TEKST NR 306 1995

Mogens Niss:

Goals of mathematics teaching

Preprint of a chapter for the forthcoming
International Handbook of Mathematics Education
(Alan J. Bishop, ed.)

TEKSTER fra

IMFUFA **ROSKILDE UNIVERSITETSCENTER**
INSTITUT FOR STUDIET AF MATEMATIK OG FYSIK SAMT DERES
FUNKTIONER I UNDERVISNING, FORSKNING OG ANVENDELSER

IMFUFA, Roskilde Universitetscenter, Postboks 260, Dk-4000 Roskilde, Danmark

Mogens Niss:
Goals of Mathematics Teaching
Preprint of a chapter for the forthcoming *International Handbook of Mathematics Education* (Alan J. Bishop, ed.)

IMFUFA tekst nr. 306/95 32 pages ISSN 0106-6242

Abstract:

The present text will appear as a chapter in the forthcoming *International Handbook of Mathematics Education* (Alan J. Bishop, ed.). This chapter attempts to analyse the justification and the goals of mathematics education from both theoretical and historical perspectives. Methodological issues related to the identification and reconstruction of the justification and the goals of mathematics education are discussed as well.

The chapter begins by setting the stage, also as far as terminology is concerned, and by asking 'what are the issues?', in order to discuss the relevance of studying the justification and goals of mathematics education. Particular emphasis is being placed on the essential distinction between descriptive/analytic and normative issues.

The main part of the chapter consists in a descriptive/analytic, internationally orientated, survey of the development of the goals of mathematics education during the past century or so, as manifested in major contributions to and documents of curriculum change as well as in contributions to the didactics of mathematics. Attempts are being made to relate the development of the goals and justification of mathematics education to the changing roles of mathematics and mathematics education in society.

The chapter concludes by placing the discourse at issue within the broader context of contemporary preoccupations and concerns in the didactics of mathematics.

Goals of Mathematics Teaching

by Mogens Niss,
Roskilde University, Denmark

A discussion of mathematical education, and of ways and means of enhancing its value, must be approached first of all on the basis of a precise and comprehensive formulation of the valid aims and purposes of such education. Only on such basis can we approach intelligently the problems relating to the selection and organization of material, the methods of teaching and the point of view which should govern instruction, and the qualifications and training of the teachers who impart it. Such aims and purposes of the teaching of mathematics, moreover, must be sought in the nature of the subject, the rôle it plays in practical, intellectual, and spiritual life of the world, and in the interests and capacities of the students (The Reorganization of Mathematics in Secondary Education, 1923 (1970), p. 390)

I. Introduction. What are the issues?

Remarks on scope and terminology

In a way, most of the considerations in the present Chapter pertain, I believe, to mathematics education in general, i.e. across the whole range of mathematics education from kindergarten to graduate studies. However, in order to avoid futile discussions about the extent to which the considerations are relevant to the teaching and learning of mathematics for various special categories of pupils and students, perhaps mainly at advanced levels, let me confine this Chapter to dealing with the majority of pupils and students in mainstream primary and secondary education, and in tertiary education not preparing for specifically mathematical professions.

Terminological issues are mostly tedious and boring. Nevertheless, for a scholarly or scientific discourse to be serious, in fact possible, it is essential that at least the key entities and concepts of that discourse are reasonably clear to those involved. This is particularly true with a field like mathematics education in which transparency and clarity are not easily achieved, let alone to be taken as a matter of course. So, please bear with me during the few pages it takes to set the terminological stage.

Before dealing with the goals of mathematics education we need to spend a few remarks on some closely related notions which we shall also be using in this Chapter: 'reason', 'justification', 'argument'. By a (real) **reason** for providing mathematics education to students within some segment of the educational system we understand a driving force, typically of a general nature, which in actual fact has motivated and given rise to the existence (i.e. the origination or the continuation) of mathematics teaching within that segment, as determined by the bodies which make the decisions (including non-decisions) in the system at issue.

