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# Dynamics of Complex Quadratic Correspondences

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Dynamics of Complex Quadratic Correspondences

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**Abstract**

This thesis summarizes some important results from the theories of Riemann surface geometry and complex dynamics in order to apply them in the analysis of quadratic correspondences. The example of the arithmetic-geometric mean value is used as a starting point to expose some properties of qadratic correspondences. Attempts to define Fatou and Julia sets as well as regular and limit sets for correspondences are presented.

**Resumé**

Denne afhandling opsummerer nogle vigtige resultater fra teorierne om geometri på Riemann flader og kompleks dynamik og behandler deres anvendelse i analysen af kvadratiske korrespondencer. Den aritmetisk-geometriske middelværdi benyttes som udgangspunkt til en undersøgelse af egenskaberne af kvadratiske korrespondencer. Försøg på at definere Fatou og Julia mængde såvel som regulær og grænsemængde for korrespondencer præsenteres.

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## Guidelines for the Reader

This report is a Master of Science thesis, and though the first chapters are written so that most readers with some background in mathematics should be able to read it, the level of difficulty will increase throughout the text. In accordance with the *Roskilde University Mathematics Department* rules, this thesis covers an equivalent of two mathematical courses, and the contents of these shall be represented in this thesis. The subjects I have chosen are the following

- *Geometry of Riemann surfaces* in particular the geometric tools that are suitable for considering the covering surfaces of the complex mappings.
- *Complex Dynamical systems*, some of which is from the "Intensive Programme on Complex Dynamics" in University of Göttingen, summer 1998. The programme consisted of different mini-courses taught by some of the researchers in the field.

Many of the results in this thesis are built on some basic concepts which will be known to some readers but not all. I have chosen to include some of these useful concepts in the beginning of the thesis. The outline of the dynamics of rational functions is presented on a somewhat easier level for the interested non-specialist or student. Some readers may want to skip these parts if they are familiar with the basics of topology, geometry or complex dynamics. I apologize for the decreased readability of the thesis on this account.

I will now give a brief outline of the thesis:

1. We will draw heavily on the theory of *Riemann Surfaces*, and the theory will be presented as the first of the two defined subject areas
2. The second subject area is *complex dynamical systems.* We will begin with an apetizer of complex dynamical systems with an introduction to the problems of rational iteration and some examples of what a dynamical system can be.
3. With the advent of the notion of *normal families* there was a vast development in dynamical systems theory. We will explore this theory and the *dynamical dichotomy* of the *Julia* and *Fatou* sets as well as that of the regular and limit set of the Fuchsian groups.
4. Starting with the *arithmetic-geometric mean* we will move into general *quadratic correspondences* and try to find parrallels to the themes of rational iteration, in particular the dynamical dichotomy.

I would like to thank my supervisor, Carsten Lunde Petersen, for many hours of explanations and my wife, Jeannette, for support.

Jacob S. Jalving

# Preface

Some of the most publicized images of mathematics through the last few decades are the fractal images exposed by BENOIT MANDELBROT [Man82]. One of these, The **Mandelbrot set** shown in figure 1, is important in this thesis, and we shall return to it in section 3.3. It is obvious that mathematics with

**Figure 1** The Mandelbrot set.

such a pretty face would become known (at least the face of it) in a great part of the general public. The field that the Mandelbrot set became a symbol of, however, is capable of a lot more than creating pretty pictures. The reason why the mathematics of dynamical systems have become a key field is that the computer power which enabled mathematicians to create the pictures of fractal sets also enables physicists and mathematicians to consider such problems where computation leads to unpredictable results, or where the results are so complicated that "chaotic" is almost a suitable word for them. Not quite though, since the results in this case are deterministic, governed by the equations of the system. This has led to the term "deterministic chaos" which is something of an oxymoron.

It has been known for centuries that some mathematical computations give very diverse and unpredictable results. One of the early examples is the arithmetic-geometric mean of C.F. GAUSS which is found in chapter 4. The above picture of the Mandelbrot set shows just how intricate the results can be. If you look closer on the boundary of the Mandelbrot set you will encounter smaller structures similar to the original cardioid-and-circle-with-antennas shape. This type of self-similarity and periodicity are features which also appear in the physicists' theories of dynamical systems.

Chaos is one of the great inventions in modern physics, and the methods of dynamical systems are used in theoretical physics to shed light over the relationships between small perturbations of initial conditions and large scale differences in system evolution. The commonly known example of the effect is known as the butterfly effect proposed by the meteorologist EDWARD N. LORENZ: A butterfly flapping its wings in Asia sets a breeze in motion which is amplified by the atmospheres instantaneous wind patterns and a few days later the flapping of the butterfly's wings in Asia has turned into a storm on the US West Coast. This scenario is now becoming more of a natural law in the public eye than the paradox it was originally posed as.

The part of theoretical physics which deals with theoretical equilibrium problems builds on statistical theory and measure theory, and it is beyond the scope of this thesis to deal with this branch of dynamical systems, but we can hint to where the problems lie. The **ergodicity** theory has its roots in the ergodic hypothesis which in one of its many forms has the essence that

> *Any point in a system's phase space will be approximated arbitrarily close by the system in finite time.*

This basically means that the variation we can expect from the system as time goes by is all the possible variation the system is capable of. The usual thought experiment which illustrates this is that we consider a small box with some air in it and place it in a much larger box that is empty. We open the small box and let the air distribute in the larger space and ask ourselves: Can the gas again, by its own movement end up in the smaller box again? We can by statistical methods determine the probability of any given distribution of air in the box; and of course, the probability of the gas returning to the small box is very small but positive. With this in mind, the ergodic hypothesis, that the gas *will* at some point end up in the smaller box again is perhaps plausible.

It is not difficult to realize that when everything is predictable, there is no chaos in this system. The only type of chaos we can have in the system is the ignorance of the exact distribution of particles due to the diversity of the system. It isn't chaotic in the sense that we have no possibility of knowing what happens. We can by the laws of mechanics determine the exact position of any particle if we know its initial position and velocity. We can be unable to measure the exact positions of any one particle, but the behaviour of the system is well understood. We can call the condition of the system "deterministic chaos" and this is a property that is present in many mathematical models of real-life systems. The existence of the "chaotic" behaviour in mathematical systems can even be what essentially makes the system a model of the real-life chaotic dynamics. This is why fractals and chaos are often mentioned in one sentence, and it is why there is good sense in studying such systems from other than a purely mathematical points of view.

It is customary to state some characteristics of chaotic behaviour in deterministic systems. The following characteristics are adapted from ROBERT L. DEVANEY [Dev92], [Dev86].

1. The system is critically dependent on initial conditions. Two arbitrarily close points* will eventually end up arbitrarily far from each other ("butterfly effect" discovered by Lorenz).

2. The system has dense periodic orbits, so that anywhere in the system, we encounter points which in a finite number of steps map to themselves.

3. the system is ergodic as described above, any point can be approximated arbitrarily close in a finite number of steps. Some authors use instead of ergodicity the notion of **topological transitivity**, which is largely equivalent, see e.g. [Dev86]†.

Recently it has been proved that any topologically transitive system with dense periodic orbits is critically dependent on initial conditions, i.e. the two latter properties imply the first [BBD$^+$92].

This work is in the tradition of complex iteration, where the goal is to completely classify the dynamics of rational functions under iteration. The problem I have chosen to work on is that of generalizing the results of rational dynamics to quadratic correspondences. To pose it as a question:

*How do some of the important features of rational dynamics translate into quadratic correspondences?*

The applications of the mathematical theory in physical systems we will leave behind, but it is briefly discussed in an article of Shaun Bullett [Bul88].

*These points can be mathematical points under repeated use of a function or points in phase space of a physical system, where time evolution is determined by the physical system.

†ergodicity has its roots in the work of BIRKHOFF and KOLMOGOROV and is used mostly in the theoretical physics branch of dynamical systems

# 1 Riemann Surfaces

The theory of Riemann surfaces gives a good basis for examining complex dynamics. There is much sense in beginning with this since it will facilitate many results. Riemann surfaces are topological spaces which meet some extra demands. Given a map which is holomorphic on some domain in $\mathbb{C}$, we can analytically continue the map to a maximal (Riemann) surface on which the mapping is holomorphic, thereby extending the domain of analyticity. This line of thought is the original idea of the Riemann surfaces from which the present notion is a generalization which we shall define in the following.

## 1.1 Some preliminaries

In the following we will draw on the concept of a **topological space**. A topological space is a pair $(X, \mathcal{T})$ consisting of a set $X$ and a topology $\mathcal{T}$ on $X$. It is assumed that the reader is familiar with the fundamentals of topology. Some properties will however be so important that we will mention them briefly. All of the topological spaces we will treat will have the **Hausdorff** property.

**Definition 1.1**
*A topological space $X$ is a* **Hausdoff space** *if for two arbitrary points $a, b \in X$ there exist disjoint, open neighborhoods $U$ of $a$ and $V$ of $b$.*

The reason why this property is important is that a sequence in a Hausdorff space can have at most one limit point. We will assume all topological spaces in the following to be Hausdorff spaces. Let's now move on to the property which ensures that a sequence has at least one limit point, the notion of compactness.

**Definition 1.2**
*Let $X$ be a topological space. A subset $K$ is* **compact** *in $X$, if for any open covering $\{U_i : i \in I\}$ of $K$, there can be found a finite open subcovering $U_1, U_2, \ldots, U_k$ of sets from $\{U_i : i \in I\}$ that covers all of $K$.*

Continuous mappings on compact sets are particularly nice to deal with. They send compact sets to compact sets, and any mapping $f : K \to \mathbb{R}$

attains minimum and maximum values. We will also deal primarily with connected topological spaces.

**Definition 1.3**
*A topological space $X$ is* **connected**, *if there is no partition of $X$, $X = U_1 \cup U_2$, consisting of disjoint, nonempty open sets $U_1, U_2$.*

The class of functions which conserve topology are the homeomorphisms.

**Definition 1.4**
*Let $X$ and $Y$ be topological spaces. A* **homeomorphism** *$f: X \to Y$ is a one-to-one and onto continuous map with continuous inverse.*

If two spaces are homeomorphic, we also say that they are **topologically equivalent.**

## 1.2 Definition of Riemann Surfaces

We review the basic concepts of Riemann surfaces with emphasis on definitions and theorems. For more details, the interested reader may look to [For81], [FK80] and [AS60], where most of the following is found.

An **n-dimensional topological manifold** is a Hausdorff topological space on which there exists an open covering by sets homeomorphic to open sets in $\mathbb{R}^n$. These homeomorphisms are called **charts** and they define a **local coordinate** on the manifold by the corresponding coordinates in $\mathbb{R}^n$. In the following we will consider two-dimensional manifolds – surfaces. When a collection of charts covers the surface and any two charts $\phi_1$ and $\phi_2$ have holomorphic overlap, that is, the change $\phi_2 \circ \phi_1^{-1}$ from one coordinate to the other is holomorphic in any point where both are defined, then the family of charts is called an **atlas.** A **complex structure** is an equivalence class of atlases defined by the equivalence relation where two atlases are equivalent if the overlapping charts from the two atlases have holomorphic coordinate change.

**Definition 1.5**
*A* **Riemann surface** *is a connected two-dimensional manifold $S$ together with a complex structure of local charts $\varphi : S \to \mathbb{C}$.*

Functions on Riemann surfaces can be said to inherit the properties of complex functions in the following way. A continuous mapping between Riemann surfaces $S$ and $T$ is holomorphic if for every pair of charts, $\varphi : S \to \mathbb{C}$ and $\psi : T \to \mathbb{C}$, the complex function $\psi \circ f \circ \varphi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic. In this fashion, all the well-known complex function theory applies to the Riemann surfaces. A singularity for a function $f : S \to T$ between Riemann surfaces is removable if the corresponding singularity is removable in $\mathbb{C}$. Two Riemann surfaces $S$ and $T$ are said to be **isomorphic** if there exists a biholomorphic map $f : S \to T$.

Charts and atlases are usually of no interest to us. We assume from now on that we can use the maximal atlas on the Riemann surface. The maximal atlas contains, losely put, all possible charts from the atlases in the equivalence class.

The simplest example of a Riemann surface is the complex plane itself. The identity map is an obvious chart. Any subset of the complex plane is also a Riemann surface, using the restriction of the identity map as chart. The extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is also a Riemann surface, and is called the **Riemann sphere**. The simplest atlas we can use in this case consists of the identity chart for $z \in \mathbb{C}$, and the chart

$$z \mapsto \begin{cases} {}^1\!/_z & \text{for } z \in (\mathbb{C} \cup \{\infty\}) \setminus 0 \\ 0 & \text{for } z = \infty \end{cases} \tag{1.1}$$

The overlap $\mathbb{C} \setminus \{0, \infty\}$ has holomorphic change of chart $z \mapsto 1/z$, so there is a complex structure. The Riemann sphere is a simple example of a compact Riemann surface, it is also called the one-point compactification of $\mathbb{C}$.

**Definition 1.6**
*Let $S$ be a Riemann surface and $X \subseteq S$ an open subset. A* **meromorphic** *map $f : X \to \mathbb{C}$ fulfills the following:*

1. *$f$ is holomorphic on a subset $X' \subseteq X$.*
2. *$X \setminus X'$ contains only isolated points.*
3. *$\forall p \in X \setminus X' : \lim_{z \to p} |f(z)| = \infty$.*

When $S = \overline{\mathbb{C}}$, infinity is just like any other point on this surface, and therefore all functions $f : \mathbb{C} \to \mathbb{C}$ which have poles which are removable singularities in the chart $z \mapsto {}^1\!/_z$ are meromorphic. An example is the polynomials. They are meromorphic in a neighbourhood $U$ of $\infty$ since they are holomorphic in all of $\overline{\mathbb{C}}$ except $\infty$ which then is an isolated point in $U \setminus (\mathbb{C} \cap U)$. Furthermore, any polynomial $p(z)$ maps $\infty$ to itself $d = \deg(p)$ times. To realize this, look at

$$\frac{1}{p({}^1\!/_z)} = z^d(1 + \ldots) \tag{1.2}$$

which maps 0 to itself $d$ times, and therefore 0 is a removable singularity.

In complex analysis we have LIOUVILLE's theorem which states:

**Theorem 1.7 (Liouville's theorem)**
*Any bounded holomorphic function* $f : \mathbb{C} \to \mathbb{C}$ *is constant.*

For compact Riemann surfaces we have the corresponding theorem as an immediate consequence

**Theorem 1.8**
*Any holomorphic function* $f : X \to Y$ *from a compact Riemann surface* $X$ *to an open Riemann surface* $Y$ *is constant.*

Evidently, holomorphic maps on $\overline{\mathbb{C}}$ are not very interesting, but the meromorphic maps give rise to many interesting situations, just as holomorphic mappings on open Riemann surfaces.

## 1.3 Geometry on surfaces

In order to treat some of the geometrical issues of dynamics, we shall need some basic tools, such as curves and fundamental groups.

**Definition 1.9**
*By a* **curve** *in a topological space* $X$, *we mean a continuous map* $u : I \to X$, *where* $I$ *is the closed unit interval* $[0, 1]$ *in* $\mathbb{R}_+$.

The point $a = u(0)$ is called the initial point of the curve, and $b = u(1)$ the end point. Note that the definition of the curve gives an orientation of the curve. The curve runs from $a$ to $b$. We can define the curve with the opposite orientation $u^-(t) = u(1 - t)$.

