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Two chapters on the teaching, learning and assessment of geometry

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Two chapters on the teaching, learning and assessment of geometry
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Abstract

This text consists of two chapters that will be included in a forthcoming ICMI Study *Perspectives on the Teaching of Geometry for the 21st Century - An ICMI Study*, edited by Vinicio Villani and Carmelo Mammana. The book will be published by Kluwer Academic Publishers, Dordrecht, the Netherlands, in the ICMI Study Series. The book is expected to appear late in 1997 or early in 1998.

The first text, *Dimensions of assessment and geometry*, pp 1-10, (forming the introductory section of Chapter 8: Assessment in Geometry) focuses on the role of assessment in geometry but addresses also the role of geometry in assessment. The second text, *Teacher qualifications and the education of teachers*, pp 1-17, (forming Chapter 9 of the book) attempts at providing an overview of the matters indicated in the title with particular respect to the teaching and learning of geometry.

Chapter 8
Dimensions of assessment and geometry

Mogens Niss

I. Introduction

The term 'assessment' in mathematics refers to the identification and appraisal of learner's knowledge, insight, understanding, skills, achievement, performance, and capability in mathematics. Assessment does not concern the judging of educational or instructional systems or programmes, curricula, teachers' competence, teacher training, etc. For such activity 'evaluation' has become the standard term (see, e.g. Niss, [7]). Evaluation will not be considered in this chapter.

The constellation 'geometry and assessment' relates to two quite different issues. The first issue may be given the following condensed and over-simplified formulation: "what can assessment do for geometry?". This formulation covers questions such as "in the teaching and learning of geometry, what is the actual and potential role of assessment?", "what specific assessment problems manifest themselves with respect to geometry?", "what modes of assessment are particularly well-suited (respectively ill-suited) as regards teaching and learning of geometry?", "what aspects of learners' geometric knowledge, insight, and skills are traditionally assessed in the classroom, at tests or at examinations?", and so forth. In other words, here the focus is on geometry as an educational topic, and assessment occurs as a specific set of glasses through which this topic is being viewed at the moment. The second issue is the converse: 'what can geometry do for assessment?'. In this context the focus is on assessment in mathematics, and geometry is considered a (potential) vehicle for the pursuit of assessment interests. For instance, it may happen that geometry possesses specific properties or qualities that make it especially helpful for the assessment of certain aspects of mathematical knowledge, insight and skills.

Both issues are interesting and deserve attention, but they should not be mixed up, let alone be confused. Since the present book deals with the teaching and learning of geometry, it is primarily the former issue that should, and will, preoccupy us in this context. However, as we are able to argue that geometry does offer valuable and special opportunities for the assessment of mathematics in general, we shall in fact consider the latter issue as well, albeit fairly briefly.

II. The role of assessment in geometry

Basically, there are three general purposes of assessment which pertain to assessment in mathematics as well. The fundamental one is the provision of *information* - whether to the individual learner, to the teacher, or to the educational 'system' in which the learner is situated - about the learner's performance, achievements and capabilities. Such information is typically designed to serve a further (ultimate) purpose, namely to make *decisions* and to take *actions* regarding the future of the learner. Some decisions or actions will be on the part of the learner, with respect to matters such as controlling his/her own learning strategies and activities, or choosing and preparing for future education or career, while others implicate the teacher or the institution at issue and deal with pupils' and students' passing/non-passing of tests or exams, and the filtering, selection and placement of each learner, whereas still others involve the education system at large in relation to pupils'/students' obtaining a licence or a certification, qualifying for various sorts of jobs etc. The final purpose regards the *shaping of reality*. This is divided into two parts, the reality of teaching, and social reality in a broad sense. Firstly, it is a classical purpose of assessment to control, at least partly, the content, the format, and the outcome of teaching and learning, including teachers' and learners' activity and behaviour. In this respect assessment serves as an instrument for educational authorities to control, perhaps on behalf of society, the agents of the classroom. Proceeding along this line we come to the second part (which is often more a tacit and indirect than an explicit purpose of assessment): the shaping of social reality in the school/institution, in the education system, in the labour market, or in society at large, brought about by the way in which assessment induces norms, attitudes and behaviour in the social environment.

As specifically regards assessment in geometry, the provision of information to the individual pupil or student, and to his or her teacher(s), comes to the fore as the more significant purpose, together with the purpose of establishing a basis for such decision making or action taking that affect the learner's controlling of his/her own learning behaviour in geometry or his/her choice of future educational path. In some cases also the purpose of exerting an influence on teachers' teaching and learners' learning is put on the agenda by curriculum authorities. The remaining purposes are hardly connected to geometry in particular but rather to mathematics (and education) in general.

Now, what types of information about a pupil's or a student's performance, achievements, and capabilities in the area of geometry may be wanted or needed? The answer to that question depends strongly on the goals (ranging from ultimate ends, over general aims, to concrete objectives) which are set for the teaching and learning of geometry, for it is primarily the attainment of these goals about which information is sought by means of assessment. Furthermore, once it has been settled what information is to be requested, the issue emerges, of choosing or designing assessment modes which are compatible with and appropriate for the goals involved and the types of information demanded. In the sections that follow we shall deal with the goals pursued, the information sought, and the assessment modes adopted, all in relation to geometry.

III. The goals of geometric teaching and learning

The goals for teaching and learning geometry fall into three categories which are, then, reflected in the information that assessment is meant to provide.

The *first* category consists of goals related to the acquisition of knowledge, insight, and skills attached directly to the *geometric subject matter* belonging to the topic(s) taught in the context under consideration, whether this topic is synthetic Euclidean or non-Euclidean geometry, analytic, projective, algebraic, differential or other type of geometry. Here, the goals typically regard the acquisition of knowledge about such notions, concepts, definitions, methods and theorems - and the skills needed for dealing with these elements - that are characteristic of the topic at issue; understanding of the foundation and theoretical architecture of the topic; and insight into its interrelations to other areas of geometry in particular and of mathematics in general. Goals of this category may also, however, regard knowledge of the historical genesis and development of the topic, or appreciation of philosophical (or psychological) issues related to its foundation, interpretation or application.

The *second* category of goals address geometry as a unique instance of intersection between mathematics as a pure, abstract and general theoretical edifice, on the one hand, and physical space, objects and phenomena in nature or in the man-made world, on the other hand. Put differently, these goals concern what we have called, elsewhere in this book, the *dual nature of geometry*, i.e. geometry as both a branch of mathematics (or several branches if you like) as well as a natural science and a design 'science'. Goals which may be relevant in this context range from the very understanding of this dual nature of geometry, over the ability to understand and activate the interplay between geometric topics, viewed as pure mathematics, and geometric aspects of 'natural or artifactual nature', to the ability to build, analyse and utilise specific geometric models (in the modern sense, not in the old sense of plaster or wood-and-wire models of abstract geometric objects) in concrete application situations.

The goals of these two categories are specific to geometry and cannot be equivalently replaced by goals connected with other branches of mathematics. The *third* and largest category of goals contains those which are special cases of goals related to the teaching and learning of mathematics at large, just restricted to the area of geometry. These goals occur in the context of geometry, either because geometry may be considered as a domain that is particularly well suited to pursue the goals in question, or simply because the teaching and learning of geometry, being on the agenda for independent reasons, happen to offer an opportunity to pursue mathematical goals of a general nature at the same time. In other words, in this category of goals geometry acts on behalf of mathematics at large. This implies that a list of these goals would be almost identical to the list of goals that one could establish for the teaching and learning of mathematics in general, only specialised to geometry. Such a list would be very long indeed. So, we have to confine ourselves to indicating the most important sub-categories of general mathematical goals.

One group of goals focus on learners' gaining knowledge of and insight into matters inherent in all mathematics: the nature and role of definitions, theorems and proofs; the relationship between general statements and special cases; the range and exemplification of concepts and propositions; the kinds of questions asked and the kinds of answers given in mathematics ("if-then" assertions; existence; "how many?"; uniqueness; equivalent characterisation; (exhaustive) classification; algorithmic computation; transformation; invariants; and so forth). A second group of goals are concerned with the learner's knowledge and mastery of mathematical techniques and methods, such as: understanding, manipulation, syntactic and semantic interpretation of symbolic representations and expressions; rules of logical deduction; various general methods of proof and proving (computational, logical equivalence, direct, indirect, induction, etc.). Another group of goals consists of pupils' and students' ability to pose, analyse, and solve mathematical problems, with respect to both routine exercises and open-ended, fuzzy, and perhaps intriguing, problems. Heuristic capabilities, creativity, and inventiveness are closely linked to this group of goals and to the following one as well. A group of goals addresses learners' ability to activate and apply mathematics in extra-mathematical contexts in order to describe, understand, or master aspects of the situation under consideration, again both in routine and in non-routine contexts. This includes the ability to identify potential mathematical content and structure in the situation, to construct a mathematical model, to solve the mathematical problems formulated within the framework of the model, to interpret the outcomes in relation to the extra-mathematical context, to make inferences about this context, and, last but not least, to critically analyse and assess the model from mathematical as well as extra-mathematical points of view. A fifth group of goals deals with learners' overall feel for and appreciation of mathematics as a discipline, its history and development, its philosophical nature and characteristics, its place and role in culture and society, its use and misuse, its relationships with other disciplines, etc. The sixth group of goals pay attention to fostering, in learners, attitudes and beliefs about mathematics and about their own relationships with the subject and its exercise. Finally, we may add a seventh group of goals which used to be in focus of mathematics education and which is still maintained as essential by many mathematics educators: formative education of the individual, i.e. the provision and training of general mental and personal capabilities to be put into practice in all sorts of context, also outside of mathematics. Traditionally, (axiomatic and deductive) geometry has been viewed as a pre-eminent training ground for exactly such education, even if, in modern times, considerable skepticism towards the justification of the underlying belief in its effectiveness has gained momentum.

IV. Assessment of the achievement of goals

It is a fundamental purpose of assessment in geometry to generate information about the extent to which learners of a given category have achieved some subset of the goals outlined above. This subset typically varies with the type of learners considered. However, the goals involved are quite often only in the air, i.e. they are not made clear, let alone stated explicitly and articulately, neither by (or to) the assessor nor to the learner assessed. This fact is probably one - but not the sole - factor responsible for another fact: the traditional assessment modes adopted in mathematics tend not to be matched with the goals officially or semi-officially pursued in its teaching. And this lack of matching is not the least manifest in relation to geometry. It is a classical observation that irrespective of the explicit or implicit goals underpinning the teaching of mathematics, the content and modes of assessment themselves give rise to immanent goals, namely those which can be derived from what it takes to succeed in assessment. Learners and teachers quickly begin to decode and see these goals as the real ones, and this has a decisive impact on teaching as well as on learning.

