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by Johnny Ottesen

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Abstract

The present text consists of two papers which will appear as chapters in *Derek Holton et al. (eds.): The Teaching and Learning of Mathematics at University Level: An ICMI Study*, to be published by Kluwer Academic Publishers in 2001.

The first paper *University mathematics based on problem-oriented student projects: 25 years of experience with the Roskilde Model*, by Mogens Niss, presents and discusses the mathematics programme at Roskilde University within the context of project-oriented project work which characterises the entire study structure at that university. In order to illustrate the points made, a specific example of a student project (on the notion and role of proof in mathematics) is explained in some detail.

The second paper *Do not ask what mathematics can do for modelling. Ask what modelling can do for mathematics!*, by Johnny Ottesen, focuses on how mathematical modelling can help deepening and consolidating students' conceptual knowledge and insight in mathematics proper. It does so by means of a presentation and discussion of a course on mathematical modelling which addresses (mainly) those students in the basic programme in the sciences who do not, in general, intend to pursue further studies in mathematics or physics.

Mogens Niss[1]

University mathematics based on problem-oriented student projects: 25 years of experience with the Roskilde model

1. Introduction

Since its inception, and inauguration in 1972, as a small university designed to innovate higher education, Roskilde University (RUC), Denmark, has based all its programmes on problem-oriented project work performed by students. This is especially true for mathematics and the sciences. In these programmes, half of the students' time is devoted to project work, whereas the other half is devoted to systematic courses taught by university teachers, in more or (sometimes) less traditional ways. By a ***problem-oriented project*** in the RUC sense, henceforth simply referred to as a ***project***, we understand a ***study activity*** in which a small group of students, typically 2-8, gather in accordance with their interests in order to investigate ***a problem***, or a complex of problems, of their own choice, i.e. identified, specified, and studied by themselves, under the supervision of a faculty member. So what we are dealing with is a combination of two components which in principle are independent: a certain form of study, i.e. project work (which could be put to use towards any sort of content, including systematically organised subject matter), and a perspective on content, i.e. problem-orientation (which could be pursued by any kind of study activity, including one-way lectures). The normal duration of an RUC project is half the time of a semester (roughly four working months), but sometimes a project takes two semesters to complete. The project students themselves do all the work. They discuss and formulate the problems they want to deal with, find and read relevant literature, conduct empirical or theoretical analyses, do experiments or make calculations, interview resource persons, visit institutions or companies, and so forth, but of course under guidance of their project supervisor, with whom they meet on average once per 1-2 weeks. In an idealised description, a problem-oriented project typically contains the following phases

* identification and specification of the problem field to be investigated, leading to a formulation of the final problem to which an answer is going to be sought;
* establishment of the modes of collaboration within the project group and with the supervisor;
* the finding, selecting and reading of potentially relevant literature;
* the establishment of the theoretical core of the project and the acquisition of the knowledge necessary to deal with it, the latter usually involves the assistance of the supervisor;
* the working with external contacts, if relevant;
* the carrying out of analyses of the specified problem;
* the synthesis and analysis of the findings of the project problem;
* the production the project product, which will always include a written report but may also contain the design of an exhibition, a text book, a video programme, etc.

As stated, one visible outcome of a project is a written report, of 50-100 pages, to which all students in the group have contributed. As well as the report, the project work itself is assessed (in various ways, according to the circumstances) at the end of term by one or more independent assessors from inside or outside the university. The supervisor is always involved in this assessment.

In principle the students themselves decide when a project is finished, and they are responsible for all the consequences that follow from this decision. In practice, however, they consult their supervisor to obtain his or her opinion on the degree of completion of the project. If, for some reason, a project is not finished (which happens very seldom), or the students do not pass, they will have to improve the project, or to undertake another one, and have the new outcome tried before assessors. So, it is not possible for a student to omit or miss out a project which is required in the programme.

2. Position and structure of the mathematics programme

We shall begin by outlining the structure and organisation of the mathematics programme and its embedding in the general study structure of the university.

Firstly, the normal road for a mathematics student at RUC (the deviations from 'normality' are of a technical nature and are omitted for the sake of simplicity and clarity), is to enrol in the university's two-year ***basic programme in the sciences***. During this programme each student will choose and take 8 'ordinary' courses in the sciences and mathematics, the choices being subject to certain constraints. The mathematics courses offered in the programme are calculus, linear algebra, complex numbers and functions, differential equations, statistics, and mathematical modelling in the sciences (as to the latter course, see Ottesen, pp. ?? this volume). Students with (evolving) mathematical inclinations will typically choose to take two or three of these courses. Moreover, in the basic programme, each student will carry out four projects, each of a duration of one semester. At the end of the two-year programme each project group will defend its (normally) final project at an oral examination conducted by their supervisor, with the involvement of an external examiner from another university, institution, or organisation. This examination will be based on the project report which both examiners will have read in advance.

The two-year basic programme in the sciences does not lead to a degree but is a common platform from which students embark on so-called further studies to obtain a bachelors or, much more frequently, a masters degree. These programmes were instigated in continuation of the basic programme in 1974. Only after having completed the basic programme do students have to choose which subjects to pursue further. It is a remarkable and recurrent empirical fact that the majority of those students who at this stage decide to study mathematics did not have that option in mind when they enrolled at university. A student who wants to go on to study mathematics will have to choose one more subject to be studied on a par with mathematics: physics, chemistry, computer science, biology, geography, history, philosophy, psychology, english, german, or french, etc. As, for historical reasons, almost all students in Denmark want to obtain a masters degree, we shall confine ourselves to considering these programmes. Here, each of the two subjects has to be studied for nominally 1.5 years, which yields a nominal total of 5 years for a masters degree. Thus, in Roskilde we don't distingush between majors and minors, the two subjects have a parallel position. The masters programme is completed with a thesis to be written in either of the two subjects, or across them if this makes sense and the student so prefers. The masters thesis takes the form of yet another project, typically carried out by students individually or in a group of two, but usually much more depth is expected than is the case with the other projects. Finally, it is a peculiar feature of the Danish – including the Roskildean – academic tradition, which again has historical roots, that almost no students in classical academic disciplines complete their studies within the nominal five years. Usually they spend 1-2 years more before graduation.

Also in the further mathematics programme, students spend half of their time taking courses in systematically organised subject matter, and half of their time on problem-oriented projects. Without going into judicial details, students are typically required to become familiar with the following topic areas (but not necessarily by taking courses), in continuation of the ones they encountered in the basic programme: linear structures from algebra and analysis, advanced mathematical analysis 1 & 2, discrete mathematics, geometry, probability theory and statistics, basic structures in mathematics, and at least one 'advanced topic' (such as optimisation theory, measure and integration, partial differential equations, functional analysis, fluid dynamics, wavelets, stochastic processes).

Each student has to carry out three projects in mathematics (and three in his/her other subject, giving a total of 10 projects throughout an entire masters programme). It is important to note that the sorts of problems that students' projects deal with certainly need not be specifically mathematical problems in the classical sense. They can be, of course, but normally the problems dealt with are intellectual problems, of a complex and general nature. More specifically, each student has to take part in one project which focuses on one (or more) ***mathematical model(s)*** in an area outside of mathematics, either by investigating and examining existing models from mathematical perspectives, e.g. with respect to their properties and behaviour, or – less frequently – by constructing a new/modified model. Naturally, if possible and appropriate, students will tend to study or construct models which are relevant to their other subject, whether, say, physics, biology, chemistry or computer science. Secondly, each student has to perform one project on the nature of ***mathematics as a discipline***. Such a project tends to be fairly theoretical. Typical projects in this category deal with aspects of the foundations and architecture of mathematics, with the history of mathematics, or with philosophical issues related to the position and nature of mathematics. Finally, each student in his/her third project has to choose between three major options: a second ***modelling*** project, in which an original mathematical model is to be constructed in cooperation with professionals in other disciplines or fields of practice; a project on and within ***pure mathematics***; or a project on aspects of ***mathematics education***. These options will typically reflect students' career expectations or intentions, thus serving to prepare them for their future professions as users of mathematics, researchers or teachers.

