TEKST NR 399 2001

Johan Lithner:

Undergraduate Learning Difficulties and Mathematical Reasoning

PhD Thesis

TEKSTER fra

IMFUFA **ROSKILDE UNIVERSITETSCENTER**
INSTITUT FOR STUDIET AF MATEMATIK OG FYSIK SAMT DERES FUNKTIONER I UNDERVISNING, FORSKNING OG ANVENDELSER

IMFUFA – Roskilde Universitetscenter
Postboks 260 – DK 4000 Roskilde – Denmark
tlf. +45 46742263 – fax +45 46743020

Johan Lithner: *Undergraduate Learning Difficulties and Mathematical Reasoning.*
PhD Thesis 167 pages (5+41+14+23+27+57)

IMFUFA tekst nr. 399/2001 **167 sider** **ISSN 0106-6242**

This PhD thesis consists of five parts: (1) A literature survey and an overview of a larger research project with the same title, including background and framework. Then follow the four main papers of the thesis, presenting completed studies within the larger project: (2) A study on four students' task solving difficulties, indicating that the students were more focused on what is familiar and remembered than on mathematical reasoning and accuracy. (3) An extension of the former study by developing an analytical framework, and focusing on the quality of their reasoning. It was found that the reasoning was more 'superficially experience-based' than mathematically based. (4) A study describing in detail how most textbook exercises may be solved without considering the core mathematics of the textbook, mainly by copying solved examples, and how this may lead to the behaviour above. (5) A study of the ways studetns conduct their homework that, among other things, confirms that they are restricted to using the superficial procedures found in (4).

Johan Lithner, Umeå University, Sweden, July 2001

Undergraduate Learning Difficulties and Mathematical Reasoning: Thesis Preface

Johan Lithner

July 10, 2001

Abstract

This PhD thesis consists of five parts: (1) A literature survey and an overview of a larger research project with the same title, including background and framework. Then follows the four main papers of the thesis, presenting completed studies within the larger project: (2) A study on four students' task solving difficulties, indicating that the students were more focused on what is familiar and remembered, than on mathematical reasoning and accuracy. (3) An extension of the former study by developing an analytical framework, and focusing on the quality of their reasoning. It was found that the reasoning was more ' superficially experience-based' than mathematically based. (4) A study describing detail how most textbook exercises may be solved without considering the core mathematics of the textbook, mainly by copying solved examples, and how this may lead to the behaviour above. (5) A study of the ways students conduct their homework that, among other things, confirms that they are restricted to using the superficial procedures found in (4).

The starting point for this thesis is a severe educational problem that is easy to formulate but difficult to resolve: We are, as organisers of undergraduate mathematics courses, not able to help sufficiently many students reach a desired level of mathematical competence [see the Background section in part 1 of this thesis (Lithner, 2001b) for some approximate data on dropout rates etc.]. Larger and larger groups are studying undergraduate mathematics but many of the students have severe learning and achievement difficulties, and the dropout rates are far too high. This is a severe problem, not only for many students as individuals but also for the society since there is a shortage of labour force in many areas were mathematically intense academic qualifications are required. It is also a severe problem for our universities, who are not able to provide a learning environment that can cope with the situation.

It is important to try to improve the learning environment and in order to do so on a well-founded basis it is crucial to study the following two research

questions, which is the purpose of this thesis:
Q1: What are the *characteristics* of the undergraduate students' main learning and achievement difficulties in mathematics?
Q2: What are the main *reasons* behind these difficulties?
The main purpose behind Q1 and Q2 is to provide a foundation for answering the following question, which is not primarily within the scope of this thesis:
Q3: What *measures* should be taken in order to reduce the difficulties?

It is far from possible to provide complete answers to the very difficult and complex questions Q1, Q2, and Q3, so the thesis will actually treat some subquestions of central importance. The thesis is divided in the following five parts, which are all related through the research questions above, but written as separate articles that stand by themselves.

Part 1. Undergraduate Learning Difficulties and Mathematical Reasoning: A Literature Survey and Project Overview.

This paper (Lithner, 2001b) contains four sections:

Section 1: Background. A short background to the research questions Q1, Q2, and Q3.

Section 2: Literature survey. A literature survey related to these questions, where the results are briefly summarised as:
- Though one of the main curricula goals is conceptual understanding, this seems hard to reach for many students, especially at a global, general level.
- A large number of research articles show the severe unbalance towards rote learning of algorithmic procedures and the inability to solve non-routine problems. This seems to be related to weak conceptual understanding.
- There is a pressure on students and teachers to reduce the mathematical complexity in the learning environment, for example to work in a 'rote learning mode'.
- It is possible to help students develop better understanding and problem solving abilities, but this often require more engagement, expertise from teachers, and significant change in practice.
- Students' reasoning is not only based on mathematical thinking: the need to cope (e.g. pass exams) in situations that are difficult for them to handle may lead them into reasoning of other types.

Section 3: General framework. A framework for the larger project mentioned above. The learning and achievement difficulties described above are not in accord with the explicit or implicit goals of the course organiser, and there are several junctures where discrepancies may exist. One way to structure the study of possible differences between goals and outcomes is provided by Robitaille and Garden (1989). They have characterised discrepancies between the components in the following framework:

> "The *intended curriculum* as transmitted by national or system level authorities; the *implemented curriculum* as interpreted and translated by teachers according to their experience and beliefs for particular classes; and the *attained curriculum*, that part of the intended

curriculum learned by students which is manifested in their achievements and attitudes" (Robitaille and Garden, 1989, p.4).

To make this study of the potential discrepancies between the different aspects of the curriculum more precise a fourth aspect is added: The *received curriculum*, the part of the implemented curriculum that influences the students.

Section 4: The project components. A brief description of the components in the larger project, of which four completed studies constitute the main papers of this thesis and are described below:

Part 2. Students' general difficulties in task solving

In this study, 'Mathematical reasoning and familiar procedures' (Lithner, 2000a), four first-year undergraduate students were videotaped while working with two tasks. The underlying question treated was 'what are the characteristics and background causes of their difficulties when trying to solve these tasks?' The purpose was to give a general survey of their main difficulties, rather than to go deeply into details. One of the common characteristics was that the students were more focused on what is familiar and remembered, than on (even elementary) mathematical reasoning and accuracy.

Part 3. Students' reasoning in task solving

This study, 'Mathematical reasoning in task solving' (Lithner, 2000c), was based on the same data as (Lithner, 2000a) but aimed at focusing on, and extending, the part of the earlier study that concerned task solving strategies. This included the development of an analytical framework. The results indicated that focusing on what was familiar and remembered at a superficial level is dominant over reasoning based on mathematical properties of the components involved, even when the latter could lead to considerable improvement in progress. The main difference between (Lithner, 2000a) and (Lithner, 2000c) is that the former is 'wider' (all their main difficulties) and the latter is more limited to treating, on a firmer theoretical foundation, certain types of mathematical reasoning.

The studies (Lithner, 2000a) and (Lithner, 2000c) indicated (together with studies of research literature) that students focus on routines and superficial reasoning, and one of the main reasons behind their difficulties is their inability and/or reluctance to consider the mathematical properties involved in the reasoning. The studies below aimed at searching for possible reasons behind these indications.

Part 4. Strategies and reasoning possible to use when solving textbook exercises.

The aim of this paper, 'Mathematical reasoning in Calculus Textbook Exercises' (Lithner, 2000b), was to study some of the strategies that are possible to use in order to solve the exercises in undergraduate calculus textbooks. The reason behind this choice of study was that students in general spend most of their study time trying to solve textbook exercises (Lithner, 2001a). It was described in detail how most exercises may be solved by mathematically superficial strategies. Strategy choices and implementations can usually be based on identifying similar solved examples and copying, or sometimes locally modifying, given solution procedures. One consequence, which is analysed, is that exercises may often be solved without actually considering the core mathematics of the book section in question.

Part 5. Strategies and reasoning applied by students when solving textbook exercises.

The study 'Students' Mathematical Reasoning in Textbook Exercise Solving' (Lithner, 2000d) investigated the ways three students conducted their study work, in particular their mathematical reasoning when working with textbook exercises. The results indicated that: (i) Most strategy choices and implementations were carried out without considering the intrinsic properties of the components involved in the solution work. This in turn lead to different difficulties. (ii) It was crucial for these students to find solution procedures to copy. (iii) There were extensive attempts, often successful, to understand each step of the copied solution procedures, but only locally. (iv) The students made almost no attempts to construct their own solution reasoning, not even locally. (v) The main situations where the students' work were not just straightforward implementations of provided solution procedures, were where careless mistakes were made in minor local solution steps when implementing provided procedures.

An informal summary of the studies

The research methods used in the studies above are mainly qualitative: Relatively fine-grained analyses of a small number of students' reasoning characteristics in limited task solving situations, where the analyses included the development of analytical frameworks. These types of analyses can not determine with a high degree of accuracy the reasoning characteristics of students in general, but can a) show the *existence* of some reasoning types and b) *indicate* plausible characteristics of larger student groups. The latter may also be supported by studying similar or related aspects from other theoretical perspectives or by other methods, hereby finding reasonable and general explanations behind the indicated behaviour. One example of this is the study (Lithner, 2000b), which is partly quantitative (600 textbook exercises were classified), where possible reasons behind the students' behaviour in the other studies are investigated.

Though the studies above treat only limited aspects of students' competence and limited aspects of the learning environment, and though the work of rather few students is investigated, the overall picture emerging is coherent: It seems like the students are founding their work mainly on superficial reasoning, and that the reasons behind this originates to a large extent from the learning environment provided by the educational system. This is (at least partly) already known, as exemplified by the literature review in (Lithner, 2001b), and it also seems to be experience-based knowledge familiar to many teachers. The motivation for carrying out as research the studies above is: (i) The studies (Lithner, 2000c), (Lithner, 2000b), and (Lithner, 2000d) explicitly and primarily address the ways that the students' reasoning is based on mathematical properties or not, something that is not done by many other studies. The studies (Lithner, 2000a) and (Lithner, 2000c) indicated that the domination of 'non-mathematical' reasoning is one of the main causes behind task-solving difficulties. (ii) The reasoning is studied in rather fine-grained detail. A framework for this type of studies is one of the outcomes. (iii) There are surprisingly few studies on textbook structure (Love and Pimm, 1996) (especially from the perspective (i)) and on students' actual learning strategies and textbook usage. (iv) The achievement difficulties of mathematics students at all levels have been known for many years, but the difficulties mainly remain. Extensive research on the questions Q1, Q2, and Q3 above is still required in order to be able to construct well-founded measures for improvement of the learning environment.

References

J. Lithner. Mathematical reasoning and familiar procedures. *International Journal of Mathematical Education in Science and Technology*, 31:83–95, 2000a.

J. Lithner. Mathematical reasoning in calculus textbook exercises. Research Reports in Mathematics Education 2, Department of Mathematics, Umeå University (27 pages), 2000b. Submitted for publication.

J. Lithner. Mathematical reasoning in task solving. *Educational studies in mathematics*, 41:165–190, 2000c.

J. Lithner. Students' mathematical reasoning in textbook exercise solving. Research Reports in Mathematics Education 4, Department of Mathematics, Umeå University (57 pages), 2000d. A shorter version is to be submitted for publication.

J. Lithner. Students' efforts in learning mathematics, 2001a. In preparation.

J. Lithner. Undergraduate learning difficulties and mathematical reasoning: A literature survey and project overview. Research Reports in Mathematics Education 5, Department of Mathematics, Umeå University (41 pages), 2001b.

E. Love and D. Pimm. "This is so": a text on texts. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 371–409. Dordrecht: Kluwer, 1996.

D.F. Robitaille and R.A. Garden. *The IEA study of mathematics 2: Contexts and outcomes of school mathematics.* Oxford: Pergamon Press, 1989.

Undergraduate Learning Difficulties and Mathematical Reasoning: A Literature Survey and Project Overview.

Johan Lithner

July 10, 2001

Abstract

This paper provides background for and connects a series of completed, ongoing or planned studies within a project that treats undergraduate students' learning difficulties and the influence from the learning environment on these difficulties. A particular focus is on students' ability to use different kinds of mathematical reasoning in task solving. The results from the studies carried out so far indicate that focusing on what is familiar and remembered at a superficial level is dominant over reasoning based on mathematical properties of the components involved, even when the latter could lead to considerable progress. One of the main reasons behind this seems to be that the main part of the student's study work consists of solving textbook exercises by mathematically superficial reasoning.

Contents

1 Introduction

This paper presents the background and structure of a project containing a series of studies on learning and achievement difficulties among undergraduate mathematics students, with a particular focus on the types of mathematical reasoning students apply when solving different tasks. Section 1 starts by a general background discussion and some fairly broad research questions are formulated. Several examples of related research are given in Section 2. In Section 3 an comprehensive project structure is presented, and Section 4 contains short abstracts of the, at this date, completed, initiated, and planned subprojects.

1.1 Background

The number of students entering mathematical studies at university level in Sweden has increased dramatically in the last 10 years, and probably also in large parts of the whole world. In a longer perspective one can say that undergraduate mathematics was an élite education 30-40 years ago, but is today a mass education. As many as 98 % of each Swedish age group (about 100 000 persons) study mathematics in upper secondary school, for one to three years. About 15 % of each age group study undergraduate mathematics, mainly as a service subject within programs in technology, natural science, and computer science but also in mathematics and teacher programs. Even in the last 5 years the changes has been so extensive that, together with what some call the 'crisis of mathematics' in the whole Swedish school system, the universities are not able to cope with the new situation. A severe problem for us who are arranging undergraduate courses in mathematics, and probably for many of us engaged in teaching mathematics at any level and at any place in the world, is:

> **Problem:** We are not able to help sufficiently many students reach a desired level of mathematical competence.

This seems to be a severe problem regardless if one considers students in programs where mathematics is the main subject or a service subject. There is a large and increasing demand from the society for persons with different kinds of mathematically intense academic educations, but our educational system is not able to provide them in spite of the fact that many want to study mathematics.

In Sweden 10-40 % of the undergraduate students drop out, though they fulfil the entering qualifications. Of those who pass very roughly 20-50 % do this with severe difficulties and many reexaminations. The learning and achievement difficulties seem to be similar in the Swedish upper secondary school (though the examination system is different). Even among the majority who pass there are clear signs of severe weakness in competence, and it also seems clear that it is an international problem (see the literature survey below). Central questions in relation to all this are:

1.2 General research questions

This paper describes a project with purpose to study the following two questions related to learning undergraduate mathematics:
Q1: What are the *characteristics* of the students' main learning and achievement difficulties?
Q2: What are the main *reasons* behind these difficulties?

The main purpose behind Q1 and Q2 is to provide a foundation for answering the following question, which is not primarily treated within this project:
Q3: What *measures* should be taken in order to reduce the difficulties?

It is far from possible to provide complete answers to the very difficult and complex questions Q1, Q2, and Q3, so this project will actually treat some subquestions, but the aim is still to treat parts of decisive importance.

1.3 The role of research

In the literature, in some of my pilot studies, and in discussions with students, teachers, researchers, administrators, etc., there is a great variety in the proposed answers to the questions above. For example: The students are lazy, unintelligent, they do not learn anything sensible at the earlier educational levels. Or that the mathematics learning environments provided by schools and universities are badly adapted to the real situation: There are too much, or too little, of tests, grades, algorithms, calculators, lectures, small-group work, projects, exploratory work, individualisation, real-life math, hard exercises, easy exercises, etc. Or that the problems are caused by mass education, budget reductions, changed attitudes, social factors, etc. This variety in the proposed answers leads to two hypothetical conclusions:
(i) The answers are actually very complex. This assertion is strengthened by Niss [Nis99] when summarising the results of research in mathematics education: "The astonishing complexity of mathematical learning. An individual student's mathematical learning often takes place in immensely complex ways, along numerous strongly winding and frequently interrupted paths, across many different sorts of terrain. Many elements, albeit not necessarily their composition, are shared by large classes of students, whereas others are peculiar to the individual."
(ii) Little is known (this is of course a relative statement): There are very few

clear answers to the questions above found in mathematics eduction research, and thousands of teachers and researchers all over the world have worked hard with this for a long time but many difficulties remain.

If (i) and (ii) are true, then a consequence is that well-founded measures are hard to suggest. At the same time, the situation for mathematics teaching and learning at all educational levels is troublesome today. Students, parents, teachers, and, not to forget, politicians are all eager to see radical improvements, but man seem to have a weakness for quick and easy solutions. A thinker (I forgot who) once said something like: 'To every complicated question there is a simple answer that is completely wrong'. Taken together, the speculative discussion in this section may be one reason why educational development in mathematics seems to be rather 'trend-sensitive', and why the development seem to progress rather slowly in many areas and often (but far from always) takes fruitless directions. The most well-known large scale radical shift was perhaps the 'new mathematics' of the 1970:s, which in many aspects was a failure. There of course is no doubt that important and influential progress is actually taking place, for example by teachers, students, administrators, politicians, etc. It is also reasonable tho assume that most of the educational development is and will be carried out by devoted teachers. The point to make is that mathematics education research about the issues above could be a valuable complement to the development work that is carried out in other ways today, and hopefully make the development a bit more well founded, stable and less trend-sensitive. For general overviews on mathematics education research, see for example [Gro92], [BSSW94], [BCK+96], [SK94], and [Nis99].

Many researchers have described the severe learning and achievement difficulties of large groups of students, often in relatively elementary situations: [HD92], [HT96], [Sch85], [Tal96], and [Tho94b]. These difficulties are of different kind and lead to different consequences. Some are related to passive and dependent learning strategies in general, see for example [Ant96], while other are related to specific areas of mathematics education. Recent research in some of these areas will be exemplified below.

2 Literature survey

The purpose of this section is to give a non-exhaustive literature survey on research, structured in relation to the general research questions in Section 1.2: What does research have to say about characteristics, reasons and measures concerning learning and achievement difficulties, in general and in particular with respect to different types of mathematical reasoning? There are many different ways to structure this survey, and all structures will give categories were many of the research examples will fit in several categories. For example, some of the studies described treat characteristics of learning difficulties, which may often be seen as reasons behind achievement difficulties. In addition to this, some studies actually treats two or all three of the research questions in Section 1.2. Most of the examples will concern upper secondary and undergraduate mathematics, but some will be from earlier school levels.

2.1 Q1: What are the characteristics of the students' main learning and achievement difficulties?

2.1.1 Conceptual understanding difficulties

Many learning and achievement difficulties are directly related to inherent mathematical difficulties within specified concepts, and some concepts seem to be harder to master than other. One of the more specific results of research in mathematics education is according to Niss [Nis99]:

> "The key role of domain specificity. For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student. [..] For example, even if students who are learning calculus or analysis are presented with full theoretical definitions [..], and even if it is explicitly stated in the textbook and by the teacher that the aim is to develop these concepts in a general form [..], students actual notions and concept images will be shaped, and limited, by the examples, problems, and tasks on which they are actually set to work."

One aspect of this is treated by Vinner and Tall ([TV81] [Vin91] [Tal92]), when introducing the notion of "concept image" and describing that these may differ substantially from the corresponding concept definitions. Selden and Selden [SS95] introduced the "statement image" as "a unifying extension of the idea of concept images which we regard as statement images corresponding to definitions". These notions have clearly helped researchers to structure and analyse the deep and influential difficulties that arise when students' representations and understandings of mathematical ideas are not in accordance with the actual contents of these ideas and definitions. For example, many students believe that all functions $y = f(x)$ are continuous (see e.g. [HD92]).

Artigue [Art96] discusses and summarises research on students' difficulties with the conceptual field of analysis. One of the main problems is that the basic objects of the field (real numbers and functions), are not stabilised for the students when they enter the field, though they have studied these basic objects earlier. Another problem is the students' difficulties in fully understanding the central concept of limit, where primitive 'pre-understandings' may have been sufficient in earlier social or scholar contexts but may in fact hinder the necessary development towards deeper insights. An extensive discussion about educational research on (among other things) students' difficulties with functions and calculus is provided by Tall [Tal96]. It is probably not a coincidence that Tall also focuses on difficulties related to functions, real numbers, and limits (see also [Tal90]). These concepts are fundamental in calculus and analysis, and also seem very difficult for many students to understand and master. In a study by Williams [Wil91] students were presented with alternative models of limit and with anomalous limit problems:

> "Individual models of limit varied widely even among students who initially described limits in similar ways. The dynamic aspect of these models was extremely resistant to change. This resistance was influenced by students' belief in the a priori existence of graphs, their prior experiences with graphs of simple functions, the value they put on conceptually simple and practically useful models, and their tendency to view anomalous problems as minor exceptions to rules. These factors combined to inhibit students' motivation to adopt a formal view of limit."

Other aspects of calculus have also been studied: Mamona [Mam90] studies "Sequences and series-sequences and functions: students' confusions", and finds vivid evidence of the confusion between sequences and series and a resistance to regarding a sequence in any sense as a function. In a study on "Images of rate and operational understanding of the fundamental theorem of calculus", Thompson's [Tho94a] findings suggest that students' difficulties with the theorem stem from impoverished concepts of rate of change and from poorly-developed and poorly coordinated images of functional covariation and multiplicatively-constructed quantities. White and Mitchelmore [WM96] found that concept-based calculus instruction helped students symbolise rate of change in noncomplex situations but not in modelling or in complex situations, and that variables were treated as symbols to be manipulated rather than as quantities to be related. Sierpinska [Sie92] lists several epistemological obstacles, which are defined as inherent difficulties connected with complex concepts, for example the concept of function.

MacGregor and Stacey [MS97] describes that (several references are provided):

> "Research studies have found that the majority of students up to age 15 seem unable to interpret algebraic letters as generalised numbers or even as specific unknowns. Instead, they ignore the letters, replace them with numerical values, or regard them as shorthand names. The principal explanation given in the literature has been a general link to levels of cognitive development."

MacGregor and Stacey found additional origins of misinterpretation that had been overlooked in the literature: "Intuitive assumptions and pragmatic reasoning about a new notation, analogies with familiar symbol systems, interference from new learning in mathematics, and the effects of misleading teaching materials."

Ferrari [Fer97] studied advanced algebraic problem solving among undergraduate students and found a focus on action-based strategies, i.e. on strategies depending on physical manipulations which are performed with little semantical control. He also found that problems requiring relational knowledge or impredicative reasoning may result difficult to a number of students even if only elementary concepts and methods are involved.

One central type of difficulty is the inability to reach global understandings of general concepts and their mutual relations. Love and Pimm [LP96, p.387] claim that:

> "Examples are, in some sense, intended to be 'paradigmatic' or 'generic', offering students a model to be emulated in the exercises which follow. The assumption here is that the student is expected to form a generalisation from the examples which can then be applied in the exercises".

It seems like these generalisations are hard to make. Kahn et al. [KAA+98] considers the extent to which students are acquiring an understanding of mathematics as a whole and of the relative significance of different parts of mathematics to that whole. Their study indicates that "even after two years of undergraduate mathematics, many of the students involved had not developed such an understanding".

There are also many studies of conceptual difficulties related to more elementary mathematics. For example Tirosh and Graeber studied [TG90] preservice elementary teachers' beliefs about multiplication and division. Beliefs like 'multiplication always make bigger and division makes smaller' were found, and the beliefs were also inconsistently related to each other and to their more correct counterparts. For more general survey articles on (among other things) conceptual difficulties at different educational levels, see for example [BS96], [BHPL92], [BC96], [CB92], [Gre92], [Fus92], [Kie92], and [Sow92].

2.1.2 Task solving

The research area of learning difficulties is enormous, the examples given in this survey only indicate a small subset, and the research project described in Section 4 treats mainly a limited component: The mathematical foundations of students' task solving reasoning.

One of the more influential learning environment components consists of the mathematical tasks. Secondary and undergraduate students normally spend the main part of of their study time trying to solve mathematical tasks, mainly from the textbooks [Lit01c]. This is the way students are supposed to practice and learn mathematics in order to be able to apply their knowledge in other situations, for example in their further studies, in their future professional life, or in their everyday life as members of the modern society. In addition, task solving in exams is often the main tool through which students' mathematical competence is assessed. Some examples of research related to different types of task solving will be described below.

Genuine problems and routine tasks

There are many different types of mathematical tasks and it is essential to clarify the central distinction between routine tasks and 'genuine' or 'creative' problems (see [Sch85] and [Sch92] for extensive discussions on this distinction) before presenting examples of research results. A routine task is one where a complete solution method is available to the solver, and the solution is carried out in an algorithmic way by following a set of wellknown procedures. The term 'algorithmic' includes all kinds of sequential well-defined procedures, not only calculational ones. For example to find the zeros of a function by drawing it on a graphing calculator and zooming in the function's intersections with the x-axis. The term 'problem' has been used in the literature with many different meanings, ranging from any mathematical task to the type of tasks only encountered by research mathematicians in frontline research [Sch92]. Schoenfeld [Sch83] found that college mathematics departments' goals in courses labelled as 'problem solving courses' varied considerably:

> "- to train students to 'think creatively' and/or 'develop their problem solving ability' (usually with a focus on heuristic strategies);
> - to prepare students for problem competitions such as the Putnam examinations and or national or international Olympiads;
> - to provide potential teachers with a narrow band of heuristic strategies;
> - to learn standard techniques in narrow domains, most frequently in mathematical modelling;
> - to provide a new approach to remedial mathematics (basic skills) or to try to induce 'critical thinking' or 'analytical reasoning' skills."

In the text below (except perhaps in some quotations), the meaning of the term 'problem' is adopted from Schoenfeld [Sch85, p. 74]:

> "The difficulty with defining the term *problem* is that problem solving is relative. The same tasks that call for significant efforts from some students may well be routine exercises for others, and answering may just be a matter of recall for a given mathematician. Thus being a 'problem' is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person. The word *problem* is used here in this relative sense, as a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one. (For example, inverting a 27×27 matrix would be an arduous task for me, and I would most likely make an arithmetic error in the process. Even so, inverting a given matrix is not a *problem* for me.) To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem."

Thus the classification of a task as routine or problem is not determined by properties of the task alone, but is determined by the relation between the task in question and the solver. Some additional examples from more elementary mathematics are: Dividing 56 marbles evenly among 4 children is probably a genuine problem for a first-grader, but a routine task to most ninth-graders. Finding the maximum of a second-degree polynomial may be a problem for a ninth-grader who just encountered algebra, but should be a routine task for an undergraduate who has studied calculus. In fact, any problem can be turned into a routine task once you have studied the problem type and its properties sufficiently. The starting point (which is described in [Sch92]) of the more systematic discussions on mathematical problem solving is by many seen as Pólya's famous book "How to solve it" [Pól45] and his following work, for example [Pól54].

In [Les94] an overview of research on problem solving is provided, and though problem solving is perceived as important Lester describes an apparent decline of research in this area:

> "Although conference reports, curriculum guides, and textbooks insist that problem solving has become central to instruction at every level, the evidence suggests otherwise. WE may have learned quite a lot over the past 25 years or so about how students learn to solve problems and how problem solving can be taught, but we have not learned enough. And yet there are signs that problem solving has begun to receive less attention from researchers."

Routine tasks and memorisation

Two of the more central, and recurrent, findings in research on problem solving are: (i) Students' focus on rote learning of routine procedures. The rote learning of routine procedures would not have been so severe if this strategy had been complemented by the development of other task solving approaches. (ii) Students' extensive difficulties in solving non-routine problems. This unbalance seem to fit very poorly with the mathematics curricula goals of most countries. Though this has been 'wellknown' for quite a while, in particular from the large body of mathematical problem solving research from the 1980:s, this unbalance seems persistent. This issue is treated in for example [Sch85], [Sch92], and [Les94]. Some examples of research related to rote learning of routine procedures are mentioned in this section.

The unbalance described above is exemplified by the research of Selden, Selden and Mason: In a study titled "Can average calculus students solve non-routine problems?" [SMS89] the researchers found that students with mathematics grade C had extremely limited problem solving abilities. A natural question to pursue then was how students with higher grades, A and B, managed to solve creative problems. The disappointing result was summarised in the title of the follow-up study [SSM94]: "Even good calculus students can't solve nonroutine problems". The researchers concluded that traditional methods of teaching calculus are insufficient in preparing even good students to apply calculus creatively. Routine tests confirmed that the students possessed an adequate knowledge base of relevant calculus skills. This is in accordance with the findings in [Sch85], that students' problem solving difficulties often had other background than lacking basic resources.

Schoenfeld, in [Sch85] which is a summary of a series of studies, investigated what it means to think mathematically, in particular with respect to nonroutine problem solving. As a result of his studies he distinguished between the following four aspects of problem solving competence [Sch85, p. 15]:

> "*Resources*: Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand. Intuitions and informal knowledge regarding the domain. Facts. Algorithmic procedures. 'Routine' nonalgorithmic procedures. Understandings (propositional knowledge) about the agreed-upon rules for working in the domain.
> *Heuristics*: Strategies and techniques for making progress on unfamiliar and non-standard problems: rules of thumb for effective problem solving, including: Drawing figures; introducing suitable notation. Exploiting related problems. Reformulating problems; working backwards. Testing and verifications procedures.
> *Control*: Global decisions regarding the selection and implementation of resources and strategies. Planning. Monitoring and assessment. Decision-making. Conscious metacognitive acts.
> *Belief Systems*: One's 'mathematical world view', the set of (not necessarily conscious) determinants of an individual's behaviour. About self. About the environment. About the topic. About mathematics."

Schoenfeld very clearly shows not only that all four aspects are central in successful problem solving, but also why and in what sense: Traditionally, problem solving proficiency has often been considered equal to mastery of Resources. One could, very simplified, say that according to Schoenfeld problem solving failure is often caused by that the solver (e.g. student): i) does not master the Heuristic strategies necessary to make progress in unfamiliar situations; ii) does not evaluate the potential utility or the progress made concerning the different solution strategies that are or could have been attempted (Control); and/or iii) has the Belief that all problems should be possible to solve in similar ways as routine exercises, essentially by recalling from the memory a short algorithm and therefore sees no point in attempting other approaches. Schoenfeld also claims that if teaching is restricted to treat only Resources, which often is the case, then students will not develop the other three aspects.

2.1.3 Reasoning

The NCTM Commission on the Future of the Standards posed some questions concerning proof and mathematical reasoning. Ross [Ros98] responds, on behalf of the MAA, in the following way:

> "One of the most important goals of mathematics courses is to teach students logical reasoning. This is a fundamental skill, not just a mathematical one. [..] It should be emphasised that the foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning. [..] If reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense."

It is probably not controversial to accept Ross' position, at least in the interpretation that logical reasoning is a fundamental component in mathematics. What different types of reasoning are or should be included in school mathematics, where proof is only one type, and how do students handle these types of reasoning? Some examples will be discussed below.

Task solving reasoning

There seem to be many ways to solve school tasks by superficial reasoning. One of the more fundamental strategies in lower school levels is described by Hegarty et al. [HMM95] as a 'keyword strategy' in the context of arithmetic word tasks:

> "In the short-cut approach, which we refer to as direct translation, the problem solver attempts to select the numbers in the problem and key relational terms (such as 'more' and 'less') and develops a solution plan that involves combining the numbers in the problem using the arithmetic operations that are primed by the keywords (e.g., addition if the keyword is 'more' and subtraction if it is 'less'). Thus, the problem solver attempts to directly translate the key propositions in the problem statement to a set of computations that will produce the answer and does not construct a qualitative representation of the situation described in the problem."

Their study shows that in tasks where keyword strategies are unsuitable the unsuccessful task solvers use keyword strategies, whereas the successful task solvers base their solution plans on models of the situations in the tasks.

Verschaffel [VCV99] found that elementary school pupils' modelling and non-routine problem solving errors often was caused by superficial, stereotyped work without considering the appropriateness of the actions in relation to the problem context.

At university level the tasks and the reasoning involved are more complex than in arithmetic. Szydlik [Szy00] compared university students' content beliefs about limits and their sources of conviction. The students were enrolled in a traditional calculus course using a traditional textbook. The data suggested that students with external sources of conviction (the authority of a teacher or a textbook) gave more incoherent definitions, held more misconceptions, and were less able to justify their calculations than those with internal sources of conviction (appeals to empirical evidence, intuition, logic, or consistency).

Cifarelli [Cif98] interviewed 14 college students solving algebra word tasks, with the aim of identifying and describing their cognitive actions in resolving the genuinely problematic situations they faced while solving the tasks. One outcome of the study was the formulation of three increasingly abstract levels of structural knowledge, and a framework for a theory of representation that is activity-based and consistent with a view of knowledge based on the idea of viability.

Tirosh and Stavy [TS99] describes (references are given) that many responses which the literature describes as alternative conceptions could be interpreted as evolving from common, intuitive rules. They had noted that students react similarly to a wide variety of conceptually unrelated situations. Tirosh and Stavy found an intuitive 'rule' (Same A – same B) that could predict some of the irrational behavior: When two systems are equal with respect to a certain quantity A but differ in another quantity B, students often argue that 'Same amount of A implies same amount of B'.

A very natural question to study is 'why do successful students succeed and why do unsuccessful fail'? Due to the complexity of mathematical learning mentioned in Section 1.3, exhaustive answers to this seemingly simple question seem far away.

Tall and others ([FT96], [GT93], [GT94], [TR93]) reaches the somewhat unexpected conclusion that one of the main characteristics of unsuccessful students is that they are actually performing a more difficult kind of mathematics than those who succeed. The successful ones have a flexible way of thinking which makes mathematics easier to do and to think about, and are able to compress knowledge into an easy-to-handle and flexible form. The less successful learn isolated techniques which make higher levels increasingly difficult, and focus procedures on physical objects. The authors consider the duality between process and concept in mathematics and define a 'procept' to be a combined neutral object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both. I.e. $3 + 2$ is either the process of addition of the numbers 2 and 3, or the concept of sum. The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema. The authors hypothesise that the successful mathematical thinker uses a mental structure that is manifest in the ability to think proceptually, and give empirical evidence to support the hypothesis that there is a qualitatively different kind of mathematical thought displayed by the more able thinker compared to that of the less able one.

Some of the characteristics of the successful graduate students in Carlsson's [Car99] study of successful students were that they were very confident and persistent when solving complex mathematical tasks. They frequently attempted to classify the task as one of familiar type, and their answers appeared to have a logical foundation.

Proof reasoning

Many studies on mathematical reasoning are restricted to reasoning related to the strict reasoning in mathematical proof. In [Pól54] Pólya discusses the relation between formal demonstrative reasoning (deductive proof) and the more informal and intuitive plausible reasoning:

> "We secure our mathematical knowledge by demonstrative reasoning, but we support our conjectures by plausible reasoning. A mathematical proof is demonstrative reasoning, but the indictive evidence of the physicist, the circumstantial evidence of the lawyer, the documentary evidence of the historian, and the statistical evidence of the economist belong to plausible reasoning. [..] In strict reasoning the principal thing is to distinguish a proof from a guess, a valid demonstration from an invalid attempt. In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess. If you direct your attention to both distinctions, both may become clearer."

Pólya mainly treats one type of plausible reasoning, inductive reasoning, but one of his main points is that other types of reasoning than the strict proof are central in mathematics.

There are many studies on different aspects of learning, understanding, and implementing proof. In a study of students' notion of proof, focusing on how they arrive at their conviction of the validity, Balacheff [Bal88] singled out two types: Pragmatic 'proofs' are about 'showing' that the result is true because 'it works'. These do not actually establish the truth of an assertion but are often believed to do so by their producers. Conceptual proofs concern establishing the necessary nature of the truth by giving reasons. Though both types could be based on mathematical properties, Balacheff found a clear break between the two concerning levels of sophistication.

Other researchers have also described students' difficulties in differing proofs from other less rigourous types of argumentation [Cha93], [Hoy97]. In [HJ96] Hanna and Jahnke discuss the distinction between proofs which prove and proofs which explain, and criticise some attempts to reduce the role of proof in mathematics education. Blum and Kirsch [BK91] discuss preformal proving ("a chain of correct, but not formally represented conclusions which refer to valid, nonformal premises") among grade 12 students.

A crucial prestage to mastering proofs is the ability to move between informal and formal statements. Selden and Selden [SS95] studied undergraduate students' ability to unpack informally written mathematical statements into the language of predicate calculus:

> "We discuss this data from a perspective that extends the notion of concept image to that of statement image and introduces the notion of proof framework to indicate that part of a theorem's image

which corresponds to the top-level logical structure of a proof. For simplified informal calculus statements, just 8.5% of unpacking attempts were successful; for actual statements from calculus texts, this dropped to 5%. We infer that these students would be unable to reliably relate informally stated theorems with the top-level logical structure of their proofs and hence could not be expected to construct proofs or validate them, i.e., determine their correctness."

Moore [Moo94] studied the transition to formal proof and found three major sources of student difficulties: (a) concept understanding, (b) mathematical language and notation, and (c) getting started on a proof. A similar topic, the relation between thought experiment and formal proof, is studied by Tall [Tal99]. He starts by structuring proof and students development and learning of proof, and the outcomes of the article is summarised in the rather long but informative abstract:

> "This presentation will address the conceptual demands placed on students attempting to deal with formal proof for the first time and present empirical evidence that reveals the subtlety of this transition. It transpires that there is more than one route to move from informal experience of proof to formal proof. Informal proof often occurs in the style of a thought experiment, using a variety of imagery to infer that, when a certain situation occurs, then another must also occur as a consequence of the first. Formal proof, on the other hand, is based on verbal/symbolic definitions and focuses only on those results that can be deduced logically from the definitions. The presentation will show that there are (at least) two cognitively different routes from informal to formal. One builds on imagery and constantly reconstructs it to fit new formalisms. Another starts from the definitions and develops only those properties that can be built by formal deduction. Empirical evidence will be given, collected in longitudinal studies from students in their first year of university mathematics, to demonstrate how both of these routes can lead to success, but that each involves a different array of cognitive difficulties that can lead to failure. For instance, the image of thought experiments may include subtle elements at variance with the formalism that causes serious blockages of understanding. On the other hand, formal proof may also lead to structure theorems which have their own mental images that can then be used in informal thought experiments to predict new directions for the formal theory. The results suggest that different students may benefit from different kinds of teaching strategies and what may help one may be of a hindrance to another."

2.2 Q2: What are the main reasons behind the learning and achievement difficulties?

2.2.1 Reduction of complexity

A large part of the research results dealing with the reasons behind the difficulties discussed above can be characterised as an unwarranted and far too extensive reduction of complexity of mathematical concepts, processes and other ideas. This seems to be done in different situations by teachers, textbook writers, and/or students in order to cope with curricula goals that are (too?) hard to reach.

Schoenfeld [Sch91a] described that students are inclined to answer questions with suspension of sense-making, and that they often use short-cut strategies. It is likely that this may also be the case with textbook exercises. According to Doyle [Doy86], [Doy88], there is a pressure from students to reduce ambiguity and risk, and to improve classroom order, by reducing the academic demands in tasks.

Dreyfus [Dre99] argues that students are in textbooks rarely given explicit instructions or indications concerning the required quality of reasoning. In a historical perspective McGinty et al. [MVZ86] analysed grade 5 arithmetic textbooks from 1924, 1944, and 1984, and found that the number of word problems had decreased, the number of drill problems had increased, and that word problems had also become shorter and less rich.

A brief comparison between some older calculus textbooks, for example [CJ65] and [dLVP54], and some newer textbooks indicates that the proportion of exercises that have more or less complete solution methods provided (e.g. worked examples that are very similar to the exercises) have increased considerably. All this may be part of a self-deceptive way in the present mass-education situation to continue, at the surface, to deal with advanced concepts in our mathematics courses. It is at the same time important to stress that this does not imply that the older books where 'better'. Love and Pimm [LP96, p. 397] claim that "While teachers' perceptions of textbooks have received some attention, there is a dearth of research into the use of texts in class".

Stacey and MacGregor [SM99] described how algebra instructions and exercises are actually reduced to easier arithmetic, though they are still supposed to treat algebra. The central but sometimes difficult transition from arithmetic to algebra is avoided by allowing, and sometimes encouraging, students to keep using familiar ways of operating based on arithmetic instead of learning the algebraic way of operating with unknowns.

A study by Cox [Cox94] suggests that many first-year university students obtain good A-level grades by strategic learning concentrating on routine topics at a superficial level, rather than a deep understanding of fundamental topics. Cox argues that "this learning approach appears to be encouraged by the excessive breadth and content of A-level syllabuses".

Vinner [Vin97] suggests a theoretical framework where two of the main notions are ‘pseudo-conceptual’ and ‘pseudo-analytical’. They are defined as thought processes that are not conceptual and analytical respectively, but which in routine task solving might give the impression of being so and could even produce correct solutions. One of Vinner’s main points is that students’ difficulties in solving routine tasks may often be better understood if they are interpreted within this ‘non-cognitive’ framework, than if they are seen as misconceptions within the domain of meaningful contexts: What may be a true learning and problem solving situation for the teacher may not be so for the student. Because of the didactic contract [Bro97] students may, consciously or not, try to please the educational system with behaviour that, perhaps only superficially, is considered acceptable by the system. Leron and Hazzan [LH97] also argue that analyses of task solving behaviour should not only consider attempts to understand the task, and successes and failures in such attempts. They emphasise additional non-cognitive means of trying to cope: attempts to guess and to find familiar surface clues for action, and the need to meet the expectations of the teacher or researcher. The position substantiated in [Vin97] and [LH97], that it may be of limited use to always study students’ behavior as if it is cognitve, is highly relevant for the research project described below in Section 4. There, among other issues, task solving reasoning that is not ‘mathematically based’ is studied.

2.2.2 A procedural focus

The perhaps most frequent type of reduction of complexity seems to be to focus the teaching and learning on algorithmic procedures that can be carried out in order to solve advanced tasks without the need for conceptual understanding or constructive reasoning.

The reasons behind the students’ focus on learning and applying routine procedures is discussed by Tall [Tal96] in an article on functions and calculus under the heading “Procedural consequences of conceptual difficulties” where he argues in the following way:

> “When faced with conceptual difficulties, the student must learn to cope. In previous elementary mathematics, this coping involves learning computational and manipulative skills to pass exams. If the fundamental concepts of calculus (such as the limit concept underpinning differentiation and integration) prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. At least this resonates with earlier experiences in arithmetic and algebra in which a sequence of manipulations are performed to get an answer. The problem is that such routines become just that - routine - so that students begin to find it difficult to answer questions that are conceptually challenging. The teacher compensates by setting questions on examinations that students can answer and the vicious circle of procedural teaching and learning is set in motion.”

Other research has also indicated that weak conceptual understanding is related to a procedural focus [TG90] [WM96].

The claimed inability of the 'common' and 'traditional' learning environments to help students to satisfactory develop some central mathematical competencies, like conceptual understanding and problem solving abilities, is discussed in the research literature. In an article with the somewhat provoking title "We're crippling our kids with kindness!", Chatterly and Peck [CP95] claim that one of the mistakes we do as teachers is the following:

> "We actually cripple our students mentally by feeding them too many hints and by trying to push them algorithmically beyond their ability without the development of proper referents for the mathematics being taught. It, too often, leads to rote memorisation and prevents the students' development of a proper conceptual understanding. Concrete referents are essential before the establishment of a conceptual background can be firmly developed in the minds of each student."

Hiebert and Carpenter [HC92] also claim that introductory procedural teaching and learning may prevent students from being able to later develop a deeper conceptual understanding.

The studies mentioned above concluded that improper teaching strategies may prevent students' mathematical development. McNeal [McN95] found that in some situations, exemplified by one child's learning of the standard addition algorithm, even a *regression* with respect to understanding may be caused by the learning environment: Changes in the child's mathematical beliefs and constructions were analysed as he moved from an experimental 2nd-grade mathematics class characterised by inquiry mathematics to a textbook-based third grade. The analysis shows that he had abandoned his self-generated computational algorithms in favour of less understood conventional procedures.

Henningsen and Stein [HS97] set out to identify, examine, and illustrate the ways in which classroom factors shape students' engagement with high-level mathematical tasks in middle school classrooms. They found that when students' engagement is successfully maintained at a high level [of mathematical thinking], a large number of support factors are present. Another result was that, though the tasks themselves were identified as being set up to encourage doing high-level mathematics, one major obstacle was a decline into using procedures without connection to concepts, meaning, and understanding. This in turn was mainly caused by three factors: (i) Challenges became nonproblems, for example by successfully pressuring the teacher to provide explicit procedures. (ii) A classroom-based shift in focus away from meaning and understanding toward the completeness and accuracy of the answer. (iii) Too much or too little time.

It is often emphasised that both in learning, understanding and applying mathematics the ability to visualise is central. Still, it seems like students are generally too focused on algebraic algorithmic approaches. Eisenberg [Eis94] set out to analyse why there is such widespread reluctance on the part of both teachers and students to choose visual methods in problem solving and in establishing a basic understanding of fundamental notions. He found that visualisation techniques are cognitively more demanding of the learner than analytical techniques which are more algorithmic in nature, and also hypothesised that another reason behind the reluctance is that visual techniques are not accepted in mathematical proof. Aspinwall [ASP97] came to the somewhat surprising (in relation to the widespread view that visualisation increases understanding and problem solving performance) conclusion that imagery might be a disadvantage on certain tasks, that persistent limited visual concept images can be a hindrance for development:

> "One of the limitations of imagery found in the literature comes to bear on a unique aspect of mathematics teaching and learning. This is the notion of an uncontrollable image, which may persist, thereby preventing the opening up of more fruitful avenues of thought, a difficulty which is particularly acute if the image is vivid."