Let us sum up, in condensed form, the gross set of kinds of possible goals for the teaching and learning of geometry, from which corresponding assessment goals are derived. These goals concern: geometric subject matter and theory; the dual nature of geometry; building and applying geometric models. And with respect to geometry on behalf of mathematics: characteristic features of objects, issues and statements in the discipline; ways of thinking, methods and techniques; problem posing and problem solving; creativety and inventiveness; the nature of modelling and applications; philosophy and history of the discipline in society and culture; learners' attitudes and beliefs; training of general mental capacities. (In order to avoid misunderstandings, it should be underlined that these goals are not (necessarily) advocated by the author of this section. Instead, they are established by analytic examination of the field.)

V. Modes of assessment

However, let us assume, here, that the teaching of geometry serves some subset of goals such that each kind of goals is represented in the subset, and let us further assume that it is intended to attempt to assess, in a serious way, the achievement of the goals in the subset. Such an attempt requires the determination of the assessment modes to be adopted.

By an assessment mode we understand a 'vector' that includes the following components (Niss [7], p 12ff): The *subject* of assessment (who is assessed?); the *objects* (what is assessed, in terms of subject matter content and learner ability?); the *items* (what sorts of output are assessed?); the *occasions* (when, in relation to the curriculum, does assessment take place?); the *procedures and circumstances* (what happens, and who does what?); the *judging and recording* (what is emphasised, and what is recorded?); the *reporting* (what is reported to whom?). A specific assessment mode is established through the specification of each of these generic components. Perhaps the most significant components giving rise to apparent differences between specific assessment modes are 'the objects', 'the items', and the 'procedures and circumstances'.

As to 'objects', what is traditionally in focus is learners' knowledge of (geometric) facts (mainly names, definitions and properties of concepts); mastery of standard methods and techniques (elementary geometric operations and their combination (e.g. bisecting an angle by ruler and compass, computing quantities in a figure by means of trigonometry, determining the curvature of a curve in 3-space given in parametric form), proving that the diagonal of the unit square has irrational length); performance of standard applications (like finding the area of a piece of land with geometrically "tame" boundaries, taking measures that allow for the estimation of the height of a flag pole). It is much more seldom to encounter assessment objects such as visualisation; open-ended problem solving; geometric modelling of complex, extra-mathematical situations; rigorous and heuristic reasoning; generation and exploration of hypotheses; explaining the structure of a geometric theory; establishing links between different geometric topics; interpreting an abstract geometric theory in relation to a specific object domain.

When it comes to 'items' and 'procedures and circumstances', traditional assessment primarily employs timed written tests or exams, in many places of a multiple choice type. Such tests are typically composed of several mutually unrelated closed-form exercises/problems that refer to different parts of the syllabus. Reproduction of definitions or proofs without the support of the textbook may be included as well. No matter whether we are talking about teaching-learning oriented, classroom bases assessment (often called 'formative') or about end-of-course ('summative') assessment, it is customary that students are required to "sit the test" or the exam in a room in which complete silence rules and where communication with others is forbidden. At the end of the test session learners hand in what they have completed and their papers are scrutinised and marked, perhaps corrected too, by the teacher/assessor. Hence, the time constraint is a key component in the test. In addition to written tests, assessment may also make use of oral interviews with learners, in which learners may be asked to solve exercises/problems in front of the assessor, quote facts or recite definitions of concepts or proofs of theorems, or (more rarely) present or explain a topic or a piece of theory.

VI. The need for innovation in the assessment of geometry

Presumably it seems quite clear that traditional assessment modes, based on components specified in the manner outlined in the preceding paragraphs, only allow for reliable and valid assessment of a fairly limited subset of the kinds of goals of teaching and learning of geometry which were listed above. More specifically, such assessment modes can deal with (aspects of) learners' knowledge of geometric subject matter and theory and mastery of corresponding basic techniques, their ability to solve closed or almost closed exercises and problems and to put (mostly standard) applications and models into use in not too complex situations. As far as the remaining types of goals are concerned - in particular those of a higher order, complex, comprehensive, and general nature - the modes traditionally adopted do not really allow for an adequate assessment of their achievement. For instance, suppose we agree (at least I do!) that we need to be able to assess learners' ability to translate and pass from one 'code' of, or perspective on, geometry to another, i.e. transfer from a visual situation to a verbal description and further on to one of several formal types of description (in, say, an axiomatic, algebraic, or vector space setting) and back again. By definition, assessment based on single-step tasks cannot capture transfers between different geometric codes and perspectives. To such and similar ends other assessment modes are needed, both in classroom practice and at the end of course or of a section of schooling. The remarkable thing is that such assessment modes actually do exist and are being/have been implemented in various places but mainly on relatively small scale.

Some of these modes are based on items such as learners' written reports of larger projects, or of extended investigations. In addition to having their report as such assessed, pupils or students are sometimes assessed at an oral defense of the report as well. It may also be the case that instead of producing a written report learners design a set of posters, an exhibition, a piece of computer software, a video programme, a teaching sequence, and soforth, which then form the item of assessment. In projects and extended investigations learners work, alone or in groups, for a longer period of time on complex themes, issues or problems. They may deal with the structure and organisation of geometric theory, the history and philosophy of geometry, geometric applications and modelling. Projects or investigations can be put into practice at all educational levels, but naturally the content, format and level of treatment will vary accordingly. At Roskilde University (Denmark) students of science and mathematics perform group projects (each occupying roughly half of the time in one semester) from the very beginning of their studies. To give just a few examples of projects on geometry: the geometry of map-making; the impact of the appearance of non-Euclidean geometry on the perception and development of mathematics; sphere packing in theory and applications; the history of angle trisection.

Other - less radical - forms of assessment are variations of the traditional, test type, ones. Here the main problem is that the time constraints inherent in most traditional assessment are so tight that they have to be disposed of or circumvented if more profound and complex geometric knowledge, understanding, insight, and skills are to be assessed. One way of circumventing these constraints is to give the learner a set of take-home questions that call for non-routine in-depth insight and capablity to be answered adequately. After a fair number of days, the learner hands in his/her paper which is then scrutinised and assessed by the teacher/assessor. Of course it is not possible to really prevent the learner from seeking advice or help from others before handing in the paper. This is no problem, however, as there is a second stage in which the learner has to defend his/her solutions in an oral interview with the teacher/assessor. At that interview, the learner is asked to explain the solutions adopted, and the theoretical bases of these, and to react to "what if instead?"-questions, and so forth. It is mainly certain sorts of mathematical capabilities that can be assessed this way. For instance, the ability to find 'aha!-type' solutions to challenging problems - so abundant in geometry - cannot be appropriately assessed in this way if it is crucial to make sure that the learner found the solution all by him/herself. However, if this requirement is given up the format just described allows for both valuable learning activity and valid assessment.

Another assessment mode is the so-called two-stage test (see, e.g., de Lange, [3]).

In the first stage, the pupil or student sits an ordinary timed written test. The resulting paper is scrutinised by the teacher, the most serious mistakes are indicated/corrected and the paper is returned to the learner. In the second stage, the learner is given a fair amount of time, e.g. two weeks, to revise the paper which is then handed in again and assessed.

As a simple example of an item which can be a component of a variety of realistic assessment modes, let us consider the following problem of applied geometry. Draw a curve - and then make a cut along it - on a plane, rectangular piece of soft cardboard such that, if rolled into a cylinder with a circular orthogonal cross-section at its base, the cardboard will form a circular cylinder cut by an oblique plane. Although at first glance it may appear slightly exotic, this is in fact an authentic problem posed by an architect who wanted to construct a cardboard model of a building designed to have this shape. In a slightly more advanced form the problem also arises in plumbing if two cylindrical tubes (not necessarily of the same diameter) are to be fitted together at a certain angle. This problem can be approached in several different ways and at a variety of educational levels, ranging from a combination of guessing and physical experiment to exact computation in 3-dimensional descriptive geometry. The problem also allows for generalisation in different directions and thus may give rise to flexible learning and assessment activities for use in various contexts. This is just one example of an abundance of applied (authentic) geometric problems with similar properties.

The assessment modes briefly sketched here are certainly not meant to be exhaustive. Many other relevant modes exist and are described and discussed in recent literature on assessment (see, for example, Gravemeijer et al. [1], Kulm, [2], Leder, [4], Lesh & Lamon, [5], Niss, [9], Webb & Coxford, [11]). Most of all these modes are not specific to geometry, but they are highly relevant for geometry nevertheless. A closer investigation of the assessment of cognitive growth in geometric insight and understanding, based on the so-called van Hiele levels and the SOLO-taxonomy, is given in the paper by Pegg, Gutierrez, and Huerta in the next section of the present chapter.

The main point here is not to produce or discuss an inventory of suitable assessment modes but to emphasise that different aspects of geometric knowledge, understanding, insight, and skills need to be assessed through different means. There is no single mode which can meet all the requirements to rich, representative, valid and reliable assessment in geometry. What is needed is balanced collections of assessment modes which in total can cover the diverse kinds of major goals of the teaching and learning of geometry. It was indicated in the preceeding paragraphs that there are, already, quite a few relevant assessment modes at our disposal for inclusion in such collections. It is mainly a matter of putting them into actual practice (cf. an example of this described in Stephens & Money, [10]). However, the obstacles for this to happen are many. Some of them reside in the politico-economic sphere (the "alternative" modes are time consuming, resource intensive and hence expensive, and are also met by conservative skepticism in society at large). Other obstacles can be encountered in the mathematics education community itself which, in addition to being fairly firmly rooted in the established assessment traditions, is not in general well acquainted with new developments in the philosophy, theory and practice of assessment. At any rate, in most places there is plenty of unexploited room for improvement in the assessment of geometry, and it is not the lack of reasonable assessment modes that is responsible for this situation.

