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The Geometry of Cauchy Data Spaces

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This report is dedicated to the memory of Jean Leray (1906-1998)

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ABSTRACT. First we summarize two different concepts of Cauchy data ('Hardy') spaces of elliptic differential operators of first order on smooth compact manifolds with boundary: the L^2–definition by the range of the pseudo–differential Calderón–Szegö projection and the 'natural' definition by projecting the kernel into the (distributional) quotient of the maximal and the minimal domain. We explain the interrelation between the two definitions. Second we give various applications for the study of topological, differential, and spectral invariants of Dirac operators and families of Dirac operators on partitioned manifolds.

CONTENTS

Introduction

0.1. Topological, geometric, and physics motivation. After Jean Leray's pioneering work of the 40s, topologists have been studying the cutting and pasting of manifolds using Betti numbers, long exact (co)homology sequences, and spectral sequences. Roughly speaking, their goal is to control the replacement of simple parts by complex ones and vice versa when constructing or decomposing a manifold. The corresponding hypersurfaces appear as a kind of 1–codimensional fixed points. The shift between manifold and hypersurface is a tenet of this branch.

Similar ideas have been around in differential geometry for long; e.g. the celebre Gauss–Bonnet Theorem of the curvature integrals of manifolds with boundary and the Morse Theory for the decomposition of manifolds. The shift to codimension 1 is also widely exploited in modern differential geometry, e.g. in Donaldson's work on 4–manifolds as boundaries of 5–manifolds and in the Seiberg–Witten Theory (for a recent survey see [**7**], Part IV, and [**19**]). The same goes for modern quantum field theory of gauge invariant fields which are pure gauge at the boundary of e.g. a 4–ball, or when one sums over all compact (Euclidean) four–geometries and matter field configurations on both sides of a fixed three–surface Σ (see e.g. [**11**] and in particular [**23**]).

Here we shall restrict ourselves to the study of the index, the determinant, and the spectral flow of Dirac operators and families of Dirac operators on partitioned manifolds. These invariants can be coded by the intersection geometry of the *Cauchy data spaces* ('Hardy classes' in complex and Clifford analysis) along the partitioning hypersurface. Various kinds of gluing formulas can be obtained for them.

0.2. Various concepts of Cauchy data spaces. The literature treats the Cauchy data spaces in slightly different ways. One is to establish the Cauchy data spaces as L^2–closures of smooth sections over the partitioning hypersurface Σ, coming from the restriction to the boundary of all smooth solutions ('monogenic functions' in Clifford analysis) over one of the parts M_j of a partitioned manifold $M = M_0 \cup_\Sigma M_1$. As for the Dirac operator, this Cauchy data space can be represented as the range of the L^2–extension of the pseudo–differential Calderón projection ('Szegö projection' in Clifford analysis) and established as a Lagrangian subspace of the symplectic Hilbert space $L^2(-\Sigma) + L^2(\Sigma)$.

The pseudo–differential approach involves a machinery which makes it too heavy and inflexible for an adequate study of continuous families of operators and other topological considerations. It imposes unnecessary limitations and assumptions in spite of the fact that L^2–spaces are more amenable than distributional spaces. However, this approach also has its merits: it leads to operational parametrices and boundary integrals in the theory of elliptic boundary value problems with important applications. In Section 3 we summarize the details of the pseudo–differential (L^2–)approach.

A more elementary approach to the Cauchy data spaces is to establish them as subspaces of the symplectic Hilbert space $\boldsymbol{\beta} := D_{\max}/D_{\min}$ of natural boundary values, i.e. as the boundary values of sections belonging to the maximal domain $D_{\max}$ of the operator. One can embed $\boldsymbol{\beta}$ as a non–closed subspace into the distribution space $H^{-1/2}(-\Sigma) \dot{+} H^{-1/2}(\Sigma)$ and get a surprisingly simple proof of the closedness of the corresponding (distributional) Cauchy data space - without resorting to pseudo–differential operator calculus. In this way it is also easy to obtain the continuity of the Cauchy data spaces for continuous families of operators. In Section 4 we explain how the results of the technically more simple distributional approach can be transferred to the more customary L^2–approach.

0.3. The function of symplectic geometry. The 'complementarity' of the *skew–symmetry* of spin–geometry and Clifford multiplication on the one hand and the *symmetry* of the induced differential operators on the other were a puzzle until Leray in his famous monograph **[26]** *explained* why aspects of symmetric differential operators can be best understood in the framework of symplectic geometry.

In Part 2, we shall elaborate the conjured *Leray's view* and use the Lagrangian property of the Cauchy data spaces to discuss the gluing formulas for spectral invariants in this setting. Section 5 shows how index theory may be considered as expressing the index of a non–symmetric elliptic differential operator over a closed partitioned manifold (which is a 'quantum' variable, defined by the multiplicity of the 0–eigenvalues) by the index of the Fredholm pair of Cauchy data spaces from both sides of the separating hypersurface, measuring their 'non–Lagrangianess' (which is a 'classical' variable, defined by the geometry of the solution spaces). This is the Bojarski Conjecture, proved in **[14]** for chiral Dirac type operators on even–dimensional partitioned manifolds. Various generalizations for global (elliptic) boundary conditions will be discussed.

In Section 6 we give a family version of the Bojarski Conjecture: the Yoshida–Nicolaescu Theorem. It relates the spectral flow of a continuous one–parameter family of (total) Dirac type operators to the Maslov index of the corresponding family of Cauchy data spaces. This is another case of quasi–classical approximation, which is further underlined by the fact that the 'quantum' variable *spectral flow* has a topological meaning only in infinite dimensions, whereas the 'classical' variable *Maslov index* has a non–trivial topological meaning already in finite dimensions.

In Section 7 we explain the function of the Cauchy data spaces in the boundary reduction and in the adiabatic gluing formula for the determinant regularized by the ζ–function. Even more than the index and the spectral flow, the determinant has remained a puzzle and it is still a challenge to understand the different character of the invariants *over the whole manifold* and in terms of the *Cauchy data spaces.*

An example: the determinant of a Dirac operator over a closed manifold or a manifold with boundary is clearly not defined as a true product of the eigenvalues (which go to $\pm\infty$) but requires a kind of regularization. A beautiful result is due to Scott and Wojciechowski, [**34**]. Formally, it is almost identical with a variant of the Bojarski Conjecture for the index and the Yoshida–Nicolaescu Theorem for the spectral formula: the Scott–Wojciechowski Formula (our Theorem 7.1) says that the ζ–regularized determinant of the Dirac operator over a compact smooth manifold with boundary, subject to a Lagrangian global pseudo–differential boundary condition 'of Atiyah–Patodi–Singer type', is equal to a Fredholm determinant defined over the Cauchy data space along the boundary, i.e. a true infinite product of eigenvalues which go rapidly to 1.

Part 1. The Functional Analysis of Cauchy Data Spaces

1. Symplectic Functional Analysis

We fix the following notation.

Let $(\mathcal{H}, \langle .,. \rangle)$ be a separable real Hilbert space with a fixed symplectic form ω, i.e. a skew–symmetric bounded bilinear form on $\mathcal{H} \times \mathcal{H}$ which is non–degenerate. Let $J : \mathcal{H} \to \mathcal{H}$ denote the corresponding almost complex structure defined by

$$\omega(x,y) = \langle Jx, y \rangle \tag{1.1}$$

with $J^2 = -\mathrm{Id}$, ${}^tJ = -J$, and $\langle Jx, Jy \rangle = \langle x, y \rangle$. Here ${}^tJ$ denotes the transpose of J with regard to the (real) inner product $\langle .,. \rangle$. Let $\mathcal{L} = \mathcal{L}(\mathcal{H})$ denote the set of all Lagrangian subspaces of $\mathcal{H}$ (i.e. $\lambda = (J\lambda)^\perp$, or, equivalently, let λ coincide with its annihilator λ^0 with respect to ω). The topology of $\mathcal{L}$ is defined by the operator norm of the orthogonal projections onto the Lagrangian subspaces.

Let $\lambda_0 \in \mathcal{L}$ be fixed. Then any $\mu \in \mathcal{L}$ can be obtained as the image of $\lambda_0^\perp$ under a suitable unitary transformation

$$\mu = U(\lambda_0^\perp)$$

(see also Figure 1a). Here we consider the real symplectic Hilbert space $\mathcal{H}$ as a complex Hilbert space by J. The group $\mathcal{U}(\mathcal{H})$ of unitary operators of $\mathcal{H}$ acts transitively on $\mathcal{L}$, i.e. the mapping

$$\begin{array}{rccc} \rho: & \mathcal{U}(\mathcal{H}) & \longrightarrow & \mathcal{L} \\ & U & \mapsto & U({\lambda_0}^\perp) \end{array} \tag{1.2}$$

is surjective and defines a principal fibre bundle with the group of orthogonal operators $\mathcal{O}(\mathcal{H})$ as structure group.

EXAMPLE 1.1. (*a*) In finite dimensions one considers the space $\mathcal{H} := \mathbf{R}^n \oplus \mathbf{R}^n$ with the symplectic form

$$\omega\big((x,\xi),(y,\eta)\big) := -\langle x,\eta\rangle + \langle \xi, y\rangle \quad \text{for } (x,\xi),\ (y,\eta) \in \mathcal{H}.$$

To emphasize the finiteness of the dimension we write $\mathrm{Lag}(\mathbf{R}^{2n}) := \mathcal{L}(\mathcal{H})$. For linear subspaces of $\mathrm{Lag}(\mathbf{R}^{2n})$ one has

$$l \in \mathcal{L} \iff \dim l = n \text{ and } l \subset l^0\,,$$

i.e. Lagrangian subspaces are true half–spaces which are maximally isotropic ('isotropic' means $l \subset l^0$).

One finds $\mathrm{Lag}(\mathbf{R}^{2n}) \cong \mathrm{O}(n)/\mathrm{U}(n)$ with the fundamental group

$$\pi_1(\mathrm{Lag}(\mathbf{R}^{2n}),\lambda_0) \cong \mathbf{Z}.$$

The mapping is given by the 'Maslov index' of loops of Lagrangian subspaces which can be described as an intersection index with the 'Maslov cycle'. There is a rich literature on the subject, see e.g. the seminal paper [**1**], the systematic review [**17**], or the cohomological presentation [**21**].

(*b*) Let $\{\varphi_k\}_{k\in\mathbf{Z}\setminus\{0\}}$ be a complete orthonormal system for $\mathcal{H}$. We define an almost complex structure, and so by (1.1) a symplectic form, by

$$J\varphi_k := \mathrm{sign}(k)\,\varphi_{-k}\,.$$

Then the spaces $\mathcal{H}_- := \mathrm{span}\{\varphi_k\}_{k<0}$ and $\mathcal{H}_+ := \mathrm{span}\{\varphi_k\}_{k>0}$ are complementary Lagrangian subspaces of $\mathcal{H}$.

In infinite dimensions, the space $\mathcal{L}$ is contractible due to Kuiper's Theorem (see [**7**], Part I) and therefore topologically not interesting. Also, we need some restrictions to avoid infinite–dimensional intersection spaces when counting intersection indices. Therefore we replace $\mathcal{L}$ by a smaller space. This problem can be solved as first suggested in Swanson, [**38**]: by relating symplectic functional analysis with the space $\mathrm{Fred}(\mathcal{H})$ of Fredholm operators, we obtain finite dimensions for suitable intersection spaces and at the same time topologically highly non–trivial objects.