Often students will overgeneralise properties of a set of examples and draw faulty conclusions about properties of a whole concept, and thereby construct faulty concept images (see Section 2.1.1 for a discussion on concept images). One very influential and frequently occurring example of a 'visual-based' overgeneralisation is the concept of function [HD92]. Many students believe that all functions have 'smooth' and continuous graphs since their concept images are not based on the abstract and difficult definition of the function concept, but on the numerous examples of function graphs they have met and almost all of them have been 'smooth'.

2.2.3 Other research examples

Christiansen [Chr97] found that the school-system's exercise-oriented perspective may have serious hindering effects on the development of true modelling abilities. For example, concerning the exercises' reference to reality, one student adopted a reality-oriented perspective but this was suppressed by the teachers (maybe unconsciously) exercise-oriented perspective. Dahlberg and Housman [DH97] studied concept formation (initial understanding of advanced undergraduate mathematical concepts) within the theory of concept definition, concept image, and concept usage, in relation to the three strategies example generation, definition reformulation, and memorising: "We infer that the students in our study who employed an example generation learning strategy were more effective in attaining an initial understanding of the new concept than those who primarily employed other learning strategies such as definition reformulation or memorisation."

There are specific difficulties related to the transition from the upper secondary school to the university [FL00] [Tal92]. A general, qualitative step in this transition is with respect to an increased level of abstraction. This level is in a sense increasing continuously through the whole educational system, but is by many seen as a crucial difference between the upper secondary school and the university (see for example several of the articles in [Tal91]). In addition to this, at the university level there are higher requirements on the students independence, which many claim to be one of the main reasons behind the learning and achievement problems [GHRV98].

It has in the later years been recognised that other factors than the cognitive and conceptual aspects described above have deep impact on students' learning and achievements, for example affective factors. Leder [Led98] found that the proportion of students who found mathematics enjoyable dropped from 60 % at upper secondary school to 35 % at university level. An example of a study on mathematics anxiety in higher-level students is [Bes95]. Social and gender related factors also influences the achievements of mathematics students at all levels, see for example [Led96]. A wide review on research on affect at all school levels can be found in [McL92].

2.3 Q3: What measures should be taken in order to reduce the learning and achievement difficulties?

The research on the characteristics of and reasons behind the learning and achievement difficulties indicates the direction for the measures to be taken in order to improve the learning environment.

2.3.1 Learning environments that promotes conceptual understanding and problem solving competence

Thompson and Thompson [TT94] studied a teacher's struggle with helping one student to learn (discover) fundamental aspects of rates and speed. The difficulties originated in the two persons' different conceptual bases and representations. The teacher was using a (for him, not for the student) powerful calculational 'language' (language in a wide meaning) and the student a limited 'language' closer to a primitive concept image, based on the idea of speed as a distance (covered in one second). In a second study by Thompson and Thompson [TT96], the setup was similar but the teacher was replaced by one of the researchers with the ambition to provide a solid conceptual foundation based on the covariation of time and distance, before introducing the more difficult questions. The goal was that the student came to understand motion in relation to speed, distance and time sufficiently well so that her ability to solve problems became a consequence of that understanding - as distinct from having the goal that she learns how to solve such problems. Conceptually-oriented teachers try to focus students attention away from thoughtless application of procedures, towards a rich conception of situations, ideas and relationships.

Several studies describe that different types of 'non-traditional' learning environments may improve students' learning and performance. In a study on elementary school childrens' multidigit addition and multiplication task solution strategies, Kamii and Dominick [KD97] found that those who had not been taught any routine algorithms produced significantly more correct answers. They also found that:

> "If children made errors, the incorrect answers of those who had not been taught any algorithms were much more reasonable than those found in the 'Algorithms' classes. It was concluded that algorithms 'unteach' place value and hinder children's development of number sense."

Boaler [Boa98] compared student (age 13-16) experiences and understandings in two different learning environments: 'traditional' and 'open project-based'. Students from the latter developed a conceptual understanding that provided them with advantages, both in school and nonschool settings. Students from the former developed a procedural knowledge (including "rule-following behavior" and "cue-based behavior") that was of limited use to them in unfamiliar situations. These students had not experienced unfamiliar demands in their mathematics lessons:

> "For their textbook questions always followed from a demonstration of a procedure or method, and the students were never left to decide which method they should use. If the students were unsure of what to do in the lessons, they would ask the teacher or try to read cues from the questions or from the contexts in which they were presented."

There are additional studies describing that 'reform' students outperform 'traditional' students, e.g. [BCFF+98].

In comparing distinctions between 'novice' and 'expert' teachers, there are probably very many complex factors to consider. One such study was made by Livingston and Borko [LB90] who contrasted two review lessons of two secondary mathematics student teachers with those of their high school cooperating teachers:

> "Despite extensive preparation, the novices' review lessons were less comprehensive than those of the experts, and their explanations were less conceptual. The experts more skilfully improvised activities and explanations around student questions and comments. These differences are explained by the assumption that novices' cognitive schemata for content and pedagogy are less elaborated, interconnected, and accessible than those of the experts."

Schoenfeld (e.g. [Sch85], [Sch91b], [Sch94], [Sch98]) has constructed a problem solving course that aims at developing students' Resources, Heuristics, Control, and Belief (see Section 2.1.2). One of the main purposes with the series of studies presented in [Sch85] was to systematically evaluate the effects of the course (which were found to be positive). Schoenfeld's general goal is to help students to learn to think mathematically:

"In sum, my goals for the students are that they develop appropriately mathematical predilections, knowledge, and skills. I want them to be aggressively mathematical - to see mathematics where it can be seen, to pursue mathematical connections, extensions, generalisations; to know how to make good conjectures, and how to prove them; to have a sense of what it means to understand mathematics and good judgement about when they do. And, I want them to have the tools that will enable them to do so. That means having a rich knowledge base, a wide range of problem solving strategies, and good metacognitive behaviour [Sch91b]."

To get a flavour of the contents of the course, consider the following 'list of properties' of the problems that students work with [Sch91b, p. 94]:

"- The problems are (relatively) accessible. I like problems that are easily understood and that do not require a lot of vocabulary or machinery in order for the students to make progress on them.
- The problems can be solved, or at least approached, in a number of ways. This leads to discussions of mathematical richness, of connections, and of strategy choice.
- The problems should serve as introductions to important mathematical ideas. The topics and mathematical techniques involved in the problem solutions should be of agreed importance, or the solutions to the problems should illustrate important problem solving strategies.
- The problems should, if possible, serve as good starting points for honest-to-godness mathematical explorations. Good problems lead to more problems. If the domain from which the problem comes from is rich enough, students can start with the problem that has been posed to them and proceed to make the domain their own."

If many of the severe student performance difficulties and learning environment inadequacies discussed above have been known for at least a couple of decades, and if teachers and researchers have shown fruitful ways of improvement, why are not the changes more profound? One of the reasons is according to Artigue [Art98] that research seldom shows extensive improvements via simple changes: "On the contrary, most research based designs require more engagement, expertise from teachers, and significant changes in practices (Dubinsky et al. [DMR97])."

2.3.2 Discrepancies between teachers intentiones and practices

The reference [Art98] above summarises some of the difficulties in creating learning environments that fosters problem solving abilities and conceptual understanding to a greater extent than today. Another indication of these difficulties is the research describing major discrepancies between teachers intentiones and their actual practices.

Eisenhart et al. [EBU$^+$93] explored a student teacher's ideas and practices for teaching procedural and conceptual knowledge, and also the (potential) influences on these aspects from the student teacher's education program and placement school:

> "We reveal a pattern in which the student teacher, her mathematics methods course instructor, her cooperating teachers, and the administrators of her placement schools expressed a variety of strong commitments to teaching for both procedural and conceptual knowledge; but with these commitments, the student teacher taught, learned to teach, and had opportunities to learn to teach for procedural knowledge more often and more consistently than she did for conceptual knowledge. We find that the actual teaching pattern (what was done) was the product of unresolved tensions within the student teacher, the other key actors in her environment, and the learning-to-teach environment itself."

Barnard and Morgan [BM96] described how a teacher actually focuses more in computational aspects (knowledge and justification) than on his aims for understanding and culture. Eley and Cameron [EC93] found that university teachers appreciate global explanations, but use only local explanations when teaching.

2.4 Summary of the literature survey

In an attempt to summarise the examples of mathematics education research described above, the following seem to be of central importance in relation to the general research questions presented in Section 1.2:
- Though one of the main curricula goals is conceptual understanding, this seems hard to reach for many students, especially at a global, general level. Students often lack a comprehensive view of what mathematics is and the ability to move flexibly between and relate different types of mathematical representations and knowledge.
- A large number of research articles, many more than the ones mentioned above, shows the severe unbalance (even among many high-achieving students) towards rote learning of algorithmic procedures and an inability to solve non-routine problems. This is also related to the understanding difficulties mentioned above, both in the sense that weak conceptual understanding leads to (perhaps even forces the students into) rote learning, and in the sense that rote learning does not develop conceptual understanding. It is still possible to obtain good grades by strategic learning of routine topics, probably because exams often, to a large extent, are adapted to suit rote learning.
- Several studies concludes that mathematical learning without algorithms leads to better results. One should be cautious not to draw the conclusion that algorithms should be banned from the classroom. It is probably not the algorithms, which actually often are powerful mathematical tools, that are bad but the ways they are handled: Superficial and without firm enough connections

to the underlying mathematical ideas.
- There is a pressure on students and teachers to reduce the mathematical complexity in the learning environment, for example to work in a 'rote learning mode'. This seems to be partly caused by the extensive courses in combination with the inherent difficulties in reaching deep understanding. At the same time, it is possible to help students develop better understanding and problem solving abilities, but this often require more engagement, expertise from teachers, and significant change in practice.
- Students' reasoning is not only based on mathematical thinking: the need to cope (e.g. pass exams) in situations that are difficult for them to handle may lead (force?) them into reasoning of other types. These types of reasoning could often be better analysed in 'non-cognitive' frameworks.

The relation to the research project described below

In short, the relation between the outcomes of the research survey described in Section 2 above and the ongoing research project presented in Sections 3 and 4 below is that the subprojects in the latter so far indicate the following:
i) The results are in line with and confirms the earlier research in the sense that among the main reasons behind the students' learning and achievement difficulties seem to be rote learning, a narrow procedure focus, and lacking problem solving ability in non-routine situations. It is also found that a large part of the students' study work consists of solving exercises by mimicking solved examples, with little opportunity to develop conceptual understanding and problem solving ability.
ii) The results add to the earlier research in the sense that the base for different types of student reasoning is studied in detail. In particular if and in what way their strategy choices and strategy implementations are based on 'true' mathematical and logical properties of the components involved in the solution reasoning, or if they are based on something else. The task solving situations studied in the project below concern calculus tasks where the solutions are more complex than in for example arithmetic and elementary algebra, and therefore simple memory-based strategies (e.g. keyword strategies) can often not be applied. In non-routine situations (which includes both non-routine problems and routine tasks where some mistake is made in the solution procedure) the students' main strategies seem to be based on trying to combine different familiar subprocedures without general considerations or understanding, which often lead to failure. There are very few situations where the students complement the application of familiar procedures by trying to construct their own reasoning, even where this probably relatively easy (for these students) could have lead to considerable progress. Another subproject describes in rather fine-grained detail how about 90% of the exercises in common textbooks can be solve by completely or essentially mimicking solved examples, and that this can be done without considering the mathematics that the exercises are supposed to treat. It is also found that students may be extremely inclined to use essentially only

this superficial exercise solving strategy, and that this leads to a very narrow type of rote learning. Some of the conclusions in i) above are also supported by detailed and perhaps partly new types of data.

3 A framework for a series of research projects on learning and achievement difficulties

The purpose of this section is to provide a general structure for a set of research projects aiming at studying some aspects of the questions introduced in Section 1.2. This structure is described as a 2-dimensional matrix, where one dimension address the discrepancy between goals and actual outcomes, and the other dimension contains some central learning environment components.

3.1 Discrepancies between goals and outcomes: The Intended, Implemented, Received, and Attained curricula

We, as course organisers and undergraduate mathematics teachers, are not able to help sufficiently many students to learn mathematics sufficiently well. This is not in accord with the explicit or implicit goals of the course organiser, and there are several junctures where discrepancies may exist. One way to structure the study of possible differences between goals and outcomes is provided by Bauersfeld [Bau79] and Robitaille and Garden [RG89]. They have in similar (but not identical) ways characterised discrepancies between the components in the following framework:

> "The *intended curriculum* as transmitted by national or system level authorities; the *implemented curriculum* as interpreted and translated by teachers according to their experience and beliefs for particular classes; and the *attained curriculum*, that part of the intended curriculum learned by students which is manifested in their achievements and attitudes" [RG89, p.4].

3.1.1 The intended curriculum

The general goal for the educational system when arranging undergraduate mathematics courses is in Sweden, and probably more or less in any country, to provide the society with a sufficient number of persons with appropriate education in mathematics at sufficient quality levels (for an extensive discussion on general national curricula goals, see [Nis96]). The quantity of students is (partly) controlled by national and local economical means of control, and manifested in the number of admissions to the different undergraduate programs. The quality is supposed to be controlled by the local exams and occasional national system evaluations. There are on one hand the *national and local formal intentiones*, and on the other hand the *teachers' intentiones*. It could seem reasonable that the former should have be more influential than the latter, but the university structure is so decentralised that the system level authorities of undergraduate mathematics are in reality the teachers (lecturers), perhaps not as individuals but as groups of teachers. The national descriptions of the goals are very concise and general. The local university goals are more specified, but still very brief and mainly content-oriented (as opposed to for example competence-oriented). These local goals are normally formulated by individuals or small groups of lecturers, and formally accepted by the mathematics departments' executive committees.

3.1.2 The implemented curriculum

The implemented curriculum may be seen as the learning environment that we as course organisers provide to the students, mainly manifested in lectures, textbooks and exams. There are (at least) two junctures in the transition from the intended to the implemented curriculum were 'distortions' may occur. Firstly, the individual teacher may interpret the national and/or the local goals differently than what was intended. This juncture is perhaps not surprising, since the national and local goals are so very sparsely specified and in a sense the teachers have to 'fill in the gaps' by themselves. Secondly, which is perhaps a bit more unexpected, as described in some of the research examples above there may be major discrepancies between teachers intentiones and their actual practices.

3.1.3 The received curriculum

To make this study of the potential discrepancies between the different aspects of the curriculum more precise a fourth aspect will be added in addition to the three above, namely the *received curriculum*, the part of the implemented curriculum that influences the students. It is not necessarily the case that what students achieve, for example in task solving situations, is a subset of neither the implemented nor intended curriculum: A student is entering a learning situation, for example a lecture, where the implemented curriculum is put forward by the teacher. The student will, consciously or not, focus on and receive the main influences from a subset of the implemented curriculum, this subset is the received curriculum. This subset may then from the student's point of view be developed, complemented, transformed, or misunderstood to fit with earlier concept images, or altered in other ways before it is 'learnt' (or 'constructed') as the attained curriculum. This modification of Robitaille's and Garden's framework makes it possible to study the question 'what influences may the students receive in a learning situation', independent from questions concerning both what is implemented by the teacher and what is achieved by the students.

3.1.4 The attained curriculum

It is difficult to measure what students learn and their mathematical competence at different educational stages. The most common method to measure undergraduate students' achievements is by written tests. It seems like written tests may often fail [Sch85, p.4] to measure important aspects of mathematical competence, and are instead often focused on memorised routine procedures (see e.g. [Tal96]). Other aspects that are difficult to measure with written tests are exemplified by heuristics, metacognition, and belief, all of which have been shown by Schoenfeld [Sch85] [Sch92] to be central competence aspects. It is beyond the scope of this text to discuss the research on assessment of achievement, see for example [Nis93a] and [Nis93b] for general surveys and further references.

The four aspects above will not be considered in relation to the curricula as a whole, but to a selection of central learning environment components which are described in the next section.

3.2 The Internal Learning Environment: Central influences on students' task solving reasoning

The *learning environment* for an individual student can be seen as everything in the student's life that affects the learning of mathematics at a certain period of time. Since factors outside school may have substantial influence, the learning environment may in a wide sense include very large parts of a student's whole living environment. This paper will be restricted to the *internal learning environment*, which here is defined as the part of the learning environment that is explicitly or implicitly provided by the mathematics course or program organiser.

The purpose of the project is to investigate some components of the internal learning environment, and their influence on students' learning and achievement difficulties. Different components of the internal learning environment have influence of different type and magnitude, and some of these components may be less relevant to this study.

Schoenfeld [Sch85] has described that some aspects of students' behaviour will not be changed quickly, and one may assume that students' ways of reasoning in task solving will be fairly stable. Therefore it seems reasonable that first year undergraduate students' difficulties are also affected by their prior (secondary school and earlier) internal learning environment, but this is outside the scope of this study to investigate. As a consequence, this study is not really about the causes behind students' difficulties but rather concerns in what ways the present (first-year undergraduate) internal learning environment will reinforce or counteract such difficulties, regardless if they originate from undergraduate education or elsewhere. The internal learning environment components that this project will focus on are *the syllabus*, *the teaching*, *the textbook*, and *the examination.*

3.2.1 The syllabus.

The syllabus contains the goals of the course. The *formal syllabus* is a written rather concise description of the mathematical contents of the course including the main time frames. The *informal syllabus* complements and adds details to the formal syllabus, and may differ between universities and also between individual teachers. It is mainly based on a tradition that is carried forward by the teachers' experiences as students, teachers, and mathematicians, by possible experiences from teacher education and other pedagogical programs, and interactions with other mathematicians and teachers. One important carrier of the tradition is the use of written course instructions (which may include detailed content descriptions, task suggestions, time tables and exam structures) that the earlier teachers of the same course have used. Another important influence on the informal syllabus is the textbook, which most undergraduate courses are mainly designed to follow.

3.2.2 The teaching

In a full-time Swedish undergraduate mathematics course there are about 10-20 scheduled lecture hours per week, normally most of them consist of lecturing by the teacher. Another common scheduled class activity is when the teacher or a teaching assistant solves exercises at the blackboard, and also helps students individually while other students are working by themselves or in small groups. Outside scheduled time the students normally work by themselves at home, and sometimes in small groups that are not organised by the teacher. Roughly half of the teachers have positions as senior lecturers, where the main formal qualification requirement is a PhD degree in mathematics. Most of the other teachers are junior lecturers, and there are also different types of teaching assistants. Very few undergraduate lectures are given by full professors.

3.2.3 The textbook

Mainly American textbooks are used in undergraduate mathematics courses in Sweden, for example [Ada95] or [EP94], which are relatively inexpensive and essentially all students have their own copy. They contain many more exercises and solved examples than older textbooks, like [CJ65] and [dLVP54].

3.2.4 The examination.

The examination is probably very influential on what the students will attempt to learn. In Sweden, there is normally a written exam after each 5-week full-time course, often consisting of questions that are fairly similar to the exercises in the textbooks. The students are usually given a couple of old exams during the course before the real one, and after a few courses it seems likely that they have learnt what a normal exam will look like. This means that exams may affect students' learning in the sense that they know what to expect from the exam and that they try to meet with these expectations.

4 Research project components

In order to provide a specified structure for the project, each of the four internal learning environment components from Section 3.2 will be matched to each of the four curricula aspects in Section 3.1, which yields a 4 by 4 matrix with different research areas. Each position will contain the subproject number (A1, A2, etc., with completed subprojects in **boldface**) of the article that treats the particular research area, and a reference to the related subsection below (e.g. (4.1.1)). All subprojects are related to the research questions in Section 1.2.

	Syllabus	Teaching	Textbook	Exam
Intended	A8 (4.3.1)	A8 (4.3.1)	A5 (4.2.1)	A8 (4.3.1)
Implemented	A10 (4.3.3)	A5 (4.2.1)	**A3** (4.1.3) A5 (4.2.1)	A6 (4.2.2)
Received	A9 (4.3.2)	A7 (4.2.3) A9 (4.3.2)	**A4** (4.1.4) A7 (4.2.3)	A9 (4.3.2)
Attained	**A1** (4.1.1) **A2** (4.1.2)	**A1** (4.1.1) **A2** (4.1.2)	**A1** (4.1.1) **A2** (4.1.2)	A6 (4.2.2)

Below follows a very concise presentation of the subprojects and their specific research questions.

4.1 Completed studies

Since the completed studies are available and references are given, the presentations contain only the abstracts from the papers and brief comments. An informal summary of all completed studies is also provided at the end of this subsection.

4.1.1 A1: Students' general difficulties in task solving

Completed report: 'Mathematical reasoning and familiar procedures' [Lit00a].

Abstract

Four first-year undergraduate students are working with two tasks. The underlying question treated is 'what are the characteristics and background causes of their difficulties when trying to solve these tasks?' The purpose is to give a general survey of their main difficulties, rather than to go deeply into details. It seems like one of the common characteristics is that the students are more focused on what is familiar and remembered, than on (even elementary) mathematical reasoning and accuracy.

4.1.2 A2: Students' reasoning in task solving

Completed report: 'Mathematical reasoning in task solving' [Lit00c].

Abstract

An earlier study [Lit00a] treated the question 'what are the main characteristics and background of undergraduate students' difficulties when trying to solve mathematical tasks?' This paper will focus on, and extend, the part of the earlier study that concerns task solving strategies. The results indicate that focusing on what is familiar and remembered at a superficial level is dominant over reasoning based on mathematical properties of the components involved, even when the latter could lead to considerable progress.

The main difference between A1 and A2 is that the former is 'wider' (all their main difficulties) and the latter is more limited to treating, on a firmer theoretical foundation, certain types of mathematical reasoning.

4.1.3 A3: Strategies and reasoning possible to use when solving textbook exercises.

Completed report: 'Mathematical reasoning in Calculus Textbook Exercises' [Lit00b].

Abstract

The aim of this paper is to study some of the strategies that are possible to use in order to solve the exercises in undergraduate calculus textbooks. It is described how most exercises may be solved by mathematically superficial strategies. Strategy choices and implementations can usually be based on identifying similar solved examples and copying, or sometimes locally modifying, given solution procedures. One consequence is that exercises may often be solved without actually considering the core mathematics of the book section in question.

The studies A1 and A2 indicated (together with studies of research literature) that students focus on routines and superficial reasoning, and one of the main reasons behind their difficulties is their inability and/or reluctance to consider the mathematical properties involved in the reasoning. The studies A3 and A4 aim at searching for possible reasons behind these indications.

4.1.4 A4: Strategies and reasoning applied by students when solving textbook exercises.

Completed report: 'Students' Mathematical Reasoning in Textbook Exercise Solving' [Lit00d].

Abstract

This study investigates the ways students conduct their study work, in particular their mathematical reasoning when working with textbook exercises. The results indicate that: (i) Most strategy choices and implementations are carried out without considering the intrinsic properties of the components involved in the solution work. This in turn leads to different difficulties. (ii) It is crucial for these students to find solution procedures to copy. (iii) There are extensive attempts, often successful, to understand each step of the copied solution procedures, but only locally. (iv) The students make almost no attempts to construct their own solution reasoning, not even locally. (v) The main situations where the students' work are not just straightforward implementations of provided solution procedures, are where mistakes are made in minor local solution steps.

4.1.5 The studies A1-A4: An informal summary

The research methods used in the completed studies above are mainly qualitative: Relatively fine-grained analyses of a small number of students reasoning characteristics in limited task solving situations, including the development of analytical frameworks. These types of analyses can not determine with a high degree of accuracy the reasoning characteristics of students in general, but can a) show the *existence* of some reasoning types and b) *indicate* plausible characteristics of larger student groups. The latter may also be supported by studying similar or related aspects from other theoretical perspectives or by other methods, hereby finding reasonable and general explanations behind the indicated behaviour. One example of this is the study A3, which is partly quantitative (600 textbook exercises were classified), where possible reasons behind the students' behaviour in the other studies are investigated.

Though the studies A1-A4 treats only limited aspects of students' competence and limited aspects of the learning environment, and though the work of rather few students are investigated, the overall picture emerging is coherent: It seems like the students are founding their work mainly on superficial reasoning, and that the reasons behind this originates to a large extent from the learning environment provided by the educational system. This is (at least partly) already known, as exemplified by the short literature review above, and it also seems to be experience-based knowledge familiar to many teachers. The motivation for carrying out as research the studies A1-4 is: (i) The studies A2, A3, and A4 explicitly and primarily address the ways that the students' reasoning is based on mathematical properties or not, something that is not done by many other studies. The studies A1 and A2 indicated that the domination of 'non-mathematical' reasoning is one of the main causes behind task-solving difficulties. (ii) The reasoning is studied in rather fine-grained detail. A framework for this type of studies is one of the outcomes of the studies A2-4. (iii) There are surprisingly few studies on textbook structure [LP96] (especially from the perspective (i)) and on students' actual learning strategies and textbook usage. (iv) The achievement difficulties of mathematics students at all levels have been known for many years, but the difficulties mainly remain. Extensive research on the questions in Section 1.2 is still required in order to be able to construct well-founded measures for improvement of the learning environment.

4.2 Ongoing studies

Short summaries of ongoing studies are presented in this section.

4.2.1 A5: Reasoning put forward by the teacher and textbooks

Report under preparation: 'Mathematical Reasoning put forward in Undergraduate Teaching' [Lit01b].

Abstract

The purpose of this paper is to study the types of mathematical reasoning put forward in the learning environment by teachers and in the textbooks' theory and examples. The primary focus is on the mathematical foundations in task solving reasoning.

The ways the teacher acts in lectures, lessons, seminars, supervision, discussions, etc. are probably, but not necessarily, very influential on the internal learning environment. The reasoning put forward in textbooks, mainly in the solved examples, may be influential on the student's learning and achievements. One could expect that it is obvious that the teachers' and textbooks' reasoning will influence students' reasoning, but there are indications that this does not influence students to the intended extent. For example, it seems like few students actually read the textbook's theory text and that many spend most of their time working with textbook exercises [Shi89] [Shi91] [Lit00b] [Lit00d]. Here will also be considered the written material like schedules and extra tasks provided for the students by the teacher.

4.2.2 A6: Examination.

Report under preparation: 'Mathematical Reasoning in Exams' [Lit01a].

What types of reasoning are required, encouraged and practised in exams? Are there discrepancies between what is intended, implemented, received and attained? Some of the methods and frameworks from the papers above are used to analyse exam tasks and students' reasoning when solving these tasks. This is complemented by interviews with teachers and students.

4.2.3 A7: The aspects of the internal learning environment that are focused by the students.

Report under preparation: 'Students' efforts in learning mathematics' [Lit01c].

What types of activities do students prioritise when studying undergraduate mathematics? This study is mainly a quantitative study of a large number students learning behavior, and complements [Lit00d] which was a qualitative study of only three students' studying activities.

4.3 Planned studies

The studies below are planned within the project, but not yet initiated.

4.3.1 A8: The teachers', syllabus constructors', and others' intentiones about mathematical reasoning when constructing the syllabus, the teaching, and the exam.

Planned report: 'intentiones about mathematical reasoning in the learning environment'.

What are the teachers' explicit and implicit intentiones when planning and implementing the syllabus (here is included all written relevant syllabuses, not only the ones written by the teachers), the teaching and the examination, in particular with respect to mathematical reasoning?

4.3.2 A9: How the students receive the syllabus, the teaching, and the exam?

Planned report: 'The learning environment as received by undergraduate students'.

This study is not about the students mathematical competence, which is studied in other parts of the project, but about their apprehension of the syllabus, the teaching and the exam. Included are also studies of the students' beliefs concerning what the proper learning strategies are in order to reach the apprehended goals, in particular with respect to different types of mathematical reasoning in task solving.

4.3.3 A10: The implementation of the syllabus, and its relation to the other studies A1-9 above.

Planned report: 'The implementation of the syllabus: A summary of a series of studies on undergraduate mathematical reasoning'.

This study is planned to summarise the other nine studies in the project, and to relate their outcome to the implemented formal and informal syllabus.

References

[Ada95] R. Adams. *Calculus: A Complete Course.* Addison - Wesley, third edition, 1995.

[Ant96] G. Anthony. When mathematics students fail to use appropriate learning strategies. *Mathematics Education Research Journal*, 8:23–37, 1996.

[Art96] M. Artigue. Teaching and learning elementary analysis. In C. Alsina, J.M. Alvarez, B. Hodgson, C. Laborde, and A. Perez, editors, *8th International Congress on Mathematical Education. Seville (Spain). Selected lectures*, pages 15–29, 1996.

[Art98] M. Artigue. What can we learn from Didactic Research carried out at University Level? In *Pre-proceedings, ICMI Study Conference*

On the Teaching and Learning of Mathematics at University Level, Singapore, 1998.

[ASP97] L. Aspinwall, K. Shaw, and N. Presmeg. Uncontrollable mental imagery: graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33:301–317, 1997.

[Bal88] N. Balacheff. Aspects of proof in pupils' practice of school mathematics. In D. Pimm, editor, *Mathematics, teachers and children*, pages 216–235. London: Hodder and Stoughton, 1988.

[Bau79] H. Bauersfeld. Research related to the mathematical learning process. In *New trends in mathematics teaching IV*, chapter X. Unesco, Paris, 1979.

[BC96] A. Bodin and B. Capponi. Junior secondary school practices. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 615–646. Dordrecht: Kluwer, 1996.

[BCFF+98] D. Ben-Chaim, J. Fey, W. Fitzgerald, C. Benedetto, and J. Miller. Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics*, 36:247–273, 1998.

[BCK+96] A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors. *International Handbook of Mathematics Education.* Dordrecht: Kluwer, 1996.

[Bes95] K. Bessant. Factors Associated with Types of Mathematics Anxiety in College Students. *Journal for Research in Mathematics Education*, 26:327–345, 1995.

[BHPL92] M. Behr, G. Harel, T. Post, and R. Lesh. Rational number, ratio, and proportion. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 296–333. New York: Macmillan, 1992.

[BK91] W. Blum and A Kirsch. Preformal proving: examples and reflections. *Educational Studies in Mathematics*, 22:183–203, 1991.

[BM96] T. Barnard and C. Morgan. Theory and practice in undergraduate mathematics teaching: a case study. In L. Puig and A. Gutierrez, editors, *20th Conference of the International Group for the Psychology of Mathematics Education (PME 20). Proceedings. Vol. 2. Valencia (Spain)*, pages 43–50, 1996.

[Boa98] J. Boaler. Open and closed mathematics. Student experiences and understandings. *Journal for Research in Mathematics Education*, 29:41–62, 1998.

[Bro97] G. Brousseau. *Theory of didactical situations in mathematics.* Dordrecht: Kluwer, 1997.

[BS96] J. Becker and C. Selter. Elementary school practices. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 511–564. Dordrecht: Kluwer, 1996.

[BSSW94] R. Bieler, R. Scholz, R. Strässer, and B. Winkelmann, editors. *Didactics of mathematics as a scientific discipline.* Kluwer, 1994.

[Car99] M. Carlson. The mathematical behavior of six successful mathematics graduate students: influences leading to mathematical success. *Educational Studies in Mathematics*, 40:237–258, 1999.

[CB92] D. Clements and M. Battista. Geometry and spatial reasoning. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 420–464. New York: Macmillan, 1992.

[Cha93] D. Chazan. High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24:359–387, 1993.

[Chr97] I.M. Christiansen. When negotiation of meaning is also negotiation of task. *Educational Studies in Mathematics*, 34:1–25, 1997.

[Cif98] V. Cifarelli. The development of mental representations as a problem solving activity. *The Journal of Mathematical Behavior*, 17:239–264, 1998.

[CJ65] R. Courant and F. John. *Introduction to Calculus and Analysis*, volume 1. Wiley International, 1965.

[Cox94] W. Cox. Strategic learning in a-level mathematics? *Teaching Mathematics and its Applications*, 13:11–21, 1994.

[CP95] L.J. Chatterly and D.M. Peck. We're crippling our kids with kindness! *The Journal of Mathematical Behavior*, 14:429–436, 1995.

[DH97] R. Dahlberg and D. Housman. Facilitating learning events through example generationk. *Educational Studies in Mathematics*, 33:283–299, 1997.

[dLVP54] C. de La Vallée Poussin. *d'Analyse Infinitésimale.* Paris: Gauthiers - Villars, eigth edition, 1954.

[DMR97] E. Dubinsky, D. Mathews, and B. Reynolds. *Readings in Cooperative Learning for Undergraduate Mathematics. MAA notes 44.* The Mathematical Association of America, 1997.

[Doy86] W. Doyle. Classroom organization and management. In M.C. Wittrock, editor, *Handbook of Research on Teaching*, pages 392–431. New York: Macmillan, third edition, 1986.

[Doy88] W. Doyle. Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, 23:167–180, 1988.

[Dre99] T. Dreyfus. Why Johnny can't prove. *Educational Studies in Mathematics*, 38:85–109, 1999.

[EBU+93] M. Eisenhart, H. Borko, R. Underhill, C. Brown, P. Agard, and D. Jones. Conceptual knowledge falls through the cracks: complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24:8–40, 1993.

[EC93] M. Eley and N. Cameron. Proficiency in the Explanation of Procedures: A Test of the Intuitive Understanding of Teachers of Undergraduate Mathematics. *Higher Education*, 26:355–386, 1993.

[Eis94] T. Eisenberg. On understanding the reluctance to visualize. *Zentralblatt fuer Didaktik der Mathematik*, 26:109–113, 1994.

[EP94] C.H. Edwards and D. Penney. *Calculus with Analytic Geometry*. Prentice Hall International, fourth edition, 1994.

[Fer97] P. Ferrari. Action-based strategies in advanced algebraic problem solving. 21th Conference of the International Group for the Psychology of Mathematics Education. Proceedings. Vol. 2. Lahti (Finland), 1997.

[FL00] H. Forgasz and G. Leder. Perceptions of the tertiary learning environment: is mathematics worth the effort? *International Journal of Mathematical Education in Science and Technology*, 31:37–42, 2000.

[FT96] R. Foster and D. Tall. Can all children climb the same curriculum ladder? *Mathematics in School*, 25:8–12, 1996.

[Fus92] K. Fuson. Research on whole number addition and subtraction. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 243–275. New York: Macmillan, 1992.

[GHRV98] M Guzman, B. Hodgson, A. Robert, and V. Villani. Difficulties in the passage from secondary to tertiary education. In Gerd Fischer and U. Rehmann, editors, *Proceedings of the International Congress of Mathematicians, Berlin 1998. Invited lectures. Vol. 3*, pages 747–762, 1998.

[Gre92] B. Greer. Multiplication and division as models of situations. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 276–295. New York: Macmillan, 1992.

[Gro92] D. Grouws, editor. *Handbook for Research on Mathematics Teaching and Learning*. New York: Macmillan, 1992.

[GT93] E. Gray and D. Tall. Success and failure in mathematics: the flexible meaning of symbols as process and concept. *Mathematics Teacher. No 142*, pages 6–10, 1993.

[GT94] E. Gray and D. Tall. Duality, ambiguity, and flexibility: a 'proceptual' view of simple arithmetic. *Journal for Research in Mathematics Education*, 25:116–140, 1994.

[HC92] J. Hiebert and T. Carpenter. Learning and teaching with understanding. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 65–97. New York: Macmillan, 1992.

[HD92] G. Harel and E. Dubinsky, editors. *The Concept of Function: Aspects of Epistemology and Pedagogy.* Washington, DC: Mathematical Association of America, 1992.

[HJ96] G. Hanna and N. Jahnke. Proof and proving. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 877–908. Dordrecht: Kluwer, 1996.

[HMM95] M. Hegarty, R. Mayer, and C. Monk. Comprehension of arithmetic word problems: a comparison of successful and unsuccessful problem solvers. *Journal of Educational Psychology*, 87(1):18–32, 1995.

[Hoy97] C. Hoyles. The curricular shaping of students' approaches to proof. *For the Learning of Mathematics*, 17:7–16, 1997.

[HS97] M. Henningsen and M.K. Stein. Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5):524–549, 1997.

[HT96] G. Harel and J. Trgalová. Higher mathematics education. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 675–700. Dordrecht: Kluwer, 1996.

[KAA+98] P. Kahn, J. Anderson, K. Austin, T. Barnard, J. Jagger, and A. Chetwynd. The significance of ideas in undergraduate mathematics: a case study of the views of lecturers and students. *Teaching Mathematics and Its Applications*, 17:78–85, 1998.

[KD97] C. Kamii and A. Dominick. To teach or not to teach algorithms. *The Journal of Mathematical Behavior*, 16:51–61, 1997.

[Kie92] C. Kieran. The learning and teaching of school algebra. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 390–419. New York: Macmillan, 1992.

[LB90] C. Livingston and H. Borko. High school mathematics review lessons: expert-novice distinctions. *Journal for Research in Mathematics Education*, 21:372–387, 1990.

[Led96] G. Leder. Research and intervention programs in mathematics education: A gendered issue. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 945–986. Dordrecht: Kluwer, 1996.

[Led98] G. Leder. Panel Discussion - Mass Education. In *Pre-proceedings, ICMI Study Conference On the Teaching and Learning of Mathematics at University Level, Singapore*, 1998.

[Les94] F. Lester. Musings about mathematical problem-solving research. *Journal for Research in Mathematics Education*, 25:660–675, 1994.

[LH97] U. Leron and O. Hazzan. The world according to Johnny: a coping perspective in mathematics education. *Educational studies in mathematics*, 32:1265–292, 1997.

[Lit00a] J. Lithner. Mathematical reasoning and familiar procedures. *International Journal of Mathematical Education in Science and Technology*, 31:83–95, 2000.

[Lit00b] J. Lithner. Mathematical reasoning in calculus textbook exercises. Research Reports in Mathematics Education 2, Department of Mathematics, Umeå University, 2000. Submitted for publication.

[Lit00c] J. Lithner. Mathematical reasoning in task solving. *Educational studies in mathematics*, 41:165–190, 2000.

[Lit00d] J. Lithner. Students' mathematical reasoning in textbook exercise solving. Research Reports in Mathematics Education 4, Department of Mathematics, Umeå University, 2000. A shorter version is to be submitted for publication.

[Lit01a] J. Lithner. Mathematical reasoning in exams, 2001. In preparation.

[Lit01b] J. Lithner. Mathematical reasoning put forward in undergraduate teaching, 2001. In preparation.

[Lit01c] J. Lithner. Students' efforts in learning mathematics, 2001. In preparation.

[LP96] E. Love and D. Pimm. "This is so": a text on texts. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 371–409. Dordrecht: Kluwer, 1996.

[Mam90] J.C. Mamona. Sequences and series-sequences and functions: students' confusions. *International Journal of Mathematical Education in Science and Technology*, 21:333–337, 1990.

[McL92] D. McLeod. Research on affect in mathematics education: A reconceptualization. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 575–596. New York: Macmillan, 1992.

[McN95] B. McNeal. Learning not to think in a textbook-based mathematics class. *The Journal of Mathematical Behavior*, 14:205–234, 1995.

[Moo94] R.C. Moore. Making the transition to formal proof. *Educational studies in mathematics*, 27:249–266, 1994.

[MS97] M. MacGregor and K. Stacey. Students' understanding of algebraic notation: 11-15. *Educational studies in mathematics*, 33:1–19, 1997.

[MVZ86] R. McGinty, J. VanBeynen, and D. Zalewski. Do our mathematics textbooks reflect what we preach? *School Science and Mathematics*, 86:591–596, 1986.

[Nis93a] M. Niss, editor. *Cases of assessment in mathematics education. An ICMI study.* Dordrecht: Kluwer, 1993.

[Nis93b] M. Niss, editor. *Investigations into assessment in mathematics education. An ICMI study.* Dordrecht: Kluwer, 1993.

[Nis96] M. Niss. Goals of mathematics teaching. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 11–48. Dordrecht: Kluwer, 1996.

[Nis99] M. Niss. Aspects of the nature and state of research in mathematics education. *Educational studies in mathematics*, 40:1–24, 1999.

[Pól45] G. Pólya. *How To Solve It.* Princeton NJ: Princeton University Press, 1945.

[Pól54] G. Pólya. *Mathematics and Plausible Reasoning, Vol. I and II.* Princeton NJ: Princeton University Press, 1954.

[RG89] D.F. Robitaille and R.A. Garden. *The IEA study of mathematics 2: Contexts and outcomes of school mathematics.* Oxford: Pergamon Press, 1989.

[Ros98] K. A. Ross. Doing and proving: The place of algorithms and proof in school mathematics. *American Mathematical Monthly*, pages 252–255, 1998.

[Sch83] A. Schoenfeld. Problem solving in the mathematics curriculum: A report, recommendations and an annotated bibliography. Washington, D.C.: Mathematical Association of America, 1983.

[Sch85] A. Schoenfeld. *Mathematical Problem Solving.* Orlando, FL: Academic Press, 1985.

[Sch91a] A. Schoenfeld. On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. Voss, D. Perkins, and J. Segal, editors, *Informal Reasoning and Education*, pages 311–344. Hillsdale, NJ: Lawrence Erlbaum Associates, 1991.

[Sch91b] A. Schoenfeld. What's all the fuss about problem solving? *Zentralblatt für didaktik der mathematik*, 91:4–8, 1991.

[Sch92] A. Schoenfeld. Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 334–370. New York: Macmillan, 1992.

[Sch94] A. Schoenfeld. Reflections on doing and teaching mathematics. In A. Schoenfeld, editor, *Mathematical thinking and problem solving*, pages 63–70. Hillsdale, NJ: Erlbaum, 1994.

[Sch98] A. Schoenfeld. Reflections on a course in mathematical problem solving. In A. Schoenfeld, J. Kaput, and E. Dubinsky, editors, *Research in Collegiate Mathematics Education III, CBMS Issues in Mathematics Education*, pages 81–113. American Mathematical Society, 1998.

[Shi89] M. Shield. Mathematics Teachers' Preferences in Textbook Characteristics. *Mathematics Education Research Journal*, 1:11–15, 1989.

[Shi91] M. Shield. Mathematics textbooks. *Australian Mathematics Teacher*, 47:24–27, 1991.

[Sie92] A. Sierpinska. On understanding the notion of function. In G. Harel and E. Dubinsky, editors, *The Concept of Function: Aspects of Epistemology and Pedagogy*, pages 25–58. Washington DC: Mathematical Association of America, 1992.

[SK94] E. Silver and J Kilpatrick. E pluribus unum: challenges of diversity in the future of mathematics education research. *Journal for Research in Mathematics Education*, 25:734–754, 1994.

[SM99] K. Stacey and M. MacGregor. Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal*, 11:24–38, 1999.

[SMS89] J. Selden, A. Mason, and A. Selden. Can average calculus students solve nonroutine problems? *The Journal of Mathematical Behavior*, 8:45–50, 1989.

[Sow92] J. Sowder. Estimation and number sense. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 371–389. New York: Macmillan, 1992.

[SS95] J. Selden and A. Selden. Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29:123–151, 1995.

[SSM94] J. Selden, A. Selden, and A. Mason. Even good students can't solve nonroutine problem. In J. Kaput and E. Dubinsky, editors, *Research issues in Undergraduate Mathematics Learning*, pages 19–26. Washington DC: Mathematical Association of America, 1994.

[Szy00] J. Szydlik. Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education*, 31:258–276, 2000.

[Tal90] D. Tall. Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics*, 12:49–63, 1990.

[Tal91] D. Tall, editor. *Advanced mathematical thinking.* Kluwer, 1991.

[Tal92] D. Tall. The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 495–511. New York: Macmillan, 1992.

[Tal96] D. Tall. Functions and calculus. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, and C. Laborde, editors, *International Handbook of Mathematics Education*, pages 289–325. Dordrecht: Kluwer, 1996.

[Tal99] David Tall. The chasm between thought experiment and formal proof. In G. Kadunz, G. Ossimitz, W. Peschek, E. Schneider, and B. Winkelmann, editors, *Mathematical literacy and new technologies*, pages 319–343. Conference: 8. internationales Symposium zur Didaktik der Mathematik, Klagenfurt (Austria), 1999.

[TG90] D. Tirosh and A. Graeber. Inconsistencies in preservice elementary teachers' beliefs about multiplication and division. *Focus on Learning Problems in Mathematics*, 12:65–74, 1990.

[Tho94a] P. Thompson. Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26:229–274, 1994.

[Tho94b] P. Thompson. Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. Schoenfeld, and J. Kaput, editors, *Research in Collegiate Mathematics Education I, CBMS Issues in Mathematics Education*, pages 21–44. American Mathematical Society, 1994.

[TR93] D. Tall and M. Razali. Diagnosing students' difficulties in learning mathematics. *International Journal of Mathematical Education in Science and Technology*, 24:209–222, 1993.

[TS99] D. Tirosh and R Stavy. Intuitive rules: a way to explain and predict students' reasoning. *Educational Studies in Mathematics*, 38:51–66, 1999.

[TT94] P. Thompson and A. Thompson. Talking about rates conceptually. part 1. *Journal for Research in Mathematics Education*, 25:279–303, 1994.

[TT96] A. Thompson and P. Thompson. Talking about rates conceptually. part 2. *Journal for Research in Mathematics Education*, 27:2–24, 1996.

[TV81] D. Tall and S. Vinner. Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12:151–169, 1981.