Most of the examples put forward in the preceding sections concern the assessment of geometry at post-elementary educational levels. This is mainly due to the fact that at the primary level geometry is usually integrated with other mathematical or proto-mathematical activities rather than being constituted as a well-defined, let alone a separate, topic. Therefore the assessment of geometry at that level may tend to be inseparable from the assessment of mathematics at large. However, since the formation of geometrical experiences and knowledge ought to start already at a very young age, mathematical assessment at that level should encompass aspects of geometry as well, such as orientation in space; measurement; drawing; practical work with paper, cardboard, scissors, strings; and so forth.

VII. The role of geometry in assessment

Up till now, this paper has primarily been dealing with assessment for the sake of geometry. In this concluding section we shall, quite briefly, consider the reverse issue, the use of geometry for the sake of assessment in mathematics. In other words, assuming, here, that the goal is to assess pupils and students in mathematics in general, does geometry have anything to contribute to this end? It seems that the answer is 'yes!'. We shall restrict ourselves to offering two argument to support this answer.

Firstly, in the previous discussion of assessment referring to the goals of the teaching and learning of geometry, it was suggested that for a number of those goals geometry does, in fact, act on behalf of mathematics at large. To sum up, once again, we are talking of goals regarding the nature and the properties of basic elements of mathematical theories, mathematical ways of thinking, problem posing and solving, creativity and inventiveness, modelling and applications, history and philosophy, attitudes and beliefs, education of the mind. So, to the extent the achievement of any of these goals is assessed in the context of geometry, it is also assessed from a wider mathematical perspective. This is not only the case because geometry is a branch of mathematics but also because so many mathematical concepts, objects, and theories possess geometric aspects, representations or interpretations. Let us just give one example: the relationships between proofs (also of theorems in, say, number theory), visualisation, and geometry, as elucidated so nicely in *Proofs without words* by Nelsen [6]. In these respects geometry certainly has something to contribute to assessment in mathematics, but so has other branches of mathematics. Nevertheless, it seems that because of the special status of geometry in mathematics and its special place at the crossroad between mathematics and physical reality, geometry offers a particular richness in issues, topics, situations, and problems that lend themselves to multi-faceted assessment of general mathematical knowledge, insights, and capabilities.

The second argument relates specifically to assessment in the area of problem posing and problem solving. Here, geometry occupies a fairly unique position (perhaps with the exception of probability and statistics, which has certain features in common with geometry but at a lower order of magnitude). The point is that there are infinitely many geometric problems (in one, two or three dimensions, primarily) which can be posed without reference to or invocation of any specific theoretical setting. Quite often such a problem may make senses in a realistic everyday (extra-mathematical) context (e.g. the design problem presented in the preceding section), and even if it doesn't it may very well be formulated in such a way that a lay person can understand what it is all about. This has four important consequences.

(1) It becomes a substantial part of the solution process to specify the problem and give it a precise formulation that allows for mathematical (geometric) treatment.

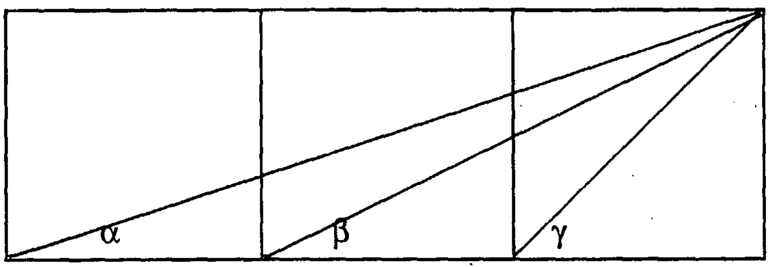
(2) The problem can usually be viewed from different perspectives, be attacked

by a multitude of different methods, and be imbedded in a variety of different frameworks of geometric theory.

(3) Because of (2) (and partly (1)), solution-approaches and solutions can be compared and discussed with respect to elementarity (degree of technical complexity of the solution method), simplicity (degree of ingenuity required for the solution), convincing power (degree of "aha!-ness"), theoretical depth and generality (or particularity, for that matter).

(4) Issues such as the possibility of generalising the problem or the solution, while links to similar or related problems may arise naturally and inspire to further exploration.

Let me quote just one example of a problem (non-trivial, I have to admit) which illustrates the points (1)-(4). In the figure below, which consists of three adjacent squares, what is the sum of the angles α, β, and γ?



I thank Andrejs Dunkels (Luleå, Sweden) for having attracted my attention to this problem. This problem can de approached in numerous different ways (more than fifty I was told), ranging from quite elementary solutions to fairly involved ones. As is always the case with good problems, the need to argue for the correctness of the solutions obtained gives rise to an excellent exercise in and assessment of proof and proving.

The arguments put forward above suggest that geometry does offer special opportunities for assessment in mathematics in general. Additional arguments could have been given on top of these, but let them suffice for the present context. Underlying all the arguments is the intrinsic and multi-dimensional role that geometry plays in mathematics as a topic and as an intimately intervowen aspect of so many other branches of mathematics.

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Chapter 9:
Teacher qualifications and the education of teachers

Mogens Niss

I. Introduction

The task of the present chapter is to consider teacher qualifications and the education of mathematics teachers with particular respect to *geometry*. This is not an easy task to accomplish, since many of the important factors and issues to be considered are associated with the teaching of mathematics in general - geometry being just a special case - or even with the profession of teacher as such. Thus, it would not make sense to insist, for this chapter, on restricting the analysis exclusively to matters specific to geometry. Firstly, this would require us to establish a clear demarcation line, not easily drawn at all, between "geometry as such" and "geometrical aspects" of other mathematical topics or branches. Secondly, although the focus of this book is on the teaching of geometry, in most places in the world geometry is, and probably will continue to be, taught as part of mathematics by teachers who teach other branches of mathematics as well. Hence their qualifications as teachers of geometry are imbedded in their qualifications as teachers of mathematics. Nevertheless, in what follows we shall attempt to concentrate our attention on geometry, as much as is possible and reasonable for the avoiding of a distorted treatment of our theme.

As is discussed elsewhere in this book, the place of geometry in mathematics curricula, at all levels, has changed dramatically more than once during the last four decades. Corresponding changes have taken place in the preparation and qualifications of mathematics teachers for primary, secondary, and tertiary levels, albeit usually with a time lag. Before, say, 1960 geometry occupied a prominent (in some places even a dominant) position in all categories of post-elementary mathematics education in most countries. Similarly, the preparation of mathematics teachers to teach geometry formed a substantial part of their pre-service teacher training. Then, in those countries in which the New Maths movement found a foot-hold in the 1960's and 1970's, the situation changed. Soon, in those places, geometry became either 'integrated away' into more general and abstract theoretical edifices, such as linear algebra or transformations groups, in which the service paid to geometry was often lip-service mainly, or it simply disappeared from curricula. This was reflected in parallel changes in teacher training programmes, agendas of in-service courses and conferences etc. It was further reflected in university programmes. In many tertiary institutions geometry disappeared altogether, or it survived under headings such as 'differential geometry', 'algebraic geometry', or 'convexity', topics which were often taught, however, without any reference to, let alone a basis in, the systematic study of specific geometrical objects or "worlds". In the same vein, research mathematicians only seldom called themselves "geometers". Rather, depending on their specialty, they preferred to see themselves as "topologists", "global analysts", or "algebraists". If geometry disappears from university mathematics, it is also likely to disappear from the preparation of those who are to teach mathematics at earlier levels of the education system. So, the situation in university departments tended to reinforce the evaporation of geometry from mathematics education at large.

It is true that this picture was not uniform across countries. Some countries, (e.g. China, Hungary) never completely severed the umbilic chord connection of their curricula to the 'pre-modern' tradition with its substantial geometrical component. So, in these countries, during thelast decade, the gradual revival of the attention paid to geometry by mathematics educators does not represent the same degree of discontinuity as is the case with countries that embarked on the 'modern mathematics' reform. The revival of geometry which began in the mid-1980's is still of a fairly modest magnitude in most countries, although the movement is gaining more momentum. The term "revival" should not be taken to imply the unchanged resumption of past traditions of the teaching of geometry. What is revived is the bringing back of geometrical objects, phenomena, problems, theories and methods as subjects of study in mathematics education. (The manifestations of, and reasons for, this development are discussed elsewhere in this book, but undoubtedly one key factor is the possibilities of visualisation offered by information technology and the ensuing opportunities and challenges for the teaching and learning of mathematics. From that perspective, what we are experiencing is not just a revival of geometry but a revitalisation as well.)

So far, the changes in the preparation of teachers seem to be scattered. No systematic innovation appears to be on the agendas of teacher trainers. However, things are moving. Thus many universities (e.g. Cornell University, USA, as reported by David Henderson[1)]) have established a variety of new courses in geometry in addition to the 'modern traditional' ones in differential or algebraic geometry. The didactics of geometry has become included in the programme of many institutions responsible for the training of mathematics teachers. But again, we are hardly witnessing a universal reform movement in the making (the experiences with the New Maths reform has probably resulted in a long lasting distrust in large-scale, top-down curriculum reform), rather a growing number of innovation initiatives which are beginning to form a pattern.

II. What are the issues?

In order to clarify the issues we shall deal with in this chapter, it might be worthwhile to say a few words about what we are *not* going to do. First of all, it is important to *avoid the 'syllabusitis trap'*[2)], i.e. the trap of discussing matters of mathematics education (in this case with respect to teachers) solely in terms of syllabi and curricula. Identifying mathematical knowledge, insight, and ability with a list of the topics, concepts, results, and methods that people ought to master, implies an unduly simplistic reduction of what mathematical competence is all about. This is quite similar to the reduction implied by the identification of linguistic competence with a vocabulary to be known and a list of grammatical rules to be mastered. So, as far as geometry is concerned, to describe the qualifications and the education of a mathematics teacher should not be equated with listing the geometrical topics, theorems, facts, and methods which the teacher has to know or, at least, has to have been exposed to. In essence, this is a question of not confusing necessary and sufficient conditions. Of course, in the same way as it is impossible to possess linguistic competence in a certain language without knowing some basic words and some basic grammar, it is crucial to know some geometry in order to teach geometry. However, as our ambitions, in this context, go beyond the mere indication of elementary necessities, we have to search deeper in order to characterise *what it takes* to be a competent mathematics teacher of geometry, as well as what it takes to educate such a person.