Reasons for mathematics education need not be explicit, well defined and articulated, let alone agreed upon and stated in public. More often than not, reasons are implicit, indirect, fuzzy and vague, and form part of a complex conglomerate of other reasons, societal or group interests, cultural and political ideals, and so on. They include also, in fact quite frequently, impersonal societal forces, of which inertial forces are particularly relevant in this context. Only in rare cases, therefore, do we have direct access to reasons for mathematics education. For such access to be possible, it is not sufficient to identify explicitly stated and formulated reasons which seem, at first sight, to express the underlying motivation of the bodies equipped with the power to establish, continue or discontinue mathematics education. The mere fact that some explicitly stated reason for mathematics education can be identified, say in official documents, does not in itself make it a *real* reason. For this to be convincingly established, a thorough analysis of the genesis, nature, rôle and status of the said reason has to be carried out. In other words, it requires in-depth research, interpretation, and analysis to uncover and elucidate the real reasons of the system for establishing or maintaining mathematics education.

It further follows from the definition of 'reason', suggested above, that a reason does not have to be "good", "correct", "well-founded", "convincing", etc., relative to certain standards. What matters is that it is found to have been activitated to support the existence of mathematics education. Moreover, if self-reference and circularity are to be avoided in the argument, only reasons which genuinely refer to matters outside mathematics education itself can be proper reasons. Thus, a statement like the following does not count as a proper reason: 'Our educational system should provide mathematics education to pupils and students because we think it is important to learn mathematics'. It can only become a reason if it is specified, in a non-circular way, *why* it is important to learn mathematics.

Analyses of mathematics education from historical and contemporary perspectives show that in essence there are just a few types of fundamental reasons for mathematics education. They include the following:

* contributing to the *technological and socio-economic development* of society at large, either as such or in competition with other societies/countries;

* contributing to *society's political, ideological and cultural maintenance and development*, again either as such or in competition with other societies/countries;

* providing *individuals with prerequisites which may help them to cope with life* in the various spheres in which they live: education or occupation; private life; social life; life as a citizen.

Here, while the other two types are likely to speak for themselves, it might be worthwhile to spend a few words to explain the second type. During the last couple of centuries, societies have often seen mathematics education as contributing to the very formation of society's political, ideological and cultural 'superstructure'. Thus, in 1915 the German schoolmaster W. von Schmiedeberg (cf. Niss, 1981), won the first prize in a public competition for a treatise in which he described the ways in which mathematics education can contribute to: Education for national defence; education for serious, diligent and conscientious labour; education for working in a community; and education for patriotism. He particularly insisted that mathematics education be conducive to 'absolute devotion to duty', 'subordination of the individual to the organism of work', and 'preparation for adaptation and obedience'. Soedijarto & Khodir explain (1980, p. 27) how in Indonesia in the mid-1970s mathematics education was meant to serve the general purpose of schools, which included 'to develop Indonesians who are healthy, mentally and physically, [...] are creative and responsible, are democratic and tolerant, [...] and love the nation and mankind consistent with the values of the 1945 Constitution'. Also in 1980, Kolyagin et al., in their a summary of the basic working principles for improving mathematics syllabuses in the Soviet Union, state (p. 79) 'The school mathematics course must be directed towards the systematic inculcation in pupils of a Marxist-Leninist world outlook, [...] and towards developing pupils' cognitive independence and reasoning powers.' During the years 1988-1993, the Danish National Research Council for the Humanities financed a fairly massive research initiative under the heading 'Mathematics Education and Democracy'.