[VCV99] L. Verschaffel, E. De Corte, and H. Vierstraete. Upper elementary school pupils' difficulties in modeling and solving nonstandard additive word problems involving ordinal numbers. *Journal for Research in Mathematics Education*, 30:265–285, 1999.

[Vin91] S. Vinner. The role of definitions in the teaching and learning of mathematics. In D. Tall, editor, *Advanced mathematical thinking*, pages 65–81. Kluwer, 1991.

[Vin97] S. Vinner. The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, 34:97–129, 1997.

[Wil91] S. Williams. Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, 22:219–236, 1991.

[WM96] P. White and M. Mitchelmore. Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, 27:79–95, 1996.

Mathematical Reasoning and Familiar Procedures

Johan Lithner

ABSTRACT. Four first-year undergraduate students are working with two tasks. The underlying question treated is "what are the characteristics and background causes of their difficulties when trying to solve these tasks?" The purpose is to give a general survey of their main difficulties, rather than to go deeply into details. It seems like one of the common characteristics is that the students are more focused on what is familiar and remembered, than on (even elementary) mathematical reasoning and accuracy.

METHOD

The students worked the tasks alone in the presence of me and a video camera, and were asked to "think aloud". After a session I tried to analyse the tape and describing not only what was taking place but also why this was happening, in a sense speculating in how the student was thinking. Not later than three days after the session I then met with the student, and we went through my written analysis. The student then had a chance to make comments, and also suggest other ways to interpret and explain the situations. The descriptions and the analyses below will focus on six examples of behaviour that seem to account for their main difficulties in this study. A fuller version of the study is found in (Lithner 1998).

SUMMARISED DESCRIPTION OF ALF'S AND TOM'S WORK

Task 1: A company shall produce x units per year, where x belongs to [400,600]. The estimated production cost is approximately $-2x^2 + 2000x - 420000$ Kr/unit, and the expected sale price approximately $-x^2 + 700x$ Kr/unit. How many units should be produced each year to maximise the yearly profit?

Alf's episode.

Minor passages omitted are replaced by [..] , pauses are indicated by .... . Alf says "It feels a bit familiar". [..] "The sale price minus the production cost is the profit" [this yields the profit per unit, not per year]. He has no difficulties in constructing the [within his faulty interpretation, correct] profit function

$$V = x^2 - 1300x + 420000$$

Alf differentiates this, solves V'(x)=0 to find x=650. "Now we can use the second derivative test to find out if it is a max or a min." He easily finds V"=2, but then gets a bit puzzled: "This is strange. It feels like it should be a minimum if I remember rightly, since the second derivative is positive."

After a while Alf decides that: "I could skip this second derivative test. There are other possibilities if I don't remember wrong. For example to check what the derivative looks like close to ... [meaning x=650]". Alf works swiftly, finds that V'(600) is negative and V' is positive to the right of x=650, and draws two arrows: ↘ ↗
"It feels like it becomes a minimum! No, wait a minute, what am I doing?" Alf spends a few minutes not being able to decide whether to examine the derivative or the function.
"Normally I would just accept it [x=650] as an answer, it feels like a rather good answer in some way. ... Maybe I should check this."
Alf uses a calculator and finds

$$V(650) = -2500$$

"I'll be damned if it was not negative! [..] Then we can assume that this [x=650] isn't so good." After some thinking he remembers that he has to check the endpoints of the definition interval, and after some routine work he states that "this implies that we have the maximum profit for 400 units".

After some discussion JL asks if he is finished with the task and Alf answers: "The question was how much one should produce, and I have determined this. In other words, one should produce 400... whatever it was ... units to earn as much as possible."
JL now asks him to describe, as carefully as possible, what is really asked for in the task and what question Alf really has answered. After a while Alf says: "They are asking how many units one should produce to maximise [..] the profit. One has two expressions that depend on the number of produced units. One for the production costs, the expenses. One for what you get when you sell. Consequently, the profit must be the income minus the expenses. So it feels like I have answered the question."

JL asks Alf to explain what profit his expression V(x) describes. Alf now admits that he is lead by JL to question his interpretation of "profit", and continues hesitantly: "The yearly profit ... No, it doesn't really fit ... I have missed the units [at the start he read and pronounced the units correctly] ... I have to consider what I have written. [..] The unit for the production costs is in Kr/unit, which implies that V describes the profit per unit, and I have drawn faulty conclusions."
JL asks what he has actually calculated, but Alf has difficulties in describing this: "If you want to earn ... let me think ... If you want to earn as much as possible if you want to sell one? It feels strange ..." 25 minutes have passed since start, and JL helps Alf to summarise. Then Alf spends 17 minutes to finally reach the correct solution.

<u>Tom's episode</u>
Tom makes the same misinterpretation as Alf and reaches an incorrect profit function:

$$f(x) = x^2 - 1300x + 420000$$

He finds that f'(x)=0 at x=650, f''(650)=2, and concludes that there is a minimum at x=650. "This does not tell me so much, I have to look at the endpoints." Using a calculator he finds f(400)=60000 and f(600)=0, but makes a careless mistake, mixes them and plots the points (400,0) and (600, 60000) in the xy-plane.

Tom notes immediately that this is not consistent with his argument that there should be only one extremal point (a minimum), since f(600)=60000 indicates that there should be a maximum between the zero at x=400 and the minimum at x=650. He starts his careful error analysis by reading the task again [without noting his faulty interpretation], checks that he has transcribed the given functions correctly, goes through his algebraic and arithmetic calculations, and finally finds his mistake seven minutes after mixing the values above. He corrects and completes his figure, and from this draws the conclusion that "one should produce 400 units per year to maximise the yearly profit".

JL is now trying to lead Tom into reflecting over his faulty interpretation:
"If this was an examination situation, would you check your work again, or are you convinced that you are correct?"
"I would consider this as finished, but I always check if I have the time."
"If you should check, what would you check?"
Tom describes how he would proceed in the same manner as when he searched for the mixed-values error above, but he is convinced that he is correct now.
JL says, a bit provocative: "If this was an exam question that could give 3 points, you would be given 0.5. What would you do if you had another chance?"
"[..] I would plot the two functions given in the task, to see what happens. I cannot see this now."

After 5 minutes of careful plotting in accordance with the standard method he learned in class [analysing derivatives, critical points, extremal points, asymptotes, etc.], JL interrupts: "I believe your interpretations of the functions are correct." Tom then considers the interval of definition, and then the relation between income and expense. With some assistance from JL he realises that the mistake is to be found elsewhere.

Tom considers the formulation of the task again, reads the last sentence aloud with a clear emphasise on the keywords "produce" and "maximise", and finally gives in: "I cannot find the error. Could you give me a clue?"
"What question is really asked, and what question have you really answered?"
"One asks about the number of units (he underlines "units") ... it must mean that it is the same ... that one is talking about units when referring to Kr/unit ... say one unit ..."
"This cost to produce, what does it tell you?"

"I see, so this is a function for the whole sum of units. It costs ... to produce 400 units so ... shall one ... use this formula ... for 400 units one doesn't get the price per unit ... ? Is that what you mean?"
"Well, what do you get when you insert x=400 in the [production cost] expression?"
"It says that I get price each."
"And what is that?"
"Price per unit."
"Yes, how much it costs per unit if one produces 400 units."
"Now things become clearer to me! If one produces 600 units, the profit could get lesser per unit but greater totally since one produces more units. I have found when to earn most per unit, not maximised the yearly profit". 44 minutes has passed since start. Tom has now no problems in constructing the proper profit function and proceeding with the task.

ANALYSIS OF ALF'S AND TOM'S EPISODES

A) Alf's and Tom's misinterpretations.

Alf reads the task superficially several times, stepwise searching for more information:
i) The mathematical symbols state that polynomials are involved, but nothing more.
ii) He searches for keywords (Hegarty et al. 1995). The first is "maximise" which tells him that the setting is calculus and that he through a well-known procedure shall maximise a function f(x).
iii) If there had only been one function given in the task, he had probably without further analysis tried to find its maximum. Now he has to decide which function to maximise and needs to decode more keywords.
iv) The familiar keywords he finds are "production cost", "sale price" and "yearly profit". He does not bother to try to specify the exact meaning of these keywords, but interprets them as just "expense", "income" and "profit". Now he has identified a meaningful interpretation, it makes sense as a mathematical task in accordance with his experiences.

Tom is very carefully reading the first two sentences. Then he feels that he has reached an interpretation that is meaningful, and he looses interest in continuing his careful analysis. When he reads the third sentence of the task he does not take in any additional information. This faulty but very solid interpretation guides his work for 40 minutes, even though he rereads the task several times.

The key is that their interpretations are meaningful and included in their library of familiar task types. By almost all their experience from their studies, exercises are essentially of a limited set of standard type, and careful analyses do not pay off. The consequences of this is that their interpretations are:

1) Superficial. When requested, both of them have difficulties in establishing what kind of profit they have actually treated.
2) Stable. Though they meet several difficulties, and question much of what they do, they have to be led practically all the way before they question the interpretations.

The same task was later given to 64 students. Of the 46 who made an attempt to answer, 33 made the same misinterpretation as Alf and Tom. Only 5 provided a correct interpretation, which indicates that Alf's and Tom's interpretations are not extreme cases.

B) The character of the critical point.

Alf expects the he as usual will find the answer by finding the zeros of the derivative V'(x). When he finds that V"(x)=2 he is puzzled by the two contradictions:
i) His (faulty) expectation that the answer is found where V'(x)=0.
ii) He (correctly) believes that V"(650)=2 => x=650 is a minimum.

There are two reasons behind Alf's decision to dismiss ii):
a) The expectation i) is stronger than his conviction that ii) is true.
b) ii) is just something he remembers. Alf does not know why this should be true, and has no means of completing and testing his memory by some kind of reasoning.
If he had made some miscalculations and for example found V"(650)=-2 (which would imply that x=650 yields a maximum), he would without doubt have considered himself finished with the task.

Another familiar method is the first derivative test, but again he reaches a contradiction, this time between i) above and:
iii) V(x) is decreasing to the left of x=650 and increasing to the right which implies that x=650 is a minimum.
Once again, the expectation i) is dominating and he questions here if the correct method in iii) is applicable at all. He is then rather puzzled and searches at first for other methods to verify that the answer is x=650, but cannot find any. It is not until he tests the value of V(650) and to his surprise finds that it is negative, that he questions i). Almost ten minutes have passed since he found that V"(x)=2 above.

C) Localising errors.

1) Tom notices at once that something is wrong when plotting the mixed-up values of f(400) and f(600). When searching for the error though, he simply checks it all by starting from the beginning. There is no consideration about what type of error it might be and it takes him seven minutes to find it.
2) When Tom is told that he has produced a faulty solution, he says after thinking only a few seconds that he wants to plot the two functions. There is essentially no deeper consideration over what type of error it might be. He has a rather strong affective reluctance towards analysing the text and the general content of the task. He is not used to doing this and feels more

comfortable analysing his calculations, where he spends most of the eleven minutes (which could have been longer if he had not been helped) of his error search.

In order to avoid the need to always check all types of errors, it is important to also consider where the error might be. It could have been possible for Tom to notice that his discovery of the error 1) above was founded in two contradictory statements:
i) f(x) is continuous and growing (at least partially) between x=400 and x=600 (wrong).
ii) f(x) has exactly one extremal point, a minimum at x=650 (correct).
Both of these originates from $f(x)=x^2-1300x+420000$, and thus he does not have to check anything he has done prior to writing down this function.

If Tom is checking too much in 1) he is checking too little and the wrong things in 2), but the common feature is in both cases that there are very quick decisions on how to start the error search, and essentially no consideration over what type it might be. It is noteworthy that in these students learning environment, there seems to be essentially no explicit focus on this type of considerations.

SUMMARISED DESCRIPTION OF JAN'S AND PER'S WORK

Task 2: The function f(x) has the graph below (Figure 1). a) Sketch the graph of f'(x).
b) What is f(-2), f'(0) and f''(2)? c) Sketch the graph of g(x), if g'(x)=f(x).

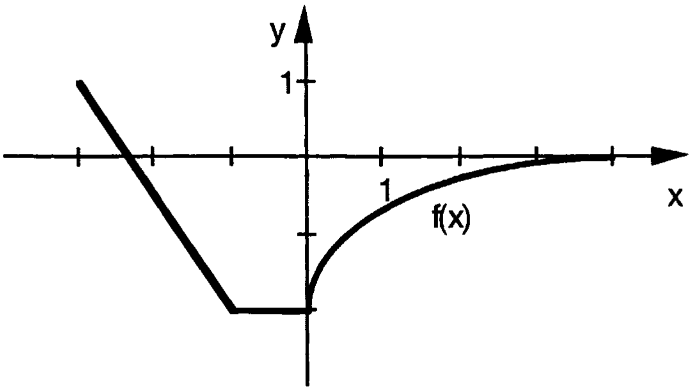


Figure 1.

Jan's episode.
When sketching f'(x) in the xy-plane, Jan swiftly draws approximately the line y=-2 for x ∈ (-3,-1), see Figure 2.

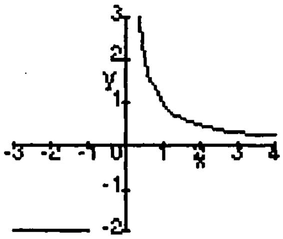


Figure 2.

Then he starts to hesitate about what to draw for $x \in (-1,0)$: "And then ... there is a jump ... it is zero here ... let's see ..." After a period of silence JL asks Jan what he is thinking.

"What the derivative looks like ... in this constant interval. Here (he points at the line y= -2 in Figure 2), I have figured out that it is constant but negative." Jan is silent for half a minute and then continues, by leaving the interval (-1,0) and turning to the next one: "And here, the derivative is like this." He draws swiftly the graph for $x \in (0,4)$, thus producing a picture that looks approximately like (he has not yet drawn any curve for $x \in (-1,0)$) Figure 2.

"Here it is ... (he points at the graph of f(x) in Figure 1 at the interval (-1,0)) it is some function that is ... this function on that interval is -2." He writes this down:

$f(x) = -2.$

"And then the derivative of this is zero" he continues and writes:

$f'(x) = 0.$

Without hesitation he adds the line y=0 for $x \in (-1,0)$ to Figure 2 and then immediately turns to verifying the shape of the curve for $x \in (0,4)$.

In part b) he has essentially no problems in estimating f(-2). Turning to f'(0) he gets more hesitant: "If one looks at the figure (he points at his graph of f'(x)) ... we have a jump ... it never gets zero." JL asks him to clarify what he means. "This (Jan's pen traces his graph of f'(x) in Figure 2 along the curve as x goes from 4 towards 0) will never cross the y-axis, this derivative, for positive x. On the other hand, this (his pen traces his graph as x goes from -1 towards 0) will be 0 all the way to ... x equal to ... zero ... because it depends ... Now I am uncertain. It should either be zero or not defined. [..]

"The question I ask myself is if this (points again at the line y=0) goes all the way into zero? ... It seems like it does not, since there is like a corner on the function ... which means that it does not exist. Jan writes down "f'(0) does not exist" and then immediately turns to the question about f''(2), which is omitted here.

In c) Jan says: "It looks approximately ... f(x) on the interval (-•, -1) ... it looks like a constant times x, a negative constant." As he talks he writes:

(-•,-1) $f(x) = -kx$ [It should be f(x)=-kx+m]

"And then the anti-derivative is one exponent higher."

$g(x) = -kx^2/2$

"If I differentiate g(x) I get f(x)." He is very uncertain and it takes him three minutes to plot (approximately) the points (-1,1) and (-2,-4), and then connect them with a curve that is bent upwards [the wrong way compared to his g(x)] (see Figure 3).

Two seconds after drawing the curve he turns to the next interval, (-1,0). This time without hesitation, he writes down

$f(x) = -2 \qquad g(x) = -2x,$

thus producing Figure 3.

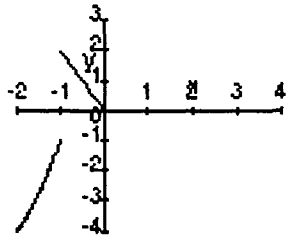


Figure 3.

"Then we have, finally, the interval (0,•). And ... there the function looks like ... something that could look like -÷x ... something ... May I take a chance on that, to use a function?" JL answers that Jan can do as he likes. After a period of silence JL asks what Jan is thinking.

"I would prefer to have an expression for f(x), it feels like. To find an anti-derivative. But it can be difficult to ... find one ... quickly. What I first thought about was ÷x, it is steeply increasing to start with, and then continues further on (he traces the graph of f(x) in Figure 1 with his pen). It could be -2÷x, it should jump down to -2 then, where it should start (he points at (0,-2) at the graph of f(x)) [Note the mistake; he implicitly claims that -2÷0 = -2]. Then it should go through ... if I insert x=1 ... It should not work, no. Because the square root of 1 times -2 is -2 (he points at (1,-0.7) at the graph of f(x)) ... Then it was probably not so wise." After a while, searching for other methods, he interrupts at 27 minutes from start his attempts to solve c).

Per's episode.

After some initial work Per swiftly draws (approximately) the line f'(x)=-1 for x ∈ (-3,-1) (see Figure 4). "Then it should be zero ... Shall I draw it like this or? ... I am a bit uncertain here." He draws the two lines connecting the points (-1,-1), (-1,0) and (0,0).

JL asks him what he is uncertain about. "How to draw this." After some silence he says "I think it is like this". Per determines f(x)=÷x, but has difficulties in sketching the graph of f'(x)=1/(2÷x) on the interval (0,4), and it takes him six minutes to complete Figure 4.

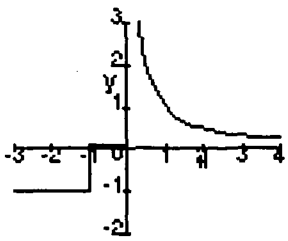


Figure 4.

"No, wait a minute! ... Isn't it so that the derivative doesn't exist in such a point (points at (-1,-2) in Figure 1) ... an edge ... I don't know ... a sharp ... turn ..."
"Why shouldn't it?"
"Well, I form associations with this ... that we had as an example [in class] ... the absolute value of x (he swiftly sketches the graph of |x|). Then the derivative doesn't exist in this point (points at origin) because there is ... can be many different (he lays his pen on his graph of |x| along the x-axis and then rotates it, centred at origin) ... one cannot simply determine the slope. Then it should also be the same at this point (points at (-1,-2) in Figure 1), I think, since this also is a ... sharp corner ... and maybe there also (points at the point (0,-2) in Figure 1)? ... And then ... well ... I don't know how to draw here (points at the vertical line at x=-1 in Figure 4) but ... maybe there shall be no line up?"

After some more thinking, once again expressing his uncertainty he finally (without explicitly motivating this) says that he guesses that f'(x) is not defined at x=-1 and x=0, deletes the vertical line at x=-1 in Figure 4, and then immediately turns to part b) which is omitted here.

In part c), Per writes after some initial work:

$y = 3/2\ x +$ [It should be y = -3/2 x +]

"If I shall estimate what ... it shall be a constant here ... I can get this by ... I can choose this point ... x=-1 and y=-2." He then writes:

$-2 = 3/2\ (-1) + m$

$m = -2/(-3/2) = 4/3$ [He divides instead of subtracting]

He says hesitantly that this does not seem to fit, because if he extends the line in Figure 1 it should cross the negative y-axis. [..] He searches but does not discover his mistakes. JL points them out to him and Per estimates m to be -3, but Per makes another mistake when writing this down:

$y = -3/2\ x + 3$ [It should be - 3]

He easily finds a primitive function to this:

$g(x) = -3/4\ x^2 + 3x$

JL shows how to draw this on a calculator, and Per produces Figure 5:

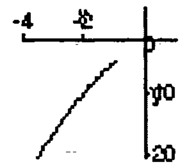


Figure 5.

About 3 seconds after he completes the graph in Figure 5, he turns to sketching g(x) on the next interval, but JL interrupts and asks if he is finished with the interval (-3,-1).
"Yes, I guess we can say that."
"Can you make some kind of estimate to see if your graph is reasonable, can you check this in some way?"
Per is a bit hesitant at first, but after convincing himself that the graph in Figure 5 corresponds to his algebraic expression for g(x) he is satisfied.
[..] JL continues:
"If you look at this function (JL points at Per's graph of g(x) in Figure 5, at approximately the point (-2,-10)), what is the derivative here? Can you say if it agrees with the derivative as given here (JL points at the graph of f(x) in Figure 1)?"
"No, it does not. The derivative is positive all the way (points at Figure 5), and it should be negative here (points at the graph in Figure 1, for $x \in (-2.3,-1)$)." Per makes some comments specifying the interval, and JL asks: "And before this the derivative is?"
Per answers without hesitation: "Positive."
"Can you from this make a very rough sketch of what the function should look like?"
It takes Per about half a minute to provide a rough but reasonable sketch of g(x) at the interval (-3,-1), see Figure 6:

Figure 6.

"The derivative is positive here (Per points at the left part of his sketch in Figure 6), and then it gets zero, and then it is negative." 61 minutes has passed since start, and JL closes the session.

ANALYSIS OF JAN'S AND PER'S EPISODES

<u>D) f'(x) on the interval [-1,0].</u>
Their knowledge bases contain mainly two methods to extract information about f'(x) from a function f(x):
1) To view f'(a) as the slope of the tangent line to f(x) at x=a.

2) To differentiate algebraic expressions of functions by familiar algorithmic rules.
Both hesitate since the situation contains, as they see it, two contradictory statements:
i) f'(x) should be zero on (-1,0).
ii) The graphs of f'(x) that they want to draw are not continuous (not even defined) on (-3,4), and therefore do not resemble any familiar graphs. Anything unusual is as they see it probably wrong. To resolve the situation, they use different approaches:

In part a) Jan doubts if i) is correct and turns to the more familiar method 2), which convinces him that f'(x)=0. In b) Jan is trying to determine the value of f'(0) from his graph of f'(x), but does not really know how to do this. He remembers after some thinking that if the graph of f(x) has a corner, then f'(x) does not exist. Per is disturbed by the vertical line at x=-1 in Figure 4. He remembers (in a similar way as Jan) almost ten minutes after drawing it that the function |x| has "a sharp turn" in a similar way as f(x). Per also remembers what the derivative of |x| look like and erases the vertical line.

The background to their hesitation is not that they initially ask themselves questions about the existence and general characteristics of f'(x) on the interval (-3,4), but that the graphs they want to draw look unfamiliar and are not included in their function concept image (Tall and Vinner 1981). I seems like posing these crucial existence questions is very seldom done by students, at least partly because it is superfluous in most textbook exercises. Per and Jan do not reflect over if and why the contradiction between i) and ii) above is resolved. They are convinced by the mere familiarity of the methods and examples. They essentially do not know why these "rules" are true but it does not matter, they are satisfied and at once turn to their next tasks.

E) Careless mistakes.

On the interval (-3,-1) in part c) their plan is:
i) f(x) is a line and thus of the form f(x)=kx+m, and the first step is to determine k and m.
ii) Integrate kx+m by the familiar procedure, which yields g(x).

Some of their careless mistakes are:
1) Jan states that "on (-•,-1) f(x)=-kx" (he misses the constant m), and finds $g(x)=-kx^2/2$ by the familiar integration algorithm.
2) Per misses to write out minus signs twice, and once he divides instead of subtracts.

Their focus is to use familiar procedures, and there are essentially no checking comparisons with other types of reasoning that might have detected the errors. The only exception is when Per notes that the function he has reached, y=3/2 x + 4/3, does not fit with the graph of f(x) in Figure 1. He makes a very superficial search for the error but cannot find it, partly because he is disturbed by the test situation.

Per has during 20 minutes worked with c), mainly with familiar elementary algorithmic methods. He has great problems caused by careless arithmetical mistakes and improper monitoring and control. When asked to check his answer, he just considers if it corresponds to his algebraic expression for g(x). When Per is mildly led into first checking if f(x) actually is the derivative of his g(x), and then making a rough estimate of g(x), he makes this swiftly and skilfully. Per cannot afterwards explain why he did not think of this himself, but it is probably caused by the fact that this type of mathematical reasoning is unusual in his studies. He says that he feels more at home when trying to apply more "exact" familiar algorithmic methods. Both Per and Jan could probably have made better progress if they have completed their present approaches with mathematical reasoning, as Per does when guided.

F) Jan's rejection of a good idea.

Jan's idea is very reasonable: If one moves the graph of ÷x two steps (units) down it resembles f(x) on (0,4) in Figure 1. He remembers wrongly that this is done by multiplying by -2 instead of subtracting 2. Unfortunately, his attempt to verify this is not well organised and fails:

Jan first says that the point (0,-2) is on the graph of -2÷x. The reason for this mistake is that he does not calculate this value, but believes that the function -2÷x is actually the translation of ÷x (which starts at (0, 0)) two steps down and therefore that the graph of -2÷x starts at the point (0,-2). Then he writes down the function's value at x=1, which he calculates; -2÷1=-2. This does not fit with the graph of f(x) in Figure 1 and he decides (correctly) that f(x) cannot be equal to -2÷x on (0,4). The function -2÷x is to him (wrongly) f(x) translated two steps down. The consequence is that he thinks that the whole idea of translating ÷x to obtain f(x) is "not so wise" and rejects a good idea without analysing why it did not work because:

1) He remembers vaguely that translating two units down is achieved by multiplying by -2 but he never really understood why (and not how translations work in general), and therefore has a weak base for asking himself why it failed.

2) As in many occasions above he is not used to, and very seldom tries to verify or test his reasoning at all. It would probably not be impossible for him to solve the sub-problem of what it takes algebraically to translate ÷x two steps down, but he does not even consider trying. He is not used to doing this in his studies, and his belief seems to be that he cannot construct his own mathematical reasoning. In general, students' beliefs about the nature of mathematics are very influential on their actual behaviour. Common beliefs are for example that ordinary students cannot expect to really understand mathematics, and cannot by themselves construct anything outside the rules and methods demonstrated by the teacher (Schoenfeld 1992).

SUMMARY

The four students often focus mainly on what they can remember and what is familiar within limited concept images. This focus is so dominating that it prevents other approaches to be initiated and implemented. There are several situations where the students could have made considerable progress by applying (sometimes relatively elementary) mathematical reasoning.

Their work and interpretations of the tasks are superficial in the sense that they are based on familiarity. At the same time they are remarkably stable. It is uncommon and takes a lot before they by themselves question and evaluate what they have done and are doing. One important component of mathematical competence is to be open and able to continuously doubt one's positions in different situations. Even skilled students and mathematicians often make minor and major mistakes, but this is not such a big obstacle if the mistakes are detected and corrected within reasonable time. The four students above have difficulties in founding their metacognitive (see e.g. Schoenfeld 1985) activities on some guiding mathematical reasoning. One common reason to activate monitoring and control is that something looks unfamiliar, not that the situation in itself might contain difficult or unclear mathematical questions that need to be addressed.

There are situations where the students above meet difficulties because they do not understand the background to the familiar procedures they apply. For example concerning Alf's problems with the second derivative test, many calculus textbooks treat the background to it as an introduction and so does probably the teacher. But when it comes to the student's part of the work (the exercises) the focus is almost entirely on applications of the test. The students are essentially not given any opportunity to practice and construct mathematical reasoning in connection with the second derivative test. Maybe this might also be the case with many other mathematical ideas at all levels in school? One may also note that there are several concepts, methods and procedures that are used without any attempts to background explanations at all.

Maybe the behaviour described above has its origins in that this usually is the best way for students to work with their studies? At short sight it might be most efficient when entering a task to (perhaps without understanding): Superficially identify the type of task, somewhat randomly choose one from the library of standard methods, apply the familiar algorithms and procedures, and finally check with the solutions section. Contrast this with Pólya's (1945) four problem solving phases: Understanding the problem, devising a plan, carrying out the plan, and looking back. It is also noteworthy that the textbook and examination questions the four students encounter that asks for the construction of some kind of mathematical reasoning, almost always also are the more difficult ones. The easier tasks, that are manageable for the students with learning difficulties, ask essentially for the application of some standard algorithm. Tall (1996) treats a related problem when discussing conceptual difficulties: "If the fundamental concepts of calculus [..] prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. [..] The problem is that such routines become just that - routine - so that students begin to find it difficult to answer questions that are conceptually challenging. The teacher compensates by setting questions on examinations that students can answer and the vicious circle of procedural teaching and learning is set in motion."

We sometimes say that "the students just do not learn what we teach them". But maybe it is actually the way of working described and criticised in this paper that we as teachers, together with textbooks and examinations, actually teach? And that the students are actually quite good at learning in accordance with the learning environment that we actually provide? If the students are basing their work more on what is familiar from a limited type of standard exercises than on solid mathematical reasoning and accuracy, and if this base is likely to lead them in the wrong direction as soon as the task is not completely familiar, then we cannot really claim that our teaching has succeeded.

REFERENCES

Hegarty, M., Mayer, R. and Monk, C.: 1995, "Comprehension of arithmetic word problems: A comparison of successful and unsuccessful problem solvers", *Journal of Educational Psychology* 87, 18-32.

Lithner, J.: 1998, "Mathematical reasoning and familiar procedures", *Research reports in mathematics education* 1, Department of mathematics, Umeå university.

Pólya, G.: 1945, *How to solve it.* Princeton: Princeton University Press.

Schoenfeld, A.: 1985, *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A.: 1992, Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In Grouws (Ed.), *Handbook for research on mathematics teaching and learning.* New York: Macmillan, pp. 334-370.

Tall, D.: 1996, "Functions and calculus". In Bishop et al. (Eds.), *International handbook of mathematics education.* Kluwer Academic Press, 289-325.

Tall, D. and Vinner, S.: 1981, "Concept image and concept definition with particular reference to limits and continuity", *Educational Studies in Mathematics* 12, pp. 151-169.

Mathematical reasoning in task solving

Johan Lithner

ABSTRACT. An earlier study (Lithner 1998) treated the question "what are the main characteristics and background of undergraduate students' difficulties when trying to solve mathematical tasks?" This paper will focus on, and extend, the part of the earlier study that concerns task-solving strategies. The results indicate that focusing on what is familiar and remembered at a superficial level is dominant over reasoning based on mathematical properties of the components involved, even when the latter could lead to considerable progress.

1. MATHEMATICAL REASONING IN SCHOOL TASKS

1.1. *Reasoning Structure.*

Ross (1998) responds on behalf of the MAA, to the NCTM Commission on the Future of the Standards' questions concerning proof and mathematical reasoning: "One of the most important goals of mathematics courses is to teach students logical reasoning. This is a fundamental skill, not just a mathematical one. [..] It should be emphasised that the foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning. [..] If reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense." It is probably not controversial to accept Ross' position, at least in the interpretation that reasoning is a fundamental component in mathematics. Authors of recent articles have expressed their concern about students' difficulties in handling mathematical proofs (Hanna and Jahnke (1996), Hoyles (1997)). Proof is indeed central in mathematical reasoning but this paper will focus on two other aspects of such reasoning: (i) Plausible reasoning, which is defined below as an extended and "looser" version of proof reasoning, but still based on mathematical properties of the involved components. (ii) Reasoning based on established experiences from the learning environment, which might be mathematically superficial.

Solving a mathematical task can be seen as solving a set of sub tasks of different grain size and character. If the sub task is not routine, one way to describe the reasoning is the following four-step structure.

(1) A *problematic situation* is met, a difficulty where it is not obvious how to proceed.
(2) *Strategy choice*: One possibility is to try to choose (in a wide sense: choose, recall, construct, discover, etc.) a strategy that can solve the difficulty. This choice can be supported by *predictive argumentation*: Will the strategy solve the difficulty? If not, choose another strategy.

(3) *Strategy implementation*: This can be supported by *verificative argumentation*: Did the strategy solve the difficulty? If not, redo 2 or 3 depending on if one thinks the problem is in the choice or in the implementation of the strategy.
(4) *Conclusion*: A result is obtained.

The term *reasoning* is defined as the line of thought, the way of thinking, adopted to produce assertions and reach conclusions. *Argumentation* is the substantiation, the part of the reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate.

1.2. *Plausible Reasoning.*

Without attempting to provide a precise definition, the type of mathematical tasks that students normally meet in their textbooks and exams will be labelled *school tasks*. One crucial distinction between school tasks and the professional use of mathematics, is that within the didactical contract (Brousseau 1984) of school one does not always have to be certain that the result is correct. One is allowed to guess, to take chances, and use ideas and reasoning that are not completely firmly founded. Even in exams, it is acceptable to have only 50% of the answers correct and, if you do not, you will get another chance later. But it is absurd if the mathematician, the engineer, and the economist are correct only in 50% of the cases they claim to be true. This implies that it is allowed, and perhaps even encouraged, within school task solving to use forms of mathematical reasoning with considerably reduced requirements on logical rigour.

A way to characterise this aspect of school task reasoning is indicated by Pólya (1954 pp. v-vi): "We secure our mathematical knowledge by *demonstrative reasoning*, but we support our conjectures by *plausible reasoning*. [..] In strict reasoning the principal thing is to distinguish a proof from a guess, a valid demonstration from an invalid attempt. In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess." In an attempt to relate Pólya's ideas to the discussion above, a version of the reasoning structure (1-4) will be called *plausible reasoning* (abbreviated PR) if the argumentation:

(i) is founded on mathematical properties of the components involved in the reasoning, and
(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

PR includes proof as a special case, with the distinction that proof requires a higher degree of certainty in (ii). For example, in a task solving situation one might make progress by PR, without formally proving ones ideas:

(1) Problematic situation: T is a calculus maximisation task. What shall be done to solve T?
(2) Strategy choice: If one sees the graph of a function as hills and valleys, a maximum is found at the top of a hill. At the top the slope is zero, and the slope is described by the derivative. So T is solved by examining the points where $f'(x)=0$.
(3 and 4) Strategy implementation and conclusion: If the task solver is familiar with this procedure, the rest is straightforward.

1.3. *Reasoning based on Established Experiences.*

Let the strategy choice above be replaced by:

(2) Strategy choice: The solution to all maximisation tasks I have solved have been found where $f'(x)=0$. So T is solved by finding where $f'(x)=0$.

Then the reasoning is not classified as PR, since the argumentation in (2) is based on established experiences from the learning environment, and not on mathematical properties of the components involved. A version of the reasoning structure (1-4) will be called *reasoning based on established experiences* (abbreviated EE) if the argumentation:

(i) is founded on notions and procedures established on the basis of the individual's previous experiences from the learning environment, and
(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

The reasoning concerns the transfer of properties from one familiar situation, to another (task solving) situation that has at least superficial resemblance to the familiar situation. It may not be possible to decide only from a person's behaviour whether the reasoning is EE or not. This is determined by the underlying thoughts of the person. It is important to stress that EE is not the same as rote learning, and solving routine exercises by following procedures and mimicking examples. An EE approach is often applied in a problematic situation, which is non routine to some extent, by trying to relate the strategy choice and implementation to something familiar.

2. RESEARCH QUESTIONS AND METHOD

2.1. *Research Questions.*

This study is based on the following questions:

Q1: In what ways do students manage or fail to engage in PR as a means of making progress in solving tasks in school? What are the roles of EE in these situations?

Q2: In the situations where the students make or could have made progress by PR or EE, what types of competencies are present and absent?

2.2. *Related Research.*

Earlier studies have described students' reluctance to base their work on "mathematical grounds" as one of the main causes behind task solving difficulties.

Vinner (1997) suggests a theoretical framework where two of the main notions are *analytical* and *pseudo-analytical* behaviour, respectively. The latter is defined as a behaviour that is not analytical, but which in routine task solving might give the impression of being analytical, and could even produce correct solutions. The examples in Vinner's article, characterised as analytical and pseudo-analytical, could essentially also be described as PR and EE, respectively. The main theoretical difference between Vinner's term analytical and PR is that the latter addresses the degree of certainty in the reasoning. EE can be seen as one type of the more general pseudo-analytical behaviour. Pseudo-analytical is defined by what it is not (analytical), but EE is more narrowly defined. One of Vinner's main points is that this kind of difficulties in solving routine tasks may often be better understood if they are interpreted within his "non-cognitive" framework, than if they are seen as misconceptions within the domain of meaningful contexts.

EE may be related to the *keyword approach* in task solving. Hegarty et al. (1995) describes this keyword strategy in the context of arithmetic word tasks: "In the short-cut approach, which we refer to as direct translation, the problem solver attempts to select the numbers in the problem and key relational terms (such as "more" and "less") and develops a solution plan that involves combining the numbers in the problem using the arithmetic operations that are primed by the keywords (e.g., addition if the keyword is "more" and subtraction if it is "less"). Thus, the problem solver attempts to directly translate the key propositions in the problem statement to a set of computations that will produce the answer and does not construct a qualitative representation of the situation described in the problem." Their study shows that the unsuccessful task solvers use keyword strategies, whereas the successful task solvers base their solution plans on models of the situations in the tasks. In more advanced mathematics like calculus the tasks are more complex than in arithmetic and there is a multitude of potential solution strategies involving a multitude of different components. Here EE might be seen as attempting to select more general "key connections" between a task and ones established experiences from (perhaps superficially) similar situations in the learning environment, in order to develop a solution strategy without constructing a qualitative representation of the task.

In a study of students' notion of proof, focusing on how they arrive at their conviction of the validity, Balacheff (1988) singled out two types: *Pragmatic "proofs"* are about "showing" that the result is true because "it works". These do not actually establish the truth of an assertion but are often believed to do so by their producers. *Conceptual proofs* concern establishing the necessary nature of the truth by giving reasons. Though both types could be based on mathematical properties, Balacheff found a clear break between the two concerning levels of sophistication. Other researchers have also described students' difficulties in differing proofs from other less rigorous types of argumentation (Chazan 1993, Hoyles 1997).

Schoenfeld (1985, p. 358) describes, in a study of geometrical problem solving, students' focus on methods that he labels *naive empiricism*: To test ideas by constructing figures, and then determine the correctness of the ideas by the shapes of the figures. This approach often caused different types of failure. Often the students did not attempt to use the mathematical properties of the objects to construct some kind of deductive reasoning, even though their resources were sufficient and proper reasoning could have helped them to make considerable progress.

2.3. *Analysis Framework.*

The framework for Q1 is given in section 1 above. A suitable theoretical tool for Q2 is provided by Schoenfeld's (1985, 1992) structuring of problem solving behaviour in the four categories *resources*, *heuristics*, *control,* and *belief.* Schoenfeld has convincingly shown that all four components have fundamental influence on progress in problem solving. The structure is summarised by Schoenfeld (1985 p. 15):
"*Resources:* Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand. Intuitions and informal knowledge regarding the domain. Facts. Algorithmic procedures. "Routine" nonalgorithmic procedures. Understandings (propositional knowledge) about the agreed-upon rules for working in the domain.
Heuristics: Strategies and techniques for making progress on unfamiliar and non-standard problems: rules of thumb for effective problem solving, including: Drawing figures; introducing suitable notation. Exploiting related problems. Reformulating problems; working backwards. Testing and verifications procedures.
Control: Global decisions regarding the selection and implementation of resources and strategies. Planning. Monitoring and assessment. Decision-making. Conscious metacognitive acts.
Belief Systems: One's "mathematical world view", the set of (not necessarily conscious) determinants of an individual's behaviour. About self. About the environment. About the topic. About mathematics."

2.4. *Method.*

The students in the examples were at the end of their first semester of university studies in mathematics. Their examination results were average or slightly above.

The two school tasks (see below) are neither pure routine tasks, nor completely non-routine genuine problems. The main purpose behind the choice of these tasks was to create task solving situations, where opportunities existed to show both competence in choosing from a multitude of familiar facts and procedures, and competence in handling novel problem solving situations by PR or other types of constructive reasoning. By doing so it was possible to study the balance between PR and EE in the student's work. The calculus context ensured that the students had met this multitude of related facts and procedures so that EE approaches were possible.

The students who volunteered worked on the tasks in the presence of a video camera, but working alone apart from my help (see the "Description" sections). They were informed in advance that they should try to "think aloud", but otherwise act as close as possible to their usual way of working. They where given as much time as they needed, the sessions normally lasted 30-60 minutes. The episodes presented in the "Description" sections are fairly complete to provide the reader with enough data to be able to question the analysis and conclusions. Some parts that are omitted here are described in (Lithner 1998). The analysis proceeded in two steps:

(1) After a session I tried to interpret the tape and describe not only what was taking place, but also why this was happening, speculating in how the student was thinking. In order to increase the reliability of these speculations, I then (not later than three days after the session) met the student, and we went through my written interpretation while viewing the videotape. Then the student could make comments, and also suggest other ways to interpret and explain the situations. The revised interpretations are presented in the "Interpretation" sections below.

(2) The second step of the analysis is focused more closely on the questions Q1 and Q2 above. These analyses are presented in the "Analysis" sections, and are based on the PR/EE and resources-heuristics-control-belief frameworks, but are not commented on by the students.

3. EMPIRICAL DATA AND ANALYSIS

The work of each student will be presented in the three parts Description, Interpretation, and Analysis. The two tasks are presented to the students in the same written form as below (translated from Swedish essentially word for word). Minor passages that are omitted in the quotations are replaced by [..] , and pauses are indicated by ... .

3.1. Description of Alf's Work with Task 1.

Task 1: A company produces x units of a commodity per year, where x belongs to [400,600]. The estimated production cost is approximately $-2x^2 + 2000x - 420000$ kr/unit, and the expected sale price approximately $-x^2 + 700x$ kr/unit. How many units should be produced each year to maximise the yearly profit?

Alf starts by reading the task aloud and shows no sign of hesitation. He says that "It feels a bit familiar" and writes down the given formulas:

$$-2x^2 + 2000x - 420000 \quad -x^2 + 700x \quad x \_ [400, 600]$$

After some thinking he states that "the sale price minus the production cost is the profit" [note that this yields the profit per unit, not the profit per year] and has no difficulties in constructing and simplifying the [within his faulty interpretation, correct] profit function

$$V = x^2 - 1300x + 420000$$

Alf differentiates this, solves V'(x)=0 to find x=650.
"Now we can use the second derivative test to find out if it is a max or a min." He easily finds V"=2, but then gets a bit puzzled:
"This is strange. It feels like it should be a minimum if I remember rightly, since the second derivative is positive." After a while Alf decides that:
"I could skip this second derivative test. There are other possibilities if I don't remember wrong. For example, to check what the derivative looks like close to ... [meaning x=650]". Alf works swiftly, finds that V'(600) is negative and V' is positive to the right of x=650, and draws two arrows [the first derivative test]:

$$\searrow \nearrow$$

"It feels like it becomes a minimum! No, wait a minute, what am I doing?" Alf spends a few minutes not really being able to decide whether to examine the derivative or the function itself.
"Normally I would just accept it [x=650] as an answer, it feels like a rather good answer in some way. ... Maybe I should check this."
Alf uses a calculator and finds that

$$V(650) = -2500$$

"I'll be damned if it was not negative! [..] Then we can assume that this [x=650] isn't so good." After some thinking he remembers that he has to check the endpoints of the definition interval, and after some routine work he states that "this implies that we have the maximum profit for 400 units". 15 minutes have passed since he started, and 10 minutes since he found that V"(x)=2.

3.2. *Interpretation.*

Alf expects to find the answer by finding the zeros of the derivative V'(x). When he finds that V"(x)=2 he is puzzled by two contradictions:

(i) His (faulty) expectation that the answer (maximum) is found where V'(x)=0, at x=650.
(ii) He (correctly) believes that V"(650)=2 implies that x=650 is a minimum.

There are two interacting reasons behind Alf's decision to dismiss (ii):
(a) The expectation (i) is much stronger than his conviction that (ii) is true.
(b) He remembers that the second derivative test states (ii) but does not know *why* it is true. Alf does not try to complete or test his memory by some kind of mathematical reasoning.
Taken together, (a) and (b) mean that the only way out, that occurs to him, is to dismiss (ii) and search for other methods to verify that (i) is correct. If he had made some miscalculations, and for example found V"(650)=-2 (which would imply that x=650 yields a maximum), he would have considered the task finished.

Another method that Alf applies in order to determine the characteristics of the critical point is the first derivative test. He again reaches a contradiction, this time between (i) above and:
(iii) V(x) is decreasing to the left of x=650 and increasing to the right, which implies that x=650 is a minimum.
Once again, the expectation (i) is dominating and he questions here if the correct method in (iii) is applicable at all. He is then rather puzzled and searches at first for other methods to verify that the answer is x=650, but cannot find any. It is not until he tests the value of V(650) and to his surprise finds that it is negative, that he questions (i).

3.3. *Analysis.*

Reasoning Structure.
The reasoning structure in section 1 can be applied to Alf's work:
(a1) Problematic situation: What shall be done to solve task 1?
(a2) Strategy choice: By his established experiences, maximisation tasks are usually solved by finding where V'(x)=0.
(a3) Strategy implementation: Alf finds that V'(650)=0, and tries to verify that x=650 is the correct answer by applying the familiar second derivative test. Alf's reasoning reveals that a contradiction between (i) and (ii) is reached, but the reasoning is too limited to reveal why and he returns to (2).
(b2) Strategy choice: If the second derivative does not verify that x=650 is the answer, other familiar methods should be searched for.
(b3) Strategy implementation: The first derivative test is recalled and implemented according to familiar procedures. A contradiction between (i) and (iii) is reached, and once again he returns to (2)
(c2) Strategy choice: He is puzzled since he does not understand why his approach does not work, and has run out of familiar procedures to apply.