DEFINITION 1.2. (*a*) The space of *Fredholm pairs* of closed infinite–dimensional subspaces of $\mathcal{H}$ is defined by

$$\mathrm{Fred}^2(\mathcal{H}) := \{(\lambda, \mu) \mid \dim \lambda \cap \mu < +\infty \text{ and } \lambda + \mu \subset \mathcal{H} \text{ closed} \\ \text{and } \dim \mathcal{H}/(\lambda + \mu) < +\infty\}.$$

(*b*) The *Fredholm–Lagrangian Grassmannian* of a real symplectic Hilbert space $\mathcal{H}$ at a fixed Lagrangian subspace λ_0 is defined as

$$\mathcal{FL}_{\lambda_0} := \{\mu \in \mathcal{L} \mid (\mu, \lambda_0) \in \mathrm{Fred}^2(\mathcal{H})\}.$$

(*c*) The *Maslov cycle* of λ_0 in $\mathcal{H}$ is defined as

$$\mathcal{M}_{\lambda_0} := \mathcal{FL}_{\lambda_0} \setminus \mathcal{FL}^{(0)}_{\lambda_0} ,$$

where $\mathcal{FL}^{(0)}_{\lambda_0}$ denotes the subset of Lagrangians intersecting λ_0 transversally, i.e. $\mu \cap \lambda_0 = \{0\}$.

Recall the following algebraic and topological characterization of Lagrangian Fredholm pairs (see [**8**] and [**9**], inspired by [**13**], Part 2, Lemma 2.6).

PROPOSITION 1.3. (a) *Let $\lambda, \mu \in \mathcal{L}$ and let π_λ, π_μ denote the orthogonal projections of $\mathcal{H}$ onto λ respectively μ. Then*

$$(\lambda, \mu) \in \mathrm{Fred}^2(\mathcal{H}) \Longleftrightarrow \pi_\lambda + \pi_\mu \in \mathrm{Fred}(\mathcal{H}) \Longleftrightarrow \pi_\lambda - \pi_\mu \in \mathrm{Fred}(\mathcal{H}).$$

(b) *The fundamental group of $\mathcal{FL}_{\lambda_0}$ is $\mathbf{Z}$, and the mapping of the loops in $\mathcal{FL}_{\lambda_0}(\mathcal{H})$ onto $\mathbf{Z}$ is given by the Maslov index*

$$\mathbf{mas} : \pi_1(\mathcal{FL}_{\lambda_0}(\mathcal{H})) \xrightarrow{\cong} \mathbf{Z}.$$

To define the Maslov index, one needs a systematic way of counting, adding and subtracting the dimensions of the intersections $\mu_s \cap \lambda_0$ of the curve $\{\mu_s\}$ with the Maslov cycle $\mathcal{M}_{\lambda_0}$. We refer to [**8**], inspired by [**30**] for a functional analytical definition for continuous curves without additional assumptions.

REMARK 1.4. (*a*) By identifying $\mathcal{H} \cong \lambda_0 \otimes \mathbf{C} \cong \lambda_0 \oplus \sqrt{-1}\,\lambda_0$, we split in [**10**] any $U \in \mathcal{U}(\mathcal{H})$ into a real and imaginary part

$$U = X + \sqrt{-1}\,Y$$

with $X, Y : \lambda_0 \to \lambda_0$. Let $\mathcal{U}(\mathcal{H})^{\mathrm{Fred}}$ denote the subspace of unitary operators which have a Fredholm operator as real part. This is the total space of a principal fibre bundle over the Fredholm Lagrangian

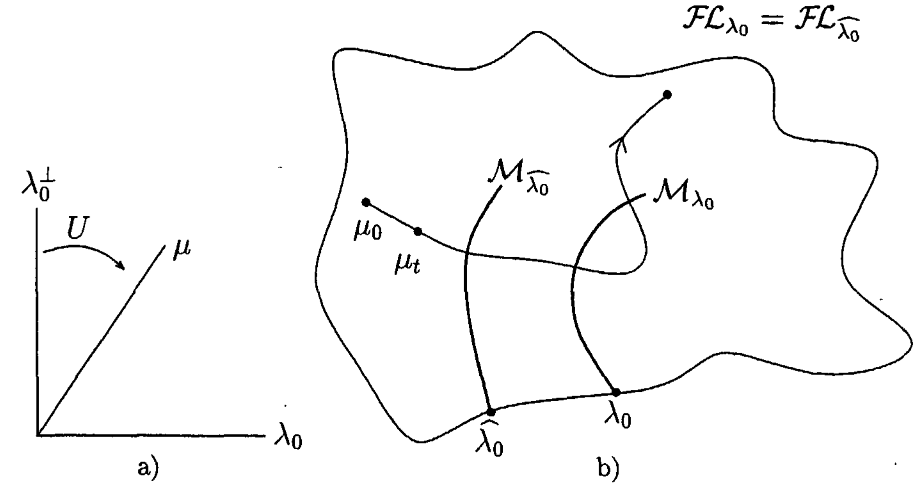

FIGURE 1. a) The generation of $\mathcal{L} = \{\mu = U(\lambda_0^\perp) \mid U \in \mathcal{U}(\mathcal{H})\}$
b) One curve and two Maslov cycles in $\mathcal{FL}_{\lambda_0} = \mathcal{FL}_{\widehat{\lambda}_0}$

Grassmannian $\mathcal{FL}_{\lambda_0}$ as base space and with the orthogonal group $\mathcal{O}(\mathcal{H})$ as structure group. The projection is given by the restriction of the trivial bundle $\rho : \mathcal{U}(\mathcal{H}) \to \mathcal{L}$ of (1.2). This bundle

$$\mathcal{U}(\mathcal{H})^{\text{Fred}} \xrightarrow{\rho} \mathcal{FL}_{\lambda_0}$$

may be considered as the infinite–dimensional generalization of the well–studied bundle $\mathrm{U}(n) \to \mathrm{Lag}(\mathbf{R}^{2n})$ for finite n and provides an alternative proof of the homotopy type of $\mathcal{FL}_{\lambda_0}$.
(*b*) The Maslov index for curves depends on the specified Maslov cycle $\mathcal{M}_{\lambda_0}$. It is worth emphasizing that two *equivalent* Lagrangian subspaces λ_0 and $\widehat{\lambda}_0$ (i.e. $\dim \lambda_0/(\lambda_0 \cap \widehat{\lambda}_0) < +\infty$) always define the same Fredholm Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0} = \mathcal{FL}_{\widehat{\lambda}_0}$ but may define different Maslov cycles $\mathcal{M}_{\lambda_0} \neq \mathcal{M}_{\widehat{\lambda}_0}$. The induced Maslov indices may also become different

$$\mathbf{mas}\,(\{\mu_s\}_{s\in[0,1]}, \lambda_0) \;-\; \mathbf{mas}\,(\{\mu_s\}_{s\in[0,1]}, \widehat{\lambda}_0) \overset{\text{in general}}{\neq} 0 \tag{1.3}$$

(see [**9**], Proposition 3.1 and Section 5). However, if the curve is a loop, then the Maslov index does not depend on the choice of the Maslov cycle. From this property it follows that the difference in (1.3), beyond the dependence on λ_0 and $\widehat{\lambda}_0$, depends only on the initial and end points of the path $\{\mu_s\}$ and may be considered as the infinite–dimensional generalization $\sigma_{\text{Hör}}(\mu_0, \mu_1; \lambda_0, \widehat{\lambda}_0)$ of the Hörmander index. It plays a

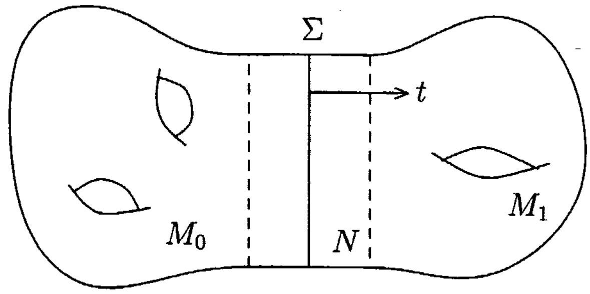

FIGURE 2. The partition of $M = M_0 \cup_\Sigma M_1$

part as the transition function of the universal covering of the Fredholm Lagrangian Grassmannian (see also Figure 1b).

2. The Analysis of Operators of Dirac Type

We fix the notation and recall basic properties of operators of Dirac type.

2.1. The general setting. Let M be a compact smooth Riemannian *partitioned* manifold

$$M = M_0 \cup M_1\ , \quad \text{where } M_0 \cap M_1 = \partial M_0 = \partial M_1 = \Sigma$$

and Σ a hypersurface (see Figure 2). We assume that $M \setminus \Sigma$ does not have a closed connected component (i.e. Σ intersects any connected component of M_0 and M_1). Let

$$\mathrm{A} : C^\infty(M; S) \longrightarrow C^\infty(M; S)$$

be an *operator of Dirac type* acting on sections of a Hermitian bundle S of Clifford modules over M, i.e. $\mathrm{A} = \mathbf{c} \circ \nabla$ where $\mathbf{c}$ denotes the Clifford multiplication and ∇ is a connection for S which is compatible with $\mathbf{c}$ ($\nabla \mathbf{c} = 0$). From the compatibility assumption it follows that A is symmetric and essentially self-adjoint over M.

For even $n = \dim M$, the splitting $C\ell(M) = C\ell^+(M) \oplus C\ell^-(M)$ of the Clifford bundles induces a corresponding splitting of $S = S^+ \oplus S^-$ and a *chiral decomposition*

$$\mathrm{A} = \begin{pmatrix} 0 & \mathrm{A}^- = (\mathrm{A}^+)^* \\ \mathrm{A}^+ & 0 \end{pmatrix}$$

of the *total* Dirac operator. The *chiral Dirac operators* $\mathrm{A}^\pm$ are elliptic but not symmetric, and for that reason they may have non–trivial indices which provide us with important topological and geometric invariants.

Here we assume that all metric structures of M and S are product in a collar neighbourhood $N = [-1,1] \times \Sigma$ of Σ. Then

$$\mathrm{A}\,|_N = \sigma\Big(\frac{\partial}{\partial t} + \mathcal{B}\Big)\,, \tag{2.1}$$

where t denotes the normal coordinate (running from M_0 to M_1) and $\mathcal{B}$ denotes the canonically associated Dirac operator over Σ, called the *tangential operator*. We have a similar product formula for the chiral Dirac operator. Here the point of the product structure is that then σ and $\mathcal{B}$ do not depend on the normal variable. We note that σ is defined by Clifford multiplication by dt. It is a unitary mapping $L^2(\Sigma; S|_\Sigma) \to L^2(\Sigma; S|_\Sigma)$ with $\sigma^2 = -\mathrm{Id}$ and $\sigma\mathcal{B} = -\mathcal{B}\sigma$. In the non-product case, there are certain ambiguities in defining a 'tangential operator' which we shall not discuss here (but see also Formula (2.3)).