If we consider these three categories of reasons as being, so to speak, affirmative, substantive (not identical to substantiated) for the establishing of mathematics education, there are, as hinted at above, reasons which are of a different nature in that they need not be related to anything substantive, i.e. anything that pertains to mathematics education as such. Many of them are to do with tradition. Thus one reason for continuing and maintaining mathematics education might be paraphrased like this: "Since mathematics education has been around for quite a while, it is probably good for something. Besides, they have it in all other countries too. Perhaps it would cause serious damage to our society if we reduced it or removed it from the curriculum". Other reasons are to do with power balance and political navigation, as exemplified in the following para-statements: "We - politicians, administrators, or institution executives - may not be convinced that mathematics education is worth its costs (in terms of economic and human resources), but riot and turmoil will break out among parents, teacher associations, employers, and other sorts of lobbyists if we made drastic changes, so we had better leave it as it is." Or: "In face of a call for reform of our educational (sub)system, we - politicians, administrators, or institution executives - have listened with much attention to a multitude of experts, including representatives of teacher associations, researchers of mathematics education, institutions of further education, etc. This has led us to conclude that mathematics education should now be given a much stronger position in the curriculum for so and so categories of recipients."

Whenever substantive or insubstantive reasons, of whatever nature, are activated in support of the existence of mathematics education we shall speak about an attempt to **justify** mathematics education. Whenever reasons are put forward or invoked by participants in discussions and debates of mathematics education we are dealing with **arguments** for mathematics education.

For a reason to make sense as such, normally it is accompanied by, or presupposes, certain corresponding explicit or implicit claims. Thus, the three fundamental substantive reasons mentioned above presuppose the following corresponding claims, respectively:

* mathematics education *can indeed* contribute to the *technological and socio-economic development* of society at large;

* mathematics education *can indeed* contribute to *society's political, ideological and cultural maintenance and development*;

* mathematics education *can indeed* contribute to providing *individuals with prerequisites which may help them to cope with life* in the various spheres in which they live.

Again, it is not essential whether a given claim is substantiated by a fair degree of evidence, or rather relies on suppositions, beliefs, or simply hopes. The important thing is that if some reason is invoked to justify mathematics education, the party invoking it has to be convinced that mathematics education can actually make a contribution as implied by the reason at issue.

It is now time to address the notion of goal. Goals of mathematics teaching and learning come in at a stage when it is given that mathematics education should exist (for whichever reasons). In what follows I shall be using the word **goal** as a comprehensive ("umbrella") term for a variety of related terms such as 'end', 'purpose', 'aim', 'objective'. These terms are supposed to be listed in increasing order of specificity and closeness. Thus, an 'end' is a final outcome - perhaps of a general, airy nature - which one intends or hopes to achieve but which may well be a point of infinity in time or space. Moreover, it is seldom clear at all how we can tell whether a given 'end' has been achieved or not. As just two examples of (different) ends we could mention 'Mathematics education should enable students to meet society's demand for a competent and flexible work force', and 'Mathematics education should enable students to master their everyday private life'. At the other extreme, 'objective' concerns fairly concrete and well defined "here and now" matters, the achievement (or non-achievement) of which may be determined relatively easily, if not necessarily automatically. For instance, one objective could be that students should be enabled to read, interpret and judge information given in graphs and tables. In summary, a 'goal' may be anything in the spectrum from 'end' to 'objective'.

Although we have attempted, in this Chapter, to make a clear distinction between 'reasons' and 'goals', by reserving the term 'reason' to concern the very existence of mathematics education vis-à-vis given categories of pupils and students and the term 'goal' to indicate the actual pursuits of mathematics teaching once it has become established, it has to be admitted that the demarcation line between the two is not always so easily drawn in practice. This is so because the goals of mathematics education are often closely related to the underlying reasons for providing it. Nevertheless, however, the relationship between reasons and goals are not deterministic, neither from a logico-philosophical nor from a pragmatic viewpoint. A given reason can give rise to, or be compatible with, several different goals. Similarly, a given goal may be pursued for several different reasons. Hence, analytically speaking, 'reasons' and 'goals' constitute relatively independent dimensions of the space we are about to explore. So, even if the title of this Chapter emphasises 'goals' we shall be dealing with 'reasons' as well.

