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Boundary Reduction of Spectral Invariants and Unique Continuation Property

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This preprint contains two draft articles of a rather expository character.

In the first article, I give some new results and various puzzles regarding the boundary reduction of spectral invariants. I consider (elliptic) operators of Dirac type and families of such operators over a compact smooth Riemannian manifold with boundary and over partitioned manifolds. I apply suitable boundary conditions and discuss three types of results regarding the involved function spaces, operators, and spectral invariants (index, spectral flow, and determinant): reductions to the boundary, correction formulas for changing the boundary condition, and pasting results.

In the second article, I summarize present basic knowledge about the Unique Continuation Property (UCP, also = Uniqueness of the Cauchy Problem) for linear elliptic operators of first order for a readership of geometers and topologists. I explain why the *weak* UCP, i.e. UCP from open subsets or, equivalently, UCP from separating hypersurfaces, is almost trivial for operators of Dirac type by simplifying a proof previously given by K.P. Wojciechowski and the author.

Boundary Reduction of Spectral Invariants – Results and Puzzles

Bernhelm Booss–Bavnbek

This report is dedicated to Sergio Albeverio on his 60th birthday

ABSTRACT. We consider (elliptic) operators of Dirac type and families of such operators over a compact smooth Riemannian manifold M with boundary Σ and over partitioned manifolds $M_- \cup M_+$, where $M_- \cap M_+ = \partial M_- = \partial M_+ = \Sigma$. We apply suitable boundary conditions and discuss three types of results regarding the involved function spaces, operators, and spectral invariants (index, spectral flow, and determinant): reductions to the boundary, correction formulas for changing the boundary condition, and pasting results. We shall emphasize the joint features of the different approaches, but focus, as well, on some puzzling differences between them.

Introduction

Philosophers and physicists have discussed the relation between *appearance* and *essence*, between *observable surface* and underlying *internal content* for centuries. Some aspects of their discussion have been formalized quite successfully by mathematicians working with boundary integral methods in numerical analysis for instance, or with long exact (co)homology sequences in algebraic topology.

In this Note we shall present and analyze various cases of *boundary reduction of spectral invariants*. They are all related to the index, the spectral flow, and the ζ–function regularized determinant of Dirac operators resp. families of Dirac operators over a compact manifold

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with boundary. We shall not attempt to give a complete list of all known boundary reduction phenomena in this field but rather restrict our attention to selected topics representative for a wide range of typical features.

Under the general heading of 'boundary reduction' we shall address three types of results: *genuine boundary reduction* when a global invariant can be recovered from expressions defined on the boundary; *correction formulas* when the difference between two spectral invariants can be localized at the boundary; and *pasting formulas* when a global invariant on a partitioned manifold can be split into components or localized along the splitting hypersurface.

In Section 1, we give general restriction results for Dirac operators and spaces of sections and distributions.

In Sections 2-3, we treat the index and the spectral flow. We do this in two slightly different and almost complementary set-ups, distinguished by differentiability and naturalness assumptions about the Cauchy data spaces. Since the index and the spectral flow are both topological invariants, the main results clearly have quite something in common. At some points one could feel tempted to put up a categorial frame work and a 'functor' to describe the boundary reduction in terms of a homology theory. The delicacy and the variability of some of the results seem, however, to speak against the productivity of such possible frame work.

In Section 4, we recall the three basic concepts of the determinant as used in quantum field theories: the quadratic functional, the Fredholm determinant, and the ζ–determinant. We present our view of the recent Scott–Wojciechowski Formula, relating the second and third concept in a satisfactory way over manifolds with boundary. It is interesting that many of the basic concepts discussed in the earlier sections also apply to the study of the determinant which is not a topological invariant but much 'finer'.

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1. Boundary Reduction of Function Spaces and Distributions

1.1. The Tangential Operator. Let M be a compact smooth Riemannian manifold with boundary Σ. We shall assume that M does not contain a connected component which is closed (i.e. with empty boundary). Let

$$\mathrm{A} : C^\infty(M;S) \longrightarrow C^\infty(M;S)$$

be an operator of Dirac type acting on sections of a Hermitian bundle S of Clifford modules over M, i.e. $\mathrm{A} = \mathbf{c} \circ \nabla$ where $\mathbf{c}$ denotes the Clifford multiplication and ∇ is a connection for S which is compatible with $\mathbf{c}$ (i.e. $\nabla\, \mathbf{c} = 0$). To begin with, we assume that all metric structures of M and S are product in a collar neighbourhood $\mathcal{N}$ of the boundary. Then

$$\mathrm{A}\,|_{\mathcal{N}} = \sigma(\frac{\partial}{\partial u} + \mathcal{B})\,, \tag{1.1}$$

where u denotes the (inward) normal coordinate, σ denotes the Clifford multiplication with du and $\mathcal{B}$ denotes the canonically associated Dirac operator over Σ, called the *tangential operator*. Here the point of the product structure is that then σ and $\mathcal{B}$ do not depend on the normal variable. We note that σ is unitary with $\sigma^2 = -\mathrm{Id}$ and $\sigma\mathcal{B} = -\mathcal{B}\sigma$. In the non–product case, there are certain ambiguities in defining a 'tangential operator' which we shall not discuss here.

1.2. General Restriction Results. We consider the Sobolev spaces $H^s(M;S)$ and $H^s(\Sigma;S|_\Sigma)$ $(s \in \mathbf{R})$. There is not one single space, one single s to pick up as a canonical choice for solving the system of Dirac equations or determining the spectral quantities: from the point of view of physics, e.g., one would be mainly interested in smooth sections. Clearly, within the smooth category, one has no problems restricting a section to the boundary. We shall write

$$\gamma_\infty : C^\infty(M;S) \longrightarrow C^\infty(\Sigma;S|_\Sigma).$$

We recall the Green–Stokes Theorem which is the model of *all* boundary reduction formulas. It takes the following form in our context.

LEMMA 1.1. (Green's Formula). *All (compatible) Dirac operators are symmetric with*

$$(\mathrm{A}\,f,g) - (f,\mathrm{A}\,g) = -\int_\Sigma \langle \sigma\gamma_\infty f, \gamma_\infty g\rangle\, d\mathrm{vol}_\Sigma$$

for any $f,g \in C^\infty(M;S)$.

From the point of view of analysis, however, one is mainly interested in L_2–sections or even in distributional sections. But γ_∞ extends to a bounded map

$$\gamma_s : H^s(M;S) \longrightarrow H^{s-\frac{1}{2}}(\Sigma; S|_\Sigma) \tag{1.2}$$

only for $s > \frac{1}{2}$. For general s we can, however, prove the following restriction theorem by a Poisson operator type argument for elliptic operators of first order. First we define:

DEFINITION 1.2. For any real s we shall distinguish the *null spaces*

$$\ker(\mathrm{A}, s) := \{f \in H^s(M;S) \mid \mathrm{A}\, f = 0\}$$

and the corresponding *Cauchy data* (or *Hardy*) *spaces*

$$\Lambda(\mathrm{A}, s) := \overline{\gamma_\infty\{f \in C^\infty(M;S) \mid \mathrm{A}\, f = 0 \text{ in } M \setminus \Sigma\}}^{H^{s-\frac{1}{2}}(\Sigma;S|\Sigma)} .$$

The null spaces consist of sections which are distributional for negative s; by elliptic regularity they are smooth in the interior; and by a Riesz operator argument they can be shown to possess a trace over the boundary in $H^{s-\frac{1}{2}}(\Sigma; S|_\Sigma)$. More precisely, we have the following *General Restriction Theorem*:

THEOREM 1.3. ([**11**]). (a) *Let $f \in H^s(M;S)$ and $\mathrm{A}\, f \in H^t(M;S)$ with $t > -\frac{1}{2}$. Then the trace of f on Σ is well-defined in $H^{s-\frac{1}{2}}(\Sigma; S|_\Sigma)$ for any real s.*
(b) *For any real s the mapping*

$$\mathcal{K} := r_+ \widetilde{\mathrm{A}}^{-1} \gamma_\infty^* \sigma : C^\infty(\Sigma; S|_\Sigma) \longrightarrow C^\infty(M;S)$$

extends to a continuous map $\mathcal{K}^{(s)} : H^{s-1/2}(\Sigma; S|_\Sigma) \to H^s(M;S)$ with range $\mathcal{K}^{(s)} = \ker(\mathrm{A}, s)$.
(c) *In fact, the restriction*

$$\mathcal{K}^{(s)}|_{\Lambda(\mathrm{A},s)} : \Lambda(\mathrm{A}, s) \longrightarrow \ker(\mathrm{A}, s)$$

is an homeomorphism (relative to the respective Sobolev norms), and we have $\Lambda(\mathrm{A}, s) = \gamma_s(\ker(\mathrm{A}, s))$ for all real s.
(d) *The Cauchy data space $\Lambda(\mathrm{A}, \frac{1}{2})$ is a Lagrangian subspace of the Hilbert space $L_2(\Sigma; S|_\Sigma)$ equipped with the symplectic form $\omega(\varphi, \psi) := (\sigma\varphi, \psi)$.*

In the preceding theorem, $\widetilde{\mathrm{A}}$ denotes the invertible double of A defined on the closed double $\widetilde{M} = -M \cup_\Sigma M$, where the bundles are glued by σ. We denote the restriction operator by $r_+ : H^s(\widetilde{M}; \widetilde{S}) \to H^s(M;S)$ and the dual of γ_∞ in the distributional sense by γ_∞^*. The composition $\mathcal{P}(\mathrm{A}) := \gamma_\infty \circ \mathcal{K}$ is called the (*Szegö–*) *Calderón projection.*

It is a pseudo-differential projection. Its extension $\mathcal{P}(\mathrm{A})^{(s)}$ to the sth Sobolev space over Σ has the corresponding Cauchy data space $\Lambda(\mathrm{A}, s+\frac{1}{2})$ as its range.

2. Boundary Reduction of the Index

Up to now we have discussed only the restriction of the section spaces to the boundary, and in particular the restriction of the null spaces. Those are all infinite-dimensional spaces. To obtain finite-dimensional null spaces we must impose suitable boundary conditions. To begin with, we restrict ourselves to the Grassmannian $\mathcal{G}r(\mathrm{A})$ of all pseudo-differential projections which differ from the Calderón projection $\mathcal{P}(\mathrm{A})$ by an operator of order -1. It has countable many connected components; two projections P_1, P_2 belong to the same component, if and only if the *virtual codimension*

$$\mathbf{i}(P_2, P_1) := \operatorname{index}\{P_2P_1 : \operatorname{range} P_1 \to \operatorname{range} P_2\} \tag{2.1}$$

of P_1 in P_2 vanishes; the higher homotopy groups of each connected component are given by Bott periodicity.

2.1. Boundary Reduction Formulas. Let $P \in \mathcal{G}r(\mathrm{A})$. We consider the extension

$$\mathrm{A}_P : \operatorname{dom}(\mathrm{A}_P) \longrightarrow L_2(M; S) \tag{2.2}$$

of A defined by the domain

$$\operatorname{dom}(\mathrm{A}_P) := \{f \in H^1(M; S) \mid P^{(0)}\gamma_1(f) = 0\}. \tag{2.3}$$

It is a closed operator in $L_2(M; S)$ with finite-dimensional kernel and cokernel. That A_P is a closed L_2 realization can be deduced from the explicit description of a left parametrix for A by

$$(r^+(\widetilde{\mathrm{A}})^{-1}e^+)\,\mathrm{A} = \mathrm{Id} - \mathcal{K}\gamma_\infty, \tag{2.4}$$

which is a direct consequence of the Calderón construction. We have an explicit formula for the adjoint operator

$$(\mathrm{A}_P)^* = \mathrm{A}_{\sigma(\mathrm{Id}-P)\sigma^*}, \tag{2.5}$$

and, in particular, *Seeley's Boundary Reduction Formula for the Index* ([**40**], see also [**11**]):

PROPOSITION 2.1.

$$\operatorname{index} \mathrm{A}_P = \operatorname{index}\{P\mathcal{P}(\mathrm{A}) : \Lambda(\mathrm{A}, \tfrac{1}{2}) \to \operatorname{range}(P^{(0)})\}.$$

To prove the Proposition, one begins with the 1-1 relation between the subspace $\ker \mathrm{A}_P$ of $H^1(M;S)$ and the space $\ker\{P\mathcal{P}(\mathrm{A}) : \Lambda(\mathrm{A},\frac{1}{2}) \longrightarrow \mathrm{range}(P^{(0)})\}$ which is contained in the kernel of the elliptic pseudo-differential operator of order zero (the 'fan') $(\mathrm{Id}-\mathcal{P}(\mathrm{A}))+\mathcal{P}(\mathrm{A})P^*P\mathcal{P}(\mathrm{A})$ and therefore only consists of smooth sections. So, the reduction to the fan is the true boundary reduction, even though the fan does not appear in the final formulation of Proposition 2.1.

Note that the bundle S of Clifford modules splits naturally into $S = S^- \oplus S^+$ on even-dimensional manifolds. Correspondingly, the Dirac operator splits

$$\mathrm{A} = \begin{pmatrix} 0 & \mathrm{A}^- \\ \mathrm{A}^+ & 0 \end{pmatrix} \tag{2.6}$$

with the chiral Dirac operators A^+ and $\mathrm{A}^- = (\mathrm{A}^+)^*$. Also the Calderón operator and the Grassmannian split, and all the preceding results remain valid in the chiral setting. Notice, however, that the Lagrangian property of the Cauchy data space (Theorem 1.3d) has to be replaced by the *chiral twisting property*

$$\sigma(\Lambda(A^+,\frac{1}{2})) = \Lambda(A^-,\frac{1}{2})^{\perp}. \tag{2.7}$$

NOTE. In the rest of this section, we shall not always distinguish between the *total* and the *chiral* Dirac operator because all the index formulas we are going to present are valid in both cases.

