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ABSTRACT. We discuss the Grassmannian of self-adjoint global elliptic boundary conditions with γ_5- and gauge-invariance of the domain for the Dirac operator over the 4-ball coupled to a gauge configuration with non-trivial curvature form arising from local chiral anomaly. We show that this space contains a variety of boundary conditions in addition to the spectral Atiyah-Patodi-Singer projection and that some of them, like the Calderón projector, imply global (strong) chiral symmetry.

We undertake to illuminate questions put to us by the physicists Gianni Morchio and Franco Strocchi (Pisa) and Andrzej Trautman (Warsaw and Trieste) and propose some possible answers to their questions.

GRASSMANNIAN AND CHIRAL ANOMALY *

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Abstract. We discuss the Grassmannian of self-adjoint global elliptic boundary conditions with γ_5- and gauge-invariance of the domain for the Dirac operator over the 4-ball coupled to a gauge configuration with non-trivial curvature form arising from local chiral anomaly. We show that this space contains a variety of boundary conditions besides the spectral Atiyah-Patodi-Singer projection and that some of them, like the Calderón projector, imply global (strong) chiral symmetry.

Key words: Atiyah-Patodi-Singer index theorem, Calderón projector, chiral symmetry, Dirac operator, elliptic boundary problems, γ_5-symmetry, gauge-invariance, pure gauge, quantum chromodynamics, pseudo-differential Grassmannian.

Introduction

It is implicit in recent literature on quantum field theory that γ_5- and gauge-invariance of a self-adjoint elliptic boundary condition P for the (massless) Dirac operator $\not{D}_A$ over the 4-ball coupled to a non-trivial gauge configuration A which is pure gauge at the boundary lead to *global* chiral anomaly, i.e. the integers n_+ and n_- of the zero frequency modes of positive and negative chirality can not be expected to be identical and that their difference will depend on the winding number $\deg A$ associated with the non-negligible external field of the gauge configuration and arising from *local* chiral symmetry breaking. The reason for this belief may be an undue prominence of the spectral Atiyah-Patodi-Singer (APS) boundary condition $\Pi_{\geq}$ for the corresponding partial Dirac operator $\not{D}_A^+$. As observed by M. Ninomiya and C.I. Tan in [20], imposing APS boundary conditions does lead to global (strong) chiral asymmetry. But the APS is far from being the only reasonable elliptic, self-adjoint, γ_5-invariant, and gauge-invariant boundary condition.

In [17], G. Morchio and F. Strocchi noticed the role played by boundary terms for the discussion of spontaneous symmetry breaking. They saw no reason to question the *local* chiral anomaly, i.e. the existence of the correction term which invalidates the continuity equation for the chiral current (reconstructed through the Matthews-

* Working paper, *January 19, 1996*, expanding discussions with Gianni Morchio and Franco Strocchi (Pisa), and Andrzej Trautman (Warsaw and Trieste)

Salam formula for the fermion correlation functions and mathematically expressed by the curvature of the underlying connection). Instead, they addressed the *global* implications of this term, essentially the question whether its volume integral vanishes or not. That must depend on the boundary conditions chosen. In that connection they questioned the belief in the necessity of imposing the APS boundary condition leading to global (strong) chiral anomaly and put forward some arguments supporting the view that

(a) the *ellipticity* and *self-adjointness* of the boundary condition, providing a nicely spaced discrete real spectrum with only accumulation point at $\pm\infty$;
(b) the γ_5-*invariance* of the boundary condition; and
(c) the invariance of the domain under any *globally defined gauge* transformation

might be compatible with

(d) *global (strong) chiral symmetry*, i.e. the vanishing of $n_+ - n_-$.

The compatibility of the four properties would lead to considerable simplifications in the calculations and a new understanding of the physical effects especially connected with the θ term in quantum chromodynamics. As explained in [17], Section 5.5, chiral symmetry would imply that the exponential

$$e^{\theta\int F^*F dx} = e^{i\theta[(n_+-n_-)(A)]} = \lim_{m\to 0} m^{-(n_++n_-)} \det \not{D}_A{}^{m,\theta} =: f_A(\theta)$$

of the effective action would become independent of the fermion angle θ of the fermion mass term, and vice versa. On the mathematical level one could simplify considerably the ζ function regularization of the determinant bundle with

$$\begin{aligned}\zeta'(s) &= - \sum_{\lambda\in\operatorname{spec}\not{D}_A} \ln\lambda e^{s\ln|\lambda|^2}\\ &= -\left[(n_+ + n_-)\ln m + i(n_+ - n_-)\theta_m\right]e^{-s\ln m^2} - \sum_{\lambda\neq 0}\ln|\lambda|^2 e^{-s\ln|\lambda|^2}.\end{aligned}$$

It follows that we must discuss the number of the zero modes. We shall show that the Morchio-Strocchi conjecture is provable and that global (strong) chiral symmetry is indeed compatible with the other requirements. It seems, moreover, that there are no gauge-theoretic arguments for selecting the chiral anomaly generating Atiyah-Patodi-Singer boundary conditions.

In Section 1 we give a review of the theory of elliptic boundary value problems for any total Dirac operator $\mathcal{D}$ and, in case of even-dimensional manifolds, for the half Dirac operator $\mathcal{D}^+$ arising from the chiral decomposition $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$. Our definition of the ellipticity is somewhat new, and we hope that it makes this concept more accessible to non-specialists. The standard definition (see for instance [6], Chapter 18) is clearly a special case of the concept we offer here, but we suspect that actually both notions of the ellipticity of a boundary problem are equivalent. We apply the classical concepts of Cauchy data spaces and of the Calderón projector $\mathcal{P}_+(\mathcal{D}^+)$ in order to investigate the ellipticity condition. Then we review the elliptic boundary problems used by physicists and mathematicians. At the end of the section we discuss the Grassmannian $\mathrm{Gr}(\mathcal{D}^+)$ of all generalized Atiyah-Patodi-Singer conditions, i.e. the space of all pseudo-differential projections with the same principal symbol as the Calderón projector. It provides a natural space of elliptic boundary conditions for the partial Dirac operator $\mathcal{D}^+$.

In Section 2 we present a theory of γ_5-invariant self-adjoint elliptic boundary problems for any total Dirac operator $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$. We show that in a natural way each $P \in \mathrm{Gr}(\mathcal{D}^+)$ defines a projection $P^\# \in \mathrm{Gr}^*_{\gamma_5}(\mathcal{D})$, the Grassmannian of γ_5-invariant self-adjoint elliptic boundary problems for the total Dirac operator.

In Section 3 we discuss the twisted Dirac operator $\not{D}_A$ over a 4-ball V (of large radius) coupled to a vector potential A (a non-trivial gauge configuration which is pure gauge at the boundary). We fix the notation, especially concerning the auxiliary vector bundles and underlying metrics and connections. We exploit the product form $\not{D}_A^+ = N(\partial_r + \not{\partial}_B)$ of the twisted Dirac operator near the boundary and investigate the effect of changes of the connection form A onto the corresponding field B over the boundary entering into the definition of the boundary (*tangential*) Dirac operator $\not{\partial}_B$. We show that not only the spectral Atiyah-Patodi-Singer projection $\Pi_{\geq}(\not{\partial}_B)$ defines a gauge-invariant element of $\mathrm{Gr}^*_{\gamma_5}(\not{D}_A)$, but also the Calderón projector $\mathcal{P}_+(\not{D}_A^+)$.

Assume that the 4-ball V is equipped with the standard metric which makes (according e.g. to J.R. Schmidt and A.M. Bincer [23]) the boundary Dirac operator invertible and its spectrum symmetric. Assume also that the vector potential A is pure gauge at the boundary. Then one obtains the well-known formula

$$\operatorname{index} \not{D}_{A,\Pi_{\geq}} = n_+(\Pi_{\geq}) - n_-(\Pi_{\geq}) = \deg(A) \neq 0 \tag{1}$$

derived by Ninomiya and Tan [20] from the Atiyah-Patodi-Singer index theorem. The proof will be discussed below.

Replacing the spectral projection $\Pi_{\geq}(\not{\partial}_B)$ by the Calderón projector $\mathcal{P}_+(\not{D}_A^+)$ we get, contrary to (1)

$$\operatorname{index} \not{D}_{A,\mathcal{P}_+} = n_+(\mathcal{P}_+) - n_-(\mathcal{P}_+) = 0, \tag{2}$$

since, by definition, $n_+(\mathcal{P}_+)$ and $n_-(\mathcal{P}_+)$ vanish. From (2) we get the main result of this paper, namely that any twisting of the Euclidean Dirac operator over the 4-ball by means of a connection in an auxiliary bundle of coefficients can be 'lifted' to a suitable section in the Grassmannian providing global (strong) chiral symmetry:

Theorem 0.1 *Let* $\mathrm{Conn}_0(V \times \mathbf{C}^2)$ *denote the affine space of smooth connections in the coefficients' bundle* $V \times \mathbf{C}^2$ *over the 4-ball* V *which are pure gauge at the boundary. Then there exists a smooth map*

$$\mathcal{R} : \mathrm{Conn}_0(V \times \mathbf{C}^2) \ni A \mapsto \mathcal{R}(A) \in \mathrm{Gr}(\not{D}_A^+)$$

which satisfies (a)-(d).

From a mathematical point of view, the preceding theorem provides the best solution available for the compatibility problem of global (strong) chiral symmetry with local symmetry breaking when the section $\mathcal{R}$ of the Grassmannian is based on the Calderón projector, since it provides

$$n_{\pm}(\mathcal{R}(A)) = 0 \tag{3}$$

for any connection A.

In Section 4 we discuss various alternative self-adjoint, elliptic, γ_5- and gauge-invariant boundary problems for the twisted Dirac operator $\not{D}_A$ with vanishing $n_+ - n_-$ to give a clear and complete picture of the variety of possibilities of obtaining compatibility of global (strong) chiral symmetry with local chiral asymmetry.

In Section 5 we make some remarks on the construction of the Calderón projector and its relation to the spectral projection (Atiyah-Patodi-Singer boundary condition).

In the Appendix we determine, based on a result by L. Nicolaescu, the adiabatic limit of the Cauchy data spaces of the twisted Dirac operator for the radius of the 4-ball $R \to \infty$ in terms of the eigenfunctions of the corresponding tangential Dirac operator over the 3-sphere.