Two categories of issues
Although the issues we shall explore in the present Chapter are multiple and complex, in essence they are of two *different categories.*

The **descriptive/analytic** category, in which the generic question is:

> For what *reasons* does society provide mathematics education to so and so categories of pupils or students? Which goals, i.e. ultimate *ends,* general *purposes,* particular *aims* and *objectives,* are actually being pursued in mathematics teaching and learning in various segments of the education system?

This question is made a little more specific by posing a number of further questions some of which are listed below. The questions posed under the descriptive/analytic category should be perceived as being of a scientific nature, in the classical sense that the fundamental intention - however futile it may be in practice - is to obtain objective, neutral answers, produced by the use of scholarly/scientific methods in which values (ideologies, tastes, positions and attitudes) are not embedded as an integral part. I am not suggesting that values can, or should, be excluded from having an influence, whether consciously or sub-consciously, on the *choice* of issues, problems, and methods etc. What I am trying to say is, simply, that what we want to know here is "what *is*?" and "*why*?" rather than "what *ought to be*?" and "*why*?". The latter questions are of a normative nature and belong to the second category which will be dealt with separately below. Once again, there is no automatic implication from society's reasons to provide mathematics education to some category of recipients, on the one hand, to the goals pursued in that education, on the other hand. However, to the extent society's reasons are of an affirmative, substantive nature, and not just a consequence of, say, the law of inertia, some goals are better in accordance with these reasons than are others.

It should be noted that the attempt to distinguish between a descriptive/analytic and a normative discourse is an endeavour that belongs to the level of theoretical investigation. It is not certain that this distinction can be maintained meaningfully in all situations. Here is a list of examples of questions which serve to further delineate the descriptive/analytic category of issues:

* How do the actual reasons for and the goals of mathematics education, *vary with society* (i.e. with socio-economic, political-ideological, cultural-historical, or geographical characteristics)? And with *time*? More specifically, which are the circumstancs, stimuli, and driving forces which give (or do not give!) rise to change of reasons and goals?

* Which *bodies* or other *instances* in a given society possess key rôles in determining - explicitly or implicitly - the goals of mathematics education? To what extent are the goals which are determined by different instances

mutually compatible? In particular, what parts do legislators, political and administrative authorities, curriculum authorities or agencies, employers, parents, mathematics educators, teachers, text book authors and publishers, testing agencies, and so forth play in this piece? Put more succinctly: *whose goals* of mathematics education and teaching are actually being pursued? And how do these goals relate to pupils' and students' goals of *learning* mathematics? How do changes in the power structure in a given country engender changes in reasons and goals?

* Where are goals of mathematics education *manifested*?

While the questions formulated thus far do, primarily, concern mathematics education as an enterprise of society carried out in and by educational institutions, we may arrive at different descriptive/analytic points by addressing the professionals of mathematics education, i.e. the researchers, developers, and practitioners. Still with the same intention to pose scientific questions and to obtain scientific answers in the classical sense, we may ask:

* Which are the *arguments put forward by mathematics educators*, in different places and at different times, for providing mathematics education to various categories of pupils and students? And what *goals* have been suggested to this end? In other words, we are looking for mathematics educators' answers to the *justification problem* of mathematics education.

* In which fora and in which contexts have mathematics educators presented their considerations of goals, and in what ways have these goals had an *impact* on the frameworks of mathematics teaching and learning?

As is the case with the uncovering of real reasons for mathematics education, it is a problem of principal significance that it is very difficult indeed to identify and locate the real goals of mathematics education in any given society. Firstly, it is often the case that these goals are not made explicit. What we can observe is the presence and the reality of mathematics education in its various formats and shapes, whereas the goals, like the underlying reasons and driving forces, are not directly observable. Of course, there *are* goals, as well as underlying reasons and driving forces. Mathematics teaching has not been provided for centuries to billions of students around the world on purely haphazard grounds.