2.2. Analysis tools: Green's Formula. For notational economy, we set $X := M_1$. For greater generality, we consider the chiral Dirac operator

$$\mathrm{A}^+ : C^\infty(X; S^+) \longrightarrow C^\infty(X; S^-)$$

and write A^- for its formally adjoint operator. The corresponding results follow at once for the total Dirac operator.

LEMMA 2.1. (Green's Formula). *Let $\langle .,.\rangle_\pm$ denote the scalar product in $L^2(X; S^\pm)$. Then we have*

$$\langle \mathrm{A}^+ f_+, f_-\rangle_- - \langle f_+, \mathrm{A}^- f_-\rangle_+ = -\int_\Sigma (\sigma\gamma_\infty f_+, \gamma_\infty f_-)\, d\mathrm{vol}_\Sigma$$

for any $f_\pm \in C^\infty(X; S^\pm)$.

Here

$$\gamma_\infty : C^\infty(X; S^\pm) \longrightarrow C^\infty(\Sigma; S^\pm|_\Sigma) \tag{2.2}$$

denotes the restriction of a section to the boundary. This is not problematic within the smooth category.

2.3. Analysis tools: the Unique Continuation Property. One of the basic properties of an operator of Dirac type A is the weak Unique Continuation Property (UCP). For $M = M_0 \cup_\Sigma M_1$, it guarantees that there are no *ghost* solutions of $Au = 0$, i.e. there are no solutions which vanish on M_0 and have non-trivial support in the interior of M_1. This property is also called UCP *from open subsets* or *across any hypersurface*. For Euclidean (classical) Dirac operators the property follows from Holmgren's uniqueness theorem for scalar

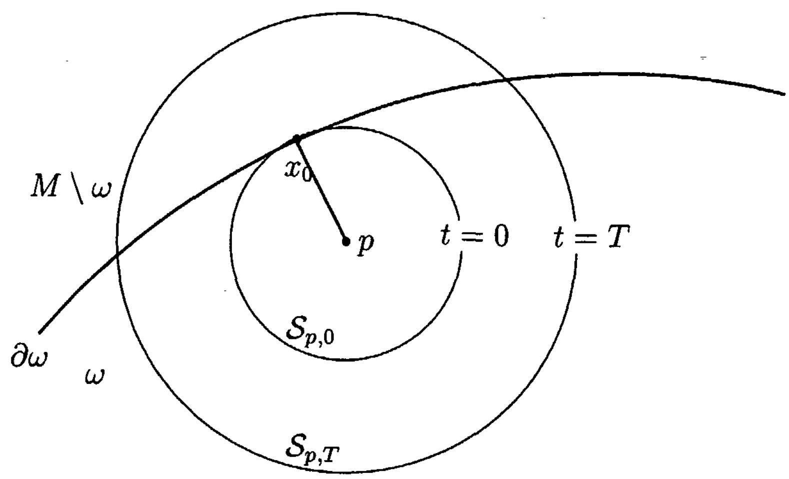

FIGURE 3. Local specification for the Carleman estimate

elliptic operators with real analytic coefficients (see e.g. Taylor [**39**], Proposition 4.3).

In [**14**], Chapter 8, the reader will find a very simple proof of the weak UCP for operators of Dirac type. We refer to [**5**] for a further slight simplification and a broader perspective (see also [**3**]). The proof does not use advanced arguments of the Aronszajn/Cordes type regarding the diagonal and real form of the principal symbol of the Dirac Laplacian, but only the following well–known generalization of the product property of Dirac type operators.

LEMMA 2.2. *Let Σ be a closed hypersurface of M with orientable normal bundle. Let t denote a normal variable with fixed orientation such that a bicollar neighbourhood N of Σ is parametrized by $[-\varepsilon, +\varepsilon] \times \Sigma$. Then any operator of Dirac type can be rewritten in the form*

$$A|_N = \mathbf{c}(dt)\Big(\frac{\partial}{\partial t} + B_t + C_t\Big), \tag{2.3}$$

where B_t is a self–adjoint elliptic operator on the parallel hypersurface Σ_t, and $C_t : S|_{\Sigma_t} \to S|_{\Sigma_t}$ a skew–symmetric operator of 0th order, actually a skew–symmetric bundle homomorphism.

To prove the weak UCP we basically follow the standard lines of the UCP literature. Let $u \in C^\infty(M; S)$ be a solution of $Au = 0$, which vanishes on an open subset ω of M. Then it vanishes on the whole connected component of the manifold. This is to be shown. First we localize and convexify the situation and we introduce spherical coordinates (see Figure 3). Without loss of generality we may assume that ω is maximal, namely the union of all open subsets where u vanishes.

If the solution u does not vanish on the whole connected component containing ω, we consider a point $x_0 \in \operatorname{supp} u \cap \partial\omega$. We choose a point p inside of ω such that the ball around p with radius $r := \operatorname{dist}(x_0, p)$ is contained in $\overline{\omega}$. We call the coordinate running from p to x_0 the *normal* coordinate and denote it by t. The boundary of the ball around p of radius r is a hypersphere and will be denoted by $\mathcal{S}_{p,0}$. It goes through x_0 which has a normal coordinate $t = 0$. Correspondingly, we have larger hyperspheres $\mathcal{S}_{p,t} \subset M$ for $0 \le t \le T$ with $T > 0$ sufficiently small. In such a way we have parametrized an annular region $N_T := \{\mathcal{S}_{p,t}\}_{t\in[0,T]}$ around p of width T and inner radius r, ranging from the hypersphere $\mathcal{S}_{p,0}$ which is contained in $\overline{\omega}$, to the hypersphere $\mathcal{S}_{p,T}$ which cuts deeply into $\operatorname{supp} u$, if $\operatorname{supp} u$ is not empty.

Next, we replace the solution u by a section

$$v(t,y) := \varphi(t)u(t,y) \tag{2.4}$$

with a smooth bump function φ with $\varphi(t) = 1$ for $t \le 0.8\,T$ and $\varphi(t) = 0$ for $t \ge 0.9\,T$. Then $\operatorname{supp} v$ is contained in N_T. More precisely, it is contained in the annular region $N_{0.9\,T}$. Moreover, $\operatorname{supp}(Av)$ is contained in the annular region $0.8\,T \le t \le 0.9\,T$.

The weak UCP follows immediately from the following lemma.

LEMMA 2.3. *Let $A : C^\infty(M;E) \to C^\infty(M;E)$ be a linear elliptic differential operator of order 1 which can be written on N_T in the product form* (2.3). *Let v denote a section made from a solution u as in* (2.4).

(a) *Then for T sufficiently small there exists a constant C such that the* Carleman *inequality*

$$R\int_{t=0}^{T}\int_{\mathcal{S}_{p,t}} e^{R(T-t)^2}\,\|v(t,y)\|^2\,dy\,dt \le C\int_{t=0}^{T}\int_{\mathcal{S}_{p,t}} e^{R(T-t)^2}\,\|Av(t,y)\|^2\,dy\,dt \tag{2.5}$$

holds for any real R sufficiently large.

(b) *If* (2.5) *holds for any sufficiently large $R > 0$, then u is equal* 0 *on $N_{T/2}$.*

3. Cauchy Data Spaces and the Calderón Projection

To explain the L^2–Cauchy data spaces we recall three additional, somewhat delicate and not widely known properties of operators of Dirac type on compact manifolds with boundary from [**14**]:

1. the invertible extension to the double;
2. the Poisson type operator and the Calderón projection; and
3. the twisted orthogonality of the Cauchy data spaces for chiral and total Dirac operators which gives the Lagrangian property in the symmetric case (i.e for the total Dirac operator).

The idea and the properties of the Calderón projection were announced in Calderón [**16**] and proved in Seeley [**35**] in great generality. In the following, we restrict ourselves to constructing the Calderón projection for operators of Dirac type (or, more generally, elliptic differential operators of first order) which simplifies the presentation substantially.

3.1. Invertible extension. First we construct the *invertible double*. Clifford multiplication by the inward normal vector gives a natural clutching of S^+ over one copy of X with S^- over a second copy of X to a smooth bundle $\widetilde{S^+}$ over the closed double $\widetilde{X}$. The product forms of A^+ and $A^- = (A^+)^*$ fit together over the boundary and provide a new operator of Dirac type

$$\widetilde{A^+} := A^+ \cup A^- : C^\infty(\widetilde{X}; \widetilde{S^+}) \longrightarrow C^\infty(\widetilde{X}; \widetilde{S^-}).$$

Clearly $(A^+ \cup A^-)^* = A^- \cup A^+$; hence index $\widetilde{A^+} = 0$. It turns out that $\widetilde{A^+}$ is invertible with a pseudo–differential elliptic inverse $(\widetilde{A^+})^{-1}$. Of course A^+ is not invertible and $r^+(\widetilde{A^+})^{-1}e^+A^+ \neq \mathrm{Id}$, where $e^+ : L^2(X; S^+) \to L^2(\widetilde{X}; \widetilde{S^+})$ denotes the extension–by–zero operator and $r^+ : H^s(\widetilde{X}; \widetilde{S^+}) \to H^s(X; S^+)$ the natural restriction operator for Sobolev spaces, s real.

EXAMPLE 3.1. In the simplest possible two-dimensional case we consider the Cauchy–Riemann operator $\bar\partial : C^\infty(D^2) \to C^\infty(D^2)$ over the disc D^2, where $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$. In polar coordinates out of the origin, this operator has the form $\frac{1}{2}e^{i\varphi}(\partial_r + (i/r)\partial_\varphi)$. Therefore, after some small smooth perturbations (and modulo the factor $\frac{1}{2}$), we assume that $\bar\partial$ has the following form in a certain collar neighbourhood of the boundary:

$$\bar\partial = e^{i\varphi}(\partial_r + i\partial_\varphi)\,.$$

Now we construct the invertible double of $\bar\partial$. By E^k, $k \in \mathbf{Z}$, we denote the bundle, which is obtained from two copies of $D^2 \times \mathbf{C}$ by the identification $(z, w) = (z, z^k w)$ near the equator. We obtain the bundle E^1 by gluing two halves of $D^2 \times \mathbf{C}$ by $\sigma(\varphi) = e^{i\varphi}$ and E^{-1} by gluing with the adjoint symbol. In such a way we obtain the operator

$$\widetilde{\bar\partial} := \bar\partial \cup (\bar\partial)^* : C^\infty(S^2; H^1) \to C^\infty(S^2; H^{-1})$$

over the whole 2–sphere.

Let us analyze the situation more carefully. We fix a bicollar neighbourhood $N := (-\varepsilon, +\varepsilon)\times S^1$ of the equator. The formally adjoint operator to $\bar\partial$ has the form:

$$(\bar\partial)^* = e^{-i\varphi}(-\partial_t + i\partial_\varphi + 1)$$

$(t = r - 1)$ in this cylinder. A section of H^1 is a couple (s_1, s_2) such that in N:

$$s_2(t,\varphi) = e^{i\varphi}s_1(t,\varphi)\,.$$

The couple $(\bar\partial s_1, (\bar\partial)^* s_2)$ is a smooth section of H^{-1}. To show that, we check the equality $(\bar\partial)^* s_2 = e^{-i\varphi}\bar\partial s_1$. We have in the neighbourhood N:

$$\begin{aligned}(\bar\partial)^* s_2 &= (\bar\partial)^*(e^{i\varphi}s_1) = e^{-i\varphi}(\partial_t + i\partial\varphi + 1)(e^{i\varphi}s_1)\\ &= \partial_t s_1 + ie^{-i\varphi}\partial_\varphi(e^{i\varphi}s_1) + s_1 = (\partial_t + i\partial_\varphi)s_1\\ &= e^{-i\varphi}e^{i\varphi}(\partial_t + i\partial_\varphi)s_1 = e^{-i\varphi}(\bar\partial s_1)\,.\end{aligned}$$

Then the operator $\bar\partial \cup \bar\partial^*$ becomes injective and $\operatorname{index}(\bar\partial \cup \bar\partial^*) = 0$.