The **product curve** of two curves is defined if the curves have a common end point. If $u : I \to S$ has initial point $a$ and end point $b$, and $v : I \to S$ is a curve from $b$ to $c$, then the product curve is defined by

$$u \cdot v(t) = \begin{cases} u(2t) & 0 \leq t \leq 1/2 \\ v(2t - 1) & 1/2 \leq t \leq 1. \end{cases}$$

If $u$ is a curve on a surface, and $f$ is a surjective continuous function $f : I \to I$ fixing the end points, then $u$ and $u \circ f$ are two different curves which consist of the same set of points, but at different parameter values. We wish to develop a way to view the two curves as identical. Also, we would like that, to some extent, curves which connect the same two points, but not necessarily through the same set of points, can be equivalent. The concept which allows this is **homotopy**. Two curves with identical initial and end points are **homotopic** if they can be continuously deformed into each other. More formally, we say

**Definition 1.10**
*Two curves $u$ and $v$ with initial point $a$ and end point $b$ are homotopic on a topological space $X$ if there exists a continuous map $A : I \times I \to X$ such that for $t, s \in I$:*

1. $A(t, 0) = u(t)$.
2. $A(t, 1) = v(t)$.
3. $A(0, s) = a$ *and* $A(1, s) = b$.

Note that the curves $u$ and $u^-$ are not, in general, homotopic.

Homotopy of curves is an equivalence relation. Given $z_0 \in X$, the equivalence classes of curves with $z_0$ as both initial and end point form a group which is called the fundamental group of the surface $\pi_1(X, z_0)$. A **simply connected** Riemann surface is a surface which has trivial fundamental group. Then all closed curves, i.e. curves with identical initial and end point, are homotopic to the constant curve, which consists of only one point. Interpreted on a simply connected surface which has a complex structure, we may say that the surface can have no holes. Figure 1.1 is an example of a surface which has a hole. There are two different paths from $a$ to $b$ which are not homotopic since the curve can't be continuously deformed into a curve on the other side of the hole. If we consider the product curve of the two curves, then we get a curve $u_1$ which begins and ends at the same point, but which loops around the hole and therefore cannot be homotopic to the constant curve. If we then consider the curve which is the product of $u_1$ with itself, we get a curve which loops twice around the hole and can't be homotopic to either $u_1$ or the constant curve. In this way, the curve $u_1$ acts as generator of the one-generator cyclic group, which is the fundamental group of the surface pictured in figure 1.1.

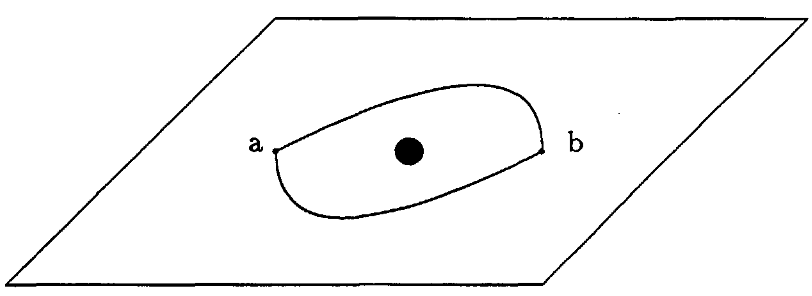



**Figure 1.1** Surface with a hole. Not simply connected.

In general, the complement of an $n$-connected domain in $\mathbb{C}$ contains $n$ disjoint components. As an example, the disc is simply ($n = 1$) connected, an annulus is doubly ($n = 2$) connected.

A topological space is called **arcwise connected** if any two points in $X$ can be connected by an arc that runs within the topological space. Clearly, the surface in figure 1.1 is arcwise connected.

## 1.4 Tangent Spaces

Let $S$ be a Riemann surface. The following is valid for any $n-$manifold, but we will assume our surface is two-dimensional. Let $p \in S$ not on the boundary of $S$. The class of functions on $S$ which are ($C^\infty$) differentiable in $p$, $\mathfrak{F}(S)_p$ is defined by the method described on page 12.

**Definition 1.11**
*A* **tangent vector** *to $S$ at $p$ is a function $V_p : \mathfrak{F}(S)_p \to \mathbb{C}$ sending $f \mapsto V_p f$ and which fulfills for all $f, g \in \mathfrak{F}(S)_p$ and all $r \in \mathbb{C}$,*

1. $V_p(f+g) = V_p f + V_p g;$
2. $V_p(rf) = rV_p f;$
3. $V_p(fg) = fV_p g + gV_p f;$ *where $fg$ is the product of functions and $fV_p g$ is the product of complex numbers.*

This definition is from [MP77], and we will build on this.

The tangent vector $V_p$ may be interpreted as the directional derivative of the function in the $V_p$ direction. Let $\gamma : I \to S$ be a differentiable curve with $\gamma(t_0) = p$. Let $V_p^\gamma$ be defined by

$$V_p^\gamma(f) = \frac{d(f \circ \gamma)}{dt}(t_0) \tag{1.3}$$

where the right hand side is the usual derivative of a complex function.

**Proposition 1.12**
*$V_p^\gamma$ is a tangent vector.*

**Proof**
We must check the three conditions stated in the definition 1.11. Given two functions $f, g \in \mathfrak{F}(S)_p$,

$$V_p^\gamma(f+g) = \frac{d(f \circ \gamma + g \circ \gamma)}{dt}(t_0) = \frac{d(f \circ \gamma)}{dt}(t_0) + \frac{d(g \circ \gamma)}{dt}(t_0)$$

since it is the real derivative of a real function, which proves that condition 1 is fulfilled. Condition 2 and 3 are also proved by reference to rules for derivatives of real functions. □

The **tangent space** to $S$ at $p$, $T_p(S)$ is the set of all tangent vectors to $S$ at $p$. It can be seen from the definition of tangent vectors that the tangent space is a vector space. Moreover, it can be proved that the tangent space of a Riemann surface is a 1-complex-dimensional vector space. Given a chart $\phi$ defined in a neighbourhood of $p$ on the Riemann surface, with $\phi(p) = x + iy$, a basis for the tangent space $T_p(S)$ is obtained by the functions

$$\frac{\partial}{\partial s}, \frac{\partial}{\partial t} : \mathfrak{F}(S)_p \to \mathbb{C}$$

given by

$$\left(\frac{\partial}{\partial s}\right)_p = \frac{\partial}{\partial x}(f \circ \phi)^{-1}|_{\phi(p)} \tag{1.4}$$

$$\left(\frac{\partial}{\partial t}\right)_p = \frac{\partial}{\partial y}(f \circ \phi)^{-1}|_{\phi(p)} \tag{1.5}$$

If we have two charts $\varphi_1 : U_1 \to \mathbb{C}$ and $\varphi_2 : U_2 \to \mathbb{C}$, both mapping $p$ to 0, the change of basis in the tangent space corresponding to the biholomorphic change of charts $\eta = \varphi_2 \circ \varphi_1^{-1} : U_1 \to U_2$ is given by the differential

$$D\eta = \begin{pmatrix} \frac{\partial \eta_u}{\partial x} & \frac{\partial \eta_u}{\partial y} \\ \frac{\partial \eta_v}{\partial x} & \frac{\partial \eta_v}{\partial y} \end{pmatrix} \tag{1.6}$$

as $\eta$ is biholomorphic, the change of charts is equivalent to multiplying with the complex number

$$\eta'(0) = \frac{\partial \eta_u(0)}{\partial x} - i\frac{\partial \eta_v(0)}{\partial y} \tag{1.7}$$

because of the Cauchy-Riemann equations.

## 1.5 Covering maps

Given two points $x \in X$ and $y \in Y$ on Riemann surfaces $X$ and $Y$, and a mapping $f : X \to Y$, we say that any point $x \in f^{-1}(y)$ **lies over** $y$. The set of points lying over $y$ is called the **fiber** of $f$ over $y$. A point $x \in X$ is called a **branch point** or **ramification point** if there is no neighbourhood $U$ of $x$ such that $f|_U$ is injective.

We may consider the mapping $w = z^d : \mathbb{C} \to \mathbb{C}$ which has branch point and value 0, since in any neighbourhood $W$ of 0, a point $w_0 \in W$ has exactly $d$ inverse images $\sqrt[d]{w_0}$.

**Definition 1.13**
*A continuous map $f$ between Riemann surfaces $X$ and $Y$ is a* **covering map** *if $\forall y \in Y$ there exists a neighbourhood $\omega$ of $y$ such that $f^{-1}(\omega) = \cup_{i \in I} U_i$ and*

1. $U_j \cap U_k = \emptyset$ *for any* $j \neq k$, $j, k \in I$.

2. *for all* $i \in I$ *the map* $f : U_i \to \omega$ *is a homeomorphism.*

If there is a covering map from $X$ to $Y$, then $X$ is said to be a covering surface for $Y$. We haven't used the fact that the spaces are Riemann surfaces. The above definitions and many of the following propositions go for Hausdorff spaces also, but we are mainly concerned with Riemann surfaces.

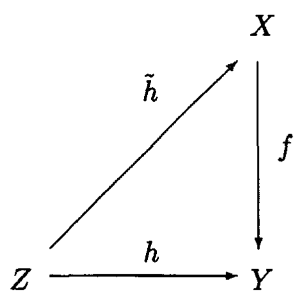
If a projection $\pi$ from a Riemann surface $X$ to a manifold $U$ satisfies

1. Each point of $X$ has a neighbourhood where $\pi$ is injective.

2. Given any curve $\gamma$ in $U$ and any point $x_0 \in \pi^{-1}(\gamma(0))$ there is a unique curve $\gamma_1 \subset X$ such that $\gamma_1(0) = x_0$ and $\pi(\gamma_1) = \gamma$.

Then $\pi$ is said to have the **curve lifting property**. Every covering map must have the curve lifting property. For proof, the interested reader may look in [For81].

**Definition 1.14**
*Let* $f : X \to Y$ *be a covering map and* $h : Z \to Y$ *be a continuous map between topological spaces. Then* $\tilde{h} : Z \to X$ *is a* lift *of* $h$ *if* $h = f \circ \tilde{h}$.




The conditions under which a lift exists can be found in [For81].

## 1.6 Uniformization of Riemann surfaces

It is also shown in [For81], that if $f : X \to Y$ is a covering map and $X$ is simply connected, then $f$ is the **universal covering map**, and $X$ is the universal covering of $Y$. There are only three distinct universal coverings.

**Theorem 1.15**
*Any simply connected Riemann surface is isomorphic to either* $\overline{\mathbb{C}}$, $\mathbb{C}$ *or the unit disc* $\mathbb{D}$.

The proof can be found in [AS60]. This result is a cornerstone in the theory of Riemann surfaces. It allows us to examine these three surfaces only. It turns out that the Riemann sufaces with $\mathbb{C}$ or $\overline{\mathbb{C}}$ as universal coverings are few and simple. The most diverse surfaces and those we shall examine in this thesis have the disc or equivalently a half-plane as universal covering.

A convenient way of treating the Riemann surfaces is by considering the automorphisms of the universal covering surface. Let $T$ be the universal covering surface, and $f$ be the universal covering map of a Riemann surface $S$. An automorphism $g : T \to T$ is called **fibre-preserving** if any point $t$ lying over $s$ maps by $g$ to another point lying over $s$, i.e.

$$f(t) = f \circ g(t) \tag{1.8}$$

The fibre-preserving automorphisms for a universal covering map of any Riemann surface form a group called the group of **deck transformations**. The particular group is determined by the particular covering map. The Riemann surface of a group of automorphisms is then the quotient surface of the covering surface modulo the deck transformation group. An example is the torus, which is isomorphic to the complex plane modulo a group with two generators.

Let $\Gamma$ represent a deck transformation group. The possible quotient Riemann surfaces of the universal covering surfaces are the following

1. $\overline{\mathbb{C}}/\Gamma = \overline{\mathbb{C}}$.
2. $\mathbb{C}/\Gamma = \mathbb{C}/\mathbb{Z}$ or $\mathbb{C}/\Gamma = \mathbb{T}$.
3. $\mathbb{D}/\Gamma$ : all other cases.

**Definition 1.16**
*A Riemann surface $S$ is called* **hyperbolic** *if it has the disc as universal covering surface, i.e. $\exists f : \mathbb{D} \to S$ a universal covering map.*

The Riemann surfaces we are mainly concerned with here are the hyperbolic ones. The *thrice-punctured sphere* $\mathbb{C} \setminus \{a, b, c\}$ has the disc as covering surface and is called the maximal hyperbolic example [Wil92].

## 1.7 The hyperbolic metric

The hyperbolic metric is a metric that allows the unit disc to become a complete (hyperbolic) metric space, whereas it in the regular metric is bounded.

We create the metric by defining a weight function $L : \mathbb{D} \to \mathbb{R}_+$ which gives the length of a curve $\gamma$ in the new metric as a function of euclidean length and position in the disc.

$$L(\gamma) = \int_\gamma \frac{2|dz|}{1-|z|^2} \tag{1.9}$$

The complex euclidean metric is adapted from the real 2-dimensional space, where the distance between two points is given by

$$d(x,y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \tag{1.10}$$

Distance between two points $z_1, z_2 \in \mathbb{D}$ in the hyperbolic metric is given by

$$d(z_1, z_2) = \inf_{\gamma \in \mathbb{D}} \{L(\gamma)\} \tag{1.11}$$

where $\gamma$ is any curve which connects the two points $z_1$ and $z_2$.

The function $ds = {}^{2|dz|}/_{1-|z|^2}$ is called the POINCARÉ or **hyperbolic** metric on the disc. A line from the center towards the edge of the unit disc has infinite length. The geodesics of the metric (curves with curvature zero with respect to the metric – "straight lines") are circular arcs with perpendicular intersection of the unit circle. The metric space itself has curvature $-1^*$. Hyperbolic space can also be represented by the upper half plane, $\mathbb{H}$, and then the Poincaré metric is

$$ds = |dw|/v \quad \text{where} \quad w = u + iv \in \mathbb{H} \tag{1.12}$$

The real axis is excluded from hyperbolic half-space. Any arc along the real axis in hyperbolic halfspace would have infinite length. Between any pair of points on the real axis, however, one can define a unique geodesic formed as a semicircle.

The reason why the hyperbolic metric is good to work with is of course that the unit disc is the covering space for many Riemann surfaces. In addition, there is a result which we shall use later on,

**Lemma 1.17** (SCHWARTZ's **lemma**)
*Any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ has $|f'(0)| \leq 1$.*

The version we shall use is sometimes called Pick's theorem.

**Theorem 1.18** (PICK's **theorem**)
*If $f : S \to T$ is a holomorphic map between hyperbolic Riemann surfaces $S$ and $T$ with hyperbolic metrics $d_S$ and $d_T$ respectively, then*

$$d_T(f(z_1), f(z_2)) \leq d_S(z_1, z_2) \tag{1.13}$$

*If equality holds, then $f$ is a covering map.*

Proofs of these theorems are in [Mil91] and [CG93].

*Some authors use the metric $ds = \frac{|dz|}{1-|z|^2}$, and then the curvature of the space is $-4$.

## 1.8 Euler characteristic

The Euler or Poincaré characteristic of a surface is deciding of the type of covering maps which exist. The characteristic is decided by the geometry or rather the topology class of the surface. A **triangulation** is a division of the surface into triangles, where one of the edges can be the edge of the surface if it is a plane surface. The disc, for example, can be triangulated with pie wedges, and the Euler characteristic is

$$\chi = F - E + V \tag{1.14}$$

where $F$ is the number of faces (triangles); $E$ is the number of edges; and finally, $V$ is the number of vertices. The trivial triangulation of the disc is obtained by simply letting three arbitrary points on the boundary be vertices and considering the intermediate arcs as edges. This gives one face, three edges and three vertices. Hence, the Euler characteristic is $\chi = 1-3+3 = 1$. Dividing the disc into three wedges, we get the same characteristic $\chi = 3 - 6 + 4 = 1$, which is unique for the disc and the topologically equivalent surfaces.

The relationship between the Euler characteristic and the **genus** $g$ of a compact surface is

$$g = \frac{2-\chi}{2} \tag{1.15}$$

The genus is only defined for compact surfaces, however.

## 1.9 Ramified covering maps

When we have the theory of covering maps then we can prove that the degree of the covering map has something to do with the number of critical points of a map on the domain being covered.