These project types have not been designed at random. On the contrary, they are meant to provide students with ***representative experiences***, based on selected cases identified and shaped by the project groups themselves, of the essential facets and properties of mathematics as a pure science, an applied science, an instrument for societal practice, an educational subject, and a field of aesthetics.

Although students have to undertake projects within certain prescribed categories, their freedom in choosing and specifying the problem field they want to investigate is infinite, and the projects actually carried out within each category are in fact very different, as will be illustrated below. Once again, each project group will have to present itself and its work at an oral examination, based on the project report, and conducted by their project supervisor and an external examiner.

3. Examples of mathematics projects

In order to illustrate the spectrum of possible projects within each category, we give a few examples of themes of projects actually undertaken over the years.

Analysis of mathematical models: Stochastic models in population genetics, Models of gonorrhoea, Mathematical models of the Belousov-Zhabotinskij reaction in chemistry, The models involved in RSA cryptography, The application of the Radon Transform to CT scanning, Models of voting and election methods, An analysis of the World Bank's global population forecast model, An analysis of terrain models, Structure models.

The nature of mathematics as a discipline: The role of the number domains in the edifice of mathematics, The trinity of Bourbaki - the general, the mathematician, the spirit, The place and role of non-standard analysis in mathematics, The mathematical foundation and position of dynamic programming, ''Is not the title of this project' is not the title of this project': Self reference in mathematical systems, Mathematical symbols and their roles in mathematics, The significance of pathological examples in the development of mathematics, Euler and Bolzano: Mathematical analysis from the perspective of science philosophy, Internal and external factors behind the genesis and development of the standard methods of statistics, The impact of hyperbolic geometry on the philosophical interpretation of mathematics, Galois' contribution to the development of abstract algebra, The genesis and development of almost periodic functions, Is mathematics a natural science?, Cayley's problem of Newton iteration %%: A%% historical analysis 1870-1918, Proofs in mathematics.

Building of mathematical models: Statistical methods to determine safe doses of carcinogenic substances, Analysis of multispectral satellite pictures, A model for periodic selection in E-coli bacteria, Modelling calcium transportation through cell membranes, Monte Carlo simulation of non-equilibrium Ising models in molecular dynamics, A linear programming approach to maintenance schemes for public bridges, Modelling the cardiovascular system with respect to neural pulse control, Modelling 'viscous fingers'.

Pure mathematics: The ellipsoid method in linear programming, Numerical methods for the solution of rigid differential equations, Contrafactual conditionals in the language HOL, Knots, links, and algebraic invariants, Wavelets, Graph theory and multidimensional contingency tables, Proof theory, exemplified by Gentzen's proof of the consistency of the theory of natural numbers, Self-avoiding random walks.

Mathematics education: The perception of mathematics amongst students in 11th grade, Around the world on flat maps – a textbook unit for the teaching of spherical geometry and map projections at the upper secondary level, Competencies of modelling – developing and testing of a conceptual framework, The justification problem in upper secondary school mathematics, Problem solving and modelling in secondary mathematics education, Rhetoric or reality? Mathematical applications in upper secondary school mathematics in Denmark, 1903-88, 'Model talk' – An upper secondary textbook on differential equation models of dynamical systems, The position of proof – Proof and proving in upper secondary mathematics teaching.

4. An illustrative case, close-up

For an illustration, let me describe in some detail one of the projects in the above list. The project, 'Proofs in Mathematics' was conducted by a group of five students (three female and two male) during the autumn semester of 1999. I have chosen to present this project for a variety of reasons. It is a recent project and one that I supervised myself, so I can report on it on the basis of first-hand experiences. It is a project which, at the final examination, was given a grade slightly above average, so it is not a special project in terms of quality. In fact it is quite typical in many respects. Three of the five students did this project as their very first in the mathematics part of the programme, i.e. after having completed the basic programme, while one student did it as her second project in the mathematics programme. For these four students, the project belonged to the category in which the nature of mathematics as a discipline is investigated. The last student was in a slightly different situation as this was his last project in mathematics, categorised as a socalled 'profession-oriented project' of the variant focusing on mathematics education. This particular student therefore had to append a small additional report of his own (on epistemological obstacles to proof and proving in the teaching and learning of mathematics) to the main report, of which he was a co-author on a par with the other students. However, for simplicity I shall confine myself to presenting only the main project. The fact that this project group consisted of students from different intake cohorts is not unusual.

This group of students responded to a general observation made by me at the introduction seminar for the autumn 1999 semester, that a multitude of project possibilities were unexploited at RUC within the domain of proof and proving. A few projects had been done in the past within this theme, but a host of aspects had been left untouched. Several students then became engaged in discussing possible projects on proof and proving. When the project group was eventually established it comprised five students.

The students spent a fair amount of time discussing which *problématique* they wanted to focus on, while reading general literature on proof and proving at a not too technical level (e.g. Brandt and Nissen: 'QED: An introduction to mathematical proof' (in Danish), Davis and Hersh: 'The Mathematical Experience', Garnier and Taylor: '100% Mathematical Proof', Hanna and Jahnke: 'Proof and proving', Howson: 'Logic with trees'), some of which were pointed out by me, and some of which were discovered by the students themselves. It was difficult for the group to identify the problem they wanted to investigate. First, they thought of focusing on different types of proof but gave this up because, on closer inspection, they found the issue too shallow. Then they thought of concentrating on investigating the difference between proofs that prove and proofs that explain, but that idea was abandoned because it seemed to lead to an educational rather than an epistemological project, which was not in accordance with the students' wishes. During their discussions, the students time and again returned to the issues: How can we know that, and when, a mathematical statement has really been proved? How much can be removed from the presentation of a proof without removing its 'proofness'? To what extent is a proof identical to a written formulation of it? Against that background, students settled on the following simple problem formulation 'What does it take for a mathematical statement to be proved?' Thus, ultimately the project concentrated on proof as a product, not on proving as a process.

From the beginning, students had decided that they didn't want to produce a philosophical treatise of a general nature but to perform a specific analysis based on concrete cases. Moreover, they wanted these cases to be drawn from mathematics that was either part of what they had already learnt or were about to learn, or was not so demanding that they would need to spend large parts of the project time on studying new mathematical topics. In other words, students saw it as essential to be able to concentrate on the analysis of proof without being distracted by too difficult mathematics. (It should be noted that this was a decision made by this particular group of students. It equally frequently happens with project groups – though usually with more experienced ones – that one criterion for the choice of a project theme is its capacity to make students familiar with mathematical topics that are new to them.)

