mathematics is all over – in our society, our surroundings and in the things we do. Therefore I do not believe mathematics to be something you invent, but on the contrary something you discover along the way. Of course, it might be difficult to say precisely, because where is the line drawn between discovery and invention?” One of the answers touch upon the question of what mathematics ‘really’ is: “Good question... very philosophical. I think there are many different standpoints to this. I personally believe that it is something you discover. Numbers and all the discoveries already made are all connected. So for me it is more a world you enter into than one you make.”

In the followup interviews the student responsible for the last remark explains further: “Well, I see it as if mathematics is kind of like there, like all natural science is, for instance, outer space. Outer space is there and now we are just discovering it and learning what it is. That’s what I think: It’s the same thing with mathematics.” When the remaining interviewees in favor of discovery were asked if the ‘exploration’ of mathematics corresponds to the exploration of the universe they all confirmed this belief. That is to say they believed mathematics to have existed always, or as one student phrases it: “Mathematics has always been there. In the form of chemistry or something like that at the creation of Earth. And then we haven’t found out about it until later.”

7.3 Fourth Questionnaire and Round of Interviews

One year and two teaching modules after the first questionnaire the students were given the fourth questionnaire, which was almost identical to the first (for the exact differences, see appendices B.1 and B.4). I shall follow the same approach in presenting the results of this questionnaire and round of interviews as I did for the first, the only difference being that I shall compare the students’ answers to the two questionnaires as I proceed. Notice that due to additional questions in the fourth questionnaire, the numbering of the questions may differ from that of the first.

The Sociologically Oriented Questions

6. *Do you believe it to be important for people in general to learn mathematics? If 'yes', for whom is it then most important and why? If 'no', why not?*

The majority of the class answers clearly “yes” to this question. The reasons given do not vary that much from those provided in the first questionnaire. Some say because it is in your everyday life, or because you use it every day. One mentions “Something with logical thinking and formulas.” Another says yes because “it ‘introduces’ a different way of thinking”. And a third says: “Yes, it is important for everyone to learn mathematics to a certain degree, so you can manage fairly well in post-modern society. Besides, there are many jobs and study programs where it is an advantage to know mathematics.” Many give a yes with modifications, saying that only more basic mathematics such as arithmetics is important to learn, and that the need to know mathematics to a high degree depends on one’s occupation: “Only some mental arithmetic when people need to plan their budget or do some shopping. But being able to differentiate functions, I hardly think my dad, for instance, has ever needed to do that.”; “To a certain degree it is necessary for everybody in order to function properly in society, just like you need to be able to speak the language spoken in society. Only some kind of basic mathematics.” Only one student answers a clear “no” to this question, and as a reason he says: “Not everybody needs it.” The answers of the first and fourth questionnaire are not too different, though a few more students provide a yes with modifications in the fourth. The followup interviews do not provide additional information.

7. *From time to time you hear that mathematics is used in many different contexts. Can you mention any places from your everyday life or elsewhere in society where mathematics is being applied, either directly or indirectly?*

The answers to this question are very similar to the answers given in the first questionnaire: shopping, sales, trading, forecasts, weather forecasts, insurance companies, banks, accounting, computers, foreign currency, bills, economics, architecture, DSB (the Danish railway system), graphs, and so on. One student writes: “The building I am sitting in right now² is built using a lot of mathematics, both directly in the construction of it, but also [indirectly] regarding the financing of it. When I shop in the canteen, I use mathematics. Math is practically everywhere in our society.” Another student: “When you see nice graphs around you there is a lot of treatment of data processing behind. You also use a lot of mathematics in, for example, computer game development.” Only few students mention things which have a direct link to the teaching modules: “Programming, coding, cryptography.”; “Debit cards, mobile phones, computers, and so on.” Nothing is added to the above from the interviews.

Thus, question 7 received similar answers in both rounds of questionnaires and interviews, the only difference being that a few students in the fourth were able to provide examples relating in some way to the two modules. Interestingly, this was something that happened in an even larger scale when giving the students the next, new question inspired by the first round of followup interviews.

² Ørestad gymnasium is located in a new building, finished in 2007, which is a very modern, high-profile piece of architecture in a new part of Copenhagen.

8. *Not counting the ordinary types of calculation (the four basic arithmetical operations, calculation of percentages, etc.) where do you then find mathematics applied in your everyday life and society in general?*

When differentiating between basic and more advanced uses of mathematics as done in this question, as opposed to question 7, more students seem to recall what they have been exposed to in the two teaching modules. Many students now answer things like: “Cryptography and coding theory used in communication, computers, and the like.”; “The Internet, home banking, mobile phones, computers.”; “In computers, encryption, decryption, in mobile phones.”; “Cryptography is being used.” Of course many students still mention computers in their answer, but in comparison to the situation in the first round of interviews, computers are not the only thing the students mention now. Some of the other answers to question 8, not referring to the mathematics of the teaching modules, are: statistics, forecasts, polls, insurance, construction, research, science, etc. and of course “Same as in question 7.” A few students simply leave the lines for answering blank. The interviews reveal nothing further in this respect.

9. *Do you think mathematics has a greater or lesser influence in society today than 100 years ago?*

The vast majority answers “greater”. Computers, advanced technology, and increase in communication are among the common reasons provided. Examples of answers are: “Greater influence because [...] it in our time is all about creating new things that makes life easier.”; “I think greater since the technology has evolved and it does that by means of mathematics.”; “I think it has a greater influence today because we use mathematics to protect ourselves, the codes developed by the mathematicians.” A few provide answers with modifications, e.g. “You apply it more, but it has always been equally necessary.” Some believe the influence to be the same, and only few answer “no”, e.g. “I don’t think so, we just use it a different way today than before.” This time round also, we get an answer saying “lesser” due to technical aids, but from a different student than in the first questionnaire: “[I’d] say the same or a little less, because some years ago people spent more time on solving stuff. Today we have many aids.”

In the followup interviews, some of the students are asked to expand or explain their answers. One of the students who thought the influence was the same says:

Well, in some way you kind of need it the same. We need it today, because we know more, but I think it had an equal importance back when you had to build the pyramids, where you also used... It’s a difficult question. Because it’s like... well of course there is a bigger need for it today in some way, but compared to the development then and now I think it is fairly equal.

In case an interviewed student had provided different answers in the two questionnaires (s)he was confronted with this and asked if (s)he had changed his/her mind, and if so why. One student who used to believe that mathematics has a lesser influence in society today than a hundred years ago now answered the opposite. When asked why, she said:

I don’t know, but I just think now that... maybe I’m coming from the things with applied mathematics and all that, right. So I think that mathematics is being applied more now as opposed to earlier, where you knew there were some arithmetic rules. And there are still some equations, and other pieces of mathematics which cannot be solved today. But back then there was more

as compared to today. So a huge development has happened, like all that about cryptography – which we saw – and the Internet, our banking systems, and so forth. Yeah, back then they didn't have all that, so mathematics has a greater influence today, because it is more in our everyday life.

Worth noticing here is that the student draws several links to elements of the teaching modules (applied mathematics, cryptography in the Internet and banking systems) as being important to her change of view concerning question 9. In fact, she seems capable of exemplifying her view by referring to elements of the historical cases of the modules. In the first round the students would sometimes refer to 'applications of mathematics' or talk about where mathematics was 'applied', since this was the phrasing of question 7, but never did they talk about 'applied mathematics' as something in itself, as opposed to (pure) mathematics. And nor did they in the second round. Thus, it seems fair to assume that this is something they have picked up from the second module and Hardy's *Apology*.

20. *Is mathematics a science? If yes, about what? If no, what is it then?*

The vast majority of the class answers "yes". The answers to what it is about are quite varying, though, examples are: a theory about numbers, theories, proofs, revealing connections, making analyses, give predictions of aspects of the future, and the real world. A few think "yes" because you can do research in it, or because it has departments in universities. One student reasons that it must be a science since it is so complex. Another that it is because you cannot analyze it like you can with a book. Some provide positive answers, but with reservations: "Yes and no. It is a necessary tool in science, but pure mathematics is also a science in itself."; "In a way yes, because it is kind of an invention." Only a few answer "no". Of these, one stick to her belief that mathematics is a language, and another says that mathematics is logic. I shall display some of the students' comments from the followup interviews in section 7.5 when discussing the focus group students individually. Also in the first questionnaire, the vast majority of students considered mathematics to be a science. However, the answers provided in the fourth round appear different in nature from those in the first (something I shall return to in section 7.6). One very clear difference is that much fewer students (only one) in round four say that mathematics is not a science because it is a language.

The question of mathematics being a natural science (*naturvidenskab*) or not had come up so often in the previous interviews that I decided to include it as a separate question in the fourth questionnaire, in order to get an overall impression of the students' beliefs about this.

21. *If you say mathematics is a science, is it then a natural science? Why or why not?*

Some students pass on this question, including a few of those who answered "yes" to question 20. The class is about equally divided into students who believe mathematics to be a natural science, and students who do not. Among the answers of those arguing that it is not we find: "It is *NOT* a natural science! Not in any way. Applied mathematics is part of other natural sciences, but mathematics in itself has *nothing* to do with nature."; "Don't know why it should be a natural science."; "Yes and no. [It] uses natural scientific methods including empirical ones. On the other hand, mathematics is most often used to underpin natural sciences. Gray area!" Arguments saying that mathematics is a natural science include: "Yes, it can describe reality."; "Yes, natural science since mathematics

is the foundation of physics, chemistry, and also a little in biology. If you had no mathematics you couldn't have formulas in physics and chemistry.”; “Of course it is a natural science since you often use empirical approaches to find out something within the subject. But mathematics does not set up any experiments as do the other natural science disciplines.” As may be noticed from the displayed answers, a few students are again able to exemplify or justify their answers, e.g. by referring to elements from the modules such as ‘applied mathematics’. The interviews reveal no further information to this question.

The Historical and Developmentally Oriented Questions

10. How do you think that the mathematics in your textbooks came into being?

Again many students believe that mathematics is due to people who have been wondering about something, people who have been looking for “connections” and people who have tried to explain their “observations”, tried to solve “mathematical problems”, or people who have been “experimenting”, e.g. by way of trial and error. Many students also speak of a profound need for mathematics. These answers are more or less summed up by the following quotes, the first of which also makes a reference to the teaching modules: “Curious people who have conducted research! And then in connection with war (coding theory). Generally just because there was a need for the mathematics.”; “I imagine that some need for a formula, a connection, or a new concept has arisen. Then a somewhat psychotic man has chosen to let himself become obsessed by the problem, devoted his life to it, and hereafter solved the riddle.” One significant difference from the first questionnaire responses, however, is the decrease in answers ascribing the mathematics of the textbooks to special, wise, and great-minded people alone. Actually, the only answer which comes close to this is the previous one mentioning the “somewhat psychotic man”. Some students also focus on the fact that the mathematics has been written down and passed on: “Mathematics, I suppose, is formulated and invented by someone who then taught it to others who then wrote the textbooks. Or the inventor wrote the book himself.”; “Research. People write it down.” A few seem more concerned with (new) mathematics building on older mathematics: “Everything builds on something else; the axiomatic and hieratic system. So basically it came into being a long time ago and since then it has been reformulated and so on.” A few do not answer the question. The interviews add nothing further to the above.

11. When do you think it came into being?

The majority of students still think “long ago”. Out of these, some emphasize that different parts of mathematics were developed at different times; that it is developed continuously throughout history; or that new things keep popping up. As opposed to the first questionnaire, some of the students seem more willing to fix dates and years, which may be seen as a sign of an increase in the students’ need to justify their answers (even though they may not be a sign of actual or certain knowledge): “Whew, many millennia ago”; “1000 B.C.”; “Once upon a time in ancient Egypt, where they constructed pyramids.”; “From before the time of the Egyptians, so over 6000 years ago.”; “Long before our time, 800-600 B.C. and a bit, or something like that. Began, amongst other, with the Babylonians and the Egyptians.” One similarity with the first questionnaire is, however, that some students think mathematics has existed always, e.g.:

“Since the dawn of time.”; “Mathematics in one form or another has existed always.” A few also point to the need for mathematics in trading. One of the more exceptional answers are: “When people defined it.” Some pass on the question. The followup interviews add nothing further.

12. *Why do you think it came into being?*

The majority believes that mathematics came into being because there was a need for it, which is in agreement with the picture provided of the students' answers to question 10 (a few students actually just refer to their answers to question 10). A few students again mention people's curiosity about and interest in mathematics as reasons. A few students believe that it is in order to be able to teach it to others. Some of the more uncommon answers are: “Because people want a model for solutions to problems!”; “Because it was found that by means of mathematics everything works better, and you have a chance of predicting some things which might be to the benefit of us all.” One of the more persistent advocates for ‘discovery’, who believed that mathematics always has existed, writes: “It did not ‘come into being’, it is a way of describing reality (or creating a fictive reality).” Overall, the answers to question 12 do not differ much from those given in the first questionnaire, and the interviews revealed nothing further.

Three new questions concerning the history of mathematics were inserted in the fourth questionnaire, questions 15, 16, and 17.

15. *When do you think people began using negative numbers? a) around 2000 B.C., b) around 300 B.C., c) around 1400, or d) around 1800.*

The majority of the class answers b, around 300 B.C., to this question. Some students answer a, around 2000 B.C., and some students answer c, around 1400 (being the approximately correct answer). No students answer d, around 1800.

16. *When do you think negative numbers became fully accepted within mathematics? a) around 2000 B.C., b) around 300 B.C., c) around 1400, or d) around 1800.*

Many answer c to this question, around 1400. Some answer b, around 300 B.C., and some answer d, around 1800. No students answer a, around 2000 B.C.

The idea of questions 15 and 16 was to ask when the negative numbers began being used as techniques, and when they were accepted as objects within mathematics, respectively. However, in retrospect, the phrasing of question 16 probably was not the most felicitous, because what is meant by ‘accepted within mathematics’? My original intention was to refer to the more formalized aspects of mathematics with the ordering of numbers according to each other (\mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C}) etc. However, surely the use of the negative numbers were well accepted before the 1800s; Euler and his contemporaries, for instance, had no problems in using them. And even earlier than this the negatives had been used as means for finding solutions to equations, although they were not accepted as solutions (e.g. with Cardan in 1545). On top of this comes the fact that questions 15 and 16 probably should have specified that they concerned European mathematics, since Chinese (and probably also the Indian) mathematicians had used negatives long before the Europeans. For these reasons it is not fair to judge the correctness of the students' answers to questions 15 and 16. Fortunately, however, it is still interesting to look at the correlation between the students' answers to the questions, i.e. whether they believe the acceptance of the negative numbers to have occurred before, after, or around the same time of the first uses of the numbers, and in case of differences then how large

they consider the ‘jump’ to have been. Out of the majority who answered 300 B.C. (b) in question 15, only a few answers b in question 16. This suggests that the students assume a difference in time between the beginning use of the negative numbers and the acceptance of them in mathematical circles. Most of the students believe this jump to be to around 1400 (c). Only a few answer around 1800 (d). Of the students who answer around 2000 B.C. (a) in question 15, only one answers around 1400 (c) in question 16. The rest answer around 300 B.C. (b). Of those who answer around 1400, only one answers the same (c) in question 16. The rest all answer around 1800 (d). Thus, the vast majority of the class only jumped one ‘period’ in their answers from question 15 to 16. A few jumped two ‘periods’, and a few answered the same in both questions. One student jumped three ‘periods’, and when asked in the followup interview why she believed it to be so, she replied:

Because I think that... firstly, I just think that you’ve had to use negative numbers for a real long time. But probably mostly related to trade and such, and then I think that the mathematics didn’t grow above what it had been earlier until around this time [1800s]. And at this time you needed it more in connections to physics. So I think that’s the reason, but I have no idea whatsoever. It’s just...

Okay. But it’s a jump of 3500 years.

Yes.

Despite this student being off ‘some years’ in question 15, her reasoning about question 16 as well as her reckoning that there had to be a time span in between the two is not too askew. This is actually something which is characteristic of the students’ answers; that they generally reason sensibly about this matter of use and acceptance concerning a mathematical concept (object). One student explains it like this:

Just as you see so many examples that it takes time for things to actually become accepted – at least accepted in society... that there is something new. So I think it’s been like that with the negative numbers as well. And I believe to recall that people kind of didn’t really want to calculate with them before late in history. So that’s why I answer differently in the two questions.

17. *From when does the coordinate system as we know it originate, do you think?*

Many students pass on this question. Some do, however, answer more or less correctly, saying between 1400-1600, or 1500s. A few believe the 1800s, or 1700-1800. One believes 1300s, and one believes 1200s. And then a few believe it to be very old answering things like: “The old Greece, approximately between 500 B.C. and year 0.”; “200 B.C. (Archimedes).” This question was not brought up in the interviews, it was primarily put in the questionnaire to get an idea of the students’ knowledge about the history of mathematics. In this sense, the question was somewhat along the lines of the meta-issue questions of the second and third questionnaires (cf. sections 5.7 and 6.7), but unlike these it did not relate to any of the historical cases of the modules. Thus, this question together with the answers of questions 15 and 16, taking into consideration the phrasing of these, may give us an idea about the students’ general knowledge of the history of mathematics. In this respect the students’ answers to these questions further add to what the answers to question 10, 11, and 12 suggest, namely that the students are not too familiar with the history of mathematics in general. Only one fifth of the class were

able to answer correctly to question 17. However, from a research perspective this is interesting, since it thus may be assumed that the majority of the students have received most of their historical knowledge of mathematics through the two modules.

22. *What do you think a researcher in mathematics (at universities and the like) does? What does the research consist in?*

In the first questionnaire many students decided to pass on this question. This is not the case this time. In fact, only one student answers “don't know”, and even she provides a few guesses: “Do research in numbers. I donno. Check formulas.” Another difference in the answers between the questionnaires is that only a few students now talk about mathematicians' primary function being to control old theories and results. Instead, students now talk about developing existing mathematics further, solving unsolved problems, finding new proofs, new formulas, new methods, new models, new relations, and finding and examining something unknown. Some of the answers may be linked to either the teaching modules, e.g. by the topics of the essay assignments, or to the topics of the previous questionnaires, e.g. invention vs. discovery or questions about what history of mathematics is (see section 7.4): “Put forward new models. Invent new techniques. Discover new objects.”; “Make proofs and develop the mathematics further and also INVENT something new, for instance. And try to solve previously unsolved ‘stuff’/formulas.”; “That they might study pure mathematics and try to solve mathematical problems which has not been solved yet.”; “Both pure and applied mathematics, I think.”; “If history, then they do history. If not, then they do applied mathematics.”

As was seen in the previous chapters, what goes down on paper is not necessarily representative for the thoughts behind the answers, or at least the written answers may be explained better. One example of this was a student who answered “investigations and further development” to the question, but when given the opportunity to explain himself in the interviews, he was able to exemplify his answer in the following way:

Well, it is kind of like, for example, when you have someone like Shannon who predicts something, and then you have Hamming who... well goes in and sees how he can use an area of mathematics to investigate, for example, the (7,4)-code, and then develop it to a different form of mathematics, which then turns into something – the process in all that. To get the idea, play with it, then arrive at something, and then investigate it afterwards, right.

The Philosophically Oriented Questions

13. *Do you think that parts of mathematics can become obsolete? If yes, in what way?*

The majority answers “no” to this question, though some provide a “no” with modifications, saying that you might discover you were wrong; that it may become more developed as time passes; and that the basic arithmetic is still being used. One student uses elements from the second teaching module to exemplify his views: “Not all of mathematics, but parts of it which you might not need anymore. Caesar encryption has for instance become obsolete today, because it is too simple. You have a need to invent a new cryptographic method.” A few are undecided: “Don't know, but don't think so.”; “It is difficult to say, but if you suddenly find a new way of describing/calculating something you may not need the previous [methods]. On the other hand, the theorem of

Pythagoras is still ‘top notch’.” Some students say “yes”, however, the arguments used to do so are actually not that different from the ones used to argue for “no”. In the first round also, most of the students had answered “no” to this question, but a difference is that a few students in the fourth round are able to exemplify or justify their answers (beliefs) using elements from the teaching modules.

18. *Are the negative numbers discovered or invented? Why?*

This time round the majority of students say “invented”, although many still say “discovered”. A few students say they do not know. One student says “both”. The arguments for discovery are amongst others: “Discovered since they have been there as much as the positives have.”; “I actually think they are discovered. Since the negative sign is part of *so* many formulas and connections today, I think that it is a part of mathematics that couldn’t avoid being discovered. A very essential part of mathematics which would act as a limiting factor did it not exist.” Examples of the arguments for invention are: “Invented because you noticed that there had to be something opposite the positive numbers.”; “It is invented since you cannot find something which isn’t there.”; “Invented. It is not real to talk about negative numbers as such. But a discovery in the already invented number system.” Some of the students still refer to a need having been present, e.g. in connection with trade, money transactions, etc.: “Invented because there was nothing named so at first. It was something you had to invent, so that there could be a negative balance on your account.” The interviews revealed nothing further.

From the first questionnaire till the fourth there is an increase in answers favoring invention of the negatives. As previously mentioned, one purpose of question 15 (18) was for it to be a more concrete ‘warm up’ for the question on discovery or invention of mathematics in general. Another purpose, however, was to test the consistency of the students’ answers to the two questions.

19. *Do you believe that mathematics in general is something you discover or invent?*

To this question many actually answer “both”, or that it is “a combination”. Examples of answers are: “Both actually, but at first you discover and from that you invent things.”; “Both. You have a problem to solve → invent. Solve the problem mathematically → discover.”; “It is a mix, and I don’t think you can give a definite answer to it.” One of the answers relates directly to the teaching modules: “It is both. You discover objects but invent techniques. So the question of discovery or invention depends on the scale of the perspective from which you look at it.” The rest of the class is equally divided between some believing “invention” and some believing “discovery”. Arguments for discovery are: “Discover, because the Earth is connected somehow, and has gravity etc. And then it is up to humans to find out, for example, how you can resist it and so on.”; “Whew... Yes, we’ve been over that one several times. But due to my answer before [answer to question 18: discovery] I might think, in this moment, that the *very* basic things were something you discovered, and that everything deduced from those are invented.”; “Discover, and afterwards you invent a solution.” What these quotes show is that even those students who answer discovery still seem to think that it is some kind of a combination. Furthermore, a general tendency seems to be that students believe that discovery precedes invention. The students who say “invention” generally do not provide arguments for their beliefs under question 19. All these students’ beliefs,

except one, are however consistent with the answers in question 18, where some of them already have provided reasons. Some students who answered “invented” in regard to the negative numbers in question 18 believe it to be a mix for mathematics in general in question 19. A few students who favored discovery in question 18 believe it to be a mix in question 19. A few students who believed in invention in question 18 answer “discover” in question 19. Only one student who is in favor of discovery in question 18 believes mathematics in general to be invented in question 19. And only one student answers “both” in both question.

One of the students who believed mathematics in general to be something “invented and defined” was asked why he thought so, and in the process confronted with his previous answer to the question:

Because I believe it to be some rules you lay down. That you kind of like build your own system.

Okay. In the first questionnaire, a year ago, you said: ‘Both, you invent formulas after discovering connections’. So have you changed your view on this?

No, well, it’s clear that if you drop a thing, then it falls with a speed, and you can observe that. And then you can describe this by means of the system which you have built within mathematics. So the mere fact that there is a connection is not something we have invented or anything, it’s just like that. *But the example you are giving there is related to physics.*

Well, but mathematics then that’s an invention. There is no meaning.

Okay.

You just describe a lot of connections which have been discovered with something you have invented. It is just a tool that you have invented, like a hammer, or...

So what this student ends up saying is that invention comes before discovery and not the other way round, like he did in the first questionnaire. But surely the question of discovery vs. invention is a very difficult question and students may often say things which contradict each other. A few times I was able to confront students with such potentially conflicting views (beliefs) that they seemed to be holding. One student who believed mathematics in general to be discovered had answered “researchers at universities, in the old days” to question 10 on how mathematics came into being. I asked her exactly what she believed the job of these researchers to be, if their job was solely to discover things:

It’s true that it’s difficult to sort out, but kind of like discover... if you discover something, then it’s exactly, has kind of already been there. If you do research in something, then... I don’t know, you find something new, I suppose. But you sort of use some rules which you’ve used before, and then it goes on to something new. But then it’s still discovered, because you build on old theories and such... No, I don’t know.

But are they like archaeologists then, digging a little into mathematics?

I don’t know, I’ve never really tried it. [...] I’m thinking a lot about that movie with the guy who gets schizophrenia and that, where they have a big project... I can’t remember the name of the movie?

‘A Beautiful Mind’.

Yes, that one. They kind of have this project where they were to find

something new, or show, or prove something new, right. And I think that is doing research.

I took the opportunity of framing my further questions around this film. The student agreed with me that John Nash, the mathematician portrayed in the film, was a creative man and that his job too was based on creativity. I asked her if she saw any conflict in believing mathematics to be discovered on the one hand, and created by a human being on the other:

I just think that no matter what... like when he is standing at his window and drawing the bird's something, or how they play ball or whatever, then he is using mathematical rules for how things are connected and that. I know that he gets new theories and such... but I don't know, it's a little difficult. Well... I suppose you could say that he invents something new.

Only I wouldn't say that because...

But you don't have to say it to please me.

Well...

Not at all.

Okay. I just still think that it is discovered.

The idea here was to apply a little pressure on the student and her beliefs, in order to see what happens when students are confronted with potentially conflicting beliefs – in this case the problem that believers in ‘discovery’ may have in accounting for the creative element of research in mathematics as a human activity. Despite the student seeing that ‘invention’ may to a larger degree be compatible with this creativity, she is not ready to commit herself to this view. I shall not be the judge of one belief being better than another, or even whether there in fact is a conflict or not. But the dialogue clearly shows that beliefs can be hard to change, even when students seem to acknowledge conflicts in their own belief systems.

For question 19 (16 in the first questionnaire) there is a shift in answers from the first to the fourth round. In the first, the majority of the students favored discovery and only a few invention, although some believed it to be a combination of both. In the fourth, many believed it to be a combination, some favoring discovery and some invention. Thus, there is an increase in students who believe it to be a combination and students who favor invention. This seems to be consistent with the increase in students answering invention to the question on the negatives. Thus, we may talk about an increase in consistency in the students' answers, which is further supported by the fact that fewer students seemed to think that mathematics has always existed (question 11). I shall examine this point further in the following sections.

7.4 Questions from the Second and Third Questionnaires

The previous sections have provided us with overall impressions of the students' beliefs and views before and after the implementation of the two teaching modules. During the presentation of the students' answers to the fourth questionnaire, a comparison was made concerning differences and similarities between the students' answers in the two rounds. In order to make it possible to confirm or reject the presence of changes in students' beliefs, or to say something about the development of the possible changes or new beliefs coming into being, it is relevant to also look at the related questions from

the second and third round of questionnaires and interviews.

Most of the questions from the intermediate questionnaires were closely connected to the treated historical cases, and are therefore not 'independent' of the teaching modules in the same way as the questions in the first and fourth questionnaire. However, as already indicated, the second and third questionnaire contained a few items from the first questionnaire. To be more precise, they contained the sociological question on mathematics being a science, plus a few other related questions, which is interesting from the viewpoint of tracing developments in students' beliefs. Also, they contained a couple of questions on the discussion of discovery versus invention, plus a few related questions. Although these more philosophical questions were quite case-specific, they are still interesting in the sense of testing for consistency and justification in the students' answers. But first the sociological questions.

Are Mathematics and History of Mathematics Sciences?

The second questionnaire included a small handful of additional questions relating to the main question of mathematics being a science (*videnskab*), for which the exact formulation was:

18. *Regardless of what you answered in the previous questionnaire, do you think that mathematics is a science? Has your opinion on this changed?*

The majority of the class says that mathematics is a science (a small decrease from the first questionnaire). Some say that it is not. A few are indecisive. Many students provide arguments for their answers, even though they were not directly asked to. These do not differ much from those shown in section 7.2, and for that reason I shall not go into details. Many students do not answer whether their position on this question has changed or not. Some say they cannot remember what they answered the last time. No one says their opinion has changed. Question 18 was not brought up in the followup interviews. The next question however was.

19. *What about the history of mathematics, is that a science?*

In the questionnaire answers, the majority thinks that history of mathematics is not a science.³ Examples of answers are: "No, history is retelling a story. But history of mathematics is a story/history about science!"; "No, it is history and not something you can explain with numbers." Some think that it is a science: "Yes, I believe so since you often hear about research in history."; "Everything is science. The question is: What kind of science? Maybe history of mathematics is history science." A few are indecisive: "If you can call history a science, you can say the same about the history of mathematics. But I'm not sure if you can do that."

In the followup interviews, one of the students who did not believe history of mathematics to be a science, but was unsure why, gave the following explanation:

If the history of mathematics is a science? Science that is, well I don't know...

I think that it isn't, it must be a... what can you say, if it is history then something within the Humanities. So I guess history can't be a science. It is more the application of it, or the useable aspect which is a science.

But what about history, is that a science?

³ In the comprehensive meaning of the Danish word *videnskab*, one would usually consider all types of history disciplines as belonging to the Humanities, and therefore as being a science.

No that's just how it is, the history is... well, it's just how Hamming and Golay developed what they developed, but what they developed is a science, their theorems, definitions, and those things.

Another student who had said no to the question provided the following argumentation:

When we work in history we have some history methods which we use to look at things, analyze them, and write assignments. But when we just get it presented as here, then that's not what we do, we just read it, kind of like reading Nordic mythology, it's just a story your reading. I wasn't analyzing and comparing and looking at different viewpoints with this, and if I did then it was because you had already provided the different viewpoints. I didn't like have to work, I just had to get through it.

What about objects and techniques?

Yeah, I didn't get too deep into that. Okay that might have been so. If you had said that this part of the world sees it as if this guy was first with this, and this other part of the world sees it as if this other guy was first, then you might have been able to work it as a history assignment in some way.

But then it is just history which you are looking at as a science and not history of mathematics, then it's just history about mathematics, right.

To the best of my knowledge, the fourth supportive essay assignment of the first module was exactly such a scenario as this student was calling for: the part of the 'world' which believes Golay should get the credit, because he was the first to publish the family of codes, and the other part which believes that Hamming should, because he was the one to develop them first. In fact, this student later admits that in his group, they split the final essay assignment between them, and that he only worked the first supportive assignment. More interesting, however, is the students' final remark. In some way he seems to think that when the history of mathematics concerns matters of attribution etc. and different views on this, it is no longer history *of* mathematics, but instead history *about* mathematics. By this the student seems to be implying that the history becomes 'detached' from the mathematics. This links to the two following questions of the second questionnaire.

20. *How does one conduct history of mathematics?*

Many students suggest that you investigate the origin or background of things, the applications, the developments, or the basic ideas. Some students provide more 'methodological' answers, a few of these relating to the topics of the main essay assignment: "You look at different articles which have been published, I suppose, and then compare them to each other, and then investigate *when what* came out."; "You look at who invented/discovered it/Why/When/How/following the surrounding world." A few students seem to think that the ways of doing history of mathematics are similar to those of doing mathematics, e.g. calculations, proofs, theorems, etc. A few of the more uncommon answers are: "It happens by itself. Mathematics needs solving, and the solution and process may turn into a history."; "Read boring history books."; "Anyway you want to." A few do not answer, or say that they do not know how. The followup interviews did not reveal much additional information about this question.

For Danish upper secondary students history (world history, national history, modern history, etc.) is a mandatory subject they must follow throughout all three years of

upper secondary school. Hence, the students now having been introduced to elements of the history of mathematics in the first module, it seemed relevant to ask them:

21. *How is research in history of mathematics different from research in mathematics?* Various different answers are given to this question. Some students say that history of mathematics belongs to the Humanities and that mathematics does not. Some students seem to have some idea of what the differences might be: "Mathematics: development within the area. History of mathematics: research *in* the development."; "History of mathematics research is to take the development of mathematics into account. Research in mathematics are new discoveries, new knowledge, and an evolution." A couple of the answers may again be related to the main essay assignment: "Because it is history, it is something people have done, it can be discussed: who/when/why. Mathematics is equations, numbers, letters, etc., it cannot be discussed."; "As opposed to in mathematics you work more with the mathematics behind the history/theory. Math. Hist: who *and* what. Math: what." Others provide answers saying: "One is history!"; "It is history and not calculations!"; "It is more text based and more loose." One says that there is focus in mathematics, insinuating that in history there is not. A few say that they do not know or cannot figure out what the difference might be.

The student who said: "Math. Hist: who *and* what. Math: what." provided the following explanation in the interview:

Well it's like, when we've had this thing, I think that the history of mathematics had more to do with the developing, I mean who did it, where they did it, and why they did it. I think that is history of mathematics while mathematics is when we sit and work with it, sit down and deduce the different things, and the like. That is more... Anyway that is how I personally divide it.

In questionnaire 3 the students were not asked any questions about the history of mathematics or research in this field, they were only given the question of mathematics being a science or not.

23. *Regardless of what you answered in the previous questionnaires, do you then consider mathematics to be a science? Has your opinion on this changed? If yes, why?*

The vast majority answers "yes" to the question this time. Only one definitively says "no" and that it is a tool. A few say "yes and no", arguing that it is a science, but also a tool. Concerning whether their view on this has changed to any degree, some say that it has not, a few that they cannot remember, but do not think so, and one says: "Changed a little. Still think that it is a language, but this can be developed to a science." The majority, however, do not answer the followup question concerning change, but instead go on to provide reasons for their view (reasons similar to the ones seen in sections 7.2 and 7.3). No further information was revealed in the followup interviews.

In the second questionnaire the majority of the class believes mathematics to be a science. In the third questionnaire there is an increase in the number of students who believe mathematics to be a science, seeing that now the vast majority thinks so, a picture which stays intact through the fourth questionnaire as well. In the first and second questionnaire the students were asked to provide arguments (justification) for their answers, but in the second and third they were not, but still they did. The arguments provided, although numerous and varied, do not differ much in nature from one questionnaire to the next. In the second and third questionnaire the students were

asked to indicate if their opinion on mathematics being a science had changed any. In the second questionnaire no one confirmed to this, which is in accordance with the answers to the first questionnaire. In the third questionnaire, however, some ought to have confirmed to having changed their minds, but only one clearly did, and a few said that they could not remember. So concerning this question, the students may not have been overly aware of their own beliefs (I shall discuss this in section 7.6). This may of course be linked to the complexity of the question, because: What is a science? Though the students do believe mathematics to be a science, they do not believe the history of mathematics to be one. From the arguments provided to this question (question 19, second questionnaire) it seems that some students consider only natural sciences as sciences – approximately the same students who consider mathematics to be a natural science (question 21, fourth questionnaire). Another thing which may also be noticed from the above presentation of students' answers is that the students exemplify their views, in particular to the questions in the second questionnaire. Sometimes this happens by referring to elements from the modules, especially the when-and-who and why-and-how accounts but also the discussion on invention and discovery – a discussion which I shall proceed with now.