**Theorem 1.19**
*Suppose $f : X \to Y$ is a non-constant holomorphic mapping between Riemann surfaces. Let $a \in X$ and $f(a) = b \in Y$. Then there exists an integer $k \geq 1$, neighbourhoods $U$ and $U'$ around $a$ and $b$ respectively, neighbourhoods $V$ and $V'$ in $\mathbb{C}$, and charts $\varphi : U \to V$ and $\psi : U' \to V'$ such that*

1. $\varphi(a) = 0$ and $\psi(b) = 0$.
2. $f(U) \subseteq U'$.
3. $\psi \circ f \circ \varphi^{-1} : V \to V'$ *maps* $z \mapsto z^k$ *for all* $z \in V$.

**Proof From [For81].**
Properties 1. and 2. are easily fulfilled by careful choices of $(U', \psi)$ and $(U_1, \varphi_1)$ (replacing $(U, \varphi)$), because $X$ and $Y$ are Riemann surfaces and $f$ is holomorphic. The function $F_1 = \psi \circ f \circ \varphi_1^{-1} : V_1 \to V'$ then has the property $F_1(0) = 0$ by construction, and can be written $F_1(z) = z^k g(z)$ if $g$ is some non-constant holomorphic function on $V$ with $g(0) \neq 0$. Then there exists a function $h$ satisfying $h^k = g$ in some neighbourhood of 0. Let $\alpha$ be the map $z \mapsto zh(z)$. $\alpha$ is a biholomorphic mapping from a neighbourhood $V_2 \subseteq V_1$ of 0 into a neighbourhood $V$ of 0. Now let $U = \varphi_1^{-1}(V_2)$. Then by using $\varphi = \alpha \circ \varphi_1 : U \to V$ we obtain the map $F = \psi \circ f \circ \varphi^{-1} : V \to V'$ satisfying $F(z) = z^k$ by construction. □

A **proper map** is a continuous map by which the inverse image of any compact set is a compact set. A map $f : X \to Y$ between Riemann surfaces is a **ramified** or **branched covering map** if every point of $Y$ has a connected neighbourhood $U$ so that each connected component of $f^{-1}(U)$ maps onto $U$ by a proper map. The **degree** $d$ of a mapping $f$ at a point $z$ is a mapping from the class of rational functions, deg: $\mathcal{R} \to \mathbb{N}_0$ to the positive integers and zero, and represents the number of inverse images of $w = f(z)$, counted with multiplicity. We also say that $w$ is taken with multiplicity $d$. A proper holomorphic map has finite degree at any point. The importance of proper maps is evident from the next theorem, which asserts that proper maps have the curve lifting property. For proof, consult [Ste93].

**Theorem 1.20**
*Suppose that $f : \mathbb{D} \to S$ is a proper map.*

1. *Every arc $\gamma$ avoiding critical values of $f$ has a lift $\Gamma$ in $\mathbb{D}$ which is uniquely determined by its initial point.*

2. *Every local inverse of $f$ can be analytically continued along any path in $S$ avoiding critical values of $f$*

Exactly which types of domains can be mapped to each other is determined by the Riemann-Hurwitz formula (1.16)

**Theorem 1.21**
*Suppose that $f$ is a proper map of degree $d$ mapping a domain $X$ of Euler characteristic $\chi(X)$ onto a domain $Y$ of Euler characteristic $\chi(Y)$, and $f$ has $n$ critical points $n_1, n_2, \ldots, n_c$ in $Y$ (counted with multiplicity), then*

$$\chi(X) = d\chi(Y) - \sum_{i=1}^{n}(deg_{c_i}(f) - 1) \tag{1.16}$$

**Proof**

Any point in $Y$ which is not a critical value has $d$ distinct preimages in $X$. Assume that $Y$ is triangulated such that all critical values of $f$ are vertices. Then each face of the triangulation in $Y$ has $d$ preimages in $X$ so the triangulation of $X$ by the preimages of the triangulation of $Y$ has $F(X) = dF(Y)$ faces, and likewise the number of edges of this triangulation in $X$ are $E(X) = dE(Y)$. Let $c$ be a critical point mapping to a critical value $v$ with multiplicity $m$, meaning that $f$ maps $d - m$ points of $X$ to $v$. Then there must be $d - m$ vertices in $X$, mapping to the one vertex $v$ in $Y$, and similarly for other critical values. Non-critical vertices in $Y$ correspond to $d$ vertices in $X$, and the Euler characteristic of $X$ is then easily calculated to what is given in (1.16). □

For a proper map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree $d$, the number of critical points $\{c \in \mathbb{C} : f'(c) = 0\}$ is found by calculation from the Riemann-Hurwitz relation, and since $\chi(\mathbb{C}) = 2$, we find the number of critical points of $f$, counted with multiplicity to be

$$\sum_{i=1}^{n} (\deg_{c_i}(f) - 1) = 2d - 2 \tag{1.17}$$

The Euler characteristic of a surface is a **topological invariant** which is conserved under any homeomorphism.

# 2 Complex Dynamics

The fascinating part about complex dynamics is that seemingly innocent maps can have an intricate behaviour, when we consider the iterates of the maps. One of the first discovered examples of this type is the "NEWTON's method", which is a numerical method of finding zeroes of differentiable functions, but which doesn't always work because of some chaotic behaviour in some regions, both with real and complex functions. This is often called CAYLEY's problem, but we shall not go into further detail here. For details, consult [Ale94] and [DJH+98]. Another example is that of the Mandelbrot set on page 7, which we shall look at in section 3.3. We will consider the problems of dynamics of quadratic functions on complex numbers. Rational dynamics is a cornerstone in this thesis, and we shall examine many details in this theory.

The first real breakthroughs in the area of rational iteration were the work of PAUL MONTEL (1876 – 1975), GASTON JULIA (1893 – 1978) and PIERRE FATOU (1878 – 1929). Montel's notion of normal families was first presented in 1901 and the work of Julia and Fatou commenced immediately before the first world war.

## 2.1 Preliminary Dynamics

We will in the following freely use any of the two terms map or function, even though some authors use the term function only for maps $f : \mathbb{C} \to \mathbb{C}$. The dynamics we are interested in is the behaviour of some point under **iteration**. Iteration is the process of mapping consecutive images by the same function. The iterates of a point $z$ by a function $f$ are the points

$$z, f(z), f(f(z)), \ldots$$

and this sequence of iterates of $z$ is called the **orbit** of $z$. We will write the composition of the function $f$ with itself as follows

$$\overbrace{f \circ f \circ f \circ \ldots \circ f}^{n}(z) = f^n(z) \tag{2.1}$$

this will also be called the **n'th iterate** of $z$ (by $f$). A **fixed point** is a point $z$ such that $f(z) = z$. A periodic point of period $p$ for $f$ is a point such that $f^p(z) = z$, and therefore $z$ is a fixed point of the function $f^p$. If it is an attracting fixed point, $z$ is called an attracting periodic point of $f$. Preimages of a periodic or fixed point are called **preperiodic points.**

Let's now turn to the dynamics of the simple linear map $g(z) = \lambda z$ where $\lambda \in \mathbb{C}$ is called the **multiplier** of $g$. Iteration of $g$ is the process

$$z \mapsto \lambda z \mapsto \lambda^2 z \mapsto \ldots \mapsto \lambda^n z \mapsto \ldots$$

The point 0 is a fixed point for any value of $\lambda$. What happens to the rest of the points $z$ in the complex plane under iteration of $g$ is strongly dependent on the parameter $\lambda$ and falls in the following categories:

1. When $|\lambda| > 1$, all iterates of $z \neq 0$ will tend to infinity as the process goes on, and we then call 0 a **repelling** fixed point (section 2.3).

2. If $|\lambda| < 1$ the iterates of all points $z \in \mathbb{C}$ will tend to zero, and we call 0 an **attracting** fixed point (section 2.3).

3. If $|\lambda| = 1$ and $\lambda = e^{i2\pi\theta}$, where $\theta$ is irrational, the family of points $\{\lambda^n\}_{n\in\mathbb{N}}$ is dense on the unit circle, and a point $z_0$ will have an infinite orbit of iterates all lying on the circle $z = |z_0|$. In this case, 0 is called an **irrationally neutral** fixed point.

4. When $|\lambda| = 1$ and $\lambda$ is a root of unity, the iteration process will give periodic orbits since for some $n$, $\lambda^n z = z$. In the general case of a rational function $f^n(z) \neq z$, and there will exist attracting and repelling directions emanating from the fixed point. It is then called a **rationally neutral** or **parabolic** fixed point (section 2.4).

The map $z \mapsto \lambda z$ is a good place to start (except in case 4), since it is as simple as they get, but still is strongly dependent on the value $\lambda$. Moreover, we will now show that many functions have the same dynamics as $g(z) = \lambda z$ in some domains.

## 2.2 Linearizing analytic functions

Given a function, we will try to examine the global dynamics, that is, the behaviour of the function in all of the domain of definition. It is reasonable to start examining the behaviour of local dynamics, for instance at fixed points.

We start by noting that any analytic mapping $f: \mathbb{C} \to \mathbb{C}$ can be expressed in a series expansion around a fixed point $z_0$*

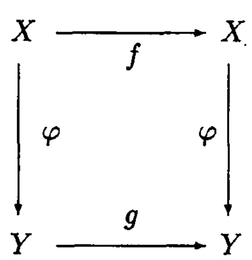
$$f(z) = z_0 + f'(z_0)(z - z_0) + \mathcal{O}((z - z_0)^2) \tag{2.2}$$

The Russian mathematician KŒNIGS proved about this familiar class of functions that in a neighbourhood of any fixed point having $f'(z_0) = \lambda \notin \{0, 1\}$, the dynamics of the functions is equivalent to the dynamics of the simple linear mapping $g : z \mapsto \lambda z$ in the following sense:

A **linearizing** map is a map $\varphi: X \to Y$ where $X$ and $Y$ are neighbourhoods in $\mathbb{C}$ of $z_0$ and $0$ respectively and $\varphi$ satisfies:

$$\varphi(f(z)) = g(\varphi(z)) \quad \text{or} \quad g = \varphi \circ f \circ \varphi^{-1} \tag{2.3}$$

This is also referred to as SCHRÖDER's functional equation after the German mathematician who was the first to introduce this conjugation. Another way of putting the linearization condition is that we want to find a map $\varphi$ so that the following diagram commutes:




A linearizing map must satisfy that it sends $z_0$ to 0. The fixed point $z_0$ can be sent to the origin by first conjugating $f$ by the affine map $\psi_0 = z - z_0$ to

$$h(z) = \lambda z + \mathcal{O}(z^2) \tag{2.4}$$

We can always perform this affine conjugation and keep dynamics intact, so we can always assume that fixed points are placed at the origin.

We can then write the linearizing map $\varphi$ which conjugates $f$ of equation (2.2) to $g(z) = \lambda z$ as $\varphi = \psi_1 \circ \psi_0$ where $\psi_1$ is the map which conjugates $h$ to $g$.

*The notation $\mathcal{O}(z^2)$ means that there exists a radius $R$ and a constant $K$ such that

$$|z| < R \;\Rightarrow\; |f(z) - \lambda z| \leq K|z|^2.$$

## 2.3 Attracting fixed points

We assume now that we have a fixed point at the origin, so the series expansion can be written

$$f(z) = \lambda z + z^2 \sum_{n=0}^{\infty} a_{(n+2)} z^n \tag{2.5}$$

where the tail is a power series and has positive radius of convergence.

**Proposition 2.1**
*If* $0 < |\lambda| < 1$, *then on an open disk of radius* $r$, $f\colon D(r) \to \mathbb{C}$ *is a contraction, and points in* $D(r)$ *will converge to the fixed point 0 under iteration by* $f$.

**Proof**
To show this, we first note that we can rewrite equation (2.2)

$$|f(z) - \lambda z| < K|z|^2 = K|z| \cdot |z| \tag{2.6}$$

Given any $\varepsilon > 0$ we can choose a radius $r$ rendering $K|z| < \varepsilon$ for $|z| < r$. This gives us

$$|f(z) - \lambda z| < K|z| \cdot |z| < \varepsilon |z| \tag{2.7}$$

Now choose a number $S$, so that $0 < |\lambda| < S < 1$, and denote the number $S - |\lambda| = \varepsilon$. Then we have

$$|f(z)| < |\lambda||z| + \varepsilon |z| = S|z| < |z| \tag{2.8}$$

That $f^n(z)$ converges to 0 clearly follows from the contraction. □

We also deduce that if the fixed point is $z_0$, and $|f'(z_0)| = |\lambda| < 1$, $f$ is a contraction in a neighborhood of $z_0$ and therefore one-to-one in the neighbourhood. This follows from the fact that 0 is a fixed point of multiplicity 1 and then theorem 1.19 states that in a neighbourhood of the origin, $f$ has degree 1 and therefore is a proper map. If 0 had multiplicity $k \geq 2$, then 0 would have $|\lambda| = 0$ and be called a *superattracting* fixed point.

Any point that converges toward $z_0$ under iteration by $f$ is said to be in the **basin of attraction** of $z_0$. The **Immediate basin of attraction** is the connected component of the basin of attraction which contains the fixed attracting point. In some cases there also exists preimages of the immediate basin which are then also components of the basin of attraction since any point in a preimage of the immediate basin of attraction maps eventually into the immediate basin and from there converges to the attracting fixed point. The basins opf attraction which do not contain the fixed point contain a preimage of the fixed point whenever $f$ is a rational function.

We wish to show as in [CG93] that

**Proposition 2.2**
*There exists a $\varphi$ that linearizes all iterates of $f$ in the immediate basin of attraction of a fixed point.*

**Proof**
Define the mapping

$$\varphi_n(z) = \frac{f^n(z)}{\lambda^n} \tag{2.9}$$

For all $n$, this inherits from $f$ the property of being a holomorphic, one-to-one map $\varphi_n : D(r) \to \mathbb{C}$, with $\varphi_n(0) = 0$ and $\varphi_n'(0) = 1$. Consider the composite of $\varphi_n$ and $f$

$$\varphi_n \circ f = \frac{f^n \circ f}{\lambda^n} = \lambda \frac{f^{n+1}}{\lambda^{n+1}} = \lambda \varphi_{n+1} \tag{2.10}$$

If it is true that $\varphi_n \to \varphi$, then $\varphi$ would be a chart linearizing $f$ in the neighborhood, and then equation (2.10) implies

$$\varphi \circ f(z) = \lambda \varphi(z). \tag{2.11}$$

For the proof of the existence of such a $\varphi$, we will use that the supremum metric on functions on a compact set completes the function space such that any Cauchy sequence converges, and find that $\varphi_n$ in this space is a Cauchy sequence:

We define the function space $C(K, \mathbb{C})$

$$C(K, \mathbb{C}) := \{f : K \to \mathbb{C} \mid f \text{ is continuous}\} \tag{2.12}$$

And equip it with the supremum metric, such that given two functions $f$ and $g$ in $C(K, \mathbb{C})$,

$$d(f, g) = \|f - g\|_\infty \tag{2.13}$$

From calculus we know that the resulting metric space $C_\infty(K, \mathbb{C})$ has the following property.

**Lemma 2.3**
*Let $K$ be a compact set. The metric space $C_\infty(K, \mathbb{C})$ is complete.* □

It is therefore sufficient to prove that $\{\varphi_n\}_n$ is a Cauchy sequence on a compact space, which we will take to be the closed disc $\overline{D(r)}$. It then follows from the lemma that $\varphi_n$ converges.