As three of the students had just started in the mathematics programme it was agreed to concentrate on subject matter that pertained to elementary number theory, linear algebra, Weierstrassian real analysis (they were taking courses in both concurrently with this project) and (Euclidean) geometry. It was not an easy task to select the cases to be investigated. In addition to the ones actually included in the project, several other cases were taken into consideration as possible candidates but were discharged for some reason or the other. The resulting cases were the following:

The irrationality of $\sqrt{2}$. Here, students came to realise that there are actually two statements involved in this. One concerning the rationals only, in which context we have an impossibility theorem: There is no rational number whose square is 2. And one concerning the reals: There is a real number whose square is 2, and that number is not rational, which is an existence theorem combined with an impossibility theorem. Many textbooks, including the one they were studying, do not attempt to make a distinction between these two statements.

The Hippokrates moon on a quarter circle chord can be squared. This theorem in classical plane geometry was investigated in the context of squaring problems at large, in particular 'where exactly does a generalisation of Hippokrates' method to the 60 degree chord fail to prove (!) that a circle can be squared?'.

Every real, bounded sequence has a convergent sub-sequence. Two proofs of this theorem were studied, one based on the notion of 'lead elements' of tail sequences, and the classical one based on bisection. In both cases the 'point of appearance' of the completeness axiom of the reals was given special attention.

The dimension theorem on linear transformations (for a linear transformation of a finite-dimensional vector space V, the dimension of V equals the sum of the dimensions of the null space and the image space of the transformation).

The complex expontial function exp satisfies $exp(z+w) = exp(z)exp(w)$.The exponential function is often introduced, as it was to these students, by means of real exponential, sine and cosine functions. The proof of the complex functional equation is then based on the well known properties of these functions, in particular the addition formulae. This example was discussed by studying (consistent) ways to introduce these real functions so as to yield the desired properties, primarily by means of power series.

The binomial formula. Here the point was to compare the nature, foundation, and scope of two proofs, a combinatorial proof and an induction proof.

The bulk of the project work was spent on analysing these cases. This was done by means of a general template that was applied to all the cases. It took a long time for the group to develop this template and specify its components. These are: What type of theorem (existence, uniqueness, impossibility, ascription of properties (including identities))? What type of proof (see below)? In what mathematical universe(s) is the theorem embedded and what conceptual foundation does it invoke? What are the basic ideas in the proof? This template was then used to guide the specific analysis of each case. It turned out that it was not always easy to clearly classify the aspects of the proofs examined into the categories anticipated by the template. The most difficult part was to identify all relevant mathematical universes actually implicated in the proof. However, by and large the template did work rather well. Moreover, as students had not taken any courses in mathematical logic, they wanted to survey the fundamentals of logic so as to make sure that all aspects of logical inference were appropriately taken into account in their analyses. Besides, they wanted this survey to serve as a basis for an initial identification of different sorts of proof (contrapositive, indirect, induction, combinatorial).

The final project report, of 86 pages, was written jointly by the students and in several steps. On average each section had been revised a handful of times before it reached its final stage. One way of achieving both collective ownership to the project report and a fair degree of homegeneity of the text is to rotate the sections between the members of the project group. This procedure, adopted here, took place through a collective editorial process. Firstly, initial drafts were produced for the individual pieces by one or two students. These drafts were then discussed in the whole group and another member was asked to take care of the revision on the basis of the comments given. Such cycles were repeated several times until no more suggestions for improvements were made.

Now what role did the supervisor, i.e. myself, have in the entire project process? A role similar to the one I usually have with projects. In this case, the initial inspiration for the project was a so-called 'project germ' that I put forward to students at the beginning of the semester. A project germ is nothing else than an indication of a potential source and generator of project issues and themes, which a group of students can develop into a project problem in infinitely many different ways. For a project germ to result in a project it has to be cultivated, developed, tailored, specified, etc. At the beginning of each semester not only the supervisors but also the students, on a par with the faculty, propose possible project germs. The project germs put forward, constitute a 'project germ market' which is used as the base for generating new projects for that particular semester.

Once the project group was formed from a common interest in the germ, my main task was to assist students in identifying the problem they wanted to investigate. This task is of a socratic nature, because it's a matter of helping students to examine their own minds. I usually ask them 'What insight do you want to have gained when this project is finished; what questions that challenge you today do you wish to be able to answer at the end?' The importance and the difficulty of arriving at a problem which is both tractable and corresponds to all project members' real wishes shouldn't be underestimated. If this process runs sloppily it is rather likely that serious problems will emerge at a later stage. In the case of this project the process was rather lengthy, in fact to the point of frustration with some of the group members. Nevertheless, the problem to be investigated was eventually found and formulated.

In addition to insisting that students reflect on what they want to do and why, it was my task to come up with initial suggestions of the literature that might serve as inspiration, give rise to ideas, and provide information that could prove useful. This literature was supplemented with literature found by the students themselves in libraries, on the internet, etc.

When it came to selecting the cases, my role was to anticipate the consequences of choosing the candidate cases and to give advice as to their eventual feasibility ***vis-à-vis*** the outcomes students wanted their project to have. As always, the final selection of cases was completely the responsibility of the students.

During the remainder of the project, I met with students to listen to what they had done, to help them understand difficult points in the literature they were using, to help them overcome obstacles in the analysis of cases, to point out traps and perspectives that they hadn't seen themselves, etc. When drafts of project report sections began to appear I commented on them in general terms, indicating weak or missing points, more or less like a referee for a research journal.

When the project report was completed copies were submitted to the supervisor and to the external examiner (in this case a colleague at the University of Copenhagen). After a couple of weeks, during which the supervisor and the external examiner read the report, students presented themselves, as a group, at an oral exam. The main point is to identify the depth and extent of each student's ownership to the project. The aim is *not* to examine students' knowledge of traditional subject matter. This is done only to the extent that such subject matter happens to form a significant part of the project. The subject of the exam is the project in its entirety, not the project report only which is just one ingredient in the project, albeit an important one. The oral exam for a group of this size typically takes 2-3 hours. At the beginning each student makes a short presentation of one aspect of the project (usually not one which has been exhaustively dealt with in the report). Then the supervisor conducts a general discussion by raising issues and posing questions concerning the project, and the external examiner joins in. Sometimes questions are addressed to the whole group sometimes to individual members, as the circumstances warrant. The idealised model of the oral exam is one of a group defence of a thesis, although there is no requirement of scientific originality in the sense that results should be produced that are publishable. Emphasis is placed on the relevance of the problem posed, correspondence between questions and conclusions, depth of analysis, choice of methodology, difficulty of the task undertaken, independence and autonomy in the project work, etc.

When the oral exam is over (i.e. when no important issues are left to be discussed), the students leave the room and the supervisor and external examiner discuss the project, the project report and the examination. Typically, they have already arrived at a common impression of the project report prior to the oral exam, and the subsequent discussion has two objectives. Firstly to determine whether the events during the examination changed that impression in a positive or a negative direction, and secondly to see if significant differences amongst students could be detected in their exam achievements. The latter is a fairly delicate matter. For if, on the one hand, too much emphasis is placed on performance, for instance activation of factual knowledge, alertness, eloquence, initiative, etc., then the project work proper gains lesser emphasis, which will have a negative backwash effect on the whole enterprise. On the other hand, it is also important to ensure that no student can hide behind the work of others and get credit that he or she does not deserve. In principle it is possible to encounter a project group in which one student has done the main or the most important part of the work while the others are in on a 'free ticket', but this happens only very seldom. Or the other way round, one can encounter a group (and this happens more often, but still not frequently) in which one or two students are in on a free ticket whereas the others have shared all the work on a fairly equal basis. It is one of the tasks of the oral examination to detect if either is the case and to assign credits accordingly. Usually the outcome is either that all students get the same mark or that they deviate by one or two marks from one another. Only pretty rarely do we see a larger spread of marks within one project group, but this does happen from time to time.