Secondly, even if official or semi-official goals are in fact established in explicit terms in accessible documents, it is far from certain that these goals are the *real* ones, those that underpin mathematics education and guide teaching and learning. Oftentimes goals are formulated "post festum" in order to embellish the curriculum or the syllabus expositions, or to provide a pursuasive preamble to politicians, administrators, employers, parents, colleagues in other subjects, or to be a *memorandum* to the teachers who are to implement the curriculum. Moreover, it frequently happens in periods of curriculum reform that the explicit goals are changed while the curriculum remains largely unchanged. Or the converse, for that matter, that the curriculum is changed while the official goals remain the same.

Altogether, to identify and unveil official and unofficial goals, as well as the real ones, and to distinguish between them is a terribly complicated task. Perhaps metaphors such as "archaeological excavations" - or "anthropological enquiry", as some prefer -into the origins and foundations of mathematics education may serve to indicate what we are faced with as far as the descriptive/analytic category of issues is concerned. The uncovering of goals (and reasons) can take place by analytic reconstruction only. For such reconstruction projects to make sense, or even to be possible, investigations into the history of mathematics education are essential.

The second category of issues in this Chapter is the **normative** one, in relation to which the generic question is:

> What *reasons* do **we** (and of course it has to have been specified who "we" are) want to put forward for providing mathematics education to so and so categories of pupils and students? And what *goals* should mathematics education strive to achieve? Once again, we have encountered the *justification problem* but this time in a normative setting.

By their very nature, the normative issues are to do with **values** (norms) in a very fundamental way. This does not imply that our discourse cannot be subject to scholarly or scientific treatment, by no means. What it does imply is that "we" have to do our utmost to (i) reveal and explain these values as honestly, explicitly and clearly as we possibly can; (ii) that the values as such have to be made subjects of examination and thus part of the scientific discourse itself; (iii) that an objective and neutral analysis of the logical, philosophical and material inter- and intra-relations between values, goals, and implications should be undertaken. So, while it is true that from a normative perspective "we" aspire to reach "our own" conclusions regarding the justification and goals of mathematics education, it is certainly not true that such conclusions cannot be valid and rely on scientifically sound theoretical and empirical investigations in which emphasis is placed on precision, consistency, correctness and so forth in exactly the same way and to the same extent as is the case with descriptive/analytic investigations.

To give just one example, if we want to obtain valid answers to the justification problem of, say, school mathematics for the general population, we have to build on (or carry out) a descriptive/analytic investigation of the rôle of mathematics in society, the present and future rôle of mathematical competence in vocations and professions, the place of mathematics in people's everyday lives and in general citizenship. The investigation of all this is a completely objective matter, although certainly not an easy task. Values come in when discussing which consequences this should have for education. Should we want mathematics education to contribute to establishing, expanding or strengthening a decentralised democratic rule of society, our answers to the justification problem would be quite different from the answers we were to give should we be proponents of a centralised, hierarchical and auhoritarian society. The relationship between values and implications for mathematics education may be established on a completely solid ground, from a scholarly point of view, and proponents of opposite values may well agree completely on the relationship between values and implications while disagreeing strongly on the values proper.

When discussing the goals of mathematics education from a normative perspective we should avoid falling into the trap of assuming that the educational system could be started to adopt and to follow new goals from scratch. The system is in place and has goals already. Formulating new sensible ones is not too difficult. What is much more difficult is to get the system to adopt and adjust itself to new goals in a controlled and effective way.

Why are these issues relevant?
To begin with, it should be noted that the title question of this section of the present Chapter is in itself of a normative nature, only at a higher (meta) level. It follows that the discussion in this section is normative at a meta level too.