Invention versus Discovery: The Two Historical Cases

The second questionnaire included three questions relating to the discussion of discovery versus invention. I shall present the data and then discuss them afterwards. Again the numbers of the questions refer to the numbering in the questionnaires (see appendix B.2).

15. Are Hamming codes and Golay codes discovered or invented?

At this point in time the class is equally split between students who believe in discovery and students who believe in invention. Arguments for discovery are amongst others: “Discovered, based on a number of premisses.”; “Discovered due to a need.”; “Discovered. The codes were there, and it was discovered that there was a certain connection, and then they were called something.” Examples of the opposite view are: “Invented due to a need.”; “Invented, I think, because it sounds awkward to discover a code.”; “Invented – if not they shouldn't be allowed to patent them.” A few students answered “both”, and one of them argue: “Both, I'd say. They discovered the codes because they used the arithmetical rules and stuff they had invented.” A few students pass on the question. The followup interviews revealed no further information.

In order to apply a little pressure on the articulation of students' beliefs, the discussion of discovery versus invention was linked to the issue of patenting the Hamming codes – as may be recalled from section 5.1, Hamming found it somewhat outrageous that a piece of mathematics could be patented.

16. Do you consider it fair that you can patent error correcting codes, as the Bell Labs did with the Hamming codes? Provide arguments to support your point of view.

Many students answer that they believe it to be fair, fewer that it is not. A few of those believing it to be fair say that it is a matter of money, e.g.: “Everything is about money, so I only think it's fair that they get the credit since they were the ones to put money into the research project.” Effort, time, and resources constitute another line of reasoning for a few of the students, e.g.: “Yes. They have put a lot of time and

resources into the development. With a patent they get additional funding to develop it further.” Getting credit in the form of a patent is something a few students touch upon, e.g.: “Yes, of course. Like artists who’s got it on their hits, lyrics, or brilliant ideas for a play. You want credit for all the work you put into it.” And finally a few students say “yes” because it was Bell Lab’s *invention*, students who had also answered invention to the previous question. For those who consider it unfair, some of the reasons are: “In a way ‘no’ since it is not an invention, as I said, but a discovery. If I discover a cave I can’t patent that one either, so that nobody can enter it.”; “No. Mathematical inventions should provide better understanding in the world, they should be accessible for all.”; “No, it is way too mathematical for anyone to patent it.” A few are a bit in doubt: “It depends on whether they get monopoly on coding theory, and if they do then ‘no’. That is not fair, and then maybe it is, since those who worked there were the ones who invented it.” A few students do not answer the question, and one student answers: “I don’t care.” The discussions of question 16 in the followup interviews did not reveal any new insights or arguments.

Of course, patenting is something which goes better hand in hand with invention than with discovery. Next, the students were asked to review their previous answers. The purpose of this was to see how they reacted in case they realized the presence of possible conflicts.

17. *How does your answer to the previous question connect to the discussion on whether mathematics is generally discovered or invented?*

Some students answer that it does not: “Huh? They are not connected at all.”; “Patent builds on inventions, so in principle it should underline that mathematics is invented. But I still believe that a lot of mathematics are discoveries, found in inventions.” Some say that their answers fit well together: “Good. Invention=patent.”; “I said so before [in question 16]. But the fact that it is discovered and not invented implies that ethically you have no right to call it your own.” A few do not put down anything. The remaining part, which is in fact the majority of the students, actually do not answer the question, or at least not directly. Instead, some of them seem to take the opportunity to express any kinds of thoughts that cross their mind in relation to the question. Some others answer indirectly by saying things as: “It is discovered, but you have to invent some things before being able to discover others.”; “Discovered, you haven’t invented that $1 + 1$ banana gives 2 bananas... Therefore discovered. Connections between different mathematical concepts and numbers are discovered.” These and other examples of indirect answers are actually in accordance with the students’ viewpoints in previous questions. But, of course, there are also examples which certainly are inconsistent, e.g. where students favor discovery and at the same time think that it only is fair to take out a patent. One such student is Andrew, and I shall discuss his views in detail in section 7.5.

In the third questionnaire only one question concerned the topic of invention versus discovery (cf. appendix B.3):

22. *Is RSA discovered or invented? Why?*

The majority of the students believe RSA to be invented. Some say that it was invented due to a very specific need of cryptography and safety in society, due to the need of solving the key-distribution problem, or due to the necessity of dealing with situations as those arising in war, for instance. Other examples of arguments for invention are: “Invented. You’ve put some theorems together and called them RSA.”; “Invented since

you've made your own system and your own codes.”; “Invented, I think, because it is a very specific way of coding and decoding.” Some students say that RSA is discovered. The arguments given may be both general or case-specific: “Discovered. All mathematics has always existed.”; “It is discovered since I don't believe that numbers can be invented: all numbers exist in all contexts, they just haven't been found yet.”; “RSA is discovered within public-key cryptography since public-key cryptography already was made then.”; “Discovered. You knew very well already that there was a one-way function to discover (public-key cryptography).” A few students say that it is a combination of discovery and invention. Another few students cannot make up their mind: “I can't seem to agree with myself upon this.”; “Well, here I might say invented... or... no... I'm not sure. You get more and more confused the more you think about it.” One student does not answer. Not much further was revealed in the followup interviews, except a few students confirming that they found it difficult to make up their minds about the question.

The above and previous displays of students' answers do indicate some development in the students' beliefs concerning the question of discovery and invention. In the first questionnaire the students favored “discovery”, in the second questionnaire they were about equally split between “discovery” and “invention” of error correcting codes. In the third questionnaire the majority believed “invention” in the case of RSA, and in the fourth questionnaire the class was split three ways, one third believing it to be “a combination”. The fact that there is an increase in students believing it to be a combination of both invention and discovery implies that some reflection has taken place, which is further supported by the fact that students say they have difficulties making up their minds. Furthermore, the development of students' beliefs on discovery or invention of the negatives and mathematics in general from the first to the fourth questionnaire suggests an increase in the level of consistency in the students' beliefs.

The students' answers to the questions of the second and third questionnaire are of course closely related to the first and second teaching modules, respectively. Thus, when students exemplify their views by referring to the cases of the modules in these questionnaires, it is not very surprising. However, in the fourth questionnaire the invention versus discovery questions did not relate directly to the modules, and an increase in exemplification in the students' answers was detected here as well – exemplifications, which also related to the modules.

Concerning the students' need to justify their views and beliefs, some development may be detected from looking at the questions on discovery and invention. The answers to the questions in the four questionnaires indicate that this need increases. Of course, in questionnaire 3 the students were more or less asked to do so, due to the ‘why?’ added in question 22 (see above), and the same goes for the questions on the negative numbers. However, in the questions on discovery and invention of mathematics in general they were not asked to justify their beliefs, and nor were they in the question on discovery or invention of error correcting codes in the second questionnaire. Still a larger number of students than in the first questionnaire chose to do so, especially in the second questionnaire.

How to Proceed From Here

The above presentation of students' answers in questionnaires and interviews has provided us with an overall picture of the possible changes in students' beliefs or views, and the

possibility of ascribing these changes to the teaching modules (students' exemplifications relating to elements of the modules); both in terms of (1) content and question specific changes and (2) changes in the way the beliefs are held. Thus, this overall picture may be seen as consisting of some findings which can be further examined by following students individually through the four rounds of questionnaires and interviews. In particular this goes for the indications of the previous subsection concerning point 2, the finding that the students' beliefs grew in consistency between the different questions, and that the students, or at least some of them, began exemplifying and/or justifying their answers to a larger extent in the later questionnaires.

As hinted to earlier, the further exploration shall be carried out by looking at the seven focus group students (Lucy, Lola, Jonathan, Andrew, Harry, Sean, and Gloria). Of course, the five other interviewees could have been used as well, but taking into consideration our knowledge of the focus group students, they seem to be a better choice. For these seven students, and especially the three 'constant' ones (Jonathan, Andrew, and Gloria), we have a reasonably good idea of their mathematical (in-issue) capabilities, their capabilities of discussing meta-issue matters, and their personal attitudes and emotions towards mathematics. And in terms of triangulation, for the focus group students it is possible to include elements of the video clips to deepen the analysis even further, something not possible for the other five interviewees.

7.5 Following the Focus Group Students

The way I shall go about presenting the beliefs of the focus group students is by considering them one by one, and in the process try to locate and discuss the major changes with the individual student. In relation to this, one thing should be mentioned: In the fourth round of followup interviews some of the students were asked, as one of the very last questions, if they themselves thought the teaching modules etc. had any influence on their idea of mathematics (as a discipline). The reason for asking this question as the last one was in order for it not to influence any of the other answers to the interview questions.

Lucy's Change in Beliefs on Mathematics as a Science

Lucy is one of the students who in the first questionnaire answers "No, mathematics is a language" to question 17 on mathematics being a science. In the followup interviews she explains her answer as follows:

Well, it's just another way to communicate. Also in physics, it's used to explain things all the time.

But is physics a science?

Yes, no, yeah, no. I don't know.

But can you give an example of something which you think is?

Not really.

Is it possible that it could be both?

Physics or mathematics? I suppose it's... well physics, I guess, is a mixture of sciences.

And what about mathematics?

I guess it's also science-like. I don't know. It's difficult.

It's no exam, you can just tell me what you think.

Yeah, yeah, but I just don't really know what I think, I think it's a difficult question.

As we know from the first module videos, Lucy was a rather quiet girl, so for that reason a lot of questions were sometimes required to get some information out of her.

In the second questionnaire Lucy takes a different viewpoint to mathematics being a science (question 18): "Yes, I think so. Can't remember what I answered the last time." In the followup interviews, Lucy is asked about this answer as well as that to the question of history of mathematics being as science:

In question 19, if history of mathematics is a science, you say that you don't think so. Can you explain why not?

We read about that in AT [a cross-curricular course], where it said that mathematics wasn't a science, but...

It said that mathematics wasn't a science?

Yeah, it wasn't as much a science as physics or biology is for example, but I guess it is anyway.

And what about history of mathematics?

I don't think that's a science, it's more like, yeah well, it's just history, right.

And history in general is not a science?

No.

What could be going on at this time is that Lucy believes science to be the same as natural science, but she was not asked explicitly about this.

In the third questionnaire her answer to the question is (question 23): "Yes, since there is more to it than just sitting down and calculating." Lucy's answer is not touched upon in the third round of interviews, but her viewpoint stays intact through the fourth questionnaire, though the justification provided the fourth time around is weaker (question 20): "Yes, I think so, but I can't explain why." In the followup interview, Lucy is confronted with the change in her answers to the question:

You say you think mathematics to be a science, but you can't explain why.

A year ago you said: 'No, mathematics is a language.'

Yes, that was something the teacher had said at one point in time.

Okay.

I was hoping I didn't have to deepen that question! I don't know... I think it's hard to define what a science is, and then say whether mathematics is one or not.

Yes.

Yes, and then I also think that it... changes a little.

But do I correctly understand that you now have some kind of idea that mathematics is a science?

Yes.

To a larger degree than you had a year ago, or what?

Yes, I think so. But I'm finding it difficult to explain why!

Okay, but why do you think your perception has changed?

Because I've gotten a better insight into what mathematics is... a wider perspective of it... and seen what it also contains.

Okay. Through your mathematics classes or what?

Yes.

In the fourth questionnaire we also get a glimpse of Lucy's view on natural science, when she answers the question of mathematics being a natural science or not (question 21): "I don't think it's a natural science since it has nothing to do with nature."

Besides the above, Lucy's answers to the four questionnaires do not indicate many changes. For example, her beliefs on mathematics being discovered or invented are rather consistent throughout the entire year. She believes the negatives (in the first and fourth questionnaire), Hamming codes and Golay codes, and RSA all to be invented, and she believes mathematics in general to be a combination of discovery and invention (in the first and fourth questionnaire). Lucy's fourth questionnaire is not particularly rich on justifications of her answers, in fact many of the answers are quite short – shorter than those she gave in the first questionnaire. She does, however, provide examples from time to time, especially in relation to the three historical questions on how, when, and why the mathematics in the textbooks came into being (questions 10, 11, and 12). She mentions the Egyptians, the Greeks, the building of the pyramids, and a need to figure out distances larger than those which could be measured 'by hand'. Concerning exemplifications relating to the modules, Lucy answers "cryptography is used" to question 8 on applied mathematics in society and everyday life, except the basic arithmetic operations etc.

Lola's Change in Beliefs on Mathematics as a Natural Science

For Lola also, I shall mainly concentrate on her view of mathematics as a science. In the first questionnaire, Lola's answer to question 17 on this topic is: "It is a science about knowledge? Natural science. Galilei and Pythagoras were scientists, I guess?" In the followup interviews, Lola is asked why she believes mathematics to be a natural science:

Because it has arisen from nature.

Okay. How?

Yeah, I don't have a ready answer. I think it's just because people say that mathematics belongs to the natural sciences, and I suspect it is so, it's connected to things like biology and physics which are also part of the natural sciences. Then I just think that mathematics must belong to the natural sciences too. Offhand I can see why things like physics and biology do, because they have something to do with nature. And then I see a connection to mathematics, because all the calculations you use and stuff, you use in those subjects.

In the second round, Lola's questionnaire answer is (question 18): "Can't remember what I answered the last time. But yes, I think it's a science." (The interview did not reveal much on the matter.) However, Lola do not believe the history of mathematics to be a science (question 19): "No, not really. But don't know why." At this time Lola still believes mathematics to be a natural science, which can be seen from her answer to question 21, asking how research in the history of mathematics is different from research in mathematics: "Research in the history of mathematics is part of the Humanities. Research in mathematics itself is natural science."

The third time round Lola provides the following answer (question 23): "A science. I can't remember if my opinion has changed. You could say that mathematics is based in logic, and not actual research, but still it belongs to the faculty of natural *science*." In the followup interview Lola was asked again why she believes mathematics to be a

natural science:

Well that's why I say that it's a little bit of both. I say that it's both a science and that it's not a science, right. But I mean that it's a science because it belongs to the faculty of natural science, I suppose there is a reason for it being exactly there. I don't know if it's because I trust it blindly, but there's a reason for me... If I had to say why it shouldn't be in the faculty of natural science, then it's because mathematics in principle is about logic. And there is not any research behind mathematics as there is with biology, for instance, so for that reason... In principle it shouldn't be placed in the faculty of natural science, but it is.

Because it's not that kind of research which is being done, it's not like, we can't take mathematics and go and observe it in reality, or?

Yes, we could do that. But I just think that the basic thoughts behind mathematics are in logic. The thing with us going from one point to another, then there must be so and so far between the two points.

Could we place mathematics in the Humanities then?

Yeah, but there's also the thing with, I believe that I at some point wrote something with deduction, something with that in mathematics then [...] you take an expression and use it instead of researching it the way you do in the other disciplines. I don't know if you see what I mean?

No, not really.

Well, that deduction means that you after calculations and intermediate calculations can deduce a conclusion or some result, and that's what you do in mathematics. But in principle it isn't really like that. Yeah, I'm not really sure how to explain it, but I do believe that you can put mathematics in the Humanities. But yeah, it's just very connected with, for example, physics and things like that, so for that reason it's probably more obvious to put it in the faculty of natural science.

Because it's connected to physics?

Because physics is applied mathematics.

In the fourth questionnaire Lola gives the following answer to the question of mathematics being a science (question 20): "No, I suppose not. I think that I have written previously that it is because it is within 'the area of natural science'. But personally I don't think so, it is because mathematics is logic. It is not really composed of theories, but on the contrary connections which are logic." To the question of mathematics being a natural science (question 21), Lola answers: "It does not build on theories as the other subjects within natural science do. On the contrary it plays a role when you use the other subjects within the same area to carry out/produce something – I can't come up with any examples right now." The discrepancies between these answers and the previous ones were explained in the following way by Lola:

Yeah, well, I remember that I wrote in the first questionnaire that it was a natural science since it is within the natural science area, right. In these... well... but now I don't really think that it's a natural science anymore since the other natural sciences are built up of theories, and mathematics in principle is not. You can use mathematics to describe many natural scientific subjects, but I don't think it's a science as such.

Okay.

Also even though you say that it's just all the scientists who have discovered or invented equations etc.

But when you say science, do you then mean natural science?

Hmm [confirming].

Or does it belong to the Humanities, do you think, philosophy etc.?

It might be so, but I can't provide any deeper reasons for it.

So to sum up, Lola goes from saying that mathematics is a science and a natural science as such, to saying that it is not a natural science and in fact not even a science, but logic. The claim to justify this is that mathematics is not composed of theories like other (natural) scientific disciplines. Where Lola gets this claim from is not clear, but that it should originate from the modules seems unlikely, since the first module concerned coding *theory* and the second included a substantial amount of elementary number *theory* – unless, of course, she means something else with the word theory. Though Lola is on to the idea of 'logic' in the third round of interviews, she still believes mathematics to be a science at this point in time, so possibly the change may have manifested itself between the third and fourth round of questionnaires and interviews. More interesting, however, is that she goes from viewing mathematics as belonging to natural science to thinking that it is not, a change which occurs after the end of the second module. At this point in time she is willing to believe that mathematics may equally well be placed within the Humanities, but more importantly, she is able to justify why it may be more 'natural' to place it in the faculty of natural sciences. A justification which may be linked to the final essay of the second module by her use of the term 'applied mathematics'.

Besides the above change, a few other changes can be observed. In the first round of questionnaires and interviews she favors discovery over invention for both the negatives and for mathematics in general (questions 15 and 16). During the second and third rounds she answers 'invented' for the Hamming and Golay codes and the RSA system, but shows some doubt, e.g. her answer on RSA (question 22): "Invented. Argh... I don't know! You can't *discover* things like that." In the fourth round she answers invented to both the negatives (question 18) and mathematics in general (question 19), even justifying the first: "Invented. Because you *figured out* that something opposite the positives was needed." This change also shows in Lola's answers to the question on what mathematicians do. The first time she answers (question 18): "Discover new things?" And in the last questionnaire she says (question 22): "Carry out proofs and develop the mathematics further. Also invent something new, for example, and try to solve previously unsolved 'things'/formulas." These two answers illustrate a general observation between Lola's first and fourth questionnaire answers, namely that those in the fourth are longer and richer on justifying answers. However, concerning exemplifications, none of those which appear in the fourth questionnaire answers may be directly related to the modules, and only rarely do Lola refer to elements (e.g. she mentions the Enigma) of the modules in the fourth interview.

Jonathan's Change in Beliefs Towards Mathematics as Invented

As early as in question 17 of the second questionnaire (see page 228), where some students take the opportunity to say whatever comes into their minds, Jonathan says: "You can say that my view has changed radically through this module. Mathematics is

generally invented and not discovered to the degree I believed before.” Jonathan had already stated this once in a class period, after having worked with the group on the final essay assignment. In the first questionnaire Jonathan had answered that he thought the negative numbers to be discovered, and to the question of mathematics in general he had simply answered (question 16): “Discovered”. To question 15 of the second questionnaire on Hamming and Golay codes being discovered or invented, Jonathan answers: “Discovered, due to a need”. In question 16 on the Bell Labs’ patent he writes: “I think it is some weird stuff since economy is very important. People are very greedy, in a way I can understand them.” And then in question 17 he provides the quote above, on his opinion having changed somewhat. Two things are noteworthy about this. First, Jonathan justifies his views in question 17, though he is not in any way being asked to. And second, his answers to the second questionnaire are actually not inconsistent.

In the first round of interviews Jonathan had given expression to mathematics as being something which has always existed: “I believe it has always been there, but I just think that humans are exploring mathematics more and more and are discovering more things.” In the first questionnaire Jonathan did not provide examples supporting his answer, but he was invited to do so in the interview:

Well, the theorem of Pythagoras is about some relations between the sides in a triangle. I mean, the sides in a triangle are not invented, or what you say, they are... when you look at a triangle you have found some relations between the different sides, and then you have proceeded from there until writing up a formula for it, right. In that sense I’d say that you have discovered that there are relationships and then taken it from there and generalized it, found that this formula fits, and that it could be applied to all other triangles, or all other right triangles.

As for the negative numbers Jonathan seemed a bit more in doubt. He said that he found this particular question rather difficult:

I think they are discovered. I don’t think it is something which is invented, because there are... damn, it is difficult to say.

You say that they will have been able to see it, but might have had difficulties explaining it. What do you mean by that?

Well, you might have... Well, if you have a parabola. There is the one side of it and that is positive, and there is the other side and that surpasses the scale, but... I don’t know how they came up with calling them negative numbers, but in some way or another you must have been able to put down a center point, or zero, and then on that side, from zero you can go in either direction. But I don’t think you’ve invented the numbers, -1 , etc. I think that people somehow found that they needed a way to indicate the other side of the scale. It’s difficult to say if it’s then discovered or invented.

In the second round of interviews, Jonathan was asked to deepen and explain his remark about his view having changed ‘radically’:

Well, the numbers are discovered, but the way you use them, the operations and such, that is something you invent. Whereas earlier I thought that it was something which was there, which you discovered or what you might say, but it’s only the numbers which have always been there or what you might say, right. Then you invent new ways of using them because you need some new mathematics for whatever. Now, coding theory is an example, right – I

don't know if I can come up with other examples – but well, you put up some equations to see if it'll fit, right, to make everything more connected. In the beginning, when we did the first questionnaire, I thought that mathematics was something you discovered, as if it was there just to be exposed, or what you might say, but it just wasn't, well, found yet or something – but it was something different.

Have you changed your opinion or?

Yes.

Okay. Well, it's not as if there is something right or wrong. It's not...

No, that's it. It's difficult. It's a question about...

I'm not even sure what I think myself, but okay.

No, for sure. But, well, I definitely think that my view on this has changed, where I thought before that everything was just there, only you hadn't found it yet, but now I more believe that everything wasn't there, but that you invent it when you need it, when there is a need for it.

I recall that in one class I heard you say, exactly, that your view on mathematics had changed. Was that in regard to this, or was it in relation to other things as well?

That was in regard to this, of course.

In the third questionnaire, Jonathan answered the following to the question of RSA being discovered or invented (question 22): “Invented on the basis of discovered mathematics, you might say.” What the discovered mathematics that Jonathan is referring to here actually is, is not quite clear. Previously he said that numbers were discovered and more or less everything else was invented, but if ‘numbers’ covers elementary number theory also, or if it only covers numbers is unclear. Whether Jonathan is withdrawing his change of view a little in the third questionnaire is difficult to say, but he is very aware of the distinction between discovery and invention, that much is certain. In a previous question, the one on RSA and basic research (question 21), Jonathan is careful in not taking a stand: “You need to have some basic research in place in order to make ground-breaking discoveries/inventions, you might say.” And when discussing this particular question in the followup interview, he keeps saying “discoveries or inventions” instead of one or the other.

In the fourth round Jonathan's views appears a bit more settled than in the third. To the question on the negative numbers being discovered or invented (question 18), he says: “Discovered in order to be able to explain different things.” And to the following question on mathematics in general (question 19): “It is a mix, and I don't think you can give a definite answer to it.” In the interviews, Jonathan was asked again about his change in views from the first to the fourth questionnaire:

Yeah, I don't know why it changes. I don't know, it's because we've worked with it, and I kind of like, I've sort of realized that I had... the way I thought about it in the beginning, it was sort of a narrow-sighted way of viewing it.

Yeah?

That it could be like that, right. Of course it is a mix of the two. Or, it's difficult to explain exactly what is discovered and what is invented, right. But...

Actually we haven't worked with it, really. You've just been asked concurrently.

Yeah, yeah, but we've... we have worked with mathematics and... with math-

ematics which was invented, or created due to, what do you say, a need for it, right. And stuff like that. So we've kind of seen examples of mathematics being invented. And in the beginning I just thought: 'Nope, it's discovered.' I know we haven't worked with it explicitly, but well... but we've still worked with mathematics which, what do you say, may provide the opportunity for you to pose that question.

Yes.

So, for that reason you might become influenced by it... what we've worked with.

Could you say then that the modules have provided you with an opportunity to change your view of what mathematics is, and how it has come into being?

Yes. Yes, they have. [...] Well, it is definitely concerning this discovery/invention that I have changed my mind... I don't know if I had any other close-knit positions from the beginning. I can't remember.

As a note on the above discussion, it should be mentioned that the teaching materials were in no way designed to favor invention over discovery or vice versa, and the matter was neither discussed directly nor indirectly in the materials. So when Jonathan says that they have seen examples of mathematics being invented this is, in fact, his own interpretation of the two historical cases. As seen from the overall picture of the questionnaire answers, other students stuck to their beliefs of mathematics being discovered. An example of this is Andrew, as we shall see below.

Another change for Jonathan concerns a first questionnaire answer where he had put the words 'new mathematics' in inverted commas (question 12, on whether parts of mathematics can become obsolete). When asked about this in the followup interview he said: "Well, I believe that mathematics exists. So in that sense, I don't believe that you can create some entirely new mathematics, not if it's already there. You can discover new mathematics." As Jonathan changes his view in direction of mathematics being invented, he apparently changes his view on 'new mathematics' as well, since he begins mentioning this without second thoughts in the later interviews, e.g. in the quote above when he says "...you need some new mathematics for whatever".

Jonathan's change of mind concerning 'new mathematics' shows some kind of consistency in his views, as these develop during the modules. Regarding justification, his answers in the fourth questionnaire are much more elaborate than those in the first. His answers to the question of mathematics having a greater influence today compared to a hundred years ago illustrates this quite well. In the first questionnaire Jonathan's answer is (question 8): "Definitely, more computer=more mathematics." And in the fourth it is (question 9): "Greater, definitely, since you have found far more areas of application, and mathematics has also evolved, and so has the need in society for mathematics." In this last questionnaire Jonathan also seems to provide more examples in his answers than in the first. Question 7 on applications of mathematics in everyday life and society illustrates this. In the first questionnaire the answer was: "The computer, in front of which I spend much time. It is used in the programming of the programs which I use." In the fourth: "Mathematical modeling; statistics; various models; informative labeling of products; my calculator. A lot of the places you do not notice, but practically all types of electronic constructions etc. contain some mathematics in the form of physics." Whereas Jonathan in his first answer refers only the use of mathematics in software, he refers to the use of mathematics in hardware in his next answer, something which was

discussed a bit in the first teaching module. But even more interesting is his comment on applications of mathematics being hidden (cf. section 3.4). As mentioned, both coding theory and cryptography are examples of 'invisible' applications of mathematics in our everyday life and in society, so it seems likely that this comment is a result of Jonathan's encounter with these two historical cases of applied mathematics. In general, however, Jonathan seldom exemplifies his fourth questionnaire answers using elements from the modules. One of the few exceptions is his answer to question 8 (applications not counting the four arithmetic rules etc.): "Cryptography and coding theory used in communication, computers, and the like."

Andrew's Beliefs on Mathematics as Being Discovered

As mentioned, Andrew was a student who stuck to his belief of mathematics as something being discovered. However, I shall argue that his view became much more reflective as a result of the modules. In the first questionnaire Andrew gave the following answer to question 15 on negative numbers: "Discovered along with the positive numbers. Why: *No* clue." On mathematics in general, he answered: "Discover. I don't believe you can invent mathematics. It is something 'abstract' you find in things already existing." In the followup interview Andrew explains what he means by this:

Well, this guy Mr. Pythagoras, for instance, he didn't invent the relations in a triangle, it was something he found. I don't think you can invent some relationship within something. Of course, the basic numbers and series of numbers, they are of course invented, but... If it was today, then I don't think you could invent mathematics. [...]

Why not today?

Well, every time you... at least all the mathematical proofs and all the different things we've seen, they build on something. And I don't really think you can invent something building on something else. You find new connections among things. It's a bit difficult to explain.

So if you had something new based on something old, then it wouldn't be new, or?

It would be something new found within something old, right.

In the second questionnaire, Andrew answers the following to the question of the Hamming and Golay codes being discovered or invented (question 15): "Discovered based on an invention (the binary number system). Discovered because they are found through work with the invention." In question 16 Andrew says that he finds it fair that the Bell Labs patented the codes, and in question 17 on the consistency of the two previous answers he says: "Patents build on inventions, so in principle it should underline that mathematics is invented. But I still believe that a lot of mathematics are discoveries, found in inventions." Andrew is confronted with the possible conflict that "patents build on inventions" in the followup interview:

That was what you asked me the last time as well, if mathematics is invented or discovered. I don't really know. I think it's because it's something which reaches so far back that I really can't see that you can invent something without it building on something already there. So I'd still see it as if mathematics is something you have and then discover new branchings in it.

And what defines 'mathematics', what do you mean by that then?

Well, I don't know, all of mathematical thought or something, far back, in history, right, the logical way of thinking or something. And then there are some branchings which are discovered, for example all that with the binary numbers that is a branching of mathematics. Therefore I believe it to be a discovery, or an extension of mathematics, I suppose you could call it. But not an invention.

And what about coding theory?

Well that is where I'm actually a little in doubt, because when you can patent it, it ought to be an invention, but for me, logically speaking, I still believe it to be a discovery within mathematics.

Apparently, the doubt that Andrew has here remains unsettled through the second module, because to the third questionnaire question of RSA being discovered or invented (question 22), he answers: "I can't seem to agree with myself upon this." In the followup interview Andrew explains:

It's the same problem I've had with all the stuff we had about coding, if it is actually something you discover or something you invent. And of course you invent a [crypto] system, but then the question is if you really have invented it, or if you've discovered it in something already discovered [invented], and that's where I really can't seem to agree with myself. I sat for several minutes trying to decide what to write, and in the end I couldn't.

But it is a, well your answer is fair enough, because it isn't...

Yeah. It's very difficult to figure out if you've been lucky to discover a relation, or if you've invented a system.

In the fourth questionnaire Andrew's view on the negative numbers is as follows (question 18): "Invented. It is not real talking about negative numbers as such. But a discovery in the already invented number system." Concerning mathematics in general (question 19), he says: "Primarily discovery, which builds on inventions/other discoveries, e.g. the number system." In the followup interview, Andrew is asked how his view on mathematics as primarily being discovered fits with his answer to the previous question on how mathematics has come into being (question 10: "Some men who once thought about connections between different things."), assuming that it takes some kind of creativity to 'discover' mathematics:

Well, that's also what I'm saying. It's a discovery building on some sort of an invention or other discoveries. Take something as a number system, for instance. That might be something you invented, but it's still connected to some discoveries which you've made. 2 being more than 1 and such, which you've figured out. And then you've just invented the number system, i.e. invented the system and from that made some new discoveries. I don't know. I think it's something very difficult to relate to, because there is something... every time you are to discuss if it's a discovery or an invention, there are a lot of contradictions coming into play. But I don't know why, it just seems to me as if most mathematics is a further development of something already there. In some strange way or another, right. All the time you find application in some already acquired systems, a number system or something for instance. And that's where you might ask yourself if something like a number system is invented or discovered... where I find it hard to say... when you talk about invention, I'm thinking that somebody should get some credit to a larger

degree than if they discovered something. I'm not sure if you can ascribe anything to anyone...

If we take error correcting codes, which we looked at, then Hamming 'discovered' his Hamming codes, the (7,4)-code that we talked about, for example. Was that one which he discovered or was it one which he invented.

There I'd say it was a, again a... it's a bit ambiguous, right. Because, of course, he probably discovered that this (7,4)-code worked. But it builds on another discovery or invention. So I think that it's all the time... kind of difficult to define what actually is a discovery or an invention within this.

And how does this fit with Shannon being the first to – you remember who Shannon was, right, he was the first to...

Yeah.

He knew that there existed some error correcting codes with good capabilities.

Yeah, but that was due to Hamming maybe looking for this, so it is a discovery which builds on a prediction, you may say. Well, I don't know why, I'm just not very fond of this word 'invention' being connected with something like mathematics. I'm actually not sure, why I don't like it.

As seen from the above, there are conflicts and discrepancies in Andrew's view on mathematics being discovered or invented. Although he generally favors discovery, he sometimes believes invention to precede discovery and sometimes discovery to precede invention, and in the fourth questionnaire he says that the negative numbers are invented. However, this may be partly explained by the fact that Andrew is struggling quite a bit with this question, something he admits himself several times in the interviews and in the third questionnaire, when he says that he cannot make up his mind about it. In this struggling process, it seems that the historical cases of the modules have provided essential input to Andrew's thoughts. Andrew is capable of justifying his answers or his doubts using elements of the modules in the second, third, and fourth round of interviews. Of course, he is invited to do this, but it appears fairly easy for him to use elements from the modules when discussing the question. Andrew is also capable of exemplifying his answers. In the fourth questionnaire, for instance, he uses number systems as an example. Although this topic may not necessarily be directly linked to the modules, number systems, and in particular the binary one, was part of the first module and number theory was essential in the second module. Andrew also provides exemplifications directly relating to the modules in the fourth questionnaire. In the question on applications of mathematics in society and everyday life (question 7) he writes: "Statistics. [...] Programming, coding, cryptography." And to question 9 on the influence of mathematics today and a hundred years ago, he says: "Greater – mathematics is used in most technical communication devices." These answers relate directly to the cases of the modules, and they are quite different from the answers which Andrew gave in the first questionnaire, where he said the following to questions 7 and 8, respectively: "The simple mathematics is used in, for example, shopping etc. But also in statistics and in economic developments, including the business world."; "Greater influence. We live in statistics and bets based on developments, tendencies, etc."

Harry's Change in Beliefs on the Application of Mathematics

In the first questionnaire Harry gave the following answer to the question of mathematics having a greater or lesser influence in society today or a hundred years ago (question 8): "The same. Because it was back then that everything began, or a least was developed a lot." One year later, in the fourth questionnaire, Harry gave a different answer (question 9): "Greater, since IT and know-how have become popular." In the followup interviews Harry was asked about this answer as well as the previous one:

Yeah, okay, there wasn't any IT back then, and long ago they didn't even have numbers, so...

No, but now... there were numbers a hundred years ago.