3.2. The Poisson operator and the Calderón projection. Next we investigate the solution spaces and their traces at the boundary. For a total or chiral operator of Dirac type over a smooth compact manifold with boundary and for any real s we define the *null space*

$$\ker(\mathrm{A}^+, s) := \{f \in H^s(X; S^+) \mid \mathrm{A}^+ f = 0 \text{ in } X \setminus \Sigma\}.$$

The null spaces consist of sections which are distributional for negative s; by elliptic regularity they are smooth in the interior; in particular they possess a smooth restriction on the hypersurface $\Sigma_\varepsilon = \{\varepsilon\}\times\Sigma$ parallel to the boundary Σ of X at a distance $\varepsilon > 0$. By a Riesz operator argument they can be shown to also possess a trace over the boundary. Of course, that trace is no longer smooth but belongs to $H^{s-\frac12}(\Sigma; S^+|_\Sigma)$. More precisely, we have the following well–known *General Restriction Theorem* (for a proof see e.g. [**14**], Chapters 11 and 13):

THEOREM 3.2. (a) *Let $s > \frac12$. Then the restriction map γ_∞ of* (2.2) *extends to a bounded map*

$$\gamma_s : H^s(X; S^+) \longrightarrow H^{s-\frac12}(\Sigma; S^+|_\Sigma). \tag{3.1}$$

(b) *For $s \le \frac12$, the preceding reduction is no longer defined for arbitrary sections but only for solutions of the operator A^+: let $f \in \ker(A^+, s)$ and let $\gamma_{(\varepsilon)}f$ denote the well–defined trace of f in $C^\infty(\Sigma_\varepsilon; S^+|_\Sigma)$. Then the sections $\gamma_{(\varepsilon)}f$ converge to an element $\gamma_s f \in H^{s-\frac12}(\Sigma; S^+|_\Sigma)$ as $\varepsilon \to 0_+$.*

(c) *For any $s \in \mathbf{R}$ the mapping*

$$\mathcal{K} := r_+ \widetilde{\mathrm{A}^+}^{-1} \gamma_\infty^* \sigma : C^\infty(\Sigma; S^+|_\Sigma) \longrightarrow C^\infty(X; S^+)$$

extends to a continuous map $\mathcal{K}^{(s)} : H^{s-1/2}(\Sigma; S^+|_\Sigma) \to H^s(X; S^+)$ with range $\mathcal{K}^{(s)} = \ker(\mathrm{A}^+, s)$.

In the preceding theorem, $\widetilde{\mathrm{A}^+}$ denotes the invertible double of A^+, r_+ denotes the restriction operator $r_+ : H^s(\widetilde{X}; \widetilde{S}^+) \to H^s(X; S^+)$ and γ_∞^* the dual of γ_∞ in the distributional sense.

The composition

$$\mathcal{P}(\mathrm{A}^+) := \gamma_\infty \circ \mathcal{K} : C^\infty(X; S^+) \longrightarrow C^\infty(\Sigma; S^+|_\Sigma) \tag{3.2}$$

is called the *(Szegö–) Calderón projection.* It is a pseudo-differential projection (idempotent, but in general not orthogonal). We denote by $\mathcal{P}(\mathrm{A}^+)^{(s)}$ its extension to the sth Sobolev space over Σ.

We now have three options of defining the corresponding *Cauchy data* (or *Hardy*) *spaces*:

DEFINITION 3.3. For all real s we define

$$\Lambda(\mathrm{A}^+, s) := \gamma_s(\ker(\mathrm{A}^+, s)),$$

$$\Lambda^{\mathrm{clos}}(\mathrm{A}^+, s) := \overline{\gamma_\infty\{f \in C^\infty(X; S^+) \mid \mathrm{A}^+ f = 0 \text{ in } X \setminus \Sigma\}}^{H^{s-\frac{1}{2}}(\Sigma; S^+|\Sigma)},$$

$$\Lambda^{\mathrm{Cald}}(\mathrm{A}^+, s) := \operatorname{range} \mathcal{P}(A^+)^{(s-\frac{1}{2})}.$$

The range of a projection is closed; the inclusions of the Sobolev spaces are dense; and range $\mathcal{P}(A^+) = \gamma_\infty\{f \in C^\infty(X; S^+) \mid \mathrm{A}^+ f = 0 \text{ in } X \setminus \Sigma\}$, as shown in [**14**]. So, the second and the third definition of the Cauchy data space coincide. Moreover, for $s > \frac{1}{2}$ one has $\Lambda(\mathrm{A}^+, s) = \Lambda^{\mathrm{Cald}}(\mathrm{A}^+, s)$. This equality can be extended to the L^2–case ($s = \frac{1}{2}$, see also Theorem 4.3 below), and remains valid for any real s, as proved in Seeley, [**35**], Theorem 6. For $s \le \frac{1}{2}$, the result is somewhat counter–intuitive (see also Example 3.5b in the following Subsection).

We have:

PROPOSITION 3.4. *For all $s \in \mathbf{R}$*

$$\Lambda(\mathrm{A}^+, s) = \Lambda^{\mathrm{clos}}(\mathrm{A}^+, s) = \Lambda^{\mathrm{Cald}}(\mathrm{A}^+, s).$$

3.3. Calderón and Atiyah–Patodi–Singer projection. The Calderón projection is closely related to another projection determined by the 'tangential' part of A: Let $\mathcal{B}$ denote the *tangential* symmetric elliptic differential operator over Σ in the product form

$$(\mathrm{A}, \text{ resp.}) \ \mathrm{A}^+ = \sigma(\partial_t + \mathcal{B}) \text{ in a collar neighbourhood of } \Sigma \text{ in } X.$$

It has discrete real eigenvalues and a complete system of L^2–orthonormal eigensections. Let $\Pi_{\geq}(\mathcal{B})$ denote the spectral (Atiyah–Patodi–Singer) projection onto the subspace $L_+(\mathcal{B})$ of $L^2(\Sigma; S^+|_\Sigma)$ spanned by the eigensections corresponding to the non–negative eigenvalues of $\mathcal{B}$. It is a pseudo–differential operator and its principal symbol p_+ is the projection onto the eigenspaces of the principal symbol $b(y,\zeta)$ of $\mathcal{B}$ corresponding to non–negative eigenvalues. It turns out that it coincides with the principal symbol of the Calderón projection.

We call the space of pseudo–differential projections with the same principal symbol p_+ the *Grassmannian* $\mathcal{G}r_{p_+}$. It has enumerable many connected components; two projections P_1, P_2 belong to the same component, if and only if the *virtual codimension*

$$\mathbf{i}(P_2, P_1) := \operatorname{index}\{P_2P_1 : \operatorname{range} P_1 \to \operatorname{range} P_2\} \tag{3.3}$$

of P_2 in P_1 vanishes; the higher homotopy groups of each connected component are given by Bott periodicity.

Example 3.5. (*a*) For the Cauchy–Riemann operator on the disc $D^2 = \{|z| \leq 1\}$, the Cauchy data space is spanned by the eigenfunctions $e^{ik\theta}$ of the tangential operator ∂_θ over $S^1 = [0,2\pi]/\{0,2\pi\}$ for non–negative k. So, the Calderón projection and the Atiyah–Patodi–Singer projection coincide in this case.

(*b*) Next we consider the cylinder $X^R = [0,R] \times \Sigma_0$ with $A_R y = \sigma(\partial_t + B)$. Here B denotes a symmetric elliptic differential operator of first order acting on sections of a bundle E over Σ_0, and σ a unitary bundle endomorphism with $\sigma^2 = -\mathrm{Id}$ and $\sigma B = -B\sigma$. Let B be invertible (for the ease of presentation). Let $\{\varphi_k, \lambda_k\}$ denote B's spectral resolution of $L^2(\Sigma_0, E)$ with

$$\ldots \lambda_{-k} \leq \ldots \lambda_{-1} < 0 < \lambda_1 \leq \ldots \lambda_k \leq \ldots$$

Then

(3.4)
$$\begin{cases} B\varphi_k = \lambda_k \varphi_k & \text{for all } k \in \mathbf{Z} \setminus \{0\}, \\ \lambda_{-k} = -\lambda_k, \ \sigma(\varphi_k) = \varphi_{-k}, \text{ and } \sigma(\varphi_{-k}) = -\varphi_k & \text{for } k > 0. \end{cases}$$

We consider

$$\begin{aligned} f \in \ker(A_R, 0) &= \operatorname{span}\{e^{-\lambda_k t}\varphi_k\}_{k \in \mathbf{Z}\setminus\{0\}} \text{ in } L^2(X^R) \\ &= \ker A_{R\max} \text{ (kernel of maximal extension).} \end{aligned}$$

It can be written in the form

$$f(t,y) = f_>(t,y) + f_<(t,y), \tag{3.5}$$

where

$$f_<(t,y) = \sum_{k<0} a_k e^{-\lambda_k t}\varphi_k(y) \text{ and } f_>(t,y) = \sum_{k>0} a_k e^{-\lambda_k t}\varphi_k(y).$$

Because of

$$\langle f, f\rangle_{L^2(X^R)} < +\infty \iff \langle f_<, f_<\rangle < +\infty \text{ and } \langle f_>, f_>\rangle < +\infty\,,$$

the coefficients a_k satisfy the conditions

(3.6)

$$\sum_{k<0} |a_k|^2 \frac{e^{-2\lambda_k R} - 1}{2|\lambda_k|} < +\infty \quad \text{or, equivalently,} \quad \sum_{k<0} |a_k|^2 \frac{e^{2|\lambda_k|R}}{|\lambda_k|} < +\infty$$

and

(3.7)

$$\sum_{k>0} |a_k|^2 \frac{1 - e^{-2\lambda_k R}}{2\lambda_k} < +\infty \quad \text{or, equivalently,} \quad \sum_{k>0} |a_k|^2/\lambda_k < +\infty\,.$$

We consider the Cauchy data space $\Lambda(A_R, 0)$ consisting of all $\gamma(f)$ with $f \in \ker(A_R, 0)$. Here $\gamma(f)$ denotes the trace of f at the boundary

$$\Sigma = \partial X^R = -\Sigma_0 \sqcup \Sigma_R\,,$$

where Σ_R denotes a second copy of Σ_0. According to the spectral splitting $f = f_> + f_<$, we have

$$\gamma(f) = (s^0_<, s^R_<) + (s^0_>, s^R_>)$$

where

$$s^0_> = f_>(0),\ s^0_< = f_<(0),\ s^R_> = f_>(R),\ s^R_< = f_<(R).$$

Because of (3.6), we have

$$(s^0_<, s^R_>) \in C^\infty(\Sigma_0 \cup \Sigma_R).$$

Because of (3.7), we have

$$(s^0_>, s^R_<) \in H^{-1/2}(\Sigma_0 \cup \Sigma_R).$$

Recall that

$$\sum a_k\varphi_k \in H^s(\Sigma) \iff \sum |a_k|^2 k^{2s/(m-1)} < +\infty$$

and $|\lambda_k| \sim |k|^{m-1}$ for $k \to \pm\infty$, where $m-1$ denotes the dimension of Σ_0.