Here, we have to beware of another trap. It is no great art for mathematics educators to formulate and to agree on a huge gross list of desirable qualifications concerning geometry and its teaching and learning, with which the mathematics teacher of the 21st century should be equipped. Similar lists of qualifications regarding the teaching and learning of algebra, calculus, probability and statistics, applications and modelling, or the history and philosophy of mathematics, etc. could be established equally easily. It is probably true that the combined list of such qualifications would characterise a marvel of a mathematics teacher whom it would take several handfuls of years to educate - and a good deal of motivation and improved working and living conditions to attract to the profession of school teacher. In other words, it is not too difficult to take off to Utopia. The difficult part is to get back to reality.

None the less, even if Utopia is hardly an appropriate habitat for permanent residence it might well be worth a visit once in a while. Occasional visits to Utopia serve one important purpose, among others (cf. Niss, [3]): When, in a real world in which not all wishes can be fulfilled at the same time, we are to design, organise and implement ways to educate mathematics teachers, it is important to be able to assign priorities to all the various desirable qualifications we can suggest, in order to identify the more crucial ones and to know what we lose with the ones we have to give up. If all decisions concerning teacher education were to take only the conditions and circumstances of the day into account, our level of ambition would tend, to an increasing extent, to be lower than should and need be the case. Ultimately, the education of teachers would be governed by a coalition of the law of inertia and political-economic power plays.

Against this background, the main issue of this chapter can be phrased as follows:

How to ensure that future teachers of mathematics will have a multi-faceted and well-founded view of geometry, an ability to stage a multitude of rich teaching-learning environments in relation to geometry, and an ability to assess pupils' and students' geometrical understanding, knowledge and ability?

For this question to make sense, we have to clarify what the key items "future teachers of mathematics", "a multi-faceted and well-founded view of geometry", "rich teaching-learning environments in relation to geometry" and "an ability to assess pupils' and students' geometrical understanding, knowledge and ability" mean, or could mean. We shall deal with this task in the sections below.

Firstly, the term "future teachers" is not well-defined. Here, it should be taken to encompass different categories of people: The teachers who are, already, in service today but will be serving in the future as well; the pre-service student teachers of today who are to function tomorrow and in a more distant future; and the yet-to-come prospective teachers who will teach future generations of pupils and students. This implies that we are not only talking about the qualifications and the pre-service teacher-training of today and tomorrow, but also about the qualifications and the in-service education of teachers today and in the future.

Although the teacher education issues we are dealing with are certainly pertinent to all sorts of teachers, from kindergarten to university, for simplicity our main focus will be on the qualifications and the education of school teachers (grades K-12). From time to time, however, we shall also address the education of tertiary level teachers, including teacher trainers and university professors of mathematics.

Our attempts to clarify what "a multi-faceted and well-founded view of geometry" means will be carried out as part of *Section IV: Knowledge and views of geometry as a mathematical discipline*. Similarly, "rich teaching-learning environments in relation to geometry" will be dealt with in *Section V: Knowledge and views of processes of learning geometry* and *Section VI: Rich teaching-learning environments in and with geometry.* A separate *Section VII: Teacher assessment in geometry* will offer a discussion of "the ability to assess pupils' and students' geometrical understanding, knowledge and ability". Perhaps the final Section of the Chapter is the key one: *Section VIII: Ways to develop a competence to teach geometry.*

III. Some background assumptions

Normally, a teacher will *function for several decades.* During his/her service, *substantial changes are likely* to take place in the political, socio-economic, technological and cultural conditions in society, in the education system, in mathematics as a pure and applied science, in the mathematics curriculum, and in our knowledge and understanding of the teaching and learning of mathematics. Probably even the very place, role, and task of a teacher will be subject to fundamental change, the nature of which is, necessarily, obscure to us today. Nevertheless, we have to form and discuss visions of the future of the teacher in general and of the teacher of mathematics in particular.

At any rate, even if changes were less drastic than might turn out to be the case, no teacher can expect his/her pre-service education, however thorough and well-founded, to provide *sufficient preparation* for his/her entire career as a teacher of mathematics (cf. Niss, [3]).

Sometimes, in some countries, society wants its teachers to be obedient instruments for the realisation of society's specific ideology, purposes and interests concerning (mathematics) education as coined in curricula, syllabi, assessment modes and so forth. In such cases, teachers are not wanted to be independent, or to have a critical and reflective attitude towards the conditions, framework, organisation and content of mathematical education. They are not expected to take initiatives on their own beyond the constraints laid down in the regulations established from above/from outside, let alone to devise curriculum segments, teaching-learning activities, or teaching materials, themselves. Succinctly put, not every society wants its teachers to have self-confidence, or to exercise it in case they have it.

In contrast, it is one of the axioms of this chapter that *teachers should be educated so as to be self-confident*, independent - yet certainly not resistant to new knowledge or reasoning - energetic, and enthusiastic *professionals in the practice of mathematics education.* Put concisely, teachers should be educated for a profession rather than for a job, and to become intellectually autonomous rather than echoes of current trends and fads. This axiom has its justification in centuries of empirical observation. Teaching of mathematics, and hence of geometry, given by teachers who are not self-confident will inevitably degenerate into stereotyped rituals which cannot be adapted or revised in response to changing conditions, circumstances or challenges, neither on a small nor on a large scale. In other words, self-confidence is a prerequisite to flexibility and multi-perspectivity in teaching. It also requires self-confidence to answer "I don't know but I can probably find out" to pupils' or students' challenging questions, and to take the risks involved in engaging in open-minded excursions to land not previously visited by the teacher. And perhaps flexibility and open-mindedness is more needed for the teaching of geometry than for most other mathematical subjects.

IV. Knowledge and views of geometry as a mathematical discipline

The nature and role of geometry as a discipline endow it with a unique position in mathematics. As mentioned in many chapters of this book, geometry is several different things at the same time. It is not just a mathematical topic. Geometry is a mathematical theory (or rather several theories) of *physical space*, at micro (the atomic world), meso (the human world) and macro (cosmological world) level. Geometry is a collection of features and properties of *physical objects*, whether they already exist in nature or are man-made constructs. In these respects, geometry forms part of *natural science* as well as of the engineering and *design sciences* in a broad sense. In both cases, geometry is an infinitely rich source for *modelling* of natural and artificial reality, ranging from ancient patterns in pottery or textiles to the intrinsic structure of the universe. Geometry is a special *separate branch* of mathematics, consisting of different inter-related theories with different sorts of foundation. These theories include axiomatic or synthetic Euclidean and non-Euclidean geometry; analytic geometry and vector spaces; projective geometry; algebraic geometry; differential geometry; finite geometries; combinatorial geometry; and so forth, in addition to various unifying "umbrella theories" of, so to speak, the second order. Geometry is further an *aspect* of, or a perspective on, almost every other branch of mathematics. This is reflected, for instance, in the very fact that so many mathematical objects are labelled with the term 'space' (e.g. 'function space' or 'measure space'), and in the fact that concepts such as 'dimension', 'orthogonal', 'volume', 'projection ' etc. are used in contexts which bear no direct relationship to geometry in its classical meaning. Finally, geometry is a constitutive component in *human vision and visualisation*, and hence in all attempts to represent, describe and understand these, for example in perspective drawings and computer graphics.

Within mathematics, hardly any other branch of mathematics occupies a similar omnipresent, complex, and significant position in the subject as does geometry, (with the exception of numbers, algebra and arithmetic). This gives rise to special requirements and challenges to teachers' knowledge and views of geometry. Regardless of the educational level at which the teacher works, or is going to work, it seems crucial to us that he or she possesses some form of experience, knowledge, and views of the fact that the dimensions just referred to are present as characteristic but different facets of geometry as a topic. It goes without saying that this does not imply that those who teach geometry, in the entire range from kindergarten to Ph.D studies, all ought to have the same degree of in-depth and detailed experience, knowledge, and views of geometry as a natural science, a design science, a multitude of theoretical structures each with its own specific basis and body, an all-pervading aspect of mathematics at large, and a fundamental component in human vision and visualisation. Of course not. In addition to being a pretty unrealistic requirement, it would also be an unreasonable exaggeration of the needs of, say, a primary school teacher. However, what *is* implied is that the dimensions as such and the differences between them should form part of the education of, in essence, any mathematics teacher irrespective of the level at which he or she teaches or will be teaching. Furthermore, at any level of education, geometrical competence takes different forms, ranging from "acquaintance", over "being knowledgeable", to "having mastery" (this goes for teachers as well as for pupils and students). The important thing is that each of these dimensions and forms of competence can be materialised in numerous ways corresponding to the teaching level at issue. Naturally, the higher the level of teaching, the more comprehensive and multi-faceted, and complex is the image of geometry that is on the agenda of teaching and learning. Accordingly, the requirements on teachers' geometrical insights increase in extent and depth with the levels on which they teach.

The teaching of geometry at primary school, in most countries, emphasises the exploration, naming, description, classification, drawing, mensuration (sometimes accompanied by simple computations) of concrete physical objects in the plane or in the space. Thus, it is the naturalistic and constructive facets of geometry that are in focus of the attention of pupils and teachers. Primary school teachers, therefore, will tend to see geometry as mainly a natural science and as a design science, by and large of an empirical nature. Generally speaking, it is certainly reasonable to concentrate on these facets in primary school geometry. However, primary school teachers should be educated so as to be aware that geometry is not merely a collection of empirical facts which form an experiental discipline. They need to know that essential parts of the geometry of physical objects in physical space can be subjected to deductive reasoning and be organised as an extensive theoretical structure in various ways. They need to know how geometrical considerations may shed light of immediate insight ('aha! experiences') on arithmetical operations, rules, phenomena, and results. They need to know that human vision is, in crucial respects, governed by geometrical laws and properties which account for the striking similarity between our visual images and photographs, pictures provided by electronic media, perspective drawings and paintings, and so forth. Already at the primary level, the relationships between 3-dimensional objects and 2-dimensional images are of key importance to the fostering of basic geometrical knowledge and understanding.

The point here is not that teachers have to know all this in an elaborate technical sense, let alone that they should necessarily teach it to their pupils. Rather, these perspectives should be considered as a surplus of knowledge and background which will help teachers to create richer and more stimulating learning environments with respect to the things they *do* teach. The more facets and connections their own geometrical backgrounds contain, the more will they be able to encourage and guide their pupils to experience, explore and examine geometrical matters, and the more will they be able to sow seeds in their pupils which, in later stages of schooling, can grow to widely ramified plants of geometrical knowledge and insight.