Hundred years ago, yes. What do you write?

If mathematics had a greater influence hundred years ago...

Hundred years ago, yeah, but still... We've had a lot about coding, and that about... computers and... where you send messages to one another and such, it's very important recently, I'd say.

Okay.

A lot of electronic devices, which weren't there back then, can be used today.

In the old questionnaire you said that you believed the influence was the same.

Yeah, but that was because we hadn't had anything about it, or what?

Well, I don't know why you said that it had the same influence.

How long ago is it that I wrote that?

A year ago.

That is before we started?

Yeah, before we started.

Well, that's probably the reason then.

Okay, so it made you change your perception?

I didn't think... I didn't think so much about that with cryptography, well, before we talked... I did know that computers were dependant on mathematics, but I didn't know precisely how, right. But now I see that there is... a great deal more to it.

In the end of the interview Harry was given the more direct question whether any changes had occurred in his view of mathematics:

Okay. I have a more direct question for you, Harry. Have the two modules on history of mathematics made you change your idea about what mathematics is, where it comes from, what it's being used for. Or do you think that they haven't?

Yes, apparently so, when I've changed my mind about it being the same a hundred years ago. But I'll probably say that... that it's a bit more complicated than I thought, with many things which you can deal with. Within computer technology, right. So of course it's, well – it's been a proof for me that mathematics can be used to many things, also within IT. And also all that we've had about modeling and such – I know that has nothing to do with this module – but also this module, you are told what you can use it for in everyday life, for example... or not only in everyday life, but also in war situations and so, right. With coding and such. And that it's

not only connected to IT, but also to coding in general, right. So of course it's had a... well an effect on me or something.

Yeah okay, but if you think back at the two modules, what would you say... or, seen in the rear view mirror, what surprised you the most then? That you had not believed to be so in the beginning.

Ohh... what I probably... that it was used so much in war, I hadn't anticipated that, or I hadn't imagined that, I didn't know that. That people were so concerned with, well, of course that you shouldn't let the enemies know, but I didn't know that you used cryptography to do that. I thought that you just used some guy who took the letters, and, well, didn't give them to the enemy. And then sent it on. I don't know, that it was a regular letter, right.

Yes.

But apparently not. And that's it, I think.

You mean that you could use cryptography for that, that it was a means to...

Yeah, and then the fight to be first within cryptography, right. In order to... so that the others couldn't, well, decode it, right. I didn't know that it was such a big race back then. Because that's not something you normally hear about in the television. There you only hear that there is a war and... what happened and stuff like that, right. Kind of like more behind the scene things.

So that mathematics in reality was an important part of that war concerning certain matters?

Yes.

The use of mathematics in war, which was discussed as part of the second module, did indeed seem to surprise and fascinate quite a few students. Like Harry, they brought this up in the interviews or touched upon it in their questionnaire answers. But that is not the only thing in the quotes above which relates directly to the modules. Also, the races within cryptography between researchers, nations, or between mathematics and technology were discussed on several occasions as part of the second module. Furthermore the use of mathematics in everyday life, as Harry mentions, was an essential part of the modules (error correcting codes in mobile phones, computers, CD-players, etc., and the use of RSA in e-mails and home banking). Thus, many of Harry's exemplifications may be directly linked to the modules. And as Harry himself states, the modules have given him 'proof' of the application of mathematics. He also ascribes his change in view on the influence of mathematics in society today versus a hundred years ago to the modules. What the above quotes show, is that Harry is capable of justifying his views, including his change in view.

Sean's Beliefs on the Influence of Mathematics in Society

For Sean as well I shall consider the influence of mathematics in society today, as opposed to a hundred years ago. In the first questionnaire, Sean's answer to question 8 is: "It has gotten a greater influence. Now it is mandatory for everybody to learn how to calculate." In the followup interview Sean explains himself in the follow way:

Just one hundred years ago it wasn't everybody who was offered a school education, it was only the rich kids and so, the more prominent people who

put their kids to school, right. At one time, all you had to do was to plow a field. Now the world is a much more complex place. I don't know, there's just more you have to learn today. For example, people demand much more from you over time.

You mean to be able to get by in society?

Yes.

And if you differ between mere calculations and mathematics, what then?

Then it's more the mathematicians, like the statisticians, those who use it in their work, who has a need for it, right.

But do their jobs in some way mean that it has a greater influence, or?

Well, I don't know. It's not, I think... Well, it might be so, because within science, if we are to evolve, then we need to – for example within computer technology – we need to be able to put up formulas in order to know how the computer is to work. And that's mathematics, very much, with codes and functions and formulas and so. How it should behave, right.

Sean's argument about school education only being for rich kids one hundred years ago is off.⁴ His later example with the computer, however, is quite good and also more historically accurate, since computers were not that much of an issue in 1907.

In the fourth questionnaire Sean takes to a somewhat different viewpoint on the same question when he answers (question 9): "The industrial revolution began 100 years ago, so mathematics must have been equally necessary for their development as it is for us today." Again Sean is invited to deepen his answer in the followup interview:

Well, I think... I think it's kind of relative, it seems to me, or I don't think that you... well, of course mathematics might have a... have more importance for us as compared to what we think it had back then. But I think it was just as important for them back then, because it's the way in which they evolved, like we are evolving now. That's why I say, okay, mathematics has always had the same impact on people.

Mm.

Especially a hundred years ago, because that was when the industrialization began, or was already going.

Okay, but in the first questionnaire you answered: Greater.

Yes. Yes, well... I don't know if I can argue against what I believed earlier, because I can't really remember it. Of course it might be because I hadn't thought along those lines, that there was kind of an historical angle to it.

We've kind of just gotten that – some history of mathematics.

As for historical accuracy, the industrialization was well on its way by 1907, but more interesting is the fact that Sean goes from initially believing mathematics to have a greater influence today to later seeing the influence relatively, and thus concluding it to be approximately the same. A change in this direction is definitely not what I would have expected – in fact, the change is the exact opposite of that of Harry, as accounted for in the previous subsection. When asked to explain the change, Sean reasons that it might be due to "an historical angle" and mentions the history of mathematics in relation to this. However, none of the two modules touched upon matters of industrialization, so

⁴ In Denmark the public school system dates back to 1814 (the first Danish law for school education dates back to 1539), so in 1907 (a hundred years ago) this system had been in function for 93 years.

the historical angle might equally well stem from Sean's history class for all we know. It is striking that Sean in the fourth round does not at all mention computers in relation to the influence of mathematics in society today, since he talked about this in the first followup interview and since both teaching modules concerned the use of mathematics in computer technology. In fact, at no time does Sean use elements from the modules to argue for his viewpoints, neither in the fourth questionnaire nor in the followup interview.

Though the examples provided by Sean do not refer to cases of the modules, Sean is generally keen on exemplifying and justifying his answers in the questionnaires and in the followup interviews, as the two quotes above illustrate. Furthermore, Sean is rather consistent in his answers (beliefs). For example, he considers mathematics to be a science in all of the four questionnaires. And he believes the negative numbers, error correcting codes, RSA, and mathematics in general to be discovered throughout the implementation of the modules as well, the only deviation being his fourth questionnaire answer to mathematics in general (question 19): "A combination. I do not want to expand on this since it's something which I've already answered." Whether this last answer is indeed an indication of his beliefs having moved a little towards invention is difficult to say for sure, since the matter was not brought up in the interview afterwards.

Apart from the above change on the influence of mathematics in society, not many changes can be spotted between Sean's answers to the first and fourth questionnaire. Nevertheless, when given the direct question on possible changes in beliefs, Sean points out a few himself:

... do you believe that the two historical modules have... or that your image of what mathematics is, how it comes into play in the world, and how it comes into being, well, has your image of that changed in any way because of these modules?

Yeah, because I didn't really have that much of an image of it before, how mathematics has come in, except in... some very short passages we've had in mathematics, and some of what I've read where it's mainly been on antiquity, so... I've discovered that there has been a much greater focus on mathematics than I would have anticipated. So... and discovered that for somebody it's their life, practically, to solve some of these proofs, which you've seen... and which you've thought to be impossible to solve, for instance, Fermat's last theorem, right.

Mm.

So... well, I do think that mathematics and the history of mathematics have a... or for me have a, well, or... I'm not sure how to phrase it. It's... at least it has a greater influence than what I had thought – at first.

Okay.

So..., so it's been kind of an eye opener for me... showing that it's a very important element.

Gloria's Beliefs on the Speed of Development in Mathematics

In the first questionnaire Gloria provides the following statement about when the mathematics in the textbooks was developed (question 10): "Long, long ago it all began and since then it has continued. But I am confident that the development goes more

and more slowly, because you eventually know quite a bit.” In the followup interviews, Gloria is invited to explain what she means by this:

Yes, but they just discovered more a long time ago, didn't they? It isn't very often you hear about someone who has discovered something new within mathematics, is it? Maybe it's just me who am not enough of a mathematics geek to be told about it, but it just seems to me that nothing is really happening. Stuff is happening more often within the natural sciences: now they have found a method to see the foetus at a very early stage by means of a new type of scanning or something.

But that might also be because it just didn't make it into the media.

Yeah, that was what I asked you.

Yes, yes.

Is it like that? Or is there nothing much happening within mathematics, is there?

I think that there's probably a lot happening within mathematics. But your answer concerns that you don't hear about it very often, and therefore you find it natural to believe that nothing much may be happening.

Yes, exactly.

Mathematics is probably one of those things that does not find its way into the media very often and the quote above touches upon the discussion of mathematics being hidden or invisible in our society and everyday lives.

In the fourth questionnaire Gloria only answers “at different times...” to the question of when the mathematics came into being (question 11). In the followup interview she is asked about her previous utterance on the speed of the development in mathematics:

Can you remember that in a previous interview you said that you were certain that the development [of mathematics] was going at a slower and slower pace?

Yeah, I do understand my reasoning behind it, because I thought that when you've gotten that far it must become more and more difficult to come up with something new. But that isn't really thought through, what I've said then, because it isn't entirely correct. Because you know, actually, that the development is going faster and faster because you get more and more opportunities. Or you might say, the long time which passes from you thought of a mobile phone till you've had one, till it became normal, and the time that then passed till you had a mobile phone where you could see each other – you understand what I'm saying, right – I mean, the steps, longer and longer steps take shorter and shorter time. The majority of the technological steps all build on mathematics. Truly speaking, right. So it was crap what I said earlier.

Exactly when Gloria has changed her view on this is difficult to say, but at least it seems to be sometime after the work on the final essay assignment of the first module. The reason for saying so is that in the video clip from this module, she still seems to hold her first view when saying the following in her argument with Andrew (appendix C.1, lines 118-120): “You can't come up with some brilliant mathematical thing now – because so much have already been created – you can't do it without really spending a lot of energy on it.”

On another of the more historical questions also, Gloria appears to have changed her mind, namely the question on what researchers in mathematics do. In the first questionnaire, Gloria was the student who provided the following answer (question 18): "I haven't got a clue! But I have actually wondered about it, and I'm quite puzzled." In the followup interview Gloria was asked if she did not have any suggestions at all:

No, I've got no clue what they do. It is so weird spending your life sitting and being nerdy with formulas. They... is the purpose of their lives to find new formulas? Or do they wonder... I don't get it. I think it's really weird. Can you tell me what they do?

Yes, I can. But I'd prefer to do it at a later time, if that's okay. So that I don't put words into your mouth. But the mathematics in the textbooks, you believe that to have come into being a long, long time ago? [question 10]

But I don't know. I really don't have an idea about the history of mathematics.

That's okay, but what do you think?

No, it might as well be something which was found recently. I haven't got a clue.

As the previously discussed video clips in chapters 5 and 6 to some extent show, Gloria does develop an idea about the activities of mathematicians during the two modules. Her fourth questionnaire answer (question 22) confirms this. She refers to her answer to the previous question (20) on mathematics being a science, and if yes then about what: "Yes, it is science to understand practically all mathematics, and it belongs at a higher and higher level. Besides, becoming better at developing and improving applied mathematics." Gloria's answers to the question of mathematics being a science are consistent throughout all four questionnaires and followup interviews, but her answers become somewhat more justified and she is able to use elements from the modules in them, as she did with the reference to applied mathematics above.

This development from not having any idea about the activities of mathematicians, to having at least some idea, is articulated by Gloria herself in the fourth interview, when she is given the direct question of whether the modules had any influence on her view of mathematics.

You having gone through these two modules and learned something about the history of mathematics, or at least something about how two parts of the history have evolved, and how they're used in society and so, has that had an influence on your image of mathematics? I mean, what mathematics is, where it comes from, what it's used for...

No, but it's implied that I've gotten an image of it. We haven't had any history of mathematics before, you know.

No, but you've had an idea about what mathematics was before, right? And has anything about that changed, and if so...

You can't really say that, because before I hadn't thought about it at all. If I had something of an image, then I hadn't been doing any deeper reflections on it or such... but it's kind as if we've been forced to now... So I'd rather say that I have developed an image of it, and that it has started some thought processes for me.

7.6 Analysis, Discussion, and Recapitulation

Gloria's comment above is an essential one. If we refer to the beliefs literature discussed in chapter 3 (section 3.8) we may recall Lester saying: "... I do not think most students really think much about what they believe about mathematics and as a result are not very aware of their beliefs" (Lester, Jr., 2002, pp. 352-353). Gloria's comment "... before I hadn't thought about it at all" supports Lester's statement, and so do other of the data displayed above. Therefore, the students' answers to the first questionnaire and round of followup interviews must be seen in this light, i.e. that not all of their answers may be thought through and thus reflect their true beliefs on the matters in question. Gloria's comment, however, tells us another important thing, namely that if provided with a proper setting and suitable tools students may indeed be able to reflect on matters concerning mathematics as a discipline, and hence be made aware of their own beliefs about mathematics. And when given the opportunity, some students even seem capable of reflecting on their own beliefs, once having become aware of these, as well as possible changes in these and possible reasons for these changes.

Despite the students not necessarily being aware of some of their own beliefs in the first questionnaire and round of interviews, the answers in this first round still provide the best picture of the students' beliefs on a virginal basis, and as such they provide a sound starting point for identifying possible changes in the students' beliefs about the discipline of mathematics. The reason for this is that not only the exact beliefs and changes in these are of interest, but also the nature of the students' beliefs and the way in which they hold them – recall the research question (RQ3): *In what way may teaching modules focusing on the use of 'history as a goal' give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs?* As already indicated by the presented data, for the class in general and the focus group students in particular, some changes do occur during or after the implementation of the modules. However, with reference to the previously given definition of beliefs as something rather persistent (page 80), to say if these changes are in fact permanent changes in the students' beliefs is difficult, if not impossible, since that would require a more longitudinal study. On the other hand, we may say that the changes occur in the students' views, if we think of (and define) *views* to be something less persistent than beliefs, but with the potential to develop into beliefs at a later point in time. In this respect, we may also consider (and define) students' *images* of mathematics (as a discipline) to be made up of their beliefs as well as their views.

In the following I shall deal with the changes in the students' beliefs/views, first the more question and content specific ones; second, changes in the way the students hold their beliefs/views; and third, the nature of these changes, exemplified by the focus group students – bearing in mind Lester's comment. In so doing I shall draw upon elements of the beliefs literature as discussed in section 3.8. Next, I shall treat the possible attribution of the changes to the teaching modules in order to discuss answers for the third research question. Lastly, I shall do a recapitulation and pose some further questions.

Question and Content Specific Changes in the Students' Beliefs

First of all, one essential prerequisite for answering the research question at all is that changes in the students' beliefs/views during the one-year period of which the study

lasted actually do occur. As mentioned above, this has been established by the display of data for the class in general and the seven focus group students in particular. Having cleared that, let us then look at (or review) the specific content changes in the students' beliefs/views for each of the three classes of questions (sociological, philosophical, and historical), both for students in the class in general and for the focus group students.

From the overall picture of the students' answers to the sociologically oriented questions it is mainly possible to identify one specific change, namely a decrease in students believing mathematics to be a language (as opposed to a science). This change is illustrated by the case of Lucy, who went from believing mathematics to be a language, and therefore not a science, to believing it a science. As the majority of students Lucy did not, however, think the history of mathematics to be a science, at least not at the time of the second questionnaire. This may have been because she somehow believed science to equal natural science, a view that about half of the students still hold at the time of the fourth questionnaire. Lucy, however, changes her view on it. In the fourth questionnaire she answers that mathematics is not a natural science, since it has nothing to do with nature – an argument which a few other students also provide. Why so many students believe mathematics to be a natural science may be illustrated by the case of Lola. Already in the first round of questionnaires and interviews Lola is very firm in her belief on mathematics being a natural science. Some of the reasons she provides are that it has to do with nature, that it belongs to the faculty of natural science, and that it is used in and therefore closely connected to physics. After having believed mathematics to be a natural science, and therefore also a science, in the first three rounds of questionnaires and interviews, Lola takes an unusual turn, as compared to the other students, in the fourth round when she says that mathematics is not a science, since it is not made up of theories but of logic – a change I shall touch upon later also.

As for the philosophical questions, the overall picture shows a clear increase in students favoring invention over discovery during the one year period, regarding the negative numbers in particular and mathematics in general – a change which could be monitored quite closely by means of the related questions in the second and third questionnaire. For the concrete versions of the discovery-invention questions (the negative numbers, Hamming and Golay codes, and RSA), the class is equally split between believers in discovery and believers in invention, in the first and second questionnaire, but in the third and fourth questionnaire there is a shift to the majority believing in invention – a change which is consistent with that for the students' answers on mathematics in general. Jonathan's change in beliefs/views in regard to this question is an almost perfect illustration of the overall change in the class: discovery for both the negatives and mathematics in general in the first round; invention for the error correcting codes and public-key system in the second and third, respectively; and a combination of discovery and invention for mathematics in general in the fourth. The deviation from the overall picture is that Jonathan in the fourth round answers that the negatives are discovered. The case of Andrew illustrates a different story, namely that of the students who more or less stick to their original belief in discovery. But Andrew's story also illustrates the struggle some of the students have with the question of invention and discovery. In the third questionnaire Andrew is in doubt regarding the case of RSA, and in the fourth he changes his view on the negatives to invention, since they are unreal, and moderates his view on mathematics in general to be something you "primarily discover".

For the historical questions, the first overall notable change is that in the question on how mathematics has come into being, not many students talk about wise men etc. in the fourth questionnaire, instead they talk about researchers in mathematics or groups of researchers. In the next question on when mathematics came into being, more students are willing to provide ‘dates’ in the fourth round than in the first, and fewer students say that mathematics has existed always. A third change is in the number of students who are able to answer the question on what researchers in mathematics do. In the first questionnaire, many students did not answer this question at all, but in the fourth everyone do. Concerning the actual answers to the question, there is a significant decrease in students believing the job of mathematicians to be controlling old results and ‘cleaning up’. In the fourth, for example, the students talk about developing existing mathematics instead. The case of Gloria illustrates this, since she in the first round expressed her wondering about what mathematicians actually do, but answers that they may become “better at developing and improving the applied mathematics” in the fourth. Thus, this is in fact an example of new beliefs/views having developed, rather than old ones having changed. Gloria’s change in beliefs about the speed of development in mathematics should also be mentioned in relation to the historical questions. Gloria goes from believing that the development takes place at a slower and slower pace in the first round to arguing the opposite in the fourth – a change she is even able to explain and account for, a matter I shall return to below.

Changes in the Nature of the Students’ Beliefs

One thing is the question and content specific changes in the students’ beliefs, another thing is the changes in the nature of the beliefs, i.e. changes in the ways in which the students hold their beliefs. By this I am mainly referring to the hypotheses based on the comparison of the students’ first and fourth questionnaire answers. Namely, that the students’ beliefs in the later questionnaires seemed to have grown in consistency between the different questions, that more of the students began exemplifying their answers and that the students to a larger extent sought to justify their answers (beliefs or views).

One of the more interesting questions in terms of consistency is that of mathematics being discovered or invented. In the first questionnaire students would not necessarily answer the same to the question on the negative numbers as they did to the question on mathematics in general. In the fourth questionnaire only one student provides inconsistent answers to these two questions. Thus, there is a significant increase in consistency between these questions in the two rounds of questionnaires. An increase in consistency would be to expect, once students have become aware of their beliefs on a certain matter, at least if we adhere to the beliefs literature. As pointed out by Op’t Eynde et al. (2002, p. 25), people always strive for a coherent belief system, since this will make them function in a more intelligible way. Beliefs which are perceived incompatible will be changed and consequently effect other beliefs in the cluster of which they are part. The case of Jonathan illustrates this. As Jonathan changes his belief towards mathematics as invented, he also changes his view on ‘new mathematics’ to be a matter which may be talked about in a more natural way, something he had rejected earlier. The fact that more students in the fourth questionnaire find mathematics to be invented also affects the number of students who answer that mathematics has existed always (to the question of when mathematics came into being) – again a sign of

consistency. Another indication of an increase in consistency in the students' beliefs is that as many of the students change their view in favor of invention over discovery, some will also begin to talk about 'research' in their answers to the question of how mathematics has come into being. But of course there are also examples of students who seem to cling to their beliefs even though they appear to be inconsistent or in conflict with other of their beliefs. An example of this is the student who eventually says that she supposes that John Nash may have invented something new, but immediately rejects it because she believes mathematics to be discovered (see page 222).

The increase in the need to justify their beliefs may be noticed in the students' answers to several of the questionnaire questions, among them also that of invention and discovery. In fact, this particular question seems very suitable for discussing an increase in justification, since it was in some way present in all four questionnaires. Once the question of discovery and invention in the first questionnaire to some extent had made the students aware of their position on this matter, there was a significant increase in the number of justifications provided to the question of Hamming and Golay codes being discovered or invented in the second questionnaire (and in questionnaire three, but here they were asked to do it). In questionnaire four this happened to an even larger extent. Another question which is interesting in regard to justification is the question of when mathematics came into being. As mentioned above, more students were willing to fix periods in time in their answers to this question in round four than they were in round one, which again may be seen as an increase in the students' need to justify their answers. And also in the question of mathematics being a science, which was also asked repeatedly throughout the four questionnaires, there is an indication of an increase in justification. The case of Lola provides an example. In the first two rounds she does not provide much justification, though she is in fact asked to, but in the third and fourth rounds she is very keen on justifying her answers, both in the questionnaires and in the interviews. In fact, Lola's fourth questionnaire is generally much richer on justifications and elaborated answers than her first is. This is something which goes for Jonathan as well, also his fourth questionnaire answers provide more justifications than his first. Also Andrew, Harry, Sean, and Gloria are students who generally are keen on justifying their questionnaire and interview answers, and for whom an increase may be spotted throughout the four rounds. Especially Harry and Gloria are interesting in this respect, since they also are capable of justifying certain changes in their beliefs/views. Harry on his change in beliefs/views on the influence of mathematics in society and Gloria on her change in beliefs/views about the speed of development in mathematics.

One way of providing justification is of course through the use of examples and exemplifications. The first question to which the students really begin to exemplify their answers is the question of mathematics being a science in the second questionnaire. The examples provided do not differ much from those given in the first questionnaire, but the students provide them without having been asked to like they were the first time. Also, the answers in the fourth questionnaire bear witness to a slight increase in exemplifications over those in the first. Some students refer to other sciences such as physics, chemistry, and biology to argue that mathematics is or is not a (natural) science. Also for the historical questions do the students provide more examples, e.g. when fixing periods of the history. In the question of mathematics having a greater or lesser influence in society today than a hundred years ago, the students also provide more examples in the fourth round, some using elements from the modules. Andrew and

Harry are examples hereof. Sean exemplifies his answer to this question as well, though not by elements relating to the modules, with his talk about the industrialization. Even Lucy, who is generally not keen on providing justification for her answers, exemplifies some of her answers in the fourth round, in particular those to the historical questions. The interviews especially seem to be a situation where the students find it necessary to exemplify their views. An example of this is Andrew, who begins exemplifying his view of mathematics as a discovery by referring to number systems. And when asked to, he is also able to include elements of coding theory in his reasonings. Gloria, in her justification of her changed view on how fast mathematics evolves, refers to the development of the mobile phone.

A more precise discussion of which exemplifications may and which may not be ascribed to the effect of the modules will be given below. But before discussing these attributions of changes to the modules, I shall deal with another matter, namely the nature of the changes in the students beliefs – which is different from changes in the nature of the beliefs as just treated.

The Nature of Changes in Students' Beliefs

By the means of the studies of the focus group students we are able to deepen some of the observations made above, in order not only to say something about changes in specific beliefs and changes in the nature of these beliefs, but also something about the nature of these changes, that is the centrality of the beliefs and how evidentially they are held.

An element of the definition of beliefs (cf. section 3.8) was that they do not necessarily change easy, especially not beliefs which appear to be central. The case of Andrew shows this very well. Of course, identifying which of a student's beliefs are central and which are more peripheral may not be straightforward, and the literature does not seem to offer a clear-cut method to go about doing this. However, a reasonable assumption would be that beliefs upon which other beliefs (or views) are 'tested' in order to be confirmed or rejected may constitute the more central beliefs within a given cluster. Surely, what are central beliefs for one student need not be central beliefs for another student. Still, though, the matter of discovery versus invention appears to be a rather qualified candidate for a central belief. The reason for this is that the students frequently refer to this matter, both in their questionnaire answers to other questions and the followup interviews, e.g. when pointing out that the job of mathematicians is to "discover, not invent". Of course, as indicated by Lester, the students need not have been very aware of their beliefs concerning the discovery or invention of mathematics before they were given the question in the first questionnaire. But this does not mean that they did not have a belief on the matter, only that they had not articulated it previously – perhaps they had not been provided with the opportunity to do so. Another thing which may suggest that a belief is of a more central nature is the amount of struggle which a student encompasses when confronted with matters in conflict with the belief. This is exactly what happens for Andrew. Once having become aware of his belief that mathematics is discovered, he finds himself struggling when confronted with the question of RSA being discovered or invented and having to account for his answer – something which supports the centrality of Andrew's belief of mathematics as discovered. In terms of Cooney's (1999, p. 172) four types of standpoints toward changes in beliefs

(see page 87), Andrew may be viewed as a reflective connectionist, since he is able to accommodate his core belief in the fourth questionnaire to mathematics as something you “primarily discover”, and since he is able to do this through reflective thinking like the fourth interview gives evidence of. And as Cooney acknowledges with his ‘four types’, reflection is a core element in changing beliefs.

Reflection is also what appears to make Lola, Jonathan and Harry change their beliefs (or views) on mathematics as a natural science, discovery versus invention, and the influence of mathematics in society, respectively. For Lola, her belief of mathematics as a natural science also appears to be a central, or at least quite strong, belief since she holds on to it throughout the first three rounds of questionnaires and interviews. Though Lola’s reasons for rejecting mathematics as a natural science, and as a science in general, in the fourth round may appear somewhat unconventional, they still show proof of some reflections having taken place. Thus, also Lola may be seen as a reflective connectionist, in regard to this particular belief at least. Jonathan may be characterized as a reflective connectionist as well, due to his reasoning on the discovery and invention of mathematics as well as the reasoning on his own change of mind. How central this belief/view actually is for Jonathan is more questionable though. First of all it changes rather fast, already during the implementation of the first module. Second, it changes back a little by the time of the fourth questionnaire. This indicates that it may only have been a change in view rather than one in belief, and if it is one in beliefs then, at least, this belief is not as central for Jonathan as it was for Andrew. Another thing this touches upon is the fact that changes in beliefs need not last (Philipp, 2007). In section 3.8 it was pointed out that the higher the level of reflection associated with a change in the belief, the larger the probability that the change would last. Although this still seems a reasonable prediction, it does not completely fit Jonathan’s change. In the second and third rounds Jonathan seems quite an advocate for invention, and in the fourth he argues for a combination of invention and discovery. Still, the interviews as well as the fourth questionnaire answer in itself (“It is a mix, and I don’t think you can give a definite answer to it.”) indicate a high level of reflection. As stated earlier, Jonathan’s view on the question of discovery versus invention changes others of his views/beliefs as well in the striving for consistency. Such consequential changes in connected beliefs would be expected for reflective connectionists. Harry appears to also have reflected somewhat upon his view of the possible applications of mathematics. He states that he has come to realize that mathematics and the use of it are more complicated than what he had previously thought, e.g. in wars, and that this has affected his view on the influence of mathematics in society today. Though Harry also may be classified as a reflective connectionist, he is not necessarily one who is very willing to change his views on things – recall the discussion between him and Gloria on inner and outer driving forces during the essay writing of the second module (cf. section 6.6).

From Green (1971) we know that evidentially held beliefs are more likely to change through reflection and reason than non-evidentially held beliefs, i.e. beliefs which are not supported by any evidence in the form of examples etc. As seen in the previous subsection, the students do become better at exemplifying their beliefs through the four rounds of questionnaires and interviews, and as we have seen for the focus group students, many of their beliefs appear to be held in a more evidentially way by the time of the fourth round. However, not every example will function as sound evidence. The interviewee with the example of John Nash believes this to be supporting her view

on mathematics as something discovered (see page 222). But the functioning of this example as evidence is vague, because the student does not know anything about the mathematics of John Nash, she only knows that Russell Crowe, the actor portraying John Nash, drew some figures on a window pane. In this sense, the belief which this student holds is a non-evidentially held belief, and is as such not very likely to be changed – Green (1971, p. 48) phrases it in terms of the familiar attitude: “Don’t bother me with facts; I have made up my mind.” To refer to Cooney’s four types again, this student appears to be an isolationist, one whose beliefs are not connected, but separated and clustered, and who is not very capable of accommodation. For an isolationist it is typical not to acknowledge and deal with a conflict in beliefs, but instead to rebuff it.

Of course some kind of weighting must be applied to the centrality of a belief on one side and the evidence supporting it on the other side. A central and non-evidentially held belief is unlikely to change, whereas a peripheral and evidentially held belief should be more likely to change by means of reason and reflection (depending, of course, on the actual connections to a larger network of beliefs), and especially if the belief is articulated. A central but evidentially held belief may also change, though sometimes only through struggles, as we saw it with Andrew. But what about the peripheral and non-evidentially held beliefs? The case of Lucy may illustrate this. For Lucy, her initial belief of mathematics being a language does not appear to have been a very central one. Her only reason for holding it appears to be that the teacher had said so at one point in time. Lucy also changes the belief rather quickly, almost during the first interview, which indicates that it is somewhat peripheral. Lucy tries arguing that mathematics is just another form of communication, but soon gives up, indicating that the belief is not very evidentially held. As we know, Lucy changes her belief (or view); in the next three rounds of questionnaires and interviews she says that mathematics is a science. However, Lucy’s new belief does not appear to be a very evidentially held one either, at least she is neither able to justify nor exemplify it in the fourth interview. In fact, Lucy appears to be what Cooney terms a naive idealist, a received knower who, unlike the isolationist, absorbs what others believe, often without analyzing her own beliefs in respect to this. At any rate, this tells us that though a belief is non-evidentially held it may still be easy to change as long as it is not a central one.

The last thing I shall touch upon in this subsection also concerns the students’ capability to reflect. It appears to be that some of the more reflective connectionists not only are able to accommodate and reformulate beliefs or views, but also to articulate and/or account for changes in these. Jonathan is the first to do this, when he in class states that he has changed his view on the question of discovery and invention. And in the second round of interviews, he is able to account for the change as well. Also Harry can account for his change on the influence of mathematics in society. And Sean too. Gloria’s change in view on the speed of development in mathematics is another good example of this. Gloria is able to first account for her previous belief, and next to justify, by means of an example, her new view in a rather concise manner. Thus it seems, perhaps not surprising, that students who hold their beliefs more evidentially and are more reflective about the beliefs they hold (i.e. are reflective connectionist⁵) are

⁵ Cooney’s third type, the naive connectionist, who also holds a more reflective stance, but fails to resolve conflicts in beliefs, was not identified among the focus group students. For that reason I am not making any predictions on such persons’ capabilities to reflect upon changes in their own beliefs.

the ones most likely to articulate on their own and/or account for any changes which may occur in their beliefs.

Attributing Changes in Students' Beliefs to the Modules

The way in which students' changes in beliefs can be attributed to the modules is by looking at those of the students' exemplifications that relate to elements of the modules. Of course, this is only relevant for the more general questions, i.e. the ones which do not ask directly about elements of the modules or are historical 'test' questions, e.g. the question on the coordinate system, etc.

In the second questionnaire the students were given four such general questions on mathematics and history of mathematics as sciences and the conduction of these. For three of the questions (19, 20, and 21) there are examples of students exemplifying their answers by elements from the modules (Hamming and Golay, the when-and-who and the why-and-how accounts, and invention versus discovery). In the third questionnaire the students also had the opportunity to exemplify their answers to the general question of mathematics being a science, however none did.

The fourth questionnaire makes up the most interesting piece of data in terms of attributing changes to the modules, since the majority of the questions in this were of a general nature, not relating directly to the modules. I shall address the relevant questions beginning with the sociological, then the philosophical, and last the historical ones.

The question on where mathematics is applied in society and everyday life received a few answers in the fourth round which may be directly related to the modules (coding, cryptography, mobile phones, debit cards), since these were not mentioned in the first round. Andrew is one of the students who provide such an answer ("... programming, coding, cryptography"). The following question, where the students were asked to disregard the uses involving only basic arithmetic operations etc., on the other hand, resulted in many students providing answers relating to the cases of the modules (coding, cryptography, communication, encryption, decryption, Internet, home banking, mobile phones, computers). As mentioned earlier, this question was inspired by the first round of interviews. It was not meant as a 'test' question to see if the students could mention elements from the modules, rather it was thought of as a way of getting the students to answer other things than just shopping, budgeting, avoid getting cheated, etc. Nonetheless, the question certainly proved effective in getting the students to refer to the elements of the modules. The questionnaires by Lucy and Jonathan illustrate this since none of them were keen on providing answers relating to the modules in the fourth round, except in this particular question.

Another of the sociological questions received answers relating to the modules in the fourth round, namely that of the influence of mathematics in society today compared to a hundred years ago (e.g. codes, applied mathematics, Internet, banking systems). One example of this is Andrew, who answers that mathematics is used in most technical communication devices. Another is Harry, who talks about IT and in the fourth interview mentions electronic devices and cryptography in relation to this question and his change in view towards mathematics having a greater, as opposed to the same, influence today – a change he himself ascribes to the modules.