A more specific boundary reduction formula can be obtained by choosing the *Atiyah–Patodi–Singer boundary condition* defined by the spectral projection $\Pi_{\geq}(\mathcal{B})$ of $L_2(\Sigma;S|_\Sigma)$ onto the space spanned by the eigensections of the boundary Dirac operator $\mathcal{B}$ corresponding to the non-negative eigenvalues. It is a pseudo-differential operator with the same principal symbol as the Calderón projection $\mathcal{P}(\mathrm{A})$. It can be shown that the difference $\mathcal{P}(\Sigma)-\Pi_{\geq}(\mathcal{B})$ is a smoothing operator ([**34**], see also [**20**], [**28**], and [**45**] for related, partially also adiabatic results). Then we have

$$\mathrm{index}\ \mathrm{A}_{\Pi_{\geq}} + \tfrac{1}{2}(\eta_{\mathcal{B}}(0)+\dim\ker\mathcal{B}) = \int_M \alpha(x)\,dx. \tag{2.8}$$

Here $\alpha(x)$ denotes the locally defined *index density* of A and

$$\eta_{\mathcal{B}}(z) := \sum_{\lambda\in\mathrm{spec}\,\mathcal{B}\setminus\{0\}} \mathrm{sign}\,\lambda|\lambda|^{-z} = \frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty t^{\frac{z-1}{2}}\ \mathrm{Tr}(\mathcal{B}e^{-t\mathcal{B}^2})\,dt \tag{2.9}$$

denotes the *η-function* of $\mathcal{B}$. It is (i) well defined through absolute convergence for $\Re(z)$ large; (ii) it extends to a meromorphic function in the complex plane with isolated simple poles; (iii) its residues are given by a local formula; and (iv) it has a finite value at $z=0$ (see e.g. Gilkey [**19**]).

Equation (2.8) is modelled after the celebre Gauss–Bonnet Theorem. It separates the contributions to the index from the whole manifold and from the structure on the boundary. A special feature is that it expresses the spectral quantities on the left side in terms of a classical integral on the right side.

2.2. Boundary Correction Formulas. Since by definition the index of the operator $\mathrm{A}_{\mathcal{P}(\mathrm{A})}$ vanishes, we may read Proposition 2.1 as providing a boundary correction formula for the difference index $\mathrm{A}_P -$ index $\mathrm{A}_{\mathcal{P}(\mathrm{A})}$. We have, however, also true *boundary correction formulas* under a change of the boundary condition. If $P_1,\ P_2 \in \mathcal{G}r(\mathrm{A})$, we obtain a version of the classical *Agronovič–Dynin formula*:

$$\text{index}\,(\mathrm{A}_{P_1}) - \text{index}\,(\mathrm{A}_{P_2}) = \mathbf{i}(P_1,P_2)\,. \tag{2.10}$$

On odd–dimensional manifolds, the chiral splitting of the bundle $S|_\Sigma$ over the (even–dimensional) boundary provides a splitting $\mathcal{B} = \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix}$ and two local boundary conditions induced by the respective chiral projections $\Pi_\pm$. The corresponding boundary correction formula becomes

$$\text{index } \mathrm{A}_{\Pi_-} - \text{index } \mathrm{A}_{\Pi_+} = \text{index}\, B^+ \tag{2.11}$$

and leads at once to the *Cobordism Theorem*:

THEOREM 2.2. (Atiyah, Singer [**3**]) *The index of a (chiral) Dirac operator $B^+ : C^\infty(\Sigma; S^+) \longrightarrow C^\infty(\Sigma; S^-)$ over a closed even-dimensional manifold Σ vanishes, if the couple (Σ, S^+) is a 'boundary', i.e. if there exists a manifold M with boundary Σ and a bundle of Clifford modules over M which, restricted to Σ, is equal to $S^+ \oplus S^-$.*

PROOF. The result follows from (2.11) since index $\mathrm{A}_{\Pi_\pm}$ vanishes by Green's formula (Lemma 1.1). □

2.3. Pasting Formulas. A third type of *reduction* results to be discussed here are pasting formulas for the index. Let $X = M_1 \cup M_2$ be an even–dimensional closed partitioned manifold with $\partial M_1 = \partial M_2 = M_1 \cap M_2 = \Sigma$. As always in this Note, we assume that no connected component of M_1 or M_2 is closed. Let A be a Dirac operator over X, let A_j denote its restriction to M_j. By combining the boundary correction formula (2.10) and the Atiyah–Patodi–Singer index theorem (2.8) with Seeley's index formula for closed manifolds

$$\operatorname{index} \mathrm{A} = \int_X \alpha(x)\,dx\,,$$

we obtain a non–additivity formula for the splitting of the index over partitioned manifolds. The formula is valid for the total and (more interesting, on even–dimensional manifolds) for the chiral Dirac operator.

THEOREM 2.3. *Let P_i be projections belonging to $\mathcal{G}r(\mathrm{A}_j)$, $j = 1,2$. Then*

$$\operatorname{index}\ \mathrm{A} = \operatorname{index}(\mathrm{A}_1)_{P_1} + \operatorname{index}(\mathrm{A}_2)_{P_2} - \mathbf{i}(P_2, \mathrm{Id} - P_1).$$

It is immediate that $\mathbf{i}(P_2, \mathrm{Id} - P_1) = \operatorname{index}(\sigma(\partial_u + \mathcal{B}); P_2, P_1)$ where the last operator is on the cylinder $[0,1] \times \Sigma$ with boundary condition P_2 at $u = 0$ and P_1 at $u = 1$. The preceding pasting formula provides an analytic explanation for the combinatorial additivity of the Euler characteristic of compact $2k$–dimensional manifolds and the *Novikov additivity* of the signature of compact $4k$–dimensional manifolds because, in both cases, the projections are the complementary spectral projections. Hence the correction term vanishes on the separating hypersurface Σ. It may appear strange that the precise additivity can be obtained topologically just by applying the two related cohomology sequences (see [**4**], p. 588) whereas the analytical proof is rather delicate. But that is part of the game with spectral invariants where the analytical gains are highest when the topology can *not* make headway.

Another pasting formula for the index, the *Bojarski Conjecture*, which relates the 'quantum' quantity index with a 'classical' quantity (coming from the Lagrangian geometry of the Cauchy data spaces), was suggested in [**5**], proved in [**11**], and generalized in [**16**].

PROPOSITION 2.4. *Let X be a partitioned manifold as before and let $\mathcal{P}(\mathrm{A}_j)$ and $\Lambda(\mathrm{A}_j, \frac{1}{2})$ denote the corresponding Calderón projections and L_2 closures of the Cauchy data spaces, $j = 1,2$. If X is connected, we have*

$$\operatorname{index}\ \mathrm{A} = \mathbf{i}(\mathrm{Id} - \mathcal{P}(\mathrm{A}_2), \mathcal{P}(\mathrm{A}_1)) = \operatorname{index}(\Lambda(\mathrm{A}_1, \tfrac{1}{2}), \Lambda(\mathrm{A}_2, \tfrac{1}{2})).$$

Recall that

$$\begin{aligned}\operatorname{index}(\Lambda(\mathrm{A}_1, \tfrac{1}{2}), \Lambda(\mathrm{A}_2, \tfrac{1}{2})) &:= \dim(\Lambda(\mathrm{A}_1, \tfrac{1}{2}) \cap \Lambda(\mathrm{A}_2, \tfrac{1}{2})) \\ &\quad - \dim(L_2(\Sigma; S|_\Sigma)/(\Lambda(\mathrm{A}_1, \tfrac{1}{2}) + \Lambda(\mathrm{A}_2, \tfrac{1}{2})),\end{aligned}$$

and that pairs of closed subspaces for which the two dimensions in the preceding definition are finite are called *Fredholm pairs* of subspaces.

The proof depends on the unique continuation property for Dirac operators and the Lagrangian property of the Cauchy data spaces (Theorem 1.3d), more precisely, the chiral twisting property (2.7).

3. Boundary Reduction of the Spectral Flow

A particular clarity of the concept of boundary reduction is required *and* can be obtained when discussing the spectral flow $\mathbf{sf}\{\mathrm{A}_{t,D}\}$ of a continuous family of (from now on always *total*) Dirac operators A_t with the same principal symbol and the same domain D. Roughly speaking, the *spectral flow* is the difference between the number of eigenvalues, which change the sign from $-$ to $+$ as t goes from 0 to 1, and the number of eigenvalues which change the sign from $+$ to $-$. It can be defined in a satisfactory way, following a suggestion by J. Phillips (see [**6**] and [**31**]; see also [**30**] where recently a topological framework has been presented for the definition of the spectral flow for families of unbounded self–adjoint Fredholm operators with *varying* domain).

3.1. Boundary Reduction of Global Sections, Revisited. We give a systematic presentation of the boundary reduction of the solution spaces, inspired by M. Krein's construction of the maximal space of boundary values for closed symmetric operators (see [**6**], [**7**]). In all this section we stay in the real category and do not assume product structure near Σ unless otherwise stated.

We denote by A_0 the restriction of the (total, compatible) Dirac operator A to the space $C_0^\infty(M;S)$ of smooth sections with support in the interior of M. As mentioned above, there is no natural choice of a Sobolev space for the boundary reduction. Therefore, a systematic treatment of the boundary reduction may begin with the minimal closed extension $\mathrm{A}_{\min} := \overline{\mathrm{A}_0}$ and the adjoint $\mathrm{A}_{\max} := (\mathrm{A}_0)^*$ of A_0. Clearly, $\mathrm{A}_{\max}$ is the maximal closed extension. We have

$$D_{\min} := \mathrm{dom}(\mathrm{A}_{\min}) = \overline{C_0^\infty(M;S)}^{\mathcal{G}} = \overline{C_0^\infty(M;S)}^{H^1(M;S)}$$

and

$$D_{\max} := \mathrm{dom}(\mathrm{A}_{\max}) = \{u \in L_2(M;S) \mid \mathrm{A}\, u \in L_2(M;S) \text{ in the sense of distributions}\}.$$

Here, the superscript $\mathcal{G}$ means the closure in the graph norm which coincides with the 1st Sobolev norm on $C_0^\infty(M;S)$. We form the space $\boldsymbol{\beta}$ of *natural boundary values* with the *natural trace map* γ in the following way:

$$\begin{array}{ccc} D_{\max} & \xrightarrow{\gamma} & D_{\max}/D_{\min} =: \boldsymbol{\beta} \\ x & \mapsto & \gamma(x) = [x] := x + D_{\min} \,. \end{array}$$

The space $\boldsymbol{\beta}$ becomes a symplectic Hilbert space with the scalar product induced by the graph norm

$$(x,y)_{\mathcal{G}} := (x,y) + (\mathrm{A}\,x, \mathrm{A}\,y) \tag{3.1}$$

and the symplectic form given by Green's form

$$\omega([x],[y]) := (\mathrm{A}\,x, y) - (x, \mathrm{A}\,y) \qquad \text{for } [x],[y] \in \boldsymbol{\beta}. \tag{3.2}$$

One shows easily that ω is non–degenerate.

We define the *natural Cauchy data space* $\Lambda(\mathrm{A}) := \gamma(\ker \mathrm{A}_{\max})$ as a Lagrangian subspace of $\boldsymbol{\beta}$.

By Theorem 1.3a and, alternatively and in greater generality, by Hörmander [**21**] (Theorem 2.2.1 and the Estimate (2.2.8), p. 194), the space $\boldsymbol{\beta}$ is naturally embedded in the distribution space $H^{-\frac{1}{2}}(\Sigma; S|_\Sigma)$. If the metrics are product close to Σ, we can give a more precise description of the embedding, namely as a *graded* space of distributions. Let $\{\varphi_k, \lambda_k\}$ be a spectral resolution of $L_2(\Sigma)$ by eigensections of $\mathcal{B}$. (Here and in the following we do not mention the bundle S). For simplicity, we assume $\ker \mathcal{B} = \{0\}$. Then $\mathcal{B}\varphi_k = \lambda_k \varphi_k$ for all $k \in \mathbf{Z} \setminus \{0\}$, and $\lambda_{-k} = -\lambda_k$, $\sigma(\varphi_k) = \varphi_{-k}$, and $\sigma(\varphi_{-k}) = -\varphi_k$ for $k > 0$. We have ([**7**], Proposition 7.15, see also [**12**] for a more general setting)

$$\boldsymbol{\beta} = \boldsymbol{\beta}_- \oplus \boldsymbol{\beta}_+ \quad \text{with} \tag{3.3}$$
$$\boldsymbol{\beta}_- := \overline{[\{\varphi_k\}_{k<0}]}^{H^{\frac{1}{2}}(\Sigma)} \text{ and } \boldsymbol{\beta}_+ := \overline{[\{\varphi_k\}_{k>0}]}^{H^{-\frac{1}{2}}(\Sigma)} .$$

Then $\boldsymbol{\beta}_-$ and $\boldsymbol{\beta}_+$ are Lagrangian and transversal subspaces of $\boldsymbol{\beta}$. Let us define two Lagrangian and transversal subspaces $L_\pm$ of $L_2(\Sigma)$ in a similar way, namely by the closure in $L_2(\Sigma)$ of the linear span of the eigensections with negative, resp. with positive eigenvalue. We have that L_+ is dense in $\boldsymbol{\beta}_+$, and $\boldsymbol{\beta}_-$ is dense in L_-. This anti–symmetric relation may explain some of the well–observed delicacies of dealing with spectral invariants of continuous families of Dirac operators.