When it comes to teachers on higher levels of the education system, the spectrum of dimensions they need to know about in specific terms obviously becomes broader. At the same time, the depth of their geometrical experience, knowledge, and views will have to be increased with the level. Furthermore, they will need to possess larger sets of examples and cases drawn from a wide variety of contexts and frameworks pertinent to geometry. At higher levels, they need to know something about different geometrical theories and their inter-relations. Especially, the transfer between different kinds of geometrical "codes" and "modes" - such as "visual observation, intuition and representation", "physically geometrical modelling", "analytic or vector geometry", "deductive Euclidean geometry", "paper folding geometry", or "transformation geometry" - becomes important at the post-elementary levels, and hence in the preparation of teachers. Even if these expectations may well appear to be self-evident in theory, they are less so in practice. Apart from the fact, discussed in the introduction, that geometry still holds a fairly humble position in many teacher training programmes throughout the world, it often so happens, in the cases where geometry *is* present, that only its theoretical dimensions are represented in the programme. Thus, at higher levels, it is not unusual that geometry is split into a (not very large) number of geometrical disciplines, each taught simply as yet another branch of theoretical mathematics.

So, if the problem with the geometrical education of primary school teachers lies mainly with the absence of knowledge of geometrical theory, the problem tends to be the reverse with teachers on secondary and tertiary levels. If they have any geometrical education it is likely to be predominantly theoretical, whereas the other dimensions of geometry, equally important in the context of education, are usually greatly under-represented both in pre-service training and in in-service programmes. Of course, it is important that teachers have a solid and varied background in different species of theoretical geometry. For instance, any post-primary mathematics teacher should be acquainted with instances of axiomatic as well of analytic geometry, and of the fact that not all geometry is Euclidean. Moreover, he or she should be able to subject pure or applied mathematical problems to different kinds of geometrical treatment as appropriate or convenient in the situation. But, in order to be able to endow their teaching of geometry with richness in quality and perspective, and to exercise it with autonomy and self-confidence, (cf. Section III), their knowledge and views of the discipline need to encompass also other dimensions of geometry than the purely theoretical ones.

In summary, we want to advocate that mathematics teachers on all levels be educated in such a way that they acquire a representative, multi-dimensional and balanced knowledge and view of geometry as a mathematical topic in all its dimensions as outlined above, with due attention being paid to the level at issue. Teachers should neither perceive geometry as only an empirical science of nature or design, nor merely as a separate theoretical mathematical discipline like so many others. Finally, teachers should be aware that geometry has a long, winding and remarkable history, as a discipline which has its origins in the earliest human communities and civilisations, and which ever since has been deeply imbedded in human culture, society, science and technology. Generally speaking, in geometry, as in any branch of mathematics, teachers need to know and understand a good deal more than their pupils or students. They need a reserve of knowledge, insight, ability, and experience, which goes beyond the immediate day-to-day requirements.

In this section we have emphasised the principal dimensions of geometry which we believe that teachers should have either insight in, knowledge or view of, or a feel for, depending on the circumstances, in order to acquire what we may call 'a geometrical culture for teaching'. We have deliberately abstained from listing specific geometrical topics, concepts, facts, methods, techniques, theorems, theories, etc. that teachers for such and such level should know or master. We have chosen to do so in order to avoid the syllabusitis trap referred to in Section II, and not because it is impossible or unreasonable to indicate specimens of this sort - for instance, it would be fairly 'safe' to require all teachers of geometry to know the area of a circle, and (the content of) the Pythagorean theorem. However, it would trivialise and distort the discussion of teachers' knowledge and view of geometry if we spent our effort on establishing a syllabus for their education.

Once the question of which dimensions of geometry should be considered essential for teachers' perception of the field at a given teaching level has been settled, the specification and implementation of these dimensions into concrete curricula and programmes can be carried out in infinitely many valid ways. Which way should be chosen in a given context depends so strongly on the actual situation, its boundary conditions and circumstances, that it does not make much sense to attempt to discuss or reach agreement concerning the design of "canonical" curricula for pre-service or in-service teacher training. Hence, it should not be taken for granted that the geometry packages designed for pre-service preparation of teachers should be restricted to containing only the classical subject matter which always used to be included.

V. Knowledge and views of processes of learning geometry

In the previous section we discussed teachers' knowledge and views of geometry as a *mathematical topic*. It is now time to turn our attention to the kinds of knowledge and views that teachers should have of the *processes of learning* geometry, at different educational levels.

Firstly, teachers' knowledge of various ways in which geometrical experiences can be gained, geometrical notions and concepts can be established, and knowledge, understanding and ability be acquired, will, roughly speaking, be specialisations to geometry of their knowledge of the ways in which mathematical concepts and insights in general are acquired and established. This is an area in which research in mathematics education has been very active during the last couple of decades, and continues to be so. Lots of reports and papers have been published to this effect. As this is not the place to present or review recent research in the area and its outcomes at a general level, let us confine ourselves to pointing out just one major finding on which there is universal agreement in the mathematics education community, no matter whether the issue is considered from some form of a so-called constructivist position or not. We should like to propose that this finding be called *The First Main Finding of the Didactics of Mathematics*: *When a pupil or student engages in learning mathematics, the specific nature, content, range, and flavour of a mathematical notion or concept that he or she is acquiring or building up are greatly influenced, if not determined, by* the set of domains *in which that notion or concept is anchored and imbedded for that particular person.*

In other words, mathematical notions and concepts and their inter-relations are immersed in, and are hence strongly coloured by, pupils' and students' personal worlds of experience. This implies that the meaning and the interpretation which an individual assigns to a mathematical notion or concept are marked by that individual's entire set of mathematical and extra-mathematical experiences. This is true not only for notions and concepts but also for the interpretation and the range of validity of mathematical statements (propositions and theorems).

Clearly, the general finding just described should be included in the luggage of any mathematics teacher. In the nature of the matter, it pertains to geometry too. However, what lessons can be learnt from this finding that are specific to geometry and its teaching and learning?

In contrast to what is the case with many other mathematical topics and branches, characterised primarily by abstractness and generality, geometry is linked to physical objects, physical space, and the perception thereof, in a multitude of different manners. So, any individual who is learning geometry, in whatever form and at whichever level, already possesses an infinitely rich fund of geometrical knowledge and experiences ("correct" or "incorrect", that is not important here) which are of a mathematical nature, albeit not necessarily expressed or represented in a mathematical language or framework. This implies that an individual's learning of 'new' topics in geometry, in a mathematical formulation, will inevitably be confronted with the geometrical intuition, insight, knowledge, and experience already present in the pupil's/student's world of experience. Naturally, this constitutes a wealth of opportunities but, as a matter of fact, significant obstacles as well. It is important that teachers are aware of both the opportunities and the obstacles to an extent which allows them to take advantage of the former and to counteract, avoid or circumvent the latter.

As far as the opportunities are concerned, the learner's everyday geometrical experience provides the main source and background for her/his forming of geometrical notions and concepts, in a more formal mathematical setting. The same is true with geometrical propositions, methods and techniques which can often be established in a dialectic interplay with corresponding propositions, methods and techniques residing in the everyday geometry of viewing, drawing, folding, cutting and pasting, constructing, measuring, moving, and so forth, with physical objects, shapes and phenomena. Furthermore, conjectures about properties of formal or abstract geometrical objects can be obtained or supported by the pupil's/student's exploration of the geometrical objects of his or her experiental world. Hypotheses may be subjected to preliminary empirical testing before one embarks on attempts at more formal derivations or deductions. Any learning of formal geometry will have to imply a reconciliation in the learner's mind of the formal representation and understanding of the topic at issue with that learner's experiental geometrical universe. This reconciliation requires the development of plane and spatial intuition along with the investigation of formal geometry.

To sum up, the epistemology of geometry (i.e. the nature, role and status of geometry and geometrical knowledge) becomes an urgent issue in the learning of geometry as soon as the conception of geometry as a solely empirical discipline is abandoned. The learning and understanding of 'post-empirical' geometry can hardly take place unless epistemological issues are brought into the open and dealt with in the learning process. Henceforth, any meaningful teaching of geometry will have to strive to create or at least help this reconciliation between empirical and theoretical geometry in the minds of each of the pupils or students to whom the teaching is being given. However, further reflections on this theme belong to the next section of this chapter.

From the 'Main Finding' referred to above, we learn that the content, meaning, and range of a geometrical notion, concept or statement, learnt or constructed by a learner, are (co-)determined by the specific experiental domains to which it is connected for that particular learner. The richer the set of domain connections, the more comprehensive and multi-faceted the concept, notion, or statement - and conversely. In other words, for pupils and students to develop a certain geometrical knowledge, that knowledge is likely to be exactly as profound and extensive as its explicit experiental foundation allows for.

In a way, what we have just said is simply a reflection of the point made in Section IV that one of the essential dimensions of geometry is that it is a natural and a design science, with a strong empirical basis. None the less, if the ambition is that the pupils'/students' understanding and perception of geometry should reach beyond its immediate naturalistic and empirical dimension, it is exactly this very dimension which can create serious obstacles to the learning of geometry.

Probably the key obstacle is the potential equating of geometry with ontology (i.e. the constituents of reality, primarily physical reality). The majority of concepts in theoretical (Euclidean) geometry (whether synthetic or analytic) have close counterparts in everyday reality. This is true with, say, line, point, circle, triangle, prism, pyramid, sphere, ball, plane, box, pentagon, length, area, volume, angle etc. In fact, as we all know, such geometrical concepts were initially formed as abstractions from corresponding 'everyday' concepts. Of course, the abstraction process implies an idealisation as well, for instance when, in axiomatic geometry, we think and speak of lines and planes as infinite, of points as having zero length and area, and so forth, but still the formal geometrical concepts inherit large parts of their meaning from the corresponding 'real' ones. This intimate correspondence does not only encompass concepts but also properties and propositions (although, and that is important, not the *verification* of propositions). In general, properties and propositions in Euclidean geometrical theory can be interpreted almost immediately as properties of and propositions concerning (suitably idealised) geometrical objects of the real world. Now, the obstacle arises primarily in two contexts.