For the philosophical questions, a few students are also able to provide examples

referring to the modules in support of their answers to the question of mathematics becoming obsolete or not. One of them is Harry, who is responsible for the answer saying "... Caesar encryption has for instance become obsolete today, because it is too simple. You have a need to invent a new cryptographic method."

For two of the historical questions in particular, answers may be related to the effect of the modules. The first of these is the question on how mathematics has come into being, for which a few are using examples relating directly to the modules (coding theory, cryptography in war). Also some more indirect attributions of changes in beliefs/views may be made to the modules concerning this question. For instance, that students begin talking about groups of researchers they saw examples of in the second module (Diffie, Hellman, and Merkle; Rivest, Shamir, and Adleman). Or when students say that research has been conducted with certain applications or societal needs in mind, which may be related both to Hamming's development of error correcting codes and to Diffie's and Hellman's creation of public-key cryptography. The other historical question which receives answers relating to the modules is that of what researchers in mathematics do and what their jobs consist in. Examples are references to the discussion of objects and techniques from the first module, pure and applied mathematics as treated in the second module, and the questionnaire questions on invention and discovery. The fact that all students in the fourth round provide answers to this question, which many did not do in the first round, may also be interpreted as a consequence of the modules. The case of Gloria supports this, since she was very clear on not having an idea as to what a mathematician does in the first round, but in the fourth round provided an answer relating to applied mathematics.

The case for Gloria, however, is not so much that the modules have resulted in changes in her beliefs as the fact that they have prompted the development of new beliefs or views, something which she expresses herself in her answer to the direct question of how the modules may have influenced her image of mathematics (as a discipline). For Gloria the modules and rounds of questionnaires and interviews appear to have 'forced' her, as she puts it herself, to reflect deeper on matters of mathematics (as a discipline) and kickstarted thought processes, resulting in her actually obtaining an image of mathematics. For Harry the modules have provided him with proof of the applications of mathematics, he says, and in particular he mentions the use of mathematics in war and IT, also as something you do not hear about in television. (Harry was the student, who in the first module introductory essay was able to provide examples on the use of binary numbers in the early industrialization, punch cards in weaving, due to a television show he had watched. See page 117.) Jonathan says, already in the second round, that the modules have made him change his standpoint 'radically' on mathematics as being discovered or invented. For Jonathan the historical cases of the modules have provided him with some kind of "... examples of mathematics being invented" as he puts it, which of course is his own interpretation. Though Andrew does not change his mind on the matter of invention and discovery, the way he holds his view does. As already discussed, his view on mathematics as being discovered becomes more reflected during the four rounds, and in the end he is able to accommodate the conflicts which he finds that he has been exposed to during the modules. Thus, the modules, in some way or another, have provided Andrew with input for reflections on the beliefs which he already held – they have become more evidentially held. On the matter of mathematics being a science instead of a language, Lucy seems to find that the modules (the mathematics classes)

have provided her with a better insight into and a wider perspective of what mathematics is, and what it contains. Also Sean, though his exemplifications and justifications in the fourth round do not relate to the modules at all, seems to believe that the modules have provided him with some kind of an image of mathematics, to a larger degree than what he had prior to the modules anyway. In particular, he says that the modules were an eye opener to him, showing that mathematics has a greater influence than what he had imagined.

Of course, many other things apart from the modules may have caused the students to change their views during the one year period of the study. Sean's change in beliefs on the influence of mathematics in society was apparently brought about by the history of industrialization. Harry mentions matters of mathematical modeling in relation to the same question, something the class was taught about in between the third and fourth rounds of questionnaires and interviews. And what brought about Lola's change in view on mathematics not being a science is still an open question to me. Besides the influence the students are exposed to in school they are, of course, liable to have their views changed by factors outside of school as well. Several students, e.g. Harry, refer to tv-shows, movies, etc. in the followup interviews when discussing their questionnaire answers. On top of this, students of this age and at this educational level find themselves in a process of maturing personally and academically as well as becoming citizens in the society they are part of.⁶ In a sense, this may be a rather impressionable age for these students, and they are exposed to a variety of influences on a daily basis, all of these adding to the amount of uncontrollable parameters in studies on students' changes in beliefs (and views). However, on the overall it seems a reasonable assumption that only a limited amount of these influences (directly) concern the students' view of mathematics (as a discipline). For that reason, and due to the impressionability of the students, the two teaching modules may have been a massive feature in regard to their views on mathematics. However, the actual attributing of changes in beliefs and views to the modules is not an easy task. But if students in their answers to the general questions in the fourth round relate to matters from the modules it appears reasonable to ascribe their views as well as any changes in these to the modules – not least because the fourth round took place almost five months after the completion of the second module.

Recapitulation

To recapitulate, the first important issues concerning an answering of the third research question are that the students' beliefs or views actually do change, that the changes can be documented, and that examples of students agreeing to these changes can be displayed. Next, it is essential that some of the changes can be attributed to the teaching modules, that the attributions to some extent can be accounted for, and, again, that examples of students ascribing their own changes to the modules can be displayed as well. Having established this, it is then possible to move on to the ways in which the modules have changed the students' beliefs or views. This was accounted for above by, first, addressing the question and content specific changes in students' beliefs/views, and, second, by addressing the changes in the nature of students' beliefs/views. Concerning the latter, three changes in the nature of the students beliefs/views were observed: a

⁶ In Denmark you become of age at 18 and is given the right to vote.

growth in consistency between the students' related beliefs/views; that the students to a larger extent sought to justify their beliefs and views; and that students provided a larger amount of exemplifications in support of the beliefs and views they held, i.e. that the beliefs appeared to be held more evidentially.

To comment on the distinction between beliefs and views, what I have detected may in many cases only be regarded as changes in views. But changes, however, which hold the potential to and perhaps later may develop into changes in beliefs. Whether this actually happens is not to say without conducting further longitudinal studies. Nevertheless, despite the changes being quite different from student to student, the changes which occurred during the implementations and which were still present five months after the second module (fourth round) and which further brought about changes in related beliefs and views, e.g. to uphold a level of consistency, suggest that some permanent changes may have taken place. At any rate, the changes being in beliefs or views, it seems fair to claim that the teaching modules did have as an outcome that some students ended up with more balanced, multifaceted, reflective, and profound images of mathematics as a discipline.

Concerning these images, the important thing is not necessarily whether the students believe one or the other, e.g. if mathematics is discovered or invented. What matters is that the students have given thought to the beliefs they hold, that they have reflected upon their beliefs and views, tried to accommodate and reformulate their beliefs in case of conflicts and contradictions, and that the beliefs they hold are evidentially held to the largest extent possible. Thus, another important finding of this study is that the students were invited to think and reflect about their views, and that they actually did so. However, as previously pointed out, for this to be possible a certain setting is needed. A scene must be set in which the students are given the opportunity to become aware of their beliefs (on mathematics as a discipline) and then are provided with tools (e.g. the general topics and issues related to the history and historiography of mathematics as used in the essays assignments) and evidence (the two historical cases) on which to test and possibly change their beliefs or views. This is supported by the KOM-report's claim that for overview and judgement on the evolution of mathematics to have any solidity and weight, it must rest on concrete examples from the history of mathematics (see page 53). Part of the definition of beliefs is that they are somewhat persistent (more persistent than attitudes and emotions) and not necessarily easy changed. Still, as pointed to earlier, evidentially held beliefs are more likely to change through reflection and reason than non-evidentially held beliefs (depending on the centrality of the beliefs, of course). That is to say, we cannot expect students' (core) beliefs to change in any way if are they not confronted with some 'evidence'. Not until students have access to evidence – or counter-evidence – are they likely to criticize rationally, reason about, and reflect upon their beliefs, and possibly accommodate and change them, should they find it necessary.

This discussion of students' reflected beliefs links to that of students' meta-issue discussions as treated in the two previous chapters on the teaching modules. And it actually raises a few new questions. For example, are those students who are more capable of carrying out meta-issue discussions also the ones who hold the more reflected beliefs, i.e. are they reflective connectionists? An important element of the analysis of the students' work on the essay assignments was that of the anchoring of their work with the meta-issues in the related mathematical in-issues of the teaching modules. Therefore,

yet a relevant topic concerns the connection between students' reflected beliefs and their understanding of mathematical in-issues. Can anything be said on this, e.g. by looking into the data of the focus group students once more?

These questions are among those which I shall deal with in the next chapter (8), the discussion chapter of the dissertation, by means of proposing a model which to some degree may operationalize what may be meant by students' reflected beliefs about mathematics as a discipline. In chapter 8 I shall also attempt to lift the discussion of the three research questions of the dissertation to a more general level, that is to say that I shall address the generality (and generalizability) as well as the importance of the more case-specific answers already given to the research questions.

8 Discussion and Conclusions

This final chapter of the dissertation contains the discussion and conclusions of the research study (empirical and analytical), and will be structured in terms of:

1. the questions asked and to some degree already answered,
2. related questions not asked but possible to answer,
3. related questions not asked and still not possible to answer.

The first group of questions, of course, make up the research questions; the three (RQ1-3) posed in chapter 3 and the preliminary one (RQ0), posed and answered in chapter 2. For these questions I shall begin by summarizing the answers already provided throughout the previous chapters (2, 5, 6, and 7).

The second group of questions consists of those which did not make up the basis for the conducted research, i.e. were not among the research questions, but which the gathered empirical data allow for an answering of anyway. Some of these questions have already been posed at the ends of chapters 5, 6, and 7, but I shall rephrase them before answering them in this chapter.

However, in order to round off the final conclusions for both the first and second group of questions some further discussions are needed. More precisely, discussions of

1. the *trustworthiness* (validity and reliability) of the findings,
2. the *generality* (and generalizability) of the findings on which the conclusions are drawn, and
3. the *importance* of the findings and conclusions.

In addressing these three dimensions of (educational) research, as discussed by Schoenfeld (2007), I shall, as promised, revisit the methodology (chapter 4) used for the empirical studies once more.

The third group of questions are those of a wider perspective, for which further (empirical) studies are required in order to provide answers.

8.1 Recapitulating Results for the Research Questions

I shall begin with the preliminary and analytical research question (RQ0) from chapter 2 on whys and hows of using history in mathematics education. Afterwards, I shall sum up the (case-specific) results for the three empirically oriented research questions (RQ1-3).

The Whys and Hows of Using History and Their Interrelations

RQ0 What different arguments for using history (the ‘whys’) and what different approaches to using history (the ‘hows’) are given in the literature, and what are the interrelations of these arguments and approaches?

As an answer to this question two sets of categories were proposed, one set for the whys and one set for the hows. Next, the interrelations between the categories in the two sets were explored, in order to point out which combinations are the most favorable if one wishes to fulfill a certain purpose (a why) of using history.

The whys were categorized into the two categories:

- history as a tool,
- history as a goal.

As explained in Jankvist (2009a) also, *history as a tool* concerns the use of history as an assisting means, or an aid, in the teaching and learning of mathematics (mathematical concepts, theories, methods, and so forth). In this respect, different roles which history may play were discussed: history as a motivational and affective tool; history as a cognitive tool (e.g. the idea of epistemological obstacles); and the role of history in the evolutionary arguments (the recapitulation argument or historical parallelism). In the category of *history as a goal*, history does not serve the primary purpose of being an aid, but rather that of being an aim in itself. In this sense, it is for instance considered a goal to show students that mathematics exists and evolves in time and space; that it is a discipline which has undergone an evolution and not something arisen out of thin air; that this evolution has been driven by outer as well as inner driving forces and mechanisms; that human beings have taken part in the evolution; that the evolution of mathematics is due to many different cultures throughout history, and that these cultures have had an influence on the shaping of mathematics as well as the other way round. Of course, the use of history as a goal may have a positive side effect of assisting the learning of mathematics, but the important thing is that it is not the primary purpose for using history here.

The hows of using history were placed into three categories (see also Jankvist, 2009a):

- the illumination approaches,
- the modules approaches,
- the history-based approaches.

In the *illumination approaches* the teaching and learning of mathematics, whether it is the actual classroom teaching or the textbooks used, is supplemented by historical information (varying in size and scope). These supplements can be thought of as ‘spices’ added to the presentation of a given mathematical topic. The *modules approaches* are instruction units devoted to history and quite often they are based on historical cases. These may also vary in size and scope, from small curriculum-tied modules, over longer and more free modules to full courses (or books) on the history of mathematics within a mathematics program. The ways in which such modules may be implemented are numerous. Examples are textbook studies, readings of original sources, student projects, historical plays, searches on the Internet, worksheets, historical problems, mechanical instruments, etc. The *history-based approaches* cover the approaches directly inspired by or based on the development and history of mathematics. Unlike the modules approaches, these approaches do not necessarily deal with the study of history of mathematics in a direct manner, but rather in an indirect fashion. The historical development is not necessarily discussed in the open. On the other hand, it often sets the agenda for the order and way in which the mathematical topics are presented. Thus, the history becomes a fully integrated part of the approaches – one might think of them as ‘historical approaches’ (examples discussed were the genetic method and guided reinvention).

As a way of deepening the idea of history as a tool and history as a goal and the differences between the two categories – and as a way of exploring the interrelations between the categories of whys and hows – the notions of in-issues and meta-issues were introduced. The term *in-issues* refers to the inner issues of mathematics, e.g. concepts, theories, methods, algorithms, etc. On the other hand, *meta-issues* (meta-perspective issues) refer to the many elements of ‘mathematics as a discipline’ (cf. section 3.8) including those related to history as a goal. Thus, where history as a goal concerns the teaching of meta-issues of mathematics, history as a tool concerns the teaching and learning of in-issues of mathematics.

When looking into and analyzing the interrelations of the whys and hows, some approaches can be said to be more suitable for bringing out the meta-issues of the history of mathematics and some for teaching the in-issues. The illumination approaches, although they may be used to illuminate some meta-issues on a small scale, are most suitable in terms of in-issues and history as a tool, especially in relation to the motivational and affective aspects of such a use. The cognitive history-as-a-tool arguments may often stand a better chance by means of a modules approach. The same appears to be the case for the history-as-a-goal arguments in general, since modules approaches open up for a variety of ways of bringing the students to work with different meta-issues. The history-based approaches are concerned with the learning of in-issues, that is the history-as-a-tool arguments. The evolutionary tool arguments, but also the cognitive, are well suited for being realized through a history-based approach.

Finally, the tool-goal categorization and its subcategories of different tool arguments also proved a means for classifying the literature on using history, at least the identified empirical research studies (adding one category of overarching studies).

Students’ Abilities to Engage in Meta-Issue Discussions

RQ1 In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of ‘history as a goal’?

From a theoretical standpoint, the answer to RQ0 told us that ‘history as a goal’ may be best realized through a modules approach. Thus, the study of RQ1 was based on two specially designed teaching modules dealing with selected cases of the history of (modern and applied) mathematics. From an existence proof perspective, the implementations of the two modules provided evidence of it being quite possible to have students discuss and reflect upon meta-issues of mathematics and its history in terms of history as a goal – and this based on some, for the students at least, quite demanding new mathematical topics. To make this existence proof a bit more constructive, I now address, in turn, the matters of sense, extent, and conditions as asked in RQ1.

More precisely, the students’ meta-issue discussions and reflections of mathematics and its history concerned elements of the following selected general topics and issues related to the history and historiography of mathematics (Jankvist, 2009c):

- inner and outer driving forces of mathematics,
- epistemic objects and epistemic techniques,
- multiple developments, e.g. in the case of ascribing credit,
- the relationship between pure and applied mathematics.

In addition to these, the students were also able to engage in discussions of mathematics being invented and/or discovered and mathematics being a science and/or a natural science (in the sense explained in section 7.1). The findings concerning the latter two are mainly due to the four rounds of questionnaires and followup interviews, whereas the findings on students' discussions of the four general topics and issues are based on the students' hand-in essays, their questionnaire answers, followup interviews, and the videos of the focus group. The discussions between the focus group students further showed that meta-issue reflections take place on the intrapersonal as well as the interpersonal communicative level. In particular regarding the interpersonal communication, the focus group students were capable of carrying out their discussions by shifting between various meta-issue discourses: historical, sociological, psychological, and philosophical (sometimes ontological). And in the process, the students were also sometimes able to resolve meta-issue discursive conflicts arising during the discussions.

Regarding the extent to which the students were able to treat the meta-issues there were some limitations. One limitation concerns the observation that at some points the students' discussions and answers to some of the essay questions became a bit home-spun and 'fluffy'. One clear example of this was the essay on inner and outer driving forces, where most students interpreted the forces to be of a personal nature for the individual researcher. This results in some limits for the quality of the students' meta-issue discussions regarding this question as well as the degree of reflections the students may have as part of their work on the essay. Partly this may be explained by the essay questions probably not always having been posed in the most remunerative way, sometimes they may have invited more to fluffiness than necessary. By 'necessary' I am referring to the fact that some of the general topics and issues indeed are difficult, and not only for students, but also for historians of mathematics, since they may require a clarification and untangling of concepts, notions, and terms applied. Thus, a minimum of fluffiness seems to be unavoidable when having students work with the above general topics and issues of mathematics. In the second module the students were asked what insights they thought could be gained by approaching the history of mathematics by inner and outer driving forces, and epistemic objects and techniques, respectively – questions which most students did not answer or only answered poorly. Of course, one possibility is that the students did not understand these general topics and issues, and for some this probably was the case. However, it may be argued that the two questions in some sense are on another cognitive level, since the students will have to reflect on their own learning and relate this to the conduction of history of mathematics in order to provide answers to them. Concerning how to do history of mathematics, the students were able to discuss aspects of this when comparing their when-and-who and their why-and-how accounts, and as we have seen some students were also able to reflect upon developments in their own beliefs and views. However, combining these capabilities by performing meta-issue reflections and articulating insights or differences in insights was apparently a more difficult task for the students.

As for the conditions on which it is possible to have the students engage in discussions of and perform reflections on meta-issues, the setting of doing essay assignments and doing these in groups played a significant role. Whether the teacher focused on the in-issues or the meta-issues, the applications or the history in the modules, the essay assignments forced the students to work with meta-issues in terms of history as a goal (Jankvist, 2008b; Jankvist, 2008j). Of course, the actual hand-in essays did not always

appear equally reflective. On the one hand, this may be taken as a sign of limitations in the students' reflections, since they were not able to articulate them in writing. On the other hand, as we saw for the focus group in the first module, a poor hand-in essay need not necessarily reflect the actual discussions which have taken place as part of the work on the essay – the discussions may be quite rich and reflective despite a poor hand-in. Anyhow, the questionnaires showed that some learning had taken place, even in the case of the more vapid hand-in essays. Again, it should be mentioned that it was potentially possible for the students to be orally examined in parts of the modules, which may have had an effect on some students' participation.

Another thing which was learned from observing the focus group discussions was that the students of their own accord would sometimes perform a small role play in order to discuss differing views against each other (the situation with Harry 'playing' G.H. Hardy). Such a setting seemed to be rather beneficial for the students in the given situation, and it could perhaps be explored as a way of having the students engage in meta-issue discussions in future designs. Part of the success of a certain group's encounter with meta-issue discussions and reflections has to do with the actual group composition, as illustrated by the recomposition of the focus group. Regarding the way in which the essay questions were posed, there was also a lesson to be learned from the first module, namely that the students seem to respond better to more open questions with somewhat open answers rather than questions where they are to 'guess the correct answer' (third supportive essay on the story of the Golay \mathcal{G}_{23} code). Of course, there must be a balance, because we do not want the questions to invite to too much fluffiness as mentioned above.

Anchoring of the Meta-Issues in the In-Issues

RQ2 In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and 'ensured' through a 'modules approach'?

I shall divide the answering of this question in two: one concerning the 'insurance' of anchoring and how this may be reached, something closely related to matters of design; and another focusing on the different levels or kinds of anchoring observed during the modules.

One way of trying to increase (ensure) the possibility of anchoring the students' meta-issue discussions and reflections in the taught and learned in-issues of the teaching modules was through the choosing of general topics and issues related to the history and historiography of mathematics (Jankvist, 2009c). Especially the issue of epistemic objects and techniques was believed to possess a high degree of potential 'insurance' of anchoring, since the students, in order to discuss the historical cases using these notions, would have to identify actual objects and techniques. The first module provides evidence of this assumption being true. In the second module the 'anchoring insurance parameters' were somewhat loosened in order to see to what degree an anchoring would still occur. As debated for RQ1, this resulted in some fluffiness of the students' meta-issue discussions, and for the essay on the general topic of inner and outer driving forces it resulted in no anchoring whatsoever in the in-issues of the module. Having asked the students to relate their discussions to what mathematics was already available on the 'shelves' and what was not, might have been one way of steering their discussions towards an

anchoring. The same idea would apply to the essay on pure and applied mathematics, since this topic could also be attacked through questions about availability on the ‘shelves’. However, the students’ discussions of this topic were generally much more anchored – they discussed the notion of primes – which can be a consequence of having them read and use Hardy’s *Apology*. A similar observation concerns the main-essays and the students’ when-and-who and why-and-how accounts of the historical cases. In both modules elements of anchoring were detected in the students’ hand-ins of these accounts. In the first module mainly the why-and-how account, but in the second also in the when-and-who account. To some degree this suggests that if the teaching material used and the teaching given have a strong focus on mathematical in-issues, then the used design will often (but not always) result in some kind of transfer to the students’ essays. Another element of the design, which should be touched upon in this first part of answering RQ2, is the use of extracts from original sources, since these from time to time served as a way of having the students work with the in-issues of the historical cases. For the historical exercises in the second module the purpose was more one of Jahnke’s replacement and reorientation (Jahnke, 2000, page 35) in terms of challenging the students’ perception by replacing the familiar with something unfamiliar, e.g. having them see that the number theoretic results in modern notation that they had been introduced to in the teaching material were, in fact, equivalent to those in the original sources. However, the real use of original sources in relation to anchoring took place in those essay assignments where the students were asked to discuss a general topic or issue related to the history of mathematics based on extracts from an original source, e.g. when discussing Hamming’s use of already established mathematical techniques in his 1950-paper. In this case, using a piece of original text together with the general issue of epistemic objects and techniques ensured, on the one hand, Jahnke’s third effect of using original sources (cultural understanding, see page 35) in terms of allowing the students to place a development of mathematics in a scientific, technological, and societal context of a given time and place, and on the other hand it did this in a way so that the discussions of these meta-issues were anchored in the mathematical in-issues of Hamming’s 1950-paper (binary numbers, metrics, the concept of linearity, n -dimensional cubes, etc.).

Still, as might be expected, the anchoring of the students’ meta-issue discussions and reflections occur on different levels. The analysis of the video data of the focus group suggests the existence of four different levels:

1. the non-anchored,
2. the anchored comments,
3. the anchored arguments,
4. the anchored discussions.

As mentioned above, some of the focus group’s more fluffy discussions during the second module were certainly *non-anchored*. Examples of *anchored comments* were Gloria’s comment on Hamming spheres in the first module and Jonathan’s comment on one-way functions in the second. An example of an *anchored argument* was when Gloria changed to a mathematical discourse bringing up Hardy and primes, in order to make the point of her meta-issue social discourse, concerning personal outer driving forces for Diffie. The difference between an anchored comment and an anchored argument is that an anchored argument uses mathematical in-issues to underpin and substantiate a meta-issue claim, whereas the anchored comment is a remark deepening the meta-issue discussion by

referring to related in-issues, the understanding of which may be confirmed by other data (hand-in exercises etc.). Anchored comments and anchored arguments may have the potential to turn into *anchored discussions*, if they are picked up by other group members, e.g. by becoming a reference point within a mathematical discourse of a discussion. Examples of this are Andrew's continuous reference to the (7,4)-code in the first module, which was picked up by Jonathan, and in the second module where the meaning and notion of primes came to function as a basis for Harry, Lola, and Andrew, when discussing Hardy's view on number theory, thus qualifying their discussion of pure and applied mathematics substantially.

Hence, the strong mathematical focus of the teaching material, the design of the modules with essay assignments to be worked on in groups, and the use of original sources in combination with carefully chosen general topics and issues related to the history of mathematics appear to have made it possible to ensure anchored discussions (the fourth level of anchoring) by enabling the students to treat the meta-issues on an in-issue basis. Thus, the students' encounter with history of mathematics was not anecdotic in terms of disconnecting the history from the mathematics, as is often the case for the textbooks used in the Danish upper secondary school (Jankvist, 2008a). Nor was the treatment of the historical mathematics anachronical or Whig, a problem pointed out by Fried (2001) (see also page 300). The latter was ensured by the use of extracts from original sources, e.g. the old number theory, as well as the choice of historical cases from modern (applied) mathematics, which for instance meant that the mathematical notation in the teaching module differed only slightly from that used by Shannon, Hamming, Golay, Diffie and Hellman, and Rivest, Shamir and Adleman.

Changes in Students' Beliefs about Mathematics as a Discipline

RQ3 In what way may teaching modules focusing on the use of 'history as a goal' give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs?

To address this research question the beliefs held by the students had to be detected first. Next, possible changes in these beliefs had to be detected. And finally, if such changes were detectable, the cause of change had to be identified. Thus, a prerequisite result for answering RQ3 is that changes and new developments actually did occur. That this was the case is seen from the analysis of questionnaires and interviews. Besides being a prerequisite for answering RQ3, this is an important and independent result in itself.

When reporting the findings on RQ3, a distinction was made between students' *beliefs* and students' *views*. Where beliefs are defined as being rather persistent and somewhat hard to change, views were taken to be less persistent, but with the potential to develop into beliefs at a later point in time (or maybe they already had, only we have no way of telling). Students' *images* of mathematics (as a discipline) were defined as consisting of both their beliefs and views. In this sense, the answers to the first round of questionnaires and interviews provided a picture of the students' beliefs on a virginal basis, whereas the answers to the fourth round to a larger extent portrayed students' views, either in the form of new views (which may or may not be or become beliefs) or as changes in old beliefs. Having settled these necessary conditions for answering RQ3, the next order of business is to describe the ways in which the changes and developments have taken place and form, and to what extent they may be ascribed to the teaching

modules.

As for the question and content specific changes in the students' beliefs or views, the fourth round indicates that much fewer students now think mathematics to have come into being because of some wise or "half-autistic" men. Instead they talk about researchers in mathematics or groups of researchers. As to the question on what these mathematicians' jobs consists in, there is a significant increase in the number of students who are willing to provide answers in the fourth round. And in the answers given, there is a decrease in the number of students saying that the job of mathematicians is to control old results or 'clean up'. Instead, the developing of already existing or new mathematics, both pure and applied, are mentioned. Also the students are more willing to suggest dates for when the mathematics came into being in the fourth round compared to the first, although these suggestions do not always seem too rooted in actual knowledge. This fits with fewer students in the fourth round believing mathematics to always have existed, which again fits with the shift from most students believing mathematics to be something you discover in the first round to believing it to be something you invent or being a combination of discovery and invention in the fourth round.

A general observation made during the fourth round was that a larger amount of the students' answers to certain questions corresponded to their answers to other (related) questions. In fact, three changes in the nature of the students' beliefs and/or views were observed throughout the four rounds of questionnaires and interviews:

- a growth in consistency in the students' related beliefs and/or views;
- an increase in the number of students who to a larger extent sought to justify their beliefs and views,
- that students provided a larger amount of exemplifications in support of their beliefs and views in the later rounds, i.e. that the beliefs appeared to be held more evidentially.

Especially the third of these, but also the second, provided a way of ascribing the changes in students' beliefs and views to the modules, since many of the examples given by students in support of their views could be related directly to elements of the modules, for example the general topics and issues of history of mathematics used in the final essays.

In addition to the observed changes in the nature of the students' beliefs, some observations on the nature of the changes were also made, in particular concerning the linking of the centrality of students' beliefs (core beliefs versus peripheral beliefs) with how evidentially the beliefs were held (evidentially held beliefs versus non-evidentially held beliefs). In regard to this, three of Cooney's four types of student standpoints toward changes in beliefs were observed among the interviewees: the isolationist, the naive idealist, and the reflective connectionist (Cooney, 1999). The observations suggest that not only are the reflective connectionists able to accommodate and reformulate their beliefs, which is what defines such students' standpoints, some of them are also able to articulate and/or account for changes in their own beliefs and views, in particular the ones who hold their beliefs and views more evidentially and are more reflective about them.

As said, the detected *views* of the students in the fourth round have the potential to perhaps later become changes in beliefs. Of course, whether this actually happens is difficult to say without conducting a further longitudinal study. Nevertheless, despite the changes being quite different from student to student, the ones which occurred

during the implementations and which still were detected again five months after the second module (in the fourth round) and which further brought about changes in other related beliefs and views, e.g. to uphold a level of consistency, seem to suggest some kind of permanence. At any rate, the changes being in beliefs or views, it seems fair to claim that the teaching modules did have as an outcome that some students ended up with more balanced, multifaceted, reflective, and profound images of mathematics as a discipline.

Regarding the fostering of such images it was argued, also from a theoretical point of view, that a scene must be set for this to happen. A scene in which the students are given the opportunity to become aware of their beliefs (on mathematics as a discipline), and next are provided with tools (e.g. the general topics and issues related to the history of mathematics as used in the essays assignments), and evidence (in this study, the two historical cases) on which to test and possibly change their beliefs or views. Part of the definition of beliefs is that they are somewhat persistent (more persistent than attitudes and emotions) and not necessarily easily changed (Philipp, 2007). Still, as pointed out by Green (1971), evidentially held beliefs are more likely to change through reflection, reason, and rational criticism than non-evidentially held beliefs (depending on the centrality of the beliefs, of course), which means that we cannot expect students' (core) beliefs to change in any way, if are they not confronted with evidence. Not until students have access to evidence, or counter-evidence, are they likely to criticize rationally, reason about, and reflect upon the beliefs they hold, and possibly accommodate, reformulate, or change them.

Summarizing the Answers and Validating the Initial Hypotheses

I shall briefly summarize the empirical findings for research questions 1-3 in order to validate the initial hypotheses posted in section 1.3.

- RQ1 It is possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history, provided that they are given a setting in which to do this. Essay assignments and the use of general topics and issues related to the history of mathematics is one way of providing such a setting. If chosen carefully, these topics and issues may also help ward off some the expected fluffiness of the students' discussions and lead these along more mathematical discourses.
- RQ2 That is to say, the use of general topics and issues proved a way to ensure an anchoring of the students' meta-issue discussions in the related in-issues, thus qualifying their meta-issue discussions and reflections. In particular, four different levels regarding anchoring of the meta-issue discussions were identified (non-anchored, anchored comments, arguments, and discussions). Thus, the students' encounter with the history of mathematics was not anecdotic, nor was it anachronical or Whig, something ensured by the use of extracts from original sources whose notations were close to those of the teaching material due to the historical cases being of newer data.
- RQ3 Changes and new developments in students' beliefs and/or views were found. Some changes were question and content specific, but others were changes in the nature of the beliefs/views and the way the students' held these. In particular the following observations were made during the one year study: a growth in

consistency between the students' related beliefs and/or views, an increase in the desire to justify their beliefs and/or views, and a larger amount of exemplifications being offered in support of their beliefs and/or views. The latter suggests that the students' beliefs/views are held more evidentially, in particular because many of the exemplifications were related directly to the cases of the modules. In this respect, it can be concluded that the modules provided some students with more profound, balanced, multifaceted, and reflected images of mathematics as a discipline.

The above answers validate the four initial hypotheses as given in the introductory chapter for the empirical research study, and as such make up the existence proof of these hypotheses being true. In this respect, the answers also serve as a way of making this existence proof into a more constructive one, in order to make it possible to reproduce the results with similar designs.

The matter of reproducibility shall be discussed as part of Schoenfeld's dimension of trustworthiness once I revisit the methodology used to gather data to answer the research questions (see section 8.3). But before doing that I shall discuss the second group of questions, those not asked but yet possible to answer, since an answering of these also may be subject to certain methodological considerations in hindsight.

8.2 Questions Not Asked but Possible to Answer

There are mainly seven questions coming from the analysis of the data or the answers to the research questions, which seem appropriate to pose and possible to answer – some with a little further display of data. The two first concern the experience of the involved students and the teacher:

- a. What were the students' experiences with and opinions about the two teaching modules?
- b. What were the teachers' initial expectations to, experiences with, and opinion about the modules, as well as any further work with these she might consider doing?

The next three questions address the positive side effects and added bonuses of the modules focusing on history as a goal:

- c. Which positive side effects, if any, did the two modules focusing on 'history as a goal' have in terms of 'history as a tool'?
- d. What effects, if any, did the choosing of a newer history of mathematics over an old or antique one seem to have on the students?
- e. Which other meta-issues of 'mathematics as a discipline', besides those concerned with 'history as a goal', did the students learn something about due to the set up of the modules and/or the research study?

The next question (f) is generated by the findings of RQ3, and the final question (g) in a sense cuts across the three research questions (RQ1-3) by addressing a possible combination of the findings for them:

- f. From the findings and conclusions to the research questions, especially RQ3, what may we understand by *students' reflected beliefs* about mathematics as a discipline?
- g. What does the research study tell us about the relationships between students' abilities to engage in discussions of meta-issues, their anchoring of these discussions in the related in-issues, and how reflected their beliefs are?

(a) Students' Experiences with and Opinions about the Modules

To answer question a, I shall begin by presenting the needed data from questionnaires and interviews, and in the end discuss these in order to provide an answer. As for the students' experiences with the modules there are many ways to address this, one being changes in beliefs and views as already discussed in length. Here, however, I shall focus on their more articulated experiences in form of their answers to question 5 in the second and third questionnaire, asking them what they considered to be the most important thing(s) they had learned from the two modules, respectively, and the similar question 23 in questionnaire four (please refer to appendix B for the exact formulations of the questions). I shall present the questionnaire data following the model used in the three previous chapters (one < few < some < many < the majority < the *vast* majority).

To question 5 in the second questionnaire, some students mention mathematical in-issues such as binary numbers and the binary number system to be the most important things they have learned. Many, however, refer to meta-issues. Some of them mention aspects of application, e.g. how transmitted information can be disturbed and errors corrected, that there are redundant data in transmissions, or the fact that we use codes to communicate. And some others provide answers such as "the theory behind", e.g. by referring to matters of history (Hamming etc.). Few say they "don't know". Some of the more expressive answers relating to meta-issues were actually given by focus group students:

Lucy: That coding theory is also a part of mathematics.

Andrew: How mathematics can be used to develop a system/a technology.