In the present case, the project was judged to be rather homogenous as was the written report, so all five students got the same mark, a little above average, as mentioned before.

No systematic research has been done on the outcomes of project studies in mathematics at RUC. However, at a recent Danish national (but non-scientific) evaluation of university studies in mathematics, physics and chemistry, conducted by a Scandinavian evaluation panel, the Roskilde model was assessed in very positive terms. Apart from that, judgments of the outcomes have to rely on impressions gained by concrete experience by those involved, including the external examiners who submit brief reports on every exam to a central agency of external examiners. The students in this project have stated in several places in the project report, at the oral exam, and on other occasions, that they felt they had learnt a lot from it. In fact, they insisted that this project not only changed their conception of the nature, role and significance of mathematical proof but also of the nature of mathematics as a discipline. After the completion of the project and the oral exam, this group decided to include their project report in the department text series. We usually encourage a project group whose project achieved a minimum level of success to do so, in order to make it generally available to present and future students (and faculty) at the department. This has proved very useful in creating a project culture in which students can get inspiration concerning what to do and what not to do, by consulting previous project reports in their area of interest.

5. Issues and comments

One of the questions which is often raised concerning the approach described above is what mathematical knowledge and skills students (need to) have when embarking on a project. How can they perform a project without possessing all the necessary prerequisites in advance?

In some cases, such as the one described in the previous section, students more or less know the mathematical subject matter which is going to be involved in the project – but they certainly do not know it from the perspectives adopted in the project. In most cases, however, students do not know all the mathematics they are going to encounter in the project, and then it is part of the project work to study it. In other words, in such cases they will have to acquire the prerequisites underway, and not the mathematical prerequisites only, also the ones that are to do with, say, subject matter knowledge in a discipline to which mathematics is being applied, the history or philosophy of mathematics, and so forth. Needless to say, this makes it very important for students to strike a balance between what they already know and what they have to become familiar with. It is a key task of the supervisor to give advice as to what is tractable and what is beyond reach for the particular project group at issue. This being said, project groups are sometimes capable of digesting a remarkable amount of difficult mathematics as part of a project. But of course, the spectrum of variation is wide and strongly depends on the capacities of group members.

It is fair to say that *on average* projects will not involve very advanced or very specialised branches of mathematics. Mostly, the topics invoked are not too distant from the topic areas studied in the courses offered. However, it should be borne in mind, once again, that the aim of project work is not to be a new means for students to learn systematically organised subject matter. Instead, the aim is that students should encounter, and gain first hand experiences with, mathematics from a variety of different perspectives with the ultimate end of deepening their mathematical understanding, insight, and competence. Thus, students in the RUC programme will typically have been exposed to a smaller set of mathematical branches, a smaller syllabus, than forms part of masters programmes in more traditional universities. We are ready to pay this price if the outcome is graduates who have a deeper reflective knowledge and a more multi-faceted competence concerning the mathematics they have actually learnt. Our experiences during the past 25 years suggest that this is the case with the majority of the graduates.

Another question which often comes up is what demands this approach puts on the supervisors. As may appear from what has been said above, the supervisor will normally be able to supervise a project on the basis of his or her general knowledge and experience in mathematics at large. This should not be taken to imply that supervisors will never encounter issues, topics, branches of mathematics, material, and so forth, which is almost as new to them as it is to the students. In fact this happens in any project. For instance, I once supervised a project on dynamic programming of which I had only little prior knowledge. This made me engage in a fair bit of reading of the literature in this field. The outcome of a project is always original and new in the sense that its particular *problématique*, its perspectives, the particular objects it investigates, or its combination of fields of knowledge, and suchlike, did not exist in that form before the project was carried through. Of course, this is not meant to say that project outcomes are new in a research sense (although occasionally they are), only that you will not be able to find the resulting 'package' on libraries' bookshelves (at least not as far as the project group and its supervisor know). So, supervisors will never be able to exert their supervision by just pouring from their taps of knowledge. Indeed, if they could the result would not be a genuine project but an extended and possibly demanding exercise. In other words, the supervisor also learns something new from any project he or she supervises.

What expertise, then, does it require to be a good project supervisor? Apart from a broad and solid knowledge of mathematics, not only from inside but from outside as well (including the history, philosophy, and sociology of mathematics and its applications), the two most important things are a broad outlook and a fair amount of 'mathematical culture'. This means the ability to perceive and reflect on mathematics from a variety of perspectives, on the one hand, and the ability to detect students' capacities, interests, potentials and limitations, on the other hand, so as to assist them in shaping their projects in ways that provide suitable challenges which match their backgrounds and ambitions. It follows that supervisor expertise grows considerably with experience. This is also reflected in the fact that novice supervisors need to spend a non-negligible amount of time to 'learn the trade', while experienced supervisors can be quite efficient without an undue investment of time. Therefore, new faculty hired to the department are expected to be willing to engage in this type of teaching, even if they may have to learn to do it – not in formal courses, though, but by actually doing it. Usually, applicants are fully aware of this when they apply for a vacant post.

Although we are not able to offer research evidence for the success of the Roskilde model – as I said before its outcomes have never been investigated from a research perspective – 'eating of the pudding' evidence based on experience abounds. Denmark has a national admission system to higher education. Students enrolling at RUC know in advance, from official information distributed to high school students, and from public debates on education, that this university differs from the traditional universities. Most of them have deliberately chosen to study at RUC as their first priority, the rest have it as their second priority. In general, the graduates from the mathematics programme have no difficulty in finding jobs within a multitude of sectors. They are well received, and not infrequently explictly sought after, by employers, and some of them have interesting and impressive careers, in research, education, industry, popularisation institutions such as science centres and museums, and administration. But, naturally, career perspectives also vary considerably with individual characteristics. Our department maintains contact with our graduates after they have left the university. Once a year we have a meeting at which graduates, present students, and staff meet to exchange information and experiences. Graduates report on their work and careers and discuss to what extent their university education prepared them for that sort of employment. It is a consistent feature of those reports that the general abilities fostered by doing project work under conditions that are meant to be fuzzy, confusing, frustrating, demanding, but also inspiring, challenging, rewarding, and gratifying, are essential to them in their jobs. The specific material they studied and dealt with is less important than the competencies they acquired by doing it.

If the RUC model is reasonably successful, as I claim it is, why has it not spread to other universities to a considerable extent? To fully answer this question would take another paper. Let me concentrate on the main points.

Firstly, there is no unique, superior way of structuring and organising studies of mathematics. Every mathematics programme is a high-dimensional vector, and there is no single programme which is 'the best' with respect to every component. Every programme has its assets and drawbacks in comparison to other programmes. So, even if for obvious reasons the Roskilde programme is very dear to my heart, I believe in diversity and would not want all university programmes in mathematics to be organised according to the Roskilde model. At other universities other priorities are given emphasis, and often the price that has to be paid for making project work a substantial component of the programme is perceived to be too high in relation to those priorities. Moreover, students are different and have different learning styles. Some thrive and prosper in a learning environment like that at RUC, in which they are expected to be active, autonomous, and take initiatives in contexts and settings which by definition are not pre-structured. Others need plans, schedules, well-defined requirements, and prefer to be told what to do. So, we need different kinds of universities. On top of that, however, there is no reason to hide the fact that many Danish universities and their faculty have held considerable skepticism, of a somewhat xenophobic nature, towards the Roskilde approach simply because of its novelty and its being at odds with traditional modes of operation. It also cannot be excluded that some of the skepticism has relied on a less than complete knowledge base.