Moreover, we have $\gamma(D_{\mathrm{aps}}) = \boldsymbol{\beta}_-$, where

$$D_{\mathrm{aps}} := \{f \in H^1(M) \mid \Pi_>(f|_\Sigma) = 0\} \tag{3.4}$$

denotes the domain corresponding to the Atiyah–Patodi–Singer boundary condition. Note that a series $\sum_{k<0} c_k \varphi_k$ may converge to an element $\varphi \in L_2(\Sigma)$ without converging in $H^{\frac{1}{2}}(\Sigma)$. So, such $\varphi \in L_-$ can not appear as the trace at the boundary of any $f \in D_{\max}$.

For all domains D with $D_{\min} \subset D \subset D_{\max}$ and $\gamma(D)$ Lagrangian, we have that the extension $A_D := A_{\max}|_D$ is self–adjoint. It becomes a Fredholm operator, if and only if the pair $(\gamma(D), \Lambda(A))$ of Lagrangian subspaces of $\boldsymbol{\beta}$ becomes a Fredholm pair. In particular, $(\boldsymbol{\beta}_-, \Lambda(A))$ is a Fredholm pair. One can show the following proposition in the product case (and may expect it to be valid also if the metric structures near the boundary are not product, see [**8**]):

PROPOSITION 3.1. *The $L_2(\Sigma)$ part $\Lambda(A) \cap L_2(\Sigma)$ of the natural Cauchy data space $\Lambda(A)$ is closed in $L_2(\Sigma)$. Actually, it is a Lagrangian subspace of $L_2(\Sigma)$ and it forms a Fredholm pair with the component L_- .*

3.2. Spectral Flow and Maslov Index. Partly, the project of understanding the topology of low–dimensional manifolds has to do with understanding the spectral flow of a family of Dirac operators with the same principal symbol on a partitioned manifold. Inspired by the surgery operations, so successful in topology, one replaces the spectral flow of a continuous 1–parameter family of self–adjoint Fredholm operators, which is a 'quantum' entity, by the *Maslov index* of a corresponding path of Lagrangian Fredholm pairs. The idea is due to Floer and was worked out subsequently by Yoshida in dimension 3, by Nicolaescu in all odd dimensions, and pushed further by Cappell, Lee and Miller, Daniel and Kirk and many other authors. For a survey, see [**6**], [**7**], [**17**].

First, we give a general result - without assuming differentiability of the path, invertibility at the ends, regular crossings, or a product structure near Σ. On a compact manifold M with boundary Σ, we consider a continuous family of Dirac operators $\{A_t\}$ (induced by a continuous family $\{\nabla_t\}$ of connections). We can fix the space $\boldsymbol{\beta}$ for the family and derive the continuity of the corresponding family $\{\Lambda(A_t\}$ of natural Cauchy data spaces from the weak unique continuation property $\ker A_{\max} \cap D_{\min} = \{0\}$ and the existence of a Fredholm extension with domain D. We obtain the *General Boundary Reduction Formula* for the spectral flow ([**6**]), which gives a family version of the Bojarski conjecture (our Proposition 2.4):

THEOREM 3.2. (a) *The spectral flow of the family $\{A_{t,D}\}$ is well defined under the preceding assumptions.*
(b) *The family $\{\Lambda(A_t) := \gamma(\ker A_{t,\max})\}$ is a continuous curve of Lagrangian subspaces of $\boldsymbol{\beta}$ which all make Fredholm pairs with $\gamma(D)$.*
(c) *The Maslov index* $\mathbf{mas}\left(\{\Lambda(A_t)\}, \gamma(D)\right)$ *is well-defined and we have*

$$\mathbf{sf}\,\{A_{t,D}\} = \mathbf{mas}\left(\{\Lambda(A_t)\}, \gamma(D)\right). \tag{3.5}$$

We have two corollaries for the spectral flow on closed manifolds with fixed hypersurface. The first corollary treats the case of a separating hypersurface, the second the case of a non–separating hypersurface. Both cases can be reduced to the situation of the preceding theorem by cutting the manifold along Σ. Then we get a manifold with two isometric boundary components in both cases (see [**7**]).

For product structure near Σ, one can obtain an L_2–version of the preceding Theorem which is closely related to the corresponding results by the aforementioned authors.

3.3. Correction Formula for the Spectral Flow. Let D, D' with $D_{\min} \subset D, D' \subset D_{\max}$ be two domains such that both $\{\mathrm{A}_{t,D}\}$ and $\{\mathrm{A}_{t,D'}\}$ become families of self–adjoint Fredholm operators. We assume that D and D' differ only by finite dimension, more precisely, that

$$\dim \gamma(D)/\gamma(D)\cap\gamma(D') = \dim\gamma(D')/\gamma(D)\cap\gamma(D') < +\infty \,. \tag{3.6}$$

Then we find from Theorem 3.2 (for details see [**7**], Theorem 6.5):

$$\begin{aligned}\mathbf{sf}\left(\{\mathrm{A}_{t,D}\}\right) - \mathbf{sf}\left(\{\mathrm{A}_{t,D'}\}\right) \\ = \mathbf{mas}\left(\{\Lambda(\mathrm{A}_t)\},\gamma(D')\right) - \mathbf{mas}\left(\{\Lambda(\mathrm{A}_t)\},\gamma(D)\right).\end{aligned} \tag{3.7}$$

On the right side of (3.7), the difference of the Maslov indices does not depend on the curve $\{\Lambda(\mathrm{A}_t)\}$, but only on the endpoints and is also called the *Hörmander index* of the four determining Lagrangian subspaces. The assumption (3.6) is rather restrictive. The pair of domains, for instance, defined by the Atiyah–Patodi–Singer projection and the Calderón projection, may not always satisfy that condition. For the present proof, however, it seems indispensable.

4. Boundary Reduction of the Determinant

The understanding of the determinant for Dirac operators is in rapid development. A formula, recently obtained by Scott and Wojciechowski, shows that the concept of boundary reduction is crucial for illuminating the relations between the competing concepts of the Fredholm determinant and the ζ–function regularized determinant. This section presents the Scott–Wojciechowski formula and puts it under the perspective of boundary reduction.

4.1. Three Determinant Concepts. Let us begin with the most simple integral of statistical mechanics, the *partition function* which is the model for all quadratic functionals:

$$Z(\beta) := \int_\Gamma e^{-\beta(Tx,x)}\,dx\,. \tag{4.1}$$

To begin with, let $\dim\Gamma = d < \infty$ and β real with $\beta > 0$ and assume that T is a strictly positive, symmetric endomorphism. In suitable coordinates we evaluate the Gaussian integrals and find

$$Z(\beta) = \pi^{d/2}\cdot\beta^{-d/2}\cdot(\det T)^{-\frac{1}{2}}\,.$$

Two fundamental problems arise when we try to take a Dirac operator for T and all sections in a bundle S over a compact manifold M for Γ according to the Feynman recipes in the Matthews and Salam program ([**25**], [**26**]. What if T is not > 0? And what if $d = +\infty$ (i.e. if M is not a finite set of points)? To get around the first problem, we reproduce a calculation made by Adams and Sen, [**1**]:

We decompose $\Gamma = \Gamma_+\times\Gamma_-$ and $T = T_+\oplus T_-$ with $T_+, -T_-$ strictly positive in $\Gamma_\pm$ and $\dim\Gamma_\pm = d_\pm$. Formally, we obtain by a suitable path in the complex plane approaching $\beta = 1$:

$$Z(1) = \pi^{\zeta/2}\,e^{\pm i\frac{\pi}{4}(\zeta-\eta)}\,(\det|T|)^{-\frac{1}{2}}\,, \tag{4.2}$$

with $\zeta := d_+ + d_-$ and $\eta := d_+ - d_-$.

We shall not discuss the various stochastic approaches to evaluate the integral when $d = +\infty$, but present two other concepts of the determinant.

From the point of view of functional analysis, the only natural concept is the *Fredholm determinant* of bounded operators acting on a separable Hilbert space of the form e^α or, more generally, $\mathrm{Id}+\alpha$ where α is of trace class. We recall the formulas

$$det_{Fr}\, e^\alpha = e^{\mathrm{Tr}\,\alpha} \quad\text{and}\quad det_{Fr}(Id+\alpha) = \sum_{k=0}^{\infty}\mathrm{Tr}\ \wedge^k\alpha\ . \tag{4.3}$$

The Fredholm determinant is notable for obeying the product rule, in difference to other generalizations of the determinant to infinite dimensions where the error of the product rule leads to new invariants, see e.g. [**22**].

Clearly, the parametrix (or Green's function) of a Dirac operator leads to operators for which the Fredholm determinant can be defined, but the relevant information about the spectrum of the Dirac operator does not seem sufficiently maintained. Note also that Quillen and

Segal's construction of the *determinant line bundle* is based on the concept of the Fredholm determinant, though without leading to *numbers* when the bundle is non–trivial.

A third concept is the *ζ–function regularized determinant*, based on Ray and Singer's observation that, formally,

$$\det T = \prod \lambda_j = \exp\{\sum \ln \lambda_j e^{-z \ln \lambda_j}|_{z=0}\} = e^{-\frac{d}{dz}\zeta_T(z)|_{z=0}},$$

where $\zeta_T(z) := \sum_{j=1}^{\infty} \lambda_j^{-z} = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} \operatorname{Tr} e^{-tT}\, dt$. By a result of Seeley, for a *positive definite self–adjoint elliptic* operator T of second order, acting on sections of a Hermitian vector bundle over a closed manifold M of dimension m, the function $\zeta_T(z)$ is holomorphic for $\Re(z)$ sufficiently large and can be extended meromorphically to the whole complex plane with $z = 0$ no pole.

The preceding definition does not apply immediately to the Dirac operator A which has infinitely many positive λ_j and negative eigenvalues $-\mu_j$. As an example, consider the operator $\mathrm{A}_a := -i\frac{d}{dx} + a : C^\infty(S^1) \to C^\infty(S^1)$ with $\mathrm{A}_a \varphi_k = k\varphi_k + a\varphi_k$ where $\varphi_k(x) = e^{ikx}$. It follows that $\operatorname{spec} \mathrm{A}_a = \{k + a\}_{k \in \mathbf{Z}}$.

Choosing the branch $(-1)^{-z} = e^{-i\pi z}$, we find

$$\begin{aligned} \zeta_{\mathrm{A}}(z) &= \sum \lambda_j^{-z} + \sum (-1)^{-z} \mu_j^{-z} \\ &= \frac{1}{2}\left\{\zeta_{\mathrm{A}^2}(\frac{z}{2}) + \eta_{\mathrm{A}}(z)\right\} + \frac{1}{2} e^{-i\pi z}\left\{\zeta_{\mathrm{A}^2}(\frac{z}{2}) - \eta_{\mathrm{A}}(z)\right\}, \end{aligned}$$

where $\eta_{\mathrm{A}}(z)$ is defined as in (2.9). Thus:

$$\zeta'_{\mathrm{A}}(0) = \frac{1}{2}\zeta'_{\mathrm{A}^2}(0) - \frac{i\pi}{2}\{\zeta_{\mathrm{A}^2}(0) - \eta_{\mathrm{A}}(0)\}$$

and

$$\det{}_\zeta \mathrm{A} = e^{-\zeta'_{\mathrm{A}}(0)} = e^{\frac{i\pi}{2}\{\zeta_{\mathrm{A}^2}(0) - \eta_{\mathrm{A}}(0)\}} \cdot e^{-\frac{1}{2}\zeta'_{\mathrm{A}^2}(0)}. \tag{4.4}$$

REMARK 4.1. **(a)** The three spectral invariants which enter in the preceding formula are of very different character. The first invariant, $\zeta_{\mathrm{A}^2}(0)$, is the most stable of the three. It is given by $\int_M \alpha(x)dx$, where $\alpha(x)$ denotes the index density which is a certain coefficient in the heat kernel expansion and is locally expressed by the coefficients of A. In particular, $\zeta_{\mathrm{A}^2}(0)$ vanishes on a closed odd–dimensional manifold.

The second invariant, $\eta_{\mathrm{A}}(0)$, is not given by an integral, not by a local formula. It depends, however, only on finitely many terms of the symbol of the resolvent $(\mathrm{A} - \lambda)^{-1}$ and will not change when one changes or removes a finite number of eigenvalues.

The third invariant, $\zeta'_{A^2}(0)$, is the most delicate of the three: even small changes of the eigenvalues will change the ζ'–invariant and hence the determinant and make it capable of detecting small *anomalies*.

(b) It should be noted that for Dirac operators in even dimensions also a vanishing determinant can provide some insight by replacing the operator A on the finite–dimensional kernel by the rotation $M(m, \theta_m) := m e^{i\gamma_5 \theta_m}$ with suitable 'mass' m and 'angle' θ_m and chiral switch $\gamma_5 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then the standardized determinant takes the form $\det'_\zeta \cdot m^{n_+ + n_-} e^{i(n_+ - n_-)\theta_m}$, where $\det'_\zeta$ denotes the rest–determinant after removing the zero–eigenvalues and $n_\pm$ denotes the dimension of the kernel of the chiral Dirac operators (see [**9**] and [**44**], based on the manifold–*with*–boundary case discussed in [**27**]).