Gloria: I thought that all the techniques were the most fun to learn/know about. But the most important thing for me has been to get an understanding of how signals/informations are transmitted and received without misunderstandings [errors]. I often think about it when I send text messages on my mobile phone.

For the second module, in the third questionnaire, some students again mention mathematical in-issues such as primes, prime factorization, relatively prime, modulo, and proof by induction to be the most important things they have learned. Many, however, focus on the meta-issues this time also. Some in relation to matters of application, e.g. the use of cryptography on the Internet, and some in relation to matters of history, e.g. who came up with what, Enigma and the development of secret cryptography keys, or the history of number theory and how it was used in RSA. One student says: "That mathematics may be useful even though it's ancient." A few of the more exceptional answers are those of Andrew and Lucy, who say, respectively: "Matters of the development of different cryptography methods/systems, and how most of these are connected to mathematics."; "Just how much mathematics there is in cryptography." Again some do not answer or say they "don't know". The followup interviews for the second and third questionnaire do not provide any additional information for question 5.

To question 23 in questionnaire four, there is a decrease in the number of students (to 'few') who give answers relating to in-issues of the modules. The number of students who relate to meta-issues has increased; this time it is the majority who do so. Of these, a few relate to matters of application. Many, however, relate to matters of history, which is an increase from the previous questionnaires. A few provide more singular answers. The focus group students' answers illustrate the situation quite well.

Lola: How the codes were used during World War 2, and how they were found in a clever way – a machine [Enigma] with so many codes.

Andrew: Investigations/further development of mathematics.

Harry: History and theory both, at least the part which you could follow.

Jonathan: Working with math in a different way, and getting insight into applied mathematics and number theory, cryptography and coding theory.

Gloria: The things with pure and applied mathematics.

Lucy: That mathematics covers a much larger area than you would think.

Sean: That I do NOT want an occupation in which history of mathematics is a part. I didn't get anything specific out of it.

Again some students do not answer, though not the same as those who did not answer question 5 in the two previous questionnaires. In the followup interviews, Gloria deepened her answer above by saying that she found the discussion on pure and applied mathematics to be the most exciting one, and when asked why she replied:

Right now I'm thinking that there are kind of two different ways of developing mathematics: if you develop it for the sake of mathematics or for the sake of application, I mean for what it's going to be used for, the purpose.

As for the students' opinions about the modules, they were given the opportunity to provide their criticism (positive and/or negative) in the second, third, and fourth questionnaire (question 25 in all questionnaires, see appendix B). In the fourth questionnaire the students were also asked to provide their view on the treatment of aspects of the historical cases in the final essay assignments (question 24).

In the second questionnaire many students were positive towards the module they had just completed. Some students were negative and some others did not answer question 25. Of the positive reactions, some say that they found it rather exciting, a few even though they sometimes found it difficult. Other more solitary answers are: "A bit dry some times, but it gives a good overview of mathematics, and not only as a mere form of calculation but also from an historical point of view."; "I thought the book to be well-written with problems on a suitable level. However, it would have been nice if more references to the book were provided in the problems [assignments]."; "The essays were difficult to grasp, and more help should have been provided. But the idea was okay." Jonathan emphasized that it had been "cool" to work with some different mathematics, and Gloria also, saying: "I found it to be a really cool module, and I'm glad to have experienced some different math, which you for once can actually link to your everyday life. Wonderful!" Of the negative reactions, a few focussed on the mathematics as having been too difficult or too abstract. Lucy, for instance, answered: "Firstly, really tough questions here at the end [final essay assignment]. I generally found coding theory difficult to understand, and I found the way in which it was explained to be kind of too abstract." A few others focussed on the fact that the module had extended over too much time. One said: "That the summer break interrupted the module was a pity, I thought, because I forgot everything I had learned before the break and therefore also lost the overview."

More students were positive towards the second module than towards the first. In the third questionnaire the majority of students gave positive reactions in question 25. Only a few provided negative criticism (all saying that it was too difficult), and some did not answer. Of the positive answers, some again referred to the fact that the mathematics was different from what they usually see, that it had a connection to everyday life and

society, or that it provided a different point of view. A couple of students (including Lucy) found it to be more interesting than the first module, and one (Gloria) found it to be less interesting. Some of the more singular answers are: “Interesting topic, too many proofs.”; “Very educational! Hate essays!”; “Stupid essays! Well, no, they were okay this time round, but it takes a long time to do them.” One student who in general was not too engaged in the class activities (nor in writing down answers to the questionnaires), and whom the teacher had expressed her concerns about on several occasions, said: “It was actually exciting. And it was very good that everyone had to present a proof in front of the class.”

In questionnaire four, the majority did not answer question 25. Out of the rest, a few were negative and said that it was too hard and difficult, and some were positive, one example being: “For coding theory the mathematics was interesting, and for cryptography the history was interesting.” However, all students except one answered question 24 on the idea of doing essays in the modules. Here the majority gave positive feedback, saying things like: “It was a very good way of obtaining an overview.”; “Good for repetition.”; “It was good, because you then write down what you’ve actually learned, historically as well as mathematically.”; “Not so many essays during the modules, [I would suggest,] but the final one was a good recap.” Some students gave negative answers, mainly saying that they did not care for the history, e.g.: “I thought it was boring and irrelevant. I take math [classes] to learn math. I have history [classes] to learn history!” Some of the students were a bit more indecisive, e.g.: “The idea was good, but couldn’t you do something else than writing essays?”; “I hate essays, but they did promote understanding. So I suppose it was positive, but a lot of work.” The followup interviews (second, third, and fourth round) confirm the impression of students’ opinions about the modules, but do not add much extra.

In summary, the students’ answers to what they found to be the most important things learned from the modules seem to suggest that the treated meta-issues have had an impact on the students’ experiences with the modules. Even though some students point to in-issues, they are outnumbered by those who mention meta-issue matters, a group which grows steadily in size during the three rounds. Although some of the mentioned meta-issues concern application and related aspects, most actually concern matters of history. Particularly in the fourth round, which indicate that once having gotten a little distance from the modules, the overall impressions which the majority of students are left with concern matters of history (as a goal). And some students even refer to matters of the used general topics and issues related to the history of mathematics. An example is Gloria, who after the first module mentioned techniques, after the second the discussion on pure and applied mathematics, and in the followup interview talked implicitly about inner and outer driving forces of mathematics – this time with what seems to be a correct understanding of these, not the previously ‘personalized’ one (cf. page 199). As for the understanding of these meta-issues, the students agree to the essay assignments having played an important role. Although they were time consuming, sometimes not straightforward, and required much work, the majority of students gave expression to their encounter with the essays as rewarding. This is a picture which seems to apply to the modules in general as well. Although question 25 was not widely answered in the fourth round, the positive criticisms in the two previous rounds still outnumbered the negative ones.

(b) The Teacher's Expectations, Experiences, and Opinions

For the teacher, the 2007-regulation (cf. section 3.5) was obviously new, but she welcomed it with a positive attitude and saw it as an opportunity to do more “experimental” and less curriculum-tied mathematics. In the first interview, I asked her how she had intended to fulfill the requirements of including elements from the history of mathematics, before she became part of my research study:

I'd probably use a little time after some modules to put things into perspective with some history, just a double lesson or so. And then I might have spent three or four double lessons, where they were to work more independently on a topic, it could be another number system, Babylonian or Egyptian mathematics or something, or the Mayan's way of working with numbers. But some relatively simple mathematics and fairly brief, where you discuss the fundamentals more, I think. (Round 1)

This led me to ask her if she had any interest in the history of mathematics herself:

It's not something I have occupied myself with, no. And it's not really a turn on for me either. No.

Why? Or why not?

It's hard to say what interests you and not, but I'm not very historically minded. I think history is a large and confusing field, and I'm not too good at keeping track of the historical periods and such. I think it's important to know some history in order to do the teaching, but mathematical history is more exciting from another point of view, namely that there might be a connection between how a mathematical concept has evolved and how a similar concept must be developed inside a person's head. [...]

So it would be more in the sense of using the history of mathematics as a tool, you might say?

Yeah.

For increasing the understanding, or?

Yes. Definitely. (Round 1)

Thus, the teacher's personal initial interest in including history was somewhat limited to the ontogenesis recapitulates phylogenesis argument, though she did not refer to it as such.

At the time of the first interview the teacher had read most of the teaching material for the first module. I asked her how she thought the implementation would come along and what her expectations and concerns were:

It depends a lot on my own effort. I hope and believe, and I really have ambitions, that they will see something in it. For the first three chapters I think that they will say ‘well, it's actually pretty cool this’, but after that I have my doubts. I don't think we'll have time enough. [...]

I'm also quite excited about what they'll gain from the historical part. Because in terms of mathematics there's a lot where I think it can give them something. There's a lot of different things in play, e.g. the historical mathematics with the binary number system and the more modern history with coding theory, there're some proofs here which actually resemble the more modern mathematics which they are used to. This is where I see the point, whereas for the more historical sections I'm a bit more lost as to how

I shall communicate this, if I say something about it at all, but I guess I will.
(Round 1)

Concerning the students' outcome from the module, she expressed:

I'm excited to see if they'll reach some of the abstractions which they have to in order to answer the final questions, the essays. I'm simply excited. It might very well be, because there are also clever kids in between. [...]

It's also cool that the material is so solid-cast... but I'm afraid that it might not unfold completely for them, with the relationships among the different things... I'm a bit nervous about the last essay, because we want them to learn something from this, so for our part we want to round off some things. And especially if they don't do enough of an effort. Really the essay should be something wrapping it up for them, right. Do you get what I mean?

You mean the final essay?

Yeah. Or more in general... The second time I read it I got a feeling that it kind of wrapped itself up in the end and became very finished in a way.

And I think the essay sort of spoils that. We need to give some response to the essay, which gives them the feeling that 'well okay, like that'. (Round 1)

As mentioned at the end of section 5.5, this resulted in a session in which I went over the hand-in essays with the students.

After the first module was completed, the teacher was more at ease with the final essay and how the students had tackled it:

It went a lot better than what I had feared, which among other has to do with the fact that they had time to work with the essays, and where they felt this, which results in a more calm situation. And they worked in a fairly serious manner, and the groups functioned reasonably well. (Round 2)

She did, however, express her concerns that some students had found parts of the content to be too abstract (like some of the students pointed out themselves, e.g. Lucy), and that the teaching perhaps should have varied more between the somewhat abstract topics (packing spheres etc.) and concrete examples. In particular she referred to Sean, whom she otherwise thought to be a quite capable student. Some of this she blamed on the summer vacation splitting of the module, which she saw as a huge disadvantage.

At the time of the second interview the teacher had read most of the material for the second module, and I again asked her about her expectations and possible concerns:

I think it'll be more fun.

More fun. Why?

Because the point of departure is more concrete. Of course you can imagine noise on a transmission and that something must be done about it, but it's not very romantic. On the other hand, ohh we have a secret, right. That in itself is a bit more cool. And I also think we get to see some of the more classical things, such as proving the number of primes to be infinite, a proof which is both simple and beautiful. And then I know some of the things better, than I did for coding theory. (Round 2)

From watching the videos of the first module it is rather clear that the teacher did not focus as much on the history part as she did on the application part, something she commented on herself:

I'm not the greatest expert in communicating this history stuff, also because I don't know what to focus on. Hopefully practicing will help. I hope I'll be

better at it in the next module, but now there's also more history in it, or at least there are more options for letting one's hair down. (Round 2)

Nevertheless, after the second module the teacher was still not satisfied with her efforts concerning the history, despite the fact that the videos show her to focus more on it the second time than the first. Part of her complaint concerned lack of time, that she had to skip history in order to focus on the mathematical necessities for the proof of correctness of RSA. I asked her if she was more satisfied with her communication of the aspects of application:

Not really, I think I've explained both equally good/bad, but of course it's easier, it's more obvious, and for that reason I don't see it as a big problem that I haven't said so much about it. I think they, I actually think that all of them got this point. There are simply more aspects of the history which I'm not sure that I've covered, and that's why I'm not satisfied with it. (Round 3)

To the question of how things had worked out with the final essay assignment this time, she replied:

Better right, well no, it's been a harder task for them. Partly because they had to read Hardy, that wasn't always easy, I think. And then they haven't had time enough to do it. But they were very good at administering the time. From what I've seen, I think it's gone well. (Round 3)

She then elaborated a bit more on the reading of Hardy's *Apology* and concluded: "but considering how difficult it actually is, and probably has been for them to read this, it has worked out remarkably well." As mentioned, the second module was also concluded with a session, after the round of interviews and questionnaires, in which I discussed the students' hand-in essays – and questionnaire answers – with them.

The fourth teacher interview took place about a year after the third. One reason for this was to be able to ask questions about any further work the teacher might have done with the modules, and if not then if she would consider using them again, and what she would do differently. But, of course, also how she viewed the two implementations in hindsight, and what she believed the students (and herself) to have learned from them. To the questions of whether she would run the modules again with a different class, she replied: "I don't have an advanced class right now, and I think it would be a bit too much for the ones I got. But when I get advanced mathematics again, then I plan to do at least one of the modules." In her answer to the question of how she thought the essays had functioned, she mentioned what she would consider doing differently in any future implementations:

I thought they [the essays] functioned well, because the students were forced to think about certain matters which I did not accomplish in making them reflect upon during class. But at the same time it was a considerable burden for them. I think that if I repeat it, I will definitely reduce the number of essays. [...] I'd do that to make sure that if things weren't running with enough available time, and if the teacher was thinking 'I don't feel completely dressed for this', because, yeah, that was how I felt a couple of times, and then you take the easy way out, because you don't know the stuff you just been reading well enough, you're not convinced, and then it becomes something you just tell. I did that a few times, and that's why I'm in doubt if the students actually got any wiser [about the history]. But possibly some

of them got a lot wiser than I am already now, that's the impression I have from reading the essays. I don't think that I'm good enough myself for the history part. (Round 4)

So the impression that the teacher is left with is still that she is lacking historical knowledge and training to do the part of her job which concerns the inclusion of history properly. On the bright side, however, this does not seem to affect her willingness to run one of the modules again. I asked her next what she thought about the use of the general topics and issues related to the history of mathematics. To this she replied that she was already familiar with and would sometimes use the pure and applied mathematics one as well as the one on multiple developments which she knew from, for instance, Newton and Leibniz. The one with objects and techniques was also somewhat familiar to her, e.g. from development of concepts, though she had not used these exact terms before. Inner and outer driving forces, however, were a different matter:

What I really thought was good and new, and also a bit difficult, was the one with the driving forces. I hadn't thought about it in that way before. Not that I'm having trouble in realizing it, but I still think that it's complicated to see: how does it work in this system, what is actually what, and then trying to distinguish them from each other. That's a very fine intellectual exercise. It's good, so of course I'd use it. The more you can add on, the better. Because, for example, someone like Jonathan, he caught on to a lot of these things very fast, but there were some others who only got the difference between pure and applied math, and others only a sense of the dynamics [of objects and techniques]. And that's what's good about providing different ways of interpreting these things, because then there's a chance that they get at least one of the points. (Round 4)

As to what she believed the students to have learned from the modules, she said: They definitely learned something about what mathematics also is, because that's not something you learn today as part of the upper secondary level curriculum, where the type of mathematics often has the concept of functions as the focal point. Nothing's wrong with that, but mathematics is more, and I think they got a real good sense of that. That was one of the things which I thought was really good.

So what you're saying is that they may have gotten a different image of what mathematics is?

Mm, a broader image. And they have become better at proving [theorems], they've become more used to it, and they've got a better understanding for it. It's a pity that they were not examined orally, because I actually think that many of them would have had a chance to do really well – a lot of them. (Round 4)

Concerning her own image, or more precisely her interest in the history of mathematics, this was not something which necessarily increased due to the modules. Her favored approach was still one of philosophy she said. However, to the question of whether her attitude towards using history in mathematics education had changed, she replied:

Definitely, definitely, yes definitely. Also due to the fact that I felt the students [...], in certain situations and toward some of the things, reacting positively in a different way than they do in the ordinary teaching. [...] Of course, the modules were 'too long' and there was 'too much obligation',

that was the negative reactions, but despite these there were some things which surprised some of the students in terms of ‘okay, in that way’ – within the more narrow mathematics and the history; getting some pictures of people who had done this and come up with that. For example the multiple development, that someone had come up with it before, and that it was kept secret. I mean these more banal ‘tabloid magazine’ stories, which I noticed hit home. (Round 4)

From the teacher’s statements it is clear that any success regarding the modules on the students’ behalf do not stem from the fact that the teacher was some history enthusiastic expert, but rather from the actual design of the material, activities, etc. To sum up the teacher’s personal experiences, after the implementation of the modules she was still positively minded toward these as well as any future implementations. Although she still did not feel completely at ease with teaching history of mathematics, she appeared comfortable enough with the modules to plan alterations in future uses of them. Also, her approach to including history was no longer limited to ancient mathematics (Babylonians, Egyptians, and Mayans) and she had taken in the idea of considering the development of mathematics through inner and outer driving forces. She had interpreted positive effects of the modules on the students’ engagement in class. And even though she did not develop a genuine interest in the history of mathematics, her attitude towards the inclusion of it had changed. Also in that her personal interest was no longer limited to the recapitulation argument and ‘history as a tool’:

I’ve seen now that you can do something else also. In that way I’ve broadened my outlook, you could say. Because now I wouldn’t immediately opt for the concepts and their development to be the center of attention. I can come up with something else now, right. And I find that to be rather nice. (Round 4)

(c) Meta-Issues as a Means for Dealing with In-Issues

There are at least two ways to answering question c, if meta-issues concerning the history of mathematics have something to offer relating to students’ understanding of mathematical in-issues. One is to look at the mathematical in-issues which the students learned something about while participating in the modules, the main purpose of which concerned history as a goal. The other way is to address the students’ in-issue learning in terms of the KOM-report’s eight mathematical competencies (cf. section 3.5, page 54).¹

Since we are already quite familiar with the mathematical topics the students were to learn as part of the two modules (cf. chapters 5 and 6), I shall only do a brief answering of question c in the first way, and then spend more time on the second. In the first module, Hamming’s way of developing his codes served as an introduction to the concept of distance in general. Most students did not realize there could be more to this than euclidian distance. The fact that there is a ‘spatial’ meaning in the binary case also surprised many (cf. e.g. figure 5.2). Hamming’s use of elements of linear algebra and the concept of linearity were also useful in introducing matters related to these topics. The historical case of module two offered the possibility of introducing number theoretic concepts to the students and showing them how these play together in a real

¹ The second way is also applied by Kjeldsen (2009b).

life application. For instance, properties of prime numbers, the euclidian algorithm, calculating modulus, congruence, and linear congruence are all needed to get RSA to function. The students knew practically nothing about number theory prior to the module, so cryptography and RSA served as a way into this discipline.

Concerning the second way of answering the question, it can be argued that the students due to the modules developed or trained seven of the eight competencies in the KOM-report. These seven are the:

- mathematical thinking competency,
- problem solving competency,
- modeling competency,
- reasoning competency,
- representation competency,
- symbols and formalisms competency,
- communication competency.

(The missing one is the aids and tools (ICT) competency.) I shall address the training of each of the seven competencies in turn.

The *mathematical thinking competency* comprises an awareness of the types of questions that characterize mathematics and an insight into the types of answers that can be expected (Niss and Jensen, 2002, p. 47). Among other things, this has to do with the nature and scope of such questions and answers and the different kinds of mathematical statements which can be made. During the two modules the students experienced various types of mathematical statements and assertions. Examples are the various definitions, conditional statements, and theorems in both modules, some of which were if-and-only-if statements, while existence and uniqueness both had to be considered for others. But also phenomenological statements about single cases, e.g. in the form of specific prime numbers and conjectures such as Goldbach's conjecture, were considered. On several occasions the students were faced with results extending the scope of a concept by abstracting or generalizing, e.g. when abstracting error detection and correction into terms of packing spheres, packing radii, and perfect codes, or when generalizing Fermat's little theorem into Euler's theorem. Furthermore, the students were introduced to the idea of a one-way function, a notion closely related to mathematical thinking, but without it being possible to define it in strict mathematical terms.

The *problem solving competency*, or problem tackling competency, involves the ability to be able to detect, formulate, delimitate, and specify different kinds of mathematical problems, pure and applied both, as well as being able to solve mathematical problems in their already formulated form, whether posed by oneself or by others (Niss and Jensen, 2002, p. 49). The important thing to notice about this competency is that the word 'problem' is relative to the person who is trying to solve it, what is a routine task for one person may be a problem for another and the other way round. Mainly it was the students' ability to solve the mathematical problems and exercises, which I had posed in the teaching material, that was trained during the two modules. However, due to the historical cases, they were also introduced to ways in which real life (applied mathematics) problems had arisen, been detected by mathematicians, reformulated in mathematical language, and delimited and specified qua their solutions (Shannons's mathematical system of communication, Hamming's error correcting codes, and Diffie's and Hellman's public-key cryptography system).

The *modeling competency* consists, on the one hand, of being able to analyse the foundations and properties of existing models and being able to assess their range and validity and, on the other hand, it involves being able to perform active modeling in given contexts, i.e. mathematizing and applying it to extra-mathematical situations (Niss and Jensen, 2002, p. 52). Regarding the first part of the modeling competency, the students were to understand and follow Shannon's model for a system of mathematical communication and the function of error correcting codes in it, as well as the systems of private-key and public-key cryptography. The second part of the modeling competency was not in focus during the modules, since the students were not to come up with, formulate, or solve any modeling problems of error correcting codes or cryptography themselves. Only in one situation, when they were to incorporate the act of data compression in a mathematical communication system involving error correction did they touch slightly upon active modeling (cf. page 136).

The *reasoning competency*, firstly, consists of being able to follow and assess mathematical reasoning, that is a chain of arguments in support of a claim – orally and in writing both. In particular it concerns the understanding of what a mathematical proof is, the basic ideas of a proof, and when claims do or do not constitute a proof. Secondly, it consists of the ability to actually devise and carry out actual and valid mathematical proofs (Niss and Jensen, 2002, p. 54). In both teaching materials there was a strong focus on the development of mathematical theories in the form of theorems, lemmas, corollaries, and proofs for these, often building upon and drawing from each other. In the first module, the students were given proof exercises as part of their hand-in mathematical exercises, and in the second module every student had to present and account for a proof in front of the class. The latter, as also emphasized by the teacher in the answer to question b, was something which really seemed to develop the students' understanding for the notion of proofs, e.g. the difference between direct and indirect proofs, proof by induction, showing both ways in if-and-only-if theorems, having to take care of existence and uniqueness both, etc. In addition to this, as illustrated by the focus group videos, the students also trained their reasoning competency when having to convince the other group members (and themselves) about the validity of various in-issue matters, e.g. when discussing Shannon's $\alpha\beta\gamma$ -decoding versus syndrome decoding (page 139 and following).

On the one hand, the *representation competency* comprises being able to understand, i.e. decode, interpret, and distinguish between, as well as utilize different representations of mathematical objects, phenomena, problems, or situations (including symbolic, algebraic, visual, geometric, graphic, diagrammatic, tabular, verbal, or material representations) (Niss and Jensen, 2002, p. 56). On the other hand, the competency includes being able to understand the mutual relations between different representational forms of the same object, knowing about their strengths and weaknesses, and being able to choose and switch between them in given situations (Niss and Jensen, 2002, p. 57). The most clear example of this comes from the first module. Here, the students were to work with error correcting (or detecting) codes as, on the one hand, strings (n -tuples) of binary symbols and, on the other hand, points in n -dimensional (binary) spaces. Algebraic manipulations could be performed by binary calculations on the n -tuples to reveal the error correcting capabilities, but could also be spatially interpreted in terms of balls (Hamming and packing spheres) to be placed around the code points in the n -dimensional spaces. In the second module the representations were almost all of a

symbolic or algebraic nature, so switching between different representations was not as big an issue. However, the amount of symbolism introduced in this module developed the next competency to be discussed.

The *symbols and formalisms competency* deals with being able to decode symbol and formal language, translate back and forth between mathematical symbols and natural language, and being able to treat and utilize mathematical symbolism. Furthermore, it comprises having an insight into the nature of the rules of formal mathematical systems, e.g. axiomatic systems (Niss and Jensen, 2002, p. 58). In both modules the students were introduced to a substantial amount of new symbolism and formal language, as compared to what they normally see. For example modulo two addition denoted as \oplus as well as the notion of floor of a number $\lfloor n \rfloor$ in the first module, and in the second, $a|b$, $a \equiv b \pmod{m}$, $\gcd(a, b)$, etc., in addition to various uses of latin and Greek letters. Definitions, theorems, and algorithms often relied on formal language using such symbolism and translating between this and natural language was required as part of the exercises and essays as well as in the presentation of proofs on the blackboard. In the first module, the students were to act within a formalism of strings, vector spaces, and a metric based on these, that is to say a formal system. And they were to carry out various calculations with inequalities, for example the triangle inequality. Concerning axiomatic systems, the second module in particular illustrated how number theory is built up from the ground with Euclid's definition of a unity and a number etc. The students' understanding for symbols and formalism, and translations to and from natural language were also required when working with the extracts from original sources. For example, when they were to understand the different notation of Shannon and Hamming and translate this to the one of the teaching material, or when they were to read the extracts from Sun Zi, Fermat, and Goldbach, using a very different formal language and hardly any symbolism, and translate this into a modern natural language and from there into modern symbolism.

From the point of view of this empirical research study, the *communication competency* is an interesting one. The competency, firstly, consists of being able to study and interpret others' written, oral, or visual mathematical expressions or texts. Secondly, it consists of the capability to express oneself in different ways and at different theoretical or technical levels about mathematical matters, again either in written form, orally, or visually (Niss and Jensen, 2002, p. 60). The reason that this competency is interesting for the present study has to do with the students' making of the essay assignments as well as their participation in the interviews. When making the essays, the students had to read, study, and interpret the material's presentation of the mathematics (my presentation), the extracts from original sources (past mathematician's presentations and expressions) and pay attention to each other's orally and sometimes visual expressions about the texts, also bearing in mind the teacher's presentation of this. And as part of this, they, of course, were to express themselves, explaining and adjusting their own expressions according to other group members (to some extent this also goes for the students' presentations of proofs on the blackboard). Furthermore, the interviewees were given the opportunity to practice expressing themselves orally about meta-issues as well as in-issues in the four rounds of interviews.

So, in summary, even though the two modules were primarily concerned with using history as a goal (meta-issues), there do seem to have been some positive side effects concerning the learning of in-issues as well (history as a tool).

(d) Possible Effects of Choosing a Newer History over an Older

In terms of history as a tool, the above discussion (question c) concerns the cognitive history-as-a-tool arguments. However, positive effects in terms of the motivational and affective ones were also detected, in particular concerning the use of a more modern history of mathematics over an older or antique one (question d).² To answer question d, I shall rely on interview data from the third round of followup interviews. As the reader knows, the second module actually concerned two histories of mathematics: one of mathematics proper with the old (and antique) development of number theory, and one of applied mathematics with the recent development of public-key cryptography, relying on the old number theoretic results. In the interviews, the students were asked which of these two histories they found to be the more interesting and why. The majority of answers provided related to the motivational and/or affective side of using history as a tool.

The fact, that the history was a newer and fairly recent history of mathematics seemed to make it easier for the students to relate. To illustrate this, one student said:

The newest, because it was most 'high tech' or whatever you may call it. It's alright that this guy Euclid did something too... but these three guys, or six, or how many they were, that I thought to be much more exciting because they are still alive. It's almost 'just around the corner' that they invented it and yet it is so widely used now. [...]

Is it easier to relate to?

I think so. It's kind of difficult imagining a guy doing something before Christ was born...

Besides being more recent, it apparently also matters that the characters were either still alive or have lived during the students' or their parents' lives.

Some students seemed to find it more interesting to work with such a history of applied mathematics, and possibly even more so if they were able to recognize elements from their everyday life:

I think the new one, because you touch much more upon this applied mathematics and learn that mathematics is being used for something... because many in the old history, they were hobby mathematicians and such. They just did mathematics to do mathematics, to show that they could. I don't find that as exciting, just doing something for the sake of doing it. So I think the new one was the most exciting one.

Because it was applied?

Yes, because you were being told stuff about how it was put to use.

This may also have helped in revealing some of the hidden uses of mathematics (cf. section 3.4) in society in general and in the students' own everyday life in particular. For that reason too, some of the students seemed to find the newer history more relevant. One student related:

I think because the development of the Internet is somewhat more relevant to us, and it's just a little more fun when it takes place in more recent time. ... Well, with Euclid and Fermat it wasn't like 'Yeah!'. But I thought that these newer researchers and their ways of doing things, the order in which

² The following discussion has also been presented in Jankvist (2009b).

events took place, and the fact that several people invented it at the same time, that was exciting.

Perceiving the newer history as more exciting was common. One student made the point this way:

The closer it comes to our present time, the easier it becomes to draw parallels, right, then it becomes more exciting. And then if it is something you do every day and then suddenly are being told, well okay that's where it comes from. I think it is very exciting knowing how things are.

When discussing mathematics in everyday life, the same student remarked: "I didn't realize at all that the messages we are sending over the Internet and such, that it was mere mathematics."

So to answer question d, some of the effects of choosing a newer history over an older one appear to be that it may be easier to relate to, because some of the elements in it are more recognizable for the students. Some students find it more interesting or relevant, which, however, also seems connected to the fact that the histories concerned applied mathematics. And several students referred to the newer history as simply being more "exciting".

One problem with modern mathematics, however, is that it often is more difficult and inaccessible to students than older mathematics. But if the above display of quotes show anything, then it is that looking for the modern pieces of mathematics which actually can be explained to students is a task worthwhile.

(e) Added Bonus Meta-Issues of Mathematics as a Discipline

Question e, which meta-issues the students also learned something about besides those concerned with history as a goal may, like question c, also profit from a look at the KOM-report. In particular, I am referring to the three types of overview and judgement which the KOM-report mentions (cf. section 3.5):

- historical evolvement of mathematics, internally as well as in a societal context;
- the actual application of mathematics in other subject and practice areas;
- the nature of mathematics as a subject.

The first of these concerns history as a goal, and is therefore not interesting in the light of question e. But the other two are.

The second form of overview and judgement concerns actual application of mathematics to extra-mathematical purposes within areas of everyday life, society, or other scientific disciplines. The application is brought about through the creation and utilization of mathematical models. Where the modeling competency also deal with active modeling skills, this type of overview and judgement is of a more broad and generalized form, almost of a sociological or science-philosophical nature (Niss and Jensen, 2002, p. 68). As examples of questions to be considered in relation to this, the KOM-report mentions:

1. Who, outside mathematics itself, actually uses it for anything?
2. What for?
3. Why?
4. How?
5. By what means?
6. On what conditions?

7. With what consequences?
8. What is required to be able to use it? Etc. (Niss and Jensen, 2002, p. 68, translated from Danish, numbers added)

In both the first and second module, the students saw examples of actual uses of mathematics, examples of uses which they themselves relied on in their everyday life (1). They were presented with the historical and present societal needs for such applications (2 and 3), something they discussed as part of their essay assignments, in which they also dealt with much of the mathematics (in-issues) required to make the solutions (models) function (4 and 5). Some of the limitations of the applications were also discussed (6), e.g. what it requires to actually break a RSA encryption and how many errors a given code can correct. Regarding the latter, the students calculated the error-detecting and error-correcting capabilities for several different codes. Some of the consequences of using mathematics were also touched upon (7), positive and negative both, the latter e.g. in relation to war, which the students discussed as part of the essay assignments of the second module and their reading of Hardy's *Apology*. As for what is required to be able to use the two applications (8), the technical necessities were not dealt with in detail and none of the exercises or essays included any forms of programming relating to Hamming codes or RSA.

The third type of overview and judgement concerns the fact that mathematics as a subject area has its own characteristics, as well as the characteristics themselves. Some of these, mathematics has in common with other subject areas, while others of them are unique (Niss and Jensen, 2002, p. 69). As examples of questions dealing with this, the KOM-report mentions:

1. What is characteristic of mathematical problem formulation, thought, and methods?
 2. What types of results are produced and what are they used for?
 3. What science-philosophical status does its concepts and results have?
 4. How is mathematics constructed?
 5. What is its connection to other disciplines?
 6. In what ways does it distinguish itself scientifically from other disciplines?
- Etc. (Niss and Jensen, 2002, pp. 69-70, translated from Danish, numbers added)

The two modules provided the students with examples of how extra-mathematical problems may be translated into mathematical language and solved by mathematical methods (1). The first module exemplified this through a development within the area of modern applied mathematics drawing on previously well-established mathematical results (general concept of distance, linearity, and n -dimensional spaces), results which the students had to identify themselves in Hamming's original paper. Also, the second module illustrated the historical development of such already well-established mathematics (number theory), as well as its use in solving a societal problem of the 20th century (the key-distribution problem), the latter by means of the mathematical idea of a one-way function. The modules also provided the students with an insight into the types of results (both in pure and applied mathematics) produced and what they are used for (2), as well as how mathematics is constructed (4). The first module discussed an example of a non-constructive existence proof, in the form of Shannon's proof that good error correcting codes exists, and how this led mathematicians into a 'wild treasure hunt' for such codes, exemplified by the creation of Hamming codes, out of which the students in

particular worked with the (7,4)-code. The second module did a similar thing with the system of public-key cryptography working in theory but lacking a concrete one-way function in practice. In addition to this, the second module illustrated various types of proofs, as mentioned under question c, and, qua elementary number theory, showed how a piece of mathematics is constructed with results building upon and drawing from each other, something the students had hands-on experiences with as part of their blackboard proofs. The modules also showed the students elements of the use of mathematics in other disciplines (5), in particular mathematics in engineering and computer science. In the second questionnaire and round of interviews the students were asked what effect they thought it to have on the development of error correcting codes that Hamming was a mathematician and not an engineer. And in the second module the students saw the role of mathematics and mathematicians in bringing about the needed one-way function to make public-key cryptography function.

The modules as such did not explicitly deal with the scientific theoretical status of mathematics in relation to other disciplines (6) or the philosophical status of mathematical objects (3). However, the activities surrounding the modules, i.e. the student questionnaires and followup interviews, to some degree did. The scientific theoretical status of mathematics, and other disciplines, came up when discussing the questionnaire questions asking if mathematics is a science, and if so then a natural science (in the sense explained in section 7.1). Although, no questionnaire questions asked specifically about the scientific philosophical, or ontological, status of mathematical concepts and results, many students engaged in this discussion due to the questions on discovery versus invention. For example, in the clip of the focus group doing the second module essay on pure and applied mathematics (see section 6.6), where the students' discussion takes on a quite philosophical discourse, touching upon inner structures of (pure and applied) mathematics as well as the nature and origin of mathematical objects (primes).