Consider the difference

$$|\varphi_{n+1} - \varphi_n| = \left| \frac{f(f^n(z)) - \lambda f^n(z)}{\lambda^{n+1}} \right| \leq \frac{(K|f^n(z)|)^2}{|\lambda|^{n+1}} \tag{2.14}$$

$f$ is a contraction, so we can find an $S$ such that $0 < |\lambda| < S < 1$

$$\frac{(K|f^n(z)|)^2}{|\lambda|^{n+1}} \leq \frac{K^2 S^{2n}}{|\lambda||\lambda^n|} \tag{2.15}$$

We may as well assume $S < \sqrt{|\lambda|}$ such that when $n \to \infty$,

$$|\varphi_{n+1} - \varphi_n| \leq \frac{K^2}{|\lambda|}\left(\frac{S^2}{|\lambda|}\right)^n \to 0 \tag{2.16}$$

which implies that $\varphi_n$ is a Cauchy sequence. □

Contrary to the attracting fixed points, the repelling fixed points where $|\lambda| > 1$, such as 1 for the function $f : z \mapsto z^2$, are unstable fixpoints. Any point not quite of modulus 1 will end up at zero or infinity under iteration by $f$, so we can find no iterative subsequence which is uniformly convergent in a small neighbourhood of 1. Linearization, however, is still possible, since the repelling fixed point is attracting for any branch of $f^{-1}$: If $D$ denotes the differentiation operator, then the determinant is $\det(Df) = |\lambda|^2 > 1$ and any inverse branch will have $\det(Df^{-1}) = 1/|\lambda|^2 < 1$, so the repelling fixed point for $f$ will be attracting for $f^{-1}$.

## 2.4 Parabolic Fixed Points

A parabolic fixed point arises when the multiplier $\lambda$ is some root of unity, such that $\lambda^q = 1$, but $f^q \neq \text{id}$. Then there will be directions in which the periodic or fixed point will be attracting and other where it will be repelling. Around attracting directions, sets which map into themselves exist. We can define an **attracting petal** $U$ as an open set with compact closure $\overline{U}$ which satisfies:

$$f(\overline{U}) \subseteq U \cup \{0\} \quad \text{and} \quad \bigcap_{k \geq 0} f^k(\overline{U}) = \{0\} \tag{2.17}$$

A **repelling petal** for $f$ can be defined as an attracting petal for $f^{-1}$. The attracting petals make up the basin of attraction of the fixed point, and the LEAU-FATOU **flower theorem** describes the dynamics near the fixed point

**Theorem 2.4 (Leau-Fatou)**
*If the origin is a fixed point of multiplicity $n+1$, then there are $n$ disjoint attracting petals alternating with $n$ disjoint repelling petals, each attracting petal intersecting the two adjacent repelling petals. The union of these $2n$ petals together with the origin itself form a neighbourhood of the origin.*

A proof of this theorem is found in [Mil91]. The figure 2.1 depicts the construction.

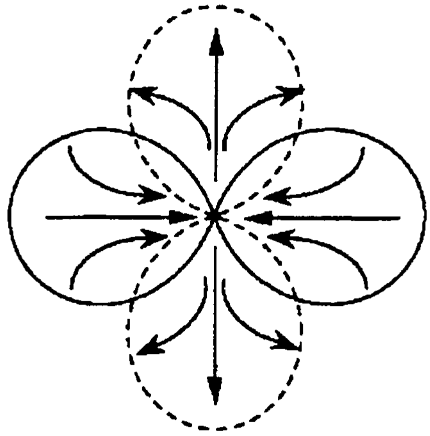


**Figure 2.1** The Leau-Fatou flower for a parabolic fixed point of degree 3 (multiplicity 2). Two attracting petals and two repelling petals are shown. From [Ale94].

Any point in a small neighbourhood of the parabolic fixpoint is both in an attracting petal and in a repelling petal. The actual orbit of the point under iteration is dependent on the distance to the repelling and attracting axes. If it is close to a repelling axis, it may escape, or it may map further away only to return along or close to an adjoining attracting axis.

## 2.5 Automorphisms of $\overline{\mathbb{C}}$

Returning to the dynamics of our function $z \mapsto \lambda z$, we find that if $|\lambda| \neq 1$, the iterates converge uniformly to either 0 or $\infty$, but if $|\lambda| = 1$ the whole complex plane is rotated by $f$ around the origin. Here we have an example where it is convenient to consider the compactification of $\mathbb{C}$, the extended complex plane: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, also called the Riemann sphere, since we then can consider the mapping a rotation of the sphere (like a globe) with the two fixed points 0 and $\infty$ as poles. Also, we can consider the point $\infty$ a fixed point of the mapping $z \mapsto \lambda z$ in exactly the same way as 0.

We are in general interested in linearizing maps of higher degree, and we want to use conjugation to send interesting dynamics to a neighbourhood of the origin, so we will now treat an example where we see that we can move around with our functions and still keep the dynamics.

Polynomials have the property that they fix infinity. General rational functions of degree two are fractional quadratic functions

$$f(z) = \frac{az^2 + bz + c}{a'z^2 + b'z + c'} \tag{2.18}$$

which map the Riemann sphere twice onto itself with three fixed points, counting multiplicity (one is placed at $\infty$). The automorphisms or bijections of the Riemann sphere $\overline{\mathbb{C}}$ are the Möbius transforms.

**Definition 2.5**
*A* Möbius transform *is a* fractional linear *transform, that is, a transform of the form:*

$$M(z) = \frac{az+b}{cz+d} \tag{2.19}$$

*with* $a, b, c, d \in \mathbb{C}$.

These are also known as the regular linear transformation matrices for rotation and displacement in $\mathbb{R}^2$:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.20}$$

with $a, b, c, d \in \mathbb{R}$. These transforms will be key elements in the rest of this work since conjugation with a Möbius transform can send any two points of the extended complex plane $\overline{\mathbb{C}}$ to the origin and infinity respectively. For example the two points which are fixed by the fractional quadratic map (2.18). Since the Möbius transforms are isomorphisms of $\overline{\mathbb{C}}$, they conserve the dynamics of a function defined on $\overline{\mathbb{C}}$, and therefore we can treat dynamics for classes of functions which are conjugate by Möbius transforms as one case instead of investigating each function explicitly.

For any polynomial of degree 2, it is especially simple to conjugate to $f_t(z) = z^2 + t$ using a subgroup of the Möbius transforms, the affine maps.

**Proposition 2.6**
*Given any polynomial* $P: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$P(z) = az^2 + bz + c \tag{2.21}$$

*the affine map* $A = az + {}^b\!/_2$ *conjugates* $P$ *to* $f_t(z) = z^2 + t$*, where* $t = ac + {}^b\!/_2 - ({}^b\!/_2)^2$.

**Proof**
Consider the conjugacy condition

$$f_t \circ A = A \circ P \tag{2.22}$$

Using the definition of the mappings this becomes

$$(az + {}^b\!/_2)^2 + t = a(az^2 + bz + c) + {}^b\!/_2 \tag{2.23}$$

expanding and solving for $t$ we get the desired result. □

The dynamics of the function $f_t$ and thereby the dynamics of degree two polynomials are treated in section 3.3.

## 2.6 Superattracting fixed points

We saw that the dynamics of a polynomial of degree 2 was conjugate to the dynamics of $z^2 + t$. More generally, we wish to know the dynamics of polynomials with critical points of higher multiplicity, that is, with roots of order $d > 1$. Assuming again that the critical point is at the origin, the expansion for $f$ is of the form

$$f(z) = z^d + \mathcal{O}(z^{d+1}) \tag{2.24}$$

As in the case of the linearizable maps (p. 27), it is possible to conjugate any map of the form (2.24) to the map $z \mapsto z^d$, a result due to the Polish mathematician L.E. BÖTTCHER. We call the number $d$ the degree of the map $f$, since it maps the unit disc onto itself $d$ times, except at the critical point, also called the **ramification point**†. We may visualize the image space of the map as a collection of $d$ unit discs joint at the common point $z = 0$, spiralling one into the other. The unit disc is then called the ramified covering of this surface. Everywhere, except at the removable singularity $z = 0$, there is a locally one to one correspondence between the cover and the Riemann surface. The action of the map $z \mapsto z^d$ on a point on the unit circle is that of multiplying the argument of the point by $d$. The points on the unit circle which satisfy $(d^n - 1)\arg(z) = p2\pi$ for some $n$ and $p$ are thus repelling periodic points of period $n$ for the map, and 1 is a repelling fixpoint. Furthermore, $z^d$ maps a pie wedge of angle $2\pi/d$ onto the whole disc.

Just as in the case of $z \mapsto \lambda z$, the origin and $\infty$ are fixed points for $z \mapsto z^d$. All points in the open unit disc are eventually mapped arbitrarily close to 0 under iteration, and all points not in the closed unit disc escape to infinity under iteration, hence both 0 and $\infty$ are attracting fixed points. In the case where $f$ is of degree $d$ and is conjugate to $z^d$, the multiplier $\lambda$ is zero and we call the fixed point superattracting.

†For an equivalent and more formal definition refer to p. 22.

# 3 The Dynamical Dichotomy

As we encountered in the cases of the functions $z \mapsto z^2$ and $z \mapsto z^d$, there can exist basins of attraction where the iteration contracts the domain such that all points converge to the fixed point, or alternatively sends points to an attracting basin. The domains where the function has this nice behaviour are called **Fatou components** for the function.

## 3.1 Normal Families

The notion of normality, which is due to PAUL MONTEL, is defined as follows.

**Definition 3.1**
*A family $\mathfrak{F}$ of functions defined on a domain $U$ is called* **normal family** *if for every subfamily there exists a locally uniformly convergent subsubfamily.*

We are interested in the situation where we are dealing with the repeated composition of one function with itself as the family of functions, and in that case we can define

**Definition 3.2**
*A domain of normality for a function $f$ is a domain $U$ in which for every subsequence of the sequence of iterates, $\{f^n|_U\}_{n\in\mathbb{N}}$ there exists a subsubsequence which converges locally uniformly.*

For families of holomorphic functions it is a necessary and sufficient condition for normality that the family be uniformly bounded, which is the statement of Montel's little theorem:

**Theorem 3.3 (Montel's little theorem)**
*Let $U \subseteq \mathbb{C}$ be a domain and let $\mathfrak{F}$ be a family of analytic functions on $U$. If $\mathfrak{F}$ is uniformly bounded, then $\mathfrak{F}$ is a normal family.*

An immediate corollary of Montel's little theorem is:

**Theorem 3.4**
*The family of all analytic maps from the unit disc to itself is normal.*

The disc endowed with the hyperbolic metric is the covering surface of hyperbolic surfaces. The "big" theorem of Montel says that as long as the family of maps omits at least three points of the Riemann sphere, then the family is normal. In the theory of covering surfaces we have mentioned that the universal cover of the thrice-punctured sphere is the disc, so the little theorem of Montel is a corollary of the big theoremand the covering statement. As you can see, Montel's theorems are powerful tools for categorizing the behaviour of functions on domains.

**Theorem 3.5 (Montel's big theorem)**
*The family of analytic maps from any domain $U$ into the thrice-punctured Riemann sphere $\overline{\mathbb{C}} \setminus \{a, b, c\}$ is normal.*

Note that the points $a, b, c$ can be placed anywhere on the sphere as long as they're distinct ($a \neq b \neq c$). Möbius transforms can move them around arbitrarily, and composition with Möbius transforms does not change analyticity.

**Proposition 3.6**
*If $U$ is a domain in $\overline{\mathbb{C}}$ which omits at least three points, then the universal cover of $U$ is the disc $\mathbb{D}$*

The theorem of Picard deals with this type of domain:

**Theorem 3.7**
*Any holomorphic map from $\mathbb{C}$ to $\overline{\mathbb{C}}$ which omits at least three distinct values is constant.*

Which follows from Liouville's theorem and the proposition 3.6.

## 3.2 The Fatou set

For $z \mapsto z^d$ then, the interior and the exterior of the closed unit disc belong to the domain of normality. These domains are also called **Fatou components.** Any basin of attraction or preimage of a basin of attraction is a Fatou component. The union of all Fatou components forms the Fatou set which has some nice properties. Given a mapping $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, we define

**Definition 3.8**

1. *A point $z$ is called* **normal** *(with respect to $f$) if the sequence of iterates $\{f^n\}$ is normal in some neighborhood of $z$.*

2. *The* **Fatou set** *$\mathcal{F}_f = \mathcal{F}$ is the set of all points $z \in \overline{\mathbb{C}}$ that are normal with respect to $f$.*

3. *The* Julia set $\mathcal{J}_f = \mathcal{J}$ *is the complement of the Fatou set:* $\mathcal{J} = \overline{\mathbb{C}} \setminus \mathcal{F}$.

An alternative definition of the Fatou set used in [Bea91b] involves the notion of **equicontinuity**. A family $\mathfrak{F}$ of functions between metric spaces $X$ and $Y$ is equicontinuous at $x_0$ if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ and for all $f \in \mathfrak{F}$

$$d_X(x_0, x) < \delta \Rightarrow d_Y(f(x_0), f(x)) < \varepsilon$$

The Fatou set of $f$ can then be defined as the maximal open set for which the family of iterates of $f$ is equicontinuous.

These two sets are named after the french mathematicians PIÈRRE FATOU and GASTON JULIA who explored the properties of these sets. We shall also examine these sets closely since they are important in any iterative setting. Let's first look at an example.

## 3.3 The Mandelbrot family

Since we started this thesis talking about the Mandelbrot set and we are dealing with quadratic maps in general, we should probably take a look at the quadratic family $f_t(z) = z^2 + t$.

The Mandelbrot set $\mathcal{M}$ is defined as the set of parameter values $t$ such that iteration of $f_t(0)$ does not escape to infinity.

In fact, it can be shown that

$$\mathcal{M} = \{t : |f_t^n(0)| \leq 2 \ \forall n \in \mathbb{N}\} \tag{3.1}$$

This means that we can compare norm of iterates with 2 as an easy bailout test for creating computer images of the Mandelbrot set such as the one in the preface.

For any polynomial, $\infty$ is a superattracting fixed point, and consequently the immediate basin of attraction $A(\infty)$ is in the Fatou set and has as its boundary the Julia set of the polynomial. Furthermore, Böttcher's theorem states that $A(\infty)$ is either simply connected or of infinite connectivity. This means that there are two possibilities for the Julia set for polynomials and thereby for $f_t$. It is either connected or consists of infinitely many components. A simple test of which case applies is

$$0 \notin A(\infty) \iff A(\infty) \text{ simply connected}$$

Hence, the Mandelbrot set contains the parameter values $t$ for which the Julia set is connected. The proof is omitted.

**Figure 3.1** Julia set for $t = 0.25$. The "cauliflower".

As an example of a conected Julia set, we have the below figure, the Julia set at $t = 0.25$ (on the boundary of the Mandelbrot set) where the dark part is the basin of attraction of 0 together with the Julia set and the white region is $A(\infty)$. For $t = 0.26$, outside the Mandelbrot set, 0 escapes to infinity under iteration by $f_t$, and then $\mathcal{F}$ has infinite connectivity, and $\mathcal{J}$ therefore consist of infinitely many disjoint components, which is indicated in the figure below where the black points is the Julia set of $f_{0.26}$. It consists of a Cantor set of points.

Another example of the connected Julia set is for $t = i$, known as a **dendrite**.

## 3.4 Properties of the Fatou and Julia sets

There are a number of properties of the two sets that need to be mentioned. From the definition of the Fatou set, it is clear that it is an open set. The Julia set is then closed, and since $\overline{\mathbb{C}}$ is compact, the Julia set is compact also. Moreover, if we take any domain $D$ which intersects the Julia set, we know that $\{f^n\}$ is not normal on $D$. If we form the union set,

$$\bigcup_{n \in \mathbb{N}} f^n(D),$$

then Montel's theorem (Theorem 3.5) tells us that this set can omit at most two points of the Riemann sphere. Hence the Julia set displays the chaotic characteristic that arbitrarily close initial points can end up arbitrarily far from each other.