Secondly, as is probably evident from this presentation, the approach taken by RUC is not restricted to mathematics but is a full-scale scheme permeating the entire university. It is not easy to copy one component, the mathematics programme, and insert it into a completely different setting. That being said, after an initial one-and-a-half decade of skepticism towards the Roskilde model, several Danish universities have adopted in their mathematics programmes, in a modified form, content and structural elements similar to the ones at RUC.

6. Concluding remarks

Problem-oriented project work has proved to be a an excellent *pedagogical* tool to foster and develop: students' initiative and independence; responsible activity in mathematics; analytical and reflective approaches and attitudes; ability to pose mathematical questions and to think creatively in, with and about mathematics; ability to communicate on matters mathematical in a non-rudimentary language; collaborative abilities. It follows that such work serves to generate and further, students' *enthusiasm*.

Furthermore, problem-oriented project work has proved to be an excellent *strategic* tool to help students create a multi-faceted and balanced perception of mathematics and its roles and functions in nature, society, and culture. The different types of projects students carry out provide them with first hand experiences of different representative dimensions of and perspectives on mathematics which can hardly be accessed through traditional modes of teaching and study. In other words, project work is not primarily a new motivational gadget to teach students standard material from the text book shelves, but a means to accomplish new ends. Such project work cannot stand alone, however. Only seldom does it seem to be an appropriate means for the acquisition of usual, systematically organised subject matter. For that purpose, classical courses organised and taught by academic teachers remain relevant – which certainly does not imply that the way such courses are taught is of minor importance; on the contrary, but that's not on our agenda in this context.

These remarks show that the mathematics programme at RUC is designed to grasp the essence of mathematics by means of a pair of pinchers, with one jaw consisting of problem-oriented studies, cast in the form of project work, while the other consists of the study of systematically organised subject matter, typically through participation in courses. As was said before, this way of structuring and organising the programme has costs. Measured in terms of the amount of systematically organised subject matter students are exposed to, this type of programme will contain a smaller component of such subject matter than is the case with traditional programmes. However, any mathematics programme can contain only a small selection of the branches and topics of contemporary mathematics. So, a selection has to be made anyway, and what matters at the end of the day is not so much how much food is being served but rather the selection and quality of the food, and how much is digested and how well.

1) The author was a member of the founding staff of Roskilde University and has remained their ever since, today as a professor of mathematics and mathematics education, thus being (co-)responsible for a non-negligible part of the programmes and their underlying ideas.

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25 October 2000

Johnny T. Ottesen

Do not only ask what mathematics can do for modelling. Ask what modelling can do for mathematics !

1. Introduction.

Frequently people discuss which parts of mathematics students should learn to become good 'scientists' in various disciplines. However, this is not the principal question one should pose. A more fundamental question is which competencies related to mathematics are of benefit to these disciplines. More specifically, since mathematics is applied to other disciplines through mathematical modelling, the question should be which competencies related to mathematical modelling are of benefit to these disciplines. This is not an easy question to answer. Indeed, to obtain an answer one has to be familiar with which competencies the teaching of mathematics, and especially of mathematical modelling, may serve to develop, and to possess a deep insight into the various disciplines that may benefit from such competencies.

However, to ask what mathematics can do for modelling in some domain is certainly interesting, but it is not the only interesting question related to the interaction between mathematics and the real world. In this paper, emphasis is on the reverse question, what can modelling do for mathematics and its teaching and learning? The first answer may be that modelling may motivate some students. Of course, this is true, but as we shall argue below, it is not the whole truth, nor the most important truth.

In section 2, the first question "what can mathematics do for modelling?" is discussed rather briefly. The second question "what can modelling do for mathematics?" is discussed in section 3. The discussion of the first question serves to put the second and, in this paper, the main question into perspective. In section 4 a summary is given. Of course the programme outlined here is very ambitious, thus this paper should be seen as an attempt to initiate a debate rather than as an exhaustive analysis in its own right.

Before continuing, it is important to comment on some central concepts used throughout this presentation. The terms model and modelling are used in the sense of *mathematical* models and *mathematical* modelling, which implies that mathematics and mathematical competencies are involved to some degree. Another keyword is 'competency', which is used to indicate a set of intellectual and other mental powers related to abilities and skills, that may well involve subject matter knowledge but involves much more than that. The notion of competencies is used in contrast to a usual mathematical syllabus, i.e. a list of mathematical topics and methods known from most mathematical textbooks. A further specification of this term will appear in section 3 but until then the definition just given will suffice.

2. What can mathematics do for modelling?

As an associate professor in mathematics I became professionally involved in mathematical modelling in physiology several years ago. During this involvement I became more and more interested in examing precisely how mathematics does contribute to furthering knowledge and development in other disciplines such as physiology. In my opinion, by and large the primary goal of university education is to develop the students' intellectual powers to (or above) the level of most academic researchers. Thus, the analysis of what mathematics can do for modelling may be applied directly, essentially without changes, to analyse what the teaching of mathematics can do for other disciplines that apply modelling at the university level. Hence, I will solely address the former issue here. To examine how mathematics contributes to the furthering of knowledge and development in physiology is indeed more important than ever. While theoretical physics has long enjoyed the status of a well-established discipline, mathematical biology, for example, is far from having gained such universal esteem. Despite this relative lack of status, important and meaningful biological experimentation can rarely be performed without theory and concepts that permit hypotheses to be formulated by means of experimentally testable mathematical models. Nonetheless, most biologists still believe that mathematical models are of no use in their disciplines. Non-mathematicians applying mathematics in their field have, in general, different perceptions as to how mathematics interacts with their discipline from mathematicians. More specifically, they do not necessarily have a single perception, but the ones they have surely deviate from those of mathematicians. These perceptions influence their conceptions of how mathematics should be taught, and thus of which competencies students should learn. In my experience, it is rather common to view disciplines other than one's own, as disciplines of service, which should be taught rapidly and by rote as a technique, assuming that it could be obtained by students in some intravenous way.

However, recently some interest has been shown in the development of theory and mathematical modelling by scientists in biology and medicine, especially in physiology. Leading biological journals have actually begun to publish papers in which the identification and study of mechanisms that govern the functioning and the dynamics of biological systems are based on rigorous mathematical modelling, see P. Tracqui, J. Demongeot, P. Auger and M. Thellier (2000). The reason for this growing interest is not just the rapid spread of fast computers which make access to simulations of complex systems easier. What has also contributed to the increasing interest in modelling has been the increasing availability of continuously sampled clinical data. From this, new insight into the dynamics of physiological systems (and not only into their steady state behaviour pattern), can be gained.

Yet another and even more important factor is the attempt to focus on precise definitions of physiological concepts in order to avoid ambiguity, confusion and misunderstandings, see Ottesen and Danielsen (2000). When building a model, a physiologist is often forced to make specific unambiguous statements about relationships that previously might have been only loosely collected thoughts. This challenge sharpens the thinking, often requires a thorough review of the literature, and reveals areas in which quantitative information is missing. Once a mathematical model has been constructed, its predictions can be used to identify and design more meaningful laboratory protocols and to avoid protocols that have little overall value. Furthermore, mathematics may provide a tool for structuring ideas and thoughts, a point that is gaining increasing attention these years.