So, in summary, the set up of the modules (the questionnaires, interviews, design of the teaching material, choice of historical cases, selection of exemplary general topics and issues related to the history and historiography of mathematics, etc.), although primarily concerned with history as a goal, did have some form of added bonus in terms of illustrating other aspects of mathematics as a discipline. Of course, question e is closely connected to the students' beliefs and views about mathematics as a discipline, which I shall discuss next.

(f) Proposing a Small Model for Students' Reflected Beliefs

To answer question f, what may be understood by *students' reflected beliefs* about mathematics as a discipline, I shall rely on the findings for RQ3. More precisely, I shall use the findings concerning changes in the nature of students' beliefs saying that by the time of the fourth round of questionnaires and interviews:

- some of the students' beliefs had grown in *consistency*,
- some students felt a greater need for *justification* when expressing and explaining their beliefs,
- some of the beliefs seemed to be held more evidentially by means of *exemplifications*.

An prerequisite condition for this is, of course, that the students are explicit about their beliefs, i.e. that they are able to express them. Taking this as an underlying assumption, the three dimensions – consistency, justification, and exemplification – may be used as a

model and/or definition of what is to be understood by *students' reflected beliefs* about mathematics as a discipline. The situation is illustrated in figure 8.1.

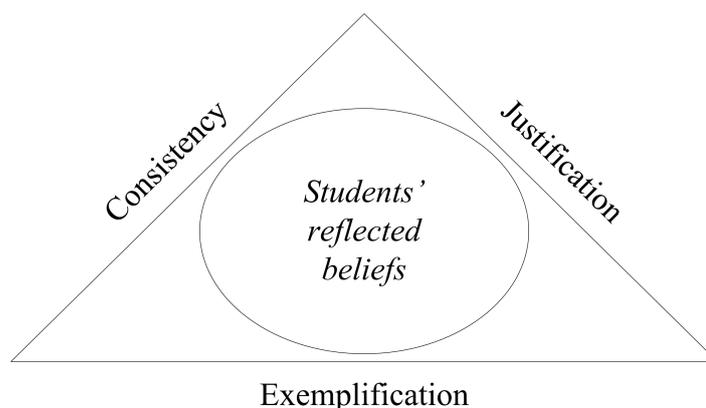


Figure 8.1 Students' reflected beliefs about mathematics as a discipline as consisting of three dimensions on a basis of explicitness: consistency, justification, and exemplification (evidence).

So, on the one hand, we may define students' reflected beliefs about mathematics as a discipline to be beliefs which are held on a basis of explicitness with some degree of consistency, justification, and exemplification. On the other hand, we may consider a model of students' reflected beliefs telling us that in order to understand the degree of reflection in them, we need to consider each of the three dimensions.

This, in fact, also suggests a way to access students' beliefs about mathematics as a discipline in case of some of the more goal-oriented descriptions of students' beliefs, images of mathematics, and awareness or appreciation of mathematics, which are often found in curricula descriptions or regulations for mathematics programs, e.g. the Danish 2007-regulation's description of 'the identity' of mathematics. But also more normative texts like, for instance, the KOM-report could benefit from such a model of students' reflected beliefs. Because how are we to assess whether students possess the three types of overview and judgement? And if so, then to what degree? Since these three types all concern matters of mathematics as a discipline, a model for students' reflected beliefs about this could serve as a tool in such assessments. And thereby perhaps become a first step in operationalizing the KOM-report's three types of overview and judgement.

Of course, any new proposition brings with it new questions. Some of the questions which the proposed model raises are, for example, what different kinds of justification that may be provided for a belief, if different kinds or degrees of exemplifications can be spotted, e.g. according to the level of in-issue anchoring of these. These two questions can not be answered by reanalyzing the available data from the present research study alone, it seems they require new empirical studies.

(g) Meta-Issues, In-Issues, and Students' Reflected Beliefs

The last of the questions (g) which it is possible to address is the one tying together research questions 1, 2, and 3, as well as question f, by asking about the relationships

between students' abilities to engage in meta-issue discussions, their in-issue anchoring of these discussions, and how reflected their beliefs (about mathematics as a discipline) are. A full answering of question g is not within reach, but the gathered data and the already made data analyses may provide some insights as to what is afoot. The way I shall go about addressing the question is by looking, one last time, at the focus group students.

Based on the data displayed in section 7.5 and yet another look at the students' questionnaire answers and followup interviews, it is possible to classify the consistency, justification, and exemplification, respectively, of the focus group students' beliefs according to four degrees:

none < occasional < some < much.

This has resulted in the conclusions depicted in table 8.1. Since the reader has no way of controlling these conclusions, (s)he will have to take my word for it and hopefully stay with me to see where this will lead.

Focus group student	Consistency	Justification	Exemplification
Gloria	much	some	much
Jonathan	much	some	much
Andrew	some	some	much
Harry	much	much	much
Lola	some	much	some
Sean	much	much	much
Lucy	much	occasional	occasional

Table 8.1 Focus group students' reflected beliefs, measured by the degrees of consistency, justification, and exemplification.

Having table 8.1 at our disposal, next step is to consider the students' capabilities in terms of discussing meta-issues and understanding of the in-issues, the latter being a prerequisite for potential anchoring. In order to do so, I shall have to grade the students' performances, something I have avoided to do previously. I shall grade them according to four levels of capabilities and understanding:

poor < mediocre < occasionally good < good.

Based on the displayed video clips, my watching of the focus group videos in general, and the students' answers to meta-issue questions in questionnaires and followup interviews, Gloria, Harry, and Jonathan appear to be the better students at engaging in the meta-issue discussions and reflections. Gloria was able to follow various different discourses in the discussions, shift between these, and she was the one responsible for the example of an anchored argument. Harry was also good at engaging in these discussions, which was seen from his participation in both clips from the second module, one of which contained an anchored discussion. Jonathan was quite capable in discussing the meta-issues, good at resolving meta-issue conflicts which arose, and in the second module he was responsible for the anchored comment on the one-way function. Although

Andrew was responsible for the anchored discussion of the (7,4)-code in the first module, his capabilities in discussing the meta-issues appears to me only occasionally good, since he sometimes would pose viewpoints which he could not always propose a justification of. Something similar was the case for Lola, who was good in the anchored discussion from the second module, but at other times seemed somewhat confused as to what was at stake, e.g. when asking if RSA was published today or not. Lucy's and Sean's participations in the meta-issue discussions of the first module may be viewed as downright poor. Lucy's because she hardly intervened in the discussions, and Sean's because he only intervened in those concerning in-issue tasks.

From looking at the students' individual hand-in exercises, the videos of them doing these, their presentation of proofs in the second module, and their discussions of in-issues as part of the essay assignments, an idea may be obtained about the focus group students' understanding of the involved mathematical in-issues. To some extent the reader again will have to trust me on my word, but I shall provide some examples from data to illustrate my findings.

A look at these gathered data suggest that Harry and Jonathan are the ones with the better understanding of the involved in-issues in the modules. For example, Harry's identification of the misprint in the Chinese remainder theorem, his presentation of this theorem at the blackboard, his hand-in exercises as displayed and discussed in chapter 6 (but also those from the first module), and his way of dealing with the extracts from original texts bear evidence hereof. Also Jonathan gave excellent answers to the historical exercises on Sun Zi and Fermat, as well as other exercises from both modules, and his proof of Fermat's little theorem as well as his comments to Andrew's proof of Euler's theorem clearly indicate a higher level of in-issue understanding. Although Gloria catches some of the points which the others do not, e.g. her pointing out that \mathbf{f} is chosen because it has the same syndrome as \mathbf{v} in the first module (see page 139), and despite her seemingly abilities to quickly grasp a proof (the presentation of her proof to the teacher outside class), her hand-in exercises still suggest that her in-issue understanding is only occasionally good. The same seems the case for Sean, who in general has decent hand-ins, in particular for the second module, but clearly has problems grasping some of the more abstract notions in the first module. Andrew's, Lola's, and Lucy's understanding of the in-issues appears to be on a lower level than that of Gloria and Sean, i.e. mediocre, however certainly not poor. Andrew has a few good points in some of the videos where the group discuss in-issues as part of the essays, e.g. in their work with Shannon's $\alpha\beta\gamma$ -decoding in the first module, but his hand-in exercises are not always up to standard. Something similar seems to be the case for Lola, whereas for Lucy some of her hand-ins are more than decent (e.g. the test in the first module), but the lack of answers to the in-issue questionnaire questions weigh down.

From the above brief analyses and/or displays of the focus group students' reflected beliefs, meta-issue capabilities, and in-issue understanding, table 8.2 may be made. On the one hand, this suggests that the better a student's understanding of the involved mathematical in-issues, the better the possibility seems that this student is also good at engaging in the meta-issue discussions (and reflections), or the other way round, depending on how you look at it. The cases of Harry and Jonathan in particular support this, and to some extent also the case of Gloria. On the other hand, this may also be because these students are generally good. This is further suggested by the observation that these three all have fairly reflected beliefs as well, i.e. perhaps indicating that if

Student	Reflected beliefs	Meta-issue capabilities	In-issue understanding
Gloria	much-some-much	good	occ. good
Jonathan	much-some-much	good	good
Andrew	some-some-much	occ. good	mediocre
Harry	much-much-much	good	good
Lola	some-much-some	occ. good	mediocre
Sean	much-much-much	poor	occ. good
Lucy	much-occ.-occ.	poor	mediocre

Table 8.2 A measure for the focus group students' reflected beliefs (consistency, justification, and exemplification), their capabilities concerning meta-issue discussions and reflections, and their in-issue understanding.

a student possesses an understanding of the in-issues as well as capabilities regarding the meta-issues, the more likely it appears that this student also has a higher degree of reflected beliefs about mathematics as a discipline.

However, in a sense, the 'askew' students may provide us with a more important observation, since a look at the combinations of their rows in table 8.2 suggest that no causality can be claimed; any combination may appear. A student may very well be good in one column and poor in the others. The most clear example is Sean, who has a 'top score' in the first column on reflected beliefs, but scores only slightly better than Lucy in the next two columns, even though she has the lowest score on reflected beliefs in the first column.

So the answer to question g is in a sense table 8.2, which shows us that there is not clear causality between the degree of a students' reflected beliefs, his or her meta-issue capabilities and in-issue understanding (and potential anchoring). However, it should be taken into account that a more thorough, full, and in-depth answering of question g would require display of data, a good deal more analysis, as well as developing a more refined methodology for how to do the evaluations best. In such considerations, a study clarifying different types and degrees of justification and exemplification in the students' reflected beliefs, as suggested for question f, might deepen the understanding of them. Also, the students' in-issue understandings might be explored further by looking into their development of the seven competencies during the two modules. To address the students' meta-issue capabilities, Sfard's theory might be looked at again, since an extension of it to include students' *meta-issue word use* could reveal more of their understanding of the meta-issues. For now, however, I shall consider a different methodology, namely the one used to answer the (asked) research questions of this dissertation.

8.3 Trustworthiness Revisited

As previously announced, the discussion of methodology shall be arranged according to Schoenfeld's three dimensions of trustworthiness, generality, and importance for educational studies. In the following sections, I shall deal with some of the factors which were difficult to deal with at the time of the methodology chapter, when the data

analysis had not been presented and the case-specific conclusions drawn. The present section is devoted to trustworthiness of the findings for research questions 1, 2, and 3. Generality for these will be taken up in the following section 8.4, research question 0 will be discussed in a separate section (8.5), and the importance shall be addressed in section 8.6.

In order to structure the discussion of trustworthiness (validity and reliability) of the empirical research findings, I shall consider in turn:

- the trustworthiness of the gathered and displayed data,
- the trustworthiness of the performed analyses,
- the trustworthiness of the (case-specific) findings and conclusions,

and in doing so, whenever appropriate, discuss Schoenfeld's eight elements of trustworthiness (cf. section 4.3): (1) descriptive power; (2) explanatory power; (3) falsifiability; (4) predictability; (5) rigorousness; (6) specificity (and detail); (7) replicability; and (8) multiple lenses or lines of evidence.

The Gathered and Displayed Data

As discussed in chapter 4, and as evident from chapters 5, 6, 7, and the previous section 8.2, the amount of gathered data strongly outweighs the amount of used and displayed data, especially in the case of the video recordings. The philosophy of 'better too many than too few data' was applied to increase the level of detail in the data, and thereby have the possibility of increasing the specificity when needed. Other important reasons were to have multiple lines of evidence on which to perform methodological triangulations and to have backup data sources. For instance, if a student did not do his or her blackboard proof, the students' hand-in exercises could be used as a backup to still get an insight into the student's understanding of the in-issues in question. Although having these positive features, operating with such a vast amount of data also poses certain problems. In particular, it may become a problem to choose which data to use and display. Of course, one tries to select as exemplary and illustrative samples as possible, but in doing so there are many pitfalls. Brown (1992, p. 162) mentions that the "selection is obviously going to buttress our theoretical stance", and that the "problem is how to avoid misrepresenting the data, however unintentionally". Brown (1992, p. 173) implies that researchers often favor the "golden moments", moments where it all works out, but that it is important to realize "that they are selected from a stream of events that are not so encouraging." She continues:

[T]here is a tendency to romanticize research [...] and rest claims of success on a few engaging anecdotes or particularly exciting transcripts. One of the major methodological problems is to establish means of conveying not only the selective and not necessarily representative, but also more important general, reliable, and repeatable. (Brown, 1992, p. 173)

Thus, the selection of data from a large database is a non-trivial task. In my selection of data, I have tried to be as inclusive as possible, showing not only the 'golden moments' but also some of the more dim or lackluster ones. By displaying and using data from the students' questionnaire answers, their interview statements, and hand-in essays from all six groups in both modules, I have attempted to paint an overall picture of the class, illustrating the golden as well as the dull moments. As for the displayed and analyzed video clips, these may be considered 'golden moments' in one sense, namely in

the sense that they make up the more illustrative moments from the focus group's essay discussions. However, this is to be considered a necessary evil in terms of methodology, because why display video sequences, or other data, in which nothing is happening just because they are representative for the gathered data? In particular, the long clip from the second module was a 'golden' and, in fact, a quite unusual one in terms of how long the students actually kept focus on the assignment at hand. Nevertheless, the clip still illustrated general features of the students' meta-issue discussions, e.g. the students' capabilities to shift between different discourses, this being a positive finding, and individual students' long periods of non-participation, a more negative one, but both being general features for all of the displayed and analyzed clips.

Needless to say, selecting and using some video clips, or any other kind of data, implies that others were not used and displayed. Thus, a reader might wonder what the content is of the non-used data, and if these tell a different story. As for the video clips of the focus group, the student's hand-in essays, the questionnaires and followup interviews, all of which I have watched and read several times (every single interview was transcribed), my conviction is that; no, these cannot tell a different story. However, this is not the same as saying that there are not *other* stories to be told, based on these data. Because, as the answers to questions a-g in section 8.2 illustrate quite well, surely the data offer answers to other questions also. A similar question to ask could be how the topics for changes in the focus group students' beliefs (and views) in section 7.5 were chosen. To this question it should be acknowledged that I, of course, had hoped and imagined beforehand that some changes would be detectable. When looking at the individual focus group students, the detectable changes concerned different topics and to a large degree these dictated the presentation in section 7.5. But of course I also wanted to cover as wide a spectrum of topics as possible, while at the same time being on the alert for possible changes deepening the understanding of *students' reflected beliefs*, as discussed in the previous section. Nevertheless, I still attempted to illustrate elements of non-changes, e.g. Andrew's belief in mathematics as being discovered. Despite not being able to display all data, I have tried, to the extent possible and whenever meaningful, to display the students' questionnaire answers and their interview statements in their full and non-edited length in order to, in this way, make it possible for the reader to 'breathe over my shoulder'. Of course, from time to time, language editing has been unavoidable due to the fact that all answers and statements had to be translated from Danish. But also in that respect I have tried not to impose any censorship, including when the students' were using foul language.

The Data Analyses

As indicated, the various gathered data sources provided several possibilities for methodological triangulations in the data analyses. For example, the answering of research questions 1 and 2 were based on findings in the focus group video clips, the focus group hand-in essays, the focus group students' hand-in mathematical exercises, their blackboard proofs, their questionnaire answers and followup interview statements, as well as being broadened by the hand-in essays and questionnaire answers of the class in general. For research question 3, I followed a strategy of first providing more quantitative measurements for the class in general and afterwards deepening these by qualitative studies of the individual focus group students. The quantitative measurements based

themselves on the four rounds of questionnaires and followup interviews, which made it possible to do some triangulations between the different rounds, and the qualitative studies of the focus group students, besides questionnaires and interviews, also included elements from the video clips.

In conducting the actual data analyses (triangulations etc.), I have tried to be as specific and rigorous as possible in explaining the words, notions, and theoretical constructs which were used. This goes for those defined as part of the research (e.g. history as a goal, history as a tool, meta-issues, in-issues, beliefs about mathematics as a discipline, students' reflected beliefs), those which were modified and/or extended from other theoretical constructs (e.g. those based on Sfard's theory of commognition: in-issue discourse, meta-issue discourse, and meta-issue discursive conflict), and those which were adapted without any modification (e.g. Sfard's notion of word use, the epistemic objects and epistemic techniques, inner and outer driving forces, Cooney's four types of standpoints toward changes in beliefs, Green's notion of evidentially and non-evidentially held beliefs, those from the KOM-report, and others).

Concerning the use of theoretical constructs in the data analyses, a few further comments may be in order. First of all, not every theoretical construct explained or discussed in chapter 3 found its way into the data analyses and discussions. Still their presence may be justified, either in the form of providing the reader with background knowledge (e.g. when discussing some of the general topics and issues related to the history and historiography of mathematics, the problems of internalism and externalism, Whig and anti-Whig history, discovery and invention, or the ideas for construction and understanding of mathematical concepts), providing the reader with an insight into the earlier and foundational notions for newer theoretical constructs (e.g. Tall and Vinner's cognitive conflicts and Sfard's earlier idea of reification as necessary components in explaining and understanding Sfard's newer theory of commognition), or in setting a scene for discussing some of the underlying issues of the research study (e.g. the relevance paradox in mathematics education and mathematical appreciation and awareness). Secondly, having explained all these underlying theoretical constructs and notions in a somewhat detailed and rigorous manner may help in increasing the transparency of the study and its results for the reader. In a way, this may actually also increase the level of falsifiability of the research study, because how can one falsify the results of a study, if the underlying theoretical constructs and methodologies making up the basis for the study have not been made explicit. Also, the use of theoretical constructs in general may do this, since other researchers may be familiar with the theoretical terms and notions used. Thirdly, for the theories which were actively used in the data analyses, it may be asked if anything was gained from using these that could not have been gained without. To answer this question I shall draw on a few concrete examples. For research questions 1 and 2 the theory mainly applied was the newer one by Sfard. This theory being so large and philosophically founded, one could rightfully ask if I have not taken a musket to kill a butterfly. My answer to this is; perhaps so, but still the few elements used from Sfard's theory, e.g. her idea of word use, have come in very handy, especially because her theory is concentrated around communication, and my first research question concerns students' discussions. Furthermore, the general setting of the theory not being limited to mathematics enabled me to apply it to the aspects concerning history and other meta-issues of my study. But yes, some of the results from the data analyses could probably still have been obtained without the theory. However,

things would also have been lost, e.g. the idea of the discursive approach to analyzing the students' discussions. For research question 3, the beliefs literature provided me with an extensive vocabulary for discussing the students' possible developments in beliefs. Also, Cooney's four types of standpoints made it possible to classify the focus group students when discussing the nature of their changes in beliefs and/or views. And Green's notion of beliefs being held evidentially or not, assisted in developing the idea of exemplification in students' reflected beliefs. For all three research questions, as well as RQ0, the KOM-report provided me with a frame for carrying out my studies, as well as extending these a bit further to answer questions c and e in section 8.2.

Case-Specific Findings and Conclusions

As already mentioned on several occasions, the findings of the research study first of all provide existence proof. The findings making up this existence proof may then be addressed in terms of their descriptive power, i.e. identifying the phenomena which 'count', and their explanatory power, i.e. why and how these phenomena occur and function. In the existence proof metaphor, the descriptive power takes care of the existence part, whereas the explanatory power concerns the laying down of 'the necessary and sufficient conditions', preferably to make the existence proof into a constructive one of the kind. As an example of this, from the findings of the research study, it was observed that the focus group students would follow and shift between different discourses, or different tracks, when carrying out their meta-issue discussions in the group (RQ1). Thus, the study was able to describe a phenomenon occurring repeatedly throughout the data. In order to be able to explain why this phenomenon occurred and how it functioned, the different discourses occurring were identified and the 'intersections between different tracks' were studied. Due to RQ2, special attention was paid to intersections between in-issue discourses and meta-issue discourses, the so-called potential anchoring points, and by means of triangulations with other data sources the potential of these intersections was evaluated. Analyses showed that potential anchoring points may remain either anchored comments or arguments, or may develop into anchored discussions. It also showed that anchoring may occur regardless of the level of 'insurance' of this in the design of the essays, and that the students' may have different agendas for referring to in-issues, e.g. when Gloria used an argument anchored in in-issues to make a meta-issue point within a somewhat separate discourse. Concerning research question 3, the study provided evidence of some changes in students' views having taken place, changes which to some extent could be justified as effects of the modules – again describing the existence of a phenomenon and 'what counts' in order for this to occur. In terms of explanatory power, it was observed that these changes were detectable on different levels; a question and content specific level and a level concerning the degree of reflection having taken place (consistency, justification, and exemplification), the latter which led to a definition of (or model for) students' reflected beliefs (question f, section 8.2). The questions of why the different kinds of changes occurred were addressed on an individual student level by referring to the theoretical constructs of Green (1971) and Cooney (1999), as mentioned above.

According to Schoenfeld (2007), if a piece of research possesses descriptive and explanatory power both, it stands a fair chance of being able to predict as well. As seen from chapters 5 and 6 some predictions were possible from the first module to

the second. For example, two different kinds of anchoring were observed in the first module, anchored comments and anchored discussions. A reasonable prediction would therefore be that these two kinds would also occur in a second module with a similar design and based on a similar research methodology. And, in fact, this was the case. The second module ‘reproduced’ data showing these two types of anchoring, and in addition revealed a third kind, the anchored arguments as mentioned above. Thus, in terms of prediction and replicability both, it therefore seems likely that these kinds of anchoring would also occur in any future implementations of the modules – unless, of course, these findings appear to be specific for the population in question, something I shall consider when addressing the dimension of generality next.

8.4 Generality and Generalizability Revisited

The answers provided to research questions 1, 2, and 3 have so far been considered as more or less case-specific. Now the time has come to address the generality and generalizability of these answers. Such an addressing might also tell us something about the generality of the answers provided to questions a-g in section 8.2, since the empirical data and circumstances for answering these were the same, and the methods to some extent also. The discussion of generality and/or generalizability will be structured according to:

- the research design (potential and limitations),
- the implementation (population and circumstances),
- the assessment (data, analyses, findings, and conclusions).

Whenever meaningful, I shall relate to Schoenfeld’s four types of generality (cf. section 4.3): (1) the claimed; (2) the implied; (3) the potential; and (4) the warranted generality of a research study.

Research Design

As for the generality of the design we may consider, in turn, the choice of historical cases, the use of general topics and issues related to the history and historiography of mathematics, and the teaching material.

As already discussed in section 4.2 (page 94) of the methodology chapter, the choice of the historical cases concerning history of modern applied mathematics had to do with, amongst other: the illustration of inner and outer driving forces in the development of mathematics; touching upon elements of the KOM-report’s two other types of overview and judgement (cf. question e); broadening the students’ images of mathematics as a discipline (RQ3, question f); that such histories might be easier for the students to relate to, and that they might uncover some of the hidden mathematics of the students’ everyday life (cf. question d). Although the two historical cases of error correcting codes and public-key cryptography seem quite suitable in these ways, they are surely not the only cases possessing such features. Other examples could be the historical development and use of linear programming, which might be closer to the original curriculum of upper secondary level mathematics, or the historical development of graph theory and its later uses in shortest-path algorithms like, for example, Dijkstra’s algorithm or variants of the traveling salesman’s problem, elements of which possibly could be explained, even to first year upper secondary students.

Of course, the choosing of the historical cases also had to do with the general topics and issues related to the history and historiography of mathematics which these could illustrate. The fact that each of these cases could illustrate at least three of the discussed general topics and issues also has a say for the generality of the design. Even though the histories of mathematics which the students were presented with were somewhat case-specific, their work on the general topics and issues for each of the cases possessed the possibility of providing them with insights into the evolution and development of mathematics in general. Some of the students' interview statements also seem to suggest such an outcome, the latest example being when Gloria mentions development for the sake of mathematics or for the sake of application (cf. question a).

Concerning the teaching material, as discussed previously, there is certainly room for improvement, in particular concerning some of the phrasings of the essay questions. Regarding the level of difficulty of the mathematics, this was also a feature which was taken into consideration when choosing the historical cases, i.e. that the in-issues should be explainable to the students. As mentioned in question d, the history of modern mathematics imposes a problem of the mathematics often being inaccessible for the students, which limits the choices. For the concrete cases, this was dealt with by stripping the presentation of error correcting codes for the uses of matrices, and the presentation of number theory for the use of groups and other such general algebraic structures. In terms of making the history Whig, this did not really pose problems, because Hamming did not use matrices much in his own presentation anyway, and Sun Zi, Fermat, and Euler did not use the notion of groups, since this was only introduced later, and for RSA there is no problem in limiting the theorems to being concerned with integers only, since this is what the algorithm uses. The fact that it was possible to go about introducing the in-issues in such a fashion may again be assumed not to be unique for these two cases. A key feature in the design of the teaching material was to have a strong focus of the in-issues, since one purpose of the research was to see if the students could reach meta-issue insights in ways which could not have been reached without knowing the related in-issues. This is not compromising the generality of the design, since other designs for integrating history may have a similar purpose. The same can be said about the observation that the essay assignments to some extent ensured that the students worked with meta-issues of history as a goal concerning the two cases, regardless of how much focus the teacher actually put on these in her classroom teaching. Although this was a positive outcome of the design, it is still something that can be done with other designs.

When discussing generality of the research design, then not only the potential of this should be considered, the limitations of the design should be considered as well. Out of these, there are probably several pertaining to the given design, but I shall concentrate on two. The first concerns the uncovering of the students' mathematical skills. The data only tell us about the students' understanding of the in-issues of the two modules, not anything about their mathematical capabilities measured against the whole of the upper secondary mathematics curriculum. Of course, the students' capabilities on the mathematics of the modules may imply various things, but the picture is limited to the treated mathematical topics, and just because a student is good at number theory, it does not mean that the student is also good at calculus, trigonometry, or probability theory. The other limitation concerns the general understanding of the maturity of the students. As discussed previously, students at this age may be quite impressionable

because they are in a process of maturing on a personal as well as on an academical level. This maturing may be an important factor in making their beliefs more reflected. From the point of view of the research design, it may be unclear whether a detected increase in a student's consistency, justification, and exemplification of his or her beliefs is due to the general maturing process which this student is undergoing or due to the effect of the teaching modules. The only way to deal with this, I suppose, is to consider the specific exemplifications made to see if these may be related to the modules; the approach taken in chapter 7.

Having considered the potential and limitations of the research design, my answer to the posed question of whether I have had a 'lucky hand' in my design (cf. chapter 4) shall be: not in any particular way, no.

Implementation

Concerning the generality of the implementation of the modules, I shall consider, in turn, the students, the teacher, and the circumstances for carrying out the modules.

At the end of chapter 6, I asked if the involved class of students in the research study could in any way be considered to be particularly bright students. To this question, the answer is that the class did take mathematics at advanced level, which means that they have mathematics during all three years of upper secondary school. So when the teacher says that she might find the modules a bit out of reach for the classes she is presently teaching, she is referring to students who only have mathematics for one or two years of upper secondary school. But in comparison with other students at the same level, the particular class was not in any way special. Although it was not possible for me to see the students' final grades for the national third year written exam, apparently for political and administrative reasons at the school, the teacher told me that the students' grades were close to the national average for that level, perhaps a bit below, which again suggests that the students were not particularly special. On a different level, it may of course be discussed whether the focus group students were in any way special compared to the rest of the class. As explained earlier, these students were chosen to be as representative for the class as was possible (cf. section 5.3), by which I am referring to the other constraints of selecting students for a group which could function together and were not too far apart, mathematically speaking. For this reason, the group did not include any students whose mathematical skills were thought to be downright poor, but nor was it supposed to be consisting of 'geniuses' only. When looking at the focus group's hand-in essays these are not outstanding compared to other groups, in fact some were of a poorer quality as pointed out for the first module. The focus group students' questionnaire answers and their followup interviews, as displayed in chapters 5, 6, and in particular in chapter 7, do not suggest that the focus group students provided better or very different answers than the other students did. The other students' questionnaire answers also indicated an increase in consistency, justification, and exemplification, and by means of the other interviewees I could have told more or less equivalent stories of changes or non-changes in beliefs and views. Only, had I done so, I would not have been able to draw on the video material and the displayed data of the focus group students' mathematical exercises.

Also the teacher of the class was a somewhat typical upper secondary school mathematics teacher. She was not chosen by any specific criteria, other than she should be

interested in taking part in the research study, in return for which she could be sure of fulfilling some of the requirements of the 2007-regulation. Other than being ethically committed to her students' learning outcome, she did not have any personal 'stocks' in the teaching material and the teaching modules. In fact, not even in the history of mathematics, as evident from her statements in the answering of question b above. Her preferred meta-issue topics were philosophy of mathematics and applications of mathematics. This was also reflected in her actual teaching, in which she otherwise concentrated mostly on the in-issues of the modules.

Often when integration of history in mathematics education is discussed, it is noted that the effectiveness and the fulfillment of ambitions are closely related to the teacher's enthusiasm. This does, of course, make it difficult to transfer experiences of using history in mathematics education from one situation, population, or teacher, to another – one of the problems stated in the earlier quotation of Gulikers and Blom (2001) (cf. section 1.1). However, due to the typicality of the teacher and the students both, this may not necessarily be the case for the present research study. To conclude this, though, we must consider also the situation, i.e. the circumstances of the empirical study. The outer circumstance of the study was, of course, the school. Although this was a new school, as discussed in chapter 4, it was not in any way an elite school among the Danish upper secondary schools. The school was new and may, due to its profile, have been more open towards engaging in teaching experiments, and the school's scheduling of classes sometimes made it easier to plan the modules, but that is about it. As for the 'inner' circumstances, one question which may be asked is how the students reacted to the atypical situation of being video filmed in class and taking part in a research study. In the fourth round of interviews, I asked the students if my presence in class had affected their level of activity. To illustrate their perception of things, I shall display the focus group students' answers, since they were the ones being exposed to the camera the most:

Gloria: No, it hasn't. [...] But a negative effect might have been that I thought it to be difficult. I'm not good at being very active when I find something hard. Of course, this is a bad thing, but... It's probably because I don't find very much in school to be hard, and then when something pops up, then I can't really cope with it, and I get like: no, no, no...

Lola: No, I haven't thought about that at all. I've just thought that it was math class as usual. So it's just been normal, neither more nor less, really.

Harry: No, it's been the same as if the teacher had given us the tasks. There was not really any difference.

Andrew: In the beginning it was... it was a bit pressure-like. But you get used to it. I didn't find it that bad. And I didn't think that it made me change my mind or behavior or something, just because it was something you were to use in your research.

Jonathan: I think, I've been more engaged in it. [...] As opposed to, well... and the teacher, she is there in order to teach us, right... and that is for our own sake. This was for my own sake AND for... what's it called? To help you, right.

Lucy: Hmm, no, I don't think so. [...] Well, I might have been more active while in the focus group... before I got in the other group. But otherwise I don't think that it's had any effect.

Sean: Well... maybe, I don't know. Maybe I would have thought it to be an

even bigger drag if it was something we had been presented to as part of the [ordinary] teaching. I think, I might have made an effort of trying to keep up, because somebody needed this. But at the same time it's been like: Okay, something is actually required from me, because there's someone who needs this. That's also been a... a stress factor in some way.

In addition to Gloria there were also a few other interviewees who thought they had done less than normally. For example, one student replied:

Actually, I think I've done less for that reason.

You've done less?

Because I knew that it was outside the curriculum, and there was a very little chance of being examined in it, only orally, not in the written exam.

So it was kind of... well.

Okay.

Yeah, to be completely honest.

So, the students' own perceptions seem to cover the possible spectrum fairly well, from more engagement due to the situation over unaltered to lesser. From watching the videos of the focus group, it can be mentioned that the students after a while seemed to have gotten used to being filmed, they would begin cutting up more frequently, not watch their language, etc., which may also be seen from the displayed clips in chapters 5 and 6. In any case, whether the atypical setting of being filmed etc. did have an effect or not, this is a typical circumstance for any empirical research study dealing with classroom teaching and students' learning. And as such, this goes for a teacher involved in a research study of this kind as well, also she may be somewhat affected, but in terms of such studies, this is as typical as it is unavoidable.

Assessing the Generalizability of Findings and Conclusions

As indicated in the beginning of the section, the last matter of this section concerns the generality of the gathered data, the data analyses, and the generalizability of case-specific findings and conclusions.

The fact that it can be argued that the student population and the teacher were rather typical for upper secondary advanced mathematics level, and the circumstances to some degree as well, suggest some kind of generality of the gathered data. The question is whether this implies generalizability for the case-specific findings as well. Or to ask differently: to what extent the identified and uncovered phenomena and mechanisms, as well as their interrelations, are independent of the research design, and may therefore, perhaps, be considered to be more general phenomena and mechanisms.

In a sense, the research design of the empirical study may be compared to lowering a test probe into a petri dish containing a substance to be investigated. Such a probe will be sensitive to certain phenomena, while it will be insensitive to others, as was illustrated by the discussion of potential and limitations of the research design. For the data analysis, the nontrivial problem of selecting representative data samples is tightly connected to the discussion of generality: If data are selected in a reasonable and responsible way then the empirical 'probe' will most certainly possess a much greater potential in terms of generalizing the findings. As discussed when dealing with trustworthiness, an effort was taken in the selection of data not only to favor the 'golden moments'. And as argued when dealing with the research design in the present section, similar findings may

be assumed possible with other designs focusing on history as a goal. Thus, with the claimed typicality of the gathered data in mind, and due to the typicality of population and circumstances, this implies some generalizability of the case-specific findings. The remaining question now is: What kind?