(c) Our choice of the branch $e^{-i\pi z}$ for $(-1)^{-z}$ does not coincide with Singer's original choice $e^{i\pi z}$ in [**42**] which gives an extra term in the Scott–Wojciechowski Formula (our Theorem 4.2, see [**38**], Theorem 7.1), unless the argument for the Fredholm determinant is also suitably reversed.

4.2. The Scott–Wojciechowski Formula. In a recent paper by Wojciechowski it was shown that the ζ–regularized determinant can also be defined for certain self–adjoint Fredholm extensions of the Dirac operator on a compact manifold with boundary, namely when the domain is defined by a projection belonging to the *smooth, self–adjoint* Grassmannian

(4.5)
$$\mathcal{G}r^*_\infty(\mathrm{A}) = \{P \in \mathcal{G}r(\mathrm{A}) \mid P \text{ is self–adjoint, } P - \mathcal{P}(\mathrm{A}) \text{ is smoothing and range}(P^{(0)}) \text{ is Lagrangian in } L_2(\Sigma; S|_\Sigma)\}.$$

We refer to [**43**] for the details of the delicate estimates needed for establishing the three involved invariants in that case.

Scott and Wojciechowski have now established a *boundary correction formula* which relates the ζ–determinant and the Fredholm determinant (see [**37**], [**38**]).

THEOREM 4.2. (a) *Let* A *be a Dirac operator over an odd–dimensional compact manifold M with boundary Σ and let $P \in \mathcal{G}r^*_\infty(\mathrm{A})$. Then the range of $\mathcal{P}(\mathrm{A})$ and the range of P can be written as the graphs of unitary, elliptic operators of order 0, K, resp. T which differ from the operator $(B^+B^-)^{-1/2}B^+ : C^\infty(\Sigma; S^+|_\Sigma) \to C^\infty(\Sigma; S^-|_\Sigma)$ by a smoothing operator. Moreover,*

$$\det\nolimits_\zeta \mathrm{A}_P = \det\nolimits_\zeta \mathrm{A}_{\mathcal{P}(\mathrm{A})} \cdot \det\nolimits_{Fr} \tfrac{1}{2}(\mathrm{Id} + KT^{-1}). \tag{4.6}$$

(b) *If we pick the alternative sign for the phase of the determinant and define*

$$\det_\zeta \mathrm{A}_P = e^{\frac{i\pi}{2}\{\eta_{\mathrm{A}}(0)-\zeta_{\mathrm{A}^2}(0)\}} \cdot e^{-\frac{1}{2}\zeta'_{\mathrm{A}^2}(0)},$$

we obtain

$$\det_\zeta \mathrm{A}_P = \det_\zeta \mathrm{A}_{\mathcal{P}(\mathrm{A})} \cdot \det_{Fr}\tfrac{1}{2}(\mathrm{Id} + KT^{-1}) \cdot \det_{Fr}(g_P)\,, \tag{4.7}$$

where g_P denotes the unique unitary elliptic pseudo-differential operator of order zero acting on $C^\infty(\Sigma; S^-|_\Sigma)$ which differs from the identity by a smoothing operator, such that

$$P = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & g_P \end{pmatrix} \mathcal{P}(\mathrm{A}) \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & g_P^{-1} \end{pmatrix}.$$

In geometric terms, the key to the understanding of the preceding Theorem is that the determinant line bundle, parametrized by the projections belonging to the smooth self–adjoint Grassmannian, is *trivial* so that one can attribute complex numbers (up to a multiple) to the canonical determinant section. This may explain why earlier attempts to relate the concept of the ζ–determinant with the Fredholm determinant (see for instance [**15**]) had to be content with discussing the metric of the determinant bundle in terms of the ζ–determinant, and why the break–through in understanding the mutual relation required a concept of boundary reduction.

4.3. A 1–Dimensional Toy Model. Some basic ideas of the proof of the Scott–Wojciechowski Formula can be best understood by analyzing a simple example on the interval $M = [0, 2\pi]$. We follow [**10**] which gives a setting consistent with the Scott–Wojciechowski Formula, though with the alternative sign choice in the definition of the ζ–determinant. For a review of other and more general approaches we refer to [**23**] (see also [**13**] and [**34**]).

Let T be a unitary $N \times N$–matrix and $\mathrm{A} := -i\frac{d}{dx} \oplus \cdots \oplus -i\frac{d}{dx}$ be acting on

$$\operatorname{dom} \mathrm{A}_T := \{f \in H^1([0, 2\pi]; \mathbf{C}^N) \mid f(2\pi) = Tf(0)\}.$$

The natural space $\boldsymbol{\beta}$ of boundary values is now finite–dimensional and the Grassmannian of boundary conditions of Atiyah–Patodi–Singer type is not really defined. Anyhow, we can let $-\mathrm{Id} \in U(N)$ play the role of the Calderón projection. We find $\operatorname{spec} \mathrm{A}_{-\mathrm{Id}} = \{\frac{2k+1}{2}\}_{k\in\mathbf{Z}}$. By the Hurwitz ζ–function we find $\det_\zeta \mathrm{A}_{-\mathrm{Id}} = 2^N$. Then the Scott–Wojciechowski Formula would imply:

PROPOSITION 4.3.

$$\det_\zeta \mathrm{A}_T = 2^N \det \frac{\mathrm{Id}_{\mathbf{C}^N} - T^{-1}}{2}. \tag{4.8}$$

This is precisely what we obtained in [**10**].

PROOF. The proof of (4.8) in [**10**] uses a variational argument. Let $T_r := e^{ir\alpha}T$ be such a variation with α a self-adjoint $N \times N$–matrix. For the phase and the modulus of the boundary term (the right side) in (4.8) we find at once

$$\Im\Big(\frac{d}{dr}\ln 2^N \det \frac{\mathrm{Id}_{\mathbf{C}^N} - T_r^{-1}}{2}\Big) = -\frac{\operatorname{tr}\alpha}{2} \quad \text{and} \tag{4.9}$$
$$\Re\Big(\cdot\Big) = -\frac{i}{2}\operatorname{tr}\Big(\alpha(\mathrm{Id}-T)^{-1}(\mathrm{Id}+T)\Big).$$

The variation of the phase and modulus of the ζ–determinant is more delicate. First we fix the domain under the variation by replacing the operator A_{T_r} by the unitarily equivalent operator $\mathrm{A}_r := (U_r\,\mathrm{A}\,U_r^{-1})_T$, where $U_r(x) := T^{-1}e^{ir\chi(x)\alpha}T$ with a smooth cut–off function χ being constant equal 1 near 0 and constant equal 0 near 2π. Then $\zeta_{\mathrm{A}_r^2}(0)$ is constant vanishing. By Duhamel's principle we can replace the heat kernel for $\chi'(x)e^{-\varepsilon\,\mathrm{A}_0^2}$ by the standard heat kernel and we obtain

$$\frac{\pi}{2}\frac{d}{dr}\eta_{\mathrm{A}_r}(0)|_{r=0} = \frac{\pi}{2}\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\operatorname{Tr}\dot{\mathrm{A}}_0 e^{-\varepsilon\,\mathrm{A}_0^2} = -\frac{\operatorname{tr}\alpha}{2} \tag{4.10}$$

and, plugging in the standard integral kernel for A^{-1},

$$\frac{d}{dr}\Big(-\frac{1}{2}\zeta'_{\mathrm{A}_r^2}(0)\Big)|_{r=0} = \lim_{\varepsilon\to 0}\operatorname{Tr}\dot{\mathrm{A}}_0\,\mathrm{A}_0^{-1}\,e^{-\varepsilon\,\mathrm{A}_0^2} \tag{4.11}$$
$$= -\frac{i}{2}\operatorname{tr}\Big(\alpha(\mathrm{Id}-T)^{-1}(\mathrm{Id}+T)\Big).$$

□

Comparing the middle terms in equations (4.10) and (4.11) we see that the variation of the ζ'–term is much more delicate than the variation of the η–invariant.

We may rest assured that Proposition 4.3 will not lead to new number theory insights since the relevant integrals of one variable have probably all been checked before in analytic number theory. We may, however, exploit the simple relation $\zeta_{-\frac{d^2}{dx^2}|S^1}(z) = 2\zeta_{Riem}(2z)$ to develop an approach which in higher dimensions, applied to Theorem 4.2 for suitable symmetric spaces, may lead to new insight in the Dirichlet ζ–function. The following calculation (suggested by K.P. Wojciechowski) shall serve not only as a pilot for future, hopefully more relevant number theoretical calculations but also to check the validity and in particular the signs of the reduction formula of Proposition 4.3.

We consider the case $N = 1$ and a path $\{T_r := e^{2\pi i r}\}$ close to $r = \frac{1}{2}$. We find

$$\operatorname{spec} \mathrm{A}_{T_r} = \operatorname{spec}(-i\frac{d}{dx} + r)_{\mathrm{Id}} = \operatorname{spec}(-i\frac{d}{dx} + r)|_{S^1} = \{k + r\}_{k \in \mathbf{Z}}\,.$$

Note that $\dot{\mathrm{A}}_r = 1$. We find

$$\begin{aligned}
(4.12)\quad \frac{d}{dr}(\ln \det\nolimits_\zeta \mathrm{A}_r^2)|_{r=\frac{1}{2}} &= \frac{d}{dr}\Big(-\int_0^\infty \frac{1}{t}\operatorname{Tr} e^{-t\mathrm{A}_r^2}\, dt\Big)|_{r=\frac{1}{2}} \\
&= 2\int_0^\infty \operatorname{Tr} \dot{\mathrm{A}}_{\frac{1}{2}}\, \mathrm{A}_{\frac{1}{2}}\, e^{-t\mathrm{A}_{\frac{1}{2}}^2}\, dt \\
&= 2\frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty t^{\frac{z-1}{2}} \operatorname{Tr} \mathrm{A}_{\frac{1}{2}}\, e^{-t\mathrm{A}_{\frac{1}{2}}^2}\, dt|_{z=1} = 2\eta_{\mathrm{A}_{\frac{1}{2}}}(1).
\end{aligned}$$

We expand like in [**24**], but at $r = \frac{1}{2}$, and get for the right side of (4.12)

$$\begin{aligned}
\eta_{\mathrm{A}_{\frac{1}{2}}}(1) &= \frac{1}{r} - 2\sum_{n=0}^{\infty} r^{2n+1}\zeta_{Riem}(2(n+1)) \\
&= 2\Big(1 - \sum_{n=0}^{\infty} 2^{-2n-1}\zeta_{Riem}(2(n+1))\Big).
\end{aligned}$$

By elementary summation, the difference vanishes in the preceding equation in the parentheses on the right side. Alternatively, we could determine it by using our boundary reduction formula for the determinant of Proposition 4.3. We recall from [**19**] that $\eta_{\mathrm{A}_{T_r}}(0) = \operatorname{sign} r - 2r$ and find

$$\det\nolimits_\zeta \mathrm{A}_{T_r} = 2^1\frac{1 - e^{-2\pi i r}}{2} = e^{\frac{i\pi}{2}(1-2r)} \cdot 2\sin \pi r.$$

So, $\ln(2\sin \pi r) = -\zeta'_{\mathrm{A}_{T_r}^2}(0)$ and we get for the left side of (4.12)

$$\frac{d}{dr}(\ln \det\nolimits_\zeta \mathrm{A}_{T_r}^2)|_{r=\frac{1}{2}} = \frac{1}{2\sin \pi r}2\cos \pi r|_{r=\frac{1}{2}} = 0$$

in nice agreement with our previous completely elementary result.

Remark 4.4. As seen in (4.11), the variation of the modulus of the ζ–determinant contains a truly global term and can not be localized near the boundary. This may explain why the proof method, which was successful in the 1–dimensional case, can not literally be imitated in higher dimensions. It works with the phase as demonstrated in [**36**], but not with the modulus. In [**38**], the authors get around that problem by varying a quotient of determinants. That idea has been applied before by Forman [18] for a boundary reduction of the determinant of the Laplacian with local boundary conditions.

Then the main ingredients in the Scott–Wojciechowski proof are, first to assume that A_P is invertible or, equivalently, that the boundary integral operator $P\mathcal{P}(\mathrm{A}) : \Lambda(\mathrm{A}, \frac{1}{2}) \to \operatorname{range} P^{(0)}$ is invertible. Then they determine a parametrix for A_P and find

$$\mathrm{A}_P^{-1} = \mathrm{A}^{-1} - \mathcal{K} \circ (P\mathcal{P}(\mathrm{A}))^{-1} \circ P \circ \gamma \circ \mathrm{A}^{-1},$$

where A^{-1} denotes the parametrix for A introduced in (2.4). Then they show that the difference $\mathrm{A}_{P_1}^{-1} - \mathrm{A}_{P_2}^{-1}$ is a smoothing operator and they prove the variation formula

$$\frac{d}{dr}\left(\ln \det\nolimits_\zeta \mathrm{A}_{P_1,r} - \ln \det\nolimits_\zeta \mathrm{A}_{P_2,r}\right)|_{r=0} = \operatorname{Tr} \dot{\mathrm{A}}_0(\mathrm{A}_{P_1}^{-1} - \mathrm{A}_{P_2}^{-1}),$$

from which the boundary reduction follows.