1. A Brief Review of the Theory of Elliptic Boundary Problems for Dirac Operators

Let M be a compact smooth oriented Riemannian manifold with boundary Y, and let $S \to M$ be a bundle of Clifford modules with compatible Hermitian structure and connection (covariant derivative) D. The *(total) Dirac operator*

$$\mathcal{D} : C^{\infty}(M; S) \longrightarrow C^{\infty}(M; S).$$

is obtained by suitably composing the connection $D : C^{\infty}(M; S) \to C^{\infty}(M; T^*M \otimes S)$ with the Clifford multiplication $\mathbf{c} : C^{\infty}(M; TM \otimes S) \to C^{\infty}(M; S)$.

Clearly $\mathcal{D}$ is an elliptic differential operator. It is formally self-adjoint and *Green's formula* holds for all spinors s and s':

$$(\mathcal{D}s, s') - (s, \mathcal{D}s') = -\int_Y \langle N(y)(s|_Y), s'|_Y \rangle, \tag{4}$$

where $N := \mathbf{c}(\mathbf{n}) : S|_Y \to S|_Y$ denotes the unitary bundle isomorphism given by Clifford multiplication by the inward unit tangent vector.

To proceed further we assume that M is an even-dimensional manifold. Let γ_5 denote the global section of $\mathrm{Hom}(S, S)$ defined locally by

$$\gamma_5 := \mathbf{c}(e_1) \ldots \mathbf{c}(e_k),$$

where $\{e_\mu\}$ is any positively oriented orthonormal local frame of tangent vectors and k denotes the dimension of the manifold M. Since k is even, S splits into subbundles $S^{\pm}$ spanned by the eigensections of γ_5 corresponding to the eigenvalue ± 1, if k is divisible by 4, or $\pm i$ otherwise; the Clifford multiplication N switches between $S^{\pm}|_Y$ and $S^{\mp}|_Y$; and the Dirac operator splits correspondingly into components

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$$

such that the *right partial (half) Dirac operator* $\mathcal{D}^+ : C^\infty(M; S^+) \to C^\infty(M; S^-)$ is formally adjoint to the left partial operator $\mathcal{D}^- : C^\infty(M; S^-) \to C^\infty(M; S^+)$.

To simplify the exposition we assume that the Riemannian metric and the Hermitian structure are product near the boundary. Let us point out that the results presented here remain true also for non-product structures. Admitting non-product structures, however, makes the analysis more complicated especially when one wants to discuss the ζ regularized determinant and related asymptotic expansions.

Close to the boundary, the total Dirac operator splits into the following *product form*

$$\mathcal{D} = \Gamma(\partial_r + \mathcal{B}) = \begin{pmatrix} 0 & -N^{-1} \\ N & 0 \end{pmatrix} \left(\partial_r + \begin{pmatrix} \not{\partial} & 0 \\ 0 & -N\not{\partial}N \end{pmatrix} \right), \tag{5}$$

where r denotes the inward oriented normal (radial) coordinate in a collar neighbourhood of the boundary and $\not{\partial} : C^\infty(Y; S^+) \to C^\infty(Y; S^+)$ denotes the *(tangential) Dirac operator* over the boundary. Notice that in fact

$$\Gamma^2 = -1 \qquad \text{and} \qquad \Gamma\mathcal{B} = -\mathcal{B}\Gamma$$

as required by the formal self-adjointness of $\mathcal{D}$.

We also discuss the operator $\mathcal{D}^+$ alone. It has the following form on the collar:

$$\mathcal{D}^+ = N\,(\partial_r + \not{\partial}) \tag{6}$$

The Dirac operator on an odd-dimensional manifold has the same form $\Gamma(\partial_r + \mathcal{B})$ on the collar. In this case the total operator $\mathcal{D}$ does not split, but the tangential operator is a Dirac operator on an even-dimensional manifold and has therefore the following form:

$$\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}^- \\ \mathcal{B}^+ & 0 \end{pmatrix}$$

Now we want to discuss the properties of a Dirac operator over a compact manifold with boundary. In the beginning we shall not distinguish between the cases of even- and odd-dimensional manifolds and whether we treat the total or the half Dirac operator. So, let $\mathcal{A} \in \{\mathcal{D}, \mathcal{D}^+\}$ with $\mathcal{A} : C^\infty(M; E) \to C^\infty(M; F)$, $E, F \in \{S, S^\pm\}$, and product form $\mathcal{A} = \Gamma(\partial_r + B)$ near the boundary Y.

Contrary to the case of a closed manifold, the space

$$\mathcal{H}(\mathcal{A}, \infty) = \{ s \in C^\infty(M; E) \mid \mathcal{A}s = 0 \}$$

of solutions of the operator $\mathcal{A}$ is an infinite-dimensional subspace of $C^\infty(M; E)$. There is also a question of regularity of the solutions. Let s denote a solution of $\mathcal{A}$, which is an element of the space $L^2(M; E)$ (or more generally of $H^k(M; E)$ the k-th Sobolev space). In general, it does not follow that s is a smooth section of S. This leads us to the following general definition of the ellipticity.

Definition 1.1 Let $\mathcal{A}$ be a Dirac operator on M and let $\mathcal{A}_\mathcal{R}$ be a closed extension of $\mathcal{A}$ in $L^2(M, S)$ with domain $\mathcal{R}$. We call $\mathcal{A}_\mathcal{R}$ an *elliptic boundary problem* for the operator $\mathcal{A}$, if and only if the following two conditions are satisfied:

(I) The extension $\mathcal{A}_{\mathcal{R}} : \mathcal{R} \to L^2$ of $\mathcal{A}$ is a Fredholm operator.
(II) The spaces $\ker \mathcal{A}_{\mathcal{R}}$ and $\operatorname{coker} \mathcal{A}_{\mathcal{R}}$ *are* (respectively in the case of the cokernel: *can be represented as*) finite-dimensional subspaces of the spaces of smooth sections.

Remark 1.2 Let $\mathcal{H}(\mathcal{A})$ denote the space of all L^2 solutions of the operator $\mathcal{A}$. Then we may reformulate condition (II) in the definition as follows:

$$\mathcal{R} \cap \mathcal{H}(\mathcal{A}) \subset C^\infty \quad \text{and} \quad \mathcal{R}^* \cap \mathcal{H}(\mathcal{A}^*) \subset C^\infty ,$$

where $\mathcal{R}^*$ denotes the domain of the adjoint operator.

It seems at first sight that the boundary Y does not appear in the definition, but usually the domain $\mathcal{R}$ is defined by a condition posed on the spinors on the boundary: Let γ_0 denote the restriction map $\gamma_0(s)(y) := s(0, y)$. It gives a continuous map $\gamma_0 : H^1(M; E) \to L^2(Y; E|_Y)$. The condition which determines $\mathcal{R}$ is usually given in the form:

$$\mathcal{R} = \{s \in H^1(M; E) \mid T(\gamma_0(s)) = 0\} ,$$

where $T : L^2(Y; E|_Y) \to L^2(Y; G)$ is a 0-th order pseudo-differential operator. Of course T has to satisfy certain additional assumptions to guarantee fulfilment of conditions (I) and (II) from the definition. We have to introduce the Calderón projection in order to explain those conditions.

The *Calderón projector* $\mathcal{P}_+(\mathcal{A})$ is defined as the (without loss of generality orthogonal) projection of $L^2(Y; E)$ onto the *Cauchy data space*, also called *Hardy space* in Clifford analysis:

$$\mathcal{H}_+(\mathcal{A}) := \overline{\{s|_Y \mid s \in C^\infty(M; E) \text{ and } \mathcal{A}s = 0 \text{ in } M \setminus Y\}}^{L^2(Y; E|_Y)} .$$

We discuss the construction of the Calderón projector in the last section of this paper. Let us only point out that the principal symbol $p_+(y; \zeta)$ of $\mathcal{P}_+(\mathcal{A})$ is equal to the orthogonal projection onto the direct sum of the eigenspaces of the automorphism $b(y; \zeta)$ corresponding to the positive eigenvalues. Here b denotes the principal symbol of the tangential operator $\not{\mathcal{B}}$. Now we are ready to formulate the conditions which the operator T has to satisfy:

Definition 1.3 Let $T : C^\infty(Y; E|_Y) \to C^\infty(Y; E)$ be a pseudo-differential operator of order 0. We call T an *elliptic boundary condition* for the operator $\mathcal{A}$, if the following conditions are satisfied:

(I') For any real r, the extension $T^r : H^r(Y; E|_Y) \to H^r(Y; E)$ of T has a closed range.
(II') Let $\sigma(T)$ denote the principal symbol of T. Then

$$\operatorname{range}(\sigma(T)) = \operatorname{range}(\sigma(T) \circ p_+).$$

In particular the restriction $\sigma(T)|_{\operatorname{range}(p_+)} : \operatorname{range}(p_+) \to \operatorname{range}(\sigma(T))$ is an isomorphism of vector bundles.

Condition (I') implies that $\mathcal{N}_T$, the orthogonal projection onto the kernel of T, is a pseudo-differential operator (see [6], Proposition 18.11). For the ease of notation we shall denote the kernel of T by the same letter $\mathcal{N}_T$. Condition (II') implies that the couple $(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A}))$ is a *Fredholm pair of subspaces*, i.e. a pair of closed subspaces with finite-dimensional intersection and with sum of finite codimension (then the difference of these two dimensions is called the *index* of the pair; see [6]). More precisely we have the following result:

Proposition 1.4 *Let T be as in Definition 1.3. Then the couple $(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A}))$ is a Fredholm pair of subspaces in $L^2(Y; E|_Y)$ with*

$$\operatorname{index}(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A})) = \operatorname{index}\{T \circ \mathcal{P}_+(\mathcal{A}) : \mathcal{H}_+(\mathcal{A}) \to \operatorname{range}(T)\} = \operatorname{index} \mathcal{A}_T\,,$$

where $\mathcal{A}_T$ denotes the realization of the operator $\mathcal{A}$ with the domain

$$\{s \in H^1(M; E) \mid T(\gamma_0(s)) = 0\}.$$

Proof We show that $\operatorname{index}(\mathcal{N}_T, \mathcal{H}_+(\mathcal{A})) = \operatorname{index}\{T\mathcal{P}_+(\mathcal{A}) : \mathcal{H}_+(\mathcal{A}) \to \operatorname{range}(T)\}$ and refer to [6], Theorem 20.8 for the proof of the second equality.