Frequently mathematical models of physiological hypotheses are preferred to verbal descriptions, and not only in order to clarify thoughts as stated above. For example, physiological homeostasis usually involves many different biological control and feedback processes that occur simultaneously and often at various time scales. The sequential nature of ordinary language is far inferior to a mathematical description when it comes to this simultaneity, and to the course of biological events that are aggregate results of many parallel and simultaneously occurring processes, see Ottesen (2000a). Many of the most important attributes of biological systems, such as non-linearity, redundancy, and hysteresis, cannot be properly described in verbal terms. In contrast, a mathematical description of concepts such as these is often both compact and precise, see Ottesen, (1997a, 1997b).

In addition to its scientific significance, the importance of the interdisciplinary field of mathematical physiology, which is based on mathematical models, has received growing recognition as an applied mathematical activity in modern industry, see Ottesen (1997c, 2000b).

An extended analysis of the reflections just presented, which serve as a basis for the discussion in this section, will appear elsewhere, Ottesen (in preparation). Let us summarise the elementary yet non-trivial findings from this analysis in a condensed form.

- Models may be used to falsify hypotheses. By means of models the range of concepts and underlying assumptions may be tested.
- The use of models may serve to reveal *ad-hoc* elements, assumptions, and vaguely or even ambiguously defined concepts.
- Models may improve the structuring of ideas and thoughts, and increase the understanding of the system being modelled. Moreover, the influence of various parameters on the model system may be investigated along with a corresponding interpretation thereof with respect to the system being modelled.
- The use of models may resolve conter-intuitive observations and paradoxes, and give insight into the causal functioning of the system modelled, with particular regard to cause and effect relationships.
- Models provide the possibility of asking not only new questions but also *new types* of questions that could not be stated without the involvement of models. Furthermore, new useful concepts may be proposed and defined, and ideas and thoughts may be tested.

Caution has to be taken when applying mathematics to the world outside mathematics. Not everything can be put into a mathematical framework in a meaningful manner. Terrible examples appear in many fields, for example in psychiatry, see Rodin (2000). A very common mistake is to use descriptive models involving many parameters that often cannot be estimated in a sensible way, or do not have a well-defined interpretation in the reality domain at issue. These flaws are serious, especially when models are applied in the making of decisions with a great impact on people or society. Some examples are given in Niss, Blum and Huntly (1991). However, the misuse of models is beyond the scope of this presentation and will not be pursued further.

In my opinion, the above points, mainly referring to the research process itself, may essentially be transferred without change to what students should be exposed to during their education. The knowledge of what researchers from other disciplines in principle (should) demand - more or less consciously - from the teaching of mathematics may serve as a background and a perspective for analysing what modelling can do for the teaching of mathematics itself. It follows from this and the next section that the points of views of mathematicians and non-mathematicians applying mathematics, need not be mutually contradictory but may well complement each other in some symbiotic way.

3. What can modelling do for mathematics?

A few years ago I became involved in designing and developing a special first-year course (study unit), called BASE (which stands for Basic Analysis, Modelling and Simulations), for the multidisciplinary two-year basic programme in the sciences at my university. Participants in this course are mainly students whose primary interests are not in mathematics or physics. Two thirds of the science and maths students at our university are supposed to take the course. A faculty member is solely responsible for a single class consisting of a maximum of 25 students. The faculty takes care of the teaching as well as of evaluating the students' homework etc. In class, emphasis is put on discussing the goals of the course as a whole and of each component in particular. Similarly, a lot of emphasis is spent on guiding the students individually regarding which competencies they should seek to improve and how this may be done. Hereby the faculty gets in rather close contact with the students. It should be emphasised that in this presentation the BASE course serves merely as an example, from which more general conclusions will be drawn. Likewise, my judgement of the students' cognitive levels and development is not a result of systematic investigations but is, rather, based on concrete experiences in teaching this course (as well as numerous other courses). Instead of basing the course on a fixed mathematical syllabus, a list of competencies, necessary for someone who is to apply mathematics to another scientific field, such as biology, chemistry or geography, was produced. As mentioned in the introduction, the term 'competency' is used to indicate intellectual and other mental powers related to abilities and skills. Based on this list of competencies, a huge number of smaller real-life problems and about five two-week projects using realistic data were developed for the course. Here the term 'two-week projects' (not to be confused with the larger one-semester projects at our university, described by Mogens Niss elsewhere in this book) denotes the investigation of a partly open but limited problem. Most of the smaller problems deal with the mathematisation part of the modelling process (i.e. the translation of verbal descriptions into mathematical representations), together with an analysis of the resulting mathematical system, and a discussion of both. The two-week projects are on modelling at large. Hereby, the students are supposed to develop motivation, realising that they will benefit from such a course by obtaining an idea of how and why they themselves can and should use mathematics. Lecture notes give the necessary mathematical background, thus making the course material largely self-contained.

The substance of the competencies mentioned above is drawn from an expanded list due to Mogens Niss, among others, see Niss (1999) (in Danish only). In compressed and translated form it reads: mathematical way of thinking; problem posing and solving; modelling; mathematical representation; mathematical formalism and symbols; reasoning; communication; and IT-competencies.

Since I am mainly dealing with modelling competencies here, I shall restrict myself to elaborating on this point. The modelling competency may be subdivided into components. They consist in the ability to:

- analyse the foundation and quality of given models, and estimate the validity and scope of such models;
- structure the field or situation to be modelled;
- mathematise, i.e. translate extra-mathematically given problems, objects and relations into mathematical representations;

- handle models, including solving the mathematical problems arising in models;
- validate models, internally as well as externally;
- reflect on, analyse and exert a critique of models both in their own right and with respect to possible alternative models;
- interpret, i.e. translate from mathematical to extra-mathematical statements, and decode elements of models and results in relation to the field or situation under consideration;
- communicate about models and results based on models;
- control the entire modelling process, i.e. to master all of the above elements actively on the basis of an insight into the interrelationships between the various elements involved.

Of course, not all of the individual elements of the previously mentioned extensive and ambitious list of competencies enjoy the same degree of attention in the BASE course all the time. Besides, it seems naïve to believe that the students can fully achieve those competencies or that these will all be achieved at a uniform level in just one year. Nonetheless, all competencies are frequently discussed in the course. However, because of the emphasis on problems and two-week projects, students have a tendency to focus mainly on those competencies closely related to modelling.

Selected experiences from the competencies-based BASE course that are hoped to be of general interest are described in the examples below. In these examples many if not all of the general competencies mentioned above are involved. However, those through which students experience a deeper understanding of mathematics or of the strength of mathematics attract the greatest attention.

Example 1. Interpreting the concept of derivative. In high school, students learn about the concept of the derivative of a function of a real variable, usually based on the familiar analytic and geometric constructions. For various reasons, some students were not able to understand this concept sufficiently well. So, in the course, we redo this introduction with emphasis on a discussion of the interrelations between the analytic and the geometric approaches. Hereby students become better able to find the derivatives of elementary functions in the usual operative manner. Before the students 'are allowed' to go on and use this concept, various possible ways to read 'verbal meaning' into the concept are discussed. While students often do not have severe difficulties in grasping the individual verbal statements as such, they can be expected to have a hard time understanding the limitations in the *interpretations*, i.e. various translations from a mathematical to an extra-mathematical form. Surprisingly, most students in the BASE course have a very hard time understanding the interpretation of mathematical statements. For example, given the derivative, $y'(t)=kt^2$, of the function y, where k is a constant and t the time, the following interpretations may be relevant;

> The function y *changes* at any given time t with a *speed* proportional to the square of that time.
> At any $t=t_1$, the *difference quotient* of the function y tends to k times t_1 squared.
> The *derivative of the function* y is equal to k times the function $f(t)=t^2$.
> The *slope of the tangent* to the graph of the function y at any point is k times the square of the time.
> *Locally* near a point $(t_1,y(t_1))$ the graph of the function y may be *approximated* with the straight line of slope $y'(t_1)$ passing through the point $(t_1,y(t_1))$.