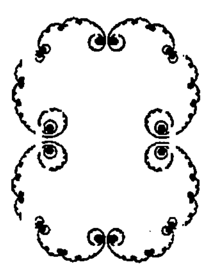


**Figure 3.2** Julia set for $t = 0.26$.

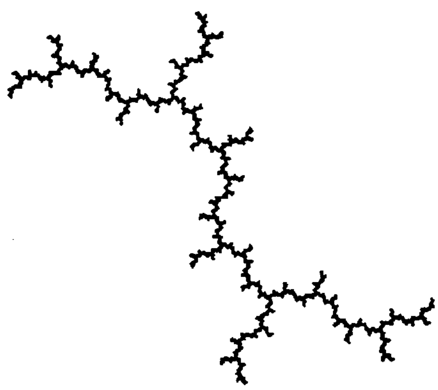


**Figure 3.3** Julia set for $t = i$, a dendrite.

Another reason why the Julia and Fatou set are so interesting in iteration theory is that they are invariant under iteration

**Proposition 3.9**
*The Fatou set and the Julia set of $f : \mathbb{C} \to \mathbb{C}$ are* **completely invariant**, *that is,*

$$f^{-1}(\mathcal{F}) = \mathcal{F} = f(\mathcal{F}), \quad \text{and} \quad f^{-1}(\mathcal{J}) = \mathcal{J} = f(\mathcal{J}). \tag{3.2}$$

**Proof**
It is only necessary to prove that $\mathcal{F}$ is completely invariant. The identity $\mathbb{C} = \mathcal{F} \cup \mathcal{J}$ then ensures that also $\mathcal{J}$ is completely invariant.

Let $D \subseteq \overline{\mathbb{C}}$ be any domain and let $\Delta$ be any component of $f^{-1}(D)$. Assume $D \subseteq \mathcal{F}$. That means that there exists a subsubsequence $\{f^{n_{k_j}+1}\}$ which converges in $D$, let's say that $f^{n_{k_j}+1} \to g$ on $D$ as $n_{k_j}+1 \to \infty$. Then $f^{n_{k_j}} \to g \circ f^{-1}$ on $\Delta$ as $n_{k_j} \to \infty$. Then there exists a function $h : \Delta \to \mathbb{C}$ defined by $h \circ f = g$ such that $f^{n_{k_j}} \to h$ on $\Delta$, which proves that $\Delta \subseteq \mathcal{F}$.

Conversely, a similar argument shows that if $\{f^{n_{k_j}-1}\}$ converges, then so does $\{f^{n_{k_j}}\}$, such that $\Delta \subseteq \mathcal{F}$ implies that $D \subseteq \mathcal{F}$. □

Since $f(\mathcal{F}) = \mathcal{F}$, we also have $f^n(\mathcal{F}) = \mathcal{F}$, and recalling the definition of the Fatou set we see that

**Corollary 3.10**
*All iterates of $f$ have identical Julia and Fatou sets.*

Another way of defining the Julia set that perhaps more directly shows us what the set contains is the following identification, which is valid for any polynomial $f$, and where we use the immediate basin of attraction of $\infty$.

$$\mathcal{J} = \partial A(\infty) \qquad \text{with} \quad A(\infty) = \{z \in \overline{\mathbb{C}} \mid n \to \infty \Rightarrow f^n(z) \to \infty\} \tag{3.3}$$

That means that if there exists repelling periodic points, that is points where $f'(z) > 1$ such that a nearby point is sent towards infinity, then the Julia set contains these points. In figure 3.1 then, the Julia set is the boundary of the black set, and all of the interior of the set is $A(0)$.

The following is a characterization of the Julia set which is equivalent to the ones we have seen until now [MR91]. *The Julia set is the set of accumulation points of $\cup_n f^{-n}(z)$ for almost all $z \in \overline{\mathbb{C}}$.* Or equivalently, The Julia set is the attracting set for $f^{-1}$. Note that $f^{-1}$ is a multivalued map and therefore iteration of the Julia set using this method involves a choice for each inverse image. This involves no practical difficulty, since a random choice of inverse image does the trick.

One of the early examples we have discussed is the map $f : z \mapsto z^d$. We know that the family $\{f^n\}_n$ is normal on the two domains $|z| < 1$ and $|z| > 1$, since any subsequence of maps from the family converge to the constant functions 0 and $\infty$ respectively. What is left over is the unit circle, and this is precisely the Julia set for this map. For instance, 1 is a fixed point of $z \mapsto z^d$, and the multiplier is $d1^{d-1} = d > 1$ so 1 is repelling, and it is easy to realize that any neighbourhood of 1 will be mapped to $\mathbb{C} \setminus \{0, \infty\}$ by iteration.

When we iterate a rational function we will inevitably bump into some chaotic-like behaviour, as is clear from the following theorem from [Ste93].

**Theorem 3.11**
*The Julia set is non-empty for any rational function $f$ of degree $deg(f) \geq 2$.*

**Proof**
Given a function $f$ of degree $d \geq 2$, assume the Julia set $\mathcal{J}_f$ is empty. Then any infinite subsequence of iterates $\{f^{n_k}\}_{n_k \in I}$ has a uniformly convergent subsequence $\{f^{n_{k_j}}\}_{n_{k_j}}$ on all of $\mathbb{C}$. From some $N$, all $f^{n_{k_j}}$ have the same degree as the limit function, which follows from the uniform convergence of the sequence and the continuity of the degree function (see [Bea91b] Theorem 2.8.2). However, $\deg(f^n) = (\deg(f))^n$, which implies that $\deg(f) = 1$, which contradicts the assumption. □

Moreover, the Julia set always has infinitely many points, so for any rational function there exists infinitely many points where an iteration sequence is not convergent. An important theorem we will not prove here (see e.g. [Bea91b]) but which relates to the big theorem of Montel, 3.5 is the following

**Theorem 3.12**
*The Julia set is the smallest closed set with at least three points which is completely invariant under the action of a rational function $f$.*

We have claimed that the Julia set is a chaotic set, and one of the properties of chaotic sets we mentioned in the preface, is that any point can be approximated arbitrarily close, and this is true about the Julia set.

**Proposition 3.13**
*The Julia set is perfect. That is, it has no isolated points.*

**Proof**
Let $\mathcal{J}_0$ be the set of accumulation points of $\mathcal{J}$. $\mathcal{J}$ is infinite and compact, so $\mathcal{J}_0$ is not empty. Since $f$ is continuous and of finite degree, $f(\mathcal{J}_0) \subseteq \mathcal{J}_0 \Rightarrow \mathcal{J}_0 \subseteq f^{-1}(\mathcal{J}_0)$. Since $f$ is also an open map, $f^{-1}(\mathcal{J}_0) \subseteq \mathcal{J}_0$. This shows that $\mathcal{J}_0$ is forward and backward invariant. From theorem 3.12 we then know that it consists of the whole Julia set $\mathcal{J}$. □

## 3.5 Fuchsian Groups

We will now look at the Fuchsian group theory of the dichotomy of the Riemann sphere, since the theory of correspondences can benefit from both theories, even though we have as starting point the rational iteration theory. A **topological group** is a group which is also a topological space. The topology of a group $G$ is induced by demanding that the maps $x \mapsto x^{-1} : G \to G$ and $(x, y) \mapsto xy : G \times G \to G$ are continuous maps.

Let $M$ be the group of isomorphisms of the Riemann sphere. As mentioned, these are the Möbius transforms of equation (4.28). In algebra, this group is known as the group $PSL(2, \mathbb{C})$. The Möbius transforms form the group generated by even numbers of reflections over lines and circles in $\mathbb{C}$.

We say that $G \subseteq M$, is **discrete** if every $g \in G$ is isolated with respect to the topology of the group $G$. That the group acts discretely means that there are no accumulation points for a sequence of elements from $G$. Discrete subgroups of $PSL(2, \mathbb{C})$ are called Kleinian groups.

Let $G \subseteq M$. We say that $G$ acts **discontinuously** on an open set $U \subseteq \overline{\mathbb{C}}$, if for all compact subsets $K$ of $U$:

$$\sharp\{g \in G: g(K) \cap K \neq \emptyset\} < \infty. \tag{3.4}$$

A group which acts discontinuously on an invariant disc in $\mathbb{C}$, is called a **Fuchsian group**. Fuchsian groups are subgroups of $PSL(2, \mathbb{R})$

The region of discontinuity of a Fuchsian group, $\Omega(G)$, is the set of all points $z$ in $\overline{\mathbb{C}}$ where $G$ acts discontinuously on a neighbourhood of $z$, and the complement $\Lambda(G) = \overline{\mathbb{C}} \setminus \Omega(G)$ is called the limit set of $G$. Here we have a division of the Riemann sphere into two disjoint sets just as we did in the case of the Julia and Fatou sets, and there is a very close connection between the two. The region of discontinuity of the group corresponds to the Fatou normality set of the elements of the group of Möbius transforms. Furthermore, there is a main theorem of uniformization connected to the Fuchsian groups, which we will not prove here.

**Theorem 3.14**
*Any hyperbolic Riemann surface $S$ has a Fuchsian group as deck transformation group $\Gamma$. In particular, $S = \mathbb{D}/\Gamma$.*

For detailed analysis of topological groups on Riemann surfaces, please refer to [Bea83] and [FK80].

A fundamental domain $F$ of the group $G$ is an open subset of the disc with the properties:

1. $\bigcup_{g \in G} g(\overline{F|_{\mathbb{D}}}) = \mathbb{D}$
2. $\forall g, h \in G, g \neq h : g(F) \cap h(F) = \emptyset$

so that every point in the disc is in the closure of some image of $F$, and no two images of the fundamental domain overlap. For more details, consult [Bea91a].

We call a Fuchsian group $G$ **geometrically finite** if there exists a fundamental domain $F$ of $G$, with finitely many sides. About a geometrically finite Fuchsian group we can say the following.

**Theorem 3.15**
*A Fuchsian group $G$ is geometrically finite if and only if $G$ is finitely generated.*

Again, we omit the proofs and refer to texts such as [FK80].

We will introduce a way to find the limit set by construction rather than exclusion, as we did earlier with the Julia set.

**Proposition 3.16**
*The set* $\Lambda(G) = \overline{G(0)} \setminus G(0)$ *is the limit set of* $G$.

This means that it is possible to have a computer generate the limit set of a Fuchsian group. The parallel between the limit set of the Fuchsian group and the Julia set is that where the Julia set is the complement to the set of points in $\mathbb{C}$ where the iterates of a function form a normal family, the limit set is the complement to the set of points in $\overline{\mathbb{C}}$ where the group forms a normal family. In an iterational setting however, we are not dealing with a function and a point, and finding backward images. We investigate how a set of points is treated by a group of functions which means that there is an actual choice between images in each step of the iteration. For practical purposes, the choice is made randomly. This is called *stochastic iteration.*

The Julia - Fatou set dichotomy is based on functions on Riemann surfaces, where the Fuchsian group limit - regular set dichotomy is based on group permutations, but in essence they treat the same behaviour difference. The Julia and limit sets are both sets of accumulation points, as can be seen from proposition 3.13 and from the definition of the limit set.

# 4 Dynamics of Quadratic Correspondences

The dynamics of quadratic correspondences have been treated by Shaun Bullett [Bul91] [Bul88], Christopher Penrose [BP98a], [BP98b], and Münzner and Rasch [MR91], and a review of the arithmetic-geometric mean and its history can be found in an article by David A. Cox [Cox84]. The quadratic correspondences have as a subset the polynomials of degree 2, so therefore we shall refer to the overview of the field of rational iteration, and give some indications of which theorems don't hold for the general case of quadratic correspondences. We will begin with an example of great importance.

## 4.1 The Arithmetic-Geometric Mean

Though the method has been known for very long, the first thorough investigation of the relationship between the arithmetic and the geometric mean values of complex numbers was made by Gauss. The arithmetic mean value between numbers $a$ and $b$ is half the sum,

$$M_a = \frac{a+b}{2} \tag{4.1}$$

and the geometric mean is the square root of the product.

$$M_g = \sqrt{ab} \tag{4.2}$$

That the arithmetic mean and the geometric mean usually do not coincide, we know from real numbers, and therefore the same goes for the moduli, when we use the two methods on two complex numbers. Geometrically, the arithmetic mean finds the midpoint on a straight line between the two numbers in the complex plane. The geometric mean of two complex numbers with same modulus $|z| = |w|$ is again a complex number with same modulus $\sqrt{|z||w|} = |z|$, while its argument is the mean value of the arguments of $z$ and $w$. Here the troubles begin, for the angle between two vectors has two distinct values (using angles modulo $2\pi$). We can no longer choose the "positive" root since they are in general complex. For instance, the geometric mean of 1 and $-1$ is $\sqrt{-1} = \pm i$, and there is no obvious reason for choosing one over the other.

If we define the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ recursively:

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \sqrt{a_n b_n} \tag{4.3}$$

where we set $a_0 = a$ and $b_0 = b$ with $a, b \in \mathbb{R}$, then the sequences converge rapidly to a common limit, $M(a, b)$, called the arithmetic-geometric mean (agm). This method gives exponentially fast, precise approximations, and has early been used to find expansions of $\pi$ and $\sqrt{2}$. When $a$ and $b$ are complex and we allow both values of the square root in equation (4.3), we cannot automatically rely on the convergence of the method, since we are faced with a two-valued map with a two-valued inverse: $(a_n, b_n) \mapsto (a_{n+1}, b_{n+1})$ from $\mathbb{C}^2 \to \mathbb{C}^2$, of which we can only be sure that one choice of value for each $n$ will make the sequences converge to a mutual limit. On the other hand, Gauss gave examples where, even with a number of "wrong" choices for the square root for some $n$, the sequences may still converge. In the following we will assume that $a \neq 0$, $b \neq 0$ and $a \neq \pm b$, since these values will make the limit the trivial $a$ or 0.

**Definition 4.1**
*Let $a, b \in \overline{\mathbb{C}}$, with the limitations above. The value $b_n = \sqrt{a_{n-1}b_{n-1}}$ is called the* **right choice** *if $|a_n - b_n| \leq |a_n + b_n|$.*

With the right choice of $b_n$ for all $n \in \mathbb{N}$ the sequences converge, but it is not necessary to make the right choice every time. In fact, we will prove that every pair of sequences which contain only a finite number of wrong choices is a convergent pair of sequences. To this end we define a "good" pair of sequences

**Definition 4.2**
*A pair of sequences $\{a_n\}$ and $\{b_n\}$ is called* **good** *if $b_{n+1}$ is the right choice for all, except finitely many, $n \in \mathbb{N}$.*

To rationalize the term "good pair" we will show the proposition of [Cox84] that any pair of sequences converges, but only the good sequences have a non-zero limit.

**Proposition 4.3**
*Let $a, b \in \mathbb{C}\setminus\{0\}, a \neq \pm b$. Any pair of sequences $(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$ defined as in (4.3) converges to a common limit, which is non-zero if and only if $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are good sequences.*

**Proof**

*Claim 1: Bad sequences tend to zero.* Let us define the sequence $M_n = \max\{|a_n|, |b_n|\}$. Notice that for the right choice of square root $b_{n+1} = \sqrt{a_n b_n}$,

$$|a_{n+1}-b_{n+1}||a_{n+1}+b_{n+1}| = |a_{n+1}^2-b_{n+1}^2| = \left|\frac{(a_n+b_n)^2}{4} - a_n b_n\right| = \frac{|a_n-b_n|^2}{4} \tag{4.4}$$

If $b_{n+1}$ is the right choice, then $|a_{n+1} - b_{n+1}| \leq |a_{n+1} + b_{n+1}|$, and we get from the equation above that $|a_{n+1} - b_{n+1}| \leq \frac{1}{2}|a_n - b_n|$ for the right choice of $b_{n+1}$. Then for the wrong choice, we have that $|a_{n+1}+b_{n+1}| < 1/2|a_n - b_n|$, and then,

$$a_{n+2} = \frac{|a_{n+1}+b_{n+1}|}{2} < \frac{|a_n - b_n|}{4} \leq 1/2 M_n \tag{4.5}$$

and since $|b_{n+2}| \leq M_n$ and $\sqrt{1/2} < 3/4$ we have for a wrong choice of $b_{n+2}$,

$$M_{n+3} \leq \frac{|a_{n+2}| + |b_{n+2}|}{2} \leq 3/4 M_n \tag{4.6}$$

Which must occur infinitely often for bad sequences proving that $M_n \to 0$ for $n \to \infty$.