References

[1] D.H. ADAMS AND S. SEN, "Partition Function of a quadratic functional and semiclassical approximation for Witten's 3-manifold invariant", Preprint, Dublin 1995.

[2] M.F. ATIYAH, V.K. PATODI, AND I.M. SINGER, Spectral asymmetry and Riemannian geometry. I, *Math. Proc. Cambridge Phil. Soc.* **77** (1975), 43–69.

[3] M.F. ATIYAH AND I.M. SINGER, The index of elliptic operators on compact manifolds, *Bull. Amer. Math. Soc.* **69** (1963), 422–433.

[4] —, —: The index of elliptic operators. III, *Ann. of Math.* **87** (1968), 564–604.

[5] B. BOJARSKI, The abstract linear conjugation problem and Fredholm pairs of subspaces, *in* "In Memoriam I.N. Vekua", Tbilisi Univ, Tbilisi, 1979, pp. 45–60 (Russian).

[6] B. BOOSS-BAVNBEK AND K. FURUTANI, The Maslov index – a functional analytical definition and the spectral flow formula, *Tokyo J. Math.* **21** (1998), 1-34.

[7] —. —. Symplectic functional analysis and spectral invariants, in: B. Booss-Bavnbek, K.P. Wojciechowski (eds.), "Geometric Aspects of Partial Differential Equations", Amer. Math. Soc. Series *Contemporary Mathematics*, vol. 242, Providence, R.I., 1999, pp. 53–83.

[8] B. BOOSS-BAVNBEK, K. FURUTANI, AND N. OTSUKI, "Symplectic Reduction: From Maximal to L_2-Boundary Values", Note, Roskilde and Tokyo, 1999 (*multiplied*).

[9] B. BOOSS-BAVNBEK, G. MORCHIO, F. STROCCHI, AND K.P. WOJCIECHOWSKI, Grassmannian and chiral anomaly, *J. Geom. Phys.* **22** (1997), 219–244.

[10] B. BOOSS-BAVNBEK, S.G. SCOTT, AND K.P. WOJCIECHOWSKI, The ζ-determinant and C-determinant on the Grassmannian in dimension one, *Letters in Math. Phys.* **45** (1998), 353–362.

[11] B. BOOSS-BAVNBEK AND K.P. WOJCIECHOWSKI, "Elliptic Boundary Problems for Dirac Operators", Birkhäuser, Boston, 1993.

[12] J. Brüning and M. Lesch, Spectral theory of boundary value problems for Dirac type operators, in: B. Booss-Bavnbek, K.P. Wojciechowski (eds.), "Geometric Aspects of Partial Differential Equations", Amer. Math. Soc. Series *Contemporary Mathematics*, vol. 242, Providence, R.I., 1999, pp. 203–215.

[13] S.E. Cappell, R. Lee, and E.Y. Miller, On the Maslov index, *Comm. Pure Appl. Math.* **47** (1994), 121–186.

[14] —, —, —, Selfadjoint elliptic operators and manifold decompositions Part I: Low eigenmodes and stretching, *Comm. Pure Appl. Math.* **49** (1996), 825–866. Part II: Spectral flow and Maslov index, *Comm. Pure Appl. Math.* **49** (1996), 869–909. Part III: Determinant line bundles and Lagrangian intersection, *Comm. Pure Appl. Math.* **52** (1999), 543–611.

[15] X. Dai and D.S. Freed, η–invariants and determinant lines, *J. Math. Phys.* **35** *(10)* (1994), 5155–5194.

[16] X. Dai and W. Zhang, Splitting of the family index, *Comm. Math. Phys.* **182** (1996), 303–318.

[17] M. Daniel and P. Kirk, with an appendix by K.P. Wojciechowski, "A general splitting formula for the spectral flow", Bloomington preprint, 1999.

[18] R. Forman, Functional determinants and geometry, *Invent. Math.* **88** (1987), 447–493.

[19] P.B. Gilkey, "Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theory" (Second Edition), CRC Press, Boca Raton, Florida, 1995.

[20] G. Grubb, Trace expansions for pseudodifferential boundary problems for Dirac–type operators and more general systems, *Ark. Mat.* **37** (1999), 45–86.

[21] L. Hörmander, Pseudo–differential operators and non–elliptic boundary problems, *Ann. of Math.* **83** (1966), 129–209.

[22] M. Kontsevitch and S. Vishik, Geometry of determinants of elliptic operators. In: Gindikin, S. (ed.) et al., *Functional Analysis on the eve of the 21st century'*. Vol. I. In honor of the eightieth birthday of I. M. Gelfand. Boston. Birkhäuser, Prog. Math. 131, 1993, pp. 173–197.

[23] M. Lesch and J. Tolksdorf, On the determinant of one–dimensional elliptic boundary value problems, *Comm. Math. Phys.* **193** (1998), 643–660.

[24] M. Lesch and K.P. Wojciechowski, On the η–invariant of generalized Atiyah–Patodi–Singer problems, *Illinois J. Math.* **40** (1996), 30–46.

[25] P.T. Matthews and A. Salam, The Green's function of quantized field, *Nuovo Cimento Series 9* **12** (1954), 563–565.

[26] —, —, Propagators of quantized field, *Nuovo Cimento Series 11* **2** (1955), 120–134.

[27] G. Morchio and F. Strocchi, Boundary terms, long range effects, and chiral symmetry breaking, in: Mitter, H., and Schweifer, W. (eds.), *Fields and Particles*, Proceedings Schladming, Austria, 1990, Springer–Verlag, Berlin–Heidelberg–New York, pp. 171–214.

[28] L. Nicolaescu, The Maslov index, the spectral flow, and decomposition of manifolds, *Duke Math. J.* **80** (1995), 485–533.

[29] —, "Generalized Symplectic Geometries and the Index of Families of Elliptic Problems", Mem. Amer. Math. Soc. vol. 609, Providence, 1997.

[30] —, "On the Space of Fredholm Self–adjoint Operators", Note, University of Notre Dame, 1999 (*multiplied*).

[31] J. Phillips, Self–adjoint Fredholm operators and spectral flow, *Canad. Math. Bull.* **39** (1996), 460–467.

[32] D.G. QUILLEN, Determinants of Cauchy-Riemann operators over a Riemann surface, *Funkcionalnyi Analiz i ego Prilozhenya* **19** (1985), 37–41.

[33] D. RAY AND I.M. SINGER, R-torsion and the Laplacian on Riemannian manifolds, *Adv. Math.* **7** (1971), 145–210.

[34] S.G. SCOTT, Determinants of Dirac boundary value problems over odd-dimensional manifolds, *Comm. Math. Phys.* **173** (1995), 43–76.

[35] —, Splitting the curvature of the determinant line bundle, *Proc. Am. Math. Soc.*, to appear.

[36] S.G. SCOTT AND K.P. WOJCIECHOWSKI, Determinants, Grassmannians and elliptic boundary problems for the Dirac operator, *Letters in Math. Phys.* **40** (1997), 135–145.

[37] —, —, ζ-determinant and the Quillen determinant on the Grassmannian of elliptic self-adjoint boundary conditions, *C. R. Acad. Sci., Serie I,* **328** (1999), 139–144.

[38] —, —, "The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary", Indianapolis and London, 1999, (*preprint*).

[39] R.T. SEELEY, Complex powers of an elliptic operator, AMS *Proc. Symp. Pure Math. X.* AMS Providence, 1997, 288–307.

[40] —: Topics in pseudo–differential operators, in: *CIME Conference on Pseudo–Differential Operators (Stresa 1968).* Ed. Cremonese, Rome, 1969, pp. 167–305.

[41] G.B. SEGAL, "The definition of conformal field theory", Oxford preprint, 1990.

[42] I.M. SINGER, Families of Dirac operators with applications to physics, *Asterisque, hors série*, 1985, 323–340.

[43] K.P. WOJCIECHOWSKI, The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian, *Comm. Math. Phys.* **201** (1999), 423–444.

[44] K.P. WOJCIECHOWSKI, S.G. SCOTT, G. MORCHIO, AND B. BOOSS–BAVNBEK, Determinants, manifolds with boundary and Dirac operators. In: V. Dietrich et al. (eds.), *Clifford Algebras and their applications in mathematical physics*, Kluwer Academic Publishers, 1998, pp. 423–432.

[45] T. YOSHIDA, Floer homology and splittings of manifolds, *Ann. of Math.* **134** (1991), 277–323.

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Unique Continuation Property for Dirac Operators, Revisited

Bernhelm Booss–Bavnbek

ABSTRACT. We summarize present basic knowledge about the Unique Continuation Property (UCP, also = Uniqueness of the Cauchy Problem) for elliptic operators of first order for a readership of geometers and topologists. We explain why the *weak* UCP, i.e. UCP from open subsets or, equivalently, UCP from separating hypersurfaces, is almost trivial for operators of Dirac type by simplifying a proof previously given by K.P. Wojciechowski and the author.

Introduction

One of the basic properties of an (elliptic) Dirac operator A is the weak Unique Continuation Property (UCP). Let $M = M_- \cup_\Sigma M_+$ be a closed connected partitioned manifold of dimension m with a separating hypersurface $\Sigma = M_- \cap M_+ = \partial M_- = \partial M_+$. The weak UCP guarantees that there are no *ghost* solutions of $Au = 0$, i.e. there are no solutions which vanish on M_- and have non–trivial support in M_+. This property is also called UCP *from open subsets* or *across any hypersurface.* For Euclidean (classical) Dirac operators the property follows from Holmgren's uniqueness theorem for scalar elliptic operators with real analytic coefficients (see e.g. Hörmander [**14**], Theorem 5.3.1).

In [**10**], Booss–Bavnbek and Wojciechowski gave a rather simple proof of the weak UCP for operators of Dirac type. Various geometric consequences are established.

1. Any non–trivial global solution leaves a non–trivial *trace* on the separating hypersurface Σ (and, in fact, on any hypersurface of a closed manifold with orientable normal bundle).

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2. If m is even, then the index

$$\operatorname{index} A = \dim\ker A - \dim\operatorname{coker} A$$

of the operator A over M is equal to the Fredholm index

$$\operatorname{index}(\mathcal{H}_-, \mathcal{H}_+) = \dim \mathcal{H}_- \cap \mathcal{H}_+ - \operatorname{codim}(\mathcal{H}_- \oplus \mathcal{H}_+)$$

of the Fredholm pair of the Cauchy data spaces $\mathcal{H}_\pm$. The spaces $\mathcal{H}_\pm$ consist of the traces at the boundary of the solutions in suitable Sobolev spaces on the two sides of the partitioned manifold. This formula is also called the *Bojarski Conjecture.*
3. If m is odd, then the families $(\{\mathcal{H}_{-,t}\}, \{\mathcal{H}_{+,t}\})$ of Fredholm Lagrangian pairs of Cauchy data spaces in a suitable symplectic Sobolev space over Σ are continuous for a continuous family $\{A_t\}$ of symmetric operators of Dirac type over M. To prove the continuity of the Cauchy data spaces, the weak UCP is required (see Booss-Bavnbek and Furutani [**9**], Theorem 3.8). Moreover, the weak UCP implies the *Yoshida-Nicolaescu Theorem*, namely that the spectral flow $\operatorname{sf}\big(\{A_t\}\big)$ is equal to the Maslov intersection index $\mathbf{m}\big(\{\mathcal{H}_{-,t}\}, \{\mathcal{H}_{+,t}\}\big)$.
4. Any operator of Dirac type over a smooth compact manifold with boundary can be extended to an invertible operator of Dirac type over the closed double.
5. The kernel of the maximal extension A^* of a (symmetric) operator of Dirac type over a smooth connected compact manifold M_+ with boundary Σ intersects the minimal domain

$$\operatorname{dom}_{\min}(A) = \overline{C_0^\infty(M_+ \setminus \Sigma; E)}^{H^1(M_+;E)}$$

transversely. It follows that A^* maps the first Sobolev space $H^1(M_+; E)$ *onto* $L_2(M_+; E)$.

Motivated by the classical mechanics of a vibrating membrane, Bär [**5**] and other authors use the notion of a *nodal set* for the zero locus $\{x \in M \mid u(x) = 0\}$ of a solution of $Au = 0$. Then, we may restate item 1 and item 5 of the preceding list by claiming that neither a hypersurface with orientable normal bundle of a closed connected partitioned manifold nor the boundary of a compact connected manifold are contained in the nodal set of a solution.

Using an unpublished system version of the Aronszajn–Cordes Theorem on the *hard* UCP, i.e. the UCP from a point (see below), Bär [**5**] obtains a sharper version of item 5, namely that no single connected component of the boundary is contained in the nodal set of a non–trivial solution.

In special cases, one can obtain much sharper results, namely relations between the nodal set of solutions of Dirac equations and the geometry of the underlying manifold. The model case is the Riemann–Roch Theorem giving a relation between a *divisor*, i.e. a weighted (discrete) set of zeros and poles; the dimensions of the solution spaces of meromorphic differential forms with the prescribed zeros and poles; and the genus of the underlying Riemann surface. See also Kotschick **[21]** for a survey of recent results by C. Taubes exploiting precise knowledge about the nodal set of solutions in the theory of 4–dimensional symplectic manifolds.