Therefore, his approach is more open and he turns to exploring properties of the expected answer.
(c3) Strategy implementation: One of the easier properties to explore is the size of the profit. By Alf's verificative argument, a negative profit is not reasonable, and here he questions (a2).
(d2) Strategy choice: By (c3), the answer is not found where V'(x)=0. Alf tries to recall from his experiences if there are any other familiar types of answers.
(d3) Strategy implementation: He remembers that maxima may be found at the interval endpoints, and implements the familiar method for testing this.
(d4) Conclusion: The answer is found at x=400.

There are instances that may be characterised as PR, for example (a3), when Alf realises that there is a contradiction between (i) and (ii). This reasoning is founded on mathematical properties of second derivatives. At the same time, the base for the reasoning is superficial (memory, not understanding) and limits its range: it reveals a contradiction but does not lead forward. Another instance is (c3), when he sees a negative profit as unreasonable. It is noteworthy that both the contradictions between (i) and (ii), and between (i) and (iii) *are* logical contradictions, but he *does not* question (i). To do so, he would have to apply PR and consider the mathematical properties of (i), (ii) and (iii). The "contradiction" between (i) and the statement "x= 650 gives a negative profit" *is not* a logical contradiction since profits may be negative, but here he *does* question (i). This time he does not have to consider mathematical properties, but can apply EE: profits are usually positive.

The main characteristic of his work is that the PR instances are all of very limited range. The overall reasoning that guides his strategy choices and implementations is mainly based on EE, without applying constructive reasoning based on mathematical properties of the components involved. This seems to be one of the main reasons for his difficulties. Alf's work might have progressed more steadily and effectively if the role of PR had been more influential. This could have been achieved in many ways, see "heuristics" below for examples. One approach could be to try to construct a deeper understanding of the second derivative test. This is unnecessarily laborious if the purpose is merely to solve task 1, but in a wider perspective the gained understanding may be useful outside the task. An imagined PR approach might be:

(1') Problematic situation: The uncertainty about the second derivative test. What does V'(650)=0 and V"(650)=2 imply concerning the shape of the graph of V(x) close to x=650? In particular, is there a max, a min, or something else?

(2') Strategy choice: V'(x) can be visually interpreted as describing the slope of V(x), and V"(x) as describing the slope of V'(x). Combine this with the information about V'(650) and V"(650) to describe the shape of V(x).
(3') Strategy implementation: V"(650) is positive, so V' is increasing. Since V' is zero at x=650, V' is negative to the left and positive to the right of x=650. The graph of V is U-shaped, since V' describes its slope.
(4') Conclusion: There is a minimum at x=650.

V(x) is a specific function but it is used as a generic example, and the reasoning concerns general principles of second derivatives.

Resources.
Alf is very familiar with the action parts of the maximisation procedure as they are treated in the textbook of the course: He swiftly differentiates his function V(x), solves V'(x)=0 and applies the second derivative test. So far it is purely routine work, which he masters. It is not until Alf meets a situation that lies outside his established experiences, the contradiction between (i) and (ii), that he starts hesitating about how to proceed. There are several possibilities to resolve this situation, where Alf has the required resources. He could for example have questioned (i), or drawn a graph.

Alf's insufficient understanding of the underlying concepts can be seen as lacking resources. He says in the interview, after the task solving session, that one reason why he was uncertain about the second derivative test, was that he never understood why it was true and therefore he could not check (ii) by some kind of reasoning. It is possible that his concept image (Tall and Vinner 1981) of the second derivative contains no visual component, and this makes it difficult for him to immediately "see" if his formulation (ii) seems correct. It seems like his statement image ("a unifying extension of the idea of concept images which we regard as statement images corresponding to definitions", Selden & Selden 1995) of the second derivative test consists only of the action of applying the test to an algebraically represented elementary function. Perhaps no visual component is included, for example like the visual interpretation in the imagined PR example (1'-4') above.

Could Alf by himself have constructed a more solid statement image of the second derivative test by PR? At a first glance, the statement image of the second derivative test and the concept image of derivatives needed in the imagined example (1'-4') may seem much more solid than Alf's, but possibly they are not: In (1'-4') PR and "old knowledge" is used to construct new knowledge. The resources required in (1'-4') may be exactly the same as Alf's when entering task 1. After finishing task 1 Alf's resources are not increased, except in the sense that he will more likely remember to check the endpoints of the definition interval. In (1'-4'), on the other hand, the resources are increased by developing both the concept image of derivatives and the statement image of the second derivative test. Of course, (1'-4') is more complicated than Alf's approach, but in another episode below, the student Jan is able to make quite a jump towards more advanced PR when guided.

Heuristics.

One of the main heuristic strategies in many calculus tasks is to draw a graph of the function involved. The ideas about maxima and minima of functions are very well represented in a graphical exposition, at least as a complement to the algebraic representation. A graph of V(x), especially if restricted to the interval of definition, could be a source for discovery and a base for PR. The graph would show an U-shaped parabola with a minimum, not a maximum as Alf expects, at its only critical point x=650. He had the resources to do this by hand, and he knew how to use the graphic calculator that lay beside him. One reason for his reluctance to draw a graph might be that his experience of these types of tasks is mainly based on practising the maximisation routine procedure, where drawing graphs may be seen as a waste of time. Another heuristic strategy is to look at the more general case: What is known about second degree polynomials, in particular with positive second degree coefficients? What types and how many different extremal points may they have? Alf has met many second degree polynomials in different contexts during his secondary and tertiary studies.

Control.

In one way Alf's control is very efficient, if one accepts the position that Alf's EE strategy choice is to apply his interpretation of the familiar maximisation procedure. He immediately notices when something comes up that does not fit with his expectation (i). He curtails his unsuccessful attempts to resolve the contradictions between (i) and (ii), and then also between (i) and (iii). He searches in his resources for familiar methods to apply, and probably also finds all the available ones. The two methods he tries to apply are in fact the main ones in the textbook.

In other ways Alf's control is poorer: The main problem is that he does not question his faulty expectation (i) earlier. There are no signs found of planning or evaluation of progress that could lead to other approaches, for example PR.

Belief.

Many teachers and calculus textbooks treat the background of the second derivative test as an introduction to it, but when it comes to the students' part of the work (the exercises) the focus is almost entirely on applications of the test. The students are not given many opportunities to practice PR in connection with the underlying properties of the second derivative test, and this could lead to the belief that this type of PR is not useful in task solving. Maybe this is also commonly the case with other mathematical ideas at school? Schoenfeld (1992 p. 359) describes that students' beliefs about the nature of mathematics are very influential on their behaviour. To carry through (1'-4') one has to see PR as a useful task solving tool.

3.4. *Description of Jan's Work with Tasks 2a and b.*

Task 2: The function f(x) has the graph below.
a. Sketch the graph of f'(x).
b. What is f(-2), f'(0) and f''(2) ?
c. Sketch the graph of g(x), if g'(x)=f(x) .

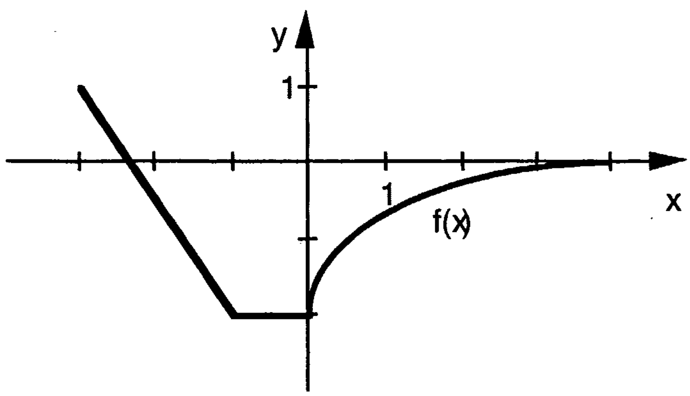


Figure 1.

When sketching f'(x) in the xy-plane, Jan swiftly draws approximately the line y=-2 for x _ (-3,-1), see Figure 2. Then he starts to hesitate about how to draw the graph for x _ (-1,0):
"And then ... there is a jump ... it is zero here ... let's see ..." Jan is moving his pen around and he is drawing the line y=0 in the air, just above the paper with the picture of Figure 2. After a period of silence JL asks Jan what he is thinking of.
"What the derivative looks like ... in this constant interval. Here (he points at the line y=-2 in Figure 2), I have figured out that it is constant but negative." Jan is silent for half a minute and then continues, by leaving the interval (-1,0) and turning to the next one:
"And here, the derivative is like this." He draws swiftly the graph for x _ (0,4), thus producing a picture that looks approximately like Figure 2 (he has not yet drawn any curve for x _ (-1,0)).

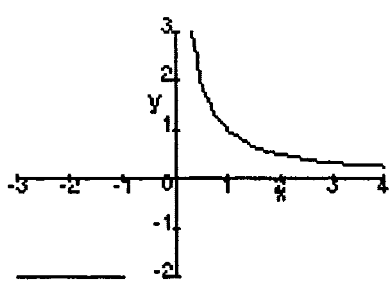


Figure 2.

"Here it is ... (he points at the graph of f(x) in Figure 1 at the interval (-1,0)) it is some function that is ... this function on that interval is -2." He writes this down:

$f(x) = -2.$

"And then the derivative of this is zero" he continues and writes:

$f'(x) = 0.$

Without hesitating he adds the line y=0 for x _ (-1,0) to Figure 2, and then immediately turns to verify the shape of the curve for x _ (0,4).

In part b he has essentially no difficulties in estimating f(-2). Turning to f'(0) he becomes more hesitant:
"If one looks at the figure (he points at his graph of f'(x)) ... we have a jump ... it never gets zero." JL asks him to clarify what he means.
"This (Jan's pen traces his graph of f'(x) along the curve as x goes from 4 towards 0) will never cross the y-axis, this derivative, for positive x. On the other hand, this (his pen traces his graph as x goes from -1 towards 0) will be 0 all the way to ... x equal to ... zero ... because it depends ... Now I am uncertain. It should either be zero or not defined. [..] The question I ask myself is whether this (points again at the line y=0) goes all the way into zero?" After half a minute of thinking he continues:
"It seems that it does not, since there is sort of a corner on the function ... which means that it does not exist." Jan writes down "f'(0) does not exist" and then immediately turns to the question about f''(2).

3.5. *Interpretation.*

Jan's resources contain two methods of extracting information about f'(x) from f(x):
(I) To view f'(a) as the slope of the tangent line to f(x) at x=a.
(II) To differentiate algebraic expressions of elementary functions by algorithmic rules.
At first, Jan uses (I) in a dynamic translation from f(x) to f'(x) on the separate intervals, and demonstrates his skill on the difficult interval (0,4). Jan hesitates when trying to determine f'(x) on the interval (-1,0). The situation contains, as he sees it, two contradictory statements:
(i) f'(x) should be zero on (-1,0), since the slope of f(x) is zero.
(ii) The graph of f'(x) that he wants to draw is unfamiliar since it is not continuous on (-3,4). Anything unusual is to Jan probably wrong.
Jan doubts if (i) is correct and turns to the more familiar method (II) above. He has no difficulties in implementing it, and considers himself finished. In b Jan is trying to determine the value of f'(0) from his graph of f'(x), but does not really know how to do this. He searches his memory and remembers after some thinking that if the graph of f(x) has a corner, like $|x|$ at x=0, then f'(x) does not exist.

Jan does not reflect over whether his problem with the two contradictory statements (i) and (ii) above is resolved. The mere familiarity of the algorithmic method is enough to convince him that he is correct. It is clear that Jan has not actually resolved the contradiction, this can be seen in his difficulty with task 2b. This difficulty with determining the existence of f'(0) is resolved when he remembers that derivatives do not exist at corners. Later he says that he does not really know why this "rule" is true, but it does not matter since he now can relate to something familiar and is fairly convinced that he is correct.

3.6. *Analysis.*

Reasoning Structure.

There are few PR instances included in Jan's work (one example is when method (I) above is applied in task 2a on the interval (0,4)). If instead of asking himself what the derivative looks like, he had first considered whether the derivative was defined, he could have proceeded by theoretically based PR:

(1) Problematic situation: Is f'(x) defined on (-3,4)?
(2) Strategy choice: Apply mathematical properties of the definition of the derivative, to see where on (-3,4) f'(x) exists.
(3) Strategy implementation: The derivative is determined pointwise. That is, f'(x) exists iff lim h_0 (f(x+h)-f(x))/h exists. This means in particular that the quotient should approach the same value when h_0- and h_0+. This condition is fulfilled everywhere on (-3,4), except at x=-1 and x=0. (As an alternative, a similar argument may be produced in a visual, less formal mode by considering the line through the points (x,f(x)) and (x+h,f(x+h)) that approaches the potential tangent line as h_0- and h_0+).
(4) Conclusion: f'(x) is defined everywhere on (-3,4) except at x=-1 and x=0.

The EE approach that convinces Jan that he has found the solution of task 2a can be structured as:

(1') Problematic situation: Consider Figures 1 and 2. What does f'(x) look like on (-1,0)? It ought to be zero, but the graph of f'(x) will then look unusual.
(2') Strategy choice: The safest way to determine derivatives is by the familiar differentiation algorithm.
(3') Strategy implementation: The function is estimated to be f(x)=-2. Then by the differentiation algorithm f'(x)=0.
(4') Conclusion: f'(x)=0 on (-1,0).

In task 2b, Jan initially takes a PR approach. He tries to determine the value and existence of f'(0) by considering what happens to his graph from task 2a, when x tends to zero from left and right. However, the attempt fails since his reasoning is not founded firmly enough in mathematical properties of the derivative concept. To just look at the graph and see if it "goes all the way" is not a feasible approach. Instead, the task is solved by a EE approach:

(1") Problematic situation: Task 2b, f'(0).
(2") Strategy choice: Search for similar familiar situations, compare with familiar graphs.
(3") Strategy implementation: At corners, derivatives do not exist.
(4") Conclusion: f'(0) does not exist.

The two EE approaches above result in fairly correct answers and Jan seems convinced that his work is satisfactory, at least in the sense that he does not spend any time trying to check his result. It does seem clear though, that he has not tried (or managed) to construct any reasoning that addresses the basic considerations concerning existence and continuity that underlie his difficulties.

Resources and Belief.
A function that is not defined or not continuous on the whole interval (-3,4) is peripheral in Jan's learning environment, and Jan suspects that f'(x)=0 on (-1,0) is wrong. This situation can be related to (Niss 1998): "For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which the concept has been concretely exemplified and embedded for that particular student."

To construct the theoretically based PR above, Jan needs experience in handling theoretical reasoning. He has in his undergraduate calculus courses met the theoretical tools needed, but they are often mainly used as definitions when introducing concepts or proving theorems, and seldom used to solve the textbook exercises which constitutes the bulk of Jan's study work. As Niss (1998) puts it: "For example, even if students who are learning calculus or analysis are presented with full theoretical definitions [..], and even if it is explicitly stated in the textbook and by the teacher that the aim is to develop these concepts in a general form [..], students actual notions and concept images will be shaped, and limited, by the examples, problems, and tasks on which they are actually set to work". In Jan's learning environment there is a very limited need for using the formal theoretical definition of the derivative to solve tasks. This may cultivate the belief that theoretically based PR is important for proving theorems but not for solving tasks.

Heuristics and Control.
Perhaps one could call control strategies heuristic which "start by investigating what kinds of answers may exist, to avoid searching for a type that does not". Jan's initial strategy is to search for the answer, without any explicit existence considerations. When he later finds that the answer he wants to construct leads to a contradiction between (i) and (ii), his attempts are founded on far too weak theoretical grounds to investigate how the existence is determined. The strategy that Jan actually applies in addition to methods (I) and (II) above, could be called "compare with familiar function graphs". Another instance that may be characterised as insufficient control, is that Jan does not consider whether his doubts concerning the contradiction between (i) and (ii) above have been resolved. He is convinced by the familiarity of the EE method.

His difficulties are in a sense actually caused by active control, but on weak grounds: If he had not bothered about that the graph he first wants to draw looks unfamiliar, and just drawn the line f'(x)=0 on (-1,0), he would probably have received a fairly good grading if the task had been included in an examination. There are no signs found that his hesitation is grounded in mathematical reasons, or that he searches for mathematical reasons to clarify it.

3.7. *Description of Per's Work with Task 2c.*

Per starts by looking at the graph of f(x) in Figure 1, saying "if this is the derivative, I am supposed to find the function to it". After a while he continues: "Here we have a decreasing derivative, between x=-3 and x=-1 ... The question is what that yields? ... One could estimate the function here (points at the graph of f(x) in Figure 1 for x _ (-3,-1)), in a similar way as I did when I had ... the second derivative. The slope here ..." He writes:

y = 3/2 x + [It should be y = -3/2 x +]

"If I shall estimate what ... it shall be a constant here ... I can get this by ... I can choose this point ... x=-1 and y=-2." He then writes:

-2 = 3/2 (-1) + m

m = -2/(-3/2) = 4/3 [He divides instead of subtracting]

He says hesitantly that this does not seem to fit, because if he extends the line in Figure 1 it should cross the negative y-axis. After some silence JL asks what he is doing.

"I am trying to discover my mistake, but I can't figure out what it is that doesn't fit ... OK, we'll say that it shall be -4/3." He writes

y = 3/2 x - 4/3

He notes that this does not fit with the graph of f(x) in Figure 1, from which he estimates m to be "less then -3". He searches, but does not discover his mistakes. After a while JL points them out to him, and says that it is OK to estimate m ª -3.

When writing down the corrected formula for y, Per makes yet another mistake:

$y = -3/2\, x + 3$ [It should be -3]

He easily finds a primitive function to this:

$g(x) = -3/4\, x^2 + 3x$

JL assists Per in sketching g(x) on a graphic calculator, and Per produces Figure 3.

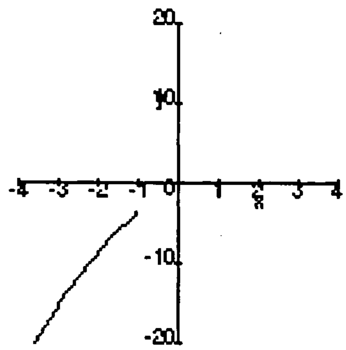


Figure 3

Three seconds after Per has completed Figure 3, he turns to sketch g(x) on the next interval.

"And then ... let's see ..." JL interrupts and asks if he is finished with the interval.

"Yes, I guess we can say that."

"Can you make some kind of estimate to see if your graph is reasonable?"

Per hesitates a little at first, but after convincing himself that the graph in Figure 3 corresponds to his g(x) he is satisfied.

When asked by JL, Per can without hesitation describe which one of the graphs (Figure 1 or 3) that is the derivative, and which one that is the function. JL continues:

"If you look at this function (JL points at Per's graph of g(x) in Figure 3, at approximately the point (-2,-10)), what is the derivative here? Can you say if it agrees with the derivative as given here (JL points at the graph of f(x) in Figure 1)?"

"No, it does not. The derivative is positive all the way (points at Figure 3), and it should be negative here (points at the graph in Figure 1, for x _ (-2.3,-1))."

Per makes some additional comments specifying the interval, and JL asks:

"And before this the derivative is?"

Per answers without hesitation: "Positive."

"Can you from this make a very rough sketch of what the function should look like?"

It takes Per about half a minute to provide a rough but reasonable sketch of g(x) at the interval (-3,-1), see Figure 4.

Figure 4.

He motivates his figure without hesitation: "The derivative is positive here (Per points at the left part of Figure 4), and then it gets zero (points at the maximum at Figure 4, and at the point (-2.3,0) at the graph of f(x) in Figure 1), and then it is negative."

3.8. *Interpretation.*

Per's plan is as follows: Since f(x) on (-3, -1) is a line and thus of the form f(x)=kx+m, determine k and m. Integrate kx+m by the familiar procedure, which yields g(x). Per makes several careless mistakes, his focus is to use familiar procedures and algorithms, and there are essentially no checking comparisons with other types of reasoning that might have detected the errors. The only exception is when Per notes that the function he has reached, y=3/2 x + 4/3, does not fit with the graph of f(x) in Figure 1. He makes a very superficial search for the error but cannot find it, partly because he is disturbed by the test situation. As a consequence of the mistakes Per produces an incorrect graph, but he does not consider checking this. Per has, up to this point, worked for 20 minutes with task 2c, mainly with familiar elementary algorithmic methods. When asked to check his answer, he just searches for possible errors in the *translation* from the algebraic representation of g(x) to its graphical representation, and not in the *construction* of g(x) where the main part of his work and the mistakes lies. Per does not consider what is asked for, how his graph relates to the information in the task, or if his methods are correctly chosen and implemented.

When Per is "mildly guided" first into checking his answer against the graph of f(x) in Figure 1, and then into making a rough estimate of g(x), he makes this swiftly and skilfully. Per cannot afterwards explain why he did not think of this himself, but it is probably caused by the fact that this type of mathematical reasoning is unusual in his learning environment. He says that he feels more at home when trying to apply more "exact" familiar algorithmic methods.

3.9. *Analysis.*

Reasoning Structure.
Per's work can be structured as:
(1) Problematic situation: Task 2c on the interval (-3,-1).
(2) Strategy choice: Primitive functions are found by the familiar integration procedure. It is applicable on algebraic representations of

functions, and this representation of the line f(x) can be obtained by the familiar procedure for determining k and m in y=kx+m.
(3) Strategy implementation: The implementation could have been straightforward, but many careless mistakes were made. There is no attempt at verification.
(4) Conclusion: Figure 3.

The strategy choice is founded on constructive reasoning, but the range of the reasoning includes only how to combine familiar procedures, not how to use mathematical properties of the involved components in a wide range PR approach. Per's approach is rather reasonable, but his work would probably have progressed more accurately if he had continuously compared his algebraic expressions of f(x) and g(x) with their graphical representations. One exception (of limited range) when he does this, is when he notes that m=4/3 does not seem to fit. As an example of PR with wider range, Per could from the start have made a rough estimate of the shape of g(x) (Figure 4). This could either have been developed to a more precise estimate, or could have served as a guide for his arithmetic manipulations. The PR Per produces when guided can be structured as:

(1') Problematic situation: Is Figure 3 the correct answer?
(2') Strategy choice: Assume Figure 3 is correct, then check if the derivative of Figure 3 is given by Figure 1.
(3') Strategy implementation: The derivative of the graph in Figure 3 is positive on (-3,-1), but f(x) is not.
(4') Conclusion: Figure 3 is not a correct answer to task 2c.

(1") Problematic situation: Task 2c.
(2") Strategy choice: Reformulate the task to be able to apply reasoning similar to the above: What should the graph of a function g(x) look like that has the function f(x) as its derivative?
(3") Strategy implementation: f(x) which describes the slope of g(x) is positive at x=-3, then decreases to zero, and then decreases to become negative. In other words, g(x) is first increasing, then horizontal, then decreasing.
(4") Conclusion: Figure 4.

Resources and Heuristics.
Per's resources are sufficient to carry through everything he does above, both in the first EE part by himself and in the second PR part when guided. His first approach could have succeeded, the methods he chooses are appropriate and he knows the algebra and the familiar integration procedure. The problem is that there are so many steps in the calculations, and so many opportunities to make careless mistakes. The guidance given in the second part does not help Per with the resources, it only leads him into another heuristic strategy:

The approach taken when guiding Per and formulating the strategies (2') and (2") above can be labelled "Reformulating the problem by assuming you have a solution and determining its properties", (Schoenfeld 1985 p. 109). It is very difficult, probably not within Per's resources, to look at Figure 1 and dynamically construct g(x) by estimating the accumulated increase of f(x) as x goes from x=-3 to x= -1. It is much easier for him to look at a function and estimate the slope, the derivative. Therefore it is also much easier for to him to answer questions like (2') and (2") because they concern slope, instead of accumulated increase.

Control and Belief.
Per's first twenty minutes' EE approach to solve task 2c could have succeeded, and in less time, if it had been complemented by appropriate control in three aspects: More careful calculations, continuous comparisons with figures and verification of conclusions. There are no indications that he asks himself if there are other ways of obtaining information about g(x), or of active attempts to use wide range PR. This is perhaps not seen as useful here by Per, an assumption strengthened by Per's comment in the discussion afterwards: He feels more at home when trying to apply more "exact" familiar algorithmic methods, than the type of PR he is guided into.

4. DISCUSSION

All three students meet extensive difficulties of different kind. Their difficulties, and their progress, can be related to the questions Q1 and Q2 of section 2.1:

EE guides the global strategy choices. PR is local, and dominated by EE.
At a global level the students' strategy choices are mainly based on EE: to apply methods they know from similar tasks. This causes problems when the familiar routines do not work for different reasons. Alf's reasoning is limited by his expectations and narrow approach, Jan's methods cannot resolve the existence problems, and Per's strategy leads to many careless mistakes. The influence from PR on this global level is minimal. Alf does not attempt any PR. Jan tries, but his graphical approach is too superficial. Per does not initiate any PR by himself, but when "mildly guided" he is able to make good progress in short time by wide range PR. There are also imagined examples provided of how wide range PR could have been used.

At a local level the students show examples of progress being made by PR approaches, for example: Alf notes the contradictions between his strategy implementation and his expectation, but he cannot reveal the underlying causes. Jan is skilled in his dynamical translation from f(x) to f'(x), but can not treat the existence problem by this approach. Per once notes there is a misfit between his algebraic expression and the graph, but there are many other similar mistakes that are not detected. These PR instances have in common that they are all too limited to address or resolve the students' main difficulties.

Resources seem to be sufficient.
All three students have the resources needed to correctly ("almost correctly" in Jan's case) solve the tasks with their EE based strategies. As described above, the reasons for their difficulties with the strategy implementations are other than lacking resources.

This study does not say if they master the resources needed to carry through anything like the imagined PR examples or not, since there are no wide range PR attempts detected. One exception is Jan, who fails because his attempt is not feasible at all. The other exception is Per, who clearly has the resources to carry through the guided PR. For the imagined PR examples described, the resources required are well within the scope of the curriculum. One of the main results in Schoenfeld's (1985) problem solving studies, is that students often fail for reasons other than lacking resources.

Heuristics are often applied in limited and inflexible ways.
Sometimes EE strategies are appropriate in task solving, and there is nothing wrong with trying to recall familiar facts, procedures, typical exercises, etc. The particular case of trying to remember similar tasks and use related structures is described as an important heuristic approach by Pólya (1945) under the headline "Do you know a related problem?" One of the reasons behind the three students' difficulties is that their EE focus is not balanced by a greater emphasis on other complementary approaches, for example PR.

One heuristic strategy that is very often applied in calculus in books and by many teachers is "Draw a figure", (Pólya 1945). Alf could have detected his unreasonable expectation with PR based on a rough sketch of V(x). Jan and Per could also have used graphs (in Jan's case completed by theoretical considerations) as a base for PR, but all three seem unwilling to make use of this strategy. Another heuristic strategy is the "reformulation", which is the most crucial aspect of the guiding in Per's PR work. It is also noteworthy that there are many situations where testing and verification procedures are absent or too limited.

Control is mainly founded on familiarity, not mathematical properties.
The main reason for reacting is that the methods used do not provide answers of the expected type, or that the result is unfamiliar. Meeting two logical contradictions, Alf does not question his faulty expectation when PR is requested in order to do so. He reacts stronger to a statement that is actually possible, but unfamiliar. Also Jan's reason for reacting is that his graph looks unfamiliar. Per does not produce anything unfamiliar, and

there are very few control reactions detected. Control that included better evaluation of the EE strategy itself could have led to better progress or to other approaches, for example PR, but then other control questions are necessary: Alf could have asked himself what the second derivative really meant. Jan could have asked for the mathematical reasons behind his hesitation. Per could have tried to search for other ways of obtaining information about g(x).

Belief may be that PR is not central in task solving.
If one supposes that PR is not perceived by the students as one of the main tools when solving tasks, perhaps because PR is not emphasised in the teaching practice they experience, and that the students hence obtain very limited experience in constructing PR, then their behaviour is perfectly natural. This study does not provide evidence to support this assumption, but some "weak" arguments may be given:

• There are few short range and no wide range PR attempts noted.
• Per's comment that he feels more at home with "exact" familiar algorithmic methods, though he was able to construct PR when guided.
• Schoenfeld (1992 p.359) describes that common beliefs are for example that ordinary students cannot expect to really understand mathematics, and cannot by themselves construct anything outside the rules and methods demonstrated by the teacher.
• It seems that the textbook and examination tasks that request the construction of PR are few and almost always the most difficult ones, while the tasks manageable for "normal" students more often ask for the application of some standard procedure.

5. ACKNOWLEDGEMENT

I am grateful to Mogens Niss and Hans Wallin for their valuable comments on earlier versions of this paper.

6. REFERENCES

Balacheff, N.: 1988, 'Aspects of proof in pupils' practice of school mathematics', in D. Pimm (ed.), *Mathematics, teachers and children*, London: Hodder and Stoughton, pp. 216-235.

Brousseau, G.: 1984, 'The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics', in H. G. Steiner (ed.), *Theory of mathematics education*, (occasional paper 54) Bielefeld, IDM. pp 110-119.

Chazan, D.: 1993, 'High School geometry students' justification for their views of empirical evidence and mathematical proof', *Educational Studies in Mathematics* 24, 359-387.

Hanna, G. and Jahnke, N.: 1996, 'Proof and Proving', in A. Bishop et al. (eds.), *International Handbook of Mathematics Education*, Kluwer Academic Publishers, Dordrecht, pp. 877-908.

Hegarty, M., Mayer, R. and Monk, C.: 1995, 'Comprehension of arithmetic word problems: A comparison of successful and unsuccessful problem solvers', *Journal of Educational Psychology* 87, 18-32.

Hoyles, C.: 1997, 'The curricular shaping of students' approaches to proof', *For the Learning of Mathematics*, 17, pp. 7-16.

Lithner, J.: 1998, 'Mathematical reasoning and familiar procedures', to appear in *International Journal of Mathematical Education in Science and Technology.*

Niss, M.: 1998, 'Aspects of the nature and state of research in mathematics education', Documenta Mathematica, Extra volume, ICM 98, III.

Pólya, G.: 1945, *How to Solve it*, Princeton: Princeton University Press.

Pólya, G.: 1954, *Mathematics and Plausible Reasoning,* Vol. I and II, Princeton: Princeton University Press.

Ross, K. A.: 1998, 'Doing and Proving: The place of Algorithms and Proof in School Mathematics', *American Mathematical Monthly*, 252-255.

Schoenfeld, A.: 1985, *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A.: 1992, 'Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics', in D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning*. New York: Macmillan, pp. 334-370.

Selden, J. and Selden, A.: 1995, 'Unpacking the logic of mathematical statements', *Educational Studies in Mathematics* 29, pp. 123-151.

Tall, D. and Vinner, S.: 1981, 'Concept image and concept definition with particular reference to limits and continuity', *Educational Studies in Mathematics* 12, pp. 151-169.

Vinner, S.: 1997, 'The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning', *Educational Studies in Mathematics* 34, 97-129.

4

Mathematical Reasoning in Calculus Textbook Exercises

Johan Lithner

July 10, 2001

Abstract

The aim of this paper is to study some of the strategies that are possible to use in order to solve the exercises in undergraduate calculus textbooks. It is described how most exercises may be solved by mathematically superficial strategies. Strategy choices and implementations can usually be based on identifying similar solved examples and copying, or sometimes locally modifying, given solution procedures. One consequence is that exercises may often be solved without actually considering the core mathematics of the book section in question.

1 Introduction

In two earlier studies [14], [15], first year undergraduate students' reasoning in mathematical task solving were examined. The students worked with mathematical tasks with no aids at hand except a graphics calculator, a situation similar to the one in examinations. The results indicated that a strategy to focus on what is familiar and remembered at a superficial level was dominant over reasoning based on mathematical properties of the components involved, even when the latter could lead to considerable progress. It seemed like the students' beliefs did not include the latter type of mathematical reasoning as a main approach, even though they mastered the necessary knowledge base. Their behaviour seemed to be quite far from the educational goals. At the same time, there were indications that this was the way they were used to working with mathematics in their studies. One question that arises is: In what way (if at all) is this way of working a reasonable outcome of some unbalance in the learning environment?

One of the main components of the learning environment to investigate is the textbook exercises. One reason is that normally at least half of the students' study time is spent working with these (this assertion is based on local unpublished surveys). Another reason is provided by Love and Pimm [16]: "The book is still by far the most pervasive technology to be found in use in mathematics classrooms. Because it is ubiquitous, the textbook has profoundly shaped

[0]The author thanks Mogens Niss and Hans Wallin for their comments on an earlier draft.

our notion of mathematics and how it might be taught. By its use of the 'explanation - example - exercises' format, by the way in which it address both teacher and learner, in its linear sequence, in its very conception of techniques, results and theorems, the textbook has dominated both the perceptions and the practices of school mathematics."

Many authors have described students' focus on memorising procedures, without understanding the underlying central ideas, as an important reason behind learning and achievement difficulties in mathematics, see for example [11], [22], and [24]. It is in a sense 'wellknown' that textbook exercises can often be solved by copying solved examples, but it is still important to: (i) Study some of the 'mathematically superficial' strategies and reasoning possible to use when solving exercises in greater detail in order to learn more about this phenomenon. Even though it is perhaps 'wellknown', it seems like the proportion of these types of exercises are increasing (see below) in textbooks. (ii) Prepare a framework that can be applied to study students actual work with textbook exercises (which is the subject of an ongoing study [13]). The setting is here a 'model situation', where no other learning environment components than the textbook are considered. Most real life learning situations are much more complex and students will be influenced by several other factors, for example other written material, the teachers, peer students, and technology. Therefore it may be too complicated to prepare the framework when studying real life situations.

2 Framework and Research Questions

In [15], the framework below was used in order to analyse the observed students' task solving reasoning. For a more complete discussion, see [15], where related references also may be found.

2.1 Reasoning structure

Solving a mathematical task can be seen as solving a set of sub tasks of different grain size and character. If the (sub)task is not routine, one way to describe the reasoning is the following four-step structure:
(1) A *problematic situation* is met, a difficulty where it is not obvious how to proceed.
(2) *Strategy choice*: One possibility is to try to choose (in a wide sense: choose, recall, construct, discover, etc.) a strategy that can solve the difficulty. This choice can be supported by *predictive argumentation*: Will the strategy solve the difficulty? If not, choose another strategy.
(3) *Strategy implementation*: This can be supported by *verificative argumentation*: Did the strategy solve the difficulty? If not, redo (2) or (3) depending on if the problem is in the choice of the strategy or in the implementation.
(4) *Conclusion*: A result is obtained.

The term *reasoning* is defined as the line of thought, the way of thinking, adopted to produce assertions and reach conclusions. *Argumentation* is the substantiation, the part of the reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate.

2.2 Reasoning characteristics

The following distinction was found to be central in [15]:

A version of the reasoning structure (1-4) is called *plausible reasoning* (abbreviated PR) if the argumentation:
(i) is founded on mathematical properties of the components involved in the reasoning, and
(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

The term *component* includes all mathematical concepts, actions, processes, objects, solution procedures, facts, heuristics, etc. that may be explicitly or implicitly involved in the reasoning. In short, the idea behind (ii) is that in school task solving it is allowed and encouraged to use mathematical reasoning with less requirements on rigour than for example in proof or in professional life. This study can not be restricted to reasoning that is required to be accepted as logically complete and correct (mathematical proof), since this is very seldom produced by students in normal learning situations.

The reasoning structure is called *reasoning based on established experiences* (abbreviated EE) if the argumentation:
(i) is founded on notions and procedures established on the basis of the individual's previous experiences from the learning environment, and
(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

Here the attempt to resolve the problematic situation is based on trying to transfer and combine (possibly incomplete) solution procedures from familiar situations, perhaps superficially and without considering the mathematical properties of the components involved. It may not be possible to decide only from a person's behaviour whether the reasoning is EE or not, this is determined by the underlying thoughts of the person. It is important to stress that EE does not only include rote learning and solving routine exercises by following procedures and mimicking examples. One reason is that the simple keyword strategies that are possible to use in elementary arithmetic (e.g. subtracting if the exercise contains the keyword 'less' [10] [21]), are most often not applicable in more complex settings such as calculus. An EE approach is often applied in a problematic situation, which is non routine to some extent, by trying to relate the strategy choice and implementation to something familiar.

In [15] there was in the examined students' task solving behaviour a distinction between superficial EE approaches and mathematically well-founded PR. PR approaches were relatively rare and of limited range, and this was one of the main reasons for the students' difficulties.

2.3 Research questions

Formulated in relation to the discussion above, the questions treated in this study are:
Q1: In what ways is it possible to solve textbook exercises without considering the mathematical properties of the components involved? This question leads to a qualitative analysis in section 3, and to the distinction between three solution types.
Q2: What proportions of a textbook's exercises can be solved by these solution types? 600 exercises from different textbooks are classified in section 4.

The PR-EE framework can not be directly applied when studying Q1, since an EE approach requires that the solver has formed established experiences from some set of solution procedures during a longer period of time. For example, one established experience could be that 'maximisation exercises are usually solved by finding the zeroes of the derivative of a function', an approach that is often correct but may be wrong. Another similar experience is that 'an irrational solution to a second degree equation is probably wrong', since second degree equations in textbooks are often arranged in order to have easily manageable solutions. This type of experiences are not developed when a student reads a new textbook section for the first time, it probably takes several weeks or months. Students may of course be influenced by established experiences from earlier levels. An undergraduate student may for example try to apply EE reasoning learnt at upper secondary school, but the textbook exercises usually treat mathematics that to some extent is new to the solver. Thus she or he seldom has sufficient experiences to base solutions solely on EE approaches, and other strategies than EE need to be applied. One possibility is to base a solution on PR reasoning, but are there other strategies? Below it will be described how solution procedures can be partially or completely copied from information in the textbook.

3 Q1: A Qualitative Classification of Exercises

3.1 Data source

A calculus coursebook [1] is studied, and all exercise references in this section are to this book. The main reason for this choice is that the study described in [15] treats calculus and the students there have read this book as a coursebook. Browsing through some American (mainly American books are used in Sweden) calculus textbooks gives the impression that they are in many ways quite similar to [1], both concerning content and pedagogy. This one can notice, in particular when it comes to the main part of the solved examples and the exercises. Therefore it seems like the one chosen is fairly representative. This assumption is strengthened in section 4 by applying the same classification methods to some randomly selected exercises from other books. The aim of this study is not to investigate these particular books, but a common tradition that is represented by them. There are of course other types of (both American and others) textbooks and traditions.

3.2 Solution conditions

After analysing possible solutions to several exercises the six exercises below were chosen to represent three solution types that differ with respect to the role of PR. Each exercise is presented and analysed under the following four subheadings:
Exercise formulation: The exercise quoted from the textbook.
Possible solution: There are of course many different ways to solve an exercise, and this can in turn be done more or less correctly. A solution presented below should: (1) Be logically reasonable and realistic in the sense that a real-life exercise solver could produce it. It would be desirable to state this condition in a more precise definition, but this is beyond the scope of this paper and not necessary for its purpose. Solutions based only on guesses are not considered since every exercise is possible, but seldom likely, to be solved by wild guesses and random applications of procedures. (2) Be meant to guide towards what probably is the truth, without necessarily having to be complete or correct (cf. section 2.2). (3) Produce a correct result. (4) If possible, avoid being based on PR. This condition is central to this paper since every exercise is possible to solve by a PR approach (cf. Q1 in section 2.3).
Reasoning structure: The solution reasoning is structured by the framework from section 2.1.
Reasoning characteristics The characteristics of the reasoning structure, in relation to question Q1 above, are summarised.

3.3 Detailed solutions of exercises

Textbook Section 1.2, Exercise 14.

Exercise formulation:

> "In Exercises 3-40, evaluate the limit or explain why it does not exist.
> **14.** $\lim_{t\to 2} \frac{t^2+3t-10}{t^2-4}$."

Possible solution: Try to identify the solved example in the textbook Section 1.2 that is most similar to Exercise 1.2.14. There are 11 examples (due to space limitations, they are not quoted here) in the section and all concern finding different types of limits for different functions by numerical, graphical, or algebraical methods. There are several superficial properties to consider:
(I) Examples 4 ac, 5, 6, 7, 8 and 9 b are the only ones that contain the expression $\lim_{t\to a} f(x)$ where a is numerical and f is an explicit function.
(II) In Examples 1, 4 a and 9 a f is a rational function.
(III) Only in Example 4 both the numerator and denominator are second-degree

polynomials.
(IV) In some examples the limit can be evaluated by simply calculating $f(a)$. The only examples where $f(x)$ is a combination of elementary functions and $f(a)$ can not be evaluated are 1, 2, 4 abc and 7.

After considering some or all similarities described above, it is reasonable to conclude that Example 1.2.4 a is similar to Exercise 1.2.14. The example and its solution are quoted from the textbook:

"**EXAMPLE** 4(a) Evaluate:

$$\lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$$

SOLUTION Each of these limits involves a fraction whose numerator and denominator are both 0 at the point where the limit is taken.

$$\lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$$

Undefined at $x = -2$. Factor numerator and denominator.

$$= \lim_{x \to -2} \frac{(x+2)(x-1)}{(x+2)(x+3)} \qquad \text{Cancel common factors.}$$

$$= \lim_{x \to -2} \frac{x-1}{x+3} \qquad \text{Evaluate this limit by substituting } x = -2.$$

$$= \frac{-2-1}{-2+3} = -3. \qquad []"$$

After this similar example is found, the next step is to solve the exercise by copying the solution procedure from the example in every detail. This results in the following solution to the exercise:

$$\lim_{t \to 2} \frac{t^2 + 3t - 10}{t^2 - 4} \qquad \text{Undefined at } t = 2. \text{ Factor numerator and denominator.}$$

$$= \lim_{t \to 2} \frac{(t-2)(t+5)}{(t-2)(t+2)} \qquad \text{Cancel common factors.}$$

$$= \lim_{t \to 2} \frac{t+5}{t+2} \qquad \text{Evaluate this limit by substituting } t = 2.$$

$$= \frac{2+5}{2+2} = \frac{7}{4}. \qquad []$$

Reasoning structure:
(1) *Problematic situation*: The exercise.
(2) *Strategy choice*: Search for an example or a theorem where the components in the exercise can be inserted. The identification of a similar example is based on the mathematically superficial properties (I-IV) above.
(3) *Strategy implementation*: Copy the solution of the example. To carry out this it is necessary to: (A) Conclude from the example that the number α appearing in the expression $\lim_{x \to \alpha}$ is the one that should be inserted in the function. (B) Insert a number into a first or second degree polynomial and obtain a function value. (C) Recall that division by 0 is not allowed. (D) Recall that the properties of a function is independent of the name of the variable (x or t). (E) Factor a second degree polynomial.
(4) *Conclusion*: The answer to the exercise.

Reasoning characteristics: The key feature here is that none of the identification similarities (I-IV) or the solution procedure steps (A-E) consider the underlying *meaning* of the expression $\lim_{t \to 2} \frac{t^2+3t-10}{t^2-4}$. The *reasons* behind these steps do not have to be known in order to carry out the solution. In fact, no mathematical properties central to the section (limits and continuity) are used, or need to be understood. In order to discuss these aspects more precisely it is suitable at this point to extend the framework from Section 2.2:

Intrinsic and surface properties: There will be a distinction between *intrinsic* and *surface* mathematical properties of the components involved in the reasoning. An intrinsic property is deep and central to the component. For example, an intrinsic property of the solution procedure above is that cancelling common factors does not alter the function if $0 < |t-2| < \delta$ for some δ. A surface property may be a consequence of an intrinsic property but carries with itself no or little mathematical meaning, for example that the number α appearing in the expression $\lim_{x \to \alpha}$ is the one that should be inserted in the function expression.

Past and current properties: The exercises of a section are most often related to the subject matter introduced in the section. One of the purposes of an exercise is to introduce, learn, practice, and consolidate subject matter in the section: concepts, methods, and other ideas. A mathematical property of a solution component is called *current* if it concerns subject matter introduced in the same (or a close) chapter as the exercise, and *past* if it concerns subject matter treated much earlier. The label 'past' may also be complemented by the approximate time, in educational system years, passed since the subject matter was treated.

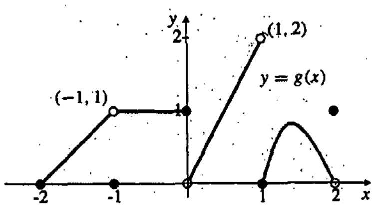
The strategy choice of the solution above is not based on intrinsic mathematical properties, it concerns identifying surface key components of this particular exercise type. The strategy implementation is based on copying the solution procedure from the example, and the mathematical competence needed here concerns recalling past mathematical facts and procedures from the beginning of upper secondary school (three years earlier in the students' perspective). Since the solution does not include any reasoning based on intrinsic current mathematical properties, the reasoning is not characterised as PR.

Textbook Section 1.4, Exercise 1

Exercise formulation:

> "State whether g is (i) continuous, (ii) left continuous, (iii) right continuous, and (iv) discontinuous at each of the points -2, -1, 0, 1, and 2."