The issues raised in the previous section are, from my perspective, highly *relevant for three different reasons.*

Firstly, for a *scholarly* reason. As mathematics teaching and learning is such a massive and complex phenomen which consumes a lot of material and non-material resources (including time, effort, commitment, and - yes - pain) in society it is as interesting and relevant an object of scientific study as any object of a similar magnitude. Since it is reasonable to assume that the goals of mathematics education are likely to play a fairly important part in the whole enterprise, these goals are worthy of scientific and scholarly investigation. This is a perfectly legitimate reason which places the goals of mathematics education on a par with any research object that has attracted researchers' attention. In other words, from this point of view 'the goals of mathematics education' can be perceived as just another research specialty. However, to me, this is not the most important reason for attributing great importance to the issues put forward here. Instead, I should like to emphasise

the *implications* reason. Let us begin in about the same way as with the previous reason. Since, as has been said, mathematics teaching and learning is such a massive and complex phenomen which consumes a lot of material and non-material resources, and which affects so many people's lives so deeply, for better and for worse, we ought to know what we are doing, why we are doing it, what consequences the things we are doing have, and whether we should do something else, in general or in detail. Identifying, uncovering and discussing goals of mathematics education serves to make the fundamental matters of mathematics open to scrutiny and debate and to keep the agents and forces of the educational system, including mathematics educators, honest. Evidently, this invokes, once again, the justification problem and assigns a crucial rôle to the issue of goals.

The very framework, structure and organisation of mathematics education, and the ways in which mathematics is taught, are all strongly dependent on the justification and the goals of mathematics education, whether implicit or explicit. No course of teaching and learning in mathematics ever takes place without being, in one way or another, a reflection of some goals of mathematics education. Or to be more precise: a reflection of a *set* of goals, namely the goals involved, directly or indirectly, in shaping the entire reality of mathematics education. The goals of the political and administrative system and of the specific institution are reflected in the framework and conditions for mathematics education (number of lessons, number of pupils in the classroom, the pre- and in-service preparation of the teachers, the teaching materials and other resources, to name just a very, very few). The goals of mathematics educators are reflected in the way the curriculum has been designed and organised and the text books written and produced. The parents' goals are reflected in the degree of stimulation and support they give to their children's pursuit of mathematical studies, and - very importantly - the teacher's goals are reflected in the way (s)he arranges and organises her/his teaching, selects and presents material, approaches the individual pupil or student in the classroom and so forth. Most importantly, however, the goals of mathematics teaching exert a great influence on pupils' and students' attitudes and learning activities. To sum up: The goals of mathematics education have crucial implications for its shape and for its outcomes. (Cf. also The Reorganization of Mathematics in Secondary Education, 1923, p. 390)

Finally, for the *discussion preparedness* reason. As mathematics educators we are not accustomed to spend much time reflecting on or discussing the existence and goals of mathematics education. These are taken for granted most of the time, and our attention and commitment are directed towards carrying out mathematics teaching, innovating curricula, improving the conditions of teaching and learning, understanding the processes that take place in the individual student or in the classroom, and so forth and so on. None the less, we ought to be equipped and prepared to engage in discussions with others concerning the reasons for and goals of mathematics education. These 'others' encompass, perhaps above all, our own pupils and students who, in the honour of truth, do not always feel completely enlightened about the reasons and goals of the mathematics education offered to them. But also, we ought to be prepared to discuss, in a non-defensive manner, these issues with people outside the mathematics education community, including politicians, employers, parents, general curriculum authorities, educational institutions, and - last but not least - colleagues in other subjects. Discussion preparedness seems to be of particular relevance in times of curriculum reform when the rôle and place, as well as the interior affairs of mathematics education are on the agenda. In saying this, I am not advocating mathematics educators' discussion preparedness as the only prerequisite for responding adequately, yet passively, to periods of reform generated 'from outside'. Discussion preparedness is no less crucial as a prerequisite for active instigation of reform 'from within' the mathematics education community.

To offer but one example to illustrate the significance of discussions of reasons and goals of mathematics education, let us take 'information technology'. The proliferation of 'information technology' into the institutions of education throughout the world can be expected to have a deep impact on the reasons for and goals of mathematics education. In order for us to understand, and influence, what is happening and how mathematics teaching and learning is being/should be changed, we need to consider reasons and goals from a scholarly perspective, from an implications perspective, and - last but not least - from a discussion preparedness perspective.