In this Note we shall give an expository, almost self–contained presentation of the weak UCP for operators of Dirac type. Our goal is to explain *how* simple the arguments for the weak UCP are and *why* other types of the UCP possibly need different types of arguments. Our point is that the symmetry of the principal symbol of the *tangential* operator for any hypersurface is sufficient to prove the weak UCP. One needs not to recur to the property of the Dirac Laplacian that its principal symbol is scalar and real.

In Section 1 we fix the notation and give the precise form of our Main Theorem on the weak UCP and of the lemmata on which its proof is built. The first lemma establishes a Carleman type inequality for operators of Dirac type. Here the point is that we use only a simple property of operators of Dirac type, namely that their *tangential* operator along any hypersurface is always an elliptic differential operator of first order with self–adjoint principal symbol. The second lemma explains how we obtain the weak UCP from our Carleman inequality. That is standard. We then summarize questions raised by the "suspicious" simplicity of our UCP proof.

In Section 2 we explain the relation between the weak UCP and various other UCP concepts, approaches, and puzzling aspects and riddles.

In the Appendix we present the details of the proofs of the two lemmata. We follow **[10]** with two simplifications: we do not need the compatibility of the connection and we refrain from any deformation of the metric.

1. The weak UCP for operators of Dirac type

Let (M, g) be a compact smooth Riemannian manifold (with or without boundary), $\dim M = m$. We denote by

$$C\ell(M) = \{C\ell(TM_x, g_x)\}_{x \in M}$$

the bundle of the Clifford algebras of the tangent spaces. Let $E \to M$ be a smooth vector bundle of Clifford modules. To keep the presentation as simple as possible, we assume that E is a complex vector bundle. Then the *Clifford multiplication* is a bundle map $\mathbf{c} : C\ell(M) \to \mathrm{Hom}(E,E)$ which yields a representation $\mathbf{c} : C\ell(TM_x, g_x) \to \mathrm{Hom}_{\mathbf{C}}(E_x, E_x)$ in each fibre. We may assume that the bundle E is equipped with a Hermitian metric which makes Clifford multiplication skew–symmetric

$$(1) \qquad \langle \mathbf{c}(v)s, s' \rangle = -\langle s, \mathbf{c}(v)s' \rangle \qquad \text{for } v \in TM_x \text{ and } s \in E_x.$$

DEFINITION 1. Any choice of a smooth connection

$$\nabla : C^\infty(M; E) \to C^\infty(M; T^*M \otimes E)$$

defines an *operator of Dirac type* $A := \mathbf{c} \circ \nabla$ under the Riemannian identification of the bundles TM and T^*M.

In local coordinates we have $A := \sum_{j=1}^m \mathbf{c}(e_j)\nabla_{e_j}$ for any orthonormal base $\{e_1, \dots, e_m\}$ of TM_x. Actually, we may choose a local frame in such a way that

$$\nabla_{e_j} = \frac{\partial}{\partial x_j} + \text{ zero order terms}$$

for all $1 \le j \le m$. So, locally, we have

$$(2) \qquad A := \sum_{j=1}^m \mathbf{c}(e_j) \frac{\partial}{\partial x_j} + \text{ zero order terms.}$$

It follows at once that the principal symbol $\sigma_1(A)(x,\xi)$ is given by Clifford multiplication with $i\xi$. Therefore any operator A of Dirac type is elliptic with symmetric principal symbol. Actually, if the connection ∇ is *compatible* with Clifford multiplication (i.e. $\nabla \mathbf{c} = 0$), then the operator A itself becomes symmetric. We shall, however, admit non–compatible metrics. Moreover, the *Dirac Laplacian* A^2 has principal symbol $\sigma_2(A^2)(x,\xi)$ given by the Riemannian metric $||\xi||^2$. So, it is scalar real (i.e. a real multiple of the identity) and elliptic.

In the literature, the last mentioned property of the Dirac Laplacian, namely that the principal symbol is real, is usually considered as the key property to establish the *weak* Unique Continuation Property, i.e. UCP from open subsets, for any real elliptic equation of second order (see e.g. Taylor [**26**], Chapter XIV, Corollary 2.9). In fact, Aronszajn [**4**] and Cordes [**12**] derived a much stronger UCP result from this property (*real, second order, elliptic, scalar* symbol), see below Theorem 7.

As shown in Booss–Bavnbek, Wojciechowski [**10**] (Theorem 8.2, pp. 43-49), there is an alternative shorter line of arguments for deriving the weak UCP directly for operators of Dirac type, using only the following well–known property.

LEMMA 2. *Let Σ be a closed hypersurface of M with orientable normal bundle. Let t denote a normal variable with fixed orientation such that a bicollar neighbourhood $\mathcal{N}$ of Σ is parametrized by $\Sigma \times [-\varepsilon, +\varepsilon]$. Then any operator of Dirac type can be rewritten in the form*

$$A|_{\mathcal{N}} = \mathbf{c}(dt)\Big(\frac{\partial}{\partial t} + B_t + C_t\Big), \tag{3}$$

where B_t is a self-adjoint elliptic operator on the parallel hypersurface Σ_t, and $C_t : E|_{\Sigma_t} \to E|_{\Sigma_t}$ a skew-symmetric operator of 0th order, actually a skew-symmetric bundle homomorphism.

PROOF. Let (t, y) denote the coordinates in a tubular neighbourhood of Σ. Locally, we have $y = (y_1, \dots, y_{m-1})$. Let $\mathbf{c}_t, \mathbf{c}_1, \dots, \mathbf{c}_{m-1}$ denote Clifford multiplication by the unit tangent vectors in normal, resp. tangential, directions. By (2), we have

$$\begin{aligned} A &= \mathbf{c}_t \frac{\partial}{\partial t} + \sum_{k=1}^{m-1} \mathbf{c}_k \frac{\partial}{\partial y_k} + \text{ zero order terms} \\ &= \mathbf{c}_t\Big(\frac{\partial}{\partial t} + \underbrace{\sum_{k=1}^{m-1} - \mathbf{c}_t\, \mathbf{c}_k \frac{\partial}{\partial y_k} + \text{ zero order terms}}_{=:\mathcal{B}_t}\Big). \end{aligned}$$

We shall call $\mathcal{B}_t$ the *tangential* operator component of the operator A. Clearly it is an elliptic differential operator of first order over Σ. From (1), we have

$$\begin{aligned} \Big(\mathbf{c}_t\, \mathbf{c}_k \frac{\partial}{\partial y_k}\Big)^* &= \Big(-\frac{\partial}{\partial y_k}\Big)(-\mathbf{c}_k)(-\mathbf{c}_t) = -\mathbf{c}_k\ \mathbf{c}_t \frac{\partial}{\partial y_k} + \text{ zero order terms} \\ &= \mathbf{c}_t\ \mathbf{c}_k \frac{\partial}{\partial y_k} + \text{ zero order terms}. \end{aligned}$$

So,

$$\mathcal{B}_t^* = \mathcal{B}_t + \text{ zero order terms}.$$

Hence, the principal symbol of $\mathcal{B}_t$ is self–adjoint. Then the assertion of the lemma is proved by setting

$$B_t := \frac{1}{2}(\mathcal{B}_t + \mathcal{B}_t^*) \qquad \text{and} \qquad C_t := \frac{1}{2}(\mathcal{B}_t - \mathcal{B}_t^*) \tag{4}$$

□

REMARK 3. (a) If all structures are product near Σ, the collar becomes a true cylinder and, moreover, the dependence of $\mathbf{c}_t$ and $\mathcal{B}_t$ on the normal variable can be removed on the collar. If, additionally, the tangential operator $\mathcal{B}$ is self–adjoint (which is the case for a compatible connection), we obtain a trivial case of UCP on the cylinder. The reason is that on the cylinder any solution can be expanded in the form $\sum h_j(t)\psi_j(y)$, where $\{\psi_j\}$ is a complete orthonormal system of eigenfunctions of $\mathcal{B}$. Even if we restrict ourselves to product structures near Σ or near the boundary, we must, however, admit non–product structures elsewhere. Then the preceding expansion argument breaks down.

(b) The point of the preceding lemma is not the splitting into a normal part $\frac{\partial}{\partial t}$ and a tangential part $\mathcal{B}_t$. That is natural for any differential operator of first order. Moreover, under that splitting one always obtains an elliptic tangential operator when starting with an elliptic operator. Finally, also the splitting of the tangential part $B_t + C_t$ into a symmetric part and an anti–symmetric part in (4) is canonical. The special property of Dirac type operators which distinguishes them from many other elliptic differential operators of first order is that the tangential part has elliptic *and* symmetric principal symbol. In general, the symmetrization made in (4) will destroy the ellipticity. That would be the case for a tangential operator with elliptic principal symbol of e.g. Jordan form $\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$ when $k = \frac{1}{2} + i\tau$ with $\tau \neq 0$.

(c) Note that the symmetry of an elliptic differential operator of first order $A = G(\frac{\partial}{\partial t} + \mathcal{B})$ only implies $\mathcal{B}^* = G\mathcal{B}G$ modulo zero order terms, whereas according to the preceding Lemma we have $\mathcal{B}^* = \mathcal{B}$ modulo zero order terms for any operator of Dirac type.

(d) A different product formula is used in Grubb, Seeley [**13**],

$$A|_{\mathcal{N}} = U\Big(\frac{\partial}{\partial t} + B + tP_t^{(1)} + P_t^{(0)}\Big), \tag{5}$$

where the identification of all parallel hypersurfaces Σ_t with Σ is induced by the metrics; U is a unitary morphism not depending on the normal variable t; neither B depends on t: it is a fixed self–adjoint elliptic operator of first order on Σ; so, when t moves, the change of the coefficients in front of the partial derivatives is coded away; the operator $P_t^{(1)}$ is of first order and the operator $P_t^{(0)}$ is of order zero.

Surprisingly, the preceding Lemma 2 implies almost directly the following theorem.

THEOREM 4. *Any operator A of Dirac type has the weak unique continuation property.*

Here, weak unique continuation property (weak UCP) means the *UCP from open subsets*, i.e. any solution of $Au = 0$, which vanishes on an open subset ω of M, vanishes on the whole connected component of the manifold.

Basically, the proof of Theorem 4 follows the standard lines of the UCP literature. First we localize and convexify the situation and we introduce spherical coordinates. Without loss of generality we may assume that ω is maximal, namely the union of all open subsets where u vanishes. If the solution u does not vanish on the whole connected component containing ω, we consider a point $x_0 \in \operatorname{supp} u \cap \partial\omega$. We choose a point p inside of ω such that the ball around p with radius $r := \operatorname{dist}(x_0, p)$ is contained in $\overline{\omega}$. We call the coordinate running from p to x_0 *normal* coordinate and denote it by t. The boundary of the ball around p of radius r is a hypersphere and will be denoted by $S_{p,0}$. It goes through x_0 which has normal coordinate $t = 0$. Correspondingly, we have larger hyperspheres $S_{p,t} \subset M$ for $0 \le t \le T$ with $T > 0$ sufficiently small. In such a way we have parametrized an annular region $\mathcal{N}_T := \{S_{p,t}\}_{t\in[0,T]}$ around p of width T and inner radius r, ranging from the hypersphere $S_{p,0}$ which is contained in $\overline{\omega}$ to the hypersphere $S_{p,T}$ which cuts deeply into $\operatorname{supp} u$, if $\operatorname{supp} u$ is not empty.

Next, we replace the solution u by a section

$$v(t,y) := \varphi(t)u(t,y) \tag{6}$$

with a smooth bump function φ with $\varphi(t) = 1$ for $t \le 0.8\,T$ and $\varphi(t) = 0$ for $t \ge 0.9\,T$. Then $\operatorname{supp} v$ is contained in $\mathcal{N}_T$. More precisely, it is contained in the annular region $\mathcal{N}_{0.9T}$. Moreover, $\operatorname{supp}(Av)$ is contained in the annular region $0.8\,T \le t \le 0.9\,T$.

Theorem 4 follows immediately from the two following lemmata.

LEMMA 5. *Let $A : C^\infty(M;E) \to C^\infty(M;E)$ be a linear elliptic differential operator of order 1. Let us assume that A can be written on $\mathcal{N}_T$ in the product form*

$$A = \sigma(t,y)\left(\frac{\partial}{\partial t} + B_t + C_t\right), \qquad t \in [0,T],\ y \in S_{p,t},$$

where $\sigma(t,y)$ is invertible, B_t is a symmetric elliptic differential operator of order 1 over $S_{p,t}$, and C_t a skew-symmetric bundle homomorphism over $S_{p,t}$. Let v denote a section made from a solution u as in (6). Then for T sufficiently small, there exists a constant C such that *the* Carleman *inequality*

$$(7)\quad R\int_{t=0}^{T}\int_{S_{p,t}} e^{R(T-t)^2}\,\|v(t,y)\|^2\,dy\,dt \leq C\int_{t=0}^{T}\int_{S_{p,t}} e^{R(T-t)^2}\,\|Av(t,y)\|^2\,dy\,dt$$

holds for any real R sufficiently large.

LEMMA 6. *If* (7) *holds for any sufficiently large* $R>0$, *then* u *is equal* 0 *on* $\mathcal{N}_{T/2}$.

We shall give a short–cut presentation of the proofs of the two preceding lemmata in the Appendix to this Note. One will see that the underlying argument for the weak UCP for operators of Dirac operators truly consists only of a short series of "cheap tricks".