Let us assume that z is an element of $\mathcal{N}_T \cap \mathcal{H}_+(\mathcal{A})$. This implies that

$$Tz = 0 \quad \text{and} \quad \mathcal{P}_+(\mathcal{A})z = z, \tag{7}$$

which shows that z is an element of the kernel of the operator $T\mathcal{P}_+(\mathcal{A})$. Let us also observe that the second equality of (7) shows that there exists a uniquely determined s, such that $\mathcal{A}s = 0$ and $\gamma_0(s) = z$. Therefore we have shown:

$$\ker \mathcal{A}_T \cong \ker T\mathcal{P}_+(\mathcal{A}) = \mathcal{N}_T \cap \mathcal{H}_+(\mathcal{A}).$$

Now let us assume that w is an element of $(\mathcal{N}_T + \mathcal{H}_+(\mathcal{A}))^{\perp}$. It means that w is perpendicular to $\mathcal{H}_+(\mathcal{A})$, hence $\mathcal{P}_+(\mathcal{A})w = 0$ and that w is perpendicular to the kernel of T. Therefore there exists q such that $w = T^*q$, which provides the identification of $\ker \mathcal{P}_+(\mathcal{A})T^*$ with the orthogonal complement of the sum and thus ends the proof of the proposition. □

Now we review some examples of boundary value problems for Dirac operators. We begin with the theoretically most obvious example:

Example 1.1 We put $T := \mathcal{P}_+(\mathcal{A})$, the *Calderón projector* of $\mathcal{A}$. This is an elliptic boundary condition and in the case of the total Dirac operator $\mathcal{A} := \mathcal{D}$ it provides us with a self-adjoint, elliptic boundary problem for the operator $\mathcal{D}$. In the case of $\mathcal{A} = \mathcal{D}^+$ we obtain a closed (unbounded) Fredholm operator $\mathcal{D}^+_{\mathcal{P}_+(\mathcal{D}^+)}$ with index equal to 0.

We have to choose different boundary conditions in order to obtain a non-trivial index.

Example 1.2 We still discuss the operator $\mathcal{D}^+$. Then the tangential Dirac operator $\not{\partial}$ is a self-adjoint elliptic differential operator over the closed manifold Y and it has an orthogonal complete system of eigenspinors providing a spectral decomposition of $L^2(Y; S^+|_Y)$. Let $\Pi_{\geq}(\not{\partial})$ and $\Pi_{<}(\not{\partial})$ denote the orthogonal projections of $L^2(Y; S^+|_Y)$ onto its subspace spanned by the eigenspinors corresponding to the non-negative, respectively negative eigenvalues of $\not\partial$. Those *spectral projections* are pseudo-differential operators and the principal symbol of $\Pi_{\geq}(\not\partial)$ is equal to p_+. Therefore $\Pi_{\geq}(\not\partial)$ is an elliptic boundary condition for the operator $\mathcal{D}^+$. The problem $\mathcal{D}^+{}_{\Pi_{\geq}(\not\partial)}$ was studied by Atiyah, Patodi and Singer in [2], where they gave the famous index formula for the operator $\mathcal{D}^+{}_{\Pi_{\geq}(\not\partial)}$. We will discuss this formula later.

Example 1.3 Let us discuss the Atiyah-Patodi-Singer problem for the total operator $\mathcal{D}$ in the case of an *odd*-dimensional manifold M. In this case their index formula gives

$$\operatorname{index} \mathcal{D}_{\Pi_{\geq}(\not\partial)} = \dim\ker \not\partial^+ ,$$

i.e. for non-vanishing kernel of the tangential operator the Atiyah-Patodi-Singer problem for the total, symmetric Dirac operator is not self-adjoint, and its index is not stable under small deformations.

On the other hand, Green's formula shows that in the case $\ker(\not\partial) = \{0\}$ the operator $\mathcal{D}_{\Pi_{\geq}(\not\partial)}$ is a self-adjoint operator.

Example 1.4 For odd-dimensional M we have two natural *local elliptic boundary conditions* $\pi_\pm$ for the total Dirac operator defined by the chiral projection of $S|_Y$ onto $(S|_Y)^\pm$. We then get $\operatorname{index}\mathcal{D}_{\pi_-} - \operatorname{index}\mathcal{D}_{\pi_+} = \operatorname{index}\not\partial^+$. But $\operatorname{index}\mathcal{D}_{\pi_\pm}$ vanishes by Green's formula so that we get the illustrious cobordism theorem from the preceding equality, namely the vanishing of the index of any (half) Dirac operator over a closed even-dimensional manifold Y, if the operator can be written as the (half) tangential operator of a (total) Dirac operator over an odd-dimensional manifold M with $\partial M = Y$.

Example 1.5 Also on any odd-dimensional manifold M we have the *chiral bag model*: Let S be a bundle of Clifford modules and $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$ a compatible Dirac operator over M. For any natural n and any smooth map $g : Y \to U(n)$ we get a self-adjoint elliptic operator $\mathcal{A}_g$ acting like

$$\mathcal{A}_g := \begin{pmatrix} n\mathcal{D} & 0 \\ 0 & -n\mathcal{D} \end{pmatrix}$$

with

$$\operatorname{dom}\mathcal{A}_g := \left\{ \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in H^1(M; (S\otimes \mathbf{C}^n) \oplus (S \otimes \mathbf{C}^n)) \mid s_2|_Y = (\Gamma \otimes g) s_1|_Y \right\}$$

where $n\mathcal{D} := \mathcal{D} \otimes \mathrm{Id}_{\mathbf{C}^n}$.

Example 1.6 Now we return to the case of even-dimensional M and replace the spectral projection (i.e. the Atiyah-Patodi-Singer boundary condition for the partial Dirac operator $\mathcal{D}^+$) by projections belonging to the *Grassmannian* $\mathrm{Gr}(\mathcal{D}^+)$ of generalized Atiyah-Patodi-Singer boundary conditions for $\mathcal{D}^+$. The space $\mathrm{Gr}(\mathcal{D}^+)$ is defined as the space of pseudo-differential projections with principal symbol equal to the orthogonal projection p_+. Here 'projection' means 'idempotent' (i.e. $P = P^2$). The Grassmannian is locally pathwise connected and has countably many connected components; two projections P_1, P_2 belong to the same component, if and only if the *virtual codimension*

$$\mathrm{i}(P_2, P_1) := \operatorname{index}\{P_2 P_1 : \operatorname{range} P_1 \to \operatorname{range} P_2\} = \operatorname{index}(\mathrm{Id} - P_2, P_1) \qquad (8)$$

of P_1 in P_2 vanishes. Here $\operatorname{index}(\mathrm{Id} - P_2, P_1)$ denotes the index of the Fredholm pair (of ranges of the projections). The higher homotopy groups of each connected component are given by Bott periodicity. According to Proposition 1.4 we have $\operatorname{index}\mathcal{D}_P^+ = \mathrm{i}(P, \mathcal{P}_+(\mathcal{D}^+))$ for all $P \in \mathrm{Gr}(\mathcal{D}^+)$.

Note: Important elements of $\mathrm{Gr}(\mathcal{D}^+)$ are the *weighted spectral projections* $\Pi_{\geq a}(\not{\partial})$ with a cut of the spectrum at an arbitrary real a. More precisely, they are defined as the orthogonal projections onto the direct sum of the eigenspaces of the tangential Dirac operator $\not{\partial}$ for eigenvalues $\geq a$.

Another important example is the *Calderón projector* $\mathcal{P}_+(\mathcal{D}^+)$ discussed in Example 1.1. Note that the Calderón projector is defined in global data whereas the spectral projections are defined by the data of the tangential Dirac operator, i.e. by data that live on the boundary. In any case, they have the same principal symbol p_+, but in general belong to different connected components of the Grassmannian.

Example 1.7 Let us now assume that M is an odd-dimensional manifold. In this case we have also an important *self-adjoint Grassmannian* $\mathrm{Gr}^*(\mathcal{D})$ of self-adjoint boundary conditions of Atiyah-Patodi-Singer type. This is the subspace of $\mathrm{Gr}(\mathcal{D})$, which consists of those (orthogonal) projections P, which satisfy the condition

$$-\Gamma P \Gamma = \mathrm{Id} - P.$$

Any element of $\mathrm{Gr}^*(\mathcal{D})$ defines a self-adjoint elliptic boundary value problem for the operator $\mathcal{D}$.

2. γ_5-Invariant Elliptic Boundary Problems

In this section we discuss the boundary value problems related to the physical situation described in the Introduction. We now assume that the manifold M is even-dimensional. We begin with two typical examples of elliptic boundary problems:

Examples 2.1 *(a)* The tangential Dirac operator $\not{\partial}$ is a self-adjoint elliptic differential operator over the closed manifold Y. As in Section 1, $\Pi_{\geq}(\not{\partial})$ and $\Pi_{<}(\not{\partial})$ denote the orthogonal projections of $L^2(Y; S^+|_Y)$ onto its subspace spanned by the eigenspinors corresponding to the non-negative, respectively negative eigenvalues of $\not{\partial}$. Choosing

$$\Pi := \begin{pmatrix} \Pi_{\geq}(\not{\partial}) & 0 \\ 0 & N\Pi_{<}(\not{\partial})N^{-1} \end{pmatrix} \qquad (9)$$

as boundary condition we obtain an operator $\mathcal{D}_{\Pi}$ which is a self-adjoint Fredholm operator with smooth kernel and cokernel and nicely spaced spectrum, such that the invariants

$$\eta_{\mathcal{D}_\Pi}(0), \quad \zeta_{\mathcal{D}_\Pi}(0), \quad \text{and} \quad \zeta'_{\mathcal{D}_\Pi}(0)$$

are defined in exactly the same way as in the closed case. Here the product structure near the boundary is important. Actually it turns out that the spectrum is symmetric due to γ_5-symmetry, hence the η-invariant vanishes, see Proposition 2.2b below.
(b) If we set (following [14])

$$T := \frac{1}{2}\begin{pmatrix} 1 & N^{-1} \\ N & 1 \end{pmatrix} \tag{10}$$

we obtain a realization $\mathcal{D}_T$ with the same properties as listed for $\mathcal{D}_\Pi$ in (a).

The reason why the listed properties are in line with each other in these two cases is that $P \in \{\Pi, T\}$ satisfy the same two basic conditions (for details see [5] and [6]):
- The first condition is the *ellipticity* (well-posedness) of the projection P defining the boundary condition explained in Definitions 1.1, 1.3 above.
- The second condition is the *self-adjointness* of the L^2-extension. As discussed in Example 1.7 this is a symmetry condition in the normal derivatives, more precisely demanding $\mathrm{Id} - P = -\Gamma P \Gamma$.

One decisive difference between the two boundary conditions Π and T defined in equations (9) and (10) lies in the γ_5-symmetry: In view of the chiral splitting γ_5 takes the form $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for dimension of M divisible by 4 (otherwise we multiply by the imaginary unit i), hence $\gamma_5 \Pi \gamma_5 = \Pi$, i.e. Π *is* γ_5-invariant, but $\gamma_5 T \gamma_5 = \mathrm{Id} - T$, i.e. T is *not* γ_5-invariant.