One notices that the estimate (3.6) for the coefficients of $s^0_<$ is stronger than the assertion that $\sum_{k<0}|a_k|^2|\lambda_k|^N < +\infty$ for all natural N. Thus our estimates confirm that not each smooth section can appear as initial value over Σ_0 of a solution of $A_R f = 0$ over the cylinder.

To sum up the example, the space $\Lambda(A_R, 0)$ can be written as the graph of an unbounded, densely defined, closed operator $T : \operatorname{dom} T \to H^{-\frac{1}{2}}(\Sigma_R)$, mapping $s^0_< + s^0_> =: s^0 \mapsto s^R := s^R_< + s^R_>$ with $\operatorname{dom} T \subset H^{-\frac{1}{2}}(\Sigma_0)$. To obtain a closed subspace of $L^2(\Sigma)$ one takes the range $\Lambda(A_R, \frac{1}{2})$ of the L^2–extension $\mathcal{P}(A_R)^{(0)}$ of the Calderón projection. It coincides with $\Lambda(A_R, 0) \cap L^2(\Sigma)$ by Proposition 3.4. In Theorem 4.3 we show without use of the pseudo–differential calculus why the intersection $\Lambda(A_R, 0) \cap L^2(\Sigma)$ must become closed in $L^2(\Sigma)$. See also Theorem 7.1 for another description of the Cauchy data space $\Lambda(A_R, \frac{1}{2})$, namely as the graph of a unitary elliptic pseudo–differential operator of order 0.

Since $\Sigma = -\Sigma_0 \sqcup \Sigma_R$, the tangential operator takes the form $\mathcal{B} = B \oplus (-B)$ and we obtain from (3.4)

$$\operatorname{range} \Pi_>(\mathcal{B})^{(0)} = L_+(\mathcal{B}) = \operatorname{span}_{L^2(\Sigma)}\{(\varphi_k, \sigma(\varphi_k))\}_{k>0}\,.$$

For comparison, we have in this example

$$\operatorname{range} \mathcal{P}(A_R)^{(0)} = \Lambda(A_R, \tfrac{1}{2}) = \operatorname{span}_{L^2(\Sigma)}\{(\varphi_k, e^{-\lambda_k R}\varphi_k)\}_{k>0}\,,$$

hence $L_+(\mathcal{B})$ and $\Lambda(A_R, \frac{1}{2})$ are transversal subspaces of $L^2(\Sigma)$. On the half–infinite cylinder $[0, \infty) \times \Sigma$, however, we have only one boundary component Σ_0. Hence

$$\operatorname{range} \Pi_>(B)^{(0)} = \operatorname{span}_{L^2(\Sigma_0}\{\varphi_k\}_{k>0} = \lim_{R\to\infty} \operatorname{range} \mathcal{P}(A_R)^{(0)}.$$

One can generalize the preceding example: For any smooth compact manifold X with boundary Σ and any real $R \geq 0$, let X^R denote the stretched manifold

$$X^R \;:=\; \big([-R, 0] \times \Sigma\big) \cup_\Sigma X.$$

Assuming product structures with $A = \sigma(\partial_t + \mathcal{B})$ near Σ gives a well–defined extension A_R of A. Nicolaescu, **[28]** proved that the Calderón projection and the Atiyah–Patodi–Singer projection coincide up to a finite–dimensional component in the *adiabatic limit* ($R \to +\infty$ in a suitable setting). Even for finite R and, in particular for $R = 0$, one has the following interesting result. It was first proved in Scott, **[32]** (see also Grubb, **[22]** and Wojciechowski, **[20]**, Appendix who both offered different proofs).

Lemma 3.6. *For all $R \geq 0$, the difference $\mathcal{P}(A_R) - \Pi_\geq(\mathcal{B})$ is an operator with a smooth kernel.*

3.4. Twisted Orthogonality of Cauchy Data Spaces. Green's formula (in particular the Clifford multiplication σ in the case of Dirac type operators) provides a symplectic structure for $L^2(\Sigma; S|_\Sigma)$ for linear symmetric elliptic differential operators of first order on a compact smooth manifold X with boundary Σ. For elliptic systems of second–order differential equations, various interesting results have been obtained in the 70s by exploiting the symplectic structure of corresponding spaces (see e.g. [**25**]). Restricting oneself to first–order systems, the geometry becomes very clear and it turns out that the Cauchy data space $\Lambda(A, \frac{1}{2})$ is a Lagrangian subspace of $L^2(\Sigma; S|_\Sigma)$.

More generally, in [**14**] we described the orthogonal complement of the Cauchy data space of the chiral Dirac operator A^+ by

$$\sigma^{-1}(\Lambda(A^-, \tfrac{1}{2})) = (\Lambda(A^+, \tfrac{1}{2}))^\perp . \tag{3.8}$$

We obtained a short exact sequence

$$0 \to \sigma^{-1}(\Lambda(A^-, s)) \hookrightarrow H^{s-\frac{1}{2}}(\Sigma; S^+|_\Sigma) \xrightarrow{\mathcal{K}^{(s)}} \ker(A^+, s) \to 0.$$

For the total (symmetric) Dirac operator this means:

Proposition 3.7. *The Cauchy data space $\Lambda(\mathrm{A}, \frac{1}{2})$ of the* total *Dirac operator is a Lagrangian subspace of the Hilbert space $L^2(\Sigma; S|_\Sigma)$ equipped with the symplectic form $\omega(\varphi, \psi) := (\sigma\varphi, \psi)$.*

The preceding result has immediate applications in index theory, see Section 5 below.

4. Cauchy Data Spaces and Maximal and Minimal Domains

We give a systematic presentation of the boundary reduction of the solution spaces, inspired by M. Krein's construction of the maximal space of boundary values for closed symmetric operators (see [**8**], [**9**]). In this section we stay in the real category and do not assume product structure near $\Sigma = \partial X$ unless otherwise stated. The operator A needs not be of Dirac type. We only assume that it is a linear elliptic symmetric differential operator of first order.

4.1. The natural Cauchy data space. Let A_0 denote the restriction of A to the space $C_0^\infty(X; S)$ of smooth sections with support in the interior of X. As mentioned above, there is no natural choice of the order of the Sobolev spaces for the boundary reduction. Therefore, a systematic treatment of the boundary reduction may begin with the minimal closed extension $\mathrm{A}_{\min} := \overline{\mathrm{A}_0}$ and the adjoint $\mathrm{A}_{\max} := (\mathrm{A}_0)^*$ of A_0. Clearly, $\mathrm{A}_{\max}$ is the maximal closed extension. This gives

$$D_{\min} := \operatorname{dom}(\mathrm{A}_{\min}) = \overline{C_0^\infty(X;S)}^{\mathcal{G}} = \overline{C_0^\infty(X;S)}^{H^1(X;S)}$$

and

$$\begin{aligned} D_{\max} &:= \operatorname{dom}(\mathrm{A}_{\max}) \\ &= \{u \in L^2(X;S) \mid \mathrm{A}\,u \in L^2(X;S) \text{ in the sense of distributions}\}. \end{aligned}$$

Here, the superscript $\mathcal{G}$ means the closure in the graph norm which coincides with the 1st Sobolev norm on $C_0^\infty(X;S)$. We form the space $\boldsymbol{\beta}$ of *natural boundary values* with the *natural trace map* γ in the following way:

$$\begin{array}{ccc} D_{\max} & \xrightarrow{\gamma} & D_{\max}/D_{\min} =: \boldsymbol{\beta} \\ x & \mapsto & \gamma(x) = [x] := x + D_{\min} \,. \end{array}$$

The space $\boldsymbol{\beta}$ becomes a symplectic Hilbert space with the scalar product induced by the graph norm

$$(x,y)_{\mathcal{G}} := (x,y) + (\mathrm{A}\,x, \mathrm{A}\,y) \tag{4.1}$$

and the symplectic form given by Green's form

$$\omega([x],[y]) := (\mathrm{A}\,x, y) - (x, \mathrm{A}\,y) \qquad \text{for } [x],[y] \in \boldsymbol{\beta}. \tag{4.2}$$

It is easy to show that ω is non–degenerate.

We define the *natural Cauchy data space* $\Lambda(\mathrm{A}) := \gamma(\ker \mathrm{A}_{\max})$. Let us assume that A has a self-adjoint L^2–extension with a compact resolvent. Then it is a Fredholm operator. Let us choose such an extension and denote its domain by D. Such an extension always exists. Take for instance $A_{\mathcal{P}(A)}$, the operator A with domain

$$\operatorname{dom} A_{\mathcal{P}(A)} := \{f \in H^1(X;S) \mid \mathcal{P}(\mathrm{A})^{(\frac{1}{2})}(f|_\Sigma) = 0\},$$

where $\mathcal{P}(\mathrm{A})$ denotes the Calderón projection defined in (3.2).

PROPOSITION 4.1. (a) *The Cauchy data space $\Lambda(\mathrm{A})$ is a closed Lagrangian subspace of $\boldsymbol{\beta}$ and belongs to the Fredholm–Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0}$ at $\lambda_0 := \gamma(D)$.*
(b) *For arbitrary domains D with $D_{\min} \subset D \subset D_{\max}$ and $\gamma(D)$ Lagrangian, the extension $\mathrm{A}_D := \mathrm{A}_{\max}|_D$ is self-adjoint. It becomes a Fredholm operator, if and only if the pair $(\gamma(D), \Lambda(\mathrm{A}))$ of Lagrangian subspaces of $\boldsymbol{\beta}$ becomes a Fredholm pair.*
(c) *Let $\{C_t\}_{t\in I}$ be a continuous family (with respect to the operator norm) of bounded self-adjoint operators. Here the parameter t runs*

within the interval $I = [0,1]$. We assume the non-existence of inner solutions ('weak UCP') for all operators $A^* + C_t$, i.e.

$$D_{\min} \cap \ker(A^* + C_t, 0) = \{0\} \quad \textit{for all } t \in [0,1]. \tag{4.3}$$

Then the spaces $\gamma(\ker(A^* + C_t, 0))$ of Cauchy data vary continuously in $\boldsymbol{\beta}$.

Here, for $t \in I$ the spaces $\boldsymbol{\beta}_t$ are all naturally identified with $\boldsymbol{\beta}$.