2. Different Concepts of UCP

Now we sort out which aspects of the UCP are intricate and which are simple; and which aspects are well–understood and which are seemingly still unsettled.

2.1. UCP from a point versus UCP from open subsets. One reason for the intricate character of the UCP literature is the emphasis on the *hard* UCP, i.e. UCP from a point. That was the main achievement by N. Aronszajn [**4**] and H.O. Cordes [**12**] in their legendary parallel papers on elliptic equations. Independently of each other, they proved the following (and a little more).

THEOREM 7. (Aronszajn, Cordes, 1956). *Let L be a linear scalar elliptic operator of second order with smooth coefficients and with real principal symbol. A sufficient condition for the constant vanishing of a solution u of* $Lu=0$ *in a connected domain is that u vanishes at a point* x_0 *with all its derivatives.*

REMARK 8. (a) As pointed out by Aronszajn [**4**], Remark 3 on p. 248, the Theorem remains valid for second order elliptic *systems*, if the principal symbol is in diagonal form, real, and scalar, i.e. all diagonal elements coincide. That is exactly the case for the square of operators of Dirac type, i.e. the Dirac Laplacian discussed above (see also Kazdan [**20**] for an alternative approach).
(b) There is an interesting explanation for the fact that almost all UCP literature is on equations and not on systems (besides for Kazdan [**20**] and the few publications dealing with Dirac operators, see also

e.g. the delicate articles Berthier and Georgescu [**7**], Jerison [**18**], Kalf [**19**], Roze [**24**], and Vogelsang [**28**] who treat Dirac operators with non–smooth coefficients).

The UCP for an elliptic first order equation is well–established because of simple characteristics. From that point of view only elliptic *systems* are interesting. Anyhow, all real (resp. complex) elliptic differential equations of first order are only in one (resp. two) variables. For higher order systems, however, one can have the impression that equations are more complicated than systems because one has "less space" to reduce to standard cases.

(c) Contrary to a common belief among geometers, it turns out that operators of Dirac type actually provide the most simple case of UCP and that the Dirac Laplacian is a far more subtle object than the original Dirac type operator regarding UCP.

The UCP from an open subset suffices for many geometric applications. One reason for that is the following simple property of elliptic operators of first order, seemingly first observed in Palais [**22**].

PROPOSITION 9. *Let M be a closed partitioned Riemannian manifold, $M = M_- \cup M_+$ with $M_- \cap M_+ = \Sigma = \partial M_- = \partial M_+$, and let $A : C^\infty(M;E) \to C^\infty(M;F)$ be an elliptic differential operator of first order acting between sections of vector bundles E and F. Then any section $u_+ \in C^\infty(M_+; E|_{M_+})$ with $Au_+ = 0$ over M_+ and $u_+|_\Sigma = 0$ can be continued to a smooth solution for the operator A over the whole manifold M by setting*

$$u := \begin{cases} u_+ & \text{on } M_+ \\ 0 & \text{on } X \setminus M_+ . \end{cases}$$

PROOF. We show that u is a weak solution for A over the whole of M. Apply Green's formula on M_+ with $A = \sigma(t,y)(\frac{\partial}{\partial t} + \mathcal{B}_t)$ close to Σ.

$$\begin{aligned} \langle u; A^* v\rangle_E &= \int_M (u(x); A^*v(x))dvol(x) \\ &= \int_{M_+} (u_+(x); A^*v_+(x))dvol(x) + \int_{M_-} 0 \\ &= \int_{M_+} (Au_+(x); v_+(x))dvol(x) \\ &\qquad + \int_\Sigma (\sigma(0,y)u_+(y); v_+(y))dvol(y) = 0 \end{aligned}$$

for any $v \in C^\infty(M;F)$ with $v_+ := v|_{M_+}$. By the regularity of the solutions of elliptic equations over closed manifolds it follows that $u \in C^\infty(M;E)$ and $\operatorname{supp} u \subset M_+$. □

Applying Theorem 4 yields the UCP from hypersurfaces with orientable normal bundle for operators of Dirac type. More precisely, we have:

COROLLARY 10. *(a) Let A be an operator of Dirac type over a closed connected manifold M and Σ a hypersurface with orientable normal bundle. Let $u \in C^\infty(M;E)$ satisfy $Au = 0$ and $u|_\Sigma = 0$. Then $u = 0$ on M.*
(b) If M is a compact connected manifold with (not necessarily connected) boundary $\partial M = \Sigma$, A an operator of Dirac type over M, and $u \in C^\infty(M;E)$ satisfies $Au = 0$ and $u|_\Sigma = 0$. Then $u = 0$ on M.

PROOF. Assertion (a) for separating hypersurfaces follows at once from the preceding proposition. Then Assertion (b) follows by passing from M to the closed double $\widetilde{M} = M \cup_\Sigma (-M)$ and extending A to the invertible double Dirac type operator $\widetilde{A} = A \cup_{\sigma(0,y)} A^*$. We refer to [**10**], Chapter 9 for the details of this construction.

To prove (a) for a non–separating hypersurface Σ of M we cut M along Σ and obtain a compact manifold $M_\sharp := M \setminus \Sigma \cup (-\Sigma \sqcup \Sigma)$ with boundary $-\Sigma \sqcup \Sigma$. Then we apply (b) to $M_\sharp$. □

As mentioned in the Introduction to this Note, Bär [**5**] has obtained a sharper version of the preceding Corollary. In (b), it suffices that the solution vanishes on one connected component of the boundary. Moreover, the compactness of the underlying manifold is dispensable. More precisely, he obtains the following result by combining a system version of the Aronszajn–Cordes Theorem with a special case of Malgrange's Preparation Theorem.

THEOREM 11. (Bär, 1997). *Let M be a connected m–dimensional Riemannian manifold and A an operator of Dirac type over M. Then the nodal set of any non–trivial solution u of $Au = 0$ is a countably $(m-2)$–rectifiable set and thus has Hausdorff dimension $m-2$ at most.*

REMARK 12. Clearly, the UCP from submanifolds of codimension > 1 is a more complicated story than the UCP from hypersurfaces. For elliptic differential operators of first order, the weak UCP, i.e. the UCP from open subsets, is equivalent to the UCP from hypersurfaces or from the boundary. That was shown in Proposition 9. At present, however, it seems not to be clear whether one can obtain the "UCP from one connected component of the boundary" directly from the "UCP from open subsets" or whether one depends (as in Bär's proof) on the UCP from points.

2.2. Local UCP versus global UCP. The question raised in the preceding remark leads us to broader questions regarding the relations between local and global aspects of the UCP.

What we have in Theorem 4 or Theorem 7 and what else is in the literature (see e.g. Alinhac [**2**]) is always the local UCP. Then the global UCP follows. And that is what we need in geometry. In principle, however, the UCP for global solutions (on closed manifolds or UCP for L_2 solutions on open manifolds) should be more simple to establish than the UCP for local solutions.

In [**8**] that extra–ingredient (namely that "infinity" is bounded away) was exploited for proving a kind of global UCP. Following a suggestion by K. Jänich, a fairly standard diffeotopy argument (due to R.S. Palais) was combined with the trivial observation that, to any finite–dimensional vector space $\mathcal{F}$ of smooth sections over M there exists a finite subset $M_0 \subset M$ such that $\dim \mathcal{F}|_{M_0} = \dim \mathcal{F}$, to show the invariance of the relative index $\dim(\ker A)|_\omega - \dim(\ker A^*)|_\omega$ of elliptic operators of a certain class of *translation–* and *homotopy–invariant* elliptic operators under variation of the open submanifold ω of the underlying closed manifold. So, when we compare the loss of linear independent solutions of $Au = 0$ and $A^*v = 0$ due to the lack of the weak UCP we find that it is the same for A and the formally adjoint operator A^* for that class of operators. In particular, for that class it follows that the weak UCP for the operator A implies the weak UCP for A^*, as conjectured by L. Schwartz [**25**] for all elliptic operators.

Note that the Bojarski Conjecture (now a Theorem and one of the consequences of the weak UCP for Dirac type operators listed in the Introduction to this Note) remains valid for all elliptic differential operators of first order under the assumption of the Schwartz conjecture. Alinhac [**3**], however, has an example for a non–elliptic equation

$$\frac{\partial u}{\partial t} + a(t,x)\frac{\partial u}{\partial x} + bu = 0$$

with the UCP, but *without* the UCP when reversing the sign of b . This supports the idea that there might also exist a counter–example of an elliptic system of first order (of course, not equation) with UCP but without UCP for the formally adjoint system against the Schwartz Conjecture. To the best of my knowledge, the literature has only two examples of elliptic systems with smooth coefficients of first order without UCP, Pliś [**23**] and Bär [**6**] (based on Kazdan [**20**] which again was based on Alinhac [**1**]). It might be interesting to check the Schwartz Conjecture on these examples.

2.3. Conclusions. This is what one gets from a careful analysis of the details of the proof of Theorem 4 (see the Appendix for details):

There are technical details in the proof which are correct but seemingly without meaning. One example is the factor R on the left side of the Carleman estimate (7) which is not needed for proving UCP. But for any reasonable adiabatic inequality (i.e. with a scaling number R going to infinity), the sum of the power of R and the degree of the highest derivative must coincide on both sides of the estimate according to Fourier analysis philosophy.

The simplicity and transparency of the present UCP proof may be attributed to various factors. Certainly, the restriction to weak UCP is most important for simplifying the arguments. Also the separation (taken from Treves [**27**]) between the symmetric and skew–symmetric terms is a particularly simplifying factor for operators of Dirac type due to Lemma 2. Another trick is the absence of boundary conditions (also taken from Treves [**27**]). The trick is first that the sections vanish near $t = 0$ and $t = T$ (t is the normal coordinate), and second that the transversal hypersurfaces $S_{p,t}$ for the *tangential* integration are closed. Finally, a substantial short cut is due to using the first Sobolev norm; then ellipticity implies its equivalence with the graph norm for sections with compact support.

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Appendix

PROOF OF LEMMA 5. First consider a few technical points. The Dirac operator $\mathcal{A}$ has the form $G(t)(\partial_t + \mathcal{B}_t)$ on the annular region $[0,T] \times S_0$, and it is obvious that we may consider the operator $\partial_t + \mathcal{B}_t$ instead of $\mathcal{A}$. Moreover, we have by Lemma 2 that $\mathcal{B}_t = B_t + C_t$ with a self-adjoint elliptic differential operator B_t and an anti-symmetric operator C_t of order zero, both on S_t. Note that the metric structures depend on the normal variable t.

Now make the substitution

$$v =: e^{-R(T-t)^2/2} v_0$$

which replaces (7) by

(8)
$$R\int_0^T\int_{S_t}\|v_0(t,y)\|^2\,dy\,dt \le C\int_0^T\int_{S_t}\left\|\frac{\partial v_0}{\partial t}+\mathcal{B}_t v_0+R(T-t)v_0\right\|^2\,dy\,dt\,.$$

We shall denote the integral on the left side by J_0 and the integral on the right side by J_1. Now we prove (8). Decompose $\frac{\partial}{\partial u}+\mathcal{B}_t+R(T-t)$ into its symmetric part $B_t+R(T-t)$ and anti-symmetric part ∂_t+C_t. This gives

$$\begin{aligned}
J_1 &= \int\int\left\|\frac{\partial v_0}{\partial t}+\mathcal{B}_t v_0+R(T-t)v_0\right\|^2\,dy\,dt\\
&= \int\int\left\|\frac{\partial v_0}{\partial t}+C_t v_0\right\|^2\,dy\,dt+\int\int\|(B_t+R(T-t))\,v_0\|^2\,dy\,dt\\
&\qquad+2\Re\int\int\left(\frac{\partial v_0}{\partial t}+C_t v_0;B_t v_0+R(T-t)v_0\right)\,dy\,dt\,.
\end{aligned}$$

Integrate by parts and use the identity for the real part

$$\Re\langle f;Pf\rangle=\frac{1}{2}\langle f;(P+P^*)f\rangle$$

in order to investigate the last and critical term which will be denoted by J_2. This yields

$$\begin{aligned}
J_2 &= 2\Re\int\int\left(\frac{\partial v_0}{\partial t}+C_t v_0;B_t v_0+R(T-t)v_0\right)\,dy\,dt\\
&= 2\Re\int\int\left(\frac{\partial v_0}{\partial t};B_t v_0+R(T-t)v_0\right)\,dy\,dt\\
&\qquad+2\Re\int\int\left(C_t v_0;B_t v_0\right)\,dy\,dt\\
&= -2\Re\int\int\left(v_0;\left\{\frac{\partial}{\partial t}\left(B_t+R(T-t)\right)\right\}v_0\right)\,dy\,dt\\
&\qquad-2\Re\int\int\left(v_0;C_t B_t v_0\right)\,dy\,dt\\
&= \int\int\left(v_0;-\frac{\partial B_t}{\partial t}v_0+Rv_0\right)\,dy\,dt+\int\int\left(v_0;[B_t;C_t]v_0\right)\,dy\,dt\\
&= R\int_0^T\|v_0\|_0^2\,dt+\int\int\left(v_0;-\frac{\partial B_t}{\partial t}v_0+[B_t;C_t]v_0\right)\,dy\,dt\\
&= RJ_0+J_3\,,
\end{aligned}$$

where $\|\cdot\|_m$ denotes the m-th Sobolev norm on $E|_{S_t}$ and J_3 requires a careful analysis. It follows from the preceding decompositions of J_1

and J_2 that the proof of (8) will be completed with $C = 1$ when $J_3 \geq 0$. If $J_3 < 0$, it suffices to show that

$$(9) \quad |J_3| \leq \frac{1}{2}\left(R\int_0^T \|v_0\|_0^2\, dt + \int_0^T \|(B_t + R(T-t))v_0\|^2\, dt\right).$$