Firstly, as regards the methods of verification of assertions in the theoretical versus the empirical domain. In the theoretical domain, verification is based on some form of reasoning, the specific format of which depends on the nature of the geometrical theory involved. (In axiomatic geometry *à la* Euclid reasoning is based on deduction from the axioms and the givens, in analytic geometry it is based on the real number system and its properties.) In contrast, verification in the empirical domain is primarily based on empirical inspection followed by induction (in the philosophical sense, of course). It is well known that many learners of geometry have severe difficulties in understanding and appreciating this difference and in accepting the mode of verification adopted in geometrical theory. Why prove assertions the correctness of which are intuitively evident to any observer, or can be substantiated convincingly by a suitable number of empirical tests? To understand and accept that the purposes and goals, and hence the methods of justification, in the two domains are different is non-trivial and requires a substantial intellectuel effort on the part of the learner. We believe, therefore, that it is highly important that mathematics teachers have insights into the origin and nature of these learning obstacles, and into ways in which they can be overcome. A possible starting point for this could be, for example, to consider the problem of obtaining certainty as regards general statements, like *the sum of the angles of a plane triangle is always 180⁰*, or *the medians of a triangle always intersect each other in one point*. These are claims which cannot be justified universally on empirical grounds only. Similarly with the fact that geometry (at higher levels, that's true) offers 'theorems of complete classification' and 'theorems of impossibility', such as *the only convex regular polyhedra that exist (in 3-space) are the five platonic solids*, or: *it is impossible to trisect any arbitrary, given angle by means of ruler and compass in Euclid's sense*. Claims of this kind may form a point of departure from which the very nature and tasks of geometry can be considered.

Secondly, if the objects of Euclidean geometry are being assigned a direct status in physical reality, it is very difficult to come to grips with other sorts of geometry in which, say, the objects called 'points' and 'lines' are of a different kind to those in empirically based Euclidean geometry, and where notions like 'parallel' and 'perpendicular' have unexpected meanings and implications, as is the case, for instance, in spherical geometry. Evidently, this difficulty is reinforced if we enter the realm of absolute geometry, inhabited by completely abstract objects which might as well be called "tables", "chairs" and "beer mugs", to quote Hilbert. The really subtle thing is that when this non-naturalistic status of the objects of Euclidean geometry (of whichever brand) is emphasised, the problem pointed out here not only remains but is aggravated by the fact that the ontological status of naturalistic Euclidean geometry "butts in", so to speak, with a competing interpretation of the objects and phenomena encountered in the geometrical theory at issue.

Again, if pupils and students are to acquire understanding and insight into these intrinsically difficult aspects of geometry, these have to be rooted in experiences which learners gain from exploring and investigating a variety of geometrical domains, both empirical and theoretical. And this takes time. If the learning process, for one reason or another, does not require the learner to explore and investigate a sufficient number of representative domains, then that learner's knowledge and view of geometry are fairly certain to become either superficial (geometry is a sort of strange game with no external meaning) or to suffer from severe misconceptions.

In conclusion, for a mathematics teacher to become able to adequately teach geometry to his or her pupils/students, it is necessary that he or she posseses knowledge of how the learning of geometry may take place and of the pitfalls that might impede such learning. It is a *conditio sine qua non* for meaningful and fruitful learning of post-elementary geometry, to establish viable links in the mind and experience of the individual learner, between geometry as a naturalistic discipline and geometry as some form of theory. This process requires a lot of time - and, naturally, learning opportunities - to be developed. The Scylla of viewing geometry as merely a naturalistic, empirical discipline, and the Carybdis of reducing geometry to just a formal game played according to sets of arbitrary rules, are always present as a challenge to the learning and teaching of geometry. Teachers of geometry have to be continually engaged in counteracting them. This takes us to the next section.

VI. Rich teaching-learning environments in and with geometry

On the basis of the conclusions we have obtained in the previous sections, this section will consider how teachers should be able to design and orchestrate rich teaching-learning environments that allow pupils and students to acquire knowledge, insight, and ability concerning those dimensions of geometry which are on the agenda at their level of education. As has been the case so often in this chapter, much of what is going to be said addresses teaching-learning environments in mathematics education in general, with geometry as a special case. However, a number of deliberations that may be seen as specific to geometry will be offered as well.

First of all, teachers have to deal with the overall purposes of, and the specific objectives for, the teaching of geometry to a given category of pupils or students. The complexity of geometry as outlined in Section IV, as well as of the learning processes sketched in Section V, gives rise to a large variety of different purposes and goals - not necessarily mutually compatible - that might be pursued at a given teaching level and in a given context. The point is not only that teachers will have to know the official goals for the teaching and learning of geometry in a given curriculum as established by the relevant authorities. More importantly, teachers should reflect, with critical analysis, on these goals as well as on the goals which they themselves would like to put forward and pursue as professionals in the teaching of mathematics. Moreover, they should reflect on the degree of compatibility between these categories of goals, and finally make their own decisions within the "room of goal autonomy" which is left to them by the curriculum.

Next, it follows from the discussion in Section V that the acquisition of geometrical knowledge, insight and ability depends strongly on the specific domains in which the geometrical experiences of the learner are gained and rooted. Therefore the most significant task of the teacher of geometry is to stage teaching-learning environments, situations, and activities in which the learner can study, explore and investigate the objects, phenomena, problems, properties, and structures of a broad spectrum of geometrical situations and 'worlds'. These teaching-learning contexts, and the domains they invoke, should be as rich and many-sided as possible. Sometimes pupils/students should work on their own, individually or in small groups, exercising free exploration of, typically, an object world or a problem situation. Sometimes, pupils should be engaged in activities designed and guided by the teacher, and sometimes they should be recipients of stimulation, information, or explanation, supplied by the teacher, or by various sorts of materials or media, including information technology. Sometimes work should take place in the classroom, sometimes in out-of-school activities. At any circumstance, the teacher need to be able to design curricula and to construct, select and implement written, concrete or electronic teaching materials suitable for the specific teaching purposes on the agenda.

The point made in the preceding paragraph is not special to geometry but concerns the teaching of mathematics in general. However, it is especially important with respect to geometry, because of the fundamental duality between geometry as a natural and design science and as a theoretical discipline. The reconciliation of these two dimensions of geometry requires a multitude of experiences gained from work in both dimensions. At the primary level, the teacher should be able to design and organise teaching-learning contexts where the pupils can explore concrete, tactile physical objects; some of the latter exist in nature, while others are artefacts presented to the pupils, and still others are constructed, under teacher guidance, by the pupils themselves of paper, cardboard, wood, and so forth. Some objects should be solid, others transformable in various manners. Geometrical objects created by computer graphics should be included in the teaching-learning environment as well. The pupils should investigate the shapes, sizes, and properties of the objects they study, solve problems, pose and answer questions, etc.

At lower secondary level, the sorts of activities mentioned for the primary level - many of which are equally relevant at higher levels *mutatis mutandi* - should be complemented with teaching-learning situations where relations between geometrical experience of physical space and physical objects, and formal geometrical theory can be established, and the interplay between them investigated by pupils. This requires activities through which the pupils can study and investigate the conceptual aspects of introductory geometrical theory (of whatever kind), with a focus on the geometrical concepts as resulting from idealisations of certain features of everyday objects and phenomena. As this concept formation represents a step of far reaching impact on the foundation of the geometrical knowledge and insight of the individual pupil, it is essential that the teaching-learning situations are sufficiently many and sufficiently rich to allow for this foundation to be extensive and firm. Teachers need to be able to flexibly create such situations in accordance with their pupils' needs, backgrounds and situations.

If the teaching-learning contexts at the lower secondary level are to emphasise the geometrisation of aspects of pupils' physical environment, the relationship between physical reality and geometric formalism is often reversed at the upper secondary level. That is to say that *in addition to* what has been proposed for the primary and lower secondary levels, quite a few teaching-learning situations for the upper secondary level should provide opportunities for students to go in the opposite direction, i.e. investigate to which extent it makes sense to interprete concepts, phenomena, and propositions from theoretical geometry in terms of physical reality. This includes inquiries into the range of correspondence between concepts and propositions of geometric theory, and physical reality. Also, students should be engaged in activities that focus on the validation of geometrical assertions within the formal frameworks at issue. Again, it is a task for the teacher to design and orchestrate teaching-learning environments that can support students' acquisition of insight into the nature of theoretical geometry and its interplay with physical reality. Numerous examples of teahing-learning environments are given in previous chapters (e.g. 24, 4, and 5) of this book.

As finally regards the teaching of geometry at tertiary levels, which typically focuses on various forms of abstract and general geometrical theory, the critical task for the teacher is to provide encouragement, stimulus and environments for students so as to allow them to examine the relationship between advanced geometrical theory and elementary geometrical theory. Special emphasis should be placed on recalling and interpreting of concepts and propositions from elementary geometry in advanced settings. For instance, which aspects of the elementary concept of angle (or plane) are preserved in the definition of angles (or hyperplanes) in vector spaces with inner products, and which are not? Or, to what extent does advanced projective geometry generalise the geometry of, say, perspective constructions? And to what extent does projective geometry give rise to phenomena which have no counterparts in elementary geometry of perspectivity? Of course, the reasoning and deduction modes, and their basis, in different geometries also deserve close attention by teachers and students at tertiary levels. Far too often, in tertiary-mathematics teaching, the bridging of the interpretation gap between advanced theories of geometry and elementary geometry is left to students' own endeavour and effort, which in actual practice amounts to disregarding the need for such links. How can we imagine future teachers of mathematics to see and establish such links in their own teaching of geometry if they have never had an opportunity to encounter these links as students themselves? Probably much of the problem stems from the fact that a non-negligible number of tertiary-mathematics professors do not take teaching seriously but confine themselves to lecturing. The time seems to be ripe for a general call on university mathematicians to invest not only time but also intellectual effort in quality teaching.

For all the levels considered, three points are of particular significance in relation to the teaching of geometry.

(*) Because of the special nature of geometry as a mathematical discipline, geometry is unique in providing opportunities for "getting much out of little", didactically and pedagogically speaking. Probably any geometrical situation, however elementary, gives rise to a rich variety of possibilities for exploration, posing of questions and problems ("what if..?", "could it happen that ...?"), formulation of conjectures, experimentation with special cases, attempts at convincing others, at proving, and so on and so forth.