*Claim 2: Good sequences are Cauchy sequences.* From some point on, the good sequence will have the right choice for every $b_n$, so we may as well assume that we have the right choice for all $n \in \mathbb{N}$. Then we have

$$|a_n - b_n| \leq 2^{-n}|a-b| \quad \text{and} \quad \theta_n = 2^{-n}\theta_0 \tag{4.7}$$

where $\theta_n$ is the angle between $a_n$ and $b_n$ which is halved in the square root of $b_n$, which for the right choice is $0 \leq \theta_n \leq \pi$. From (4.7) we can conclude that the two sequences converge toward each other, and since $a_n - a_{n+1} = 1/2(a_n - b_n)$ we get

$$|a_{n+1} - a_n| \leq 2^{-(n+1)}|a-b| \tag{4.8}$$

which is a Cauchy sequence, which proves that both sequences converge to a common limit.

*Claim 3: The common limit isn't zero if we are dealing with a good pair.* Let us again assume we have made the right choice for every step, and let

$$m_n = \min\{|a_n|, |b_n|\} \tag{4.9}$$

where $a_n$ and $b_n$ are both non-zero. Obviously, $|b_{n+1}| = |\sqrt{a_n b_n}| \geq m_n$, so it remains only to show that $a_n$ does not tend to 0 for $n \to \infty$ to complete the proof. Consider

$$\begin{aligned}(2|a_{n+1}|)^2 &= |a_n|^2 + |b_n|^2 + 2|a_n||b_n|\cos\theta_n \\ &\geq 2m_n^2(1+\cos\theta_n) = 4m_n^2\cos^2\left(\frac{\theta_n}{2}\right)\end{aligned} \tag{4.10}$$

We can replace $a_{n+1}$ with $m_{n+1}$, and for the right choice, $\cos\theta_n \in [0,1]$ so

$$m_{n+1} \geq m_n \cos\frac{\theta_n}{2} \tag{4.11}$$

using the second part of (4.7) we find that

$$m_n \geq m_0 \prod_{i=0}^{n-1} \cos\frac{\theta_0}{2^{i+1}} \tag{4.12}$$

where the product is an expansion for ${}^{\sin\theta_0}/_{\theta_0}$ which is greater than 0 for $\theta_0 \in [0, \pi[$, which completes the proof. □

So now we are finally ready to define the arithmetic-geometric mean values, which is the set of limit points for the sequences.

**Definition 4.4**

1. *The* **arithmetic-geometric mean,** *$M(a,b)$ is a set of limit points $\mu$ satisfying*

$$\mu = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{4.13}$$

*where the pair of sequences $(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$ is a good pair.*

2. *The value of $\mu$ obained with only right choices for $b_n$ is the* **simplest value** *of $M(a,b)$.*

Hence there is a countable amount of values of the arithmetic-geometric mean. David A. Cox [Cox84] proves the following result, which has also been obtained by von David [vD28], and which Gauss hinted at in his mathematical diary.

**Theorem 4.5**
*Let $a, b \in \overline{\mathbb{C}}$ with $|a| \geq |b|$, both non-zero, and let $\mu, \lambda$ be the simplest values of $M(a,b), M(a+b, a-b)$ respectively. Then all non-zero values $\mu'$ of $M(a,b)$ are given by*

$$\frac{1}{\mu'} = \frac{d}{\mu} + \frac{ic}{\lambda} \tag{4.14}$$

*with $d \equiv 1$ (mod 4) and $c \equiv 0$ (mod 4).*

The proof of the theorem is quite involved, and for the complete proof, please consult the article of David A. Cox [Cox84]. However, we will discuss some of the steps involved because they are interesting applications of the notions of *the universal cover* and *fundamental domains.*

## 4.2 Iteration of the *agm*

We will first reduce the dimension such that we only deal with one sequence. The arithmetic and geometric mean maps (4.3) are linear with respect to multiplication with a scalar, so we can without loss of information project the maps onto the Riemann sphere by using the projection $\pi$:

$$\pi : (\mathbb{C} \times \mathbb{C})^* \to \overline{\mathbb{C}} \qquad \pi(a, b) = \frac{a}{b} \tag{4.15}$$

We utilize the fact that the projective space for $\mathbb{C}^2 \setminus \{(0,0)\}$ is the Riemann sphere $\overline{\mathbb{C}}$ using the equivalence relation

$$(z_1, z_2) \sim (w_1, w_2) \iff \exists \lambda \in \mathbb{C}^* : \lambda z_1 = w_1,\ \lambda z_2 = w_2$$

to identify pairs which are mapped to the same point. It is then clear that the projection $\pi$ sends any pair in $\mathbb{C}^2 \setminus \{(0,0)\}$ to a point in $\overline{\mathbb{C}} = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$.

Any mapping $f$ respecting $\sim$ in $(\mathbb{C} \times \mathbb{C})^*$, i.e. $f(\lambda z_1, \lambda z_2) = \lambda f(z_1, z_2)$, can be projected onto $\overline{\mathbb{C}}$ requiring that the projected mapping $\pi \circ f = \hat{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fulfills that given any $z \in \overline{\mathbb{C}}$ there exists a $z' \in \overline{\mathbb{C}}$ such that $f(\pi^{-1}(z)) \subseteq \pi^{-1}(z')$. This is indeed fulfilled by the mapping (4.15).

The projetion defines the sequence $\{z_n\}_{n\in\mathbb{N}}$:

$$z_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \sqrt{\frac{(1+z_n)^2}{4z_n}} \tag{4.16}$$

which has the arithmetic-geometric mean of 1 and $z_n$ as limit values.

Let us describe the dynamics of the *agm* iteration function

$$f_+(z) = \sqrt{\frac{(1+z)^2}{4z}} \tag{4.17}$$

$f_+(1) = 1$, and it is a superattractive fixed point since $f_+'(1) = 0$. However, 1 also maps to $-1$, which is critical in the sense that it has only one forward and backward image where other points have two. The same is the case with 0, and $-1 \mapsto 0 \mapsto \infty$. The point at infinity is also fixed and is repelling. Hence all of the critical points for the mapping (4.16), $\{-1, 0, 1, \infty\}$, are on a common orbit, sometimes called the critical orbit.

$$1 \overset{2:1}{\longmapsto} 1 \overset{2:1}{\longmapsto} -1 \overset{2:2}{\longmapsto} 0 \overset{1:2}{\longmapsto} \infty \overset{1:2}{\longmapsto} \infty$$

Other fixed points of $f$ are the remaining roots of the cubic polynomial $4z^3 - z^2 - 2z - 1 = 0$ which are $R = {(-3\pm i\sqrt{7})}/{8}$. The eigenvalues at these fixed points are $f_+'(R) = {(-3\pm i\sqrt{7})}/{4}$. These values are on the unit circle with irrational rotation number, and hence the fixed points are neutral.

In [Bul91] Bullett shows that an infinite number of symmetrical periodic orbits exist, i.e. orbits which are invariant as sets under the involution $J$ (which also leaves the pair $R$ invariant)

$$J : z \mapsto \frac{\bar{z}+1}{\bar{z}-1}$$

Such periodic orbits are neutral, since points which are invariant under the involution have derivative $\zeta$ which satisfy $\zeta^{-1} = \bar{\zeta}$, and therefore $|\zeta| = 1$ [Bul88]. He furthermore conjectures that all periodic orbits except the fixed points 1 and $\infty$ are symmetric.

We shall prove the following proposition:

**Proposition 4.6**
*Every periodic orbit of the agm iteration function is irrationally neutral.*

**Proof**
Assume we have a repelling periodic point $z$ of period $n$ that is not in the critical orbit. By this we mean that there exists a branch of $f_+^n$ which sends $z$ to itself. When we omit the critical orbit and lift $f_+^n$ to the unit disc, which is the universal cover of the sphere with four punctures, such that the lift $\tilde{f}_+^n$ fulfills $\tilde{f}_+^n(0) = 0$, then we can analytically continue $\tilde{f}_+^n$ to the whole disc since the critical points are omitted. This contradicts Schwarz's lemma which says that $|(f_+^n)'| \leq 1$. The same argument applies to $f^{-1}$, proving that a periodic point is neutral.

Given an integer $n$, the number of periodic points with period dividing $n$ is at most $2^{n+1}$ since they are solutions to a polynomial equation of degree $2^{n+1}$. Assume that the derivative at $z$ is $e^{i2\pi p/q}$. Then $f_+^{nq}$ would lift to the identity on the unit disc, and there would be an infinite number of periodic points with period dividing $nq$. This contradiction shows that any periodic point has irrational derivative. □

This means that the periodic points of the agm correspondence are surrounded by Siegel discs and a computer generated picture of iteration of the agm is displayed in figure 4.1 where the siegel discs are coloured black. The computer programme takes a point $z$ and iterates using

$$z \mapsto \sqrt{\frac{(1+z)^2}{-4z}}$$

and compares iterated values with $z$. If the iterated point is sufficiently close to $z$, then $z$ is coloured black. We use this Möbius conjugated version of the map $f$ in order to have the discontinuity line of the square root on the line $y = x$ for reasons of numerical stability in the second quadrant. Some of the siegel discs are not quite filled by the iteration, but increased number of

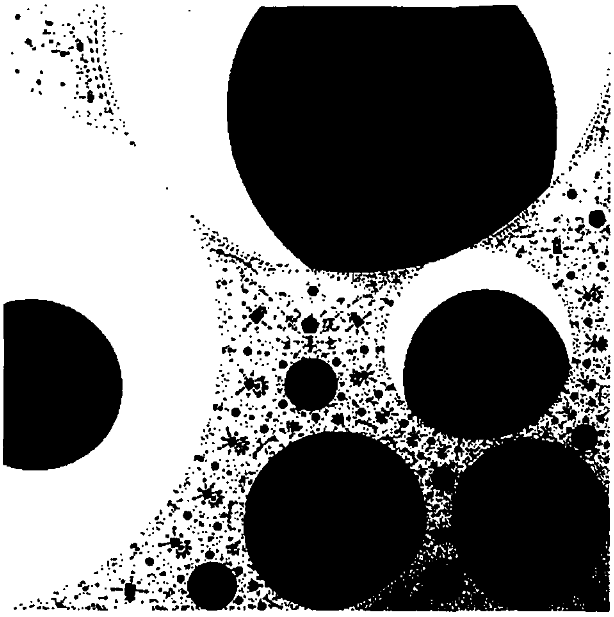


**Figure 4.1** Iteration of the agm in the domain $\mathrm{Re}(z) \in ]-2, 0[$ and $\mathrm{Im}(z) \in ]0, 2[$.

iterations would fix that. Other discs display sharp as opposed to rounded edges (near imaginary axis), which are also present in [Bul91]. They are due to the discontinuity. Other choices of the square root and discontinuity axis gave less round discs.

## 4.3 The covering of the *agm*

The universal cover of the arithmetic-geometric mean when we remove the four critical points is the unit disc, or equivalently the hyperbolic half-plane. For technical reasons I will use the latter:

$$\mathbb{H} = \{\tau \in \mathbb{C}: \mathrm{Im}(\tau) > 0\} \tag{4.18}$$

In [Cox84] it is shown, that the functions which are the covering maps of the arithmetic and geometric means, are built from two of the "Jacobi Θ-functions" (in the notation of [Cox84]):

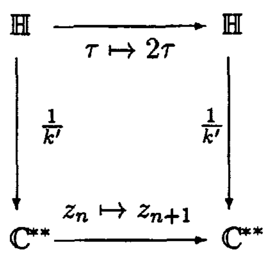
$$p(\tau) = 1 + 2\sum_{n=1}^{\infty} e^{i\pi\tau n^2} \tag{4.19}$$

$$r(\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{i\pi\tau n^2} \tag{4.20}$$

If $\tau$ is a point in $\mathbb{H}$, then $e^{i\pi\tau}$ is a point which is inside the unit circle, so the functions $p$ and $r$ are well defined since the series converge. It takes some lengthy calculus, but it is possible to show that $(p(2\tau))^2$ and $(r(2\tau))^2$ are the arithmetic respective geometric mean of $(p(\tau))^2$ and $(r(\tau))^2$. We will accept this without proof and again refer to [Cox84] and [vD28]. With the above notation, the projection function from $\mathbb{H}$ to $\mathbb{C}$ we will call

$$\frac{1}{k'(\tau)} = \frac{(p(\tau))^2}{(r(\tau))^2} \tag{4.21}$$

and the lifting of (4.16) from $\mathbb{C}^{**} = \mathbb{C} \setminus \{-1, 0, 1\}$ to the hyperbolic upper halfplane is then the simple map $\tau \mapsto 2\tau$.




Then the following lemma from [Cox84] gives us a limit to the sequence, provided we can find a point lying over $b/a$ in the covering space.

**Lemma 4.7**
*Let $a, b \in \mathbb{C}^*$ satisfy $a \neq \pm b$, and suppose there is a $\tau \in \mathbb{H}$ such that $k'(\tau) = b/a$. Set $\mu = a/(p(\tau))^2$ and, for $n \in \mathbb{N}$, set $a_n = \mu \cdot (p(2^n\tau))^2$ and $b_n = \mu \cdot (r(2^n\tau))^2$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \mu$.*

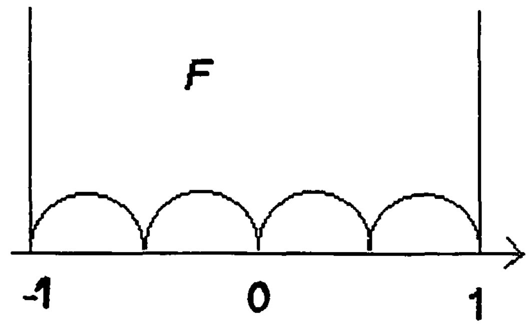
Thus, if we are able to find $p(\tau)^2$, we are able to find a value for the *agm*.

The next lemma we will discuss, but not prove (refer again to [Cox84]), involves the fundamental domain of the *agm*. This is the domain in which the value of the *agm* mentioned in the previous lemma is the simplest value of $M(a, b)$.

**Lemma 4.8**
*Let $\tau \in F$ where $F$ is the domain:*

$$F = \{\tau \in \mathbb{H} : -1 \leq Re(\tau) \leq 1, \quad |\tau \pm 1/4| > 1/4, \quad |\tau \pm 3/4| > 1/4\} \tag{4.22}$$

*then $\mu = a/(p(\tau))^2$ is the simplest value of $M(a, b)$*




Recalling from our discussion of Fuchsian groups, we suspect that this is a fundamental domain of a Fuchsian group. This is correct, and the group is geometrically finite, and therefore finitely generated since the fundamental domain has finitely many sides as discussed on page 42. The generators can be determined from the shape of the fundamental domain.

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} a \equiv d \equiv 1 \pmod 4 \\ b \equiv 0 \pmod 2, c \equiv 0 \pmod 4 \end{array} \right\} \tag{4.23}$$

Such that the upper halfplane modulo $G$ is the universal covering surface for the *agm*, since

$$k', 1/k' : \mathbb{H} \longrightarrow \mathbb{C} \setminus \{-1, 0, 1\}$$

are universal covering maps with deck transformation group $G$. The mapping 4.16 lifts to $\tau \mapsto 2\tau$ by $1/k'$ such that

$$z_{n+1} = 1/k' \circ 2\tau \circ 1/k'^{-1}(z_n) \tag{4.24}$$

In [BP94] it is proven that any **critically finite** correspondence, i.e. any correspondence with finitely many critical points and values in a common orbit, can be **resolved** by removing the critical orbits and lifting to the universal cover as described above. Furthermore it is shown that the lift is the (Fuchsian) group action of a group $G$ generated by the free product between an infinite cyclic group and an order two cyclic group $G = C_\infty \times C_2$. The results in proposition 4.6 transfer to any critically finite correspondence with three or more critical points.

## 4.4 General Quadratic Correspondences

As we have seen in previous chapters, the rational iteration and the Kleinian group iteration have many common features. The correspondences are a generalization of both of these fields. We will mostly limit ourselves to quadratic correspondences, but further generalizations are possible, and we will touch upon them when appropriate.