Whereas the two previous considerations on research design and implementation concern the implied generality of the research study, and after dealing with them possibly also a claimed generality, the generality of the actual findings are of a more potential nature. That is to say that they demonstrate the existence of the previously discussed phenomena, which seem worthy of further investigations, but as Schoenfeld (2007, p. 88) points out, “findings of studies with existence proofs are not yet general – but there may be the potential for them to be”. Thus, there is no warranted generality of the present findings, nor the conclusions drawn. In this sense, the empirical research study of this dissertation may be seen as a ‘pilot’ of a somewhat exploratory nature for future studies, qualitative as well as quantitative.

Some suggestions for possible topics of such future studies have already been made under questions f and g in section 8.2, but I shall reenter the discussion in section 8.7, after having dealt with the trustworthiness and generality of the categorizations of whys and hows (section 8.5), and the importance of the research findings (section 8.6).

8.5 The Whys and Hows Categorizations Revisited

Research question 0 has not been exposed to as many methodological considerations as research questions 1, 2, and 3. This has a natural reason that I shall discuss now in terms of trustworthiness and generality.

Trustworthiness

First of all, RQ0 is different in nature from RQ1-3, i.e. the answering of it does not rely on an experimental design and a line of empirically gathered data. In this sense the answering of RQ0 is purely analytical armchair research. However, it did require some ‘data’, namely the posed arguments for and approaches to the inclusion of history in mathematics education given by researchers in published papers etc. From a reading of various older literature, a first set of categories were proposed (a little different from those in chapter 2). These were then modified into the ones we know and tested on the ICMI Study on *History in Mathematics Education* (Fauvel and van Maanen, 2000), the most comprehensive publication available on the topic. Finally, they were tested before publication in ESM on the scanning of ten years (1998-2007) of publications in four major mathematics education journals (ESM, JRME, FLM, and ZDM). In a way, all these publications served as different lines of evidence or multiple lenses, to refer to some of Schoenfeld’s elements of trustworthiness, and added to the replicability of the given study, as well as the predictability of the use of history in any future studies.

Also in this study (RQ0), I have attempted to be as rigorous and specific as possible when defining and using new terms to introduce and discuss the proposed categorizations and the links between them. For example, much effort was taken to keep the categorization of the whys separated from that of the hows, and vice versa, since any mixing was believed to possibly blur the interrelations between these, and consequently any analysis of these interconnections.

Again, the attempted level of clarity ought to make it possible for others to attempt to falsify the categories by testing them against other and, possibly, future publications on history in mathematics education. Concerning descriptive and explanatory power, the study of arguments for and approaches to history in mathematics education, as described in the literature, made it possible to analyze the different interrelations between these whys and hows, and to propose some guidelines for using history in mathematics education based on this analysis. (For the latter, see also section 8.6.)

Generality

How general are then the proposed categorizations of whys and hows? Well, for using history in mathematics education, I would argue that they are quite, if not completely, general. And due to the way they have been tested, a warranted type of generality actually. Whether they may be applied to matters outside the field of using history in mathematics education is more uncertain. However, I do see one place where they ought to be applicable, namely when it comes to using applications of mathematics in mathematics education. This also may be used in terms of either a tool or a goal (the hows would be somewhat different, though, due to the history-based approaches).

The notion of meta-issues and in-issues apply to mathematics as a whole. As should already be evident, the history as a goal meta-issues are only a subset of the mathematics as a discipline meta-issues, since these concern matters of application, sociology, philosophy, epistemology, etc. Thus, other studies than those concerned with history in mathematics education may very well benefit from discussing matters as being concerned with either meta-issues or in-issues of mathematics. An example could be the field of using philosophy in mathematics education.

8.6 Importance Revisited

The contributions of the empirical research study presented in this dissertation and their possible importance were already discussed in section 1.3. In this section I shall revisit the discussion, since we are now more familiar with the actual findings of the research and the conclusions to the four research questions 0, 1, 2, and 3 and the seven additional questions (a-g). I shall structure the discussion in two parts:

- the new territory reclaimed by the study,
- possible importance in terms of pedagogical considerations.

New Territory Reclaimed

As mentioned in section 1.3, this study contributes to five out of the seven topics (1, 2, 5, 6, and 7, to be explained again below) for the CERME6 working group (WG15) on the role of history in mathematics education (cf. page 5).

First of all, the study contributes to the development of theoretical and conceptual frameworks (WG15 topic 1) by the proposed categories of whys and hows. Of course this is not the first categorization ever of the whys and hows of using history, as discussed in section 2.1, but it is one which takes into account a quite comprehensive amount of literature, also because it was tested upon the ICMI-Study and thus hopefully reflects the comprehensiveness of this, and in addition surveys the literature published after the ICMI-Study. Unlike some of the previous categorizations, and as discussed above, the

proposed ones of this study tries to separate strictly the categorization of whys from that of the hows. One purpose of this being to illuminate and analyze the interrelations of these, something previous studies have not focussed specifically on.

Secondly, the research study is an example of an investigation of the *effectiveness* of using history in mathematics education (WG15 topic 2 on the role of history), although this effectiveness concerns history as a goal and not history as a tool, which seems to be the normal agenda for discussing this.³ As pointed out in the methodology chapter (section 4.1), since there are no prior empirical experimental studies on the effectiveness of using history as a goal (cf. section 2.8), an investigation of this calls for the development of a methodology. Developing such a methodology for both designing and assessing the teaching materials for the two modules (WG15 topic 7) has played an important role in carrying out this research study. But the present empirical research study not only concerns the effectiveness of using history (as a goal), it also concerns the *efficacy* of using history, something seldom discussed explicitly in the literature.

As mentioned, there is a lack of empirical studies in the field of using history in mathematics education; studies which test – empirically – some of the more advocating ‘armchair research’ assumptions and proposals, and as seen from the survey in chapter 2, this is especially true for empirical studies on history as a goal. When visiting Luis Radford, author of chapter 5 of the ICMI-Study (Radford, 2000a), in Sudbury, Canada in June 2008 we discussed the purpose of such empirical studies. Radford being acquainted with the terminology of my research study revealed the following when interviewed:

We need to refine the theoretical principles of a conception of mathematics that takes seriously into account the role of history and culture. Such a conception, if clearly defined, should allow us to envision new ways in which to use and integrate the history of mathematics in the classroom. The problem, however, is not merely a theoretical one. We also need to have empirical studies to inform us of the effectiveness of didactic designs – like teaching modules, working with primary sources, etc. First of all, we need to design modules and teaching sequences that will not only integrate the history of mathematics but also open spaces for classroom reflection and discussion. Resorting to history to improve the teaching and learning of mathematics is not just a question of providing information that cubic equations were solved in a certain year or that a certain event happened in the 18th century, or that somebody called *X* did this and that. In resorting to history in the teaching and learning of mathematics, we have to go beyond historical facts. We live in a historical world – a world populated by historically formed concrete and abstract objects, modes of thinking and values. The main point is that the reality in front of us is intrinsically historical and its understanding and disclosure can only be achieved through a critical stance. As far as I can see, a critical, historical approach is the only key to penetrate the deepness of reality. In this context, one of the fundamental questions for us mathematics educators is how to offer the students contexts for critical action, discussion, and reflection. And history, for the reasons already mentioned, must play a fundamental mediating role here. But we need empirical studies to see how

³ The matter of effectiveness of history as a goal has also been discussed in some detail in Jankvist (2009c).

it works, how critical and empowered the students become as a consequence of being exposed to a space of discussion that instead of stopping at the ephemeral present reveals its historical nature. (Radford, 2008b)

In this sense, the present research study has surely provided an example of an empirical study that “inform us of the effectiveness of didactic designs” and “open spaces for classroom reflection and discussion” without “stopping at the ephemeral present” as well as attempting to understand the rhetoric of mathematical discourses for students at upper secondary level. And in terms of the development of students’ images of mathematics, consisting of beliefs and/or views, the study has provided some insights into the empowering effects of these on the students’ abilities to be critical about matters of mathematics as a discipline.

Concerning the design of the modules (WG15 topic 7, again), this also suggests a way of including history of mathematics in the teaching and learning of mathematics without making it Whig. When discussing this, Fried says:

That mathematics teachers are committed to teaching modern mathematics and modern mathematical techniques naturally makes their relationship to the history of mathematics quite different from that of an historian of mathematics. (Fried, 2001, p. 395)

By modern mathematics Fried is referring to modern presentations of curriculum mathematics, and on this basis he further argues that “the history of mathematics becomes something not studied but something *used*”, and continues:

However, when *history* is being *used* to justify, enhance, explain, or encourage distinctly modern subjects and practices, it inevitably becomes what is ‘anachronical’ [...] or ‘Whig’ history. (Fried, 2001, p. 395)

In this criticism by Fried there are two underlying assumptions: first, that the history is mainly used as a tool; and second, that the historical cases are concerned with more normal curriculum mathematics. These assumptions do not apply to the two modules of this research study. Furthermore, if one is operating within a mathematics program which bases itself not only on a content curriculum, but also on a description of competencies, as the one for Danish upper secondary to some degree does, then the second underlying assumption becomes less relevant. Another point which can be made is that Fried mentions the use of original sources (WG15 topic 5) as one of the only possible ways of avoiding anachronism when using history. But again, this to a larger degree seems to apply to the use of history as a tool than to the use of history as a goal. Because when using history as a goal, you need the background story of the evolution and development of the given piece of mathematics. Just having the original sources alone will not do, since the students may find these somewhat detached from the historical, societal, and cultural background and circumstances, if they are not introduced to these as well. The use of general topics and issues related to the history and historiography of mathematics in the essays seems to be a promising way of attaching such background and circumstances to extracts of original sources.

Beside what is mentioned above, the research study may have gained yet another piece of territory, namely by talking about a more goal-oriented dimension of students’ beliefs (WG15 topic 2, again). As discussed in chapter 3, when surveying the beliefs literature, beliefs are most often considered tools to access students’ learning abilities or disabilities. Only rarely are beliefs about mathematics as a discipline considered a goal in themselves, and when this is the case, they are usually not referred as beliefs.

Instead terms such as images or mathematical appreciation and awareness are often used, which may have the undesirable effect of detaching studies on these matters from the beliefs literature. Nevertheless, if surveying existing uses of history in curriculums etc. (WG15 topic 6), as I have done for the Danish upper secondary level, descriptions like the 2007-regulation's talk of 'identity' of mathematics and the KOM-report's three types of overview and judgement more or less explicitly address such a goal-oriented (or normative) dimension of students' beliefs. However, such a dimension about 'beliefs about desirable beliefs' – *meta-beliefs* we could call them – can only be addressed properly if the meta-beliefs are articulated as such, i.e. as goals in themselves. The present research study has attempted such an articulation by providing a definition or model of students' reflected beliefs, and in doing so also attempted to take a first step at operationalizing the goal-oriented dimension of students' beliefs.

Some Pedagogical Considerations

From a pedagogical perspective, knowing about the possible ill-suited relations between whys and hows could benefit mathematics teachers, curriculum developers, and material designers, since choosing an ill-suited 'how' may obstruct the fulfilling of an intended 'why'. The material designer who for example wants to use history as a goal and a tool at the same time, and possibly also as an affective as well as a cognitive tool, can use the knowledge about the interrelations of the whys and hows to navigate between the different approaches in order to fulfill his or her purpose. In this respect, I shall allow myself to propose a few guidelines for potential users of history in the teaching of mathematics. To begin with, try to get a clear idea about why history should be used in a given situation, i.e. which 'whys' the use of history should fulfill. If the whys concern history as a tool, there are several suitable options of 'hows' depending on the kind of tool argument: motivational and affective tool arguments might very well be realized through the illumination approaches; cognitive tool arguments may demand additional time and space and are therefore better matched with modules approaches; a realization of evolutionary tool arguments may be obtained through the history-based approaches. If the whys concern history as a goal, the best match seems by far the modules approaches, since these can allow for a deepening of the meta-issues in question.

Concerning how to go about dealing with the latter, the present research study offers a line of ideas on how to meet requirements of history as a goal, again for material designers, curriculum developers, and teachers. Examples are the idea of general topics and issues related to the history and historiography of mathematics, the idea of essay assignments in mathematics class, the doing and handing in of these in groups, and having a strong mathematical focus in the teaching material, etc. For designers and teachers who would be interested in applying the idea of essay assignments, the research study illustrates the importance of trying to ensure some kind of mathematical anchoring of the meta-issue discussions that the students are to carry out, as well as suggests some ways of doing this. However, the study also shows that if a strong mathematical focus is kept in the used material and the teaching, there is likely to be some kind of transfer into the essays.

If it within a given curriculum is considered important to illustrate historical, societal, sociological, philosophical, or epistemological elements of mathematics as a discipline to the students, then designers and teachers may also get inspiration from the present

research study. Firstly, it shows how modules focusing on history as a goal may have added bonuses in the form of some of these other elements, thus providing the students with some kind of ‘overview and judgement’ in the way that the KOM-report talks about, an idea of the ‘identity’ of mathematics as described in the Danish 2007-regulation, and a deeper mathematical appreciation and awareness as described by Ernest (1998). Or in the words of the present study, to provide the students with more balanced, multifaceted, profound, and reflected images (beliefs and/or views) of mathematics as a discipline. Secondly, the study suggests some elements to look at if one wishes to actually assess students’ developments in these – qua the proposed definition or model of students’ reflected beliefs. Thirdly, the study illustrates that even though the used historical cases may be outside and far away from the original mathematics curriculum, they may still develop and train the students’ mathematical competencies. This also shows a positive side effect in terms of cognitive history-as-a-tool arguments for using history as a goal. Fourthly, it illustrates some of the more motivational and affective factors of using history (as a tool), which designers and teachers may profit from. I am thinking of the finding that many students seem to find more recent historical cases easier to relate to, more interesting or exciting, and this especially so if they have a touch of application, because the students may recognize elements from their own everyday life in such applications, which again serves the purpose of revealing some of the otherwise hidden uses of mathematics in society, in order to possibly contribute to the students’ critical citizenship (cf. section 3.4). Fifthly, as discussed above, the use of more recent historical cases suggest a way for teachers and designers, at least to some extent, to avoid making the use of history Whig or anachronical.

A quite different question is how to actually cope with all the above in a given teaching situation, if, say, you are a teacher and you wish to use history in your class. As seen from the teacher interviews (question b), our teacher did not feel at ease with the situation, despite the fact that she was handed already prepared materials. This is a matter which I, among others, shall touch upon in the next section.

8.7 Perspectives: Questions Not Yet Possible to Answer

Like the gathered data of the research study and the performed data analyses in the dissertation revealed questions which to some extent were possible to answer (questions a-g), so they also give rise to questions which do not seem possible to answer on the basis of the existing investigations. I shall address such questions related to RQ1-3 first, and next the further questions which the answers to questions a-g in section 8.2 have generated.

As seen in chapter 5, the students’ hand-in essays did not always reflect the discussions which had actually taken place in the group (illustrated by the focus group’s work on the fourth supportive essay). What went down on paper was not representative for the meta-issue discussions of the group. Potentially speaking there was a learning profit, but what would it take for the students to actually be able to put this into writing in their hand-in essays? The study does not seem to provide an answer to this. Of course, practice could be one answer, and the focus group did seem to do a better job in the second module, being a bit more experienced. Still, I doubt that does it alone. It seems that the students need to be brought to realize that they have to change some of their

more unfavorable working methods. Time and again the students would minimize their workload on the tasks at hand by splitting the essay questions between them, dividing themselves into subgroups, etc., and as seen from the interviews, this was not only the case for the focus group. For some students this meant that they lost the overall picture of the essay assignments (cf. the interviewee on page 225) and therefore also some of the more important points of them.

The analysis of the hand-in essays from the second module raises another question. In the main essay of the final essay assignment, practically no groups were able to answer questions d and e, asking the students which insights they believed could be obtained by approaching the history of mathematics through inner and outer driving forces and objects and techniques, respectively. As discussed in the recapitulation of chapter 6, these questions are on another cognitive level, since the students will have to reflect upon their own learning regarding these general topics and issues in terms of the two historical cases and relate this to the conduction of history of mathematics in general. As also pointed out when first discussing this, the students were, in fact, able to say something about the conduction of history when comparing the when-and-who and the why-and-how accounts, in particular in their first module essays, and from chapter 7 we know that at least some of the focus group students were able to reflect upon matters of their own learning in the form of changes in their views or beliefs. However, what it takes to get the students to combine these capabilities and perform meta-issue reflections on possible insights or differences in insights concerning questions like d and e is an open question.

In the posing and answering of question b in section 8.2, we encountered one of the more pressing questions within the field of using history in mathematics education (one related to topic 3 in the CERME6 WG15 list, see page 5). Namely, how in-service teachers may be brought to cope with the integration of history in their practice. As mentioned at the end of the previous section, even after having participated in the implementation of both modules, our teacher still felt on shaky ground regarding the meta-issues of history in them, as well as concerning the inclusion of history in general. She revealed:

I need to read more. I haven't really had time to read the book you gave me [Katz (1998)], unfortunately. But that is one of the things you could use to obtain some kind of understanding of this. Because generally I think that I lack some continuing education and further training before I really can begin to get some substance into these things. (Round 4)

Often when I have talked to upper secondary teachers at conferences, meetings, etc. they have shown great interest in my two teaching materials, have wanted copies, or in the case of non-Danish teachers wanted me to translate them into English. However, if the experiences with the teacher of the present research study shows anything, then it is that having already prepared materials available may not do the job alone, not even if you are being coached by the author of these along the way. That it is so, is not a new finding. Already when discussing this with Man-Keung Siu in Hong Kong in 2006, he revealed similar hypotheses:

Some teachers hope that one day someone will write all these materials and distribute them, or they can get it in the bookstore and use it directly in the classroom. But I don't think that would work, because you need the enthusiasm of the teacher himself or herself in order to use this kind of

material well. Just having the material there is not enough. (Siu, 2006)
But how can such enthusiasm be mobilized? One suggestion is to have the teachers construct their own materials, which of course requires arousing some kind of awareness that this would be helpful in order to get teachers to cooperate. Siu explained:

I hold workshops for large groups of teachers now and then. If I can arouse the interest of a few teachers to begin this kind of collaborative activities then it will be a good start. You have to have this self-motivated initiative to do the thing. And usually their excuses are: 'I do not know history, I'm not an historian of mathematics, I don't know how to do it', but I always give my own example, that I didn't know it at the beginning either. But it's not very persuading or convincing, because they say, well, you have been doing it for thirty years. My answer is that you have to start somewhere! I don't like to paint a rosy picture for them, letting them think that they don't have to do anything, just sit there and then they have the material and then it works wonders. I don't think it ever works that way. But one has to start somewhere. It is hard work and it is going to be hard work. In my case, or in my days, I did it alone. But one of the things we can achieve is, I think, to have a group collaborating in a collective effort, which will make the task easier. (Siu, 2006)

Similar initiatives have been taken in other parts of the world. In Taiwan, for instance, researchers have followed and coached teachers who met for three hours a week and discussed texts on the history of mathematics, and based on these discussions prepared materials to use in their teaching (Su, 2007; Horng, 2007). At Roskilde University, courses have been offered where in-service upper secondary teachers could receive assistance from professionals to develop their own small modules on topics from the history of mathematics, as well as in evaluating their implementations of them. The produced materials for the modules and the evaluations were then posted on the web page of the course, so other teachers could benefit as well. Such activities are of course much more demanding for the teachers than just grappling prepared material from the bookshelf. However, it seems likely that a teacher should feel more at ease with a material he/she had taken part in preparing himself/herself, and the enthusiasm, of which Siu talks, might come easier too. At any rate, much more empirical research on such matters are required before any parts of this territory can be reclaimed.

As for questions f and g in section 8.2 some further questions have already been mentioned when dealing with these. In particular, concerning the answer to question f, it was mentioned that further studies should be done on what different types of justification may be given for a belief, and if it is possible to talk about different degrees of exemplification, e.g. according to anchoring in in-issues. A wider question which may be asked is if the idea of viewing students' reflected beliefs about the discipline of mathematics as a 'trinity' of consistency, justification, and exemplification may also be applied to the other dimensions of students' mathematics-related beliefs in figure 3.1: mathematics education, social context (the class), and the self (Op't Eynde et al., 2002). Concerning question g, it was mentioned that a full answer to this probably would require the development of a more refined methodology, and that such a methodology could rely on answers to the further questions related to question f, as well as a use of the seven discussed mathematical competencies (cf. question c) and an extended use of Sfard's theory for approaching the students' in-issue and meta-issue understanding,

respectively. Regarding the answer for question g, an important comment should be made. The answer given to this question is also an existence proof, but unlike the ones provided to research questions 1, 2, and 3, it is not a very constructive one. The answer to question g deals with correlation, i.e. if the phenomena x , y , and z co-occur (reflected beliefs, meta-issue capabilities, and in-issue understanding), it does not provide what Schoenfeld (2007, p. 83) refers to as “explanation at the level of mechanism”, which would include saying more about how x , y , and z are linked, but more importantly *why* they are or are not linked. An answering of this question, as well as the other ones mentioned in this section, can not be based on a reanalysis of the already available data from the present research study alone, it requires new empirical studies. Studies designed with the purpose of uncovering exactly these matters.

8.8 Epilogue: Future Hopes for the Field

In section 3.9 I accounted for my personal position on the use of history in mathematics education. Having presented the empirical research study and discussed the findings of this, it now seems appropriate to revisit some of these viewpoints.

One of the things I mentioned was that I am a little skeptic towards the use of history as a cognitive tool and not least the evolutionary arguments for using history. On the one hand, this, of course, has to do with my personal belief that students should be introduced to history as a goal. But on the other hand, it also has to do with the relatively little impact that use of history appears to have had in mathematics education in general. Of course, I am new in the business, but in none of the general mathematics education conferences I have attended have any of the plenary lectures concerned the use of history in mathematics education. And this despite the fact that the HPM has operated since the 1970s. As seen from the survey of empirical studies (section 2.8), the majority of these actually do concern the use of history as a cognitive tool. But still, around 50 studies is a small number and maybe such scattered studies of fragmented mathematical topics are not enough to make a real impact. Having observed this, the idea of addressing the use of history as a tool through the notion of mathematical competencies appears to me a more promising road to follow. As empirically documented in question c and in Kjeldsen (2009b), students’ training and development of mathematical competencies seem to go naturally hand in hand with a use of history. And maybe such an approach could be an eye opener for mathematics educators regarding the effectiveness and efficacy of using history as a tool in mathematics education.

Convincing the mathematics education community at large that students need more profound, balanced, multifaceted, and reflected images of mathematics as a discipline may be an even more difficult task. Nevertheless, I believe it is in this respect that history as a goal has its finest role to play. Throughout this dissertation, history as a goal has mainly been concerned with showing the students mathematics as a cultural, societal, and human endeavor, that its evolution is driven by both inner and outer forces and mechanisms, its dependance of time and space, etc. However, as pointed out in section 3.9 another side of history as a goal has to do with empowerment and the making of students into 21st century citizens (Radford, 2008a). But how can the history of mathematics make students better citizens? I discussed this with Radford in Sudbury and his idea is that students will have to be made “aware that the reality that we have in

front of us is a reality that can't really be understood if it is not comprehended through an analysis of the history that lies behind the processes of contemporary mathematics" (Radford, 2008b). In a sense, the argument is similar to that of Skovsmose (1990), who argues that knowledge about mathematical modeling will provide students with a democratic competence and allow them to practice critical citizenship. Surely knowledge of the history of mathematics will also add to such a democratic competence, but unlike mathematical modeling it will do so from a more cultural point of view, contributing to students' understanding of "other rationalities and other ways of thinking about the world" (Radford, 2008b). And maybe this empowering role of history in mathematics education is one which will also be easier to disseminate to the mathematics education community, instead of just a plea for more reflected images of mathematics as a discipline.

In any case, whether promoting history as a tool through mathematical competencies or history as a goal through empowerment and critical citizenship, the propagation of history in mathematics education must be made through a significant increase in the body of empirical research studies. And the HPM has a pivotal role to play in the nurturing and promotion of such studies.

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A Time Line of Empirical Study

In this appendix a time line for the empirical study is provided, covering the implementation of the modules as well as the pre-implementation and post-implementation activities.

A.1 Pre-Implementation Activities

Week	Day	Time	Activities
Week 12	Mon 03.19.07	10.00-11.40	Questionnaire 1.
Week 15	Fri 04.13.07	10.00-11.40	Followup interviews.
Week 16	Thur 04.19.07	10.00-11.40	Followup interviews, interview with the teacher, composition of focus group.
Week 17	Mon 04.23.07	10.00-11.40	Test filming, handing out teaching material.

A.2 Implementation of Module 1

Week	Day	Time	Activities
Week 18	Tue 05.01.07	10.00-11.40	Chapter 1 to the section on binary numbers, exercises 1, 2 (individually) exercises 6, 8, 9, 10, 11, 12 and perhaps 7 (groups).
		12.10-13.50	Rest of chapter 1, work with exercises 16, 17 (individually) and first introductory essay assignment (groups).
	Thur 05.03.07	12.10-13.50	Chapter 2, exercise 21 (individually) and second introductory essay assignment (groups), hand in first introductory essay assignment.
Week 19	Thur 05.10.07	10.00-11.40	Summary of chapter 2, exercise 22 (individually) and second introductory essay assignment (groups), hand in second introductory essay assignment.
	Fri 05.11.07	10.00-11.40	Chapter 3, exercise 24 (individually), begin exercises 32, 33 and finish at home.
Week 20	Tue 05.15.07	10.00-11.40	Summary of chapter 3, summary of exercises 32, 33, exercise 34 (individually). Begin chapter 4, exercises 35, 36, 39 (group), hand in third introductory essay assignment.
	Wed 05.16.07	14.00-15.40	Continue chapter 4, exercises 40, 41, 42, 43 (groups).
Week 21	Tue 05.22.07	14.00-15.40	Continue chapter 4, exercises 46, 48, 51(a,b) (groups).

Week	Day	Time	Activities
Week 33	Wed 08.15.07	08.00-09.40	Discussion of test, including the questions on coding theory.
	Thur 08.16.07	12.10-13.50	Work on missing essays, most the second and third (groups). Hand-in exercises given: 13, 18d+f, 23, 25, 34, 37, 45.
Week 34	Tue 08.21.07	09.50-11.30	Small summary of last part of chapter 4 and the historical figures. Begin work on the first supportive essay (groups), the entire final essay is assigned to the students, hand in of missing introductory essays.
	Wed 08.22.07	09.50-11.30	Repetition of the mathematics of the module.
Week 35	Tue 08.28.07	12.10-13.50	Continuous repetition of the mathematics of the module, further work on the final essay (groups).
		15.45-16.45	Study-camp where students are given individual help on the hand-in exercises by the teacher.
	Wed 08.29.07	08.00-09.40	Continuous work on final essay assignment (groups).
Week 36	Thur 08.30.07	12.10-13.50	Continuous work on final essay assignment (groups).
	Tue 09.04.07	09.50-11.30	Continuous work on final essay assignment (groups). The entire final essay is to be handed in Thursday this week.
	Wed 09.05.07	09.50-11.30	Questionnaire 2.
	Week 38	Thur 09.20.07	12.10-13.50
Week 39	Mon 09.24.07	09.50-11.30	Followup interviews.
	Wed 09.26.07	08.00-09.40	Followup interviews and discussion of final essay assignment in class.
Week 43	Mon 10.22.07	12.10-13.50	Interview with the teacher.

A.3 Implementation of Module 2

Week	Day	Time	Activities
Week 43	Mon 10.22.07	9.50-11.30	Chapter 1, exercises 3 and 5. Exercises 1 and 2 to be done for next class.
	Tue 10.23.07	8.00-9.40	Continuous work on chapter 1, exercises 5 and 10.
	Wed 10.24.07	8.00-9.40	Chapter 2, section 2.1, students prove theorems.
	Thur 10.25.07	12.10-13.50	Chapter 2, section 2.2, students prove theorems.
Week 44	Tue 10.30.07	9.50-11.30	Chapter 2, section 2.3, students prove theorems.
	Wed 10.31.07	9.50-11.30	Chapter 2, section 2.4, exercises 17, 19, 21 and 22. Hand-in exercises given: 24a,b and 26b.
Week 45	Tue 11.06.07	8.00-9.40	Chapter 3, section 3.1, students prove theorems.
Week 46	Tue 11.13.07	9.50-11.30	Chapter 3, section 3.2 and 3.3, exercise 35. Hand-in exercises given: 28, 29 and 30.
	Wed 11.14.07	9.50-11.30	Chapter 3, section 3.3 continued, (section 3.4), students prove theorems.
Week 47	Mon 11.19.07	9.50-11.30	Chapter 4, exercise 64.
	Tue 11.20.07	8.00-9.40	Rest of chapter 4 including the proof of correctness of RSA, begin work on final essay assignment (groups). Hand-in exercises given: 41b, 42b, 45 and 46.
	Wed 11.21.07	8.00-9.40	Continuous work on final essay assignment (groups).
	Thur 11.22.07	12.10-13.50	Continuous work on final essay assignment (groups).
Week 48	Tue 11.28.07	9.50-11.30	Continuous work on final essay assignment (groups). The final essay is to be handed in Friday this week.
	Wed 11.29.07	9.50-11.30	Questionnaire 3.

Week	Day	Time	Activities
Week 49	Mon 12.03.07 Tue 12.04.07 Wed 12.05.07 Thur 12.06.07	9.50-11.30 8.00-9.40 8.00-9.40 12.10-13.50	Followup interviews. Followup interviews. Followup interviews. Followup interviews and discussion of final essay-assignment in class.
Week 51	Tue 12.18.07	9.50-11.30	Interview with the teacher.

A.4 Post-Implementation Activities

Week	Day	Time	Activities
Week 14	Wed 04.02.08	—	Questionnaire 4.
Week 16	Tue 04.15.08	—	Followup interviews.
Week 17	Wed 04.23.08	—	Followup interviews.
Week 07	Fri 13.02.09	—	Interview with the teacher.

B Questionnaires

In this appendix a translation of the four questionnaires given to the students is provided. Questionnaire 1 was given to the students before they had been exposed to any of the teaching material on the history of mathematics. Questionnaire 2 was given after the completion of the first teaching module. Questionnaire 3 was given after the completion of the second teaching module. And questionnaire 4, which closely resembles questionnaire 1, was given one year after the students answering of questionnaire 1 had taken place.

B.1 Questionnaire 1

Date: 19th of March 2007. Number of students to answer: 26.

1. On a scale from 1 to 5, where 5 is the subject you like the most and where 1 is the subject you like the least, where would you place mathematics?
2. Are you good at mathematics?
3. Regardless of what you answered to the previous questions, mention the things you like the most about the subject mathematics.
4. Regardless of what you answered to the previous questions, mention the things you like the least about the subject mathematics.
5. Do you believe it to be important for you to learn mathematics? If yes, why? If no, then why not?
6. Do you believe it to be important for people in general to learn mathematics? If 'yes', for whom is it then most important and why? If 'no', why not?
7. From time to time you hear that mathematics is used in many different contexts. Can you mention any places from your everyday life or elsewhere in society where mathematics is being applied, either directly or indirectly?
8. Do you think that mathematics has a greater or lesser impact on society today than 100 years ago?
9. *How* do you imagine that the mathematics of your textbooks came into being?
10. *When* do you think it came into being?
11. *Why* do you think it came into being?
12. Do you think that parts of mathematics can become obsolete? If yes, then in what manner?
13. Have you at any point in time during your schooling come into contact with the history of mathematics? If yes, when and in what way?
14. Is the history of mathematics something which interests you or which you may think could be of interest to you? If yes, *what* about the history of mathematics? If no, why not?
15. Were the negative numbers discovered or invented? Why?

16. Do you generally believe that mathematics is something which you discover or invent?
17. Is mathematics a science? If yes, about what? If no, what is it then?
18. What do you think a researcher in mathematics (at universities and the like) does? What does the research consist in?
19. Have you ever heard about error correcting codes? If yes, where? And for what are these codes used and where are they used?
20. Have you ever heard about cryptography? If yes, where? And in what connection have you seen cryptography used?

B.2 Questionnaire 2

Date: 5th of September 2007. Number of students to answer: 23.

1. What do you think was *most* interesting in the module on the early history of error correcting codes? (You may mention more than one thing.)
2. Was there anything about the early history of error correcting codes which surprised you? If yes, what and why?
3. What do you think was the *least* interesting in the module on the early history of error correcting codes? (You may mention more than one thing.)
4. Was it the historical aspects in the module or the aspects of application in the module which you found to be the most interesting? Why?
5. What do you think was the *most important* which you learned due to the module on the early history of error correcting codes?
6. From where does the binary number system originate?
7. What is meant by *binary representation*? Give an example.
8. Explain with your own words what is meant by the concept of *Hamming distance* and provide a concrete example.
9. Explain what is meant by a *Hamming sphere* and explain in terms of spheres, Hamming spheres or more precisely packing spheres, what is meant by a *perfect* code.
10. What perfect codes are known?
11. What motivated Hamming to develop his error correcting codes?
12. Explain what is meant by *objects* and *techniques*.
13. What significance may we with reasonableness assume for Hamming's work with the codes that he was a mathematician and not, for instance, an engineer?
14. What other area of mathematics has the theory of perfect codes contributed to?
15. Are Hamming codes and Golay codes discovered or invented?
16. Do you consider it fair that you can patent error correcting codes, as the Bell Labs did with the Hamming codes? Provide arguments to support your point of view.
17. How does your answer to the previous question connect to the discussion on whether mathematics is generally discovered or invented?
18. Regardless of what you answered in the previous questionnaire, do you think that mathematics is a science? Has your opinion on this changed?
19. What about the history of mathematics, is that a science?
20. How does one conduct history of mathematics?

21. How is research in history of mathematics different from research in mathematics?
22. Can you give a short description of how you construct an area of mathematics?
23. Why do we prove mathematical theorems?
24. What is meant by a mathematical model and the conduction of mathematical modelling?
25. If you, in addition to what you have already answered to the previous questions, have any praise and/or criticism to the module on the early history of error correcting codes and/or the teaching material used, then you are welcome to state your honest and well argued viewpoints here below.