REMARK 4.2. (a) It is an astonishing aspect of symplectic functional analysis that the proof of the preceding proposition is completely elementary (see [**8**], Proposition 3.5 and Theorem 3.8).
(b) Clearly, $D_{\max}$ and $D_{\min}$ are $C^\infty(X)$–modules, and so the space $\boldsymbol{\beta}$ is a $C^\infty(\Sigma)$–module. This shows that $\boldsymbol{\beta}$ is *local* in the following sense: If Σ decomposes into r connected components $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$, then $\boldsymbol{\beta}$ decomposes into

$$\boldsymbol{\beta} = \bigoplus_{j=1}^{r} \boldsymbol{\beta}_j$$

where

$$\boldsymbol{\beta}_j := \gamma\Big(\{f \in D_{\max} \mid \operatorname{supp} f \subset N_j\}\Big)$$

with a suitable collar neighbourhood N_j of Σ_j . Note that each $\boldsymbol{\beta}_j$ is a closed symplectic subspace of $\boldsymbol{\beta}$ and therefore a symplectic Hilbert space.
(c) For elliptic differential operators of first order, the weak unique continuation property discussed in Subsection 2.3 implies unique continuation from hypersurfaces and so the property (4.3): if a section f belongs to $D_{\min}$, it vanishes at the whole boundary. So it can be extended to the closed double by 0. By elliptic regularity this extension is smooth, so $f|_X = 0$ by weak UCP.

Now consider a section f which vanishes on one connected component of the boundary. The arguments of Subsection 2.3 extend to this case and we obtain once again $f|_X = 0$ by weak UCP. For operators of Dirac type, this follows also from the early results by Aronszajn and Cordes (for a recent review and generalization see [**3**]). We combine this result with the preceding Remark b. Let us assume that the operator A satisfies the weak unique continuation property (from each connected component of the boundary as explained). For simplicity we assume that the boundary consists of three connected components. Then the natural Cauchy data space $\Lambda(A)$ intersects transversally each of the 'faces'

$$
\begin{aligned}
(4.4)\quad \Lambda(A) \cap \Big(\{0\} \times \boldsymbol{\beta}_2 \times \boldsymbol{\beta}_3\Big) &= \Lambda(A) \cap \Big(\boldsymbol{\beta}_1 \times \{0\} \times \boldsymbol{\beta}_3\Big) \\
&= \Lambda(A) \cap \Big(\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \times \{0\}\Big) = \{0\}.
\end{aligned}
$$

In finite dimensions, this would contradict the Lagrangian property of $\Lambda(A)$ in the full symplectic Hilbert space $\boldsymbol{\beta} = \boldsymbol{\beta}_1 \dotplus \boldsymbol{\beta}_2 \dotplus \boldsymbol{\beta}_3$ for dimension considerations: because of the transversality (4.4), any such space $l := \Lambda(A)$ can be written as the graph of an injective linear mapping $C : \boldsymbol{\beta}_1 \to \boldsymbol{\beta}_2 \times \boldsymbol{\beta}_3$, so $l = \operatorname{graph} C$ and $\dim l = \dim \boldsymbol{\beta}_1$. Without loss of generality we assume that $\dim \boldsymbol{\beta}_1 \leq \dim \boldsymbol{\beta}_j$ for $j = 2, 3$. That implies $2 \dim l \lneq \dim \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \times \boldsymbol{\beta}_3$. So l is not Lagrangian.

There is nothing disquieting in this remark because $\boldsymbol{\beta}$ becomes only finite-dimensional when X is an interval where the number of connected components of the boundary is limited to two.

By Theorem 3.2a and, alternatively and in greater generality, by Hörmander **[24]** (Theorem 2.2.1 and the Estimate (2.2.8), p. 194), the space $\boldsymbol{\beta}$ is naturally embedded in the distribution space $H^{-\frac{1}{2}}(\Sigma; S|_\Sigma)$. Under this embedding we have $\Lambda(\mathrm{A}) = \Lambda(\mathrm{A}, 0)$, where the last space was defined in Definition 3.3.

If the metrics are product close to Σ, we can give a more precise description of the embedding of $\boldsymbol{\beta}$, namely as a *graded* space of distributions. Let $\{\varphi_k, \lambda_k\}$ be a spectral resolution of $L^2(\Sigma)$ by eigensections of $\mathcal{B}$. (Here and in the following we do not mention the bundle S). Once again, for simplicity, we assume $\ker \mathcal{B} = \{0\}$ and have $\mathcal{B}\varphi_k = \lambda_k \varphi_k$ for all $k \in \mathbf{Z} \setminus \{0\}$, and $\lambda_{-k} = -\lambda_k$, $\sigma(\varphi_k) = \varphi_{-k}$, and $\sigma(\varphi_{-k}) = -\varphi_k$ for $k > 0$. In **[9]**, Proposition 7.15 (see also **[15]** for a more general setting) it was shown that

$$
\begin{gathered}
\boldsymbol{\beta} = \boldsymbol{\beta}_- \oplus \boldsymbol{\beta}_+ \quad \text{with} \\
(4.5)\qquad \boldsymbol{\beta}_- := \overline{[\{\varphi_k\}_{k<0}]}^{H^{\frac{1}{2}}(\Sigma)} \text{ and } \boldsymbol{\beta}_+ := \overline{[\{\varphi_k\}_{k>0}]}^{H^{-\frac{1}{2}}(\Sigma)}.
\end{gathered}
$$

Then $\boldsymbol{\beta}_-$ and $\boldsymbol{\beta}_+$ are Lagrangian and transversal subspaces of $\boldsymbol{\beta}$.

4.2. Criss–cross reduction. Let us define two Lagrangian and transversal subspaces $L_\pm$ of $L^2(\Sigma)$ in a similar way, namely by the closure in $L^2(\Sigma)$ of the linear span of the eigensections with negative, resp. with positive eigenvalue. Then L_+ is dense in $\boldsymbol{\beta}_+$, and $\boldsymbol{\beta}_-$ is dense

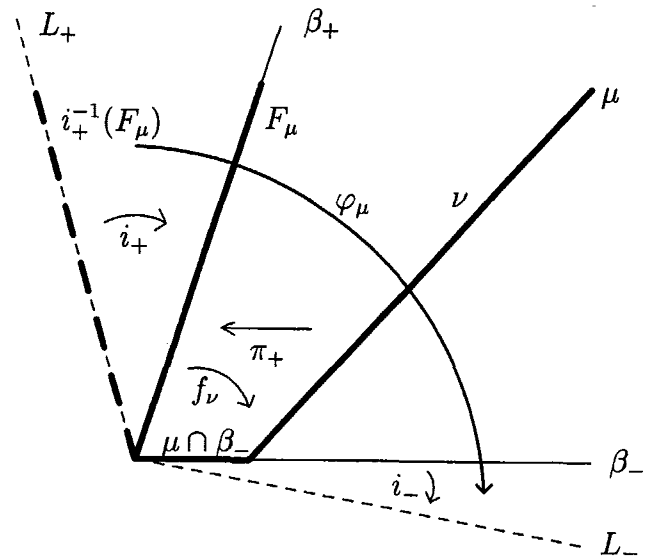

FIGURE 4. The mapping $\tau : \mathcal{FL}_{\beta_-}(\boldsymbol{\beta}) \to \mathcal{FL}_{L_-}(L)$

in L_-. This anti–symmetric relation may explain some of the well–observed delicacies of dealing with spectral invariants of continuous families of Dirac operators.

Moreover, $\gamma(D_{\text{aps}}) = \boldsymbol{\beta}_-$, where

$$D_{\text{aps}} := \{f \in H^1(X) \mid \Pi_>(f|_\Sigma) = 0\} \tag{4.6}$$

denotes the domain corresponding to the Atiyah–Patodi–Singer boundary condition. Note that a series $\sum_{k<0} c_k\varphi_k$ may converge to an element $\varphi \in L^2(\Sigma)$ without converging in $H^{\frac{1}{2}}(\Sigma)$. Therefore such $\varphi \in L_-$ can not appear as trace at the boundary of any $f \in D_{\max}$.

Recall Proposition 4.1 and note that $(\boldsymbol{\beta}_-, \Lambda(\mathrm{A}))$ is a Fredholm pair.

This can all be achieved without the symbolic calculus of pseudo–differential operators. Therefore one may ask how the preceding approach to Cauchy data spaces and boundary value problems via the maximal domain and our symplectic space $\boldsymbol{\beta}$ is related to the approach via the Calderón projection, which we reviewed in the preceding section. How can results from the distributional theory be translated into L^2–results?

To relate the two approaches we recall a fairly general symplectic 'Criss–Cross' Reduction Theorem from [**10**] (Theorem 1.2). Let $\boldsymbol{\beta}$ and L be symplectic Hilbert spaces with symplectic forms $\omega_{\boldsymbol{\beta}}$ and ω_L, respectively. Let

$$\boldsymbol{\beta} = \boldsymbol{\beta}_- \dotplus \boldsymbol{\beta}_+ \qquad \text{and} \qquad L = L_- \dotplus L_+$$

be direct sum decompositions by transversal (not necessarily orthogonal) pairs of Lagrangian subspaces. We assume that there exist continuous, injective mappings

$$i_- : \boldsymbol{\beta}_- \longrightarrow L_- \qquad \text{and} \qquad i_+ : L_+ \longrightarrow \boldsymbol{\beta}_+$$

with dense images and which are compatible with the symplectic structures, i.e.

$$\omega_L(i_-(x), a) = \omega_\beta(x, i_+(a)) \quad \text{for all } a \in L_+ \text{ and } x \in \boldsymbol{\beta}_- .$$

Let $\mu \in \mathcal{FL}_{\beta_-}(\boldsymbol{\beta})$, e.g. $\mu = (\mu \cap \boldsymbol{\beta}_-) \dot{+} \nu$ with a suitable closed ν. Let us define (see also Figure 4)

$$\tau(\mu) := i_-(\mu \cap \boldsymbol{\beta}_-) + \operatorname{graph}(\varphi_\mu) , \tag{4.7}$$

where

$$\begin{array}{rccc} \varphi_\mu : & i_+^{-1}(F_\mu) & \longrightarrow & L_- \\ & x & \mapsto & i_- \circ f_\nu \circ i_+(x) \end{array} .$$

Here F_μ denotes the image of μ under the projection π_+ from μ to $\boldsymbol{\beta}_+$ along $\boldsymbol{\beta}_-$ and $f_\nu : F_\mu \longrightarrow \boldsymbol{\beta}_-$ denotes the uniquely determined bounded operator which yields ν as its graph. Then:

THEOREM 4.3. *The mapping* (4.7) *defines a continuous mapping*

$$\tau : \mathcal{FL}_{\beta_-}(\boldsymbol{\beta}) \longrightarrow \mathcal{FL}_{L_-}(L)$$

which maps the Maslov cycle $\mathcal{M}_{\beta_-}(\boldsymbol{\beta})$ of $\boldsymbol{\beta}_-$ into the Maslov cycle $\mathcal{M}_{L_-}(L)$ of L_- and preserves the Maslov index

$$\mathbf{mas}\,(\{\mu_s\}_{s\in[0,1]}, \boldsymbol{\beta}_-) = \mathbf{mas}\,(\{\tau(\mu_s)\}_{s\in[0,1]}, L_-)$$

for any continuous curve $[0,1] \ni s \mapsto \mu_s \in \mathcal{FL}_{\beta_-}(\boldsymbol{\beta})$.