The main result of this section is the following proposition.

Proposition 2.2 (a) *The mapping*

$$\mathrm{Gr}(\mathcal{D}^+) \ni P \mapsto P^\# := \begin{pmatrix} P & 0 \\ 0 & N(\mathrm{Id} - P)N^{-1} \end{pmatrix} \tag{11}$$

provides us with the γ_5*-Grassmannian* $\mathrm{Gr}^*_{\gamma_5}(\mathcal{D})$ *of self-adjoint elliptic (well-posed) boundary conditions for the total Dirac operator* $\mathcal{D}$ *which are all* γ_5*-invariant. In particular we have a natural identification*

$$\pi_0(\mathrm{Gr}^*_{\gamma_5}(\mathcal{D})) \simeq \pi_0(\mathrm{Gr}(\mathcal{D}^+)) \simeq \mathbf{Z}. \tag{12}$$

(b) *For all* $P^\# \in \mathrm{Gr}^*_{\gamma_5}(\mathcal{D})$ *the* L^2 *realization* $\mathcal{D}_{P^\#}$ *has a discrete real spectrum. Each eigenvalue is of finite multiplicity and there are no finite accumulation points. The spectrum is symmetric around the origin of the real axis (hence there is no* η *function).*
(c) *The null space*

$$\ker \mathcal{D}_{P^\#} := \{s \in H^1(M; S) \mid \mathcal{D}(s) = 0 \text{ and } P^\#(s|_Y) = 0\}$$

consists solely of smooth spinors. It is finite-dimensional and splits naturally into the direct sum of a space of spinors of positive chirality of dimension $n_+(P^\#)$ and a space of spinors of negative chirality of dimension $n_-(P^\#)$ with

$$\text{index}\, \mathcal{D}_P^+ = n_+(P^\#) - n_-(P^\#)\,, \tag{13}$$

where $P^\#$ and P are related through (11).

Proof (a) and the main proposition of (b) follow at once from the general theory of global elliptic boundary problems for the Dirac operator. To prove that the spectrum of $\mathcal{D}_{P\#}$ is $\lambda \mapsto -\lambda$ symmetric we consider an eigenspinor $s = \begin{pmatrix} s_+ \\ s_- \end{pmatrix} \in \operatorname{dom} \mathcal{D}_{P\#}$ with $\mathcal{D}s = \lambda s$. Because of the anti-diagonal form of $\mathcal{D}$ this means

$$\mathcal{D}^- s_- = \lambda s_+ \text{ and } \mathcal{D}^+ s_+ = \lambda s_-\,.$$

Then $\begin{pmatrix} s_+ \\ -s_- \end{pmatrix}$ also belongs to $\operatorname{dom} \mathcal{D}_{P\#}$ and is an eigenspinor of $\mathcal{D}_{P\#}$ with eigenvalue $-\lambda$, since

$$\begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \begin{pmatrix} s_+ \\ -s_- \end{pmatrix} = \begin{pmatrix} -\mathcal{D}^- s_- \\ \mathcal{D}^+ s_+ \end{pmatrix} = \begin{pmatrix} -\lambda s_+ \\ \lambda s_- \end{pmatrix} = -\lambda \begin{pmatrix} s_+ \\ -s_- \end{pmatrix}$$

and, trivially,

$$N(\mathrm{Id} - P)N^{-1}(s_-|_Y) = 0 \Longrightarrow N(\mathrm{Id} - P)N^{-1}(-s_-|_Y) = 0\,.$$

To see (c), we notice that by definition

$$\ker \mathcal{D}_{P\#} = \ker \mathcal{D}_P^+ \oplus \ker \mathcal{D}^-_{N(\mathrm{Id}-P)N^{-1}}$$

with $\dim \ker \mathcal{D}_P^+ = n_+(P^\#)$ and $\dim \ker \mathcal{D}^-_{N(\mathrm{Id}-P)N^{-1}} = n_-(P^\#)$. Since $(\mathcal{D}_P^+)^* = \mathcal{D}^-_{N(\mathrm{Id}-P)N^{-1}}$, it follows that index $\mathcal{D}_P^+ = n_+(P^\#) - n_-(P^\#)$. □

We close this section with a discussion of the global (strong) chiral anomaly $n_+(P^\#) - n_-(P^\#)$ for the γ_5-invariant boundary conditions induced by

- the Calderón projector $\mathcal{P}_+(\mathcal{D}^+)$;
- the spectral projection $\Pi_{\geq}(\not{\partial})$; and
- the weighted spectral projections $\Pi_{\geq a}(\not{\partial})$ for any real a.

For (c) of the preceding proposition it suffices to determine the index of the corresponding problems for the partial (half) Dirac operator.

Examples 2.3 *(a)* As noticed before, from the definition of the Calderón projector it is immediate that $\ker \mathcal{D}^+_{\mathcal{P}_+(\mathcal{D}^+)} = 0$. From Green's formula we get that

$$\mathcal{P}_+(\mathcal{D}^-) = N(\mathrm{Id} - \mathcal{P}_+(\mathcal{D}^+))N^{-1}\,,$$

hence

$$\ker \mathcal{D}^-_{N(\mathrm{Id}-\mathcal{P}_+(\mathcal{D}^+))N^{-1}} = \ker \mathcal{D}^-_{\mathcal{P}_+(\mathcal{D}^-)} = 0,$$

hence the index of the elliptic boundary value problem $\mathcal{D}^+_{\mathcal{P}_+(\mathcal{D}^+)}$ vanishes: There is no global (strong) chiral anomaly when imposing the Calderón projector as boundary condition.

(b) Choosing the spectral projection $\Pi_{\geq}(\not\partial)$ as boundary condition we have the Atiyah-Patodi-Singer index theorem which gives

$$\operatorname{index} \mathcal{D}^+_{\Pi_{\geq}(\not\partial)} = \int_M \alpha(x) - \frac{1}{2}(\eta_{\not\partial}(0) + \dim\ker \not\partial)\,. \tag{14}$$

Here $\alpha(x)$ denotes the locally defined *index density* of $\mathcal{D}^+$ which expresses the local chiral anomaly, and

$$\eta_{\not\partial}(z) := \sum_{\lambda\in\operatorname{spec}\not\partial\setminus\{0\}} \operatorname{sign}\lambda|\lambda|^{-z} = \frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty t^{\frac{z-1}{2}}\operatorname{tr}(\not\partial e^{-t\not\partial^2})\,dt \tag{15}$$

denotes the *η-function* of the tangential Dirac operator $\not\partial$. It is

(i) well defined through absolute convergence for $\Re(z)$ large;

(ii) it extends to a meromorphic function in the complex plane with isolated simple poles;

(iii) its residues are given by a local formula; and

(iv) it has a finite value at $z = 0$,

(see e.g. Gilkey [10]).

One can not expect global (strong) chiral symmetry for the Atiyah-Patodi-Singer boundary problem; in general, none of the expressions in formula (14) will vanish. For sufficiently elementary operators and under additional assumptions the error terms $\eta_{\not\partial}(0)$ and $\dim\ker\not\partial$ will vanish (especially for symmetric spectrum and invertible tangential operator), and fairly easy expressions for $\int_M \alpha(x)$ are obtainable (see below Section 3).

(c) For any real a we consider the weighted spectral projection $\Pi_{\geq a}(\not\partial)$. From the Agranovič-Dynin theorem (see [6], p.207) we get

$$\operatorname{index}\mathcal{D}^+_{\Pi_{\geq a}(\not\partial)} = \operatorname{index}\mathcal{D}^+_{\Pi_{\geq}(\not\partial)} + \mathrm{i}(\Pi_{\geq a}(\not\partial), \Pi_{\geq}(\not\partial)) \tag{16}$$

with the virtual codimension defined in (8) as error term. For $a \geq 0$ the virtual codimension of $\Pi_{\geq}(\not\partial) = \Pi_{\geq 0}(\not\partial)$ in $\Pi_{\geq a}(\not\partial)$ becomes $\sum_{0\leq\lambda<a}\dim E_\lambda$ and for $a < 0$ it becomes $-\sum_{a\leq\lambda<0}\dim E_\lambda$ where E_λ denotes the eigenspace of the tangential operator $\not\partial$ corresponding to λ. Hence, for suitable choice of a we can obtain global (strong) chiral symmetry

$$\operatorname{index} \mathcal{D}^+_{\Pi_{\geq a}(\not\partial)} = 0,$$

even if the index of the Atiyah-Patodi-Singer problem does not vanish. If its index is $\nu \neq 0$, say $\nu > 0$, the spectral cut a must be chosen in such a way that $\sum_{0\leq\lambda<a} E_\lambda = \nu$.

To exploit the rich structure of the γ_5-Grassmannian and to investigate the passing from one connected component (*sector*) to another under change of boundary conditions we shall now apply the preceding theory to a specific 4-dimensional problem of gauge theoretic physics, the problem of global (strong) chiral symmetry in the presence of local (weak) chiral symmetry breaking for gauge-invariant boundary conditions.

3. Twisted Dirac Operators over the 4-Ball

Now we address the physics situation. As manifold M we take a 'volume' V in $\mathbf{R}^4$. We think of V as a ball of large radius R. Actually, we are interested in the asymptotic situation with $R \to \infty$. As the bundle of Clifford modules we take

$$V \times (S \otimes \mathbf{C}^2) = S \otimes \mathbf{C}^2\,, \tag{17}$$

the Clifford bundle of *Euclidean spinors with coefficients in the trivial bundle* $V \times \mathbf{C}^2$ with Clifford action $\mathrm{c}(a) \otimes 1$. As the full Dirac operator $\mathcal{D}$ we take a twisted Dirac operator defined by a connection A for $V \times \mathbf{C}^2$ which is *pure gauge* on the boundary of V.

To make all definitions precise, to fix the notation, and to check the parity and the signs we recall that the *(free) Euclidean Dirac operator*

$$\not{D} = \begin{pmatrix} 0 & -\frac{\partial}{\partial q} \\ \frac{\partial}{\partial \overline{q}} & 0 \end{pmatrix} : C^\infty(V;S) \longrightarrow C^\infty(V;S)$$

is canonically defined over $\mathbf{R}^4$ with

$$\frac{\partial}{\partial \overline{q}} = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\,, \tag{18}$$

and

$$\frac{\partial}{\partial q} = -i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\,, \tag{19}$$

where the bundle S of Euclidean spinors in 4 dimensions splits into a pair of quaternions $S = V \times (\mathbf{H} \oplus \mathbf{H})$ with Clifford multiplication $\mathrm{c} : C\ell_4 \to \mathrm{Hom}_{\mathbf{C}}(S,S)$ given by the four complex 4×4 matrices

$$\mathrm{c}(e_\mu) = \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} \quad \text{for } \mu = 1,2,3 \qquad \text{and} \qquad \mathrm{c}(e_4) = \gamma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with $\{\sigma_\mu\}$ denoting the Pauli matrices and $\{e_1, \ldots e_4\}$ a basis of $\mathbf{R}^4$. The connection defining the Euclidean Dirac operator is just the standard connection d for S.