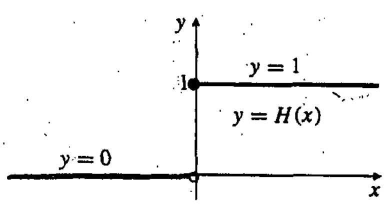
Figure 1: (Textbook fig. 1.32)



Possible solution: The only example in Section 1.4 that both (I) explicitly mentions right or left continuity, and (II) contains a figure with segments of curves connected to filled and empty dots, is Example 1.4.1:

> "**EXAMPLE 1** The Heaviside function $H(x)$ whose graph is shown in Figure 1.19 is continuous at every number x except 0. It is right continuous at 0, but is not left continuous or continuous there."

Figure 2: (Textbook fig. 1.19)



Here it is not possible to copy the solution procedure in every detail, as in the solution to Exercise 1.2.14 above. However, the procedure described in the example needs only some slight interpretation: From the example text and figure one may conclude that a function is right continuous at $x = 0$ if the filled dot is connected with a curve to the right. It seems reasonable to assume that a function is left continuous if the filled dot is connected with a curve to the the left. It does not matter if the filled dot is above or below the empty dot. The filled dots in figure 1.32 that are connected with curves to the right are at $x = -2$ and $x = 1$. The filled dot that is connected with a curve to the left is at $x = 0$. The conclusion is that g is right continuous at $x = -2$ and $x = 1$, and left continuous at $x = 0$. [The questions 1.4.1 (i) and (iv) can be solved in similar ways, relating to other examples in Section 1.4. Due to space limitations, this will not be described here.]

Reasoning structure:
(1) *Problematic situation*: The exercise.
(2) *Strategy choice*: Search for an example or a theorem where the components in the exercise can be inserted. The identification of a similar example is based on the properties (I) and (II).
(3) *Strategy implementation*: Interpret which graphical components of the picture that are connected to the term 'right continuous', and how. Then copy the solution procedure in the example.
(4) *Conclusion*: The answer to the exercise.

Reasoning characteristics: None of these steps considers the mathematical meaning of the expression 'right continuous', and no intrinsic mathematical properties central to the section are used or needed to be understood in the solution above. It is important to stress that, it is of course *possible* that the solver will actually learn something about the intrinsic properties but not *certain*, which is essential in this paper (c.f. solution condition (4) in Section 3.2).

The strategy choice, the identification of the similar properties (I) and (II), is possible to base on surface reasoning: It is merely the keywords 'right continuous' and the surface visual forms (dots and lines) that are used, not their intrinsic mathematical meaning. The strategy implementation, the identification of which graphical components of the picture that are connected to the expression 'right continuous', may be based on surface reasoning. In addition to this, it is necessary to recall a past property of graphs: in what way a position in a graph is connected to a numerical value of x, once again a topic treated in the beginning of upper secondary school. Since the solution does not include any reasoning based on current intrinsic mathematical properties, the reasoning is not characterised as PR.

Textbook Section 1.3, Exercise 9

Exercise formulation:

"Find the limit $\lim_{x\to\infty} \frac{3x+2\sqrt{x}}{1-x}$."

Possible solution: There is no example or theorem of exactly the same type. However, in the section several examples and earlier exercises have treated limits at $\pm\infty$ of rational functions, like Example 1.3.3 (which is preceded by an introductory text):

> "The following examples show how to render such a function in a form where its limits at infinity and negative infinity (if they exist) are apparent. The way to do this is to *divide the numerator and the denominator by the highest power of x appearing in the denominator.* The limits of a rational function at infinity and negative infinity either both fail to exist, or both exist and are equal.
> **EXAMPLE 3** **Numerator and denominator of the same degree**
>
> $$\text{Evaluate } \lim_{x\to\pm\infty} \frac{2x^2 - x + 3}{3x^2 + 5}.$$
>
> **SOLUTION** Divide the numerator and denominator by x^2, the highest power of x appearing in the denominator:
>
> $$\lim_{x\to\pm\infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{x\to\pm\infty} \frac{2 - (1/x) + (3/x^2)}{3 + (5/x^2)} = \frac{2 - 0 + 0}{3 + 0} = \frac{2}{3}. \,\square\text{"}$$

The outcomes of Example 1.3.3 and other similar examples are summarised in the textbook:

> "**Summary of limits at $\pm\infty$ for rational functions**
> Let $P_m(x) = a_m x^m + \cdots + a_0$ and $Q_n(x) = b_n x^n + \cdots + b_0$ be polynomials of degree m and n, respectively. Then $\lim_{x\to\pm\infty} \frac{P_m(x)}{Q_n(x)}$ (a) equals zero if $m < n$, (b) equals $\frac{a_m}{b_n}$ if $m = n$, (c) does not exist if $m > n$."

The only difference between Exercise 1.3.9 and the Example 1.3.3, is that in the former the numerator and denominator consists of sums of power functions where the exponents are rational numbers, while in the latter the exponents are whole numbers. In the exercise, after dividing the numerator and denominator by the highest power of x appearing in the denominator, the denominator tends to a constant as x tends to $\pm\infty$. Thus the limit is determined by the behaviour of the numerator, in the same way as in the example. Therefore it does not matter that there are rational exponents in the exercise, the method of the example is applicable. According to the summary, the limit is $\frac{a_m}{b_n} = \frac{3}{-1} = -3$.

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Search for a situation in the text, where the components in the exercise can be inserted.
(a.3) *Strategy implementation*: Similar examples and a method is found but it can not be copied in every detail, since $f(x)$ in the exercise is not a rational

function. This leads to:
(b.2) *Strategy choice*: Determine the properties of the method, and the differences between the examples and the exercise. Determine if these differences make the method applicable or not.
(b.3) *Strategy implementation*: Analyse the intrinsic idea in the solutions of examples treating limits at $\pm\infty$ of rational functions. Conclude that the method is applicable to fractions of power functions with rational exponents.
(c.2) *Strategy choice*: Apply the method in the summary.
(c.3) *Strategy implementation*: Straightforward.
(c.4) *Conclusion*: The answer to the exercise.

Reasoning characteristics: The strategy choice (a.2) is based on first identifying examples and other situations in the text that are similar to the exercise. This is more difficult, compared to the solutions to Exercises 1.2.14 and 1.4.1 above, to do without considering the mathematical properties of the components involved. However, it is possible since there are only ten examples to choose from, and most of them are very different from Exercise 1.3.9.

As a part of the overall strategy choice, (b.2) and (b.3) determine if the method in Example 1.3.3 is applicable. Here it is, contrary to the solutions to Exercises 1.2.14 and 1.4.1, necessary to consider (at least some of) the intrinsic mathematical properties of the components in the example. This reasoning, described in the final paragraph of the solution, can be done more or less thoroughly but is characterised as PR. Still, the main contribution to the solution construction comes from identifying and copying parts of examples and a prescribed method.

Textbook Section 1.4, Exercise 27

Exercise formulation:

> "Find the intervals on which the functions $f(x)$ in Exercises 25-28 are positive and negative.
> **27** $f(x) = \frac{x^2-1}{x^2-4}$"

Possible solution: The closest approximation to the exercise formulation is Example 1.4.10. The identification of a similar example can be based on the insight that semantically both the exercise and the example ask for 'find A where B is C', and that in both A = 'intervals', B = 'f(x)' and C = 'positive and negative'. The mathematical meaning of A, B, and C does not have to be considered when making this identification. All the other 11 examples in Section 1.4 are linguistically very different from Exercise 1.4.27, and mathematically most of them just describe different types of continuous and discontinuous functions.

> "**EXAMPLE 10** Determine the intervals on which $f(x) = x^3 - 4x$ is positive and negative.
> **SOLUTION** Since $f(x) = x(x^2 - 4) = x(x-2)(x+2)$, $f(x) =$

0 only at $x = 0$, 2, and -2. Because f is continuous on the whole real line, it must have constant sign on each of the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. (If there were points a and b in one of those intervals, say in $(0, 2)$, such that $f(a) < 0$ and $f(b) > 0$, then by the Intermediate-Value Theorem there would exist c between a and b, and therefore between 0 and 2, such that $f(c) = 0$. But we know f has no zero in $(0, 2)$.)

To find whether $f(x)$ is positive or negative throughout each interval, pick a point in the interval and evaluate f at that point.
Since $f(-3) = -15 < 0$, $f(x)$ is negative on $(-\infty, -2)$.
Since $f(-1) = 3 > 0$, $f(x)$ is positive on $(-2, 0)$.
Since $f(1) = -3 < 0$, $f(x)$ is negative on $(0, 2)$.
Since $f(3) = 15 > 0$, $f(x)$ is positive on $(2, \infty)$. []"

The entire solution can not be copied since the function in Exercise 1.4.27 is more complicated than the function in the example, so the solution needs to be somewhat modified: By Example 1.4.10, a continuous function can only change sign where it is zero. Thus the places where the function $f(x) = \frac{x^2-1}{x^2-4}$ can change sign is where it is zero (at $x = \pm 1$) or not continuous (at $x = \pm 2$ where it is undefined). From here on, the solution of Example 1.4.10 can be mimicked.

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Search for a similar example or a theorem where the components in the exercise can be inserted.
(a.3) *Strategy implementation*: The identification of the similar Example 1.4.10 can be based mainly on linguistic grounds, and that the other examples in Section 1.4 do not treat the same type of question as Exercise 1.4.27. At the same time, there are differences between Exercise 1.4.27 and Example 1.4.10. The solution can not be copied in every detail, and this leads to:
(b.2) *Strategy choice*: Determine the necessary modifications of the example solution procedure.
(b.3) *Strategy implementation*: Analyse the intrinsic properties of the example solution: $f(x)$ may change sign where $f(x) = 0$, as in the example, or where $f(x)$ is not continuous.
(c.2) *Strategy choice*: Find the values of x where $f(x)$ is zero or not continuous, then copy the rest of the solution to Example 1.4.10.
(c.3) *Strategy implementation*: Straightforward.
(c.4) *Conclusion*: The answer to the exercise.

Reasoning characteristics: The strategy choice, identifying a similar example, can be carried out without considering the mathematical properties of the components involved. But, contrary to the solutions to Exercises 1.2.14 and 1.4.1 above, the whole solution can not be copied in detail. Some of the mathematical intrinsic properties of the solution are considered, for example the consequences of the continuity of $f(x)$. A local modification of the solution procedure is needed, and this is achieved by PR in step (b.3). Still again, the main contribution to the solution construction comes from identifying and copying parts of an example.

Textbook Section 1.2, Exercise 81

Exercise formulation:

"If $\lim_{x\to 2}\frac{f(x)-5}{x-2} = 3$, find $\lim_{x\to 2} f(x)$."

Possible solution: There is in Section 1.2 no example, theorem, or other situation that is similar to the exercise and that could give guidance concerning the strategy choice. In order to understand the meaning of the expression '$\lim_{x\to 2}\frac{f(x)-5}{x-2} = 3$', consider the only definition of limit that has been introduced so far:

> **"An informal definition of limit**
> If $f(x)$ is defined for all x near (on either side of) a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a (on either side of a), we say that the function f approaches the **limit** L as x approaches a, and we write
>
> $$\lim_{x\to a} f(x) = L."$$

Applying the definition to the exercise leads to the conclusion that if x is close to 2, then $\frac{f(x)-5}{x-2}$ will be close to 3. This in turn implies that $f(x)-5$ is close to $3(x-2)$, and that $f(x)$ is close to $3(x-2)+5$. $3(x-2)$ is close to 0, so $f(x)$ is close to 5. According to the definition above, $f(x)$ can become arbitrarily close to 5 by choosing x sufficiently close to 2. Consequently, $\lim_{x\to 2} f(x) = 5$. $\square$

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Since a similar example (or another situation in the text) is not found, it is necessary to analyse the situation and construct a qualitative understanding of the exercise and the relevant definition.
(a.3) *Strategy implementation*: The meaning and consequences of the exercise and the definition are interpreted in approximation terms.
(b.2) *Strategy choice*: When x is close to 2, the limit can be seen as an 'approximate equation'. 'Solve' this 'approximate equation'.
(b.3) *Strategy implementation*: This is essentially done by replacing 'equal to', in the familiar procedure for solving ordinary equations, by 'close to'.
(b.4) *Conclusion*: The answer to the exercise.

Reasoning characteristics: None of the steps above can be 'superficially copied' from any situation in the text. In order to carry out (a.3) and (b.2) it is necessary to consider present intrinsic mathematical properties of the components involved in the exercise and in the definition of limit. The argumentation is not extensive, but the strategy choice is based on analysing the exercise. The entire solution is a new PR construction based on intrinsic mathematical properties of limits and approximations.

Textbook Section 1.5, Exercise 37

Exercise formulation:

> "Use the definition of limit twice to prove Theorem 7 of Section 1.4; that is, if f is continuous at L and if $\lim_{x\to c} g(x) = L$, then
>
> $$\lim_{x\to c} f(g(x)) = f(L) = f(\lim_{x\to c} g(x))."$$

Possible solution: The exercise formulation is similar to the task formulation in Example 1.5.4, which is contained in the textbook paragraph "Using the Definition of Limit to Prove Theorems" (the formal ϵ- and δ-definition):

> "**EXAMPLE 4 Proving the rule for the limit of a sum**
> If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, prove that $\lim_{x\to a} f(x) + g(x) = L + M$.
> **SOLUTION** Let $\epsilon > 0$ be given. We want to find a positive number δ such that
>
> $$0 < |x - a| < \delta \;\Rightarrow\; |(f(x) + g(x)) - (L + M)| < \epsilon."$$
>
> [The rest of the example solution is not described, since it cannot be copied to solve Exercise 1.5.37]

Start in the same way as Example 4: Let $\epsilon > 0$ be given. We want to find a positive number δ such that

$$0 < |x - c| < \delta \;\Rightarrow\; |(f(g(x)) - f(L)| < \epsilon.$$

This means that if x is close but not equal to c, then $f(g(x))$ shall be close to $f(L)$. This is as far as the example solution can be copied.

The next step is to analyse the conditions of the exercise: what is known in ϵ- and δ-terms? Since $\lim_{x\to c} g(x) = L$, there is (by the definition of limit) for every $\epsilon_1 > 0$ a $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |g(x) - L| < \epsilon_1.$$

This means that if x is close to c, then $g(x)$ is close to L. Similarly, since f is continuous at L, there is (by the definitions of limit and continuity) for every $\epsilon_2 > 0$ a $\delta_2 > 0$ such that

$$0 < |y - L| < \delta_2 \Rightarrow |f(y) - f(L)| < \epsilon_2.$$

This means that if y is close to L, then $f(y)$ is close to $f(L)$. Altogether, this implies (with y replaced by $g(x)$), that if x is close to c, then $g(x)$ is close to L, and consequently $f(g(x))$ is close to $f(L)$.

Finally, translate this reasoning into formal ϵ- and δ-terms: Let $\epsilon_2 > 0$ be given. Then there is a $\delta_2 > 0$ such that

$$0 < |g(x) - L| < \delta_2 \Rightarrow |f(g(x)) - f(L)| < \epsilon_2.$$

Set $\epsilon_1 = \delta_2$. Then there is a $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |g(x) - L| < \delta_2.$$

Combining the two implications above results in:

$$0 < |x - c| < \delta_1 \Rightarrow |g(x) - L| < \delta_2 \Rightarrow |f(g(x)) - f(L)| < \epsilon_2. \qquad \Box$$

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Search for an example or a theorem where the components of the exercise can be inserted.
(a.3) *Strategy implementation*: Example 1.5.4 is the only one that is similar to the exercise. It can be followed to start with, but the main part of the solution can not be copied in order to solve the exercise. Following the first part of the example solution helps to formulate and answer the heuristic question 'what is asked for' in ϵ- and δ-terms.
(b.2) *Strategy choice*: The question 'what is asked for' is followed by formulating the heuristic questions 'what is known', and 'how can this be used'?
(b.3) *Strategy implementation*: It seems unlikely that a non-expert student can construct a solution to the exercise by thinking only in formal ϵ- and δ-terms. At the same time it is hard to construct a solution without the ϵ- and δ-machinery. Therefore the solution is based on the interpretation of limits in both the formal ϵ-δ-representation, and the the informal 'closeness'-representation. The necessity and consequences of the continuity of f at L also have to be considered. The proper connections in 'closeness'-terms are made between x and c, $g(x)$ and L, and $f(g(x)$ and $f(L)$. The reasoning is first informally structured in terms of closeness, which is then translated into formal language.
(b.4) *Conclusion*: The answer to the exercise.

Reasoning characteristics: Though the exercise formulation includes the clue "use the definition of limit twice", and though Example 1.5.4 can be copied to start with, this is far from providing guidance to a complete solution. The solution of Exercise 1.5.37 requires the solver to: (A) Consider intrinsic properties of the limit and continuity concepts, including the formal definitions. (B) Consider intrinsic properties of mathematical proofs, in particular under what conditions the reasoning is accepted as a proof. (C) Construct a solution procedure, which includes the construction of the 'closeness'-reasoning but also the insight that this is not precise enough, the formal ϵ- and δ-machinery has to be applied. All the components (A-C) are known to be hard for the students, and the exercise must be seen as one of the most difficult in the section. The solution is mainly a new PR construction based on intrinsic mathematical properties of limits and proofs.

3.4 Conclusion: Definition of solution categories

The reasoning characteristics of the solutions above describe, as a partial answer to Q1 in Section 2.3, three different ways that the solution reasoning can be based on current intrinsic mathematical properties of the components involved: Not at all, locally, or globally. Some modifications of the framework of Section 2.2 are suggested in order to capture the essence of these three reasoning categories:

3.4.1 Identification of Similarities (IS)

The reasoning characteristics of the solutions to Exercises 1.2.14 and 1.4.1 are summarised in the following definition:

The reasoning in an exercise solution attempt will be called *reasoning based on identification of similarities* (abbreviated IS) if the reasoning fulfils both of the following two conditions:
(i) The strategy choice is founded on identifying similar surface properties in an example, definition, theorem, or some other situation that is described earlier in the text. This identification does not consider the current intrinsic mathematical properties of the components involved.
(ii) The strategy implementation is carried through by copying the procedure from the identified situation.

Both IS and EE (Section 2.2) concern a mathematically superficial transfer of solution procedures from a textbook (IS) or an experience-based (EE) situation. In both EE and IS reasoning, the task solver may seem to be working with advanced mathematics. An IS solution is often short and simple to carry through, but it may be long, technically tricky, and/or require a lot of past basic mathematical knowledge and skills. An IS approach can often be applied in a problematic situation, which is new and nonroutine to some extent.

3.4.2 Local Plausible Reasoning (LPR)

The reasoning characteristics of the solutions to Exercises 1.4.27 and 1.3.9 are summarised in the following definition:

The reasoning in an exercise solution attempt will be called *local plausible reasoning* (abbreviated LPR) if it differs from IS in at least one of the following two ways:
(i) The strategy choice is founded on the identification of similarities between components in the exercise and components in a situation in the text, but these components differ in one or a few local parts, and PR (Section 2.2) is used to determine whether the procedure can be copied in order to solve the exercise or

not.
(ii) The strategy implementation is mainly based on copying the solution procedure from the identified situation, but one or a few local steps of this procedure are modified by constructive PR.

What differs LPR from IS is that in the former PR is applied locally: in the strategy choice (Exercise 1.3.9) to see *if* the solution procedure can be copied, or in the strategy implementation (Exercise 1.4.27) to see *how* the solution procedure should be modified. The main part of the solution reasoning is still similar to IS. One difference from IS is a consequence of the definition of LPR: IS reasoning may be possible to carry out without considering anything of the current intrinsic mathematics treated. In LPR, since PR reasoning is applied and this can not be done arbitrarily, it may be necessary to understand large parts of the exercise and the identified textbook situation in order to make the required local decisions or modifications.

3.4.3 Global Plausible Reasoning (GPR)

The reasoning characteristics of the solutions to Exercises 1.2.81 and 1.5.37 are summarised in the following definition:

The reasoning in an exercise solution attempt will be called *global plausible reasoning* (abbreviated GPR) if at least one of the following conditions is fulfilled:
(i) The strategy choice is mainly founded on analysing and considering the current intrinsic mathematical properties of the components in the exercise. A solution idea is constructed and supported by PR.
(ii) The strategy implementation is mainly supported by PR based on current intrinsic mathematical properties.

GPR is similar to LPR in the sense that PR is applied, and therefore it is necessary to understand large parts of the exercise and the identified textbook situation. GPR differs from LPR in the range of the PR reasoning: if it concerns the whole solution (global) or a few limited components (local). If an exercise is not possible to solve by IS or LPR then GPR is required, and in that case the exercise is, to the solver, a genuine problem in the sense of Schoenfeld [20].

3.4.4 Comments to the categories

The three categories IS, LPR, and GPR are exclusive in the sense that a solution can be classified in one category only. However, the extent of PR in exercise solutions may in a sense vary along a continuum from none (IS) to extensive (GPR), and this variation is not quantifiable. Therefore the definitions do not admit an exact classification of borderline cases, which may therefore be rather arbitrarily categorised. When the framework is applied below in order to classify a larger number of exercises, the purpose is not to categorise every exercise but to study the approximate distribution among the three categories.

4 Q2: A Quantitative Classification of Exercises

4.1 Classification conditions

This section aims at answering Q2 by classifying a larger number of exercises in the three categories above. From three calculus textbooks, some exercises are chosen among the central one-variable sections. All exercises below were analysed through the same procedure as the six exercises in Section 3.3 above. The reasoning characteristics determine in which of the three categories an exercise should be classified: As 'IS' if it is possible to solve by IS reasoning, as 'LPR' if it is not possible to solve by IS but by LPR, and as 'GPR' if GPR is required. In this section the classification procedure is not described, only its outcome. The textbook number of each classified exercise is placed in one of the rows IS, LPR or GPR. The total amount of exercises in each row is summarised within square brackets [Sum].

4.2 Classification results

The textbook 'Calculus - a Complete Course' [1]

All exercises in chapter 1, 'Limits and continuity', are classified.

Section 1.1, 'Examples of velocity, growth rate and area'

IS [Sum: 8]: 1 2 3 4 5 6 10 12
LPR [Sum: 1]: 11
GPR [Sum: 4]: 7 8 9 13

Section 1.2, 'Limits of functions'

IS [Sum: 71]: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 25 26 30 31 33 34 35 39 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 79 80 83 84 85 86 87 88 89 91 92 93
LPR [Sum: 18]: 24 27 28 36 37 38 40 41 42 43 44 45 46 75 76 77 78 94
GPR [Sum: 7]: 29 32 81 82 90 95 96

Section 1.3, 'Limits at infinity and infinite limits'

IS [Sum: 45]: 1 2 3 4 5 6 7 10 11 13 14 15 16 17 18 19 20 21 22 23 27 29 30 33 34 39 40 41 42 43 44 45 46 47 48 49 51 52 53 54 56 59 60 61 62
LPR [Sum: 16]: 8 9 12 24 25 26 28 31 32 35 36 37 38 55 57 58
GPR [Sum: 4]: 50 63 64 65

Section 1.4, 'Continuity'

IS [Sum: 22]: 1 2 3 4 5 6 11 12 13 14 15 16 26 29 35 36 37 38 39 40 41 42
LPR [Sum: 13]: 7 8 9 10 19 21 22 23 24 25 27 28 30
GPR [Sum: 7]: 17 18 20 31 32 33 34

Section 1.5, 'The formal definition of limit'

IS [Sum: 9]: 1 2 13 21 22 23 24 25 26
LPR [Sum: 21]: 3 4 5 6 7 8 9 10 11 12 14 15 16 17 18 19 20 27 28 29 30
GPR [Sum: 8]: 31 32 33 34 35 36 37 38

Comment: Section 1.5 is an optional section treating the formal ϵ- δ-definition of limit. The section text, the examples, and the exercises are unusually theoretical and difficult. A large proportion of the exercises are LPR and GPR.

Classification summary: 61 % IS, 27 % LPR, and 12 % GPR.

The textbook 'Calculus: One and Several Variables' [19]

Four sections were randomly chosen among the sections in the first eight (out of 18) chapters.

Section 4.3, 'Local extreme values'

IS [Sum: 30]: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
LPR [Sum: 0]:
GPR [Sum: 22]: 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52

Comment: There are several exercise solutions among those classified as IS that include difficult equation solutions.

Section 5.2, 'The function $F(x) = \int_a^x f(t)dt$'

IS [Sum: 11]: 1 2 5 6 7 8 9 12 13 14 15
LPR [Sum: 13]: 10 11 16 17 18 19 20 21 22 23 24 29 32
GPR [Sum: 10]: 3 4 25 26 27 28 30 31 33 34

Comment: There are only three examples, and a rather theoretical and difficult text. Many of the IS solutions are difficult.

Section 5.4, 'Some area problems'

IS [Sum: 24]: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 23 24 25 26 27 28 29 30
LPR [Sum: 10]: 17 18 19 20 21 22 33 34 35 36
GPR [Sum: 4]: 31 32 37 38

Comment: There are several exercise solutions among those classified as IS that include difficult equation solutions.

Section 8.7, 'Numerical integration'

IS [Sum: 20]: 1 2 3 4 5 6 7 8 9 10 13 14 15 16 17 18 19 20 21 22
LPR [Sum: 2]: 23 24
GPR [Sum: 6]: 11 12 25 26 27 28

Classification summary: 56 %, 16 % LPR, and 28 % GPR.

The textbook 'Calculus With Analytic Geometry' [7]

Four sections were randomly chosen among the sections in the first nine (out of 16) chapters.

Section 1.4, 'A brief catalog of functions'

IS [Sum: 30]: 1 2 3 4 5 6 7 8 9 10 13 14 15 16 17 18 19 22 25 26 27 28 29 30 31 32 33 34 35 36
LPR [Sum: 4]: 11 12 20 21
GPR [Sum: 2]: 23 24

Section 2.2, 'The limit concept'

IS [Sum: 38]: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 44	
LPR [Sum: 10]: 16 18 41 42 43 45 46 47 48 49	
GPR [Sum: 5]: 17 50 51 52 53	

Section 7.2, 'The natural logarithm'

IS [Sum: 59]: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 59 60 61 64 65 66 67 71	
LPR [Sum: 8]: 52 53 54 55 56 62 63 69	
GPR [Sum: 4]: 57 58 68 70	

Section 8.3, 'Indeterminate forms and L'Hôpitals rule'

IS [Sum: 43]: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 25 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48	
LPR [Sum: 2]: 23 49	
GPR [Sum: 5]: 24 26 27 28 50	

Classification summary: 81 %, 11 % LPR, and 8 % GPR.

4.3 Comments

It is important to stress that the distribution described in the tables above can only be interpreted in approximate terms. First of all, a few of the borderline cases between IS and LPR, and between GPR and LPR were difficult to classify. The reader is invited to apply the classification structure in order to see if similar classification results are obtained. Secondly, the classification does not consider that the solutions to some exercises were short and easy, while other were longer or more difficult. The most common exercise type is represented by exercises 1.2.14 and 1.3.9 in Section 3.3. The other four exercises in Section 3.3 are of less common types.

Some of the LPR and GPR exercises are similar to each other. For example, If one considers only the information available in the text, 1.2.28 in [1] (which is presented in an LPR row above) is an LPR exercise. At the same time, if the reasoning from the preceding similar LPR Exercise 1.2.27 is copied, then 1.2.28 becomes an IS exercise. This results in (if one solves all exercises) that the proportion of LPR exercises are reduced roughly from about 20 % to 15 %, and the GPR exercises from 15 % to 10 %.

One may also note that in about 90 % of the IS and LPR exercises, similar situations were found in solved examples. The other 10 % were unmarked examples, definitions, theorems, rules, etc. This implies that it is possible in about 80 % of the exercises to base the solution not only on searching for similar situations, but on searching only the solved examples.

There are in some books a smaller number of exercises included in other types of sections, like 'chapter reviews', 'miscellaneous problems', 'challenging problems' and 'projects'. These exercises are not included in the classification in

Section 4 above, since they are relatively few and of different types in different textbooks. The main part of these are mixed exercises of the same types as in the ordinary textbook sections, and they follow approximately the same IS - LPR - GPR distribution as those classified above. Some of the exercises, for example 'challenging problems', are mainly of GPR type.

5 Discussion

The purpose of this section is to summarise the results and discuss the consequences. There are differences in Section 4, concerning the distribution among the three exercise types, between and within the textbooks studied. A comparison between different types of textbooks and sections will not be attempted since this is not the primary purpose of this paper and would require the analysis of many more exercises (it may be the topic of a future study). Roughly there are 70 % IS, 20 % LPR, and 10 % GPR exercises. This means that a majority of the exercises are possible to solve without considering the current intrinsic mathematical properties of the components involved in the exercise and in the solution, and that the strategy choice and implementation may normally be based on finding and copying a similar situation in the same textbook section as the exercise in question.

5.1 Consequences on problem solving competence

What types of problem solving competence is required, and therefore perhaps practised and encouraged, when applying IS, LPR, and GPR strategies, respectively? A suitable framework when discussing this question is provided by Schoenfeld [20, p. 15]:

> "*Resources*: Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand. Intuitions and informal knowledge regarding the domain. Facts. Algorithmic procedures. 'Routine' nonalgorithmic procedures. Understandings (propositional knowledge) about the agreed-upon rules for working in the domain.
> *Heuristics*: Strategies and techniques for making progress on unfamiliar and non-standard problems: rules of thumb for effective problem solving, including: Drawing figures; introducing suitable notation. Exploiting related problems. Reformulating problems; working backwards. Testing and verifications procedures.
> *Control*: Global decisions regarding the selection and implementation of resources and strategies. Planning. Monitoring and assessment. Decision-making. Conscious metacognitive acts.
> *Belief Systems*: One's 'mathematical world view', the set of (not necessarily conscious) determinants of an individual's behaviour. About self. About the environment. About the topic. About mathematics."

These four aspects of competence can be seen as related to IS, LPR, and GPR, respectively, in the following ways:
Resources: In IS reasoning the current intrinsic mathematical properties of the solution components are not considered. Therefore, the resources that may be developed are restricted to past and/or surface mathematical areas. The only exception from this is that the solution procedure for the particular exercise type (in a narrow sense) may be memorised. If this procedure can be remembered a couple of weeks, and if there are not too many different solution types treated in the exam, then this may be sufficient in order to pass. In LPR to some extent, and in GPR to a larger extent, the current intrinsic mathematical properties are considered. Therefore, this type of reasoning provides a better base for developing resources.
Heuristics: In IS and LPR reasoning, only the strategy 'search for a similar situation' is practised. In IS this strategy leads to a complete solution procedure that can be copied, in LPR it leads to an almost complete solution procedure that needs slight modification. In GPR reasoning, on the other hand, it is often most efficient to apply a variation of different heuristics. See [18], [20], and [22] for extensive discussions about heuristic strategies.
Control: In IS and LPR, 'surface control' is invoked in the strategy choice when identifying the example or text situation that is most similar to the exercise. There is no control needed in the IS strategy implementation since the whole solution procedure is copied. In LPR, control is activated in deciding if and how the local modifications should be implemented. Solving a GPR exercise generally requires active and continuous control, and insufficient control is often one of the main reasons for failure when trying to solve these types of tasks [20], [22].
Belief: Belief is not affected by a single exercise, but the characteristics and domination in number of IS and LPR exercises may encourage the common belief that mathematics is about following procedures developed by others. This belief may seriously affect a person's problem solving behaviour, for example that own solution constructions are not even attempted. Schoenfeld [22, p. 359] describes that common beliefs are for example that ordinary students cannot expect to really understand mathematics, and cannot by themselves construct anything outside the rules and methods demonstrated by the teacher.

In general, IS and LPR exercises do not provide sufficient practice concerning any of the four aspects described above. In particular it is a bit unexpected that, if the assertions of this paper are correct, IS solutions may not develop resources except in the narrow sense that the solution procedure related to a particular solution type may be remembered. The argument for including a large number of solved examples and related similar exercises is often to develop some aspects of resources.

If IS practice is sufficient to pass exams, then it may be possible to pass a mathematics course without having learnt much about neither the concepts treated nor general problem solving skills. Many researchers have described the severe learning and achievement difficulties of large groups of students, often in relatively elementary situations: [8], [9], [20], [23], [25], [26] and [27]. Tall [25] discusses the background to these difficulties: "If the fundamental concepts of calculus [..] prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. [..] The problem is that such routines become just that - routine - so that students begin to find it difficult to answer questions that are conceptually challenging. The teacher compensates by setting questions on examinations that students can answer and the vicious circle of procedural teaching and learning is set in motion." The IS focus in exercises may be a part of this, and may thus lead to short-term gains (passing exam) and long-term losses (weak concept understanding and reasoning construction difficulties).

Schoenfeld [21] has described that students are inclined to answer questions with suspension of sense making, and that they often use shortcut strategies. According to Doyle [4], [5], there is a pressure from students to reduce ambiguity and risk, and to improve classroom order, by reducing the academic demands in tasks. This may also be the case in textbook organisation and can be achieved by including a large proportion of IS and LPR exercises. Dreyfus [6] argues that students are in textbooks rarely given explicit instructions or indications concerning the required quality of reasoning. In a historical perspective McGinty et al. [17] analysed grade 5 arithmetic textbooks from 1924, 1944, and 1984, and found that the number of word problems had decreased, the number of drill problems had increased, and that word problems had also become shorter and less rich. A brief comparison between some older calculus textbooks, for example [2] and [3], and the ones analysed in section 4 indicates that the proportion of IS and LPR exercises have increased considerably. All this may be part of a self - deceptive way in the present mass - education situation to continue, at the surface, to deal with advanced mathematical concepts in our calculus courses. It is at the same time important to stress that this does not imply that the older books where 'better', or that textbooks should only contain GPR exercises.

In Sweden there has been a rather intense debate about the decreasing pass rates at high school and university. Many teachers claim that the main reason behind the problem is the students' insufficient prerequisite algebra skills. One possible explanation for this is that our educational system actually fails to provide a sufficient environment for developing *any* mathematical skill, but since the main mathematical competence needed in order to solve IS exercises is elementary algebra, one may wrongly draw the conclusion that the students' difficulties are caused by insufficient algebra skills only.

5.2 Double difficulties in GPR exercises

Comparing the IS, LPR and GPR exercises through the perspective of this paper, one may distinguish two reasons (that might not be apparent at a first glance) to why the latter are more difficult: (i) Solution construction in the PR sense is a difficulty in itself. (ii) In GPR the reasoning has to be based on the current intrinsic mathematical properties of the components involved, not only on some much more elementary mathematics as in the IS case. This double difficulty in GPR exercises may lead to that they become much more difficult compared to IS and LPR exercises than what was intended.

This could be hard to see for teachers and textbook writers, who are experts in the field, and for example may see the IS Exercise 1.2.14 and the GPR Exercise 1.2.81 above to be of about the same conceptual difficulty. They are, if one believes that the student's solutions will be based on conceptual understanding in the same way as the expert's solutions, but not if the student use an IS approach in Exercise 1.2.14. One may also note that Exercise 1.2.81 is not representative in the sense that most GPR exercises are much more difficult, often like Exercise 1.5.37. A likely consequence is that the GPR exercises that are reachable for the average students are much less than the 10% described above, perhaps 0% for large groups of less proficient students. Hoyles [12], when discussing the role of proof, has argued that a consequence of the hierarchically organised UK National Curriculum (for children aged 5-16 years) is that "most students have little chance to gain any appreciation of the importance of logical argument in whatever form". A similar interpretation, that reasoning construction is for the most able students, can be made from the Swedish national grading criteria for upper secondary school.

5.3 Relation to EE

Most students seem to spend the main part of their time trying to solve exercises. About 90% of the exercises are IS or LPR that can mainly be solved by searching the text for methods. Therefore students may develop strategy choice approaches where the question 'what method should be applied?' is immediately asked, instead of first trying to reach a qualitative representation [10] of the task and base the solution attempt on the intrinsic mathematical properties of the components involved. When the textbook is at hand, this question may lead to the IS or LPR strategy to search for a similar example that provides the solution method. In a task solving situation when the book is not at hand, like in exams or in the experimental situations where the students' behaviour described in Section 2.2 were studied, the same question may lead to a memory search for similar situations to base a solution method on. If the student then has difficulties in reaching a real understanding of the concepts involved and no real practice in constructive problem solving, the only thing to do may be to search for superficial similarities. That is, to use the EE approaches described in Section 2.2. This may be a connection that at least partially explains the students' EE behaviour as a consequence of the large proportion of IS and LPR exercises in the book.

It is important to stress that in this paper only one, yet important, aspect of textbook exercises is studied. The textbook author has to consider a balance between many different factors. One may also argue that IS and LPR exercises have their place in textbooks. Still, the results of this paper together with the discussion, indicates that a larger proportion of elementary GPR exercises should be included in textbooks. Otherwise the majority of the students will not get any proper opportunity to practice GPR, and this may hypothetically in the long run lead to both weak conceptual understanding and to a focus on EE solution strategies.

References

[1] R. Adams. *Calculus: A Complete Course.* Addison - Wesley, third edition, 1995.

[2] R. Courant and F. John. *Introduction to Calculus and Analysis*, volume 1. Wiley International, 1965.

[3] C. de La Vallée Poussin. *d'Analyse Infinitésimale.* Paris: Gauthiers - Villars, eigth edition, 1954.

[4] W. Doyle. Classroom organization and management. In M.C. Wittrock, editor, *Handbook of Research on Teaching*, pages 392–431. New York: Macmillan, third edition, 1986.

[5] W. Doyle. Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, 23:167–180, 1988.

[6] T. Dreyfus. Why Johnny can't prove. *Educational Studies in Mathematics*, 38:85–109, 1999.

[7] C.H. Edwards and D. Penney. *Calculus with Analytic Geometry.* Prentice Hall International, fourth edition, 1994.

[8] G. Harel and E. Dubinsky, editors. *The Concept of Function: Aspects of Epistemology and Pedagogy.* Washington, DC: Mathematical Association of America, 1992.

[9] G. Harel and J. Trgalová. Higher mathematics education. In A. Bishop et al., editors, *International Handbook of Mathematics Education*, pages 675–700. Dordrecht: Kluwer, 1996.

[10] M. Hegarty et al. Comprehension of arithmetic word problems: a comparison of successful and unsuccessful problem solvers. *Journal of Educational Psychology*, 87(1):18–32, 1995.

[11] M. Henningsen and M.K. Stein. Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5):524–549, 1997.

[12] C. Hoyles. The curricular shaping of students approaches to proof. *For the Learning of Mathematics*, 17:7–16, 1997.

[13] J. Lithner. Students' mathematical reasoning and textbook exercises. In preparation.

[14] J. Lithner. Mathematical reasoning and familiar procedures. *International Journal of Mathematical Education in Science and Technology*, 31:83–95, 2000.

[15] J. Lithner. Mathematical reasoning in task solving. *Educational studies in mathematics*, 41:165–190, 2000.

[16] E. Love and D. Pimm. "This is so": a text on texts. In A. Bishop et al., editors, *International Handbook of Mathematics Education*, pages 371–409. Dordrecht: Kluwer, 1996.

[17] R. McGinty et al. Do our mathematics textbooks reflect what we preach? *School Science and Mathematics*, 86(7):591–596, 1986.

[18] G. Pólya. *How To Solve It.* Princeton NJ: Princeton University Press, 1945.

[19] S.L. Salas et al. *Calculus: One and Several Variables.* Wiley, eighth edition, 1999.

[20] A. Schoenfeld. *Mathematical Problem Solving.* Orlando, FL: Academic Press, 1985.

[21] A. Schoenfeld. On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. Voss et al., editors, *Informal Reasoning and Education*, pages 311–344. Hillsdale, NJ: Lawrence Erlbaum Associates, 1991.

[22] A. Schoenfeld. Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 334–370. New York: Macmillan, 1992.

[23] J. Selden et al. Even good students can't solve nonroutine problem. In J. Kaput and E. Dubinsky, editors, *Research issues in Undergraduate Mathematics Learning*, pages 19–26. Washington DC: Mathematical Association of America, 1994.

[24] M. Shield. Mathematics textbooks: messages to students and teachers. In C. Kanes et al., editors, *Teaching Mathematics in New Times*, pages 516–522. 21 annual conference of the Mathematics Education Research Group of Australasia, 1998.

[25] D. Tall. Functions and calculus. In A. Bishop et al., editors, *International Handbook of Mathematics Education*, pages 289–325. Dordrecht: Kluwer, 1996.

[26] P. Thompson. Students, functions, and the undergraduate curriculum. In E. Dubinsky et al., editors, *Research in Collegiate Mathematics Education I, CBMS Issues in Mathematics Education*, pages 21–44. American Mathematical Society, 1994.

[27] S. Vinner. The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, 34:97–129, 1997.

Students' Mathematical Reasoning in Textbook Exercise Solving

Johan Lithner

July 10, 2001

Abstract

This study investigates the ways students conduct their study work, in particular their mathematical reasoning when working with textbook exercises. The results indicate that: (i) Most strategy choices and implementations are carried out without considering the intrinsic properties of the components involved in the solution work. This in turn leads to different difficulties. (ii) It is crucial for these students to find solution procedures to copy. (iii) There are extensive attempts, often successful, to understand each step of the copied solution procedures, but only locally. (iv) The students make almost no attempts to construct their own solution reasoning, not even locally. (v) The main situations where the students' work are not just straightforward implementations of provided solution procedures, are where mistakes are made in minor local solution steps.

Contents

1 Introduction

In two earlier studies [Lit00a], [Lit00c], first year undergraduate students' reasoning in mathematical task solving was examined. The students worked alone with mathematical tasks with no aids at hand except a graphic calculator, a situation similar to the one in examinations. The results indicated that a strategy to focus on what is familiar and remembered at a superficial level was dominant over reasoning based on mathematical properties of the components involved, even when the latter could lead to considerable progress. It seemed like the students' beliefs did not include the latter type of mathematical reasoning as a main approach, even though they mastered the necessary knowledge base. Their behaviour seemed to be quite far from the educational goals. At the same time, there were indications that this was the way they were used to work with mathematics in their studies. One question that arises is: In what way (if at all) is this way of working a reasonable outcome of some 'unbalance' in the learning environment?

An important and influential component of the learning environment to study is the textbook exercises, since normally at least half of the students' study time is spent working with these (this assertion is based on a local unpublished survey). There are studies describing that it is the teachers who *read* textbooks, not the students, and that students' work is often restricted to solving exercises [Mor89], [Shi98]. Still, most research on textbooks seem to concern the text itself, see e.g. [LP96] and references there. Love and Pimm [LP96, p. 397] claim that "While teachers' perceptions of textbooks have received some attention, there is a dearth of research into the use of texts in class". The present study is an attempt to learn more about how first-year undergraduate students actually work with their textbooks and exercises, in particular how they conduct and support their mathematical reasoning.

In [Lit00b] it was described in detail how 70 % of the exercises in representative undergraduate calculus textbooks are possible to solve by copying solution procedures from worked examples or other situations described in the text. This can be done without any constructive mathematical reasoning, and actually also without considering the advanced mathematics treated in the textbook section that contains the exercises. About 20 % of the exercises can be solved by mainly copying worked examples, but some local modifications are required. In about 10 % of the exercises it is necessary both to construct a solution procedure and to consider the advanced mathematics treated. The purpose of [Lit00b] was not only to study possible ways to solve textbook exercises, but also theoretical: To extend the framework from [Lit00a] and [Lit00c] in order to study the research questions of Section 3 below.

2 Framework

More extensive descriptions of the work behind the framework construction, and related references, may be found in [Lit00b] and [Lit00c]. In [Lit00c], the structure below was introduced in order to analyse the observed students' task solving reasoning.

2.1 Reasoning structure

Solving a mathematical task can be seen as solving a set of sub tasks of different grain size and character. If the (sub)task is not routine, one way to describe the reasoning is the following four-step structure (which generally is a simplification of the solver's actual reasoning):
(1) A *problematic situation* is met, a difficulty where it is not obvious how to proceed.
(2) *Strategy choice*: One possibility is to try to choose (in a wide sense: choose, recall, construct, discover, etc.) a strategy that can solve the difficulty. This choice can be supported by *predictive argumentation*: Will the strategy solve the difficulty? If not, choose another strategy.
(3) *Strategy implementation*: This can be supported by *verificative argumentation*: Did the strategy solve the difficulty? If not, redo (2) or (3) depending on if the problem is in the choice of the strategy or in the implementation.
(4) *Conclusion*: A result is obtained.

The term *reasoning* is defined as the line of thought, the way of thinking, adopted to produce assertions and reach conclusions. In the literature, the term *mathematical reasoning* is often used (implicitly or explicitly) to indicate that the reasoning taking place is of high quality in one way or another. In the present paper the term *mathematical reasoning* means only 'reasoning about mathematics', and carries with it no implications about the quality or other characteristics of the reasoning. This is described separately by the *reasoning characteristics* below. *Argumentation* is the substantiation, the part of the reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate.