B.3 Questionnaire 3

Date: 21st of November 2007. Number of students to answer: 23.

1. What do you think was *most* interesting in the module on the history of RSA and the mathematics behind the algorithm? (You may mention more than one thing.)
2. Was there anything about the history of RSA and the mathematics behind the algorithm which surprised you? If yes, what and why?
3. What do you think was the *least* interesting in the module on the history of RSA and the mathematics behind the algorithm? (You may mention more than one thing.)
4. Was it the historical aspects in the module or the aspects of application in the module which you found to be the most interesting? Why?
5. What do you think was the *most important* which you learned as part of the module on the history of RSA and the mathematics behind the algorithm? (You may mention more than one thing.)
6. What is meant by *private-key* and *public-key* cryptography, respectively? In what lies the public?
7. What is the key-distribution problem?
8. Where is private-key and public-key cryptography used, respectively? (You may provide concrete examples.)
9. What is meant by an *algorithm*?
10. What is meant by *number theory*? (What kind of numbers are we talking about?)
11. Did you find it funny to work with number theory? Why or why not?
12. Do you think that number theory is very much different from the mathematics which you are normally presented to at upper secondary level? Why or why not?
13. On what relationship concerning division of (large) integers is RSA based?
14. On what theorems from number theory does the *correctness* of the RSA algorithm in particular build?
15. Why do you think it took so long before the key-distribution problem was solved (by public-key cryptography) when the mathematics behind RSA had been available since the 18. century?
16. What is the relationship between Fermat's little theorem and Euler's theorem?
17. What is meant by *outer* and *inner driving forces*? (You may provide examples.)
18. What is meant by *pure* and *applied mathematics*? (You may provide examples.)

19. What was G.H. Hardy's viewpoint on pure and applied mathematics? And what was his viewpoint on number theory concerning this?
20. Why do you think that two parallel and independent developments of public-key cryptography and RSA took place at almost the same time (Stanford&MIT vs. GCHQ)?
21. From the viewpoint of the history of RSA and the mathematics behind the algorithm can you then say anything about the importance of basic research (here pure mathematics) and if yes, then what?
22. Is RSA discovered or invented? Why?
23. Regardless of what you answered in the previous questionnaires, do you then consider mathematics to be a science? Has your opinion on this changed? If yes, why?
24. What methods, tools, or approaches, can you mention which you may use to describe aspects of the history of mathematics?
25. If you, in addition to what you have already answered to the previous questions, have any praise and/or criticism to the module on the early history of error correcting codes and/or the teaching material used, then you are welcome to state your honest and well argued viewpoints here below.

B.4 Questionnaire 4

Date: 2nd of April 2008. Number of students to answer: 23.

1. On a scale from 1 to 5, where 5 is the subject you like the most and where 1 is the subject you like the least, where would you place mathematics?
2. Are you good at mathematics?
3. Regardless of what you answered to the previous questions, mention the things you like the most about the subject mathematics.
4. Regardless of what you answered to the previous questions, mention the things you like the least about the subject mathematics.
5. Do you believe it to be important for you to learn mathematics? If yes, why? If no, then why not?
6. Do you believe it to be important for people in general to learn mathematics? If 'yes', for whom is it then most important and why? If 'no', why not?
7. From time to time you hear that mathematics is used in many different contexts. Can you mention any places from your everyday life or elsewhere in society where mathematics is being applied, either directly or indirectly?
8. Not counting the ordinary types of calculation (the four basic arithmetical operations, calculation of percentages, etc.) where do you then find mathematics applied in your everyday life and society in general?
9. Do you think that mathematics has a greater or lesser impact on society today than 100 years ago?
10. *How* do you imagine that the mathematics of your textbooks came into being?
11. *When* do you think it came into being?
12. *Why* do you think it came into being?
13. Do you think that parts of mathematics can become obsolete? If yes, then in what manner?

14. Is the history of mathematics something which interests you? If yes, *what* about the history of mathematics? If no, why not?
15. When do you think people began using negative numbers? a) around 2000 B.C., b) around 300 B.C., c) around 1400, or d) around 1800.
16. When do you think negative numbers became fully accepted within mathematics? a) around 2000 B.C., b) around 300 B.C., c) around 1400, or d) around 1800.
17. From when does the coordinate system as we know it originate, do you think?
18. Were the negative numbers discovered or invented? Why?
19. Do you generally believe that mathematics is something which you discover or invent?
20. Is mathematics a science? If yes, about what? If no, what is it then?
21. If you say mathematics is a science, is it then a natural science? Why or why not?
22. What do you think a researcher in mathematics (at universities and the like) does? What does the research consist in?
23. What do *you* think to be the most important you learned in the two historical teaching modules last year?
24. What do you think about the idea of treating parts of the history of mathematics in the final essay assignments?
25. If you have any kind of praise or criticism to the two modules you are welcome to state them here.

C Transcripts of Video Clips

In this appendix a translated transcription of the video clips used in the dissertation is provided. The first clip is from the first module and the two following clips are from the second module.

C.1 Video Clip from the First Teaching Module

The clip from the first module contains the first focus group's (group 1) discussion of the fourth supportive essay assignment. The clip is a contraction of a discussion which ran over approximately 18 minutes. The parts where the students discussed matters not dealing with the assignment have been omitted.

Fourth Supportive Essay Assignment

- 1 **Andrew:** Well, what does the last assignment say?
- 2 **Lucy:** You have to argue your point of view, Andrew.
- 3 **Andrew:** Now I'm really annoyed with all this.
- 4 [...]
- 5 **Andrew:** Who should be given credit for the family of Hamming codes? Hamming, for
6 instance.
- 7 **Jonathan:** Yes, but he isn't the one to publish them. Golay develops them further.
- 8 **Sean:** Nope, but the funny thing is...
- 9 **Andrew:** You could say that he started it, right.
- 10 **Sean:** And when he has his name on it already, right.
- 11 **Andrew:** It's his fuck-shit that they build on.
- 12 **Jonathan:** Yeah, it is his shit they continued building on.
- 13 **Sean:** It is his theories.
- 14 **Jonathan:** Yes.
- 15 [...]
- 16 **Gloria:** Are we any closer to this... Golay? Have we moved on?

- 17 **Andrew:** Yes we have moved on. Then we'll have to get frustrated later [about the third
18 supportive assignment].
- 19 **Gloria:** Who do you think should be ascribed the honor for the family of Hamming
20 codes?
- 21 **Jonathan:** And why. Argue your point of view.
- 22 **Sean:** Hamming!
- 23 **Andrew:** I would say Hamming, for instance.
- 24 **Gloria:** Hamming-Hamming-Hamming-Hammiiiiing Hamming.
- 25 **Jonathan:** Yes, but you need to provide reasons.
- 26 **Gloria:** Yeah, but they are called Hamming spheres, right. What is there to ask about.
- 27 **Andrew:** He has already received honor for them.
- 28 **Jonathan:** Well yeah, but if it is published under Golay's name then maybe he was the
29 one to receive credit back then.
- 30 [...]
- 31 **Andrew:** We can also take this saying that it is fairly sure that Hamming knew that his
32 (7,4)-code could be generalized when he gave it to Shannon, but kind of didn't really
33 have time to publish it.
- 34 **Jonathan:** Yes that is what I mean. That is why Shannon might have gotten the honor
35 for it. And Golay...
- 36 **Andrew:** Yes he kind of put the theory higher than his own name by passing it on to
37 one who had time.
- 38 **Jonathan:** Yeah, he passed it on.
- 39 [...]
- 40 **Andrew:** Gloria, are you writing this down or what?
- 41 **Gloria:** Hmm.
- 42 **Andrew:** We're saying that we want to credit Hamming because...
- 43 **Jonathan:** That we believe he was aware of...
- 44 **Andrew:** Yes, he kind of made the basic theory, right.
- 45 **Jonathan:** And we believe that he was capable of creating the theory that Shannon and
46 Golay later published.
- 47 **Andrew:** Yeah, in this historical thing it surely says something with Hamming being
48 so busy working at Bell Labs that he passed his (7,4)-code on to Shannon. And it was
49 Shannon who made the generalizing...
- 50 **Gloria:** Yeah, it's kind of like... what do you say? He sowed the seeds, but somebody
51 else looked after the plants. Ha ha.

- 52 **Andrew:** Yes, but you can say that he didn't have time for it, so he let somebody who
53 had time do it.
- 54 **Gloria:** We believe that the honor for... what?
- 55 **Andrew:** The Hamming codes.
- 56 **Andrew:** And the award goes to: Hamming.
- 57 **Gloria:** ... should be ascribed to the sweet Hamming.
- 58 [...]
- 59 **Gloria:** It says that even though somebody else published the things...
- 60 **Andrew:** You can say that he provided the basics, right.
- 61 **Jonathan:** The debate concerns whether or not he knew of the generalization of his
62 (7,4)-code when he passed it on. Or if Golay was the first to discover it.
- 63 **Gloria:** Yes, who do we think it was?
- 64 **Jonathan:** Do we believe that Hamming knew of this when he made it...
- 65 **Andrew:** Yes, he passed it on because he was busy.
- 66 **Jonathan:** Or do we think that Golay was the first to discover it.
- 67 **Andrew:** Kind of like that we want to help you do this assignment, Gloria, but we are
68 too busy, so we pass it on to you and you can take credit for it once you have made it.
69 Aren't we being sweet?
- 70 **Gloria:** Yes, it is actually really sweet of you.
- 71 **Andrew:** We are so fantastic.
- 72 **Gloria:** This I don't know, because I want to answer 'yes he knew', but this doesn't
73 fit with what I want to answer in b. Because look at this. In b there is a question
74 which says: Why do you think people care about finding out who is responsible for
75 a mathematical result? Which personal driving forces of the mathematicians play a
76 part in mathematical research? Can you say anything about the sociology? People who
77 study mathematics, they are so nerdy. They enter completely into their own shell...
- 78 **Jonathan:** Like that movie we saw once with that guy [Andrew Wiles].
- 79 **Gloria:** Exactly. They spend their lives on it. There's nobody who spends his life on
80 something and then says: 'Ohh well, it doesn't matter, you can take the credit.'
- 81 **Jonathan:** So it is clear that it must come to someone...
- 82 **Gloria:** Well exactly that is why it doesn't make sense, if Hamming did know and he
83 spent so long time on it.
- 84 [...]
- 85 **Gloria:** We believe that the honor should be ascribed to the sweet Hamming.
- 86 **Jonathan:** Can't you not write 'the sweet Hamming'?

- 87 **Gloria:** It is written now and that will not be changed.
- 88 **Andrew:** It also says here that they believe that he did actually have... or that he did
89 know that there was some generalization of his (7,4)-code.
- 90 **Gloria:** Why didn't he have time? What was he doing at Bell Labs? How important
91 was it?
- 92 **Andrew:** He worked.
- 93 **Gloria:** Worked, right. If he was a real researcher he wouldn't bloody work.
- 94 **Jonathan:** It says here that he had to wait with publishing the codes until the thing
95 with the patent had been resolved.
- 96 **Andrew:** Okay, it has something to do with patent, it has something to do with money?
- 97 **Jonathan:** Yes.
- 98 **Andrew:** Then it is important.
- 99 **Jonathan:** Yes, he wanted the patent before he published it, right. So that no one else
100 could use it, other than himself.
- 101 **Andrew:** And then Shannon got it, and he ran riot?
- 102 **Gloria:** And then he gets the patent?
- 103 **Gloria:** How unpleasant. [Joking]
- 104 **Andrew:** I too think that this is very unpleasant.
- 105 **Lucy:** Ha ha... he said this to me three seconds ago.
- 106 **Andrew:** [Takes the essay assignment] I'm going to write my name on this!
107 [...]
- 108 **Andrew:** Why does it matter for people to get the right... well creator?
- 109 **Gloria:** Because they fucking spend their life on that shit.
- 110 **Andrew:** So we know whom to hate.
- 111 **Lucy:** We must hate this guy.
- 112 **Jonathan:** It says: Many educated people are of the opinion that the honor should be
113 ascribed to the first person to publish something. That is the dilemma with Hamming,
114 right. Because both Shannon and Golay have published their stuff before Hamming
115 himself publishes it.
- 116 **Andrew:** But it is also kind of a principle... that you don't...
- 117 **Gloria:** No, it is god damn because everybody know that such research mathematicians...
118 that they spend their lives on this. You can't come up with some brilliant mathematical
119 thing now – because so much have already been created – you can't do it without really
120 spending a lot of energy on it. And anybody who really make an effort, damn it, wants
121 to get some credit for it.

- 122 **Andrew:** Yeah yeah, that was what I was going to say that it is a huge honor for them
123 to kind like of like have their name on it...
- 124 **Gloria:** No, it is a huge...
- 125 **Jonathan:** They can't get the credit for it before it has been published.
- 126 **Gloria:** No, it is the thing with it being a huge work load and it being a huge effort they
127 make to make it happen.
- 128 **Jonathan:** That is why it must be a fucking downer to see someone else get...
- 129 **Sean:** And then you know who to go totally berserk at if you actually find an error in it.
- 130 **Gloria:** You don't do that.
- 131 **Sean:** Yeah, because if there are further things...
- 132 **Jonathan:** He also got slaughtered that guy with Fermat's... the guy who made
133 Fermat's...
- 134 **Lucy:** There *was* an error.
- 135 **Jonathan:** Yes, there was an error. Then he was just slaughtered. Then he went totally
136 down. And then he corrected it again.
- 137 **Lucy:** Imagine actually wanting to sit and go through that long proof and then finding
138 an error in it.
- 139 **Andrew:** We'll just write that there is a lot of work behind and its a bummer if someone
140 else gets the credit, right.
- 141 **Jonathan:** Bummer?
- 142 **Andrew:** I'm simply frustrated, I don't seem able to express myself right now.
- 143 **Gloria:** No, you are really being snappish.

C.2 Video Clips from the Second Teaching Module

The clips from the second module contain the modified focus group's (group 1*) discussions of the the first and second supportive essay assignments. In the clip from the second supportive essay assignments the students stayed 'focussed' discussing matters relating to the task at hand for a total time of 12 minutes. In the clip from the first supportive essay, parts have, however, been omitted in the 14 minutes time period of the clip.

Second Supportive Essay Assignment

- 1 **Harry:** Hey, we can easily agree that Diffie and Hellman were driven by inner driving
2 forces because they were the first to invent the theory while Rivest, Shamir, and Adleman
3 they only build on theirs. They saw theirs and then found a function for it, and therefore
4 that must be outer driving forces.

- 5 **Andrew:** But those guys, Diffie and Hellman, they were also of an outer driving force
6 since they also kind of had a purpose with their mathematics. Weren't it them who
7 wanted to do...
- 8 **Gloria:** I think, Diffie and Hellman... that is, those guys who wanted to produce the
9 thing with the Internet...
- 10 **Andrew:** It's both inner and outer. You of course have to have the inner before you
11 even want to do it. And the outer, there is a need for security.
- 12 **Gloria:** Yeah, we need that for part c, the thing with all mathematicians having...
- 13 **Jonathan:** ... somehow some inner. Where is the thing with Diffie and Hellman? Where
14 does it say about them?
- 15 **Harry:** Diffie and Hellman they are most certainly more inner, I'd say, than Rivest,
16 Shamir, and Adleman are. Because they only saw their article and then they wanted to
17 find a code which fits it. So if Diffie and Hellman never had done it then Rivest, Shamir,
18 and Adleman would never have been known today and there would never have been a
19 RSA algorithm. And that is only based on the outer driving force.
- 20 **Jonathan:** Oh, here it says.
- 21 **Andrew:** At least you can say, that common for them is that they have more an outer
22 driving force than the others, right?
- 23 **Gloria:** Yes, but that is just though time [history].
- 24 **Jonathan:** Yes, we can put that in part c.
- 25 **Gloria:** Yes, that's for part c.
- 26 **Andrew:** It is because you have gotten a need for it, right?
- 27 **Gloria:** Yes, exactly.
- 28 **Andrew:** It is because they also have an outer driving force, there is an external need
29 for what they do. ... I suppose that is easy since they do security.
- 30 **Jonathan:** It was Diffie who wanted to work with the Internet, wasn't it?
- 31 **Harry:** But, was that because he was head hunted by someone who worked with the
32 Internet? Wasn't it just because he was interested in it on his own, or what?
- 33 **Andrew:** I don't know if it was because he was head hunted, but Randi [the teacher]
34 has said a couple of times that it was because he could predict...
- 35 **Jonathan:** It says right here: Diffie was especially fascinated and interested in the
36 beginning Internet, the so-called ARPANet, when it at first belonged to the US military.
37 And he was one of the first to realize the possibilities, blah blah ...
- 38 **Harry:** Well, okay.
- 39 **Andrew:** 'Diffie' that's a bummer...
- 40 **Jonathan:** And then he learns about Hellman and he drives 5000km to California...

- 41 **Gloria:** ‘Diffie’, ‘Diffie’, that’s a cool name. ‘You can call me Diffie’. [laughing]
- 42 **Andrew:** There you don’t know if it’s a man or a woman. ... So Diffie should get some
43 middle names squeezed in.
- 44 **Gloria:** Yeah, he should: Diffie Uffe ... Diffie Hamming. [laughing]
- 45 **Andrew:** You’re totally cracking up.
- 46 **Jonathan:** There is also some inner due to the fact that they want to compete with the
47 American Security Service.
- 48 **Harry:** They want to compete with it?
- 49 **Jonathan:** Yes, it says: Hellman’s colleagues had on several occasions expressed the fact
50 that they didn’t understand that he wanted to work on cryptography since you as a
51 cryptographer at a university will be competing with the American national security
52 service NSA and their million dollar budget.
- 53 **Harry:** Then it’s also partly outer?
- 54 **Andrew:** Yeah, you can say that that is also an outer, because they get some kind of
55 recognition for it.
- 56 **Gloria:** Yes, exactly.
- 57 **Harry:** Yes and they want to compete with...
- 58 **Jonathan:** No, they don’t do it to compete. They do it because they themselves want
59 to do it. But the others can’t understand why they want to because the US security,
60 they have some kind of huge budget where they get loads of money to do it while they
61 don’t get anything out of it.
- 62 **Harry:** The American security... what?
- 63 **Andrew:** Service.
- 64 **Jonathan:** It doesn’t say they want to compete with them.
- 65 **Harry:** What does it then say?
- 66 **Jonathan:** It says that the others can’t understand why they want to compete with
67 them and why they work on it when the American security service have so much money
68 to do research in it. They don’t really get any money out of it.
- 69 **Harry:** Okay.
- 70 **Jonathan:** How could he imagine developing something they already didn’t have? And
71 if he did anyway, the NSA would probably...
- 72 **Harry:** But it had to be some kind of competition anyway?
- 73 **Jonathan:** He was driven by his own interests and shared Diffie’s idea that cryptography
74 should be for the benefit of all, and not just the government.
- 75 **Lola:** No, but okay...
- 76 **Jonathan:** They don’t have any money to do it.

77 **Lola:** Check this out: Diffie recalls the day in 1975 when he first got the idea for
78 public-key cryptography like this: I went downstairs for a Coke and almost forgot the
79 idea. I remembered that I had been thinking of something interesting, but I could not
80 recall what it was. Then it came back to me in an adrenaline rush of excitement. For the
81 first time while I had been working with cryptography I was aware that I had discovered
82 something really valuable. He is...

83 **Harry:** He is really interested in it himself, right?

84 **Gloria:** Yes yes, but all of them are. There is actually an idea behind him producing
85 this mathematics: it is for developing the Internet.

86 **Harry:** Yes, but nobody told him that he should do it to develop the Internet. It's like
87 in wars, there are many...

88 **Gloria:** No, nobody... but I also think... No, but the outer circumstances, it also has to
89 do with the situation of the world and its state, and how far you are and what it can be
90 used for.

91 **Andrew:** And again, the thing with you getting the credit for it.

92 **Harry:** But I also think that these three persons...

93 **Gloria:** But he also wants to do this for the sake of the world.

94 **Harry:** It's like that with all of them. I mean, it's for the benefit of the world, all the
95 things these guys are doing.

96 **Jonathan:** All the previous ones haven't said that they wanted to do this for something.
97 Here they say that they want to do it so that everyone may benefit from it.

98 **Harry:** Yes, but that still isn't an outer driving force. It is just his own thought as to
99 what his project may amount to. Isn't it?

100 **Jonathan:** Yes, write!

101 **Jonathan:** We've got 15 minutes left.

102 **Andrew:** I'm drawing you a picture, Gloria.

103 **Gloria:** Are you? That's okay, I think.

104 **Andrew:** It's a little bit out of proportion. In some places.

105 **Jonathan:** Also, there is this thing that they keep trying a million times until they
106 succeed, right?

107 **Harry:** I also think that there is a reason for us having to take these two first and then
108 the others because there is a difference between them... and they...

109 **Jonathan:** Yes, there is.

110 **Harry:** But what about Ellis, Cocks, and Williamson? They are the ones working for
111 the military, so they must definitely be outer.

112 **Jonathan:** Yes, they are. They are told that this is something we need to have a solution
113 for. That's what they call and say. That's what they tell this guy Ellis. But still Cocks
114 is driven by his...

115 **Lola:** Even though they are happy with this then the outer is still having a large effect,
116 because they are kind of forced to...

117 **Jonathan:** Well, I think that it is an outer motivation or outer driving force for the last
118 guys at least. Because they are told: We need a solution for this, figure out something!

119 **Lola:** But it is a job which is given to them by outer...

120 **Jonathan:** Yes.

121 **Jonathan:** [Asking the teacher:] Outer driving forces, right, they are only money and so,
122 right?

123 **Teacher:** Driving forces need not only be money.

124 **Jonathan:** No, but I said outer driving forces.

125 **Teacher:** Oh, outer.

126 **Jonathan:** So, they are being told: Do this task.

127 **Gloria:** But can't the state of the world also be an outer driving force: Okay, now we
128 are like this, we need some Internet which works even better?

129 **Teacher:** Yes, I think you can say that that can be an outer driving force.

130 **Lola:** So you can say that the task they are being given is an outer driving force?

131 **Harry:** Yes, but the task they are being given must be given by an outer driving force
132 before you can say that they are driven by an outer driving force. But they are not
133 being given this task.

134 **Gloria:** Listen. All mathematicians are also driven by an inner driving force, otherwise
135 they wouldn't do it. Otherwise they wouldn't care to spend their life on it.

136 **Harry:** But those three persons were not told to make a better system.

137 **Gloria:** No, no...

138 **Teacher:** Oh, you mean outer driving force as someone else posing the task.

139 **Gloria:** It doesn't have to be. It doesn't have to be somebody else who gives the task.

140 **Lola:** It could also be because they get money for it.

141 **Jonathan:** Yeah yeah, we said that already.

142 **Gloria:** Try and listen to this: Outer driving forces are understood as those forces which
143 affect the research of mathematics from the outside.

144 **Harry:** They began this not knowing if it was going to lead somewhere or not. What
145 he [Hellman] writes is that it's just idiots who keep on trying, isn't that right? They
146 continue because they are interested in it themselves.

- 147 **Gloria:** And because they want to succeed, so the world can evolve a little and the
148 Internet can come to work.
- 149 **Harry:** But then you can say that about everybody, because everybody wants the world
150 to evolve.
- 151 **Gloria:** That's not right. This guy who says that I want to play with prime numbers,
152 and I don't care if it can be used for anything.
- 153 **Harry:** There is no doubt that they all make the world evolve, and all have both an
154 inner and an outer driving force. What I'm saying is just that the three have more of an
155 inner because they are interested in the stuff and have started looking into it without
156 anything having influenced them to do it. And when they start, they don't even know if
157 they will achieve anything which means that they don't think about money.
- 158 **Gloria:** Can't we just say that you can discuss what has driven them the most because
159 we, in the group, do not entirely agree, and it isn't something which we can...
- 160 **Teacher:** But it is in fact right what Harry says, but you can still say... So, one thing
161 is that they have the inner with it being an interest and so on, but at the same time
162 you can say that there is also an outer. I mean, that Diffie can see this purpose for the
163 benefit of all. And I can't see any conflict in saying that it is a combination of inner
164 and outer in this way.
- 165 **Harry:** I also think that it is very obvious, the way it is being posted with Diffie, Hellman,
166 and Merkle first that that is mostly inner, and then Rivest, Shamir, and Adleman who
167 comes afterward and take their ideas. But they also have an inner driving force. They
168 do in fact have both. Actually, they have just as much an inner as they have an outer,
169 I'd say. And then the last one which is only outer.
- 170 **Teacher:** Do Adleman have an inner at all?
- 171 **Gloria:** All mathematicians have an inner driving force otherwise they couldn't get
172 themselves to do...
- 173 **Teacher:** Yes, but in relation to this?
- 174 **Gloria:** Otherwise they couldn't keep themselves doing it.
- 175 **Teacher:** And yet, check what you have read. Precisely about Adleman you could ask
176 yourself if he has an inner driving force in relation to cryptology.
- 177 **Harry:** He was the one who was put in because they needed him to do cryptology.
- 178 **Gloria:** You can't be a researcher if you don't have an inner driving force.
- 179 **Teacher:** Yes, but when you first have the education. He has an inner driving force -
180 and an outer -to take care of his job.
- 181 **Jonathan:** Adleman he is the one who isn't really interested in it, but whom they use to
182 figure out the mathematical calculations.
- 183 **Teacher:** 'Check this one, check this one, and check this one too... number 43, oh yeah,
184 hurray.'

- 185 **Jonathan:** *That* he isn't doing out of own interest, I think that too.
- 186 **Teacher:** He does it because he is a good pal. So he has no inner driving force whatsoever,
187 I'd say.
- 188 **Lola:** Did he ever have one?
- 189 **Teacher:** Not for this.
- 190 **Jonathan:** The others try to make him interested, but they never really succeed.
- 191 **Teacher:** One thing is that he has an inner driving force related to mathematics, oh
192 yeah, that we know nothing about, but in relation to this, no, not at all.
- 193 **Harry:** So it is a mix, right? Because Rivest, he is really interested in coding, and he is
194 the one to take initiative to it all, right?
- 195 **Jonathan:** Yes, yes, Rivist and Shamir they are... but shouldn't we finish Diffie and
196 Hellman first?

First Supportive Essay Assignment

- 1 **Harry:** Isn't it very logical that he [Hardy] says that number theory is only connected
2 to mathematics, pure mathematics that is? And if we have to compare this to RSA
3 then what we have to say is that they have to have this third guy [Adleman] in order to
4 be able to encode, right? If they didn't have him they wouldn't know how to encode.
5 And he was a pure mathematician, right.
- 6 **Andrew:** Who is that?
- 7 **Jonathan:** It's Adleman who was a number theoretician and who sits and runs all those
8 though in order to test their codes, right.
- 9 **Andrew:** But we agree that Hardy he says that number theory is pure mathematics,
10 right?
- 11 **Jonathan:** Yes.
- 12 **Andrew:** So RSA is just a...
- 13 **Harry:** Yes, but I'm just not sure where it says that. But when he says that everything
14 is pure mathematics, or that the best is pure mathematics, and that it can be created
15 based on everything, or that everything can be created based on mathematics, then it
16 has to say that.
- 17 **Andrew:** Yes, you could kind of say...
- 18 **Lola:** What was the question again?
- 19 **Jonathan:** We have to discuss...
- 20 **Andrew:** Discuss Hardy's statements about number theory based on your knowledge of
21 the development of RSA. For instance, what does the example of RSA tell us about the
22 importance of basic research in mathematics? How does this fit with Hardy's statements
23 about number theory?

24 **Andrew:** Well, at least you can say that if number theory is pure mathematics then it
25 is very basic, right? And then we have all this RSA which is being developed using this
26 number theory, and as Jonathan says, it is Adleman who's kind of the crucial link in
27 this...

28 **Jonathan:** Yeah, if they hadn't had this number theoretician guy then they wouldn't
29 have gotten anywhere. The others kind of want to develop it and are very interested in
30 it, and they have this problem. But they kind of have to work together with a number
31 theoretician.

32 **Andrew:** But that underpins that the applied mathematics must build on the pure
33 mathematics, right?

34 **Jonathan:** Yes.

35 [...]

36 **Lola:** Okay, this is a muddle, what I've written. I've written: if pure mathematics
37 according to Hardy is the best, applied mathematics should build on pure mathematics.
38 Isn't that awkward?

39 **Andrew:** That isn't muddle, it's just an awkward formulation.

40 **Gloria:** What did you write?

41 **Lola:** If pure mathematics according to Hardy is the best, applied mathematics should
42 build on the pure.

43 **Andrew:** But it must build on the pure.

44 **Jonathan:** ... applied mathematics in the form of RSA must build on the pure...

45 **Lola:** So instead I must write that applied mathematics builds on pure mathematics?

46 **Andrew:** Yeah, and then we can use RSA as...

47 **Jonathan:** ... an example.

48 **Andrew:** Yes.

49 **Jonathan:** Where they needed Adleman in order to... or needed Adleman as 'an
50 important piece in the game'.

51 **Lola:** And then we say that RSA, for example... But what does that have to...?

52 **Andrew:** Well, we've got these two RS – I can't remember what it stands for, but
53 that doesn't matter – they've got this idea that they want to develop this odd thing,
54 right. But they can't really get moving until they get this guy Adleman, who is a pure
55 mathematician who kind of... well, he...

56 **Lola:** [Writing:] For example, RSA could not be developed further before they got...

57 **Jonathan:** ... help from Adleman.

58 **Gloria:** From the number theoretician Adleman.

59 **Lola:** [Writing:] from the number theoretician Adleman.

- 60 **Harry:** Just ‘develop’, right. There was nothing before. Or invent.
- 61 **Lola:** [Writing:] ... Adleman who is a pure mathematician.
- 62 **Harry:** Can’t you just write ‘develop’ instead of what you’ve got.
- 63 **Lola:** Why?
- 64 **Harry:** Because it wasn’t developed further. RSA wasn’t there before.
- 65 **Lola:** It must have been developed a little? Otherwise it wouldn’t...
- 66 **Andrew:** No. Well, the thought was there, but it is not developed until these guys hook
67 up with Adleman.
- 68 **Lola:** Okay, but it is developed now, right?
- 69 **Jonathan:** Yeah. They’ve got this problem that they need this one-way function. So
70 they come up with some suggestions for some solutions. Then they pass them on to
71 Adleman and say ‘try and see if these ones check out’. And then Adleman sits and tests
72 them...
- 73 **Andrew:** But Adleman, he was the the guy who wasn’t interested in this thing at all,
74 right?
- 75 **Jonathan:** Yeah, he wasn’t really interested, but he helps them out.
- 76 **Lola:** So, in principle it isn’t *can* RSA but *could* RSA, because it is something which
77 they already did, right.
- 78 **Andrew:** Yes, well, it *couldn’t* have been developed if it wasn’t for...
- 79 [...]
- 80 **Andrew:** [Reading question d aloud:] What do you generally think about Hardy’s
81 *Apology* and the view of mathematics and mathematicians which...
- 82 **Lola:** I don’t think we’re done with part c, we’ve only written two lines.
- 83 **Andrew:** But we’ve discussed it in terms of RSA...
- 84 **Lola:** The only thing I’ve written is that applied mathematics builds on pure mathematics.
85 RSA, for example, could not be developed before they received help from the number
86 theoretician Adleman, who was a pure mathematician. That’s all we’ve got for c.
- 87 **Andrew:** But that also underlines...
- 88 **Jonathan:** We probably need to discuss his [Hardy’s] statements more. The problem is
89 that we can’t remember where they are written.
- 90 **Andrew:** No.
- 91 **Lola:** We have to quote it?
- 92 **Jonathan:** Yeah, that’s what I was thinking.
- 93 **Andrew:** Harry, did you find the thing with...
- 94 **Jonathan:** ... number theory in the Hardy book?

95 **Harry:** Well, no.

96 **Jonathan:** We need to locate it so that we can talk about his statements. I haven't got a
97 clue where it is. We must skim it or something.

98 [They read for about 1 minut.]

99 **Andrew:** I just didn't think that there was anywhere where he said anything specific.

100 **Jonathan:** I can't remember, but there must something since we are to discuss his
101 statements.

102 [They read for 1 minut more.]

103 **Harry:** [After reading in the *Apology*] I think we might have to read some in section 3.5
104 [in the teaching material], maybe it's something he says here.

105 **Harry:** [After about 20 seconds of reading in section 3.5:] It says [in the teaching
106 material] that one of the problems in number theory is to decide if a number is a prime
107 or not. And in this book [the *Apology*], he [Hardy] says that finding primes is pure
108 mathematics, because when you are a mathematician then you already have the frame
109 for the area you are working within, you know what a prime number is, in contrast to a
110 physicist or a chemist who works with some applied mathematics, they have to work
111 with things relatively, I would say: Here is a table [points to the table], but to you...
112 for some other person this might not be a table, or for some other thing in a different
113 universe, this might not be a table. But a prime number will always be a prime number.

114 **Andrew:** So the pure mathematics cannot be discussed, you might as well say.

115 **Lola:** That depends on how you conceive that the history of mathematics has been
116 developed, because if someone else had been sitting and thinking over a number and
117 had found some other connections, well then a prime might suddenly not have the same
118 mening.

119 **Harry:** But now they have this frame...

120 **Andrew:** You could say that, what we are kind of working with... it is our frame, it
121 is the numbers we have, it is our frame, and then within these there is a lot of pure
122 mathematics, for example prime numbers.

123 **Lola:** Yes, primes can never be different, if you look at how we look at mathematics.

124 **Andrew:** No, if you stick within our frame.

125 **Lola:** But if we imagine that the numbers had some different values or whatever you'd
126 say, then...

127 **Andrew:** Yes, but then you are changing to a new frame, and then there is a connection
128 within this frame.

129 **Lola:** Well... but then you could also say...

130 **Harry:** In that way it might be kind of relative, I can see that. But what he says is
131 that you can't discuss it. He says that in this world we live in here there can be two

132 different physicists who tell you what this is [points to the table] while two different
133 mathematicians [equally] can tell you what a prime number is.

134 **Lola:** Yes, and that is what Hardy he says, right? Do you want me to write that?

135 **Andrew:** But then you could also say that the pure mathematics is as objective as
136 anything can ever become, right, because it isn't colored by anything.

137 **Harry:** But we have to see it in connection to RSA.