Then any connection A for the trivial bundle $V \times \mathbf{C}^2$ defines in a natural way a twisted Dirac operator $\not{D}_A = \not{D} \otimes_A \mathrm{Id}_{\mathbf{C}^2}$. It is characterized by the property

$$\not{D}_A(s \otimes f)(x) = (\not{D}(s) \otimes f)(x)$$

whenever $(Af)(x) = 0$. It is a true (total) Dirac operator with regard to the induced Clifford multiplication $\mathrm{c}(a) \otimes 1$ and the induced connection $d \otimes A$.

Now we must discuss the choice of the connection A. From a physical point of view it does not suffice to consider only the trivial choice, namely the standard connection d in $V \times \mathbf{C}^2$ given by exterior differentiation $\sum_\mu \alpha_\mu e_\mu \mapsto \sum_\mu \alpha_\mu \partial_\mu$. Roughly speaking, the standard connection would correspond to the description of two non-interacting fermions. When these fermions are not considered independently, their interaction on the surface ∂V of the volume V can be described in a canonical

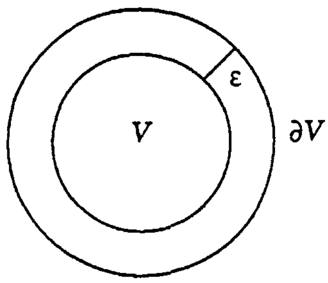

Fig. 1. The collar $[0,\varepsilon)\times\partial V$

way, but expressed in different ways, although it is very much the same thing over the contractible n-ball V (for details see e.g. B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov [7]): One way is to introduce a smooth family h of SU(2) matrices parametrized over ∂V; this is equivalent to introducing a smooth connection ∇ over the whole ball which is pure gauge at the boundary with regard to h; or, equivalently, to introduce a vector-valued field $\{A_\mu\}$ which is pure gauge at the boundary with regard to h providing $\nabla_{e_\mu} f = \frac{\partial f}{\partial x_\mu} + A_\mu f - f A_\mu$ for any $f \in C^\infty(V; V\times \mathbf{C}^2)$ and e_μ the μth basis vector in $\mathbf{R}^4$.

The geometrical idea behind demanding the interaction boundary term to behave as pure gauge is to get a non-trivial curvature form $\Omega_\nabla = \sum_{\mu<\nu} F_{\mu\nu} dx_\mu \wedge dx_\nu$ corresponding to an action $\int F_{\mu\nu}^2\, dx < \infty$ with

$$\begin{aligned} F_{\mu\nu} := [\nabla_{e_\mu}, \nabla_{e_\nu}] &= \nabla_{e_\mu}\nabla_{e_\nu} - \nabla_{e_\nu}\nabla_{e_\mu} \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in L^2 \end{aligned}$$

and $F_{\mu\nu}(x) \to 0$ for $|x|\to\infty$. This is equivalent to writing the connection $\nabla = d + \sum_\mu A_\mu$ with $A_\mu(x) \to h\partial_\mu h^{-1}$ for $|x|\to\infty$.

The different mathematical descriptions are formalized in the following definition.

Definition 3.1 Let $\mathrm{Conn}(V\times \mathbf{C}^2)$ denote the affine space of smooth connections for the bundle $V\times\mathbf{C}^2$. A connection $A\in \mathrm{Conn}(V\times\mathbf{C}^2)$ is called *pure gauge* at the boundary $\partial V = S^3$, if in a collar $[0,\varepsilon)\times\partial V$ (see Figure 3):

- A does not depend on the normal (radial) coordinate r; and
- there exists a smooth $h : \partial V \sim S^3 \to \mathrm{SU}(2) \sim S^3$ such that $A = h\circ d\circ h^{-1}$ in the collar, where d denotes the standard connection given by exterior differentiation.

We write $\mathrm{Conn}_0(V\times\mathbf{C}^2)$ for the subspace of connections which are pure gauge at the boundary.

Proposition 3.2 (a) *A connection A for $V\times\mathbf{C}^2$ is pure gauge at the boundary if it can be written in the form $A = d - (dh)h^{-1}$ in a collar of the boundary.*
(b) *If A is pure gauge at the boundary, then the tangential Dirac operator B over ∂V corresponding to the partial Dirac operator $\not{D}_A^+$ over V takes the form*

$$B = \not{\partial}_{S^3} \otimes_{-(dh)h^{-1}} \mathrm{Id} = (\mathrm{Id}\otimes h)\,(\not{\partial}\otimes \mathrm{Id}_{\mathbf{C}^2})\,(\mathrm{Id}\otimes h^{-1}) \tag{20}$$

with $\not{\partial}\otimes\mathrm{Id}_{\mathbf{C}^2} = \not{\partial}\oplus\not{\partial}$. Here $\not{\partial} = \not{\partial}_{S^3}$ denotes the tangential operator over S^3 corresponding to the Euclidean Dirac operator $\not{D}^+$.

Proof To prove (a) we find

$$Af = (hdh^{-1})f = hd(h^{-1}f) = h(h^{-1}df + d(h^{-1})f) = df - (dh)h^{-1}f \tag{21}$$

since $d(hh^{-1})$ vanishes.
To prove (b) we notice that the restriction of A to the boundary takes the form $-(dh)h^{-1}$, therefore we get such a simple form for lifting $\partial\!\!\!/$ to the auxiliary bundle. For details of the calculation see e.g. [21] and [30]. □

We shall discuss the Atiyah-Patodi-Singer index formula for the operator $D\!\!\!\!/_A$

$$\text{index } (D\!\!\!\!/^+_A)_{\Pi_\geq} = \int \alpha(x) - \frac{1}{2}\left(\eta_B(0) + \dim\ker B\right).$$

From the preceding proposition we have

$$\eta_B(0) \;=\; 2\,\eta_{\partial\!\!\!/_{S^3}}(0) \qquad \text{and} \qquad \dim\ker B = 2\dim\ker \partial\!\!\!/_{S^3}.$$

We find $\deg(h)$ for the value of the integral of the index density. This result is actually independent of the choice of the metric. Then

$$\text{index } (D\!\!\!\!/^+_A)_{\Pi_\geq} = \deg(h) - \eta_{\partial\!\!\!/_{S^3}}(0) - \dim\ker \partial\!\!\!/_{S^3}. \tag{22}$$

The two numbers on the right were found to vanish for the standard metric of $\mathbf{R}^4$, slightly modified close to ∂V in a calculation done by J.R. Schmidt and A.M. Bincer in [23], see also Schmidt [22]. In that metric the tangential Dirac operator on the 3-sphere $\partial\!\!\!/_{S^3}$ has a spectrum symmetric about $\lambda = 0$ and is invertible. More precisely:

Lemma 3.3 [J. R. Schmidt and A. M. Bincer, 1987] *The tangential Dirac operator $\partial\!\!\!/$ over ∂V corresponding to the (free) Euclidean Dirac operator over a ball $V \subset \mathbf{R}^4$ of radius R in a spherical metric has eigenvalues λ with multiplicity $M(\lambda)$ expressed as*

$$\lambda R = \pm(\frac{1}{2} + \kappa), \quad M(\lambda) = \kappa(\kappa+1), \quad \kappa = 1, 2, \dots \,. \tag{23}$$

Proof To explain the metric chosen in [23] we must repeat parts of the proof. First rewrite the operators $D\!\!\!\!/^+ = \frac{\partial}{\partial \bar q}$ and $D\!\!\!\!/^- = -\frac{\partial}{\partial q}$ defined above in (18) and (19) as 2×2 matrices

$$\frac{\partial}{\partial \bar q} = \begin{pmatrix} \partial_4 + i\partial_3 & \partial_2 + i\partial_1 \\ -\partial_2 + i\partial_1 & \partial_4 - i\partial_3 \end{pmatrix}$$

and

$$\frac{\partial}{\partial q} = \begin{pmatrix} \partial_4 - i\partial_3 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & \partial_4 + i\partial_3 \end{pmatrix}.$$

Then parametrize V by

$$x_1 = r\sin\theta\sin\varphi_1, \qquad x_2 = r\sin\theta\cos\varphi_1\,,$$
$$x_3 = r\cos\theta\sin\varphi_2, \qquad x_4 = r\cos\theta\cos\varphi_2\,,$$
$$0 \le \varphi_{1,2} \le 2\pi, \quad 0 \le \theta \le \tfrac{\pi}{2}, \quad 0 \le r \le R$$

and notice

$$\begin{aligned}
\partial_4 \pm i\partial_3 &= e^{\pm\varphi_2}\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta \pm \frac{i}{r\cos\theta}\partial_{\varphi_2}\right), \\
\partial_2 \pm i\partial_1 &= e^{\pm\varphi_1}\left(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta \pm \frac{i}{r\sin\theta}\partial_{\varphi_1}\right).
\end{aligned}$$

To cast

$$\not{D} = \begin{pmatrix} 0 & -\frac{\partial}{\partial q} \\ \frac{\partial}{\partial \bar{q}} & 0 \end{pmatrix}$$

into the required product form

$$\not{D} = \Gamma(\partial_r + \mathcal{B})$$

close to the boundary, the partial Dirac operators $\frac{\partial}{\partial \bar{q}}$ and $\frac{\partial}{\partial q}$ are replaced by

$$\widetilde{\not{D}^+} := Q\frac{\partial}{\partial \bar{q}}R^{-1}, \qquad \widetilde{\not{D}^-} = -R\frac{\partial}{\partial q}Q^{-1},$$

with

$$\begin{aligned}
R &:= (r^3\sin\theta\cos\theta)^{\frac{1}{2}}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{-i\varphi_-} & 0 \\ 0 & e^{i\varphi_-} \end{pmatrix}, \\
Q &:= (r^3\sin\theta\cos\theta)^{\frac{1}{2}}\begin{pmatrix} e^{-i\varphi_+} & 0 \\ 0 & e^{i\varphi_+} \end{pmatrix}, \\
\varphi_\pm &:= \frac{1}{2}(\varphi_1 \pm \varphi_2).
\end{aligned}$$