There are a lot of open questions about the dynamics of the quadratic correspondences. I will in the following try and show why the usual theorems of rational iteration don't apply in the case of correspondences. For instance, it is difficult to define a Julia set for this type of maps. If we decide to define the Julia set as the closure of the union of all backward orbits of the branch points of $f(z)$ as in [MR91], we obtain a dichotomy with a reasonable normal set, but the obtained Julia set is not forward invariant, and the normal set is not backward invariant.

**Definition 4.9**
*Given a quadratic equation in two complex variables with complex coefficients,*

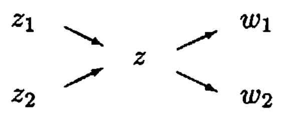
$$P(z,w) = Az^2w^2 + Bz^2w + Czw^2 + Dz^2 + Ew^2 + Fzw + Gz + Hw + J = 0, \tag{4.25}$$

*with no factors of the form* $(z+a)$ *or* $(w+b)$. *the quadratic correspondence* $f$ *arising from* $P$ *is the set of pairs*

$$f = \{(z,w) \in \mathbb{C} \times \mathbb{C} : P(z,w) = 0\} \tag{4.26}$$

The polynomial equation is regarded as an implicit 2-valued map $z \mapsto w$ of the Riemann sphere to itself with a 2-valued inverse, when $P$ has no factors of the form $(z+a)$ or $(w+b)$. Otherwise, the values $a$ and $b$ would be points where the map, respectively its inverse, would have the whole Riemann sphere as values.

In general, a quadratic correspondence has a lattice of possible values with two possibilities in both directions for each point $z$:




Another point $z'$ having a preimage in common with $z$, say $z_1$, needn't necessarily have $w_1$ or $w_2$ as image point, so a drawing of the full system would be impossible.

As a simple example of a correspondence, consider the class of maps

$$g(z,w) = czw + dw - az - b = 0. \tag{4.27}$$

Since it is not a quadratic correspondence, there are no two-value problems with these maps. They are the by now familiar Möbius transforms

$$w = \frac{az+b}{cz+d} \tag{4.28}$$

which are biholomorphic. In fact, they are the only isomorphisms of the Riemann sphere.

The quadratic map $z \mapsto z^2 + c$ which I have treated in an earlier chapter is given by the correspondence

$$g(z,w) = w - (z^2 + c) = 0 \tag{4.29}$$

hence rational iteration in the degree two case, which is covered in the preceding chapters, treats some classes of quadratic correspondences with two-value problems in one direction only ($g$ is of degree one in $w$). The question is then, how much of the rational iteration theory can be applied to the general quadratic correspondences? We shall try and define regular and limit sets and Fatou and Julia sets, but the branching of the maps shall give us a bit of extra work.

Our example of the Arithmetic-Geometric mean is a quadratic correspondence. Recall the sequence (4.16):

$$z_{n+1} = \frac{1 + z_n}{2\sqrt{z_n}} \tag{4.30}$$

Let $z_n = z$ and $z_{n+1} = w$ and rewrite to

$$4w^2 z - (z+1)^2 = 0 \tag{4.31}$$

This shows that the sequence we defined as the arithmetic-geometric mean defines a quadratic correspondence which will from now on be called the *agm-***correspondence**. This correspondence as a map (the iteration map of section 4.2) $f^+ : z \mapsto w$ is single-valued when restricted to the upper halfplane except on the set $(f^+)^{-1}(\mathbb{R})$.

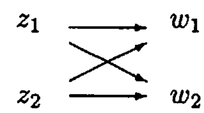
## 4.5 Orbits and Paths

In iteration of rational maps, we defined the forward and backward orbits of a point. Correspondences, however, have no predefined forward or backward map, since the two maps $z \mapsto w$ and $w \mapsto z$ are equally valid as maps derived from the correspondence. From the point of view of kleinian groups, a direction of iteration doesn't make much sense, so in general we will allow iteration in both directions of any $z$ and call the resulting sequence of points the **grand orbit** of $z$. The existence of two images in both directions gives $2^n$ possible different orbits of length $n$ for a generic point. Therefore we will need an adaptation found in [BP98b] of the notion of an **itinerary** known from rational iteration as the sequence of iterates of a point. First we need to decide which direction we will consider forward. Let $f^+ : z \mapsto w$ be the "forward" map defined by a quadratic correspondence, a forward direction, which is not intrinsic to the correspondence. It is generally branched, and so is it's inverse which we take to be the map $f^- = (f^+)^{-1} : w \mapsto z$.

## 4.6 Maps of pairs

We will now consider a special class of quadratic correspondences, which have some nice properties. The reason we deal with this class is that it contains the arithmetic-geometric mean correspondence.

We say that a quadratic correspondence $f$ defines a **map of pairs**, $\hat{f} : S \to T$, with $S, T \subset \overline{\mathbb{C}}^2$, if, given a pair of points $z_1, z_2$, the images under $\hat{f}$ is a unique pair of points. The possible actions of a map of pairs are




We now define $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which interchanges the two points $z$ which map to the same $w$. $\Phi$ is bijective since every quadratic must have exactly two roots, counting multiplicity, and it is easily seen to be holomorphic except possibly at critical points for the correspondence, ${}^{d\hat{f}}/_{dz} = 0$, which are removable singularities because the correspondence is continuous. Therefore, $\Phi$ must be a Möbius transform with $\Phi^2 = \mathsf{id}$. And hence it is an involution of the Riemann sphere. Any involution of the Riemann sphere is Möbius conjugate to $z \mapsto -z$. Consider the Riemann surface ${}^{\overline{\mathbb{C}}}/_{\Phi}$. It follows from the fact that $\Phi$ is an involution that this surface is isomorphic to $\overline{\mathbb{C}}$, and that the projection mapping $g : \overline{\mathbb{C}} \to {}^{\overline{\mathbb{C}}}/_{\Phi}$ is a fractional quadratic, since it is a $2 : 1$ holomorphic map from $\overline{\mathbb{C}}$ to itself. Similarly there exists a projection, a fractional quadratic map $h$ such that for any $u \in \overline{\mathbb{C}}$, the set $h^{-1}(u) = \{w_1, w_2\}$ forms a pair which is the image of a pair $(z_1, z_2)$ under $\hat{f}$.

**Proposition 4.10**
*A quadratic correspondence defines a map of pairs, $\hat{f} = h^{-1} \circ g$, if and only if the following equivalent conditions are satisfied:*

1. *The coefficients in equation (4.25) satisfy*

$$\begin{vmatrix} A & C & E \\ B & F & H \\ D & G & J \end{vmatrix} = 0$$

   *or $P(z, w)$ is of the form $(azw + bz + cw + d)^2$.*

2. *Equation (4.25) can be rewritten in the form*

$$g(z) = h(w) \tag{4.32}$$

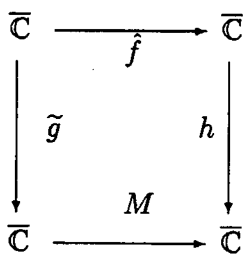
   *where $g$ and $h$ are fractional quadratic functions with complex coefficients, i.e. on the form*

$$g(z) = \frac{az^2 + bz + c}{dz^2 + ez + f}$$

**Proof**

For the proof of 1, we note that if $\hat{f}$ is the square of a Möbius transform which is one-to-one, $\hat{f}$ must be a map of pairs. For a proof of the result about the determinant, please look in [Bul88]. For the proof of 2, let $\hat{f}\colon (z_1, z_2) \mapsto (w_1, w_2)$ be a map of pairs. Consider the map $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which interchanges the two points $z$ which map to the same $w$. $\Phi$ is bijective since every quadratic must have exactly two roots, counting multiplicity, and it is easily seen to be holomorphic except possibly at critical points for the correspondence, ${}^{d\hat{f}}/_{dz} = 0$, which are removable singularities because the correspondence is continuous. Therefore, $\Phi$ must be a Möbius transform with $\Phi^2 = \mathsf{id}$. And hence it is an involution of the Riemann sphere. Any involution of the Riemann sphere is Möbius conjugate to $z \mapsto -z$. Consider the Riemann surface ${}^{\overline{\mathbb{C}}}/_{\Phi}$. It follows from the fact that $\Phi$ is an involution that this surface is isomorphic to $\overline{\mathbb{C}}$, and that the projection mapping $\widetilde{g} : \overline{\mathbb{C}} \to {}^{\overline{\mathbb{C}}}/_{\Phi}$ is a fractional quadratic, since it is a $2:1$ holomorphic map from $\overline{\mathbb{C}}$ to itself. Similarly there exists a projection, a fractional quadratic map $h$ such that for any $v \in \overline{\mathbb{C}}$, the set $h^{-1}(v) = \{w_1, w_2\}$ forms a pair which is the image of a pair $(z_1, z_2)$ under $\hat{f}$.

The map of pairs $\hat{f}$ defines a map $M\colon u \mapsto v$ which makes the diagram commutative:




Similarly to $\Phi$, $M$ can be shown to be a Möbius transform. By defining $g = M \circ \widetilde{g}$, we see that a map of pairs can be written as in (4.32).

Conversely, if we have a map satisfying (4.32), then the diagram shows that it can be considered a map of pairs. □

The maps $h$ and $g$ are double covers of the Riemann sphere branched at the critical points $dh/dw = 0$ or $dg/dz = 0$. The condition in 4.32 is called **separability**, and we shall mostly treat correspondences that are separable, i.e. are maps of pairs.

**Proposition 4.11**

*$\Phi$ maps any circle through the two critical points to itself, exchanging the two discs bounded by the circle.*

**Proof**

Consider the case where the two critical points are 0 and $\infty$ and let $\Phi(z) = -z$. Any other case can be Möbius conjugated into this. Any circle through 0 and $\infty$ is a straight line through 0, and then $\Phi$ has the desired effect. □

For a map of pairs with real coefficients and purely imaginary critical points and $\hat{f}(z,w) = 0 \iff \hat{f}(1/\bar{z}, 1/\bar{w}) = 0$, the unit circle is invariant under forward and backward iteration, and if a definition of the Julia set should make sense for quadratic correspondences, we may want the unit circle to be included. With an arbitrary initial value, iteration using a random choice of the two values will accumulate on the unit circle [Bul88].

It is then not suitable to define the Julia set for a correspondence as the points which form the boundary of $A(\infty)$ (as can be done for rational maps) since a point may escape to infinity under one sequence of choices of forward values, but with other choices, the point sequence may converge to a finite image or not at all.

In [Bul88], examples are given where Julia sets found as accumulation points of a generic point under random backward iteration of maps of pairs are difficult to obtain, since attracting and strongly repelling points in some regions are close together, making computer experiments difficult. A solution would be to restrict certain regions from the iteration in order to obtain a sharp Julia set. However, as we mentioned about the *agm*-correspondence, different single-valued restrictions of iteration domain for correspondences give different iteration pictures, and therefore different 'Julia sets' if we are not careful.

## 4.7 Zipeomorphisms

The following class of correspondences defined in [Bul88] give us the possibility of comparing dynamics with Fuchsian group dynamics. Given a map of pairs, $\hat{f} = h^{-1} \circ g : (z_1, z_2) \mapsto (w_1, w_2)$, we have from proposition 4.11 that for each $z$, the corresponding values of $w$ occur in pairs. Either with one value inside a circle $C$ through the critical points $c$ and $d$ of $h$, and one value outside it, or with both values on the circle.

Then the correspondence is single valued in either of the discs surrounded by the circle, except on the set of preimages of the circle through $c$ and $d$. If the critical points $s$ and $t$ of $g$ are also on the circle $C$, then the correspondence as a map of pairs is a homeomorphism on each of the discs, except the two arcs of preimages of the circle.

The action of $\hat{f}$ is to send the arc $c'd'$ to the circle $cd$, and conversely sending $cd$ to an arc in the disc. This "zipping" of circles into arcs and "unzipping" of arcs into circles has given rise to the term zipeomorphism. The zipeomorphisms only differ from the complex analytic maps on a halfplane by the existence of the zip discontinuity.

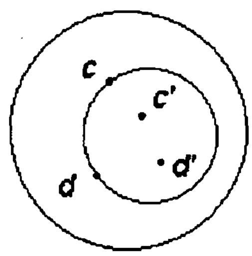



**Figure 4.2** The Riemann sphere with the circle through the critical points $c, d$ of $h$.

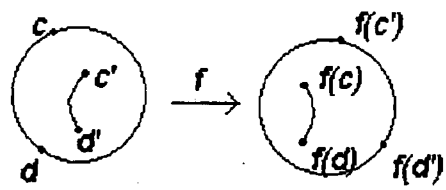


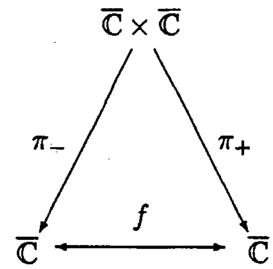

**Figure 4.3** The action of $\hat{f}$ as a zipeomorphism.

## 4.8 Desingularization of correspondences

In the quest for helpful tools in defining a set of normality for correspondences, we will now move to the covering surface of a correspondence. A quadratic correspondence $f$ can be considered as a graph, or set of pairs: $f = \{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} : \; P(z, w) = 0\}$. Let the projection $\pi_- : \overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by $\pi_-(f) = z$ be named the backward projection, and likewise the forward projection $\pi_+ : f \mapsto w$.

It is then clear, that what we until now has referred to as the forward and backward maps defined by the correspondence are the maps

$$f^+ = \pi_+ \circ (\pi_-|_f)^{-1} \tag{4.33}$$
$$f^- = \pi_- \circ (\pi_+|_f)^{-1} \tag{4.34}$$

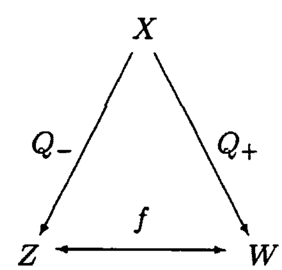



The correspondence $f$ itself can be regarded an algebraic hypersurface. Problems occur when we encounter **singular points**. A forward singular point $z$ of $f$ is a point with fewer images under $f^+$ than its immediate neighbours. Backward singular points are defined in a similar way.

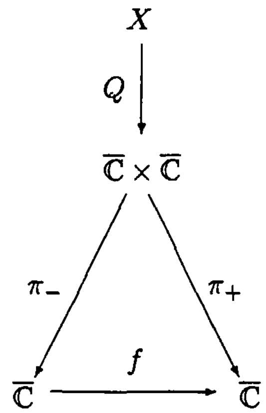
**Definition 4.12**
*A point $(z, w) \in f$ is* **forward non-singular** *if there exists a neighbourhood $U$ of $(z, w)$ in $f$ such that $\pi_-|_U$ is a homeomorphism onto its image. The composition $\pi_+ \circ (\pi_-|_U)^{-1}$, which then is single valued, is called a* **branch** *of $f$ at $z$.*

For the singular point problem, we will draw on the theory of **desingularization**, which, in the version of [BP98b], says that given a correspondence $f \in Z \times W$ between Riemann surfaces, there exists a manifold cover $X$ containing $f$ and projections $Q_-$ and $Q_+$ such that

$$Q_- : X \to Z \quad \text{and} \quad Q_+ : X \to W$$




and the product map $Q = Q_- \times Q_+ : X \to f \subseteq Z \times W$ given by $Q(x) = (Q_-(x), Q_+(x))$ has $f$ as image and is one to one except if $(z, w)$ is both forward and backward singular for $f$. The surface $X$ is called the desingularization of $Z \times W$ since (one-way) singular points in $Z \times W$ are not singular in $X$. In our work we have $f \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$, and may assume $X$ to be the desingularized graph $Q^{-1}(f)$ of $f$ since it is a closed Riemann surface and covers $f$. We shall not prove these results or go further into desingularization, but we shall enjoy the results. With these definitions we have the backward projection $Q_- = \pi_- \circ Q$ and $Q_+ = \pi_+ \circ Q$ as holomorphic maps of degree two from $X$ to $\overline{\mathbb{C}}$.