The given class exercises, reveal the students' lack of making such active interpretations in various cases. Even more troubles arise when the students have to translate *between* the various interpretations. This seems to be very provocative and usually brings the students into a state of 'constructive astonishment.' Those students who do overcome this interpretation obstacle seem to use the statements to read sense into the mathematical concept and its formulation. After a while these students develop an understanding of the mathematical concept itself, apparently without referring to the interpretations anymore. They somehow use the interpretations as a 'tin opener' to the mathematical concept. My conclusion from the course is that students who have not gained the competency of interpreting do not understand the fundamental idea of the concept of derivative, and so they will not be able to translate between this mathematical concept and corresponding aspects of the world not formulated in mathematical terms. Focussing for a while on those competencies which relate to interpretation may help remedy, to some extent, the incompleteness of their understanding.

Example 2. Understanding the concept of derivative. When re-introducing the derivative of a function of a real variable, in the manner outlined in example 1, the aim is to help the students to acquire a deeper understanding of the concept. In that way, students are expected to understand the concept so as to be able to use it within, as well as outside of, mathematics. However, in the BASE course most students, to begin with, learn the concept of derivative by heart. Thus, they are not quite able to employ the concept beyond the types of examples studied in class. By means of algebraic operations the students can find the derivative of a function, such as $f(x)=3x^2 + 7x - 13$, and can relate the results to a geometric point of view. The derivative, $f'(x)=6x+7$, is generally obtained by the students by means of rules and procedures, and totally without further reflection. Severe problems then arise for the students when they are to use the concept in mathematisation and modelling. In one of the problems, students are told that arteries of the human body can be thought of as elastic tubes filled with blood. Furthermore, the volume (V) of a segment of such an artery is often assumed to be proportional to the pressure difference (P) across the wall of that artery, $V=cP$, where c denotes the constant of proportionality. Also they are told that the flow out of the right end of the segment is supposed to be proportional to the pressure inside that segment, assuming that the flow out of the segment goes into a domain of approximately zero pressure. The situation of no flow in or out of the left end of the segment is considered. The students are then guided to find an expression for the time-derivative, $V'(t)$, of V as a function of time. The students usually solve this part of the exercise (albeit not easily for this category of students) as follows: the change of V per time unit is given by the flow into the segment minus the flow out of the segment. Thus

$$V'(t)=-k_1P(t)$$

for some constant k_1, due to the assumption made. Thereafter, it seems to be intellectually very difficult for most students to get the idea of substituting P by $\frac{1}{c}V$ on the right hand side to obtain

$$V'(t)=-kV(t)$$

for some constant k. Next, the students are asked to give an expression that describes the derivative of the pressure, P, in the segment. Indeed it is a point here that no strategy for dealing with this sub-question is shown in advance. Serious problems then begin to occur. The students take a lot of different approaches but until now none of them has been very efficient. It turns out that there are at least three obstacles. Firstly, the students are not able to deduce themselves that $V'(t)=cP'(t)$ from the explicitly given equation, $V=cP$. Secondly, they are not able to combine this expression with the earlier one, $V'(t)=-kV(t)$. The first part is very surprising, since the students do not have any problems in finding the derivative of functions like $f(x)=cx$, $f(x)=cx^2$, or even $f(x)=cg(x)$, where $g(x)$ is another function. Thus this suggests that differentiation is done procedurally by heart and is not based on any understanding of the derivative as a mathematical object. The second part is supposed to simply be due to an insufficient breadth of view and to inexperience. Of course, this may also be a factor concerning the first part of the obstacle. Thirdly, an insufficient insight into the meaning and the idea of an equation is without doubt present as well. This third part of the obstacle is closely related to the competencies of mathematical representations and that of mathematical formalism and symbols. So the students are not able to fully understand symbolic representations or to manipulate them in fairly elementary situations. After having realised this, students are generally willing to work to get rid of these obstacles in a conscious way, a precondition for gaining a deeper understanding. Finally, it should be emphasised that students capable of taking an alternative approach are encouraged to do so if possible. For example, one could imagine that some students were able to solve the differential equation $V'(t)=-kV(t)$ and afterwards use the equation $V=cP$ to obtain a solution for $P(t)$. However, this happens only rarely with this group of students.

Example 3. Enrichment of the concept of steady state solution. In introducing a single ordinary differential equation, compartmental terminology is used. The idea is to keep account of a certain quantity under study over time, say $V(t)$. The time-derivative, $V'(t)$, of the quantity may be interpreted as the change of the function $V(t)$ per time unit. If $I(t)$ denotes the amount of the quantity flowing into the compartment (account) per time unit at time t and $U(t)$ the amount flowing out per time unit at time t, the net flow into the compartment per time unit at time t equals the change of the quantity $V(t)$ at time t,

$$V'(t)=I(t)-U(t)$$

There exists a constant solution, V_0, to this differential equation if and only if $I(t)=U(t)$. This does not usually cause trouble for students' understanding of the model. However, when allowing inflow, I, and outflow, U, to be functions of $V(t)$ troubles do appear. A steady state solution $V(t)=V_0$ of the differential equation exists if and only if $I(V_0)=U(V_0)$. One might say that with respect to a steady state, explicit time having been left out of the problem, the relevant dependency is how I-U behaves as a function of V. Some students are not ready to climb over this obstacle when it is presented to them for the first time. Thus they are able neither to discuss stability of steady state solutions nor to carry the analysis further. The following example may open their minds. Consider the average volume over time (for example during the last 24 hours) of liquid in the human body. Then it seems reasonable to assume that U and I are continuous functions of how much liquid, V, the body contains. After a little thinking it becomes meaningful to state that V does not change in time if and only if $I=U$. If the students do this reasoning while drawing graphs representing I and U the general idea gradually becomes clear. In this way some students verbally interpret, read sense into, or visualise whole mathematical statements or ensembles of concepts. This interpretation differs from the earlier interpretation, discussed in example 1 and 2, due to the higher level of complexity of the subject under study. Hence, these students are able to overcome some of their conceptual obstacles by means of modelling, not primarily because they were motivated but because an applied point of view opened their minds to the meaning and significance of mathematical concepts. Other students who, in the beginning, thought they understood the concepts of steady state solution had to revise their perception based on the very same example. So, the point is not that modelling is a shortcut to understanding mathematics, on the contrary, it is a hard yet fruitful way for many students if not for all to achieve a deeper understanding. The understanding of the concept of steady state solution for a single differential equation is crucial later in the course when the qualitative analysis of systems of differential equations is put on the agenda.

Example 4. Mathematics as a key to knowledge. In the last two-week project, at the end of the course, students have to set up a model describing the situation of anaesthetising a hospitalised patient. Prior to the project the students are given three pages in which the situation is presented. In compressed form it reads:

> "During anaesthesia patients have to be given different drugs. Some drugs, for example Pancuronium, serve to relax the muscles. Since the heart is a muscle as well, the level of this drug has to be below a certain level. On the other hand, the level has to be above a certain minimum to prevent patient twitches. Pancuronium affects the musculature locally. In summary, the concentration of Pancuronium in the muscles is desired to be kept between two values, $c_{\min}$ and $c_{\max}$. Furthermore, the anaesthesiologist often divides the human body into two compartments, the bloodstream and the rest of the body. When a drug is injected directly into the bloodstream by means of a drop, it will be almost uniformly distributed in the bloodstream in less that one minute. The amount of a drug in the bloodstream is easily measured. The other compartment consists of all the various organs, fat, tissue and muscles. It is usually not possible to measure the amount of a drug outside of the bloodstream directly."