This transformation is equivalent to a slight modification of the metric of V near the boundary. One obtains

$$\widetilde{\not{D}^+} = \partial_r + \not{\partial}, \qquad \widetilde{\not{D}^-} = -\partial_r + \not{\partial},$$

with

$$\not{\partial} = \frac{1}{r}\begin{pmatrix} \frac{1}{2} + i\partial_{\varphi_1} + i\partial_{\varphi_2} & \partial_\theta + i\cot\theta\partial_{\varphi_1} - i\tan\theta\partial_{\varphi_2} \\ -\partial_\theta + i\cot\theta\partial_{\varphi_1} - i\tan\theta\partial_{\varphi_2} & \frac{1}{2} - i\partial_{\varphi_1} - i\partial_{\varphi_2} \end{pmatrix}. \tag{24}$$

Setting $r := R$ in (24) we get an explicit form of the tangential Dirac operator. For details of the eigenvalue determination of $\not{\partial}$ we refer to [23], 3996-3997. □

Remarks 3.4 *(a)* We also refer to T. Kori [15] which ensures the same result, namely a symmetric spectrum, not containing zero, for a particular metric set-up coming from a metric over the 4-ball which is product near the boundary. Actually, for Kori's metric the Calderón projector $\mathcal{P}_+$ and the Atiyah-Patodi-Singer projection $\Pi_\geq$ coincide. Clearly the Calderón projector and the Atiyah-Patodi-Singer projection coincide for the standard metric in two dimensions, but not in four dimensions; see also Section 5 and the Appendix below.
(b) Also from Hitchin [12] it follows directly that the tangential Dirac operator over the 3-sphere in standard metric is non-singular with symmetric spectrum. Explicit calculations of its eigenfunctions were given by Sudbery [27] and Sommen [26].

Recall that $\mathrm{Conn}_0(V \times \mathbf{C}^2)$ denotes the subspace of connections which are pure gauge at the boundary.

Proposition 3.5 [M. Ninomiya and C.I. Tan, 1985] *For a suitable metric we have*

$$\text{index } (\not{D}_A^+)_{\Pi_{\geq}} = \deg(h) \tag{25}$$

for any $A \in \mathrm{Conn}_0(V \times \mathbf{C}^2)$ *with* $A = h \circ d \circ h^{-1}$ *in a collar of* ∂V *and suitable smooth* $h : \partial V \to \mathrm{SU}(2)$.

Proof The proposition follows at once from the Atiyah-Patodi-Singer Index Theorem (14) and Lemma 3.3. □

In the same article [20] Ninomiya and Tan pointed out that the Atiyah-Patodi-Singer boundary condition is *natural* or *physical* in the following sense:

Corollary 3.6 *Let*

$$\begin{aligned} \mathrm{Conn}_0(V \times \mathbf{C}^2) \ni A &\mapsto \Pi_h := \Pi_{\geq}(\not{\partial} \otimes_h \mathrm{Id}_{\mathbf{C}^2}) \in \mathrm{Gr}(\not{D}^+ \otimes_A \mathrm{Id}_{\mathbf{C}^2}) \\ &\mapsto \Pi(h) := (\Pi_h)^{\#} \in \mathrm{Gr}^*_{\gamma 5}(\not{D} \otimes_A \mathrm{Id}_{\mathbf{C}^2}) \end{aligned}$$

denote the mapping provided by the Atiyah-Patodi-Singer boundary condition

$$\Pi(h) := \begin{pmatrix} \Pi_{\geq}(\not{\partial} \otimes_h \mathrm{Id}_{\mathbf{C}^2}) & 0 \\ 0 & N\Pi_{<}(\not{\partial} \otimes_h \mathrm{Id}_{\mathbf{C}^2})N^{-1} \end{pmatrix}, \tag{26}$$

where $h : \partial V \to \mathrm{SU}(2)$ *denotes the mapping corresponding to the connection* A *which is supposed to be pure gauge at the boundary. Then the family of operators* $\{\not{D}_{A,\Pi(h)}\}_{A \in \mathrm{Conn}_0(V \times \mathbf{C}^2)}$ *which act like* $\not{D} \otimes_A \mathrm{Id}_{\mathbf{C}^2}$ *with*

$$\mathrm{dom}\, \not{D}_{A,\Pi(h)}$$

$$:= \left\{ s \in H^1(V; S \otimes \mathbf{C}^2) \mid \Pi(h)(s) = \begin{pmatrix} \Pi_h & 0 \\ 0 & N(\mathrm{Id} - \Pi_h)N^{-1} \end{pmatrix} \begin{pmatrix} s_+ \\ s_- \end{pmatrix} = 0 \right\}$$

satisfies the following three fundamental conditions:
(a) $\not{D}_{A,\Pi(h)}$ *is self-adjoint;*
(b) $\Pi(h)$ *is* γ_5*-invariant; and*
(c) the domain $\mathrm{dom}\, \not{D}_{A,\Pi(h)}$ *is gauge-invariant.*

We recall the meaning of (c): Let $U : V \to \mathrm{SU}(2)$ define the *gauge transformations*

$$\begin{aligned} f(x) &\mapsto U(x)f(x)U^{-1}(x), \\ A_\mu(x) &\mapsto U(x)A_\mu(x)U^{-1}(x) - \frac{\partial U}{\partial x_\mu}|_x U^{-1}(x), \end{aligned}$$

then the connection A transforms as follows:

$$\begin{aligned} A_{e_\mu} f|_x &\mapsto U(x)(A_{e_\mu} f|_x)U^{-1}(x), \\ A &\mapsto UAU^{-1}, \text{ and} \\ \Omega_A &\mapsto U\Omega_A U^{-1} - (dU)U^{-1}. \end{aligned}$$

This motivates the following definition:

Definition 3.7 A smooth family

$$\mathrm{Conn}(V \times \mathbf{C}^2) \ni A \quad \mapsto \quad P(A) \in \mathrm{Gr}^*_{\gamma_5}(\not{D}_A) \cong \mathrm{Gr}(\not{D}_A^+)$$

is *gauge-invariant*, if we have

$$P(A_1) = U^{\#} P(A)(U^{\#})^{-1} \tag{27}$$

for all $A, A' \in \mathrm{Conn}(V \times \mathbf{C}^2)$ where $A_1 := UAU^{-1}$ with arbitrary $U : V \to \mathrm{SU}(2)$ and $U^{\#} := \mathrm{Id} \otimes_{\mathbf{C}^2} U$.

Remarks 3.8 *(1)* Clearly, for any $A, A_1 \in \mathrm{Conn}(V \times \mathbf{C}^2)$ with $A_1 = UAU^{-1}$ we have pointwise (see Palais [21])

$$\not{D}_{A_1} = \not{D} \otimes_{A_1} \mathrm{Id} = U^{\#}(\not{D} \otimes_A \mathrm{Id})(U^{\#})^{-1} = U^{\#} \not{D}_A (U^{\#})^{-1} .$$

Therefore property (c) - gauge-invariance as defined in (27) - means

$$\not{D}_{A_1, P(A_1)} = U^{\#} \not{D}_{A,P(A)} (U^{\#})^{-1}$$

and, especially,

$$\mathrm{dom}\,(\not{D}_{A_1,P(A_1)}) = U^{\#}(\mathrm{dom}\,\not{D}_{A,P(A)})\,; \tag{28}$$

i.e. we require that the boundary condition transforms in a correct way under variation of the background operator resp. of the connection.
(2) The change of the number

$$\xi(\not{B}_A; s) := \frac{1}{2}\left(\eta_{\not{B}_A}(s) + \dim\ker \not{B}_A\right)$$

for arbitrary change of the connection A, here considered solely as a connection over the closed manifold ∂V, was addressed already in [2] and further investigated in [30]. It turns out that

$$\xi(\not{B}_{A_1}; 0) - \xi(\not{B}_A; 0) \equiv \mathrm{index}\,\{\not{B}_{A_t}\}_{t\in[0,1]} \quad \mathrm{mod}\ \mathbf{Z}.$$

Here $t \mapsto A_t$ is a smooth family of connections connecting

$$A = A_0 \quad \text{with} \quad A_1 = UAU^{-1}$$

and $\{\not{B}_{A_t}\}_{t\in[0,1]}$ denotes the corresponding family of Dirac operators parametrized over S^1 or, put differently, the elliptic differential operator of first order over the torus $S^1 \times \partial V$ defined by that family.

Proof of the Corollary

The first two properties are obvious from the choices. The gauge invariance follows from the corresponding transformation law for the tangential operator

$$\not{B} \otimes_{h_1} \mathrm{Id}_{\mathbf{C}^2} = U^{\#}(\not{B} \otimes_h \mathrm{Id}_{\mathbf{C}^2})(U^{\#})^{-1}, \tag{29}$$

where the smooth families $h, h_1 := U|_{\partial V} h (U|_{\partial V})^{-1} : \partial V \to \mathrm{SU}(2)$ correspond to the connections A, A_1 which are supposed to be pure gauge at the boundary. Equation (29) implies that the eigenvalues do not change under gauge transformation and that the eigenspaces transform like $E_\lambda(\not{B}_{h_1}) = UE_\lambda(\not{B}_h)$. Hence $\Pi(h)$ satisfies (a), (b), and (c). □

Note: 1. One should keep in mind that from a geometrical point of view there is no particular reason to choose the Atiyah-Patodi-Singer boundary conditions among all the other boundary problems which, as we shall see, equally satisfy (a)-(c); in other words, it is hard to see what kind of special gauge invariance of the domain should be required to select that particular boundary condition.
2. Moreover, for our purpose it is an unfortunate aspect of the prominent Atiyah-Patodi-Singer boundary condition that its index

$$\operatorname{index} \not{D}^{+}_{A,\Pi_h} = n_+(\Pi_h) - n_-(\Pi_h)$$

does not vanish as seen above in Proposition 3.5 stating index $\not{D}^{+}_{\Pi(h)} = \deg(h)$ under suitable conditions about the metric. But there are other quite natural boundary conditions $\mathcal{R}_A$ instead of Π_A which fulfil the conditions (a)-(c) of the Corollary and additionally provide global (strong) chiral symmetry, namely

(d) the vanishing of index $\not{D}^{+}_{A,\mathcal{R}_A} = n_+(\mathcal{R}_A) - n_-(\mathcal{R}_A)$.

As announced in the Introduction:

Theorem 3.9 *There exists a smooth map*

$$\mathcal{R} : \mathrm{Conn}_0(V \times \mathbf{C}^2) \ni A \mapsto \mathcal{R}_A \in \mathrm{Gr}(\not{D}^{+}_{A})$$

which satisfies (a)-(d).