We will now restrict our attention to some correspondences where we can define completely invariant regular or normal set and limit set. We will now introduce a further restriction on our separable correspondences.

**Definition 4.13**
*A* holomorphic correspondence *is a correspondence $f$ between compact Riemann surfaces $Z$ and $W$ which has a factorization $f = Q_+ \circ Q_-^{-1}$, where $Q_-$ and $Q_+$ are holomorphic maps from a Riemann surface $X$ onto $Z$ and $W$ respectively.*

About these holomorphic correspondences, we can state the following version of the Riemann-Hurwitz relation.

**Proposition 4.14**
*An $m : n$ holomorphic correspondence on the Riemann sphere has at most $2(m-1)n$ backward singular points and at most $2(n-1)m$ forward singular points.*

**Proof (sketch)**
Let the correspondence be defined by a polynomial $P$ of degree $m$ in $z$ and degree $n$ in $w$. A point $w_0$ is backward singular if and only if $P(z,w)$ and $\partial P(z,w)/\partial z$ has a common root $z_0$. This system has $2(m-1)$ degrees of freedom in $z$. Each of the two polynomials are also of degree $n$ in $w$, and this gives the total number of backward singular points as $2(m-1)n$. Forward singular points are counted similarly. □

Then the number of backward and forward singular points are four for a 2 : 2 correspondence. We have earlier mentioned that the desingularized graph of the correspondence forms a closed hypersurface. the genus of the surface can also be computed.

**Proposition 4.15**
*If $P(z,w)$ is an irreducible polynomial of degree $m$ in $z$ and $n$ in $w$, then the graph $f$ of the holomorphic correspondence defined on the Riemann sphere by $P(z,w) = 0$ has genus at least $(m-1)(n-1)$.*

**Proof**
The graph $f$ expressed as a polynomial in $z$ is an $m$-fold cover of the Riemann sphere, and has at most $2(m-1)n$ backward singular points. Therefore, $X$ has Euler characteristic at least $2m - 2(m-1)n = 2 - 2(m-1)(n-1)$ by the Riemann-Hurwits formula, and so does $Q^{-1}(f)$. Then the genus of $f$ is at least $(m-1)(n-1)$. □

For our polynomials of degree two in both variables, the graph has genus at most 1, which corresponds to a torus. This is the case when the polynomial has four distinct roots. In [BP94] the topological possibilities of the normalization surface are listed. If there are one double or one triple root, the surface is a sphere which self-intersects or with a nonsmooth point. Two double roots or one quadrouple root makes the surface a pair of disjoint spheres.

## 4.9 Normalization of quadratic correspondences

Recall that anlytic maps can be conjugated to $z \mapsto z^d$, where $d$ is the degree of the original map. The key was finding an automorphism $\phi$ of the Riemann sphere moving the map to the origin. The same is possible for the projections $Q_-$ and $Q_+$. The degree of these projections is defined locally for any $x \in Q^{-1}(f)$, and we know that $Q^{-1}(f)$ is a covering Riemann surface with a complex structure. We can therefore choose the appropriate charts such that a neighbourhood of $x$ is mapped $d:1$ onto a neighbourhood of the origin.

**Definition 4.16**
*We say that $Q_-$ of degree $m$ and $Q_+$ of degree $n$ are* simultaneously normalizable *if there exist neighbourhoods $U$, $V$ and $W$ around $x$, $z$ and $w$ respectively and charts $\varphi : Z \to \mathbb{C}$ and $\psi : W \to \mathbb{C}$ such that*

$$\varphi \circ Q_- : x_0 \mapsto 0 \ (m:1)$$
$$\psi \circ Q_+ : x_0 \mapsto 0 \ (n:1)$$

Simultaneous normalization is possible only if $m$ and $n$ are coprime. If $\gcd(m,n) = d > 1$ then we may have some point $x$ in any neighbourhood of $x_0$, for which $\varphi \circ Q_-(x) = 0(m:d)$ and $\psi \circ Q_+(x) = 0(n:d)$, whereby $x$ aswell as $x_0$ is a singular point, contradicting that singular points are isolated in a holomorphic map.

With the definition above, the construction looks as follows:

**Definition 4.19**
*The iterates of a holomorphic correspondence $f$ at $z_0$ are called* **equicontinuous** *if $\forall \varepsilon > 0$ there exists a $\delta > 0$ such that for all branches $f_n^e$ along paths $\underline{z}$ starting at $z_0$,*

$$f_n^e(B_\delta(z_0)) \subseteq B_\varepsilon(z_n).$$

We still have difficulties defining a normal family of iterates since they are multi-valued, but since we have the equicontinuity set as Fatou set for rational functions, it is possible that future progress could come from this basis.

The complement of the equicontinuity set we will call $J(f)$. However, this set is not completely invariant, which is an important feature of the Julia set for rational functions. If $f$ is a rational map then with this definition $J(f)$ contains not only the usual Julia set but also forward and backward images of attractive and superattractive periodic points. To exclude these, one may use the forward equicontinuity set where the orbit $\underline{z}$ is constructed by forward iteration only. Grand orbits of Siegel discs, which we have encountered in the *agm* correspondence are, however, included in the equicontinuity set as it is defined above. Comparing this with proposition 4.6, we deduce that for all critically finite correspondences with a critical orbit with at least three points, all of the Riemann sphere except the critical orbit is in the equicontinuity set, leaving the critical orbit as the only elements of $J(f)$, which is then neither infinite nor perfect.

It is proved in [BP98b] that for a finitely generated kleinian group regarded as a correspondence, the equicontinuity set of $f$ is equal to the regular set $\Omega(G)$.

# Conclusion

The scope of this thesis is to survey the current knowledge of the dynamics of the quadratic correspondences, and in order to round off the subject, the volume of the subjects treated has been weighted to the detriment of depth and precision in some instances. However, we have seen that when treating the dynamics of a quadratic correspondence, we can profit from drawing on experience from both the dynamics of holomorphic functions and that of Kleinian groups.

Restricting iteration to one direction of the correspondence with a choice of domain of iteration may give good results as we have seen in the case of the *agm* correspondence, but the choice of domain and discontinuity plays an important role in the quality of the generated pictures.

Because of the nondirectionality of the correspondences, a regular – limit set dichotomy is easiest to define and determine in the (separable) multi-valued cases we have treated, whereas a Fatou – Julia dichotomy requires many arduous definitions and limitations to make sense even in the simplest multivalued cases.

It seems, though, that separability is a necessary condition to get anywhere, and good definitions of a universal dichotomy may never be found. Even in the very limited cases we have encountered, the sets we define are not as elegant as the ones found in rational dynamics.

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306/95 Goals of mathematics teaching

Preprint of a chapter for the forth-comming International Handbook of Mathematics Education (Alan J.Bishop, ed)

By: Mogens Niss

307/95 Habit Formation and the Thirdness of Signs

Presented at the semiotic symposium

The Emergence of Codes and Intensions as a Basis of Sign Processes

By: Peder Voetmann Christiansen

308/95 Metaforer i Fysikken

af: Marianne Wilcken Bjerregaard, Frederik Voetmann Christiansen, Jørn Skov Hansen, Klaus Dahl Jensen Ole Schmidt

Vejledere: Peder Voetmann Christiansen og Petr Viscor

309/95 Tiden og Tanken
En undersøgelse af begrebsverdenen Matematik udført ved hjælp af en analogi med tid

af: Anita Stark og Randi Petersen

Vejleder: Bernhelm Booss-Bavnbek

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310/96 Kursusmateriale til "Lineære strukturer fra algebra og analyse" (E1)

af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Hélène Connor-Lajambe, Bernd Kuemmel, Stefan Krüger Nielsen, Bent Sørensen

312/96 Grassmannian and Chiral Anomaly

by: B. Booss-Bavnbek, K.P.Wojciechowski

313/96 THE IRREDUCIBILITY OF CHANCE AND THE OPENNESS OF THE FUTURE
The Logical Function of Idealism in Peirce's Philosophy of Nature

By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian Cardiovascular System

By: Johnny T. Ottesen

315/96 a + b "Rejsen til tidens indre" - Udarbejdelse af et manuskript til en fjernsynsudsendelse + manuskript

af: Gunhild Hune og Karina Goyle

Vejledere: Peder Voetmann Christiansen og Bruno Ingemann

316/96 Plasmaoscillation i natriumklynger

Specialerapport af: Peter Meibom, Mikko Østergård

Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer

af: Ulla Rasmussen

Vejleder: Anders Madsen

318/96 Modelling the Respiratory System

by: Tine Guldager Christiansen, Claus Dræby

Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming Impacts

by: Bent Sørensen

320/96 Grassmannian and Boundary Contribution to the -Determinant

by: K.P.Wojciechowski et al.

321/96 Modelkompetencer - udvikling og afprøvning af et begrebsapparat

Specialerapport af: Nina Skov Hansen, Christine Iversen, Kristin Troels-Smith

Vejleder: Morten Blomhøj

322/96 OPGAVESAMLING
Bredde-Kursus i Fysik 1976 - 1996

323/96 Structure and Dynamics of Symmetric Diblock Copolymers
PhD Thesis

by: Christine Maria Papadakis

324/96 Non-linearity of Baroreceptor Nerves

by: Johnny T. Ottesen

325/96 Retorik eller realitet ?
Anvendelser af matematik i det danske Gymnasiums matematikundervisning i perioden 1903 - 88

Specialerapport af Helle Pilemann

Vejleder: Mogens Niss

326/96 Bevisteori
Eksemplificeret ved Gentzens bevis for konsistensen af teorien om de naturlige tal

af: Gitte Andersen, Lise Mariane Jeppesen, Klaus Frovin Jørgensen, Ivar Peter Zeck

Vejledere: Bernhelm Booss-Bavnbek og Stig Andur Pedersen

327/96 NON-LINEAR MODELLING OF INTEGRATED ENERGY SUPPLY AND DEMAND MATCHING SYSTEMS

by: Bent Sørensen

328/96 Calculating Fuel Transport Emissions

by: Bernd Kuemmel

329/96 The dynamics of cocirculating influenza strains conferring partial cross-immunity

and

A model of influenza A drift evolution

by: Viggo Andreasen, Juan Lin and Simon Levin

330/96 LONG-TERM INTEGRATION OF PHOTOVOLTAICS INTO THE GLOBAL ENERGY SYSTEM

by: Bent Sørensen

331/96 Viskøse fingre

Specialerapport af:

Vibeke Orlien og Christina Specht

Vejledere: Jacob M. Jacobsen og Jesper Larsen

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332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG

Specialerapport af:

Stine Sofia Korremann

Vejleder: Dorthe Posselt

333/97 Biodiversity Matters

an extension of methods found in the literature on monetisation of biodiversity

by: Bernd Kuemmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Bernd Kuemmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids

by: Jeppe C. Dyre

336/97 PROBLEM-ORIENTATED GROUP PROJECT WORK AT ROSKILDE UNIVERSITY

by: Kathrine Legge

337/97 Verdensbankens globale befolkningsprognose

- et projekt om matematisk modellering

af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen

Vejleder: Jørgen Larsen

338/97 Kvantisering af nanolederes elektriske ledningsevne

Første modul fysikprojekt

af: Søren Dam, Esben Danielsen, Martin Niss, Esben Friis Pedersen, Frederik Resen Steenstrup

Vejleder: Tage Christensen

339/97 Defining Discipline

by: Wolfgang Coy

340/97 Prime ends revisited - a geometric point of view -

by: Carsten Lunde Petersen

341/97 Two chapters on the teaching, learning and assessment of geometry

by Mogens Niss

342/97 LONG-TERM SCENARIOS FOR GLOBAL ENERGY DEMAND AND SUPPLY

A global clean fossil scenario discussion paper

prepared by Bernd Kuemmel

Project leader: Bent Sørensen

343/97 IMPORT/EKSPORT-POLITIK SOM REDSKAB TIL OPTIMERET UDNYTTELSE AF EL PRODUCERET PÅ VE-ANLÆG

af: Peter Meibom, Torben Svendsen, Bent Sørensen

344/97 Puzzles and Siegel disks

by Carsten Lunde Petersen

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345/98 Modeling the Arterial System with Reference to an Anestesia Simulator

Ph.D. Thesis

by: Mette Sofie Olufsen

346/98 Klyngedannelse i en hulkatode-forstøvningsproces

af: Sebastian Horst

Vejledere: Jørn Borggren, NBI, Niels Boye Olsen

347/98 Verificering af Matematiske Modeller

- en analyse af Den Danske Eulerske Model

af: Jonas Blomqvist, Tom Pedersen, Karen Timmermann, Lisbet Øhlenschlæger

Vejleder: Bernhelm Booss-Bavnbek

348/98 Case study of the environmental permission procedure and the environmental impact assessment for power plants in Denmark

by: Stefan Krüger Nielsen

Project leader: Bent Sørensen

349/98 Tre rapporter fra FAGMAT - et projekt om tal og faglig matematik i arbejdsmarkedsuddannelserne

af: Lena Lindenskov og Tine Wedege

350/98 OPGAVESAMLING - Bredde-Kursus i Fysik 1976 - 1998

Erstatter teksterne 3/78, 261/93 og 322/96

351/98 Aspects of the Nature and State of Research in Mathematics Education

by: Mogens Niss

352/98 The Herman-Swiatec Theorem with applications

by: Carsten Lunde Petersen

353/98 Problemløsning og modellering i en almendannende matematikundervisning

Specialerapport af: Per Gregersen og Tomas Højgaard Jensen

Vejleder: Morten Blomhøj

354/98 A GLOBAL RENEWABLE ENERGY SCENARIO

by: Bent Sørensen and Peter Meibom

355/98 Convergence of rational rays in parameter spaces

by: Carsten Lunde Petersen and Gustav Ryd

356/98 Terrænmodellering
Analyse af en matematisk model til konstruktion af terrænmodeller
Modelprojekt af: Thomas Frommelt, Hans Ravnkjær Larsen og Arnold Skimminge
Vejleder: Johnny Ottesen

*357/98 Cayleys Problem*
*En historisk analyse af arbejdet med Cayley problem fra 1870 til 1918*
*Et matematisk videnskabsfagsprojekt af: Rikke Degn, Bo Jakobsen, Bjarke K.W. Hansen, Jesper S. Hansen, Jesper Udesen, Peter C. Wulff*
*Vejleder: Jesper Larsen*

*358/98 Modeling of Feedback Mechanisms which Control the Heart Function in a View to an Implementation in Cardiovascular Models*
*Ph.D. Thesis by: Michael Danielsen*

*359/99 Long-Term Scenarios for Global Energy Demand and Supply Four Global Greenhouse Mitigation Scenarios*
*by: Bent Sørensen*

*360/99 SYMMETRI I FYSIK*
*En Meta-projektrapport af: Martin Niss, Bo Jakobsen & Tune Bjarke Bonné*
*Vejleder: Peder Voetmann Christiansen*

*361/99 Symplectic Functional Analysis and Spectral Invariants*
*by: Bernhelm Booss-Bavnbek, Kenro Furutani*

*362/99 Er matematik en naturvidenskab? – en udspænding af diskussionen*
*En videnskabsfagsprojekt-rapport af Martin Niss*
*Vejleder: Mogens Nørgaard Olesen*

*363/99 EMERGENCE AND DOWNWARD CAUSATION*
*by: Donald T. Campbell, MArk H. Bickhard and Peder V. Christiansen*

*364/99 Illustrationens kraft*
*Visuel formidling af fysik*
*Integreret speciale i fysik og kommunikation*
*af: Sebastian Horst*
*Vejledere: Karin Beyer, Søren Kjørup*

*365/99 To know – or not to know – mathematics, that is a question of context*
*by: Tine Wedege*

*366/99 LATEX FOR FORFATTERE*
*En introduktion til LATEX og IMFUFA-LATEX*
*af: Jørgen Larsen*

**367/99 Boundary Reduction of Spectral invariants and Unique Continuation Property**
**by Bernhelm Booss-Bavnbek**