The operators B_t are elliptic of first order, hence

$$\|f\|_1 \leq c\,(\|f\|_0 + \|B_t f\|)$$

for any section f of E on S_t (and $0 \leq t \leq T$). Then

$$\begin{aligned}
|J_3| &\leq \int_0^T \|v_0\|_0 \left\| -\frac{\partial B_t}{\partial t} v_0 + [B_t; C_t]v_0 \right\|_0 dt \leq c_1 \int_0^T \|v_0\|_0\, \|v_0\|_1\, dt \\
&\leq c_1 c \int_0^T \|v_0\|_0\, (\|B_t v_0\|_0 + \|v_0\|_0)\; dt \\
&\leq c_1 c \int_0^T \|v_0\|_0\, \{\|(B_t + R(T-t))v_0\|_0 + (R(T-t)+1)\|v_0\|_0\}\; dt \\
&\leq c_1 c(RT+1)\int_0^T \|v_0\|_0^2\, dt + c_1 c \int_0^T \|(B_t + R(T-t))v_0\|_0\, \|v_0\|_0\, dt\,.
\end{aligned}$$

The integrand of the second summand is equal to

$$\begin{aligned}
(10) \quad & \frac{\|(B_t + R(T-t))v_0\|_0}{\sqrt{c_1 c}}\, (\sqrt{c_1 c}\, \|v_0\|_0) \\
& \qquad \leq \frac{1}{2}\left\{ \frac{1}{c_1 c}\, \|(B_t + R(T-t))v_0\|_0^2 \; + \; c_1 c \|v_0\|_0^2 \right\}
\end{aligned}$$

with the inequality due to the estimate $ab < \frac{1}{2}(a^2 + b^2)$. By inserting (10) in the preceding inequality for $|J_3|$ we obtain

$$|J_3| \leq \frac{1}{2}\int_0^T \|(B_t + R(T-t))v_0\|_0^2\; dt + c_1 c R\left(T + \frac{c_1 c + 2}{2R}\right)\int_0^T \|v_0\|_0^2\, dt\,.$$

So the desired result holds for T and $\frac{1}{R}$ sufficiently small. □

PROOF OF LEMMA 6. We have

$$\begin{aligned} e^{RT^2/4} \int_0^{\frac{T}{2}} \int_{S_t} \|u(t,y)\|^2 \, dy \, dt &= \int_0^{\frac{T}{2}} \int_{S_t} e^{RT^2/4} \|u(t,y)\|^2 \, dy \, dt \\ &\leq \int_0^T \int_{S_t} e^{R(T-t)^2} \|v(t,y)\|^2 \, dy \, dt \\ &\leq \frac{C}{R} \int_0^T \int_{S_t} e^{R(T-t)^2} \|Av(t,y)\|^2 \, dy \, dt \\ &\leq \frac{C}{R} e^{RT^2/25} \int_0^T \int_{S_t} \|Av(t,y)\|^2 \, dy \, dt , \end{aligned}$$

hence we have

$$\int_0^{\frac{T}{2}} \int_{S_t} \|u(t,y)\|^2 \, dy \, dt \leq \frac{C}{R} e^{-21RT^2/100} \int_0^T \int_{S_t} \|Av(t,y)\|^2 \, dy \, dt ,$$

which gives the result as $R \to \infty$. □

References

[1] ALINHAC, S., Non-unicité pour des opérateurs différentiels à charactéristiques complexes simples, *Ann. Scient. Éc. Norm. Sup., 4e série,* **13** (1980), 385–393.

[2] —, Unicité du problème de Cauchy pour des opérateurs du sécond ordre à symboles réels, *Ann. Inst. Fourier* **34** (1984), 89–109.

[3] —, Personal Communication, 1998.

[4] ARONSZAJN, N., A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, *J. Math. Pures Appl.* **36** (1957), 235–249.

[5] BÄR, C., On nodal sets for Dirac and Laplace operators, Comm. Math. Phys. **188**, 709–721.

[6] —, *Zero Sets of Solutions to Semilinear Elliptic Systems of First Order*, Freiburg, 1999 (*multiplied*).

[7] BERTHIER, A.-M., AND GEORGESCU, V., Sur la propriété de prolongement unique pour l'opérateur de Dirac, *C. R. Acad. Sci. Paris Sér. I Math.* **291** (1980), 603–606.

[8] BOOSS, B., *Eindeutige Fortsetzbarkeit für elliptische Operatoren und ihre formal Adjungierten*, Diplomarbeit, Bonn, 1965 (*multiplied*).

[9] B. BOOSS–BAVNBEK AND K. FURUTANI, The Maslov index - a functional analytical definition and the spectral flow formula, *Tokyo J. Math.* **21** (1998), 1–34.

[10] BOOSS–BAVNBEK, B., AND WOJCIECHOWSKI, K.P., *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston, 1993.

[11] CARLEMAN, T., Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes, *Ark. Mat. Astr. Fys.* **26 B**, *No 17* (1939), 1–9.

[12] CORDES, H.O., Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. IIa, Nr.* **11** (1956), 239–258.

[13] GRUBB, G., AND SEELEY, R.T., Weakly parametric pseudodifferential operators and Atiyah–Patodi–Singer boundary problems, *Invent. Math.* **121** (1995), 481–529.

[14] HÖRMANDER, L., *Linear Partial Differential Operators*, Springer–Verlag, Berlin, 1964^2.

[15] —, Pseudo–differential operators and non–elliptic boundary problems, *Ann. of Math. (2)* **83** (1966), 129–209.

[16] —, Uniqueness theorems for second order elliptic differential equations, *Comm. Partial Differential Equations* **8** (1983), 21–64.

[17] —, *The Analysis of Linear Partial Differential Operators III*, Springer, Berlin, 1985.

[18] JERISON, D., Carleman inequalities for the Dirac and Laplace operators and unique continuation, *Adv. Math.* **62** (1986), 118–134.

[19] KALF, H., Non-existence of eigenvalues of Dirac operators, *Proc. Roy. Soc. Edinburgh Sect. A* **89** (1981), 309–317.

[20] KAZDAN, J.L., Unique continuation in geometry, *Comm. Pure Appl. Math.* **41** (1988), 667–681.

[21] KOTSCHICK, D., The Seiberg–Witten invariants of symplectic four–manifolds (after C.H. Taubes), Séminaire Bourbaki, vol. 1995/96, exp. 812, Paris, Société Mathématique de France, *Astérisque* **241** (1997), 195–220.

[22] PALAIS, R.S. *Seminar on the Atiyah–Singer Index Theorem*, Ann. of Math. Studies 57, Princeton University Press, Princeton, 1965.

[23] PLIŚ, A., A smooth linear elliptic differential equation without any solution in a sphere. *Comm. Pure Appl. Math.* **14** (1961), 599–617.

[24] ROZE, S.N., On the spectrum of the Dirac operator, *Theoret. Math Fiz.* **2** (1970), 275–279. (Russian, English translation in *Theoret. and Math. Phys.*).

[25] SCHWARTZ, L. *Ecuaciones differenciales elipticas*, Bogota, Universidad nacional de Colombia, 1956 (*multiplied*).

[26] TAYLOR, M.E. *Pseudodifferential Operators*, Princeton University Press, Princeton, 1981.

[27] TREVES, F., *Pseudodifferential and Fourier Integral Operators I*, Plenum Press, New York, 1980.

[28] VOGELSANG, V. Absence of embedded eigenvalues of the Dirac equation for long range potentials, *Analysis* **7** (1987), 259–274.

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306/95 Goals of mathematics teaching

Preprint of a chapter for the forth-comming International Handbook of Mathematics Education (Alan J.Bishop, ed)

By: Mogens Niss

307/95 Habit Formation and the Thirdness of Signs

Presented at the semiotic symposium

The Emergence of Codes and Intensions as a Basis of Sign Processes

By: Peder Voetmann Christiansen

308/95 Metaforer i Fysikken

af: Marianne Wilcken Bjerregaard,
Frederik Voetmann Christiansen,
Jørn Skov Hansen, Klaus Dahl Jensen
Ole Schmidt

Vejledere: Peder Voetmann Christiansen og Petr Viscor

309/95 Tiden og Tanken
En undersøgelse af begrebsverdenen Matematik udført ved hjælp af en analogi med tid

af: Anita Stark og Randi Petersen

Vejleder: Bernhelm Booss-Bavnbek

310/96 Kursusmateriale til "Lineære strukturer fra algebra og analyse" (E1)

af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Hélène Connor-Lajambe, Bernd Kuemmel,
Stefan Krüger Nielsen, Bent Sørensen

312/96 Grassmannian and Chiral Anomaly

by: B. Booss-Bavnbek, K.P.Wojciechowski

313/96 THE IRREDUCIBILITY OF CHANCE AND THE OPENNESS OF THE FUTURE
The Logical Function of Idealism in Peirce's Philosophy of Nature

By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian Cardiovascular System

By: Johnny T. Ottesen

315/96 a + b "Rejsen til tidens indre" - Udarbejdelse af et manuskript til en fjernsynsudsendelse + manuskript

af: Gunhild Hune og Karina Goyle

Vejledere: Peder Voetmann Christiansen og Bruno Ingemann

316/96 Plasmaoscillation i natriumklynger

Specialerapport af: Peter Meibom, Mikko Østergård

Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer

af: Ulla Rasmussen

Vejleder: Anders Madsen

318/96 Modelling the Respiratory System

by: Tine Guldager Christiansen, Claus Dræby

Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming Impacts

by: Bent Sørensen

320/96 Grassmannian and Boundary Contribution to the -Determinant

by: K.P.Wojciechowski et al.

321/96 Modelkompetencer - udvikling og afprøvning af et begrebsapparat

Specialerapport af: Nina Skov Hansen,
Christine Iversen, Kristin Troels-Smith

Vejleder: Morten Blomhøj

322/96 OPGAVESAMLING
Bredde-Kursus i Fysik 1976 - 1996

323/96 Structure and Dynamics of Symmetric Diblock Copolymers
PhD Thesis
by: Christine Maria Papadakis

324/96 Non-linearity of Baroreceptor Nerves
by: Johnny T. Ottesen

325/96 Retorik eller realitet ?
Anvendelser af matematik i det danske Gymnasiums matematikundervisning i perioden 1903 - 88

Specialerapport af Helle Pilemann
Vejleder: Mogens Niss

326/96 Bevisteori
Eksemplificeret ved Gentzens bevis for konsistensen af teorien om de naturlige tal

af: Gitte Andersen, Lise Mariane Jeppesen,
Klaus Frovin Jørgensen, Ivar Peter Zeck

Vejledere: Bernhelm Booss-Bavnbek og
Stig Andur Pedersen

327/96 NON-LINEAR MODELLING OF INTEGRATED ENERGY SUPPLY AND DEMAND MATCHING SYSTEMS

by: Bent Sørensen

328/96 Calculating Fuel Transport Emissions
by: Bernd Kuemmel

329/96 The dynamics of cocirculating influenza strains conferring partial cross-immunity

and

A model of influenza A drift evolution

by: Viggo Andreasen, Juan Lin and Simon Levin

330/96 LONG-TERM INTEGRATION OF PHOTOVOLTAICS INTO THE GLOBAL ENERGY SYSTEM

by: Bent Sørensen

331/96 Viskøse fingre

Specialerapport af:

Vibeke Orlien og Christina Specht

Vejledere: Jacob M. Jacobsen og Jesper Larsen

332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG

Specialerapport af:

Stine Sofia Korremann

Vejleder: Dorthe Posselt

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an extension of methods found in the literature on monetisation of biodiversity

by: Bernd Kuemmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Bernd Kuemmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids

by: Jeppe C. Dyre

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by: Kathrine Legge

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- et projekt om matematisk modellering

af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen

Vejleder: Jørgen Larsen

338/97 Kvantisering af nanolederes elektriske ledningsevne

Første modul fysikprojekt

af: Søren Dam, Esben Danielsen, Martin Niss, Esben Friis Pedersen, Frederik Resen Steenstrup

Vejleder: Tage Christensen

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by: Carsten Lunde Petersen

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by Mogens Niss

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A global clean fossil scenario discussion paper

prepared by Bernd Kuemmel

Project leader: Bent Sørensen

343/97 IMPORT/EKSPORT-POLITIK SOM REDSKAB TIL OPTIMERET UDNYTTELSE AF EL PRODUCERET PÅ VE-ANLÆG

af: Peter Meibom, Torben Svendsen, Bent Sørensen

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Ph.D. Thesis

by: Mette Sofie Olufsen

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af: Sebastian Horst

Vejledere: Jørn Borggren, NBI, Niels Boye Olsen

347/98 Verificering af Matematiske Modeller

- en analyse af Den Danske Eulerske Model

af: Jonas Blomqvist, Tom Pedersen, Karen Timmermann, Lisbet Øhlenschlæger

Vejleder: Bernhelm Booss-Bavnbek

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by: Stefan Krüger Nielsen

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350/98 OPGAVESAMLING - Bredde-Kursus i Fysik 1976 - 1998

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by: Mogens Niss