2.2 Reasoning characteristics

2.2.1 Students' work in exam-like situations

The reasoning types PR and EE were found to be central in [Lit00c], where students worked with tasks in exam-like situations (with no aids at hand except a graphing calculator):

PR A version of the reasoning structure above is called *plausible reasoning* (abbreviated PR) if the argumentation:
(i) is founded on mathematical properties of the components involved in the reasoning, and

(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

The term *component* includes all mathematical concepts, actions, processes, objects, solution procedures, facts, heuristics, etc. that may be explicitly or implicitly involved in the reasoning. In short, the idea behind (ii) is that in school task solving it is often allowed and encouraged to use mathematical reasoning with less requirements on rigour than for example in proof or in professional life. This study can not be restricted to reasoning that is required to be accepted as logically complete and correct (mathematical proof), since this is very seldom produced by students in normal learning situations. The term *plausible reasoning*, but not the PR definition above, is adopted from Pólya [Pól54]. According to Pólya plausible reasoning is used to "distinguish [...] a more reasonable guess from a less reasonable guess" while "in strict reasoning the principal thing is to distinguish a proof from a guess". The PR definition above is an attempt to relate Pólya's ideas to the results of the study [Lit00c].

EE The reasoning structure is called *reasoning based on established experiences* (abbreviated EE) if the argumentation:
(i) is founded on notions and procedures established on the basis of the individual's previous experiences from the learning environment, and
(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct.

Here the attempt to resolve the problematic situation is based on trying to transfer and combine solution procedures from familiar situations, perhaps superficially and without considering the mathematical properties of the components involved. It may not be possible to decide only from a person's behaviour whether the reasoning is EE or not, this is determined by the underlying thoughts of the person. It is important to stress that EE does not only include rote learning and solving routine exercises by following procedures and mimicking examples. One reason is that the simple keyword strategies that are possible to use in elementary arithmetic (e.g. subtracting if the exercise contains the keyword 'less' [H+95] [Sch91]), are most often not applicable in more complex settings such as calculus. An EE approach is often applied in a problematic situation, which is non routine to some extent, by trying to relate the strategy choice and implementation to something superficially familiar.

In [Lit00c] there was in the examined students' task solving behaviour a distinction between superficial EE approaches and mathematically well-founded PR. PR approaches were relatively rare and of limited range, and this was one of the main reasons for the students' difficulties.

2.2.2 The possibility of implementing superficial solutions in textbook exercises

The reasoning types IS, LPR, and GPR below were found to be central in [Lit00b], where possible (imagined) ways to solve textbook exercises (with the textbook at hand) were studied. The following two distinctions were also introduced:

Intrinsic and surface properties: There will be a distinction between *intrinsic* and *surface* mathematical properties of the components involved in the reasoning. An intrinsic property is deep and central to the component. For example, when comparing general properties of the functions $f(x) = \ln x \cdot \sin x$, $g(x) = \sin x \cdot \ln x$, and $h(x) = \ln(\sin x)$, an intrinsic property is the distinction between product ($f(x)$, $g(x)$) and composition ($h(x)$) of two elementary functions. A surface property may be a consequence of an intrinsic property but carries with itself no or little mathematical meaning, for example the semantic order of the factors (which is written first, 'ln' or 'sin'?). An assertion based on the latter property may be that $f(x)$ is similar to $h(x)$ but not to $g(x)$.

Past and current properties: The exercises of a section are most often related to the subject matter introduced in the section. One of the purposes of the exercises are to provide opportunities to introduce, learn, practice, and consolidate this subject matter: concepts, methods, and other ideas. A mathematical property of a solution component is called *current* if it concerns subject matter introduced in the same (or a close) chapter as the exercise, and *past* if it concerns subject matter treated much earlier. The label 'past' may also be complemented by the approximate time, in educational system years, passed since the subject matter was treated.

IS The reasoning in an exercise solution attempt will be called *reasoning based on identification of similarities* (abbreviated IS) if the reasoning fulfils both of the following two conditions:
(i) The strategy choice is founded on identifying similar surface properties in an example, definition, theorem, rule, or some other situation that is described earlier in the text. This identification does not consider the current intrinsic mathematical properties of the components involved.
(ii) The strategy implementation is carried through by copying the procedure from the identified situation.

Both IS and EE (Section 2.2.1) concern a mathematically superficial transfer of solution procedures from a textbook (IS) or an experience-based (EE) situation. In both EE and IS reasoning, the task solver may seem to be working with advanced mathematics. An IS solution is often short and simple to carry through, but it may be long, technically tricky, and/or require a lot of past basic mathematical knowledge and skills. An IS approach can often be applied in a problematic situation, which is new and none routine to some extent.

LPR The reasoning in an exercise solution attempt will be called *local plausible reasoning* (abbreviated LPR) if it differs from IS in at least one of the following two ways:
(i) The strategy choice is founded on the identification of similarities between

components in the exercise and components in a situation in the text, but these components differ in one or a few local parts, and PR (Section 2.2.1) is used to determine whether the procedure can be copied in order to solve the exercise or not.
(ii) The strategy implementation is mainly based on copying the solution procedure from the identified situation, but one or a few local steps of this procedure are modified by constructive PR.

What differs LPR from IS is that in the former PR is applied locally: in the strategy choice to see *if* the solution procedure can be copied, or in the strategy implementation to see *how* the solution procedure should be modified. The main part of the solution reasoning is still similar to IS. One difference from IS is a consequence of the definition of LPR: IS reasoning may be possible to carry out without considering anything of the current intrinsic mathematics treated. In LPR, since PR reasoning is applied and this can not be done arbitrarily, it may be necessary to understand large parts of the exercise and the identified textbook situation in order to make the required local decisions or modifications.

GPR The reasoning in an exercise solution attempt will be called *global plausible reasoning* (abbreviated GPR) if at least one of the following conditions are fulfilled:
(i) The strategy choice is mainly founded on analysing and considering the current intrinsic mathematical properties of the components in the exercise. A solution idea is constructed and supported by PR.
(ii) The strategy implementation is mainly supported by PR based on current intrinsic mathematical properties.

GPR is similar to LPR in the sense that PR is applied, and therefore it is necessary to understand large parts of the exercise and the identified textbook situation. GPR differs from LPR with respect to the range of the PR reasoning: if it concerns the whole solution (global) or a few limited components (local). If an exercise is not possible to solve by IS or LPR then GPR is required, and in that case the exercise is, to the solver, a genuine problem in the sense of Schoenfeld [Sch85].

In [Lit00b] 600 calculus textbook exercises were classified. About 70 % were possible to solve by IS reasoning, 20 % by LPR reasoning, and 10 % required GPR reasoning. It was analysed in detail how solutions can be carried out, in particular how many exercises may be solved without considering the current intrinsic mathematical properties of the components involved.

3 Research questions

Can some of the causes behind the domination of mathematically superficial reasoning described in [Lit00c] be found in the learning environment that we provide? The purpose of the study is to relate the student task solving behaviour treated in [Lit00a] and [Lit00c] to the possible ways of solving textbook exercises treated in [Lit00b], and will therefore focus on the balance between

PR and more superficial strategies:

Q1: What are students' main strategies when solving textbook exercises? In particular, in what ways are strategy choices and implementations based on the mathematical properties of the components involved (PR), and how is the mathematically superficial (IS) reasoning defined in Section 2.2.2 used?
Q2: How do different strategies and types of reasoning affect students' success and failure in exercise solving?
Q3: What may the consequences of working with different strategies be? In particular, how may these relate to the domination of EE and absence of PR discussed in Section 2.2.1?

4 Method

4.1 Setting

The students who volunteered worked on the tasks in the presence of a video camera, but working alone apart from my help in a few minor situations. They were informed in advance that they should try to "think aloud", but otherwise act as close as possible to their usual way of working when conducting their ordinary homework outside scheduled lecture time. The sessions lasted for 2 hours plus a post-interview. The episodes presented in Section 5 below are fairly complete in order to (i) provide the reader with enough data to be able to question the analysis and conclusions, and (ii) give the reader the 'full picture' of the students' general reasoning characteristics instead of just a few isolated quotations.

4.2 Solution conditions

Each exercise solution attempt is presented and analysed according to a similar framework as in [Lit00b] under the following four subheadings:
Exercise formulation: The exercise quoted from the textbook.
Solution work: The student's solution is described.
Reasoning structure: The solution reasoning is interpreted and structured by the framework from Section 2.1. The interpretation either follows unambiguously from the solution description, is supported by discussions with the student, or is explicitly described as a speculation. It is primarily the structure of the reasoning that substantiates the solution that is described, not what the student will or may learn.
Reasoning characteristics The characteristics of the reasoning structure, in relation to the research questions from Section 3 above, is summarised. Under this heading metacognitive actions are also included. The reasoning characteristics of all exercises are finally summarised and discussed in Section 6.

5 Data and Analysis

The discussion is translated from Swedish, but the quotations from the textbook are originally written in English. Pauses in the quotations are indicated by ... , and minor omitted passages are replaced by [...] .

5.1 Jon

Jon is studying a three-year computer engineer program. About one quarter of the two years he has studied so far has been courses in 'pure' mathematics. The grading system for this program has four grades (fail, 3, 4, and 5) and Jon's average is approximately 4.1, which is relatively high. He says that he does not spend much time studying, and that at the last course (linear algebra) he did only 15 exercises (out of some 200 recommended) but still received grade 5 at that exam. He has earlier taken some extra courses, and plans to study almost twice the ordinary pace (which is very uncommon) this present semester. Thus, at least by some measures, he seems to be a very competent student. Jon received grade 3 at the exam after the video recording of his work below, it seems like he somewhat less successful this time.

In his work with the exercises below, Jon uses information from four different text sources: (a) The textbook [Ada95]. (b) The lecture notes. Jon says that he attends almost all lectures taking notes, and if he is absent he copies them from someone else. (c) The Instructor's solution manual (abbreviated ISM below). This book, that contains solutions to all the exercises in the textbook, is actually used by very few instructors but by many students. (d) A high school formula collection with mathematical formulas.

Jon does not start with reading the text that precedes the exercises in the textbook, and he has not read this text earlier. Instead he opens the textbook's exercise section, the lecture notes, and ISM at the proper pages:
"I want the book to be open at the right place. It is impossible to solve an exercise if you don't have any idea how."
Then he turns to the exercise section:

Textbook Section 9.4 (Arc lengths and areas for parametric curves), Exercise 1

Exercise formulation:

> "Find the lengths of the curves in Exercises 1-8
> **1.** $x = 3t^2$, $y = 2t^3$, $(0 \leq t \leq 1)$"

Solution work: Jon starts by writing down the exercise formulation and looks up the corresponding lecture in his notes. He does not spend any time trying to solve the exercise without searching for IS (Section 2.2.2) information. JL asks Jon what he is doing.
"I am trying to formulate this."

"From...?"
"From the lecture notes, it is a bit hard to find in the book."
"What do you find in your notes? "
"I think it is the length of the curve, I will try to calculate this."
The arc length formula for parametric curves that Jon finds in his notes is the same as can be found in the textbook Section 9.4:

$$s = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt \tag{1}$$

JL asks Jon what he is thinking.
"I will formulate this and see if I reach a correct answer finally."
"And what are you doing?"
"I follow this formula, to see if it gets correct. And then I try to learn this formula so I will know it on the exam."
Jon first finds the derivatives and inserts them in the formula above.

$$x' = 3 \cdot 2t = 6t \quad y' = 2 \cdot 3t^2 = 6t^2$$

$$\int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\ dt = \int_0^1 \sqrt{36t^2 + 36t^4}\ dt$$

$$= \int_0^1 \sqrt{36t^2(1+t^2)}\ dt = \int_0^1 6t\sqrt{1+t^2}\ dt \tag{2}$$

Jon hesitates briefly when calculating the final integral. He rather quickly looks it up in ISM, and JL asks what he is doing.
"I became a bit hesitant, so I have to check how they have thought how to get it out... Aha, they solve it that way!"

The solution to Exercise 9.4.1 in ISM states:

"**1.** $x = 3t^2\ y = 2t^3\ (0 \le t \le 1) \quad \frac{dx}{dt} = 6t\ \frac{dy}{dt} = 6t^2$

$$\text{Length } = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\ dt$$

$$= 6\int_0^1 t\sqrt{1+t^2}\ dt \qquad \text{Let } u = 1+t^2 \quad du = 2t\ dt$$

$$= 3\int_1^2 \sqrt{u}\ du = 2u^{3/2}\Big|_1^2 = 4\sqrt{2} - 2 \text{ units}"$$

JL asks how this differs from Jon's approach.
"They did a substitution, or whatever it is called."
"Was it the integration that became tricky or...?"
"Yes, I did not realise at once how to do it, so I had to look it up."
Jon makes the same substitution as he found in ISM and writes:

$$u = 1+t^2 \quad du = 2t\ dt \qquad t=0 \Rightarrow u=1 \quad t=1 \Rightarrow u=2$$

Then he continues from (2) above, but he makes a slip and forgets the constant 3:

$$= \int_1^2 \sqrt{u}\, du = \left[\frac{2u^{3/2}}{3}\right]_1^2 = \frac{2}{3}(2^{3/2} - 1^{3/2}) \tag{3}$$

Jon wants to know in what form to give the answer. He looks it up in the textbook's solution section and discovers that his answer is not correct. Jon does not try to analyse his work in order to find the mistake, instead he immediately looks at ISM. He easily finds and corrects his slip.

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Search for IS information: an example, a theorem, or a rule where the components in the exercise can be inserted. No time is spent on other considerations.
(a.3) *Strategy implementation*: Jon immediately finds the proper formula (1) in his lecture notes. The competence required to carry out the implementation of the IS solution is: (A) Understanding that the arc length formula requires the evaluation of an integral. (B) Knowing that a main component in this evaluation consists of transforming and simplifying the function expression into one whose primitive function can be fairly easily found. (C) Finding the derivatives of simple polynomials. (D) Finding a primitive function to $6t\sqrt{1+t^2}$. This requires familiarity with the integration method of substitution, treated extensively earlier in the textbook on page 324 ff. (E) Knowing the familiar procedure of applying the fundamental theorem of calculus when evaluating integrals. (F) Being familiar with the terminology used in the earlier steps. Jon swiftly simplifies the integral expressions until he reaches one where it is not obvious to him how to proceed.
(a.4) *Conclusion*: The expression (2) above.

(b.1) *Problematic situation*: How to integrate (2)?
(b.2) *Strategy choice*: Copy this step from ISM.
(b.3) *Strategy implementation*: Straightforward.
(b.4) *Conclusion*: He completes the solution, apart from a small slip.

(c.1) *Problematic situation*: Why is not Jon's answer (3) the same as in the textbook's solution section?
(c.2) *Strategy choice*: Compare with ISM to find and correct the differences.
(c.3) *Strategy implementation*: Straightforward.
(c.4) *Conclusion*: A solution to the exercise.

Reasoning characteristics: Exercise 9.4.1 is an IS exercise, there is complete IS information in the textbook, and Jon's main solution components (a.2) and (a.3) are based on IS reasoning. This conclusion is also supported when Jon says before he starts with the exercise that: "I want the book to be open at the right place. It is impossible to solve an exercise if you don't have any idea how." No problematic situations are resolved by PR.

The IS information in the formula provides a complete solution for the strategy implementation, apart from the evaluation (D) of the integral which is a familiar, though non-trivial, procedure. The mathematical competence (A, B, C, E, and F) used at (a.3) is based only on past mathematical facts and procedures treated in school year 11 (year 2 in upper secondary), two years earlier in Jon's perspective. There are no indications, and not required, that Jon considers any intrinsic properties related to arc length.

Jon works swiftly at a high pace and the implementation of the IS strategy and the familiar procedures is straightforward, except at (b.1) and (c.1) where he very quickly turns to ISM. In all three problematic situations described above Jon's strategy choices are made very quickly. There is no time spent analysing and understanding intrinsic mathematical properties in any of the problematic situations. In (a.2) Jon's statement "I *think* (author's emphasis) it is the length of the curve, I will try to calculate this" indicates that he is not sure that he is using the proper formula. Jon resolves this uncertainty by testing the formula with the aim to compare the result with the textbook's solution section: "I follow this formula, to see if it gets correct. And then I try to learn this formula so I will know it on the exam." His goal seems to be to 'insert' the exercise into the formula, which is to be learnt for the exam. The consequences of his solution work on the development of his mathematical competence may be rather limited, perhaps only that he evaluates yet another integral (he has already evaluated hundreds of them) and better remembers the arc length formula. There are no problem solving activities like PR reasoning, heuristics, or control involved.

The IS strategy used at (a.2) is defined in section (2.2.2) above. The ISM strategy in (b.2) and (c.2) is an 'extreme' version of IS, and is therefore defined as a strategy of its own:

Available Solution: The reasoning in an attempt to resolve a problematic situation will be called *reasoning based on an available complete solution* (abbreviated AS) if:
(i) The strategy choice is to study a complete solution to exactly the same problematic situation.
(ii) The strategy implementation is carried through either (A) by copying the found solution or (B) by comparing how the found solution differs from ones own attempt in order to detect, understand, and correct mistakes.
In an IS approach the strategy choice is to compare with a *similar* one, while in an AS approach it is to compare with a solution to an *identical* problematic situation. The strategy implementation version (A) may be carried out with or without the aim to understand the provided solution. If (A) is carried out 'blindly', without any attempts to consider the mathematical properties of the components involved, then this version of AS can be seen as the 'ultimately superficial' IS strategy. The main source containing AS information is, to the

students in this study, ISM. It should be noted that AS reasoning is not necessarily easier than IS reasoning. The AS solution may be very concise and the corresponding IS information exhaustive and and very similar to the exercise that is to be solved.

Textbook Section 9.4, Exercise 3

This is an IS exercise, but two mistakes lead to 55 minutes of solution work.

Exercise formulation:

> "Find the lengths of the curves in Exercises 1-8
> **3.** $x = a\cos^3 t,\ y = a\sin^3 t,\ (0 \le t \le 2\pi)$"

Solution work, part one: Jon writes down the exercise, and then immediately opens his formula collection. JL asks what Jon is doing.
"I am uncertain about the differentiation rules for sine and cosine [his plan is the same as in Exercise 9.4.1: to find x' and y' and insert them in the arc length formula (1)]. There is always a minus sign that I miss."
Jon does not find the rule in his formula collection [The proper rule to use, the chain rule (the product rule is more laborious but also possible to apply), is actually included in his formula collection but written in a general form $(f(g))' = f'(g) \cdot g'$. There is no explicit example in the formula collection on how to differentiate $a\cos^n t$ or $a\sin^n t$], and Jon applies a faulty recalled rule:

$$x' = a3(-\sin^2 t) \qquad y' = a3(\cos^2 t) \tag{4}$$

[The correct derivatives are $x' = a3(\cos^2 t)(-\sin t)$ and $y' = a3(\sin^2 t)\cos t$. This 'stall mistake' (a mistake that stalls the progressive part of his work) leads him into unprogressive work (an integral that is impossible for Jon to evaluate) that will last for 32 minutes.] Jon inserts his faulty derivative (4) into the arc length formula (1):

$$\int_0^{2\pi} \sqrt{(a3)^2 \sin^4 t + (a3)^2 \cos^4 t} \tag{5}$$

$$= \int_0^{2\pi} \sqrt{(a3)^2(\sin^4 t + \cos^4 t)} = a3 \int_0^{2\pi} \sqrt{\sin^4 t + \cos^4 t}\, dt \tag{6}$$

Jon now wants to simplify the last expression further, in order to reach a familiar function expression that he can find a primitive function to:
"Then one can use the Pythagorean identity [$\sin^2 x + \cos^2 x = 1$], if one is lucky... maybe. If I am not mistaken... no..."
JL asks what Jon is thinking.
"I am considering if I could use the Pythagorean identity. But it is to the power 4, so I am uncertain if one can do it... like that... Perhaps it works..."
Jon expands $(\sin^2 t + \cos^2 t)^2$ and substitutes this into the integral:

$$\left((\sin^2 a + \cos^2 a)^2 = \sin^4 a + \cos^4 a + 2\sin^2 a \cos^2 a \right) \tag{7}$$

$$= 3a \int_0^{2\pi} \sqrt{\underbrace{(\sin^2 t + \cos^2 t)}_{1}{}^2 - 2\sin^2 t \cos^2 t}\, dt \tag{8}$$

Jon can not find a familiar primitive function to the expression in (8), so he searches in his formula collection for formulas that could simplify it further:
"Let's see if I can find the formula for this..."
"Are you searching for other formulas?"
"Yes. There should be another formula, but I am not certain that there is one... perhaps... yes, I can do this... perhaps I could have done this in an easier way..."
Jon finds another formula and makes a substitution:

$$= 3a \int_0^{2\pi} \sqrt{1 - \frac{1}{2} sin^2 2t} \tag{9}$$

"It must be like this... I am uncertain now."
"What are you uncertain about?"
"What it shall be in the end."
Jon opens ISM and finds the solution to Exercise 9.4.3 which states:

"**3.** $x = a\cos^3 t,\ y = a\sin^3 t,\ (0 \leq t \leq 2\pi)$. The length is

$$s = \int_0^{2\pi} \sqrt{9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\, dt \tag{10}$$

$$= 3a \int_0^{2\pi} |\sin t \cos t|\, dt = 12a \int_0^{\pi/2} \frac{1}{2} \sin 2t\, dt \tag{11}$$

$$= 6a(-\frac{\cos 2t}{2})\Big|_0^{\pi/2} = 6a \text{ units.''} \tag{12}$$

"They have arrived at almost the same thing, but not exactly. I don't really know what I have done wrong. It seems like a difficult exercise... I am considering to start from the beginning... I will try a substitution instead."
Jon writes

$$u = \sin^4 t + \cos^4 t, \tag{13}$$

but does not continue the implementation of the substitution attempt. One may note that he does not start from the very beginning, and in particular that he 'restarts' at a point in the solution after his differentiation mistake (4).
"No, I don't understand what they mean in ISM, and I don't understand how I can solve this myself."

He returns to ISM and searches for differences between his and ISM's solution.

"Oh yes... wait a minute..."
"What do you see?"
"They have done it in some other way."
"At which step in the solution?"
"It seems like I did something wrong here (points at (4)). If I understand this correctly. This (5) is not like in ISM... I have done... or they have done something different..."

[Comment: Jon finds parts of the difference between his solution and ISM's but not the central one: his stall mistake at (4) where his problems actually started. He realises that his expression (5) is not the same as ISM's (10). This is not difficult to see, but perhaps he thinks that the functions may be the same through some trigonometric formula that he does not know about? His pointing at (4) indicates that he suspects that his derivatives are wrong, but still he continues to work for a while with the integration.]

Jon continues to search for information in his lecture notes, in ISM, and in the formula collection. After a while JL asks what Jon is thinking.
"I saw that there may be a formula I can use in the formula collection... But I don't know if it will help me... perhaps I have done it right..."
He writes:

$$\frac{1}{2}\sin^2 t \Rightarrow \frac{1}{2}(\frac{1-\cos 4t}{2}) \tag{14}$$

"What are you thinking?"
"How they have got it all together. It seems impossible. I will start all over."
Jon tears off the paper that his work so far is written on.
"Do you often start all over?"
"Sometimes, sometimes I go to another exercise. But I have to learn this at some point. It should not be that difficult."

After a while Jon says:
"In ISM they have a completely different solution."
"What are you writing?"
"I check this backwards. What they have done. They have done something strange."
Jon realises that (10) in the ISM solution is obtained by inserting the derivatives into the arc length formula (1). His reasoning is then correctly that he could find x' and y' by taking the square roots of the two terms in (10), and he obtains:

$$x' = -3a\cos^2 t\sin t \qquad y' = 3a\sin^2 t\cos t \tag{15}$$

"I took the square root out of what they have written in ISM. It seems like it could be this way... or perhaps the other way around [meaning the '-' sign]..."
He is not convinced that this is the right expression, and it is clear from his work below that he is eager to know what differentiation rule that yields these derivatives from the functions $x(t)$ and $y(t)$.

Jon searches in the formula collection, and JL asks what he is looking for.
"How it really should be."

"I see that you are looking at differentiation rules?"
"Yes... I am uncertain... I have always been (laugh)."
Jon starts to search in the textbook, and JL asks what he is searching for.
"How they use these differentiation rules, what they are really doing. They have done this in these pages [in Section 9.4, which contains four pages + exercises]. They have a lot of cosine things... then it should be here... I think... Because it is only Exercise 3 and then they should have treated this in an example... I think... Aha!"
"Why do you say 'aha'?"
"They are doing it in a way I did not think was allowed (laugh)!"
"In what way?"
"Like this (points at Example 9.4.2). Here is the same exercise, almost. There is something that differs, but it is mainly the same."
Example 9.4.2 states:

> "**EXAMPLE 2** Find the area of the surface of revolution obtained by rotating the astroid curve $x = a\cos^3 t,\ y = a\sin^3 t$ (where $a > 0$) about the x-axis.
> **SOLUTION** The curve is symmetric about both coordinate axes. (See figure 9.30. [this figure is omitted here]) The entire surface will be generated by rotating the upper half of the curve, and, in fact, we need only rotate the first quadrant part and multiply by two. The first quadrant part of the curve corresponds to $0 \leq t \leq \pi/2$. We have
>
> $$\frac{dx}{dt} = -3a\cos^2 t\sin t, \qquad \frac{dy}{dt} = 3a\sin^2 t\cos t. \tag{16}$$
>
> Accordingly, the arc length element is
>
> $$ds = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\ dt$$
>
> $$= 3a\cos t\sin t\sqrt{\cos^2 t + \sin^2 t}\ dt = 3a\cos t\sin t\ dt.$$
>
> Therefore the required surface area is
>
> $$S = 2\cdot 2\pi\int_0^{\pi/2} a\sin^3 t\ 3a\cos t\sin t\ dt$$
>
> $$= 12\pi a^2\int_0^{\pi/2}\sin^4 t\ \cos t\ dt \qquad \text{Let } u = \sin t,\ du = \cos t\ dt$$
>
> $$= 12\pi a^2\int_0^1 u^4\ du = \frac{12\pi a^2}{5} \text{ square units.} \qquad []"$$

The first part of the example solution is similar to a solution to Exercise 9.4.3, in particular x' and y' are the same. As will be seen below, Jon is not satisfied with just finding x' and y', he wants to learn a rule to use to find these derivatives. Such a rule is not explicitly described in Example 9.4.2.

Jon reads Example 9.4.2 carefully. He then searches the textbook register for more differentiation examples and then studies several sections, including some about more general differentiation rules (e.g. the chain rule, which could have been applied by Jon) and one section about derivatives of trigonometric functions [Ada95, p. 116]. In the latter he finds examples of derivatives of different types of trigonometric functions (e.g. $\cos x$, $x^2 \sin x$, $\sin t \cos t$, etc.), but no example of the derivatives of $\cos^n x$ or $\sin^n x$.
"They were too easy. In that chapter... It is a bit difficult to find things in the book. It is in English and rather thick. One is moving back and forth. If one is uncertain about the foundations one has to check it up."
"What are you looking for?"
"I am looking for... something that looks like the one I had before, but a bit more well-written."
"Do you mean the whole exercise, or only the derivative?"
"When it is kind of... raised to... like sine raised to something... and then take the derivative of this. Why it became like it did... how they got it out... I don't really understand this. I don't think I have worked so much with this raised to... I think anyway... I have to find out why it is so."

Jon continues to search the textbook for more information. In Section 6.6 he finds:

> "**EXAMPLE 7** Evaluate the integrals:
> (a) $\int \sin^2 x \cos^3 x \; dx$, (b) $\int \sin^3 x \cos^8 x \; dx$, and (c) $\int \cos^5 ax \; dx$
> **SOLUTION** (a) $\int \sin^2 x \cos^3 x \; dx = \int \sin^2 x (1 - \sin^2 x) \cos x \; dx$
> (Let $u = \sin x, \; du = \cos x \; dx$)
> $= \int u^2(1-u^2) \; du = \int (u^2 - u^4) \; du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$ [The solutions to (b) and (c) are similar and omitted here]"

"What have you found?"
"I just wanted to check if I saw something interesting. Perhaps I did... They are using u here... that seems OK... It became too tricky."
Jon turns the pages rapidly and a bit planlessly for a while.
"No, I can't find it. I have to go on what I've got. I think."
Jon returns to Example 9.4.2 above.
"I will just have to assume that it is so."
"Is what?"
"That it comes out a such one automatically."
"Comes out what?"
"An extra sine thing, out of this."
"Can you point at what you mean?"
Jon points at (16).
"I will have to check this."
"What are you looking for in the formula collection?"
"That it really is so..."
Jon returns to the textbook.

"What they are doing is perhaps that... Yes, now I get what they are doing. They take this ($\cos^3 t$), then they extract this one sine. Yes... Then it should be..."
Jon writes to test his idea:

$$x = a\cos^4 t, x' = -4a\cos^3 t \sin t \qquad (17)$$

[Jon's idea is thus correctly that if $x = a\cos^n t$, then there is some rule that says $x' = -na\cos^{n-1} t \sin t$]
"So perhaps... I'll take a chance on that... I think. One has to guess a bit, and see if it adds up. If not sooner, you will find out at the exam."

Reasoning structure, part one:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: It is clear to Jon that he shall use the same IS information, the arc length formula, that he used in exercise 9.4.1.
(a.3) *Strategy implementation*: Immediate.
(a.4) *Conclusion*: Insert the exercise components in the arc length formula.

(b.1) *Problematic situation*: In order to use the arc length formula, the derivatives of $a\cos^3 t$ and $a\sin^3 t$ need to be found.
(b.2) *Strategy choice*: Search for the rule in the formula collection.
(b.3) *Strategy implementation*: The proper rule is not found since it is (of course) written on a general instead of a specific form. He makes a stall mistake by wrongly recalling:
(b.4) *Conclusion*: The expression (4) above.

(c.1) *Problematic situation*: The insertion of (4) into the arc length formula is straightforward to begin with, but at (6) he reaches a difficulty: If Jon had inserted the correct derivatives he would have obtained an easier integral and the implementation could have proceeded, but how to turn (6) into an integrable function?
(c.2) *Strategy choice*: From his experience, earlier exercises often contain expressions that should be cleverly simplified using the familiar fact that $\sin^2 t + \cos^2 t = 1$. It is a good idea to try but fruitless because of his stall mistake at (4). This can be seen as EE reasoning, since it is based on surface similarities with earlier situations and no PR is involved.
(c.3) *Strategy implementation*: Jon skilfully implements his idea at (7), but is stalled at (8).
(c.4) *Conclusion*: Jon needs to find more information.

(d.1) *Problematic situation*: Same as (c.1).
(d.2) *Strategy choice*: Jon searches the formula collection and finds a trigonometric formula that contains the term $2\sin^2 t\cos^2 t$ in (8). He inserts this and tries to simplify.
(d.3) *Strategy implementation*: Fails at (9).

(d.4) *Conclusion*: Same as (c.4).

(e.1) *Problematic situation*: Same as (c.1).
(e.2) *Strategy choice*: Jon reads the ISM solution, but he spends almost no time trying to analyse and understand it.
(e.3) *Strategy implementation*: Jon fails to discover the central difference, that ISM's solution contains other derivatives at (10), and abandons this approach.
(e.4) *Conclusion*: Same as (c.4).

(f.1) *Problematic situation*: Same as (c.1).
(f.2) *Strategy choice*: In a similar way as (c.2) Jon tries another familiar method, a substitution, and sets $u = \sin^4 t + \cos^4 t$.
(f.3) *Strategy implementation*: He interrupts the implementation at (13).
(f.4) *Conclusion*: Same as (c.4).

(g.1) *Problematic situation*: Same as (c.1).
(g.2) *Strategy choice*: Jon reads the ISM solution again.
(g.3) *Strategy implementation*: This time he reads more carefully than at (e.3), but still without understanding all central components, and notes that the expression (5) is not the same as the corresponding expression (10) in ISM.
(g.4) *Conclusion*: He seems to suspect that his derivatives (4) are wrong.

(h.1) *Problematic situation*: Same as (c.1).
(h.2) *Strategy choice*: Search in the lecture notes, ISM, and the formula collection for information.
(h.3) *Strategy implementation*: He searches fairly rapidly and finds only (14) which he is unable to use.
(h.4) *Conclusion*: He fails, so he tears of the paper and "starts all over".

(i.1) *Problematic situation*: Jon doubts that (4) is correct.
(i.2) *Strategy choice*: Analyse the ISM solution, this time more thoroughly than at (e.3), (g.3), and (h.3), and with the specified purpose of finding the correct derivatives: Since (10) in the ISM solution is obtained by inserting the correct derivatives in the arc length formula (1), the correct derivatives can be obtained by 'working backwards'. This reasoning can be classified as LPR (Section 2.2.2), a kind of 'AS (page 11) heuristics', with the aim of getting access to information that is not explicitly provided in the ISM solution.
(i.3) *Strategy implementation*: Straightforward.
(i.4) *Conclusion*: The correct expression (15).

(j.1) *Problematic situation*: Jon has found the derivatives (15), but he wants to learn the rule that led to them.
(j.2) *Strategy choice*: Search for information about differentiation rules. Jon searches first in his formula collection, then in the Section 9.4 in the textbook. It is clear that his search is based on finding information about $\sin^n t$ and $\cos^n t$, not about general principles (see also (k.3)): He searches for "How they use these

differentiation rules, what they are really doing. [...] They have a lot of cosine things... then it should be here... I think..." Jon continues by saying that there ought to be IS information provided, since it is one of the first (and therefore one of the easiest) exercises in Section 9.4: "Because it is only Exercise 3 and then they should have treated this in an example... I think..."
(j.3) *Strategy implementation*: Jon finds Example 9.4.2. His comment "there is something that differs, but it is mainly the same" indicates that he does not (try to?) understand the intrinsic properties that differs Example 9.4.2 from Exercise 9.4.3: the difference between finding length and area. The reasoning is based on considering surface properties, but leads here to a situation where useful IS info may be found (at (k.3) it does not).
(j.4) *Conclusion*: Example 9.4.2 (16) verifies that (15) is correct but does not explain the differentiation rule, and thus does not resolve (j.1).

(k.1) *Problematic situation*: Same as (j.1).
(k.2) *Strategy choice*: Search for more information in the textbook.
(k.3) *Strategy implementation*: Jon spends quite some time and it is becoming clearer that he is searching for surface similarities and disregarding intrinsic properties, there are several factors that support this claim: (I) He looks briefly in the textbook sections that treats the chain and product rules but does not realise that it is one of these rules he needs, probably because they are written in a general form, and there are in these sections no examples on how to differentiate functions of the form $\cos^n t$ or $\sin^n t$. (II) He studies more carefully Section 2.4 with elementary differentiation rules of trigonometric functions but says that "they were too easy" and that he is looking for "something like the one I had before, but a bit more well-written". This probably means that he is searching for particular rules or more examples on how to differentiate $\sin^n t$ and $\cos^n t$, "like sine raised to something", not general principles. Another indication of this is when he says "I don't think I have worked so much with this raised to". (III) After searching through many pages he studies Example 6.6.7 above because it contains the similar surface properties, the 'keywords', $\sin^2 x$ and $\cos^3 x$. Jon does not comment that this example treats completely different mathematical intrinsic properties than Exercise 9.4.3, he just says that "it became too tricky". (IV) In the post-interview he says "I did not think about the chain rule since it was cosine, I am so unused to working with cosine." It seems like Jon wants a special case rule 'served', without having to analyse the text information deeply. He then continues to search through some more pages, before he temporarily gives in:
(k.4) *Conclusion*: "No, I can't find it (the differentiation rule for $\cos^n t$ or $\sin^n t$). I have to go on what I've got." He "just has to assume" that the derivatives (16) are correct, without knowing the rule.

(l.1) *Problematic situation*: In spite of his decision at (k.4), Jon is still so eager to learn the differentiation rule that he immediately resumes his attempts to resolve (j.1).
(l.2) *Strategy choice*: Search once again in the formula collection and textbook.

(1.3) *Strategy implementation*: While Jon once again reads Example 9.4.2 he, for some reason, thinks of a possible rule:
(1.4) *Conclusion*: He finds the formulation of the correct rule that is tested at (17), but not the reason behind it (he does not seem interested in the latter). He can now restart from the beginning of the exercise by inserting the correct derivatives in the arc length formula (1).

Solution work, part two: Jon restarts from the beginning.
"What are you doing now?"
"I am writing down the exercise, and I am going to solve it. I have checked what it is like, so now I should be able to solve it without looking in the book, I think... I think."
Jon inserts x' and y', this time the correct derivatives, into the arc length formula:

$$x' = -3a\cos^2 t\sin t, \qquad y' = 3a\sin^2 t\cos t$$

$$\int_0^{2\pi} \sqrt{(-3a)^2(\cos^2 t\sin t)^2 + (3a)^2(\sin^2 t\cos t)^2}\, dt$$

$$= 3a\int_0^{2\pi} \sqrt{(\cos^4 t\sin^2 t) + (\sin^4 t\cos^2 t)}\, dt$$

$$= 3a\int_0^{2\pi} \sqrt{(\cos^2 t\sin^2 t)\underbrace{(\sin^2 t + \cos^2 t)}_{1}}\, dt = 3a\int_0^{2\pi} \cos t\sin t\, dt \tag{18}$$

Jon works swiftly and is algebraically skilled, but in the last expression he makes another stall mistake: He wrongly sets $\sqrt{\cos^2 t\sin^2 t} = \cos t\sin t$, the correct equality is

$$\sqrt{\cos^2 t\sin^2} = |\cos t\sin t| \tag{19}$$

which is the expression that is integrated at (11) in ISM.
Jon makes the same substitution ($u = \sin t$, $du = \cos t\; dt$) as in Example 9.4.2:

$$= 3a\int u\; du = 3a\left[\frac{u^2}{2}\right] = 3a\left[\frac{\sin^2 t}{2}\right]_0^{2\pi} = \frac{3a}{2}(\sin^2 2\pi - \sin 0) \tag{20}$$

When Jon compares his answer [he missed to square $\sin 0$, but this is not the central mistake] with ISM:s (12) he notes that they are different:
"Perhaps it was not possible to do this way?... They [ISM] are using a formula."
"What formula?"
"This one (points in his formula collection at $\sin 2a = 2\cos a\sin a$, which corresponds to (11) in ISM). Then they got it in another form, that I hadn't... I think... but I am uncertain... I thought I got it nicely... Perhaps it is not

possible to do like this, anyway... really difficult..."
Jon is searching for his mistake in the wrong part of his solution: He would have reached the same faulty result if he had used the same formula as ISM.
"At the exam I will not know these [trigonometric] formulas. Then I don't know if I will be allowed to have the formula collection... Perhaps, I don't know."
"What do you do if you don't have the formula collection at the exam?"
"Then I have to learn the formulas. That is the only alternative... And hope that I pass the exam. This course was much harder than I thought it would be at the beginning. At the beginning it felt easy. Then it became so much, in some way... After about half the course, it felt like much... But it should be possible to solve in my way I think."
"What is your way"
"To insert $u = \sin t$, and do du/dt, to differentiate. I believe this way should be possible, but apparently not... perhaps it works, but not as it looks right now... This means that I have to do like they have done in ISM."

Jon uses the formula $\sin 2a = 2\cos a \sin a$ that he saw in the ISM solution at (11) but once again skips the absolute value signs, and therefore misses the central change in integration interval when he starts over from (18):

$$3a\int_0^{2\pi} \cos t \sin t \, dt = \frac{3a}{2}\int_0^{2\pi} \sin 2t \, dt = \frac{3a}{2}\Big[\frac{(-\cos 2t)}{2}\Big]_0^{2\pi} = -\frac{3a}{4}(\cos \quad (21)$$

Jon interrupts his work and after a while JL asks:
"Are you following ISM?"
"I am trying to do the [ISM] substitution, to use the trigonometric formulas. And in that way be able to move... in a more correct way."
"This is the ISM method. It is not the earlier substitution method you used?"
"No, it is not. I am getting confused... It ought to work, but I did not get it correct... But I see something else here. It could have something to do with what the sine curve looks like. They have written $\pi/2$ [pointing at ISM's upper limit of integration at (11): Jon finds his mistake].

Jon draws the curve $1/2 \sin 2t$ on his graphic calculator.
"They takes it times $12a$. Otherwise they cancel each other [probably meaning integration over negative and positive parts of the function]. I am afraid that it does for me, therefore I get something wrong. In some way they have found out that they can only do in a certain way... they change the interval... they want the positive part several times... But this is really difficult. This is why sine is so difficult. They can react in different ways... and you are not prepared for all situations... since you don't now them really. Or you know how it works, but you don't know all special cases... What they do is to extract a 4 out of this... yes, now I see it... But perhaps this is allowed...? I think I have understood this exercise now... then I'll move on."
Though he is hesitant, Jon believes that he has found his mistake and understood the ISM solution. After 55 minutes of hard work he turns to the next exercise without completing his solution to Exercise 9.4.3.

Reasoning structure, part two:
(m.1) *Problematic situation*: The exercise.
(m.2) *Strategy choice*: Insert the correct derivatives (15) into the arc length formula (1). He believes that he has found sufficient IS information to "solve it without looking in the book".
(m.3) *Strategy implementation*: Straightforward until he makes the stall mistake at (18) and later notes that the faulty answer (20) does not fit with the ISM solution (12) (he does not note the contradiction that his answer actually is 0 while the curve should have positive length).
(m.4) *Conclusion*: Jon's IS strategy implementation is stalled.

(n.1) *Problematic situation*: Why is (20) wrong?
(n.2) *Strategy choice*: Jon compares his solution with ISM's.
(n.3) *Strategy implementation*: Jon finds a difference, that he made the substitution at (20) and ISM used the trigonometric formula $\sin 2a = 2\cos a \sin a$ at (11). His insight into the intrinsic properties is not deep enough to help him realise that both methods work and that the origin of his problems is elsewhere, in his stall mistake (18). Instead he wonders if his substitution method, "To insert $u = \sin t$, and do du/dt, to differentiate", is perhaps not applicable and causes his problems: "I thought I got it nicely... Perhaps it is not possible to do like this." Jon wants to understand why his method does not work, but does not put in a lot of effort to find out.
(n.4) *Conclusion*: He reluctantly gives up his attempts to carry out 'his method' (20), and decides that: "This means that I have to do like they have done in ISM [(11)]".

(o.1) *Problematic situation*: Same as (n.1).
(o.2) *Strategy choice*: Jon decides that he has to "do like they have done in ISM", "to do the [ISM] substitution, to use the trigonometric formulas". He does not want to copy the solution blindly step by step, he just applies ISM's trigonometric formula (11) and performs the calculations by himself in order to understand each local step.
(o.3) *Strategy implementation*: Jon again misses to note the central absolute value signs and change in the limits of integration at (11). He interrupts his work at (21) since it does not fit with ISM's solution and becomes more puzzled: "I am getting confused... It ought to work, but I did not get it correct." After a while he discovers that there is a change in limits of integration in ISM because "otherwise they cancel each other". The base of his discovery is that he notes the symbolic difference between his limits of integration (0 and 2π at (21)), compared to ISM's (0 and $\pi/2$ at (11)). There are no signs that he has any deeper insight into why it is so by considering the intrinsic properties of (19) or the fact that curves have positive length on every subinterval: "I get something wrong. In some way they have found out that they can only do in a certain way... they change the interval... they want the positive part several times... But this is really difficult. This is why sine is so difficult. [...] But perhaps this is allowed?" Jon seems to believe that the problem is that he is working with

sine functions (a surface property), not that he is wrongly trying to calculate the curve length by integrating a partly negative function (an intrinsic property).
(o.4) *Conclusion*: Jon believes that he has understood each step of the ISM solution and finally, after 55 minutes of hard work, leaves Exercise 9.4.3 without completing his solution.

Reasoning characteristics: Exercise 9.4.3 is an IS exercise that is very similar to Exercise 9.4.1 above, it could have been solved in a few minutes, and Jon's main strategy choice at (a.2) and implementation is based on IS reasoning. The mathematical competence required to carry out the IS solution is essentially the same as in Exercise 9.4.1 (reasoning structure step (a.3)), and, again, it is not required to consider any intrinsic properties of arc length or apply any PR. However, there are some components that are slightly more difficult than in Exercise 9.4.1. Jon makes stall mistakes in two of them, at (4) and (18), that lead to very long (55 minutes) and laborious solution work even though he is a skilled student and does not follow unprogressive approaches very far.

To find the derivatives x' and y' is supposed to be a minor part of the solution, that concerns past properties, but it takes Jon 32 minutes to resolve his stall mistake at (4). The central characteristics of his work during these 32 minutes are that:

(A) Jon is able to try very many different approaches at (c-k) above, mainly searching for information or applying familiar procedures. He seems used to search for IS information in the textbook, his lecture notes, the formula collection, and ISM. This is complemented by applications of familiar procedures at (c.2) and (f.2) to test if the changes will lead into an integrable function.

(B) He follows none of these approaches very far, in particular he is not hindered by the 'follow-no-matter-what-happens' implementation of one single non-productive approach that Schoenfeld [Sch85] [Sch92] described as one of the main reasons behind students' problem solving difficulties. In this sense his control [Sch85] is rather good.