Even if we agree that analysing and discussing goals of mathematics education are significant endeavours, which happen to form the focus of this Chapter, it should be stressed that mathematics education is not, of course, simply a matter of implementing a set of goals. There are multitudes of other important aspects of mathematics education which cannot be reduced to the realisation or non-realisation of goals. Elsewhere (cf. Niss, 1994) I have suggested that at least two other major *problématiques* are significant, the *possibility problem* (who can learn what kinds of mathematics under what circumstances and conditions?) and *the implementation problem* (how to design, organise and carry out mathematics teaching in accordance with answers to the justification and the posibility problems?). However, these *problématiques* are not of concern to us in this Chapter.

II. Goals of mathematics education - a descriptive/analytic perspective

A preliminary remark: In an internationally oriented survey such as the present one, which is made at a highly aggregate level, it has not been possible to clearly separate the contributions by official curriculum authorities and agencies from those of more or less influential mathematics educators. That would require detailed accounts and analyses of the highly diverse spectrum of countries considered. Thus throughout this section we shall refrain from attempts to carefully sort out what could be described as official policy, what could be ascribed to the mathematics education community at large, and what lies with the individual mathematics educator.

In a previous section it was claimed that there are, basically, three categories of fundamental reasons for the existence of mathematics education in the educational system at large: the technological and socio-economic development of society; the political, ideological and cultural maintenance and development of society; the provision of individuals with prerequisites which may assist them in coping with private and social life, whether in education, occupation, or as citizens. This is not the place to attempt to document this claim (in fact, by virtue of the very nature of the claim it is indeed a non-trivial task to provide conclusive evidence for it). Instead, we shall briefly review the position and significance of these reasons from a historical perspective and shall henceforth use them as spectacles through which the goals of mathematics education can be viewed and interpreted.

In most countries in the world, it was not until the 19th century that society began to provide education to wider segments of the general, common population. Before that time formal education of any generality, offered in particular institutions such as schools, was reserved for the 'happy few' who belonged to the ruling or rich classes or were recruited to special institutions, e.g. of an administrative, scientific or religious nature. The numbers of those who received formal education, including mathematics education, prior to the 19th century were very small indeed. The rest of the population, the great majority, either did not receive any formal education at all or received it in close connection with the learning of a trade, through apprenticeship or simply through practice. As society was only involved in providing education to a very small minority of the population, the reasons for so doing are perhaps of limited interest to our investigation. However, as far as mathematics education is concerned it seems plausible that the primary concern was to contribute to the political, ideological and cultural maintenance of society, whereas contribution to the technological and socio-economic development of society was, at first, of secondary importance, although it gradually gained momentum from, say, the 15th century and onwards. This is hardly surprising simply because, before the Modern Era, there were only pretty few links between mathematics and technical, industrial and economic activity (namely restricted to contexts such as mensuration, commerce and finance, currency, weight, calendars, surveying, and navigation). The idea of providing larger numbers of individuals with prerequisites which may assist them in coping with private and social life is of much more modern origin. It is closely related to the increasing importance and power gained by the merchant, financial or industrial commoner, and to the genesis and upsurge of democratic movements at the late 18th and the early 19th centuries.

The 19th century saw the establishment and dissemination of public education (in Denmark compulsory primary education was installed in 1814), also in mathematics, in a great many countries. Essentially, what we are talking about here is primary education. Secondary, not to speak of tertiary, education remained a privilege for the 'happy few', although their numbers began to increase slowly. Primary education encompassed elementary mathematics (i.e. arithmetic with applications, basic descriptive geometry with an emphasis on aspects of mensuration). Presumably, the predominant reason for including this sort of mathematics education in the primary curriculum was to contribute to the technological and socio-economic development of society, while at the same time placing some emphasis on equipping individuals with tools for the mastering of their vocational and everyday private lives, and more so in countries in which democratic movements gained impetus. In contradistinction, it does not seem to be the case that primary mathematical education was often supposed to contribute much to society's political, ideological and cultural maintenance and development.