(*) Because of its age old position as a natural and design science, geometry is intimately linked to human society, culture, history, science, technology, philosophy, arts and crafts. The teaching of geometry, therefore, offers ample opportunity to uncover and relate mathematical ideas to human and social activity. (*) Probably, there is no better place than geometry for illucidating and discussing the concept and role of proof and proving in mathematics. All possible notions and variants of reasoning, justifcation and proof are present in opulent measures in the context of geometry. The entire spectrum of seeing, believing, following, or accepting a statement or a line of thought, of being convinced, persuaded, intimidated etc. can be encountered and investigated in geometrical settings (see Chapter 2).

In their endeavours to stage rich teaching-learning environments, teachers should be able to take advantage of these three features of geometry, adapted to the level, context, and situation in question.

Finally, as far as the role of information technology is concerned, a remark in conclusion is warranted. There is no doubt that for all the levels we have been considering, information technology offers powerful and splendid tools for the furthering and support of the development of geometrical experience and intuition, and the investigation of geometrical worlds, for instance through the opportunities to formulate and test hypotheses and conjectures (cf. Chapter 4). However, it is important to us to emphasise that this does not imply that information technology can *replace* the real world of natural and man-made objects. The fundamental insight that geometry *also* is to do with the real world cannot be experientally substantiated without systematic geometrical experiences with and from the real world.

VII. Teacher assessment in geometry

A separate chapter (Chapter 8) in this book is devoted to the question of assessment in geometry. In this context, therefore, we shall confine ourselves to considering - briefly - those aspects of assessment in geometry that are to do with the qualifications and education of teachers.

If we agree that teachers should not concentrate only on "delivering" their own teaching but also on paying attention to their pupils'/students' learning, it follows that teachers should be able to observe, analyse, and monitor pupils' and student's learning, and to offer guidance and advice concerning the adjustment of learning strategies and behavior, for individual study as well as for classroom work. This requires teachers to be in possession of a variety of assessment modes that are suitable for different purposes and goals in relation to geometry, among which examinations and promotion tests occupy only a position of secondary importance, if viewed from a didactical perspective (Niss, [4]).

This implies that one main purpose of teacher assessment in geometry is to procure and provide *information* to the pupils/students concerning the progress they make and the problems they have in acquiring geometrical knowledge, insight and ability. Moreover, such information should also serve to enable the learner to monitor and assess his or her own learning of geometry. Another main purpose is to place and *guide*, on the basis of the information obtained, the learner in learning situations and activities which can help to underpin and develop knowledge and ability in those aspects of geometry in which there is a particular need for it.

In accordance with the discussion in Section V, teachers' should, in their assessment of an individual pupil's/student's learning of geometry, pay particular attention to

(a) the extent to which he or she has overcome the epistemological obstacles caused by the dual nature of geometry (as a natural and design science and as a formal discipline, respectively, cf. Sections IV and V) as far as the acquisition of geometrical notions, concepts, and propositions are concerned;

(b) his or her ability to interpret matters and situations originating in one geometrical world in terms of another such world;

(c) his or her ability to reason within a given geometrical universe;

(d) his or her ability to solve pure and applied geometrical problems;

(e) his or her ability to pose questions and problems, and to formulate hypotheses and conjectures.

In other words, the education of teachers with respect to the assessment of geometry should focus on ways in which these components of the learner's geometrical knowledge and ability can be identified, diagnosed, analysed (and ultimately improved). As to a discussion of possible means and tools for that purpose, we refer to Chapter 8. Suffice it, here, to say that timed written tests cannot stand alone in the pursuit of that purpose.

VIII. Ways to develop a competence to teach geometry

In the previous sections, our attention has been focused on ideals rather than on the everyday situations, circumstances, and constraints of the real world. Now, an obvious question poses itself fairly strongly: To what extent, and it what ways is it possible to pursue these ideals in actual practice? More specifically, can we ensure that teachers of geometry acquire knowledge and views of geometry as a mathematical topic and of the processes of learning geometry, that they become able to stage rich teaching-learning environments and situations in and within geometry, and that they become able to assess their pupils' or students' learning of geometry so as to identify, monitor, and guide the learning processes and strategies of these pupils and students, in accordance with the perspectives outlined in the previous sections?. Unfortunately, the answer to this question, as it stands, is probably 'no', in general. For the answer to the question to be 'yes', Utopia had to be invoked. But less will do. Instead, we may ask what could be done to approach the ideals sketched in this chapter. The remainder of this chapter is devoted to this issue. A remark in advance: clearly, in view of the immense diversity of circumstances across and within countries and institutions, it is not reasonable to expect that it is possible to come up with specific, concrete ideas and suggestions which can be implemented right away in any context whatsoever.

In Section III it was stated that for no teacher, irrespective of the educational level on which he or she teaches, will his/her pre-service preparation be sufficient for his/her entire career as a teacher of mathematics. This is true also for the preparation to teach geometry. Therefore, continual in-service education and professional development of the individual teacher becomes a necessity if we (i.e. society and the mathematics education community) want teachers to be able to approach the ideals discussed here. However, successful in-service professional development has to be built on a fairly solid pre-service basis of subject matter knowledge and views combined with didactical and pedagogical knowledge and insight. As regards geometry, this implies that all the main aspects dealt with in this chapter should, *in principle*, be *on the agenda* of any pre-service preparation, even if the depth to which they can be treated will necessarily differ from level to level and from place to place simply because the conditions and problems vary greatly with country, teaching level, and so forth.

In many countries it is customary to educate primarily school teachers to obtain only a rudimentary knowledge of mathematics beyond arithmetic. It is not unusual that primary school teachers of mathematics have a knowledge of geometry which can only be characterised as being below the minimum that any meaningful teaching of geometry does require. Although this chapter is not the place to discuss the overall structure and organisation of teacher-training programmes for primarily school teachers (in most countries this is a highly political socio-economic issue), it seems that in quite a few places even a small (absolute) increment in the geometrical content of these programmes will result in a substantial, and much needed, relative increase in primary school teachers' geometrical knowledge and an improvement in their prerequisites for teaching geometry. Put differently, there is a potential for development of competence here which is not fully exploited in many places. Moreover, such a solidification of their pre-service backgrounds would provide a basis for primary school teachers to later participate in and benefit from the kinds of in-service activities on the teaching of geometry which will be discussed below.

As to the secondary level, the diversity of pre-service teacher training programmes, with respect to structure and content, among countries, institutions and soforth is even greater than on the primary level. In some countries secondary school teachers are prepared to become specialists in a few subjects, for instance by being required to have a university Master's degree in those subjects, whereas other countries educate teachers to teach a broad spectrum of subjects for which their preparation is then, necessarily, less subject specific and intensive. However, it seems fair to summarise the situation as follows (in very gross terms, of course; several counter examples do exist): in programmes emphasising subject matter specialisation, teachers of mathematics are likely to be given some, although in general not substantial, preparation in geometry as a discipline, but seldom much in the didactics and pedagogy of geometry. In programmes emphasising a general preparation for the profession of secondary teacher, teaching in general tends to be in focus, and the mathematical part of the programme is usually less manifest and comprehensive, and hence the same is true for geometry, both as a mathematical and as a teaching topic. No matter which type of in-service teacher training programme we consider, there is reason to believe that there is room left for improvement with respect to the preparation of future teachers for the teaching of geometry, regarding either the subject matter side or the didactical/pedagogical side, or both.

As finally regards the background of university lecturers and professors for teaching geometry, including geometry to future teachers, the average picture is that they are fairly well versed in geometry as a mathematical discipline, but rarely in geometry as a topic for teaching and learning. Here is clearly room for progress as far as the didactics and pedagogy of geometry (or of mathematics at large, for that matter!) are concerned.

Although it is evident that it is desirable to equip future teachers with as solid as possible a pre-service preparation concerning the dimensions of geometry, previous considerations in this chapter led us to conclude that even the most splendid pre-service programme should be followed up by continual in-service education and professional development of the individual teacher on whichever level. Such continual professional development can - and should - take a multitude of different forms and formats, depending on all the needs, boundary conditions, factors, circumstances and opportunities that prevail in a given context. So, the important thing is not so much *exactly* what happens, and how it happens, but rather that *something* happens that may serve the purpose of enlarging and enriching teachers' qualifications as teachers of geometry with respect to:

(*) geometry as a mathematical topic, in all its manifestations;

(*) processes of learning geometry, viewed from the perspectives of practice and research;

(*) environments, contexts, situations, and activities for the teaching and learning of geometry.

Just a few examples of ways in which the professional development of teachers of geometry can be staged are in-service courses and workshops, summer schools, conferences, publication activities, local, regional or national informal networks or formal organisations of teachers and researchers working at various levels, teaching experiments, participation in development or research project on the teaching and learning of geometry, international exchange of ideas and experiences. For this to be furthered, "let a thousand flowers bloom".

In-service activities which go across teaching levels and institutions seem particularly worthwhile and fruitful, not only because they serve as a platform for exchange of information and inspiration. For they also stimulate teachers on a given level to articulate and reflect on and to re-consider what they are doing, if they are to communicate about it to professionals in other segments of the educational system. One example of this is the IREMs in France in which teachers and researchers on various levels collaborate within a regional context (see Douady and Henry, [1]). Another example, addressing the tertiary level only, is the so-called *Undergraduate Mathematics Teaching*

Conferences (know as the Nottingham Conferences), in the UK, where, with the goal of raising critical consciousness, small groups of university teachers are expected to provide written arguments about, e.g., curricula, to criticise the work of other groups in writing, and so forth (Brian Griffith, personal communication).

In conclusion, once again it is true that much of what has been said in this section with respect to geometry can be considered as a special case of what could be said of mathematics at large. This is because geometry is a part of mathematics, yet a very special one, and because geometry is taught by teachers of mathematics. To those teachers, and to their pupils or students, and to the mathematics education community at large, geometry should be taught for the same reasons as mathematics as a whole is taught: in order to help the recipients be able to better understand and cope with life in society, culture, and nature.

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NOTES

1. This information was given in a plenary presentation at the ICMI Study Conference *Perspectives on the Teaching of Geometry for the 21st Century*, Catania, Sicily (Italy), 28 Septemer - 2 October 1995.

2. The term 'syllabusitis' was coined in Lewis, 1972. The author is grateful to his friend and colleague Jens Højgaard Jensen for having pointed out the term and the reference.