(C) Jon's work is mainly based on considering surface properties of the components involved. There is no GPR and very little LPR involved when he tries to resolve the problematic situations. His reasoning seems to be mainly based on EE, IS, and AS. The data supporting this claim is: (I) Jon explicitly says (j.2) that the there ought to be IS information provided. (II) Jon has earlier worked with very many integration exercises where the main part of the solution work consists of simplifying integrands (the functions to integrate). These are often solved with familiar methods like the ones he tries at (c.2) and (f.2). (III) When Jon at several occasions searches for information, for example at (b, h, j, and k) he sometimes misses the texts that treat relevant general intrinsic properties, and instead focuses on text parts that treats, mainly irrelevant, surface similar properties (see in particular the data analysis at (j.2) and (k.3)). Another indication that he does not analyse the IS information deeply is at (j.3) when he says "there is something that differs" but he does not try to find out what. (IV) At (e, g, h, and i) he tries to apply AS (version (B), see page 11), and deduce from ISM how to resolve the problematic situations. First at (e) very quickly and superficially, then more carefully, and it is not until (i) that his (LPR complemented) analysis of the ISM solution is deep enough to help him find the correct derivatives. (V) At (j, k, and l) Jon is searching for the particular rule how to differentiate $\cos^n t$ and $\sin^n t$, not the reason behind it or for some more general rule. This interpretation is based on that he is very eager to find a rule but once he finds it ("Yes, now I get what they are doing") at (17) he does not spend any time considering its background. Jon never mentions anything about the mathematical foundation of the rule, why it is valid, or anything related to more general ideas (e.g. the chain rule).

(D) Since Jon's work is based on surface properties, he is unable to recognise useful IS information (like Example 9.4.2 and the general chain rule) when he reads it, and to distinguish it from unuseful (for example at (b, d, h, j, k, and l)) information. Therefore he needs to test many approaches, sometimes the same ones several times, without making progress and this is the main reason that his solution work is so very long (55 minutes). If Jon's control (see (B)) has not been efficient, he might even have spent much more time on each approach.

(E) In a similar way, since his AS attempts are very superficial to start with, Jon is first unable to make use of the AS information in ISM. This indicates that AS (version B) is often difficult to use, since the ISM solutions are often compact.

(F) It is clear that Jon does not accept to blindly use AS (version A). He wants to understand each step, both when he uses IS and AS information. The data supporting this claim is: (I) After having read the ISM solution at (11), he does not just copy it but tries to to make his own solution work and understand each local step. (II) After (14) Jon says "I have to learn this". (III) After being fairly convinced that he has found the correct derivatives at (15), he continues to search for the differentiation rule and wants to learn "How they use these differentiation rules, what they are really doing". (IV) After being completely convinced that he has found the correct derivatives at (16) he still continues to search for the differentiation rule. (V) Jon reluctantly gives up searching for the rule at (k.4), but resumes his attempts. He stops searching immediately after he finds a plausible rule at (l.4).

(G) At the same time there are no indications that he wants to learn anything but the local details of the solution to this particular exercise type, that he wants to learn global solution ideas or general principles of differentiating products or composite (of the form $f(g(x))$) functions.

Jon believes at (m.2) that he has found sufficient IS information, but makes another stall mistake at (18) that concerns both the past intrinsic properties of (19) and the present intrinsic basic property that curve length is positive. His work is again rather superficial and for similar reasons as (D) above it takes as long as 21 minutes to resolve the stall mistake. The central characteristics of his work during these 21 minutes is that:

(H) Jon's main strategy is to read and understand the AS information provided in ISM, but his analysis of the ISM solution is too superficial and 'effortless' so he does not realise: (I) That his answer at (20) is unreasonable since it is 0. (II) Why his method does not work (n.4). (III) The complete reasons behind his difficulties at (o.3), instead he focuses on the surface properties of sine and cosine. In particular, he seems satisfied with his (partial) understanding of *what* is done in ISM, and disregards *why* this is done.

(J) There is in Jon's work a general IS focus and he is worried if he will recall and be able to use the trigonometric formulas at the exam:
"At the exam I will not know these formulas. Then I don't know if I will be allowed to have the formula collection... Perhaps, I don't know."
"What do you do if you don't have the formula collection at the exam?"
"Then I have to learn the formulas. That is the only alternative... And hope that I pass the exam."
From the way he works it seems like he sees the recollection and application of proper formulas as a more central strategy base than intrinsic mathematical understanding.

(K) The search for local understanding is not PR-based. He may be interested in finding intrinsic principles, but the base for the information search is surface IS.

(L) It seems like he, as in (F) and (G) above, wants to learn and understand each step of ISM's solution on a local but not global level. Jon's goal is probably not to learn the general intrinsic ideas, which he does not once mention, but all special cases that he may meet, here represented by the case of finding the arc lengths of sine curves: "This is why sine is so difficult. They can react in different ways... and you are not prepared for all situations... since you don't now them really. Or you know how it works, but you don't know all special cases."

(M) He is unwilling (as in (F) above) to accept 'blind copying' of methods and is very reluctant at (n.4) to give up 'his method' (and blindly follow ISM) since he does not understand why his method does not work.

5.2 Ulf

Ulf is studying to become an upper secondary teacher in mathematics and Swedish. In his first semester of mathematics studies he failed all four different course exams and the corresponding re-exams. In the following semester, where the session analysed below took place, he was studying the same courses again and had managed to pass three of the exams (all at the third attempt). The data below describes his work in the fourth course ("Calculus 2", treating mainly integration and series). Ulf failed the subsequent exam but later passed at his fourth attempt. Thus his mathematics difficulties are more severe than for the average student, but by no means exceptional. Ulf has also studied five semesters of other university subjects than mathematics (mainly Swedish), and passed all course exams at first attempt.

Ulf starts by reading the theory text of Section 8.3 in [Ada95] for 30 minutes,

before turning to the exercises. He studies mainly the solved examples and tries to understand each solution step:
"I think it gives me more to read the examples than to just read the text. And they often skip so many steps that one has to make ones own calculation, to see what have happened"

Textbook Section 8.3 (Arc length and surface area), Exercise 1

Exercise formulation:

> "In exercises 1-16, find the lengths of the given curves.
> 1. $y = 2x - 1$ from $x = 1$ to $x = 3$"

Solution work: Ulf starts by writing down the exercise formulation and draws the graph. JL asks him to explain what the task is.
"One is supposed to find the length of the curve from $x = 1$. I know what the curve looks like, but I would anyway like to draw it, to have it in front of me so that I can think about what I am doing. Therefore I am drawing it on the calculator... So I am supposed to find this length (points at his correct curve, which he actually does not use for anything later)."

The exercise is on textbook page 422, and Ulf looks up the arc length formula in a blue-marked box on page 417 which states:

> "The arc length s of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by
> $$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx."$$

JL asks Ulf what he is doing.
"I go back to see what the formula looks like. I write it down so it will perhaps go into my head. If I write it down I will perhaps have it in my head."
Ulf copies the formula from the book. The implementation is essentially straightforward and he writes:

$$y' = 2 \qquad s = \int_1^3 \sqrt{1 + (2)^2}\, dx = \int_a^b \sqrt{5}\, dx$$
$$[5x]_1^3 = 15 - 5 = 10$$

Ulf makes a small slip, he forgets the square root in the last line, and says:
"Hm, this did not become a good integral."
"Why not?"
"It became too easy (laugh). One gets suspicious when it gets too easy."

Ulf looks up the textbook's solution section.
"One can check the answer on the calculator if one likes, but one can also check in the solution section."

"What do you usually do?"
"I usually check the solution section... Like here for example, here I have done something crazy (he notes that it does not fit with his answer)... Yes, it shall be a square root."
Ulf writes:

$$[\sqrt{5}x]_1^3 = 3\sqrt{5} - \sqrt{5} = 2\sqrt{5}$$

The IS exercise 8.3.2 is solved by Ulf in a similar way, but the description of this solution is omitted here.

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Search in the textbook for IS information: an example, a theorem, or a rule where the components in the exercise can be inserted. To find the formula in the blue box with complete IS information is fairly unproblematic, since both Exercise 8.3.1 on page 422 and the formula on page 417 explicitly mention "length of the curve" Ulf does not mention that since the curve is a straight line, the length may easily be found using the Pythagorean theorem.
(a.3) *Strategy implementation*: The implementation is straightforward, apart from the small slip. The competence required to carry out the implementation of the IS solution is: (A) Understanding that the arc length formula requires the evaluation of an integral. (B) Knowing that a main component in this evaluation consists of transforming and simplifying the function into one whose primitive function can be fairly easily found. (C) Finding the derivative of a first degree polynomial. (D) Finding a primitive function to a constant function. (E) Knowing the familiar procedure of applying the fundamental theorem of calculus when evaluating integrals. (F) Being familiar with the terminology used in the earlier steps.
(a.4) *Conclusion*: The faulty solution and answer "5".

(b.1) *Problematic situation*: The answer needs verification, partly because it "became too easy" to fit with his expectations.
(b.2) *Strategy choice*: For Ulf, as for most students, the main method of checking and verifying solution work is to compare with the textbook's solution section, which he does in order to (i) see if his answer is correct and (ii) if not, try to learn what the correct solution should be.
(b.3) *Strategy implementation*: Ulf (i) is immediately convinced that his answer is wrong and (ii) notes that the solution section answer contains a square root which leads him to search for and find his missing square root.
(b.4) *Conclusion*: Finding and correcting the mistake.

Reasoning characteristics: The main point here is that this is an IS exercise, and Ulf's strategy choice at (a.2) is clearly to find and use the available IS information which is found in the curve length formula. In order to identify this formula as the proper one, no intrinsic properties have to be considered since both the exercise and the formula explicitly mention the keywords "length of the curve".

The IS information in the formula provides a complete solution for the strategy implementation, apart from the evaluation of the integral which is a familiar procedure. The mathematical competence (A-F) used at (a.3) is based only on past mathematical facts and procedures treated in school year 11 (year 2 in upper secondary), two years earlier in Ulf's perspective. There are no indications, and no need for, that Ulf considers any intrinsic properties related to arc length. Even though he draws the curve from the start, he does not use it in the solution.

This is an IS exercise, and it is almost straightforwardly solved by Ulf as one. He makes a small slip, and it is noteworthy that his metacognitive reaction at (b.1) to the result is not mathematically but entirely EE based: a "too easy" solution does not fit with his established experiences. He resolves this at (b.2) by quickly turning to the solutions section, not by analysing his work trying to find the mistake. Ulf could easily have verified his answer by estimating the length of the line segment in his figure, or by the Pythagorean theorem.

Similar to Jon's work with Exercise 9.4.1, it seems likely that Ulf's mathematical competence is developed only in the sense that he has seen yet another application of the integral, and perhaps that he may better remember the arc length formula. The latter seems to be one of his main goals, since he says twice that "I write it down so it will perhaps go into my head". There is no need for any problem solving activities like construction of PR reasoning, heuristics or control. It is of course *possible* that Ulf will learn more from his work, but there are no such indications.

Textbook Section 8.3, Exercise 7

Exercise formulation:

> "In exercises 1-16, find the lengths of the given curves.
> **7.** $y = \frac{x^3}{12} + \frac{1}{x}$ from $x = 1$ to $x = 4$"

Solution work: Ulf starts by drawing the graph (which he, as above, does not use for anything), and uses his calculator's built-in program for curve lengths to find the exercise's answer (6 units) to, as he says, "work against". He then applies the same curve length formula that he used in Exercises 8.3.1 and 8.3.2, but he makes a stall mistake and integrates $1/x$ instead of differentiating it:

$$y' = \frac{3x^2}{12} + \ln x = \frac{x^2}{4} + \ln x$$

[The correct derivative is $y' = x^2/4 - 1/x^2$]
"This derivative is a bit more difficult..."
He then inserts the faulty y' into the curve length formula:

$$s = \int_1^4 \sqrt{1 + (\frac{x^2}{4} + \ln x)^2}\, dx \qquad (22)$$

"Now I shall try to simplify this in some way... so I can integrate it in the easiest possible way."

Ulf says that he is hesitant about how to proceed: "[...] just to square such a thing is not the easiest... We can go back in the book and see if there is some similar example, to compare with... Example 2 seems fairly similar."

Ulf carefully reads Example 8.3.2:

> "**EXAMPLE 2** Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from $x = 1$ to $x = 2$.
> **SOLUTION** Here $\frac{dy}{dx} = 4x^3 - \frac{1}{16x^3}$ and
>
> $$1 + (\frac{dy}{dx})^2 = 1 + (4x^3 - \frac{1}{16x^3})^2 = 1 + (4x^3)^2 - \frac{1}{2} + (\frac{1}{16x^3})^2$$
> $$= 1 + (4x^3)^2 + \frac{1}{2} + (\frac{1}{16x^3})^2 = (4x^3 + \frac{1}{16x^3})^2.$$
>
> The expression in the last set of parentheses is positive for $1 \leq x \leq 2$, so the length of the curve is
>
> $$s = \int_1^2 4x^3 + \frac{1}{16x^3}\, dx = (x^4 + \frac{1}{32x^2})\Big|_1^2$$
> $$= 16 - \frac{1}{128} - (1 - \frac{1}{32}) = 15 + \frac{3}{128} \text{ units.} \qquad \square\text{"}$$

Example 8.3.2 above is followed by a text paragraph, *which is not read by Ulf* [recall that he said above that "it gives me more to read the examples than to just read the text"], that explains the 'trick' that is supposed to be used in Exercise 8.3.7:

> "The examples above are deceptively simple; the curves were chosen in such a way that the arc length integrals could be easily evaluated. For instance, the number 32 in the curve in Example 2 was chosen just so the expression $1 + (dy/dx)^2$ would turn out to be a perfect square and its square root would cause no problems. Because of the square root in the formula, arc length problems for most curves lead to integrals that are difficult or impossible to evaluate without using numerical techniques."

[The 'trick' here is thus that if dy/dx is of the form $f(x)-g(x)$, where $2f(x)g(x) \equiv 1/2$ (which is a rather strong restriction), then

$$\sqrt{1 + (dy/dx)^2} = \sqrt{1 + (f - g)^2} = \sqrt{1 + f^2 - 1/2 + g^2}$$
$$= \sqrt{f^2 + 1/2 + g^2} = \sqrt{(f + g)^2} = |f + g|$$

which is probably easier to integrate than the original square root expression.]

After reading Example 8.3.2 Ulf says "Aha" and describes that the example solution has reached an even square [but he does not fully understand the 'trick' yet]. He then says "Let's see if we can do magic in the same way with this", and expands his faulty and tricky expression:

$$\int_1^4 \sqrt{1+(\frac{x^2}{4})^2+2(\frac{x^2}{4}\ln x)+(\ln x)^2}\ dx$$

$$\int_1^4 \sqrt{1+(\frac{x^2}{4})^2+\frac{1}{2}(x^2\ln x)+(\ln x)^2}\ dx$$

"It did not help me anything. Let's try this."

$$\int_1^4 \sqrt{1+\frac{x^4}{16}+\frac{1}{2}(x^2\ln x)+(\ln x)^2}\ dx \tag{23}$$

"I want this number 1 to disappear in some way... as they did in the book, so that I can rewrite this as... one parenthesis..."
Ulf continues hesitantly.
"Let's see what they have done here again (points at Example 8.3.2 above). They have got 1/2 here in some way... To see what they have done, I calculate the example myself."
Ulf copies Example 8.3.2 step by step. He wants to understand each step, and says that he normally works like this.
"Now I understand where they got this 1/2 from. When you multiply the factors then you can cancel common factors. Then it must be possible to do in the same way in my exercise... But it gets a bit more difficult, I get a lot of $\ln x$ and so... If I have not mixed the derivative and the integral of $1/x$? It is not necessarily $\ln x$. It [the derivative] must be $1/x^2$, have I done such a mistake!"

Ulf thus understands the 'trick', finally finds his mistake, and finds the correct derivative $y' = x^2/4 - 1/x^2$. He can now complete the solution in the same way as the solution to Example 8.3.2, but the description of this part of his work is omitted here.

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: The same IS strategy as in Exercise 8.3.1 above.
(a.3) *Strategy implementation*: The implementation could have been straightforward as in Exercise 8.3.1. There is also additional IS information in Example 8.3.2 where almost the same task as Exercise 8.3 7 is solved, and this solution could have been copied in detail. Unfortunately, Ulf makes a stall mistake and his work becomes more and more algebraically difficult since the faulty expressions are impossible to integrate by elementary methods.
(a.4) *Conclusion*: Ulf becomes hesitant about how to proceed.

(b.1) *Problematic situation*: How to simplify and integrate the expression (22)?
(b.2) *Strategy choice*: Ulf's strategy choice is clearly to search for additional IS information: "We can go back in the book and see if there is some similar example, to compare with... Example 2 seems fairly similar." It is straightforward to see that Example 8.3.2 is similar to Exercise 8.3.7, since: (i) the task in both is to find the curve length and (ii) the functions in the exercise and the example are both of the form $ax^m + b/x^n$.
(b.3) *Strategy implementation*: He analyses the example and concludes that:
(b.4) *Conclusion*: Ulf realises that in Example 8.3.2 "they have reached an even square", and he hopes that he "can do magic in the same way" with the exercise. That is, to reach a solution in a similar way. He has not yet fully understood the 'trick' in the example solution.

(c.1) *Problematic situation*: How to reach an even square?
(c.2) *Strategy choice*: Expand the parenthesis in (22) and try to simplify.
(c.3) *Strategy implementation*: Expanding is straightforward, but simplification is hard [it is actually impossible, but Ulf does not realise this].
(c.4) *Conclusion*: Ulf says that "It did not help me anything" and reaches the tricky expression (23).

(d.1) *Problematic situation*: Ulf wants "this number 1 to disappear in some way... as they did in the book, so that I can rewrite this as... one parenthesis", but *how*?
(d.2) *Strategy choice*: Search for additional information in Example 8.3.2, try to learn more about how to solve the exercise. Ulf reads the example again, but this time his strategy is to understand the details by writing down and carefully consider the intrinsic properties in each local step of the example solution.
(d.3) *Strategy implementation*: The implementation takes some time, but is straightforward.
(d.4) *Conclusion*: At (b.4) above Ulf understood a part of the 'trick' described on page 29, now he seems to have full insight. He also states that "Then it must be possible to do in the same way in my exercise".

(e.1) *Problematic situation*: Why is it so hard to use the same trick as in Example 8.3.2, that is, why is it so hard to carry out an IS solution?
(e.2) *Strategy choice*: Try to find some differences between the example solution and Ulf's attempt.
(e.3) *Strategy implementation*: Ulf finds that it is the appearance of the $\ln x$ expressions that makes it difficult to carry out the cancellation trick: "But it gets a bit more difficult, I get a lot of $\ln x$ and so..."
(e.4) *Conclusion*: The $\ln x$ expressions are probably wrong.

After this Ulf finds the stall mistake and completes an IS solution.

Reasoning characteristics: In the same way as Exercise 8.3.1 this is an IS exercise. Ulf's strategy choice at (a.2) is again clearly to find and use the available IS information in the curve length formula, and again no intrinsic properties have to be considered in order to identify this formula as the proper one. The competence required is essentially the same as in Exercise 8.3.1, though the differentiation and integration are technically a bit more difficult. The implementation of the IS information could have lead to a straightforward solution, as in Exercises 8.3.1 and 8.3.2, but Ulf makes a stall mistake at (a.3) that leads his work into an unprogressive direction.

In trying to resolve the difficulties, the central characteristic of Ulf's work is that even though he clearly sees that the function expression becomes very complicated, it takes long time before he tries to search for mistakes in his solution work or tries in any other way to understand his work. A possible reason for this is that according to his experiences, the main (and often only) difficulty in almost all of the textbook's integration exercises is to simplify a tricky expression in order to find a familiar primitive function: "Now I shall try to simplify this in some way... so I can integrate it in the easiest possible way." Therefore he may see the tricky expression at (b.1) as a 'normal' part of the solution work. A very common strategy in integration exercises is to expand parentheses and simplify, as he does at (c.2).

Ulf seems (correctly) to be so very convinced that there *should be* IS information available, that he at (d.3) reads the example solution very carefully in order to understand each detail and find the differences between the example solution and his work. This assumption is strengthened both by Ulf's comment at (d.4), and in the post-interview when JL asks:
"How did you find your mistake?"
"It was not possible to get something out of this parenthesis (points at (22)), so therefore it was not possible to have $\ln x$ there."
Ulf thus resolves the difficulty by a careful IS information search, but this work includes a local component that may be classified as LPR: It is hard to make the cancellation trick work if the square root expression includes both x^2 and $\ln x$. Here he considers intrinsic properties of the example solution, in order to determine why his work does not 'fit in' with the provided IS information.

A comment to the exercise type: The 'trick' used to solve the exercise is only applicable to a very limited family of functions (see the discussion at page 29). If the reasoning is only based on IS and if a small slip, like Ulf's above, or a minor mistake as Jon's first one in Exercise 9.4.3, is made then it becomes impossible to make any progress. It seems like students often get stuck in similar situations. This type of 'arranged' exercises are rather common in textbooks, but perhaps they should be more sparsely included? In Exercise 8.3.11 Ulf makes a mistake that leads to similar difficulties as in Exercise 8.3.7, but this description is omitted here.

Textbook Section 8.3, Exercise 17

Exercise formulation:

"Find the circumference of the closed curve $x^{2/3}+y^{2/3}=a^{2/3}$. *Hint:* the curve is symmetric about both coordinate axes (why?), so one quarter of it lies in the first quadrant."

Solution work: Ulf starts by saying:
"Yes, this was the kind of task that someone asked about at the lecture today. The circumference of a closed curve. This one can not draw... What can I do with this? Perhaps solve for y?"
He writes

$$y^{2/3} = a^{2/3} - x^{2/3} \tag{24}$$

"But this did not become much easier, y raised to 2/3... Another problem is, how shall I know between what limits I shall integrate? The curve should be closed, so perhaps it will go around (he sketches a circle that is centred at the origin in a coordinate plane). It is symmetric... around both axes, so that one-quarter is in the first quadrant (this is the clue in the exercise formulation)... So it is an ellipse, or a circle, or whatever it can be... Then one can take the limits from here to here (points at the circle segment in the first quadrant of his sketch, from x=0 to y=0). It says that the curve is symmetric, why? I do not know, I do not even know if I have guessed correctly to the right curve."
Ulf reads Example 8.3.4, which states:

"**EXAMPLE 4 The circumference of an ellipse**
Find the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{25}$$

where $a \geq b \geq 0$. See figure 8.25 [a figure of an ellipse].
SOLUTION The upper half of the ellipse has equation $y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}$. Hence

$$\frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

and so

$$1 + (\frac{dy}{dx})^2 = 1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}.$$

The circumference of the ellipse is four times the arc length of the part lying in the first quadrant, so

$$s = 4\int_0^a \frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}\,dx \quad \text{Let } x = a\sin t,\ dx = a\cos t\ dt$$

$$= 4\int_0^{\pi/2} \frac{\sqrt{a^4 - (a^2 - b^2)a^2\sin^2 t}}{a(a\cos t)} a\cos t\ dt$$

$$= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\sin^2 t}\, dt$$

$$= 4a \int_0^{\pi/2} \sqrt{1 - \frac{a^2 - b^2}{a^2}\sin^2 t}\, dt = 4a \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \sin^2 t}\, dt \text{ units,}$$

where $\epsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 9.1 for a discussion of ellipses.) Note that $0 \leq \epsilon \leq 1$. The function $E(\epsilon)$, defined by

$$E(\epsilon) = \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \sin^2 t}\, dt \tag{26}$$

is called the **complete elliptic integral of the second kind.** [the text then briefly discusses numerical methods to evaluate E] []"

"This says something about the circumference of an ellipse, I will read this. There is a formula that looks like..."
Ulf writes down the function (26).
"The ϵ is the eccentri... eccentri..., whatever it is called [Ulf has difficulties in translating the term from English to Swedish]. Perhaps it is something like that one should use? Then I must in principle rewrite this (24) to se if it is an ellipse."
Ulf tries to rewrite (24) in a similar form as (25):

$$\frac{(x^{2/3} + y^{2/3})}{a^{2/3}} = \frac{a^{2/3}}{a^{2/3}}$$

$$\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{a^{2/3}} = 1 \tag{27}$$

"But this is perhaps not so good? It should not be like this? (The difference is that the exponents are 2/3 instead of 2 in his formula)... It is close... If you can do this way, that is..."

After some silence JL asks Ulf what he is thinking.
"First, what differs this from an ellipse is that there are not squares here (points at the exponents). Secondly, the denominators are the same ($a^{2/3}$ in (27)). I am thinking about, if it says a^2 and b^2 here (points at (24)), what does it mean for an ellipse? If it can be some other kind of ellipse if they [the denominators] are equal, some kind of circle or so... Yes, there must be squares if it is an ellipse... I don't know... Perhaps it is a circle if they are equal. If this distance (points at the ellipse's major axis in the figure belonging to Example 8.3.4) is equal to this distance (the minor axis), then it is a circle... It was a difficult exercise. How shall I handle this? Raised to two thirds was trickier. I can not use this (the $E(\epsilon)$ formula), because then I have to find ϵ..."
Ulf writes $\epsilon = \sqrt{a^2 - b^2}$.
"What does it ((24)) say? It says:"

$$\frac{\sqrt[3]{x^2} + \sqrt[3]{y^2}}{\sqrt[3]{a^2}} = 1$$

"But perhaps it did not help me?"

After a longer period of silence JL interrupts:
"I think we shall break here. It is three o'clock (two hours have passed since Ulf started with Exercise 8.3.1 above)."
"Oh! Damn! I have not done much!"
"If you had not made any progress with this exercise, had you turned to the Instructor's solution manual?"
"No I had skipped this, I have other book sections to work with."
JL then guides Ulf into solving for y in (24), which he manages. That is, if this idea had occurred to Ulf he probably could have solved the exercise.

Reasoning structure:
(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Recall the solution procedure related to "circumference of a closed curve" from today's lecture.
(a.3) *Strategy implementation*: Though he seems to have some insight in the procedure, it seems rather superficial and very incomplete.
(a.4) *Conclusion*: Ulf does not really consider any intrinsic properties related to arc length or to methods of calculating it. What he says is approximately that: (i) The curve can not be drawn (wrong, it is the composition of the two curves $y = \pm(a^{2/3} - x^{2/3})^{3/2}$). (ii) Perhaps he should solve for y (correct), this is the first step but he is neither able to carry it out nor to see the necessity of it. (iii) Perhaps the curve will 'go around', like and ellipse (not really, but something like it). (iv) The limits of integration may be found by studying the first quadrant, since the curve is symmetric (correct, the clue in the exercise). (v) He is not sure that he has 'guessed correctly' about all this.

(b.1) *Problematic situation*: Ulf's recollection of the solution procedure is not sufficient, how to proceed?
(b.2) *Strategy choice*: Search for additional information in the text, that could complete the solution procedure.
(b.3) *Strategy implementation*: The identification of a similar example is straightforward since the only example in Section 8.3 that "says something about circumference" is Example 8.3.4. Ulf's understanding of the example solution is too superficial, so he does not realise that it is only the first part that can be copied, not the whole solution and in particular not the formula (26) that he tries to use. Ulf notes the symbolic difference (a surface property) between his expression (27) and the corresponding expression (25) in Example 8.3.4, but does not realise the central similarity (an intrinsic property of the solution method) between Exercise 8.3.17 and Example 8.3.4: that the function needs to be rewritten on the form $y = f(x)$. He then tries wrongly to verify that the function in Exercise 8.3.17 is also an ellipse, with the purpose of using the formula $E(\epsilon)$ from the example.
(b.4) *Conclusion*: No progress.

Reasoning characteristics: 8.3.17 is an LPR exercise. There is actually almost complete IS information in the above cited Example 8.3.4 (and probably in the method recalled from the lecture), but the example's solution leads to an integral that can not be evaluated by elementary techniques. Thus the solver of Exercise 8.3.17 has to understand parts of the example's solution, and realise that a central difference is that the integral in the exercise can be evaluated by elementary techniques. It is only this difference that turns 8.3.17 into an LPR exercise, otherwise it would just have been an (difficult) IS exercise.

From (a.3) and (a.4) it seems like Ulf has access to most of the central intrinsic components in a possible solution to Exercise 8.3.17. There are two key features behind his failure: (I) He probably could have continued if he had been able to "solve for y" in (24) as he would like to. At the same time, his insight in the (from the lecture) recalled method is so superficial that he does not see that this is *necessarily* the next step. Thus he is not searching for information on how to "solve for y", but general solution procedure information: His inability to consider intrinsic properties leads him to search for IS information at the wrong places. (II) Ulf does not really try to use his knowledge from (a.4), which combined with a careful reading of the first part of the example solution might have lead him into understanding the necessity of solving for y. Instead he makes a very superficial search for IS information. Ulf tries in (b.4) unsuccessfully to 'adjust' his solution to fit in with the solution of Example 8.3.4. It seems clear that this attempt is based on surface properties: For example when he says "Perhaps it is something like that [the eccentricity] one should use?", and later when he says: "It is close" and "First, what differs this from an ellipse is that there are not squares here. Secondly, the denominators are the same." It seems like Ulf makes some unsuccessful attempts to consider intrinsic properties, "I am thinking about, if it says a^2 and b^2 here, what does it mean for an ellipse?", but he is completely unable to do so and relate this to his solution work.

In Exercise 8.3.7 above, his strategy of trying to adjust his solution to fit with the IS information was successful: By understanding the solution of Example 8.3.2 in detail he was then able to find his mistake. Now, his understanding of Example 8.3.4 is too superficial, and the example's solution is impossible to completely copy this time. Therefore Ulf is not able to build a solution to Exercise 8.3.17 solely by considering surface properties, and since no global PR or intrinsic property considerations are invoked his whole attempt fails.

5.3 Dan

Dan is studying the same three-year computer engineer program as Jon. There are some courses that he has not yet passed and Dan's grading average on the courses he has passed is 3.2 which altogether is a relatively poor result. Dan failed the exam after the video recording of his work below. In his work with the exercises below, Dan uses information from the same text sources as Jon: (a) The textbook [Ada95]. (b) The lecture notes. (c) ISM. (d) A high school formula collection with mathematical formulas.

Dan starts with reading the text that precedes the exercises in the textbook's

Section 10.4:
"I usually start by reading the section unless it is self-evident, but during this course there has not been so much self-evident."
In spite of this statement, before turning to Exercise 10.4.1 he reads the (rather difficult) six text pages of Section 10.4 for only 14 minutes and says:
"It is a bit hard to understand the language. I understand most of the general ideas, but often not exactly all of the details."
It seems below like he understands none of the general ideas and very few of the details of the text in Section 10.4, which treats alternating series and methods for determining absolute and conditional convergence for these series.

Textbook Section 10.4 (Absolute and conditional convergence), Exercise 1

Exercise formulation:

> "Determine whether the series in Exercises 1-12 converge absolutely, converge conditionally, or diverge.
> **1.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$"

Solution work: Dan starts by saying:
"Now I do not know exactly, before I have read it, I do not recall exactly how this works. When I look at the exercise I see that it is an alternating curve, no, alternating series."
Dan writes down the first 4 terms:

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} \qquad (28)$$

"If one compares with the closest one should check, then it is this (writes)":

$$\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}} \qquad \frac{1}{2} < 1 \quad \text{Diverges} \to \infty \qquad (29)$$

[(29) is Dan's faulty answer to the exercise]
"Which means that this, I think, diverges."
[Comment: Dan recalls a result from the earlier Section 10.3 concerning 'p-series', which states that

$$\text{``}\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges to infinity if } p \leq 1.\text{''} \qquad (30)$$

This test can not be used to determine convergence for the series in Exercise 10.4.1. It could be used to determine *absolute* convergence but, as becomes clearer below, Dan does not consider the distinctions between convergence, absolute convergence, and conditional convergence. To determine convergence, Dan could have applied the alternating series test which is contained in Section 10.4 that he read 10 minutes earlier:

THEOREM 14 The alternating series test
Suppose that the sequence $\{a_n\}$ is positive, decreasing, and converges to 0, that is, suppose that
(i) $a_n \geq 0$ for $n = 1, 2, 3, \ldots$;
(ii) $a_{n+1} \leq a_n$ for $n = 1, 2, 3, \ldots$;
(iii) $\lim_{n\to\infty} a_n = 0$.
Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots$ converges." (The proof of the theorem then follows)]

Dan makes no reference to this theorem, instead he opens the textbook's solution section which only says:

> "**1.** Converges conditionally".

"This seems to be wrong... Converges conditionally..."
Dan opens ISM and reads the solution to Exercise 10.4.1:

> "**1.** $\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test (since the terms alternate in sign, decrease in size, and approach 0). However, the convergence is only conditional, since $\sum \frac{1}{\sqrt{n}}$ diverges to infinity."

"It was correct what I said, that it diverges to infinity. But it converges since it is an alternating series, and it... decreases in size, tends to zero. Condi... condi... [Dan has difficulties in translating the term from English to Swedish] It converges only conditionally."
Dan seems to consider himself finished, and turns to the next exercise.

Reasoning structure:
(a.1) *Problematic situation*: What is the exercise about?
(a.2) *Strategy choice*: Analyse the exercise.
(a.3) *Strategy implementation*: The analysis is very brief and superficial.
(a.4) *Conclusion*: Dan concludes quickly that it is an "alternating series", but one of the main characteristics of his work with all the exercises in Section 10.4 (which mainly treats alternating series) is that Dan is essentially never able to consider any of the intrinsic properties of alternating series. Dan also writes down the first four terms, which seems to be a routine for him but he seldom actually considers any properties of the terms.

(b.1) *Problematic situation*: How shall the exercise be solved?
(b.2) *Strategy choice*: Dan makes a superficial IS connection at (29) to the p-series of Section 10.3. The connection is based on the surface property that the terms (28) in the Exercise 10.4.1 are, if one wrongly disregards the numerator $(-1)^n$, on the form $\frac{1}{n^p}$ as in (30). Since Dan does not consider the intrinsic properties of alternating series, he does not realise that convergence can not be determined by (30). He misses the complete IS information provided in Theorem 14 that he read earlier in Section 10.4.
(b.3) *Strategy implementation*: Dan recalls correctly that a p-series diverges if

$p \leq 1$ (the answer is still wrong since the strategy choice is not fruitful).
(b.4) *Conclusion*: Dan concludes that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ "Diverges $\to \infty$", but notes that it does not fit with the textbook's solution section.

(c.1) *Problematic situation*: What is wrong with Dan's solution?
(c.2) *Strategy choice*: AS, read ISM.
(c.3) *Strategy implementation*: Everything Dan says after reading ISM's solution except "it was correct what I said" is (partly wrongly) quoted from ISM. He says at the same time that "it converges" and that "it diverges", without commenting the apparent contradiction or that there are different kinds of convergence involved.
(c.4) *Conclusion*: It seems like Dan believes that he, at least partially, solved Exercise 10.4.1 (this claim is strengthened at Exercise 10.4.3 (b.3) below where he, from working in a similar way as with Exercise 10.4.1, believes that he has solved Exercise 10.4.3.).

Reasoning characteristics: This is an IS exercise but Dan is unable to provide even a partial solution. Dan's work is very superficial, he is unable to use even surface properties of the components in his solution work, and there are no signs that his reasoning considers any intrinsic properties: (I) He is unable to use the relatively (series is a difficult concept) simple and clear IS information provided in the alternating series test, even though he read it just before and explicitly says at (a.4) that Exercise 10.4.1 is an "alternating series". Instead, based on superficial properties he applies at (b.2) IS information from the preceding Section 10.3 (Convergence tests for positive series), that probably worked to some extent for many of the easiest exercises in that section. In the post-interview it is clear that Dan did not understand the meaning of the alternating series test, and he also says that he can not use the comparison tests at all. There are no real signs that he even *tries* to find IS information. (II) There are no signs that he is able at (c.3) to make any use of the AS information provided by the ISM solution, apart from quoting it. (III) He does not relate his solution work to anything else that he read in Section 10.4, though the section treats only alternating series and absolute and conditional convergence. There are no signs, not even at (a.4) and (c.3), that he considers any intrinsic or surface properties related to these terms. (IV) There is no PR involved.

Dan does not, contrary to Jon and Ulf above, work hard to understand the local components of the provided IS and AS information. He says in the post-interview that "Math can be, though it has not shown today, one of my stronger sides." Perhaps his difficulties have lately become so great that he has given up his attempt to understand even local components? Dan's work above and below is sometimes very hard to follow and find meaning in. It seems like he has 'lost contact' with mathematics, that Dan's concept images [TV81] and statement images [SS95] are so distant from the corresponding concept and statement definitions that analysing his work his difficult. Dan seems to believe ((c.3) and (c.4)) that he has solved the Exercise 10.4.1, which he is very far from doing.

This is an example of that even surface IS and AS reasoning may be difficult when the understanding is very weak.

Textbook Section 10.4, Exercise 3

Exercise formulation:

> "Determine whether the series in Exercises 1-12 converge absolutely, converge conditionally, or diverge.
> **3.** $\sum_{n=1}^{\infty} \frac{\cos n\pi}{(n+1)\ln(n+1)}$"

Solution work: Dan writes (copies from the exercise formulation):

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{(n+1)\ln(n+1)} = \qquad (31)$$

and continues by recalling and once again reading Example 10.4.1:

> "**EXAMPLE 1** Test for absolute convergence:
>
> $$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n}$$
>
> **SOLUTION**
> (a) $\lim_{n\to\infty} |\frac{(-1)^{n-1}}{2n-1}|/\frac{1}{n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0$. Since the harmonic series $\sum_{n=1}^{\infty}(1/n)$ diverges to infinity, therefore $\lim_{n\to\infty}((-1)^{n-1}/(2n-1))$ does not converge absolutely (comparison test).
> (b) $\rho = \lim_{n\to\infty} |\frac{(n+1)\cos((n+1)\pi)}{2^{n+1}} / \frac{n\cos(n\pi)}{2^n}| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$.
> (Note that $\cos(n\pi)$ is just a fancy way of writing $(-1)^n$.) Therefore (ratio test) $\sum_{n=1}^{\infty}((n\cos n\pi)/(2^n))$ converges absolutely."

Dan writes (copies from Example 10.4.1):

$$\cos(n\pi) = (-1)^n \qquad (32)$$

and then decides to apply the ratio test from the earlier Section 10.3, which there is formulated as:

> "**THEOREM 11 The ratio test**
> Suppose that the sequence $a_n > 0$ (ultimately) and that $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists or is $+\infty$.
> (a) If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
> (b) If $1 < \rho \leq \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.
> (c) If $\rho = 1$, this test gives no information; the series may either converge or diverge to infinity." (The proof of the theorem then follows)

Dan continues from (31) by trying to apply the ratio test:

$$\frac{\frac{\cos n+1\pi}{(n+2)\ln(n+2)}}{\frac{\cos n\pi}{n+1\ln(n+2)}} =$$

One may note that the insertion itself of the term into the ratio test is correct [though the test can not be used to solve Exercise 10.4.3], apart from that: it is the absolute value of the expression above that should be tested, he twice misses the parentheses around $n+1$, and that it should be $\ln(n+1)$ instead of $\ln(n+2)$ in the lower denominator. However, Dan corrects these slips as he continues:

$$\frac{\cos((n+1)\pi)}{(n+2)\ln(n+2)} \cdot \frac{(n+1)\ln(n+1)}{\cos(n\pi)} = \tag{33}$$

It seems like he then decides to wrongly use the information (32) and sets both $\cos((n+1)\pi) = 1$ and $\cos(n\pi) = 1$:

$$\frac{(n+1)\ln(n+1)}{(n+2)\ln(n+2)}$$

Dan then applies the 'faulty logarithm law' $a \ln b = \ln ab$ and obtains:

$$\frac{\ln(n+1) + \ln(n^2+n)}{\ln(2n+4) + \ln(n^2+2n)} \tag{34}$$

[Comment: It is noteworthy that none of the mistakes below (32) are the main causes behind his difficulties. The main problem is that the ratio test can not be used to determine absolute convergence (since Dan would, if he carried out the calculations correctly, reach case (c) above: "$\rho = 1$, the test gives no information"), or conditional convergence (since the series is alternating). The former is determined by the integral test and the latter by the alternating series test.]

Dan's work is becoming more and more hesitant, and when it seems to have come to a standstill JL asks what Dan has done.
"I have taken a_{n+1}/a_n and then inverted [as in the ratio test]."
"OK, and then?"
"I check what remains when I multiply this, something usually disappears. But I am a bit sceptic about what really disappears... It is $\cos(n+1)$ here and only $\cos n$ here (points at numerator and denominator at (33)), but I wonder what disappears..."

Dan reads ISM's solution to Exercise 10.4.3 which states:

> "**3.** $\sum \frac{\cos(n\pi)}{(n+1)\ln(n+1)} = \sum \frac{(-1)^n}{(n+1)\ln(n+1)}$ converges by the alternating series test, but only conditionally since $\sum \frac{1}{(n+1)\ln(n+1)}$ diverges to infinity (by the integral test)."

"Yes, this remains. They have not done like this at all. They do like I did first, they checked that it is an alternating series. Same as Exercise 1... Hm, I have missed something."
Dan returns to the textbook's Example 10.4.1.
" This is almost..." [He is probably about to say "the same as the exercise".] No, they have 2^n here... Well, I'll start all over."
Dan leaves off Exercise 10.4.3 and turns to Exercise 10.4.5.

Reasoning structure:
(a.1) *Problematic situation*: How to solve the exercise?
(a.2) *Strategy choice*: Dan searches for information and finds Example 10.4.1, but the IS application of the information in the example is extremely superficial: (i) Dan completely disregards all intrinsic properties that are related to alternating series and to absolute and conditional convergence. (ii) Instead he notes the surface property that in both Exercise 10.4.3 and Example 10.4.1 (b) the numerator contains '$\cos n\pi$', and from this concludes wrongly that both can be solved in the same way. (iii) Dan shows no sign of understanding or considering any intrinsic properties of the example solution, but notes that one surface property is that it contains a reference to a test, "(ratio test)", and therefore Dan tries without considering its role in the example or the exercise solution to solve Exercise 10.4.3 by the ratio test.
(a.3) *Strategy implementation*: As commented after (34) were he reaches a standstill, Dan could not have solved Exercise 10.4.3 by the ratio test. It is anyhow a reasonable attempt to try it, especially if he could have correctly considered the relations and distinctions between absolute and conditional convergence. It seems, from his long and hesitant work with the algebraic expressions, like he believes that his problems are caused by that the algebraic manipulations and simplifications are tricky ("something usually disappears"), not that he made a superficial strategy choice at (a.2) and fails to consider the intrinsic properties of the exercise.
(a.4) *Conclusion*: A standstill.

(b.1) *Problematic situation*: Same as (a.1).
(b.2) *Strategy choice*: Search for AS information in ISM and IS information in the textbook's Example 10.4.1.
(b.3) *Strategy implementation*: Dan notes that ISM's solution has something to do with alternating series: "They do like I did first, they checked that it is an alternating series. Same as Exercise 1..." Actually, there are no signs that he uses any surface or intrinsic properties of alternating series for anything. It seems like he believes that he has achieved a partly correct solution. (This assumption is strengthened below when Dan says about Exercise 10.4.5 that: "It was not so easy. Exercise [10.4] 3 was much easier.") Dan finally says that he "missed something", and returns to Example 10.4.1. He is unable to consider the intrinsic properties of the example solution in order to determine if it can be followed, and how this relates to the ISM solution.

(b.4) *Conclusion*: Dan leaves off the exercise.

Reasoning characteristics: This is, as Exercise 10.4.1, an IS exercise that can be solved by applying the IS information in the integral test (absolute convergence) and the alternating series test (conditional convergence). Dan applies neither of them, and as described at (a.2) his faulty strategy choice is based on extremely superficial IS.

In the post-interview, Dan's description of his reasoning in this exercise is coherent with the interpretations (a.2) (ii) and (iii):
JL asks why Dan applied the method from Example 10.4.1.
"It says that $\cos n\pi$ is just a fancy way of writing $(-1)^n$, therefore ratio test, alternating series. It was therefore I went back to this."
"It was because $\cos n\pi$ was similar to the exercise?"
"Yes, one goes back to see if there is something to relate to, something that will push you in the right direction. These things one never gets to know when the teacher teaches, he just writes down his definitions and so, he never gives these small hints and tricks, so you can know in what ways to attack it."
Dan's problem here is that his identification of the 'hint' is based on surface considerations only, and leads in the wrong direction.

Later in the post-interview JL asks why Dan left exercise 10.4.3.
"What I have the greatest difficulties with is when I have to divide, when one has $\frac{\ln n+1}{\ln n+2}$. What will disappear? I assume that ... $\ln n + 2$ ... is it equal to $\ln n + \ln 2$?"
"No, [...]"
"He has never really treated ... Or did he [Dan is probably trying to recall if the teacher has treated division of logarithms]? These ln tasks are difficult."
Dan thus (see also (a.3)) seems to wrongly believe that his difficulties were caused by difficulties in simplification of expressions (local property), instead of difficulties in founding the strategy choice on intrinsic properties (global). The main part of the solution work in the textbook exercises is often to simplify tricky expressions in order to be able to apply provided solution methods. From this Dan may establish experiences that lead to EE reasoning that focuses on these algebraic simplifications. Dan also wants the teacher to give suggestions for more exercises, in order to learn each possible type:
"There are so many different ... they are not variations on the same, ... very many different, totally different exercises. And if it comes up a variation on any of them, you are stuck. There should be more easy exercises, so one can look at different variations. [...] It usually is like, this exercise is about polynomials, another is about sine, the third is about e raised to something, and then you have four of them, and there is only one on sine and on tan, and then there is something one have not learnt really. I would like the teacher to give us a paper with, say, four or five exercises each on sin, power functions, tan, arctan, etc. To have more of the same type. This is what I miss."
This is another indication that Dan is focused on considering local surface prop-

erties of series (e.g. whether the terms are sine or logarithms) instead of intrinsic global properties (e.g. wether the series are alternating or not). He also criticises that it is so much to memorise, but also says that understanding is desirable.