In the product case, the 'Criss–Cross' Reduction Theorem implies for our two types of Cauchy data that all results proved in the theory of natural boundary values ($\boldsymbol{\beta}$–theory) remain valid in the L^2–theory. In particular we have:

COROLLARY 4.4. *The $L^2(\Sigma)$ part $\Lambda(\mathrm{A}) \cap L^2(\Sigma)$ of the natural Cauchy data space $\Lambda(\mathrm{A})$ is closed in $L^2(\Sigma)$. Actually, it is a Lagrangian subspace of $L^2(\Sigma)$ and it forms a Fredholm pair with the component L_- , defined at the beginning of this subsection.*

Part 2. Cauchy Data Spaces and Spectral Invariants

5. Non–Lagrangian Half–Spaces and Index Theory

5.1. Index theory. Recall that the index of a Fredholm operator measures its non–symmetry: it is defined by the difference between the dimension of the kernel (the null space) of the operator and the dimension of the kernel of the adjoint operator (= the codimension of the range). So, the index vanishes for self–adjoint Fredholm operators. For an elliptic differential operator A on a closed manifold M the index is finite and depends only on the homotopy class of the principal symbol σ of the operator over the cotangent sphere bundle S^*M. Therefore, the index always vanishes on odd–dimensional manifolds. On even–dimensional manifolds one has the Atiyah–Singer Index Theorem which expresses the index in explicit topological terms, involving the Todd class defined by the Riemannian structure of M and the Chern class defined by gluing two copies of a bundle over S^*M by σ. It has turned out that various topological invariants of manifolds can be expressed by the index of naturally defined operators of Dirac type. In the index theory of closed manifolds one mostly studies the chiral and not the total Dirac operator (which is symmetric for compatible connections).

5.2. The Bojarski conjecture. The *Bojarski Conjecture* gives quite a different description of the index of an elliptic operator over a closed partitioned manifold $M = M_0 \cup_\Sigma M_1$. It relates the 'quantum' quantity index with a 'classical' quantity, the Fredholm intersection index of the Cauchy data spaces from both sides along the hypersurface Σ. It was suggested in [4] and proved in [14] for operators of Dirac type.

PROPOSITION 5.1. *Let M be a partitioned manifold as before and let $\Lambda(\mathrm{A}_j, \frac12)$ denote the L^2-Cauchy data spaces, $j = 0, 1$. Then*

$$\operatorname{index}\ \mathrm{A} = \operatorname{index}(\Lambda(\mathrm{A}_0, \tfrac12), \Lambda(\mathrm{A}_1, \tfrac12)).$$

Recall that

$$\begin{aligned}\operatorname{index}(\Lambda(\mathrm{A}_0, \tfrac12), \Lambda(\mathrm{A}_1, \tfrac12)) := {}& \dim(\Lambda(\mathrm{A}_0, \tfrac12) \cap \Lambda(\mathrm{A}_1, \tfrac12)) \\ & - \dim(L^2(\Sigma; S|_\Sigma)/(\Lambda(\mathrm{A}_0, \tfrac12) + \Lambda(\mathrm{A}_1, \tfrac12)).\end{aligned}$$

It is equal to $\mathbf{i}(\mathrm{Id} - \mathcal{P}(\mathrm{A}_1), \mathcal{P}(\mathrm{A}_0))$ where $\mathcal{P}(\mathrm{A}_j)$ denotes the corresponding Calderón projections.

The proof of the Proposition depends on the unique continuation property for Dirac operators and the Lagrangian property of the Cauchy data spaces, more precisely the chiral twisting property (3.8).

5.3. Generalizations for global boundary conditions. On a smooth compact manifold X with boundary Σ, the solution spaces $\ker(\mathrm{A}, s)$ depend on the order s of differentiability and they are infinite–dimensional. To obtain a finite index one must apply suitable boundary conditions (see [**14**] for local and global boundary conditions for operators of Dirac type). In this report, we restrict ourselves to boundary conditions of Atiyah–Patodi–Singer type, i.e. $P \in \mathcal{G}r(\mathrm{A})$, and consider the extension

$$\mathrm{A}_P : \mathrm{dom}(\mathrm{A}_P) \longrightarrow L^2(X;S) \tag{5.1}$$

of A defined by the domain

$$\mathrm{dom}(\mathrm{A}_P) := \{f \in H^1(X;S) \mid P^{(0)}(f|_\Sigma) = 0\}. \tag{5.2}$$

It is a closed operator in $L^2(X;S)$ with finite–dimensional kernel and cokernel. We have an explicit formula for the adjoint operator

$$(\mathrm{A}_P)^* = \mathrm{A}_{\sigma(\mathrm{Id}-P)\sigma^*}\ . \tag{5.3}$$

In agreement with Proposition 4.1b, the preceding equation shows that an extension A_P is self–adjoint, if and only if $\ker P^{(0)}$ is a Lagrangian subspace of the symplectic Hilbert space $L^2(\Sigma; S|_\Sigma)$.

Let us recall the *Boundary Reduction Formula for the Index* of (global) elliptic boundary value problems discussed in [**13**] (inspired by [**37**], see [**14**] for a detailed proof for Dirac operators). Like the Bojarski Conjecture, the point of the formula is that it gives an expression for the index in terms of the geometry of the Cauchy data in the symplectic space of all (here L^2) boundary data.

PROPOSITION 5.2.

$$\mathrm{index}\ \mathrm{A}_P = \mathrm{index}\,\{P\mathcal{P}(\mathrm{A}) : \Lambda(\mathrm{A}, \tfrac{1}{2}) \to \mathrm{range}(P^{(0)})\}\,.$$

5.4. Pasting formulas. We shall close this section by mentioning a slight modification of the Bojarski Conjecture/Theorem, namely a non–additivity formula for the splitting of the index over partitioned manifolds.

THEOREM 5.3. *Let P_i be projections belonging to $\mathcal{G}r(\mathrm{A}_j)$, $j = 0, 1$. Then*

$$\mathrm{index}\ \mathrm{A} = \mathrm{index}\,(\mathrm{A}_0)_{P_0} + \mathrm{index}\,(\mathrm{A}_1)_{P_1} - \mathbf{i}(P_1, \mathrm{Id} - P_0).$$

It turns out that the boundary correction term $\mathbf{i}(P_1, \mathrm{Id} - P_0)$ equals the index of the operator $\sigma(\partial_t + \mathcal{B})$ on the cylinder $[0,1] \times \Sigma$ with the boundary conditions P_0 at $t = 0$ and P_1 at $t = 1$.

REMARK 5.4. (*a*) In this section we have not always distinguished between the *total* and the *chiral* Dirac operator because all the discussed index formulas are valid in both cases.
(*b*) Important index formulas for (global) elliptic boundary value problems for operators of Dirac type can also be obtained without analyzing the concept and the geometry of the Cauchy data spaces (see e.g. the celebre Atiyah–Patodi–Singer Index Theorem, [**2**] and [**14**], Chapter 22, or [**31**] for a recent survey of index formulas where there is no trace of the Calderón projection). The basic reason is that the index is an invariant represented by a local density inside the manifold plus a correction term which lives on the boundary and may be local or non-local as well. However, these formulas do not explain the simple origin of the index or the spectral flow, namely that *all* index information is naturally coded by the geometry of the Cauchy data spaces. To us it seems necessary to use the Calderón projection in order to understand (not calculate) the index of an elliptic boundary problem and the reason for the non–locality resp. locality.

6. Family Versions: the Spectral Flow and the Maslov Index

6.1. Spectral flow and the Maslov index. Let $\{\mathrm{A}_t\}_{t\in[0,1]}$ be a continuous family of (from now on always *total*) Dirac operators with the same principal symbol and the same domain D. To begin with, we do not distinguish between the case of a closed manifold (when D is just the first Sobolev space and all operators are essentially self–adjoint) and the case of a manifold with boundary (when D is specified by the choice of a suitable boundary value condition).

We consider the *spectral flow* $\mathbf{sf}\{\mathrm{A}_{t,D}\}$. Roughly speaking, it is the difference between the number of eigenvalues which change the sign from $-$ to $+$ as t goes from 0 to 1, and the number of eigenvalues which change the sign from $+$ to $-$. It can be defined in a satisfactory, purely functional analytical way, following a suggestion by J. Phillips (see [**8**] and [**30**]). We want a pasting formula for the spectral flow. To achieve that, one replaces the spectral flow of a continuous 1–parameter family of self–adjoint Fredholm operators, which is a 'quantum' quantity, by the *Maslov index* of a corresponding path of Lagrangian Fredholm pairs, which is a 'quasi–classical' quantity. The idea is due to Floer and was worked out subsequently by Yoshida in dimension 3, by Nicolaescu in all odd dimensions, and pushed further by Cappell, Lee and Miller, Daniel and Kirk, and many other authors. For a survey, see [**8**], [**9**], [**20**].

Let us fix the space $\boldsymbol{\beta}$ for the family. By Proposition 4.1c the corresponding family $\{\Lambda(\mathrm{A}_t\}$ of natural Cauchy data spaces is continuous. In [**8**] we obtained the *General Boundary Reduction Formula* for the spectral flow which gives a family version of the Bojarski conjecture (our Proposition 5.1):

THEOREM 6.1. *The spectral flow of the family* $\{\mathrm{A}_{t,D}\}$ *and the Maslov index* $\mathbf{mas}\left(\{\Lambda(\mathrm{A}_t)\}, \gamma(D)\right)$ *are well-defined and we have*

$$\mathbf{sf}\{\mathrm{A}_{t,D}\} = \mathbf{mas}\left(\{\Lambda(\mathrm{A}_t)\}, \gamma(D)\right). \tag{6.1}$$

We have various corollaries for the spectral flow on closed manifolds with fixed hypersurface (see [**9**]). For product structures near Σ, one can apply Theorem 4.3 and obtain an L^2–version of the preceding Theorem which gives a new proof and a slight generalization of the Yoshida–Nicolaescu Formula (for details see [**10**], Section 3):

THEOREM 6.2.

$$\begin{aligned} \mathbf{sf}\{A_0 + C_t\} &= \mathbf{mas}\left(\{\Lambda_t^0 \cap L^2(-\Sigma) \dotplus \Lambda_t^1 \cap L^2(\Sigma)\}, L_-\right) \\ &=: \mathbf{mas}\left(\{\Lambda_t^0 \cap L^2(-\Sigma)\}, \{\Lambda_t^1 \cap L^2(\Sigma)\}\right), \end{aligned}$$

where the last expression is given by the formula of the Maslov index of Fredholm pairs of two curves.

REMARK 6.3. Nor here is it compelling to use the symplectic geometry of the Cauchy data spaces (see Remark 5.4b). Actually, deep gluing formulas can and have been obtained for the spectral flow by coding relevant information not in the full infinite–dimensional Cauchy data spaces but in families of Lagrangian subspaces of suitable finite–dimensional symplectic spaces, like the kernel of the tangential operator (see [**18**] and [**19**]).