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Vejleder: Bernhelm Booss-Bavnbek

277/94 PROJEKTARBEJDSPÆDAGOGIK - Om tre tolkninger af problemorienteret projektarbejde
af: Claus Flensted Behrens, Frederik Voetmann Christiansen, Jørn Skov Hansen, Thomas Thingstrup
Vejleder: Jens Højgaard Jensen

278/94 The Models Underlying the Anaesthesia Simulator Sophus
by: Mette Olufsen(Math-Tech), Finn Nielsen (RISØ National Laboratory), Per Føge Jensen (Herlev University Hospital), Stig Andur Pedersen (Roskilde University)

279/94 Description of a method of measuring the shear modulus of supercooled liquids and a comparison of their thermal and mechanical response functions.
af: Tage Christensen

280/94 A Course in Projective Geometry
by Lars Kadison and Matthias T. Kromann

281/94 Modellering af Det Cardiovaskulære System med Neural Pulskontrol
Projektrapport udarbejdet af:
Stefan Frello, Runa Ulsøe Johansen, Michael Poul Curt Hansen, Klaus Dahl Jensen
Vejleder: Viggo Andreasen

282/94 Parallelle algoritmer
af: Erwin Dan Nielsen, Jan Danielsen, Niels Bo Johansen

283/94 Grænser for tilfældighed
(en kaotisk talgenerator)

af: Erwin Dan Nielsen og Niels Bo Johansen

284/94 Det er ikke til at se det, hvis man ikke lige ve' det!
Gymnasiematematikkens begrundelsesproblem

En specialerapport af Peter Hauge Jensen og Linda Kyndlev

Vejleder: Mogens Niss

285/94 Slow coevolution of a viral pathogen and its diploid host

by: Viggo Andreasen and Freddy B. Christiansen

286/94 The energy master equation: A low-temperature approximation to Bässler's random walk model

by: Jeppe C. Dyre

287/94 A Statistical Mechanical Approximation for the Calculation of Time Auto-Correlation Functions

by: Jeppe C. Dyre

288/95 PROGRESS IN WIND ENERGY UTILIZATION

by: Bent Sørensen

289/95 Universal Time-Dependence of the Mean-Square Displacement in Extremely Rugged Energy Landscapes with Equal Minima

by: Jeppe C. Dyre and Jacob Jacobsen

290/95 Modellering af uregelmæssige bølger
Et 3.modul matematik projekt

af: Anders Marcussen, Anne Charlotte Nilsson, Lone Michelsen, Per Mørkegaard Hansen

Vejleder: Jesper Larsen

291/95 1st Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM
an example of using methods developed for the OECD/IEA and the US/EU fuel cycle externality study

by: Bent Sørensen

292/95 Fotovoltaisk Statusnotat 3

af: Bent Sørensen

293/95 Geometridiskussionen - hvor blev den af?

af: Lotte Ludvigsen & Jens Frandsen

Vejleder: Anders Madsen

294/95 Universets udvidelse - et metaprojekt

Af: Jesper Duelund og Birthe Friis

Vejleder: Ib Lundgaard Rasmussen

295/95 A Review of Mathematical Modeling of the Controled Cardiovascular System

By: Johnny T. Ottesen

296/95 RETIKULER den klassiske mekanik

af: Peder Voetmann Christiansen

297/95 A fluid-dynamical model of the aorta with bifurcations

by: Mette Olufsen and Johnny Ottesen

298/95 Mordet på Schrödingers kat - et metaprojekt om to fortolkninger af kvantemekanikken

af: Maria Hermannsson, Sebastian Horst, Christina Specht

Vejledere: Jeppe Dyre og Peder Voetmann Christiansen

299/95 ADAM under figenbladet - et kig på en samfunds-videnskabelig matematisk model
Et matematisk modelprojekt

af: Claus Dræby, Michael Hansen, Tomas Højgård Jensen

Vejleder: Jørgen Larsen

300/95 Scenarios for Greenhouse Warming Mitigation

by: Bent Sørensen

301/95 TOK Modellering af træers vækst under påvirkning af ozon

af: Glenn Møller-Holst, Marina Johannessen, Birthe Nielsen og Bettina Sørensen

Vejleder: Jesper Larsen

302/95 KOMPRESSORER - Analyse af en matematisk model for aksialkompressorer

Projektrapport sf: Stine Bøggild, Jakob Hilmer, Pernille Postgaard

Vejleder: Viggo Andreasen

303/95 Masterlignings-modeller af Glasovergangen
Termisk-Mekanisk Relaksation
Specialerapport udarbejdet af:

Johannes K. Nielsen, Klaus Dahl Jensen

Vejledere: Jeppe C. Dyre, Jørgen Larsen

304a/95 STATISTIKNOTER Simple binomialfordelingsmodeller

af: Jørgen Larsen

304b/95 STATISTIKNOTER Simple normalfordelingsmodeller

af: Jørgen Larsen

304c/95 STATISTIKNOTER Simple Poissonfordelingsmodeller

af: Jørgen Larsen

304d/95 STATISTIKNOTER Simple multinomialfordelingsmodeller

af: Jørgen Larsen

304e/95 STATISTIKNOTER Mindre matematisk-statistisk opslagsværk indeholdende bl.a. ordforklaringer, resuméer og tabeller

af: Jørgen Larsen

305/95 The Maslov Index:
A Functional Analytical Definition
And The Spectral Flow Formula

By: B. Booss-Bavnbek, K. Furutani

306/95 Goals of mathematics teaching

Preprint of a chapter for the forth-
comming International Handbook of
Mathematics Education (Alan J.Bishop, ed)

By: Mogens Niss

307/95 Habit Formation and the Thirdness of Signs

Presented at the semiotic symposium

The Emergence of Codes and Intensions as
a Basis of Sign Processes

By: Peder Voetmann Christiansen

308/95 Metaforer i Fysikken

af: Marianne Wilcken Bjerregaard,
Frederik Voetmann Christiansen,
Jørn Skov Hansen, Klaus Dahl Jensen
Ole Schmidt

Vejledere: Peder Voetmann Christiansen og
Petr Viscor

309/95 Tiden og Tanken
En undersøgelse af begrebsverdenen Matematik
udført ved hjælp af en analogi med tid

af: Anita Stark og Randi Petersen

Vejleder: Bernhelm Booss-Bavnbek

310/96 Kursusmateriale til "Lineære strukturer fra
algebra og analyse" (E1)

af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH
ENERGY SYSTEM

by: Hélène Connor-Lajambe, Bernd Kuemmel,
Stefan Krüger Nielsen, Bent Sørensen

312/96 Grassmannian and Chiral Anomaly

by: B. Booss-Bavnbek, K.P.Wojciechowski

313/96 THE IRREDUCIBILITY OF CHANCE AND
THE OPENNESS OF THE FUTURE
The Logical Function of Idealism in Peirce's
Philosophy of Nature

By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian
Cardiovascular System

By: Johnny T. Ottesen

315/96 "Rejsen til tidens indre" - Udarbejdelse af
a + b et manuskript til en fjernsynsudsendelse
+ manuskript

af: Gunhild Hune og Karina Goyle

Vejledere: Peder Voetmann Christiansen og
Bruno Ingemann

316/96 Plasmaoscillation i natriumklynger

Specialerapport af: Peter Meibom, Mikko Østergård

Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer

af: Ulla Rasmussen

Vejleder: Anders Madsen

318/96 Modelling the Respiratory System

by: Tine Guldager Christiansen, Claus Dræby

Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming
Impacts

by: Bent Sørensen

320/96 Grassmannian and Boundary Contribution to the
-Determinant

by: K.P.Wojciechowski et al.

321/96 Modelkompetencer - udvikling og afprøvning
af et begrebsapparat

Specialerapport af: Nina Skov Hansen,
Christine Iversen, Kristin Troels-Smith

Vejleder: Morten Blomhøj

322/96 OPGAVESAMLING
Bredde-Kursus i Fysik 1976 - 1996

323/96 Structure and Dynamics of Symmetric Diblock
Copolymers
PhD Thesis
by: Christine Maria Papadakis

324/96 Non-linearity of Baroreceptor Nerves

by: Johnny T. Ottesen

325/96 Retorik eller realitet ?
Anvendelser af matematik i det danske
Gymnasiums matematikundervisning i
perioden 1903 - 88

Specialerapport af Helle Pilemann

Vejleder: Mogens Niss

326/96 Bevisteori
Eksemplificeret ved Gentzens bevis for
konsistensen af teorien om de naturlige tal

af: Gitte Andersen, Lise Mariane Jeppesen,
Klaus Frovin Jørgensen, Ivar Peter Zeck

Vejledere: Bernhelm Booss-Bavnbek og
Stig Andur Pedersen

327/96 NON-LINEAR MODELLING OF INTEGRATED ENERGY
SUPPLY AND DEMAND MATCHING SYSTEMS

by: Bent Sørensen

328/96 Calculating Fuel Transport Emissions

by: Bernd Kuemmel

329/96 The dynamics of cocirculating influenza strains conferring partial cross-immunity

and

A model of influenza A drift evolution

by: Viggo Andreasen, Juan Lin and Simon Levin

330/96 LONG-TERM INTEGRATION OF PHOTOVOLTAICS INTO THE GLOBAL ENERGY SYSTEM

by: Bent Sørensen

331/96 Viskøse fingre

Specialerapport af:

Vibeke Orlien og Christina Specht

Vejledere: Jacob M. Jacobsen og Jesper Larsen

332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG

Specialerapport af:

Stine Sofia Korremann

Vejleder: Dorthe Posselt

333/97 Biodiversity Matters

an extension of methods found in the literature on monetisation of biodiversity

by: Bernd Kuemmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Bernd Kuemmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids

by: Jeppe C. Dyre

336/97 PROBLEM-ORIENTATED GROUP PROJECT WORK AT ROSKILDE UNIVERSITY

by: Kathrine Legge

337/97 Verdensbankens globale befolkningsprognose

- et projekt om matematisk modellering

af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen

Vejleder: Jørgen Larsen

338/97 Kvantisering af nanolederes elektriske ledningsevne

Første modul fysikprojekt

af: Søren Dam, Esben Danielsen, Martin Niss, Esben Friis Pedersen, Frederik Resen Steenstrup

Vejleder: Tage Christensen

339/97 Defining Discipline

by: Wolfgang Coy