Proof It is immediate that

$$A \mapsto \mathcal{R}_A := \mathcal{P}_+(\not{D}^{+} \otimes_A \mathrm{Id}_{\mathbf{C}^2})$$

satisfies all conditions, where $\mathcal{P}_+(\not{D}^{+} \otimes_A \mathrm{Id}_{\mathbf{C}^2})$ denotes the Calderón projector of the partial Dirac operator $\not{D}^{+} \otimes_A \mathrm{Id}_{\mathbf{C}^2}$. □

Note: 1. A nice feature of the Calderón projector is that it is by definition invariant under parity, i.e.

$$N \circ \left(\mathrm{Id} - \mathcal{P}_+(\not{D}^{+} \otimes_A \mathrm{Id}_{\mathbf{C}^2})\right) \circ N^{-1} = \mathcal{P}_+(\not{D}^{-} \otimes_A \mathrm{Id}_{\mathbf{C}^2}),$$

whereas the Atiyah-Patodi-Singer boundary condition is invariant under parity only if the tangential operator is invertible.
2. One must expect the existence of a lot of sections of the Grassmannian which lead to families satisfying (a)-(d) (see also Section 4). From a theoretical point of view, the Calderón family is the best solution available for the global (strong) chiral symmetry problems, since it does not only give global (strong) chiral symmetry but also the following proposition. It simplifies radically e.g. the calculation of $\zeta'(s)$ mentioned in the Introduction.

Proposition 3.10 *For the Calderón family the dimensions of the zero frequency modes of positive and negative chirality vanish; i.e. we have*

$$n_\pm(\mathcal{P}_+(\not{D}^{+} \otimes_A \mathrm{Id}_{\mathbf{C}^2})) = 0.$$

4. Alternatives and Further Ramifications

There are various ways of obtaining global (strong) chiral symmetry by imposing elliptic, self-adjoint, γ_5-symmetric, and gauge-invariant boundary conditions in the presence of local (weak) chiral anomaly (i.e. non-vanishing $\deg h$ for connections which are pure gauge at the boundary). One way is the Calderón projector. It removes all solutions such that kernel and cokernel become trivial. This makes many calculations easy. But since the Calderón projector depends on the gauge configuration also inside the region and not only on the boundary, this must have consequences in, for example, the derivation of identities by variation of the gauge field configuration in a subregion.[1]

Instead of removing all solutions by imposing the Calderón projector one can add further solutions to the original Dirac equation with APS boundary condition until one gets global (strong) chiral symmetry. There are three ways to do that.

Let us begin with a given connection A in the auxiliary bundle $V \times \mathbf{C}^2$ which is pure gauge at the boundary. Hence it can be expressed in a collar of the boundary by a mapping $h : S^3 \to \mathrm{SU}(2)$ which has a degree (*topological number*) $\deg h$. If $\deg h = k$ is non-trivial, the dimensions n_+ and n_- of the zero modes (subject to the APS boundary condition Π_h) do not coincide as seen in Proposition 3.5. Then, to get global (strong) chiral symmetry we enlarge the solution spaces until n_+ and n_- become equal. More precisely:

Alternative 4.1 The easiest, but physically hardly very meaningful way of doing the equalization of the solution spaces is to take a second copy of the coefficients bundle $V \times \mathbf{C}^2$ and to choose a connection A' which is pure gauge at the boundary ∂V with a unitary mapping g of opposite degree $-k$.

Then, instead of tensoring the original Euclidean Dirac operator $\mathcal{D}^+$ solely with the h-connection, we do two twistings: first with h, then with g. The resulting twisted Dirac operator

$$\mathcal{D}'^+ := \mathcal{D}^+ \otimes_A \mathrm{Id}_{\mathbf{C}^2} \otimes_{A'} \mathrm{Id}_{\mathbf{C}^2} = \mathcal{D}^+_A \otimes_{A'} \mathrm{Id}_{\mathbf{C}^2}$$

with coefficients in $\mathbf{C}^2 \otimes \mathbf{C}^2 = \mathbf{C}^4$ admits again an APS boundary condition Π' which is gauge invariant such that

$$\begin{aligned} n_+ - n_- = \operatorname{index} \mathcal{D}'^+{}_{\Pi'} &= \deg(h \otimes g) \\ &= \deg hg = \deg h + \deg g = k - k = 0. \end{aligned}$$

To get global (strong) chiral symmetry one can also apply a less trivial mirror process:

Alternative 4.2 Instead of twisting the global Dirac operator $\mathcal{D}^+_A$ over the full 4-ball V it suffices to twist the transversal (tangential) Dirac operator B_h with a connection of opposite degree over the 3-sphere. We get a new operator B'_h. Then we apply the APS spectral projection Π'_h corresponding to the twisted operator B'_h to the original operator $\mathcal{D}^+_A$. It follows that Π'_h is an admissible boundary value problem

[1] These calculations will be presented separately.

for $\mathcal{D}_A^+$. It belongs to the same Grassmannian as the standard APS projection Π_h and all the nice properties (a)-(c) are guaranteed, but Π'_h belongs to a different connected component. In fact, the index jumps by the winding number yielding global (strong) chiral symmetry.

To see this we recall from equation (20) the formula

$$\Pi_h = (\mathrm{Id} \otimes h)(\Pi_{\geq}(\not{\partial}) \otimes \mathrm{Id})(\mathrm{Id} \otimes h^{-1}) \tag{30}$$

where $\Pi_{\geq}(\not{\partial})$ denotes the APS spectral projection belonging to $\mathcal{D}^+$. We recall $\operatorname{index} \mathcal{D}^+_{A,\Pi_h} = n_+(APS) - n_-(APS) = \deg h$. Here one can prove that Π_h belongs to the same Grassmannian as $\Pi_{\geq}(\not{\partial})$ (more precisely as $\Pi_{\geq}(\not{\partial}) \otimes \mathrm{Id}$), but with the virtual codimension $\mathbf{i}(\Pi_h, \Pi_{\geq}(\not{\partial})) = \deg h$.

Then we go one tensoring further, namely from B_h to

$$B'_h := (\mathrm{Id} \otimes g)(B_h \otimes \mathrm{Id})(\mathrm{Id} \otimes g^{-1})$$

with $\deg g = -\deg h$. We generalize formula (30) and get a similar formula expressing Π'_h, the APS projection belonging to the operator B'_h, in terms of Π_h, and a new jump formula for the virtual index:

$$\begin{aligned}
\mathbf{i}(\Pi'_h, \Pi_h) &= \mathbf{i}(\mathrm{Id} \otimes g)(\Pi_h \otimes \mathrm{Id})(\mathrm{Id} \otimes g^{-1}), (\Pi_h \otimes \mathrm{Id})) \\
&= \operatorname{index} \left((\mathrm{Id} - (\Pi_h \otimes \mathrm{Id}) \oplus \mathrm{Id} \otimes g)(\Pi_h \otimes \mathrm{Id})(\mathrm{Id} \otimes g^{-1}) \circ (\Pi_h \otimes \mathrm{Id})\right) \\
&= \deg g,
\end{aligned}$$

hence

$$\begin{aligned}
\operatorname{index} \mathcal{D}_{A,\Pi'_h} &= \mathbf{i}(\Pi_{\geq}(B_h), \mathcal{P}_+(\mathcal{D}_A^+)) + \mathbf{i}(\Pi_{\geq}(B'_h), \Pi(\mathcal{D}_A^+)) \\
&= \deg h + \deg g = 0.
\end{aligned}$$

Note: An attractive feature of Alternative 4.2 is that in fact the (non-free) operator $\mathcal{D}_A$ is not changed; only the boundary condition is changed.

Alternative 4.3 One more alternative is provided by a suitable spectral cut (weighted spectral projection), see above Example 2.3c and, more generally for the problem of uniform choice of the spectral cut, Melrose and Piazza [16].

5. Some Remarks on the Calderón Projector

We begin with the construction of the Calderón projection. Let M be a compact smooth oriented Riemannian manifold with boundary like in Section 1. Let $\widetilde{M} = M \cup_Y (-M)$ denote the double of M and $\widetilde{S}^+ = S^+ \cup_N S^-$ the corresponding spinor bundle over $\widetilde{M}$. We denote by $\widetilde{\mathcal{D}}^+$ the invertible double of the operator $\mathcal{D}^+$ on $\widetilde{M}$. This is an invertible Dirac operator on $\widetilde{M}$ extending $\mathcal{D}^+$. The invertibility means that there exists an elliptic pseudo-differential operator $\mathcal{G}$ of order -1, such that

$$\widetilde{\mathcal{D}}^+\mathcal{G} = \mathrm{Id} \quad \text{and} \quad \mathcal{G}\widetilde{\mathcal{D}}^+ = \mathrm{Id}.$$

For any $f \in C^\infty(Y; S^+|_Y)$ we denote by $\delta \otimes f$ the distribution:

$$\langle \delta \otimes f; s\rangle := \int_Y (f; \gamma_0(s))dy \qquad \text{for } s \in C^\infty(\widetilde{M}; \tilde{S}^+).$$

In fact, the map $f \to \delta \otimes f$ is the adjoint map to the map γ_0. Given $f \in C^\infty(Y; S^+|_Y)$ we denote by $F(f)$ the distribution over $\widetilde{M}$ defined as

$$F(f) := \mathcal{G}(\delta \otimes \Gamma f).$$

Now we can give the formula for the Calderón projection:

$$\mathcal{P}_+(\mathcal{D}^+)f := \lim_{u \to 0} F(f)|_{u \times Y} = \gamma_0 F(f).$$

Though this formula is abstract one can see that basically it depends only on the Green's function of the operator $\tilde{\mathcal{D}}^+$. The definition extends to $f \in L^2(Y; S^+|_Y)$ by continuity defining a pseudo-differential operator and yielding the Cauchy data space $\mathcal{H}_+(\mathcal{D}^+)$ for the range of $\mathcal{P}_+(\mathcal{D}^+)$, and it turns out that $L^2(Y; S^+|_Y)$ splits into the direct sum

$$L^2(Y; S^+|_Y) = \mathcal{H}_+(\mathcal{D}^+) \oplus N(\mathcal{H}_+(\mathcal{D}^-)),$$

where N denotes Clifford multiplication by the inward unit tangent vector (as above), $\mathcal{D}^- = (\mathcal{D}^+)^*$, and $N(\mathcal{H}_+(\mathcal{D}^-)) = \mathcal{H}_-(\tilde{\mathcal{D}})$, the outer or right Cauchy data space of the invertible double $\tilde{\mathcal{D}}^+$.

Next we want to explain the relation of the Calderón projection to the spectral projection of the tangential operator.