The first half of the project is to put up and analyse a model describing an anaesthetised patient. The second half deals with the problem of whether the amount of a drug in the muscles can be controlled solely on the basis of knowledge of the amount of the drug in the bloodstream, and, if so, how can it be done? As we shall see, the second half of the project will be answered by use of mathematics. Furthermore, it can only be answered by use of mathematics. It is even more important to emphasise that the mathematical formulation and analysis related to the first half of the project gives rise to the statement of very important questions that could not be stated without the use of mathematics.

Based on some assumptions, students are able to put up a model relatively easily. It turns out that the concentration of a drug in the bloodstream, c_b, and in the muscles (i.e. the extra-bloodstream compartment), c_m, fulfil a system of differential equations such as

$$c_b'(t) = -(a_1 + a_3)c_b(t) + a_2 c_m(t) + I(t)$$
$$c_m'(t) = a_1 c_b(t) - a_2 c_m(t)$$

Here t denotes the time, a_1 describe the rate of the drug flowing from the bloodstream into the muscles, a_2 the reversed rate, and a_3 the elimination of the drug from the body due to urination etc. The amount of drug injected into the bloodstream per minute is described by the function $I(t)$. In addition to putting up the model, students have to perform a qualitative analysis of the system for the special case when the patient gets a certain amount of the drug to start with, while $I(t)=0$ thereafter. Based on such a mathematical model and the related analysis, it becomes natural to ask the question, is it possible to estimate the rate parameters from measurements and, if so, how could it be done? In fact, this question can only be stated by means of a mathematical formulation, and it is not possible in any meaningful way to state it without access to such a formulation. (By the way, the answer to the question is "yes" and it turns out that it can be done from a single time-series of the concentration of Pancuronium in the blood!) Thus students begin to see that mathematics provides them with a complementary and deeper insight into the subject modelled. Furthermore, they recognise that this insight can be gained only by means of mathematics. Frequently such insight is surprising or even counter-intuitive to students and it may throw them into a state of fascination. Moreover, most students realise that theoretical mathematical concepts and definitions, such as eigenvalues, are meaningful and relevant. Hereby these students also obtain a better overall understanding of the mathematical structure of the problem. Finally, this modelling problem forces the students to relate different mathematical concepts, for example the derivative and a corresponding differential equation, a differential equation and the corresponding compartments diagram, a differential equation and the corresponding eigenvalues, etc. In this way, students develop a stronger understanding of the mathematical concepts involved.

In answering the second half of the problem "can c_m be suitable controlled?" students may be guided to look for the steady state solution $c_m = c_0$ where, for example,

$$c_0 = \frac{c_{\min} + c_{\max}}{2}.$$

Thus from the second differential equation one obtains

$$c_b(t) = \frac{a_2}{a_1} c_0.$$

Insertion into the first differential equation yields

$$I(t) = \frac{a_2 a_3}{a_1} c_0.$$

Once the steady state level is reached, it may be maintained by injecting the constant value found above. This answer is clearly based on the use of mathematics. However, only a few students are able to handle this part of the problem with confidence.

My experience is that through such a two-week project students become familiar with parts of the real world that are influenced by mathematics, learn to express themselves clearly in mathematical terms, and learn to analyse and solve problems involving mathematics. Hence students begin to see how important it is to possess competencies related to manipulating equations and symbolic representations. Without these competencies they are not able to build a model and subsequently are unable to draw relevant conclusions from it. Furthermore, students learn to ask certain types of questions that can only be *answered* by the use of mathematics, as well as types of questions that can only be *posed* by use of mathematics. Thus the students realise some of the strengths and depths of mathematical formalism and ways of thinking. Thereby their resistance to work with and to discuss fundamental mathematical structures and arguments is significantly lowered. Moreover, the students become able to reflect on the approaches taken. All of the competencies involved in working with this two-week project are indeed closely related to the competency of understanding and manipulating with symbolic representations. In fact, a substantial portion of the competencies mentioned earlier does indeed come into play during this project.

4. Summary.

At the beginning of the BASE course , most students tend to believe that mathematics has to be learnt by heart, not by such insight and understanding that generate the mastery of the competencies of mathematics. Often students do not understand the background to, or the basic idea of, the mathematical concepts they are supposed to learn. So they are not able to translate between purely mathematical concepts and parts of the extra-mathematical world not represented in mathematical terms. Focusing on these translation competencies for a while, students' insufficient understanding reveals itself to them, which is a necessary prerequisite for progress. Of course, the students do not become perfect masters of mathematics during this one-year course, but their minds become opened to the thinking and structures of mathematics and to how these may be used within mathematics as well as outside mathematics.

When some students realise that they are experiencing difficulties in modelling, they become able to work on reducing these difficulties in a conscious way, and from here a deeper understanding is often gained. In other words, these students may be able to overcome some of their difficulties through modelling, not only because they are motivated but also because different points of view have opened their minds. In a first approach, these students use the reality being modelled as a 'tin opener' for grasping mathematical concepts. Continuing working with various interpretations and translations seems to result in an understanding of a mathematical concept that goes beyond the original, more superficial one. In roughly the same way, after some time, mathematical statements of higher complexity or ensembles of concepts can be grasped by a model-based interpretation leading to independent mathematical understanding. Furthermore, some students, who initially thought they understood a given concept, have to revise their perception on the basis of modelling examples. The point is not that modelling is a shortcut to understanding mathematics, on the contrary, it is a hard yet fruitful way for students in general to achieve a deeper understanding.

As pointed out earlier, through projects students become familiar with parts of the real world which are or can be influenced by mathematics. A large number of the competencies mentioned earlier come into play during such projects. Students learn to ask certain types of questions which can only be answered by means of mathematics, as well as types of questions which can only be posed by means of mathematics. Also they learn to express themselves clearly in mathematical terms, to analyse and solve problems involving mathematics. Moreover, these competencies are not only essential to students whose main interests do not lie with mathematics. Every student, including those mainly interested in mathematics, should acquire these competencies. Hereby a greater understanding of the role of mathematics in culture and society and of the development of mathematics as a discipline is gained. Hopefully this will also contribute to preventing too narrow-minded mathematicians to graduate from our programme.

In summary, modelling should be taught not only to motivate students, but also because it may help grasping mathematical concepts and statements, as well as serving as an enrichment of the comprehension of mathematical concepts. However, it should be emphasised that this kind of situated learning, on the one side, and learning of well known ordinary structured and conventionally organised mathematics, on the other side, in my opinion should be somehow equally represented in the educational system at all levels. Neither of these approaches should be omitted in the teaching of mathematics. The ordinary, structured and conventionally organised mathematics represents supplementary comprehension, in addition to being suitable for guiding students into various domains and perspectives that they could hardly have achieved otherwise. The two points of views mentioned, that mathematics can do something for modelling and that modelling can do something for (the learning of) mathematics, are not at all mutually contradictory. If both points of view are taken seriously they complement each other in a symbiotic way, thus giving rise to a more complete understanding of mathematics.

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