Dan is unable to correctly use or consider any surface or intrinsic properties of alternating series or absolute and conditional convergence, neither in the exercise, the textbook, nor in ISM, even though there is IS and AS information available. There is no sign of any PR.

Textbook Section 10.4, Exercise 5

Exercise formulation:

> "Determine whether the series in Exercises 1-12 converge absolutely, converge conditionally, or diverge.
> **5.** $\sum_{n=0}^{\infty} \frac{(-1)^n(n^2-1)}{n^2+1}$"

Solution work: Dan starts by saying:
"This is an alternating series, but it goes to infinity."
Then he writes down the exercise formulation and the first five terms in the series:

$$1 - 0 + \frac{3}{5} - \frac{5}{7} + \frac{15}{17} \tag{35}$$

[Dan makes two slips: The first term should be -1 and the fourth term should be -8/10, but it does not affect his solution work.]
"This goes to infinity then... Yes it goes to 0, but it will never get there really. But this is strange, since it says here that this will be 0 (points at the term 0 in (35))"
Dan calculates the second 0 term again and asks JL to confirm that the term is zero, which JL does.
"If I continue then (he points at (35)): This + this - this + this... It tends to zero.

Dan opens ISM and its solution to Exercise 10.4.5:

> "**5.** $\sum \frac{(-1)^n(n^2-1)}{n^2+1}$ diverges since its terms do not approach zero."

"Diverges... since its terms do not approach zero... Aha! No they did not... they get smaller and smaller [his statement is wrong since $|\frac{(-1)^n(n^2-1)}{n^2+1}|$ increases towards 1, which is the reason that the series diverges], but... The area, they will never reach zero, apparently."
"How do you mean?"
Dan sketches a curve that looks like $y = 1/x$ and marks the area between the curve and the x-axis, approximately between $x = 1$ and $x = 7$.
"It looks something like this. It gets smaller and smaller, but will never go towards 0... Which is a bit strange, I think personally."
Dan taps his pen at the 0 in (35).

JL asks if it is strange because of the zero.
"Because of this 0 that suddenly appears. On the other hand... that is, it is just a number in the series, 1-0. Oh yes, this 0 is just a number in the series. It is not a final 0, it does not tend towards 0.
Dan copies fragment of the ISM solution into a faulty answer:

Diverges $\rightarrow \infty$ Since it never becomes 0. The area under the curve.

Dan leaves Exercise 10.4.5 by saying:
"It was not so easy. Exercise 3 was much easier."

Reasoning structure:
Dan works rather quickly and his reasoning is very difficult to follow and structure, in particular the bases for his (sometimes contradictory) statements. He starts by quickly deciding that "This is an alternating series, but it goes to infinity", which could mean many different more or less correct things. As above, he does not use any properties of alternating series. Then follows something like:

(a.1) *Problematic situation*: The exercise.
(a.2) *Strategy choice*: Analyse the behaviour of the terms in (35).
(a.3) *Strategy implementation*: He first seems to hold on to his earlier statement "This goes to infinity then...", which seems to contradict the next one: "Yes it goes to 0, but it will never get there really." One possible interpretation, which is mathematically wrong in relation to Exercise 10.4.5, but where the two statements are not contradictory is one that he actually expresses at (b.4) below (when he compares with $y = 1/x$): That the *terms* decreases towards 0, but the *sum* diverges to infinity (which is the case with $\sum_{n=0}^{\infty} 1/x$). He then seems to change his mind when he notes that there is a 0 term in (35), and therefore wrongly makes the extremely superficial conclusion that:
(a.4) *Conclusion*: The series converges to 0.

(b.1) *Problematic situation*: Dan wants to know if (a.4) is correct.
(b.2) *Strategy choice*: Search for AS information in ISM.
(b.3) *Strategy implementation*: Contrary to (a.4), ISM says explicitly that the series diverges since the terms do not approach 0. Dan mixes consequences of several components that are involved in his solution work, without being able to make any fruitful considerations of their intrinsic properties and their relations. It seems, to the observer, like extremely superficial 'random' relations:
(b.4) *Conclusion*: (i) At first dan concludes wrongly (a.4). (ii) After reading ISM, he concludes that the series diverges since even though the terms will get "smaller and smaller (wrong)", the area under the curve (which should mean $y = \frac{(x^2-1)}{x^2+1}$ if one disregards the factor $(-1)^n$, though this curve does no resemble the graph of $1/x$ he sketched above) will not approach 0, in a similar way as the area under $y = 1/x$ (wrong). This is probably a faulty application of the integral test, or some superficial connection to its graphical representation. (iii)

Dan then correctly realises that the reasoning that led to (a.4) is wrong since the 0 in (35) is just a term in the series, not its limit. (iv) He finally concludes, incompletely and wrongly, that the series diverges since the area under the curve never becomes 0.

Reasoning characteristics: This is an IS exercise, that could be solve by applying Theorem 4 from Section 10.2: "If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$." Since $\lim_{n\to\infty} \frac{(-1)^n(n^2-1)}{n^2+1} \neq 0$ the series diverges. Dan never in his work considers anything that is related to this theorem. He is probably not aware of (the intrinsic property) that the absolute value of the terms approaches 1 and that the sum 'oscillates' as $n \to \infty$.

Dan's strategy choice at (a.2) is probably an attempt to use the heuristic strategy (that is more often elaborated by teachers than by textbooks) to study the behaviour of the first few terms in the series, in order to 'get a feeling' for how the sum behaves as more terms are added. Though Dan's reasoning is very hard to follow it is clear from (a.3), (a.4), (b.3), and (b.4) that he actually draws conclusions based on extremely weak surface properties, without considering any of the intrinsic properties of the components involved. There is no explicit mathematical reasoning to support his first statement "it goes to infinity", or his later decision that the terms tends to 0 but the area is infinite (like $y = 1/x$). Both these statements (or their correct counterparts) would take some work to verify, but Dan just formulates them, as it seems, 'out of the air'. It is also an extremely superficial connection when he wrongly states that the series converges to 0 since there is a 0 among the terms (35). The only explicitly correct part of his work is when he realises that the 0 in (35) is just one term and not the sum of the terms, but this is a very basic fact in the definition of series. No PR is involved.

It seems from his final comment that Dan believes that he has solved Exercises 10.4.1 and 10.4.3 though he is not close to producing answers to any of the three exercises above, even with the help of ISM. The rest of his work with other exercises proceeds in a similar way as with Exercise 10.4.5, but the descriptions of them are omitted here.

Dan relates Exercise 10.4.3 to Example 10.4.1, since both the exercise and the example contains the numerator '$\cos n\pi$'. It is noteworthy that there is no example in the whole Section 10.4 which has a summation term that includes the expression $\sqrt{n}$ from Exercise 10.4.1, and no example that includes a term similar to $\frac{(-1)^n(n^2-1)}{n^2+1}$ in Exercise 10.4.5. This may explain why Dan does not relate his work with Exercises 10.4.1 and 10.4.5 to something in Section 10.4: There is no *example* with similar surface properties (as in Exercise 10.4.3), and he can not find the provided IS information since his only base for reasoning is extremely superficial IS. The IS information that he is (probably) used to find in examples is here contained in *theorems* instead, and he does essentially not read the latter.

6 Discussion

According to the classification structure in [Lit00b] all exercises that the students work with above are IS exercises, except 8.3.17 which is an LPR exercise. In [Lit00b] 600 calculus textbook exercises were classified, and about 70 % were IS, 20 % were LPR, and 10 % were GPR exercises. The PR exercises were mainly found at the end of each exercise section, and the three students in Section 5 were videorecorded starting with new sections, where they were likely to meet mainly IS exercises. Their work would probably have been somewhat different if they had worked with LPR and GPR exercises, but since frequent failures in their IS implementations lead to problematic situations there are opportunities to study other types of reasoning than straightforward IS.

Though there are clear differences in the three student's reasoning characteristics there are also some clear similarities, in particular in their strategy choices, which is a bit unexpected considering their great variation in examination success: Jon passes several more exams than is required in full time studies, Ulf fails all mathematics exams at first attempt but passes later, and Dan fails so many exams that he is likely to drop out of his engineer program. The aspects that seem most central in relation to the research questions from Section 3 are discussed below.

6.1 Strategy choice: Characteristics and reasoning base

6.1.1 What are the main strategies?

All fruitful global strategy choices (9.4.1 (a.2), 9.4.3 (a.2), 8.3.1 (a.2), 8.3.7 (a.2)) are based on IS reasoning. These are all 'proper' choices, since the IS exercises can be solved by applying the found formulas. The unsuccesful global strategy choices are based on: (i) An incomplete recollection of a solution procedure described at a lecture (8.3.17 (a.2)). (ii) Faulty connections to wrong IS information (10.4.1 (b.2), 10.4.3 (a.2). (iii) Some major misconceptions (10.4.5 (a.2) and (a.3)). The strategy choice 10.4.5 (a.2) is both the only one that is not based on a (by the textbook or teacher) provided specific and complete solution method, and the one who's implementation is based on the severest misconceptions. The students' main global strategy choices are thus IS, a conclusion that is also strengthened when they explicitly say that they expect IS information to be available somewhere (9.4.1, 9.4.3 (j.2), 8.3.7 (d.4)).

Almost all of the strategy choices of a more local character are based on IS or AS reasoning. The few exceptions are: (i) 9.4.3 (c.2), (f.2) and 8.3.7 (c.2) are unsuccessful attempts to apply recalled familiar methods often used in earlier exercises. (ii) 10.4.1 (a.2) and 10.4.5 (a.2) are unsuccessful and very superficial attempts to analyse the behaviour of sums. There are only two local situations found that may be classified as LPR: (i) Jon 'works backward' at (i.2) in order to access the derivatives from ISM's solution of Exercise 9.4.3. (ii) Ulf realises in Exercise 8.3.7 that his $\ln x$ component makes the central idea in the example solution hard to carry out. It is noteworthy that both these LPR situations concern getting access to IS or AS information.

Conclusion: Most strategy choices are of IS or AS types, and several of the IS choices lead to correct solution procedures. It is crucial, for these students, to identify proper IS information in order to be able to solve exercises successfully, especially if they are going to make it without AS information.

6.1.2 Why are these strategy choices made?

It is not possible to determine what kind of reasoning and conceptual understanding that underlies the decisions in the cases where the IS strategy choices are correctly and quickly made. There are some weak indications that even these choices are not based on considerations of intrinsic properties, for example: (i) Jon says (9.4.1 (a.2)) about the formula his using that "I think it is the length of the curve". (ii) Ulf searches in the textbook and says (8.3.1 (a.2)) that: "I go back to see what the formula looks like. I write it down so it will perhaps go into my head." (iii) Ulf says (8.3.7) that " We can go back in the book and see if there is some similar example, to compare with... Example 2 seems fairly similar."

However, there are several situations where the strategy choices are clearly based on surface property considerations only, for example: (i) Most of Jon's many attempts in Exercise 9.4.3 to find the differentiation rule for $\cos^3 t$ and $\sin^3 t$ consist of studying texts that have something to do with the expressions sin or cos in a wide variety of, sometimes irrelevant, situations. At several occasions he reads, but fails to recognise, proper IS information because it is written on a more general form (e.g. the chain rule) and he does not make the (fairly basic) intrinsic property considerations that are necessary in order to find and use it. (ii) Though Jon, at Exercise 9.4.3, reads ISM's solution carefully several times he fails to to discover and consider the basic intrinsic property of the positivity of curve lengths, and its calculational consequences, in ISM's solution. (iii) Ulf has very little insight in the intrinsic properties of the solution procedure he recalls at 8.3.17.

Sometimes the surface IS strategy choices are extremely superficial and the found solution procedures have completely irrelevant intrinsic properties, for example: (i) In his search for IS information about differentiation rules for Exercise 9.4.3 Jon studies carefully the integration example 6.6.7 just because it contains the 'keywords' $\sin^2 x$ and $\cos^3 x$. (ii) At 8.3.17 Ulf tries wrongly to 'squeeze' the exercise into a formula for arc length of ellipses. (iii) Most of Dan's work. The clearest situation is perhaps when he at 10.4.3 decides to use the ratio test, just because both the exercise and an example has '$\cos n\pi$' as numerator, and the example mentions the ratio test. There are absolutely no intrinsic property connections between them and he does not consider how the ratio test has been used in the example.

All AS strategy choices can of course trivially be made without considering any mathematical properties at all: There is a provided solution to each exercise.

The AS strategy implementation may though, as seen, be tricky since the ISM solutions often are rather compact.

Conclusion: The reasons behind the strategy choices are almost exclusively based on surface considerations, sometimes extremely superficial, or trivial in AS situations.

6.1.3 What is, or could have been, the role of PR?

As argued in [Lit00c] and [Lit00b], PR can be seen as central in competent mathematical reasoning, in a sense necessary both in routine situations where things turn problematic for some reason and in creative problem solving. Without the ability to construct and carry out PR, it is impossible to solve any mathematical task unless a solution procedure has been provided by someone else. In the present study PR is very rare while AS and IS is very common. In many of the situations the students seem to want to avoid AS, it is probably seen as a bit of 'cheating', but often only until things become too difficult. Even when AS fails, there are essentially no PR attempts. Not even in Jon's solution work, though he is a high achieving student.

IS could have been complemented by PR in problematic situations, but very seldom is and then only locally. A few examples of possible PR, considering intrinsic properties, that could have lead forward may be mentioned. It is probably noncontroversial to claim that the ability to carry out this type of reasoning is central in the curricula goals:
(i) If Jon had analysed his work and noticed that his faulty answer at (20) was 0, he could have concluded something like: If the (arc length) integral of a continuous function (that is not identically 0) over an interval of positive length is 0, then the integrand function must be negative on some interval of positive length. Then the arc has negative length on that interval, which is impossible. This implies that the mistake in Jon's calculations probably is at the first place where a negative integrand appears: at (18).
(ii) Ulf could in Exercise 8.3.17, in a similar way as in Exercise 8.3.7, have tried to analyse the solution to Example 8.3.4 and found that the central intrinsic property that differed from his solution work was that his curve needed to be divided into parts expressed on the form $y = f(x)$.
(iii) Dan could have, perhaps using his calculator, actually have added some of the first terms of the series in all his exercises in order to study their behaviour.

Conclusion: PR is not required in IS exercises if the solution work proceeds smoothly. The IS implementations often go wrong for the students but there are still very few and limited PR situations found, not even in the work of the high achieving student Jon. More extensive PR could have lead to better progress (which is one of the main conclusions in [Lit00c]).

6.2 Strategy implementation: Success and failure

6.2.1 What are the results of implementing IS and AS reasoning?

It is clear, which is to be expected, that the 'idealised' IS solutions presented in [Lit00b] are very simplified. In reality, other types of reasoning are often involved and influence success and failure. There are no absolutely 'pure' IS solutions found in the present study but some that almost are (e.g 9.4.1, 8.3.1), where the strategy choice (to find IS information) is immediate and the strategy implementation (copying the found solution procedure) can be straightforwardly carried out without considering the current intrinsic properties of the components involved. In these solutions the competence required and practised is elementary (mostly algebra) and mainly based on past mathematical properties.

The implementations of the IS and AS strategy choices are, as seen, often far from trivial for the students, for different reasons:
(i) Often the students' restriction to consider only surface properties leads to difficulties in distinguishing useful IS information from unuseful, even when useful IS information is closely present (9.4.3, 8.3.17, all Dan's exercises). Jon's statement (9.4.3) that "I am uncertain... I have always been" may be an indication that considering intrinsic properties may be hard even for skilled (by exam measures) students. There are also several situations where surface considerations are sufficient in order to find useful IS information (see Section 6.1.2).
(ii) Jon, and Ulf to some extent, probably has the resources required to consider intrinsic properties to a greater extent than what they actually do. In contrast to this, Dan's apparent lack of basic conceptual understanding makes it impossible for him to consider anything but extremely superficial properties, and he misses central basic IS information in a way that Jon and Ulf does not. One could suspect that lacking conceptual understanding would lead to a more intense IS focus, here is actually the opposite case present: Dan's IS information searching is not only much less efficient than Jon's and Ulf's, he even tries to a far lesser extent. He does not even relate his work with the exercises in Section 10.4 to the text that he just read in the same section. Dan seems to have 'lost contact' with the mathematics treated in Section 10.4, and his work indicates that it is actually very difficult to carry out IS solutions by finding and 'blindly' copying IS information without any insight in the intrinsic properties of the components involved.
(iii) The students are often not able to use the AS information from ISM (9.4.3, all Dan's exercises). Jon has difficulties when he studies the ISM solution too superficially, and does not really consider its intrinsic properties. A consequence of Dan's lacking basic conceptual understanding is that the ISM solutions are not sufficiently detailed for him to follow, and Dan fails to understand or make any use of them.
(iv) The IS solutions are often seriously stalled by slips and mistakes (9.4.3, 8.3.7), which are perhaps not primarily related to the reasoning type (but to carefulness). On the other hand, these mistakes are probably less common in PR reasoning, since then the intrinsic properties need to be more or less con-

tinuously considered and controlled. The mistakes often takes most of the IS solution time to discover, but are when found usually quickly resolved. This is especially the case in 'arranged' exercises where it is hard to make anything (e.g. calculate a faulty answer, since the arc length formula requires a very special type of function in order to work smoothly) if the mistake is not found.

Conclusion: IS and AS solutions are fairly easy to implement when things proceeds smoothly, and it is possible to get very far by only or essentially only IS reasoning. At the same time, the surface focus often leads to difficulties in actually finding proper IS information. Stall mistakes are serious obstacles, and their inability to consider intrinsic properties may makes IS and AS information hard to use.

6.2.2 What are the students' goals?

As discussed above, Dan's strategy implementations include really no considerations of any mathematical properties at all, neither surface or intrinsic, due to lacking conceptual understanding. Perhaps his difficulties are so severe that he has even given up to aim for local understanding? Contrary to this, Jon's and Ulf's strategy implementations are, though essentially no PR is invoked, clearly carried out with the goal to reach local understanding of each step in the IS and AS information. Their reluctancy to accept 'blind copying' is exemplified by Jon's hard work in Exercise 9.4.3 to learn the differentiation rules behind the derivatives he found, and later his reluctancy to leave 'his' method and follow ISM while not understanding why his method does not work. Another situation is when Ulf 8.3.7 (d.2) carefully studies a solved example in order to understand the local details. At the same time, they seem prepared to accept 'blind copying' of general solution strategies, there are no signs that they attempt to learn general ideas by considering global intrinsic properties. For example, Jon wants to learn the formulation of the differentiation rule for $\cos^n x$, but seems not interested in learning or considering its general mathematical background (differentiation of products or composite functions). Dan wants (10.4.3) to see more examples of different exercise types, but only different from a local surface perspective. It seems like they aim at learning how to solve a particular, limited, exercise type, instead of learning general ideas: Jon says (9.4.3 (o.3)) that sine (local property) is difficult, but does not refer to integration of partly negative functions (the central principle). In the same way, Dan believes wrongly that his difficulties (10.4.3) are caused by problems with local strategy implementation simplifications of expressions, not by his inability to make proper strategy choices.

This aim for local understanding may be seen as a way to try to learn formulas and methods in order to pass the exam, an assumption that is supported by several statements: Jon says (9.4.1) that "I follow this formula, to see if it gets correct. And then I try to learn this formula so I will know it on the exam." At 9.4.3 he says that "But I have to learn this at some point. It should not be that difficult", and later as an answer to JL's question about what he will do without his formula collection at his exam: "Then I have to learn the formulas. That is the only alternative... And hope that I pass the exam." Ulf says when working with 8.3.1 that "I go back to see what the formula looks like. I write it down so it will perhaps go into my head."

Since the students spend little or no time reading the rather difficult but informative text that precedes the exercises, it seems like they aim at learning not only task solution procedures but also concepts and all other mathematical ideas through working with exercises.

Conclusion: Their goal seem to be to apply IS solutions, and try to learn and understand the local components of the particular solution type in order to remember it for the exam. The main part of these students' homework time is spent working with textbook exercises.

6.2.3 In what ways are control and verification conducted?

The students' main way of verifying solution work is through comparing their work with the textbook's solution section or with ISM. Jon and Ulf are often able to verify or correct their work by comparing with these information sources, though it is sometimes far from trivial. Dan is not able to make use of it, there is no situation where he understands how his work differs or coincides with the information from the solution section and ISM.

At 9.4.3 Jon tries many different approaches in short time. Mostly, he does not follow them too far (sometimes to short), he curtails approaches that does not lead forward. In that sense his monitoring and control is efficient (in the same way as one of the more proficient students in [Lit00c], where solution work in exam-like situations were studied). Schoenfeld [Sch85] have found that absence of monitoring and control is one of the main reasons behind problem solving failure (in situations where no aid is at hand), which results in that solution procedures are followed very far even if no progress is being made. This is not the case here when the students have access to different kinds of written information sources: all three students turn to these sources as soon as things get difficult. This can be compared with the study in [Lit00c] (where no aids were available), where the main reason when students reacted and questioned their work was that the results looked superficially unfamiliar (EE reasoning). This occurs once in the present study, when Ulf (8.3.1) thinks that his answer became too easy to fit with his experiences. There are no attempts at verification by PR, for example by comparing with sketches of arcs or estimating convergence by studying the behaviour of the first few terms in a series.

In other ways the control is also less efficient: The students seem to believe that the trouble lies in that the algebraic simplifications of the expressions are tricky (9.4.3, 8.3.7, 10.4.3 (a.3)), and do not question other faulty work. Instead of analysing (the intrinsic properties of) their work, their main strategy is to search for IS and AS information. In general, the search for errors is unsystematic and often takes most of the solution time, which is characteristic also of the students' work in [Lit00a].

Dan's control and monitoring is ineffective, since he is unable to relate both his own reasoning and the IS and AS information to the basic intrinsic properties of the concepts treated.

Conclusion: Control and verification is mainly based on comparing with available information sources, not on own considerations.

6.3 Consequences

What types of competence will be developed from working this way? Since this is not actually investigated in this study, this section becomes a bit speculative.

It is remarkable that, if Section 5 describes examples of the general task solving behaviour of these students, IS strategy choices are so dominating and often so successful, especially at the global level. One consequence of this is that several central components in non-routine problem solving may not be practised and developed, for example heuristics and control ([Sch85], [Sch92]). It is also likely that the students will develop the belief ([Sch85], [Sch92]) that mathematical tasks are solved by searching for a, by someone else provided, solution method, and not at all by ones own solution constructions. This coincides with the main results of [Lit00c]. It is also possible that their resources and conceptual understanding will, as discussed more extensively in [Lit00b], will only be marginally developed. If IS solutions are possible in exercises, and practised by students as main global and local strategy choices, then the *only* situations where students practice anything but copying procedures are those where mistakes and slips are made in the IS implementation.

The students' work in Section 5, together with the IS - LPR - GPR distribution discussed in Section 2.2.2, indicates a focus on local procedures and absence of more global and conceptual considerations. But could not a procedural focus at least be seen as a 'prestage' to conceptual understanding? Hiebert and Carpenter [HC92, p. 78] discuss the relation between conceptual and procedural knowledge, and claim that a procedural focus may even prevent a later conceptual development: "The evidence [several references are cited] suggest that learners who possess well-practised, automatized rules for manipulating symbols are reluctant to connect the rules with other representations that might give them meaning. [...] The tendency to persist in using procedures once they are well-rehearsed, without reflecting on them or examining them further, has been noted for some time in a variety of domains."

Henningsen and Stein [HS97] set out to "identify, examine, and illustrate the ways in which classroom factors shape students' engagement with high-level mathematical tasks" in middle school classrooms. They found that "when students' engagement is successfully maintained at a high level [of mathematical thinking], a large number of support factors are present". Another result was that though the tasks themselves were identified as being set up to encourage doing high-level mathematics one major obstacle was a "decline into using procedures without connection to concepts, meaning, and understanding. This in turn was mainly caused by three factors: (i) Challenges became nonproblems, for example by "successfully pressuring the teacher to provide explicit procedures". (ii) A "classroom-based shift in focus away from meaning and understanding toward the completeness and accuracy of the answer". (iii) Too much or too little time. In middle school, the students' main source of information and communication is the teacher. At undergraduate level, especially when working with exercises, this source is mainly the textbook. In undergraduate textbooks, factor (i) above can be seen as corresponding to the increasing (during the past 30 years or so) amount of solved examples and other types of IS information provided. There is at least one influential component in the students' work in Section 5 that encourages a shift that corresponds to (ii): The main way to verify and evaluate their work is to measure the correctness of their answer (not their solution work) by comparing it with the textbook's solution section. The counterpart to (iii) is not primarily investigated in this study, but it is clear that the students' slow progress will lead to difficulties in keeping the schedule. In addition to this, it is well-known that very many students (at least in Sweden) complain about the high pace of the courses.

Szydlik [Szy00] compares university students' content beliefs about limits and their sources of conviction. The students were enrolled in a traditional calculus course using a traditional textbook. The data suggested that students with external sources of conviction (the authority of a teacher or a textbook) gave more incoherent definitions, held more misconceptions, and were less able to justify their calculations than those with internal sources of conviction (appeals to empirical evidence, intuition, logic, or consistency). It is non-trivial to compare these sources of conviction to the reasoning types described in Sections 2.2.1 and 2.2.2, but they are probably related in the way that the sources of conviction in IS reasoning are the textbooks. The sources of conviction in PR are consequences of mathematical properties, mainly concluded from (more or less consistent) deductive or inductive logical arguments.

The students' (at least Jon's and Ulf's) aim for local understanding will probably lead to some concept and solution procedure understanding. A main problem is that this understanding may be very local, based only on surface properties, and related only to exercises of a very limited type, for example to find the arc length of $\sin^n x$ curves. This means that, in order to pass the exam, they will have to memorise solution procedures to very many different exercise types. On the other hand, if the exams are adapted to coincide with the competence developed in IS textbook exercise solving then it is probably possible to go very far, within the learning environment that we provide for our students, by memorising and (partly) understanding local procedures.

Boaler [Boa98] compared student (age 13-16) experiences and understandings in two different learning environments: 'traditional' and 'open project-based'. Students from the latter developed a conceptual understanding that provided them with advantages, both in school and nonschool settings. Students from the former developed a procedural knowledge (including "rule-following behavior" and "cue-based behavior") that was of limited use to them in unfamiliar situations. These students had not experienced unfamiliar demands in their mathematics lessons: "For their textbook questions always followed from a demonstration of a procedure or method, and the students were never left to decide which method they should use. If the students were unsure of what to do in the lessons, they would ask the teacher or try to read cues from the questions or from the contexts in which they were presented." Though these students' textbook exercises probably differ in several aspects from undergraduate textbook exercises, it is reasonable to hypothesise that the traditional students in [Boa98] applied (something similar to) IS strategies. There are additional studies describing that 'reform' students outperform 'traditional' students, e.g. [BC+98].

It is reasonable to assume that working with almost exclusively IS strategies as the three students in Section 5 does, especially on the global level, will not lead to a global understanding of mathematical ideas. Kahn et al. [K+98] "considers the extent to which students are acquiring an understanding of mathematics as a whole and of the relative significance of different parts of mathematics to that whole". Their study indicates that, "even after two years after undergraduate mathematics, many of the students involved had not developed such an understanding". Love and Pimm [LP96, p.387] claim that: "Examples are, in some sense, intended to be 'paradigmatic' or 'generic', offering students a model to be emulated in the exercises which follow. The assumption here is that the student is expected to form a generalisation from the examples which can then be applied in the exercises". In the work of the students in Section 5 there are no real signs that generalisations are considered or made. On the contrary, most of their work seem to focus local details and procedures.

It is in a sense a pity that these students, in particular Jon, do not use their competence to learn concepts and problem solving more 'solidly', less superficially, by relating their thinking more to the intrinsic properties of the mathematics they work with. Some of the characteristics of the successful graduate students in Carlsson's [Car99] study were that they were very confident and persistent when solving complex mathematical tasks. They frequently attempted to classify the task as one of familiar type, and their answers appeared to have a logical foundation. There is no support for anything but a speculative comparison between these results and the successful Jon's work in Section 5, but it seems reasonable that Jon's IS focus and his eagerness to understand and logically connect each local solution step may lead to the behaviour described by Carlsson.

A focus on surface properties and absence of intrinsic property considerations in different ways is one of the main outcomes of both [Lit00a], [Lit00b], [Lit00c], and the present paper. Perhaps this way of working will lead to short term gains (passing exam), but to long term losses like weak concept understanding and weak non-routine problem solving competence? As seen above it is possible to get really far by superficial IS strategies. It is also often possible to pick up marks on procedural exams, while having difficulties in considering definitions and proofs [Tal99] or in applying mathematics outside school [Boa98]. It seems likely that an IS focus when solving textbook exercises may lead to an EE focus when solving tasks with no aids at hand. Schoenfeld [Sch85, p. 374], when studying geometry problem solving, provides strong support for a related conclusion: "The data suggest that many of the counterproductive behaviors wee see in students are learned as unintended by-products of their mathematics instruction. A very strong classroom emphasis on performance - on memorizing constructions and practising them until they can be performed with a very high degree of accuracy - ultimately results in the students losing sight of the rational reasons for the correctness of those constructions."

References

[Ada95] R. Adams. *Calculus: A Complete Course.* Addison - Wesley, third edition, 1995.

[BC+98] D. Ben-Chaim et al. Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics*, 36:247–273, 1998.

[Boa98] J. Boaler. Open and closed mathematics. Student experiences and understandings. *Journal for Research in Mathematics Education*, 29:41–62, 1998.

[Car99] M. Carlson. The mathematical behavior of six successful mathematics graduate students: influences leading to mathematical success. *Educational Studies in Mathematics*, 40:237–258, 1999.

[H+95] M. Hegarty et al. Comprehension of arithmetic word problems: a comparison of successful and unsuccessful problem solvers. *Journal of Educational Psychology*, 87(1):18–32, 1995.

[HC92] J. Hiebert and T. Carpenter. Learning and teaching with understanding. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 65–97. New York: Macmillan, 1992.

[HS97] M. Henningsen and M.K. Stein. Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5):524–549, 1997.

[K+98] Kahn et al. The significance of ideas in undergraduate mathematics: a case study of the views of lecturers and students. *Teaching Mathematics and Its Applications*, 17:78–85, 1998.

[Lit00a] J. Lithner. Mathematical reasoning and familiar procedures. *International Journal of Mathematical Education in Science and Technology*, 31:83–95, 2000.

[Lit00b] J. Lithner. Mathematical reasoning in calculus textbook exercises. Research Reports in Mathematics Education 2, Department of mathematics, Umeaa university, 2000.

[Lit00c] J. Lithner. Mathematical reasoning in task solving. *Educational studies in mathematics*, 41:165–190, 2000.

[LP96] E. Love and D. Pimm. "This is so": a text on texts. In A. Bishop et al., editors, *International Handbook of Mathematics Education*, pages 371–409. Dordrecht: Kluwer, 1996.

[Mor89] B. Morris. Textbooks: Over-relied on but under-utilised. *Australian journal of reading*, 12:312–329, 1989.

[Pól54] G. Pólya. *Mathematics and Plausible Reasoning, Vol. I and II.* Princeton NJ: Princeton University Press, 1954.

[Sch85] A. Schoenfeld. *Mathematical Problem Solving.* Orlando, FL: Academic Press, 1985.

[Sch91] A. Schoenfeld. On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. Voss et al., editors, *Informal Reasoning and Education*, pages 311–344. Hillsdale, NJ: Lawrence Erlbaum Associates, 1991.

[Sch92] A. Schoenfeld. Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws, editor, *Handbook for Research on Mathematics Teaching and Learning*, pages 334–370. New York: Macmillan, 1992.

[Shi98] M. Shield. Mathematics textbooks: messages to students and teachers. In C. Kanes et al., editors, *Teaching Mathematics in New Times*, pages 516–522. 21 annual conference of the Mathematics Education Research Group of Australasia, 1998.

[SS95] J. Selden and A. Selden. Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29:123–151, 1995.

[Szy00] J. Szydlik. Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education*, 31(3):258–276, 2000.

[Tal99] David Tall. The chasm between thought experiment and formal proof. In G. Kadunz et al., editors, *Mathematical literacy and new technologies*, pages 319–343. Conference: 8. internationales Symposium zur Didaktik der Mathematik, Klagenfurt (Austria), 1999.

[TV81] D. Tall and S. Vinner. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12:151–169, 1981.

Liste over tidligere udsendte tekster kan ses på IMFUFA's hjemmeside: http://mmf.ruc.dk eller rekvireres på sekretariatet, tlf. 46 74 22 63 eller e-mail: imfufa@ruc.dk.

332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG
Specialerappoprt af: Stine Korremann
Vejleder: Dorthe Posselt

333/97 Biodiversity Matters
an extension of methods found in the literature on monetisation of biodiversity
by: Bernd Kuemmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM
by: Bernd Kuemmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids
by: Jeppe C. Dyre

336/97 Problem-orientated Group Project Work at Roskilde University
by: Kathrine Legge

337/97 Verdensbankens globale befolkningsprognose
- et projekt om matematisk modellering
af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen

338/97 Kvantisering af nanolederes elektriske ledningsevne
Første modul fysikprojekt
af: Søren Dam, Esben Danielsen, Martin Niss,
Esben Friis Pedersen, Frederik Resen Steenstrup
Vejleder: Tage Christensen

339/97 Defining Discipline
by: Wolfgang Coy

340/97 Prime ends revisited - a geometric point of view -
by: Carsten Lunde Petersen

341/97 Two chapters on the teaching, learning and assessment of geometry
by: Mogens Niss

342/97 A global clean fossil scenario DISCUSSION PAPER prepared by Bernd Kuemmel for the project LONG-TERM SCENARIOS FOR GLOBAL ENERGY DEMAND AND SUPPLY

343/97 IMPORT/EKSPORT-POLITIK SOM REDSKAB TIL OPTIMERET UDNYTTELSE AF EL PRODUCERET PÅ VE-ANLÆG
af: Peter Meibom, Torben Svendsen, Bent Sørensen

344/97 Puzzles and Siegel disks
by: Carsten Lunde-Petersen

345/98 Modeling the Arterial System with Reference to an Anesthesia Simulator
Ph.D. Thesis
by: Mette Sofie Olufsen

346/98 Klyngedannelse i en hulkatode-forstøvningsproces
af: Sebastian Horst
Vejledere: Jørn Borggren, NBI, Niels Boye Olsen

347/98 Verificering af Matematiske Modeller
- en analyse af Den Danske Eulerske Model
af: Jonas Blomqvist, Tom Pedersen, Karen Timmermann, Lisbet Øhlenschlæger
Vejleder: Bernhelm Booss-Bavnbek

348/98 Case study of the environmental permission procedure and the environmental impact assessment for power plants in Denmark
by: Stefan Krüger Nielsen
project leader: Bent Sørensen

349/98 Tre rapporter fra FAGMAT - et projekt om tal og faglig matematik i arbejdsmarkedsuddannelserne
af: Lena Lindenskov og Tine Wedege

350/98 OPGAVESAMLING - Bredde-Kursus i Fysik 1976 - 1998
Erstatter teksterne 3/78, 261/93 og 322/96

351/98 Aspects of the Nature and State of Research in Mathematics Education
by: Mogens Niss

352/98 The Herman-Swiatec Theorem with applications
by: Carsten Lunde Petersen

353/98 Problemløsning og modellering i en almendannende matematikundervisning
Specialerapport af: Per Gregersen og Tomas Højgaard Jensen

354/98 A Global Renewable Energy Scenario
by: Bent Sørensen and Peter Meibom

355/98 Convergence of rational rays in parameter spaces
by: Carsten Lunde Petersen and Gustav Ryd

356/98 Terrænmodellering
Analyse af en matematisk model til konstruktion af digitale terrænmodeller
Modelprojekt af: Thomas Frommelt, Hans Ravnkjær Larsen og Arnold Skimminge
Vejleder: Johnny Ottesen

357/98 Cayleys Problem
En historisk analyse af arbejdet med Cayleys problem fra 1870 til 1918
Et matematisk videnskabsfagsprojekt af: Rikke Degn, Bo Jakobsen, Bjarke K.W. Hansen, Jesper S. Hansen, Jesper Udesen, Peter C. Wulff
Vejleder: Jesper Larsen

358/98 Modeling of Feedback Mechanisms which Control the Heart Function in a View to an Implementation in Cardiovascular Models
Ph.D. Thesis by: Michael Danielsen

359/99 Long-Term Scenarios for Global Energy Demand and Supply
Four Global Greenhouse Mitigation Scenarios
by: Bent Sørensen (with contribution from Bernd Kuemmel and Peter Meibom)

360/99 SYMMETRI I FYSIK
En Meta-projektrapport af: Martin Niss, Bo Jakobsen & Tune Bjarke Bonné
Vejleder: Peder Voetmann Christiansen

361/99 Symplectic Functional Analysis and Spectral Invariants
by: Bernhelm Booß-Bavnbek, Kenro Furutani

362/99 Er matematik en naturvidenskab? - en udspænding af diskussionen
En videnskabsfagsprojekt-rapport af: Martin Niss
Vejleder: Mogens Nørgaard Olesen

363/99 EMERGENCE AND DOWNWARD CAUSATION
by: Donald T. Campell, Mark H. Bickhard, and Peder V. Christiansen

364/99 Illustrationens kraft - Visuel formidling af fysik
Integreret speciale i fysik og kommunikation
af Sebastian Horst
Vejledere: Karin Beyer, Søren Kjørup

365/99 To know - or not to know - mathematics, that is a question of context
by: Tine Wedege

366/99 LATEX FOR FORFATTERE - En introduktion til LATEX og IMFUFA-LATEX
af: Jørgen Larsen

367/99 Boundary Reduction of Spectral Invariants and Unique Continuation Property
by: Bernhelm Booss-Bavnbek

368/99 Kvartvejsraport for projektet SCENARIER FOR SAMLET UDNYTTELSE AF BRINT SOM ENERGIBÆRER I DANMARKS FREMTIDIGE ENERGISYSTEM
Projektleder: Bent Sørensen

369/99 Dynamics of Complex Quadratic Correspondences
by: Jacob S. Jalving
Supervisor: Carsten Lunde Petersen

370/99 OPGAVESAMLING - Bredde-Kursus i Fysik 1976 - 1999
Eksamensopgaver fra perioden 1976 - 1999. Denne tekst erstatter tekst nr. 350/98

371/99 Bevisets stilling - beviser og bevisførelse i en gymnasial matematik undervisning
Et matematikspeciale af: Maria Hermannsson
Vejleder: Mogens Niss

372/99 En kontekstualiseret matematikhistorisk analyse af ikke-lineær programmering: Udviklingshistorie og multipel opdagelse
Ph.d.-afhandling af Tinne Hoff Kjeldsen

373/99 Criss-Cross Reduction of the Maslov Index and a Proof of the Yoshida-Nicolaescu Theorem
by: Bernhelm Booss-Bavnbek, Kenro Furutani and Nobukazu Otsuki

374/99 Det hydrauliske spring - Et eksperimentelt studie af polygoner og hastighedsprofiler
Specialeafhandling af: Anders Marcussen
Vejledere: Tomas Bohr, Clive Ellegaard, Bent C. Jørgensen

375/99 Begrundelser for Matematikundervisningen i den lærde skole hhv. gymnasiet 1884-1914
Historiespeciale af Henrik Andreassen, cand.mag. i Historie og Matematik

376/99 Universality of AC conduction in disordered solids
by: Jeppe C. Dyre, Thomas B. Schrøder

377/99 The Kuhn-Tucker Theorem in Nonlinear Programming: A Multiple Discovery?
by: Tinne Hoff Kjeldsen

378/00 Solar energy preprints:
1. Renewable energy sources and termal energy storage
2. Integration of photovoltaic cells into the global energy system
by: Bent Sørensen

379/00 EULERS DIFFERENTIALREGNING
Eulers indførelse af differentialregningen stillet over for den moderne
En tredjesemesters projektrapport på den naturvidenskabelige basisuddannelse
af: Uffe Thomas Volmer Jankvist, Rie Rose Møller Pedersen, Maja Bagge Pedersen
Vejleder: Jørgen Larsen

380/00 MATEMATISK MODELLERING AF HJERTEFUNKTIONEN
Isovolumetrisk ventrikulær kontraktion og udpumpning til det cardiovaskulære system
af: Gitte Andersen (3.moduls-rapport), Jakob Hilmer og Stine Weisbjerg (speciale)
Vejleder: Johnny Ottesen

381/00 Matematikviden og teknologiske kompetencer hos kortuddannede voksne
- Rekognosceringer og konstruktioner i grænselandet mellem matematikkens didaktik og forskning i voksenuddannelse
Ph. d.-afhandling af Tine Wedege

382/00 Den selvundvigende vandring
Et matematisk professionsprojekt
af: Martin Niss, Arnold Skimminge
Vejledere: Viggo Andreasen, John Villumsen

383/00 Beviser i matematik
af: Anne K.S.Jensen, Gitte M. Jensen, Jesper Thrane, Karen L.A.W. Wille, Peter Wulff
Vejleder: Mogens Niss

384/00 Hopping in Disordered Media: A Model Glass Former and A Hopping Model
Ph.D. thesis by: Thomas B. Schrøder
Supervisor: Jeppe C. Dyre

385/00 The Geometry of Cauchy Data Spaces
This report is dedicated to the memory of Jean Leray (1906-1998)
by: B. Booss-Bavnbek, K. Furutani, K. P. Wojciechowski

386/00 Neutrale mandatfordelingsmetoder – en illusion?
af: Hans Henrik Brok-Kristensen, Knud Dyrberg, Tove Oxager, Jens Sveistrup
Vejleder: Bernhelm Booss-Bavnbek

387/00 A History of the Minimax Theorem: von Neumann's Conception of the Minimax Theorem - - a Journey Through Different Mathematical Contexts
by: Tinne Hoff Kjeldsen

388/00 Behandling af impuls ved kilder og dræn i C. S. Peskins 2D-hjertemodel
et 2. moduls matematik modelprojekt
af: Bo Jakobsen, Kristine Niss
Vejleder: Jesper Larsen

389/00 University mathematics based on problemoriented student projects: 25 years of experience with the Roskilde model
By: Mogens Niss
Do not ask what mathematics can do for modelling. Ask what modelling can do for mathematics!
by: Johnny Ottesen

390/01 SCENARIER FOR SAMLET UDNYTTELSE AF BRINT SOM ENERGIBÆRER I DANMARKS FREMTIDIGE ENERGISYSTEM Slutrapport, april 2001
Projektleder: Bent Sørensen
Projektdeltagere: DONG: Aksel Hauge Petersen, Celia Juhl, Elkraft System#: Thomas Engberg Pedersen##, Hans Ravn, Charlotte Søndergren, Energi 2#: Peter Simonsen, RISØ Systemanalyseafd.: Kaj Jørgensen*, Lars Henrik Nielsen, Helge V. Larsen, Poul Erik Morthorst, Lotte Schleisner, RUC: Finn Sørensen**, Bent Sørensen
#Indtil 1/1-2000 Elkraft, ## fra 1/5-2000 Cowi Consult
*Indtil 15/6-1999 DTU Bygninger & Energi, ** fra 1/1-2001 Polypeptide Labs.
Projekt 1763/99-0001 under Energistyrelsens Brintprogram

391/01 Matematisk modelleringskompetence – et undervisningsforløb i gymnasiet
3. semesters Nat.Bas. projekt af: Jess Tolstrup Boye, Morten Bjørn-Mortensen, Sofie Inari Castella, Jan Lauridsen, Maria Gøtzsche, Ditte Mandøe Andreasen
Vejleder: Johnny Ottesen

392/01 "PHYSICS REVEALED" THE METHODS AND SUBJECT MATTER OF PHYSICS
an introduction to pedestrians (but not excluding cyclists)
PART III: PHYSICS IN PHILOSOPHICAL CONTEXT
by: Bent Sørensen.

393/01 Hilberts matematikfilosofi
Specialerapport af: Jesper Hasmark Andersen
Vejleder: Stig Andur Pedersen

394/01 "PHYSICS REVEALED" THE METHODS AND SUBJECT MATTER OF PHYSICS
an introduction to pedestrians (but not excluding cyclists)
PART II: PHYSICS PROPER
by: Bent Sørensen.

395/01 Menneskers forhold til matematik. Det har sine årsager!
Specialeafhandling af: Anita Stark, Agnete K. Ravnborg
Vejleder: Tine Wedege

396/01 2 bilag til tekst nr. 395: Menneskers forhold til matematik. Det har sine årsager!
Specialeafhandling af: Anita Stark, Agnete K. Ravnborg
Vejleder: Tine Wedege

397/01 En undersøgelse af solvents og kædelængdes betydning for anomal swelling i phospholipiddobbeltlag
2. modul fysikrapport af: Kristine Niss, Arnold Skimminge, Esben Thormann, Stine Timmermann
Vejleder: Dorthe Posselt

398/01 Kursusmateriale til "Lineære strukturer fra algebra og analyse" (E1)
Af: Mogens Brun Heefelt