Proposition 5.1 *Let $\not{D}^+_A$ denote the Euclidean Dirac operator over the 4-ball twisted by a connection A which is pure gauge at the boundary with* $\deg h$ *different than 0. Then its Calderón projection $\mathcal{P}_+(\not{D}^+_A)$ and the spectral projection $\Pi_{>0}(\not\partial_h)$ of the corresponding tangential ('spherical') Dirac operator $\not\partial_h$ belong to different components of the Grassmannian.*

Proof Let $\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2}$ denote the untwisted operator. The operator $\not{D}^+_A = \not{D}^+ \otimes_A \mathrm{Id}_{\mathbb{C}^2}$ has the same principal symbol, hence $t\not{D}^+_A + (1-t)(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})$ is a path of Dirac operators. It follows from the construction of the Calderón projection that it changes in a continuous way (see Nicolaescu [19] for details). Therefore $\mathcal{P}_+(\not{D}^+_A)$ and $\mathcal{P}_+(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})$ belong to the same connected component of the Grassmannian.

The index of $(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})_{\Pi_{>0}}$ is equal to 0 (the standard connection has degree 0). This index is equal to $\mathbf{i}(\Pi_{>0}(\not\partial \otimes \mathrm{Id}), \mathcal{P}_+(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2}))$, hence those projections are in the same connected component of the Grassmannian. On the other hand:

$$\deg\ h = \mathbf{i}(\Pi_{>0}(\not\partial \otimes_h \mathrm{Id}), \mathcal{P}_+(\not{D}^+_A)) \neq 0,$$

and we see that the spectral projection and the Calderón projection of the twisted operator belong to different connected components of the Grassmannian. □

6. Appendix. Adiabatic Limit of the Cauchy Data Space

We showed that usually the Calderón projection and the Atiyah-Patodi-Singer condition (the spectral projection) belong to different connected components of the Grassmannian. There is, however, a more precise description of the difference between those two boundary conditions. We give a brief review of some results of the beautiful work of Liviu Nicolaescu [18], [19] which are valid in great generality. For explicit calculations characterizing the Cauchy data space $\mathcal{H}_+(\not{D}^+)$ of the Euclidean, untwisted Dirac operator on the 4-ball in terms of the eigenfunctions of the tangential ('spherical') Dirac operator on the 3-sphere see Sudbery, [27], and Sommen, [26].

We have to point out that in his work Nicolaescu considered the case of an odd-dimensional manifold with boundary, but the result holds also in the case of the total Dirac operator $\mathcal{D}$ on an even-dimensional manifold with boundary.

For real, positive R we define the manifold M_R as

$$M_R := ([-R,0] \times Y) \cup M.$$

The operator $\mathcal{D}$ extends to M_R in a natural way. We study the Calderón projection $\mathcal{P}_+^R(\mathcal{D})$ and the Cauchy data space $\mathcal{H}_+^R(\mathcal{D})$ of the operator $\mathcal{D}$ on M_R . We introduce the notion of the resonance set for the operator $\mathcal{D}$ on M_R :

$$N_R(\mathcal{D}) := \{t \mid H_{<t}(\not{\mathcal{B}}) \cap \mathcal{H}_+^R(\mathcal{D}) = \{0\}\}.$$

Here $H_{<t}(\not{\mathcal{B}})$ denotes the subspace of $L^2(M;S)$ spanned by eigensections of $\not{\mathcal{B}}$ corresponding to eigenvalues smaller than t. Nicolaescu proved that there exists a real number $E(\mathcal{D}) := \sup\{N_R(\mathcal{D})\} \leq 0$. One of the main technical results of his work is the following theorem:

Theorem 6.1 *There exists a positive number a and a Lagrangian subspace $\mathcal{L}$ of $H_{[-a,a]}(\not{\mathcal{B}})$ such that:*

$$\lim_{R\to\infty} \mathcal{H}_+^R(\mathcal{D}) = H_{>a}(\not{\mathcal{B}}) \oplus \mathcal{L}. \tag{31}$$

It was pointed out by Nicolaescu that the convergence in the formula (31) is not uniform. There is a more precise result in the socalled non-resonance case.

Definition 6.2 The operator $\mathcal{D}$ is called *non-resonant,* if $E(\mathcal{D}) = \{0\}$.

In this case the convergence is uniform and we have the following result:

Theorem 6.3

$$\lim_{R\to\infty} \mathcal{H}_+^R(\mathcal{D}) = H_{>0}(\not{\mathcal{B}}) \oplus \mathcal{L}^2\,,$$

where $\mathcal{L}^2$ denotes the space of the limiting values of socalled extended L^2-solutions *of the operator $\mathcal{D}$.*

Now we can reprove the result of the previous section:

Corollary 6.4 *The Euclidean Dirac operator $\not{D}_A$ on the 4-ball V coupled to any connection A which is pure gauge at the boundary is a resonant operator.*

Proof Assume that the operator $\not{D}_A$ is non-resonant. We know that in our case the tangential operator is invertible, and therefore

$$\lim_{R\to\infty} \mathcal{H}_+^R(\not{D}_A^+) = H_{>0}(\not{\partial}_h),$$

which means that the Calderón projection $\mathcal{P}(\not{D}_A^+)$ and the spectral projection $\Pi_>(\not{\partial}_h)$ onto $H_{>0}(\not{\partial}_h)$ belong to the same connected component of the Grassmannian and hence by [6], Theorem 20.8 $\operatorname{index}(\not{D}_A^+)_{\Pi_>} = 0$, but we know that it is equal to $\deg\ h \neq 0$. □

7. Addendum. Elementary Proof of the Ninomiya/Tan Result *(sketch, unfinished)*

We offer a second proof of the Ninomiya/Tan result (Proposition 3.5) based on results from [6], where the authors have shown that the index of any elliptic boundary problem for operators of Dirac type is equal to the index of a suitable Toeplitz operator which lives on the boundary (see [6], Theorem 20.1). In fact, instead of applying the Atiyah-Patodi-Singer Index Theorem we apply the general index formula for global elliptic boundary problems of Proposition 1.4 yielding

$$\operatorname{index}\ (\not{D}_A^+)_{\Pi_>} = \mathbf{i}(\Pi_>, \mathcal{P}_+) = \deg(h).$$

To derive the right equality we recall from (30) the explicit description of the Atiyah-Patodi-Singer projection of the tangential Dirac operator $\not{\partial}_h$ for the twisted Dirac operator $\not{D}_A$ coupled to a vector potential $A \in \mathrm{Conn}_0(V \times \mathbf{C}^2)$

$$\Pi_{\geq}(\not{\partial}_h) = (\mathrm{Id} \otimes h)(\Pi_{\geq}(\not{\partial}) \otimes \mathrm{Id})(\mathrm{Id} \otimes h^{-1}). \tag{32}$$

Then the virtual codimension can be written as the index of an elliptic pseudo-differential operator over the closed manifold Y and the Ninomiya/Tan formula follows from the (classical) Atiyah-Singer Index Theorem.

More precisely,

$$\begin{aligned}
\operatorname{index}(\not{D}_A^+)_{\Pi_>(\not{\partial}_h)} &= \mathbf{i}(\Pi_>(\not{\partial}_h), \mathcal{P}_+(\not{D}_A^+)) = \mathbf{i}(\Pi_>(\not{\partial} \otimes_h \mathrm{Id}), \mathcal{P}_+(\not{D} \otimes_A \mathrm{Id}))\\
&= \mathbf{i}(\Pi_>(\not{\partial} \otimes_h \mathrm{Id}), \mathcal{P}_+(\not{D} \otimes_d \mathrm{Id})) = \mathbf{i}(\Pi_>(\not{\partial} \otimes_h \mathrm{Id}), n\mathcal{P}_+(\not{D}))\\
&= \mathbf{i}(\Pi_>(\not{\partial} \otimes_h \mathrm{Id}), \Pi_>(\not{\partial} \otimes_d \mathrm{Id})) + \mathbf{i}(\Pi_>(\not{\partial} \otimes_d \mathrm{Id}), n\mathcal{P}_+(\not{D}))\\
&= \mathbf{i}(\Pi_>(\not{\partial} \otimes_h \mathrm{Id}), \Pi_>(\not{\partial} \otimes_d \mathrm{Id})) + n\mathbf{i}(\Pi_>(\not{\partial}), \mathcal{P}_+(\not{D})).
\end{aligned}$$

Note that $\mathcal{P}_+(\not{D} \otimes_d \mathrm{Id}) = \mathcal{P}_+(\not{D}) \otimes \mathrm{Id}_{\mathbf{C}^n} = n\mathcal{P}_+(\not{D})$ and $n = 2$ in our application. The third equality follows from the continuity of taking the Calderón projector: since $\not{D} \otimes_A \mathrm{Id}$ and $\not{D} \otimes_d \mathrm{Id} = n\not{D}$ are connected by a smooth path, the Calderón projectors $\mathcal{P}_+(\not{D} \otimes_A \mathrm{Id})$ and $\mathcal{P}_+(\not{D} \otimes_d \mathrm{Id})$ belong to the same connected component of the Grassmannian and the virtual codimensions coincide.

Now we determine the last sum in the preceding array of equations. The left term does not provide any problems:

$$\begin{aligned}\mathrm{i}(\Pi_>(\not{\partial}\otimes_h \mathrm{Id}), \Pi_>(\not{\partial}\otimes\mathrm{Id})) &= \mathrm{i}((\mathrm{Id}\otimes h)\Pi_>(\not{\partial}\otimes_d \mathrm{Id})(\mathrm{Id}\otimes h^{-1}), \Pi_>(\not{\partial}\otimes\mathrm{Id}))\\ &= \deg h\end{aligned}$$

by elementary index theory.

The vanishing of $\mathrm{i}(\Pi_>(\not{\partial}), \mathcal{P}_+(\not{D}))$ is trivial when one uses Kori's result (see Remark 3.4a). For general metrics it follows when one shows that $\not{D}$ on the ball is non-resonant (*must be worked out*).

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- Modul 3 fysik projekt -
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af: Bent Sørensen

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Et 1.modul fysikprojekt

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Projektrapport

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Vejledere: Gestur Olafsson og Jesper Larsen

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Specialerapport

af: Linda Szkotak Jensen og Lise Odgaard Gade

Vejledere: Petr Viscor og Niels Boye Olsen

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- LÆREPROCESSER I SKOLEN

af: Lena Lindenskov, Statens Humanistiske Forskningsråd, RUC, IMFUFA

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by: Jeppe C. Dyre

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To Sigurdur Helgason on his sixtyfifth birthday
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