6.2. Correction formula for the spectral flow. Let D, D' with $D_{\min} \subset D, D' \subset D_{\max}$ be two domains such that both $\{\mathrm{A}_{t,D}\}$ and $\{\mathrm{A}_{t,D'}\}$ become families of self–adjoint Fredholm operators. We assume that D and D' differ only by a finite dimension, more precisely that

$$\dim \gamma(D)/\gamma(D) \cap \gamma(D') = \dim \gamma(D')/\gamma(D) \cap \gamma(D') < +\infty\,. \tag{6.2}$$

Then we find from Theorem 6.1 (for details see [**9**], Theorem 6.5):

$$\begin{aligned} &\mathbf{sf}\left(\{\mathrm{A}_{t,D}\}\right) - \mathbf{sf}\left(\{\mathrm{A}_{t,D'}\}\right) \\ &\quad = \mathbf{mas}\left(\{\Lambda(\mathrm{A}_t)\}, \gamma(D')\right) - \mathbf{mas}\left(\{\Lambda(\mathrm{A}_t)\}, \gamma(D)\right) \\ &\qquad = \sigma_{\text{Hör}}\left(\Lambda(A_0), \Lambda(A_1); \gamma(D'), \gamma(D)\right) \end{aligned} \tag{6.3}$$

(c.f. Remark 1.4b). The assumption (6.2) is rather restrictive. The pair of domains, for instance, defined by the Atiyah–Patodi–Singer projection and the Calderón projection, may not always satisfy this condition. For the present proof, however, it seems indispensable.

7. The Boundary Reduction and the Gluing of Determinants

There are competing concepts of the Fredholm determinant and the ζ–function regularized determinant. Following the condensed presentation of [**6**], this section presents the recent Scott–Wojciechowski formula under the perspective of Cauchy data spaces (the 'Bojarski approach').

7.1. Three determinant concepts. Let us begin with the most simple integral of statistical mechanics, the *partition function* which is the model for all quadratic functionals:

$$Z(\beta) := \int_\Gamma e^{-\beta(Tx,x)}\, dx\,. \tag{7.1}$$

To begin with, let $\dim\Gamma = d < \infty$ and β real with $\beta > 0$, and assume that T is a strictly positive, symmetric endomorphism. In suitable coordinates we evaluate the Gaussian integrals and find

$$Z(\beta) = \pi^{d/2}\cdot\beta^{-d/2}\cdot(\det T)^{-\frac{1}{2}}\,.$$

Two fundamental problems arise when we try to take a Dirac operator for T and all sections in a bundle S over a compact manifold M for Γ according to the Feynman recipes. What if T is not > 0? And what if $d = +\infty$ (i.e. if M is not a finite set of points)? To get around the first problem, one proceeds as follows:

We decompose $\Gamma = \Gamma_+ \times \Gamma_-$ and $T = T_+ \oplus T_-$ with T_+, $-T_-$ strictly positive in $\Gamma_\pm$ and $\dim\Gamma_\pm = d_\pm$. Formally, we obtain by a suitable path in the complex plane approaching $\beta = 1$:

$$Z(1) = \pi^{\zeta/2}\, e^{\pm i\frac{\pi}{4}(\zeta-\eta)}\,(\det|T|)^{-\frac{1}{2}}\,, \tag{7.2}$$

with $\zeta := d_+ + d_-$ and $\eta := d_+ - d_-$.

We shall not discuss the various stochastic ways of evaluating the integral when $d = +\infty$, but present two other concepts of the determinant.

From the point of view of functional analysis, the only natural concept is the *Fredholm determinant* of bounded operators acting on a separable Hilbert space of the form e^α or, more generally, $\mathrm{Id}+\alpha$ where α is of trace class. We recall the formulas

$$(7.3) \qquad \det_{Fr} e^\alpha = e^{\operatorname{Tr}\alpha} \quad \text{and} \quad \det_{Fr}(\mathrm{Id}+\alpha) = \sum_{k=0}^{\infty} \operatorname{Tr} \wedge^k \alpha \ .$$

The Fredholm determinant is notable for obeying the product rule in difference to other generalizations of the determinant to infinite dimensions, where the error term of the product rule leads to new invariants.

Clearly, the parametrix (or Green's function) of a Dirac operator leads to operators for which the Fredholm determinant can be defined, but the relevant information about the spectrum of the Dirac operator does not seem sufficiently maintained. Note also that Quillen and Segal's construction of the *determinant line bundle* is based on the concept of the Fredholm determinant, but does not lead to *numbers* when the bundle is non–trivial.

A third concept is the *ζ–function regularized determinant*, based on Ray and Singer's observation that formally

$$\det T = \prod \lambda_j = \exp\{\sum \ln \lambda_j e^{-z\ln\lambda_j}|_{z=0}\} = e^{-\frac{d}{dz}\zeta_T(z)|_{z=0}},$$

where $\zeta_T(z) := \sum_{j=1}^{\infty}\lambda_j^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\operatorname{Tr} e^{-tT}\,dt$. For a *positive definite self–adjoint elliptic* operator T of second order, acting on sections of a Hermitian vector bundle over a closed manifold M of dimension m, Seeley [**36**] has shown that the function $\zeta_T(z)$ is holomorphic for $\Re(z)$ sufficiently large and can be extended meromorphically to the whole complex plane with $z=0$ no pole.

The preceding definition does not apply immediately to the Dirac operator A which has infinitely many positive λ_j and negative eigenvalues $-\mu_j$. As an example, consider the operator $\mathrm{A}_a := -i\frac{d}{dx}+a : C^\infty(S^1)\to C^\infty(S^1)$ with $\mathrm{A}_a\varphi_k = k\varphi_k + a\varphi_k$ where $\varphi_k(x)=e^{ikx}$. It follows that $\operatorname{spec}\mathrm{A}_a = \{k+a\}_{k\in\mathbf{Z}}$.

Choosing the branch $(-1)^{-z} = e^{-i\pi z}$, we find

$$\begin{aligned}\zeta_{\mathrm{A}}(z) &= \sum\lambda_j^{-z} + \sum(-1)^{-z}\mu_j^{-z}\\ &= \tfrac12\left\{\zeta_{\mathrm{A}^2}(\frac{z}{2}) + \eta_{\mathrm{A}}(z)\right\} + \tfrac12 e^{-i\pi z}\left\{\zeta_{\mathrm{A}^2}(\frac{z}{2}) - \eta_{\mathrm{A}}(z)\right\},\end{aligned}$$

where $\eta_{\mathrm{A}}(z) := \sum_{j=1}^{\infty}\lambda_j^{-z}+\mu_j^{-z}$. Thus:

$$\zeta'_{\mathrm{A}}(0) = \tfrac12\zeta'_{\mathrm{A}^2}(0) - \tfrac{i\pi}{2}\{\zeta_{\mathrm{A}^2}(0) - \eta_{\mathrm{A}}(0)\}$$

and

$$(7.4) \qquad \det_\zeta \mathrm{A} = e^{-\zeta'_{\mathrm{A}}(0)} = e^{\frac{i\pi}{2}\{\zeta_{\mathrm{A}^2}(0)-\eta_{\mathrm{A}}(0)\}}\cdot e^{-\frac12\zeta'_{\mathrm{A}^2}(0)}.$$

7.2. The Scott–Wojciechowski Formula. In the one–dimensional case various authors obtained formulas for the ζ–regularized determinant of a system of ordinary differential equations subject to linear boundary conditions in terms of the usual determinant of the (finite) matrices defining these boundary conditions, see e.g. [**12**], [**17**], and [**27**]. Here, we shall review recent progress in higher dimensions.

In 1995 it was shown by Wojciechowski that the ζ–regularized determinant can also be defined for certain self–adjoint Fredholm extensions of the Dirac operator on a compact manifold with boundary, namely when the domain is defined by a projection belonging to the *smooth, self–adjoint* Grassmannian

$$
\begin{aligned}
\mathcal{G}r^*_\infty(\mathrm{A}) = \{P \in \mathcal{G}r(\mathrm{A}) \mid P \text{ is self-adjoint, } P - \mathcal{P}(\mathrm{A}) \text{ is smoothing} \\
\text{and } \operatorname{range}(P^{(0)}) \text{ is Lagrangian in } L^2(\Sigma; S|_\Sigma)\}.
\end{aligned} \tag{7.5}
$$

We refer to [**40**] for the details of the delicate estimates needed for establishing the three involved invariants in that case.

Since then, Scott and Wojciechowski have established a formula which relates the ζ–determinant and the Fredholm determinant (see [**33**], [**34**]).

THEOREM 7.1. *Let* A *be a Dirac operator over an odd–dimensional compact manifold* M *with boundary* Σ *and let* $P \in \mathcal{G}r^*_\infty(\mathrm{A})$ *. Then the range of the Calderón projection* $\mathcal{P}(\mathrm{A})$ *(the Cauchy data space* $\Lambda(\mathrm{A}, \frac{1}{2})$*) and the range of* P *can be written as the graphs of unitary, elliptic operators of order 0,* K*, resp.* T *which differ from the operator* $(B^+B^-)^{-1/2}B^+ : C^\infty(\Sigma; S^+|_\Sigma) \to C^\infty(\Sigma; S^-|_\Sigma)$ *by a smoothing operator. Moreover,*

$$
\det{}_\zeta \mathrm{A}_P = \det{}_\zeta \mathrm{A}_{\mathcal{P}(\mathrm{A})} \cdot \det{}_{Fr} \tfrac{1}{2}(\mathrm{Id} + KT^{-1})\,. \tag{7.6}
$$

In geometric terms, the key to understanding the preceding Theorem is that the determinant line bundle, parametrized by the projections belonging to the smooth self–adjoint Grassmannian, is *trivial* so that one can attribute complex numbers (up to a multiple) to the canonical determinant section. This may explain why earlier attempts at relating the concept of the ζ–determinant with the Fredholm determinant had to be content with discussing the metric of the determinant bundle in terms of the ζ–determinant, and why the break–through in understanding the mutual relation required a concept of boundary reduction.

REMARK 7.2. (*a*) The variation of the modulus of the ζ–determinant contains a truly global term and can not be localized near the

boundary. In [**34**], the authors get around this problem by varying a quotient of determinants.
(*b*) Various modifications and generalizations of the Scott–Wojciechowski Formula are to be expected, in particular for the ζ–determinant over a partitioned manifold (i.e. a reduction formula to the hypersurface and a true pasting formula). Even so such results are not yet obtained, Scott's and Wojciechowski's formula provides an illuminating example of the meaning of the geometry of Cauchy data spaces for the study of spectral invariants.

7.3. An adiabatic pasting formula. At the end of this exposition let us present a recent adiabatic splitting formula for the determinant proved by Park and Wojciechowski ([**29**]):

THEOREM 7.3. *Let* $R \in \mathbf{R}$ *be positive, let* M^R *denote the stretched partitioned manifold* $M^R = M_0 \cup_\Sigma [-R,0] \times \Sigma \cup_\Sigma [0,R] \times \Sigma \cup_\Sigma M_1$, *and let* A_R, $\mathrm{A}_{0,R}$, $\mathrm{A}_{1,R}$ *denote the corresponding Dirac operators. We assume that the tangential operator* $\mathcal{B}$ *is invertible. Then*

$$\lim_{R\to\infty} \frac{\det_\zeta \mathrm{A}_R^2}{\left(\det_\zeta(\mathrm{A}_{0,R})^2_{\mathrm{Id}-\Pi_>}\right)\cdot\left(\det_\zeta(\mathrm{A}_{1,R})^2_{\Pi_>}\right)} = 2^{-\zeta_{\mathcal{B}^2}(0)}\,.$$

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