The opposite is true for mathematics education at secondary and tertiary level. In the 19th century, still, the connections between post-elementary mathematics and technological and socio-economic matters were few and limited and specialised in scope and range - again, with a considerable growth of such connections with time. In fact, though most members of the little minority who did obtain secondary or tertiary education were exposed to post-elementary mathematics teaching, only a small proportion of them were ever to activate mathematical substance of a higher level to matters extra-mathematical. The reason why mathematics was part of their curriculum was partly in order to contribute to society's cultural and ideological maintenance and development, partly to prepare for liberal or scientific professions, or for careers in public administration, the clergy etc. Both reasons were founded on two assumptions, (i) that mathematical training may serve to further the development of general mental discipline and capacity, in particular as regards reasoning (training of the 'thinking muscle', cf. Brooks, 1883 (1970), or the 'thinking power gymnastics' of J.F. Herbart (cf. Jahnke, 1990, p 126)), (ii) that mathematics as a pure and applied science represents a unique and superior exemplary accomplishment of humankind with which educated people ought to be acquainted.

Along with the increasing significance of the application of mathematics to other subjects and practice areas during the 20th century, and along with the concurrent and connected long-term trend of 'mathematics education for all', all the three fundamental reasons, presented in the previous section, seem to have been active in this century, albeit with different emphases in different places at different times (cf. also Niss, 1981). On the one hand, the utility-oriented reasons (technological and socio-economic development of society, and tools for individuals' mastering of everyday life) have been predominant throughout the century in most countries, but especially in the first decade of this century and again in 1930s, in the 1970s, 1980s and the early 1990s. On the other hand, from time to time also the cultural maintenance and development of society have received more weight, relatively, especially in times of economic and cultural optimism and progress (the early 1920s, the late 1950s, the 1960s).

In this attempt to describe the general global pattern, it should be kept in mind that this is not the whole story, because specific regional, national, ideological and political features interfere with the general pattern to produce strong 'local' variations.

First of all, the situation in the developing countries is not really covered adequately by the picture painted above. During the colonial era, the reasons and goals of the colonial powers were simply imposed on the colonies, with hardly any attention being paid to the particular needs of and in those countries. In post-colonial times, it often continued to be the case that the goals, framework and organisation of mathematics education were borrowed from developed countries, often, perhaps, on the basis of a belief in the automaticity of the implication 'education produces progress of development'. Unfortunately, the international literature on mathematics education has not told us much about the situation in developing countries. Still, diversity is a major characteristic of the developing parts of the world as it is of the developed parts. Yet, it does not seem unfair to claim that the fundamental patterns outlined above prevail in the developing countries, too. However, the problems of these countries in providing justification and goals of mathematics education to serve *their* needs, and in providing means for the successful pursuit of those goals, remain deep and immense.

Furthermore, societies in which traditions of democratic rule and control are old and strong and not completely subordinated to the forces of the free market economy (e.g. in the Scandinavian countries and the Netherlands) tend to place relatively more emphasis on providing individuals with prerequisites for competent, active, concerned, and critical citizenship. In contrast, societies with manifest authoritarian traditions (readers are invited to think of their own favourite examples) tend to disregard or discard - or even actively combat - the empowerment of the population at large for democratic citizenship, also as far as mathematical competence is concerned. And here it does not seem to matter whether the authority is simply constituted by a specific ruling class or supported by an economic, political or religious ideology in power. In such countries, it is often the case that the primary reason, the expectation that mathematics education could and should contribute to the technological and socio-economic development of society, is accompanied by the further demand that mathematics education should contribute to its political and ideological consolidation and maintenance as well. Finally, historical tradition also has a strong say in the distribution of relative weight to the three fundamental reasons. Thus, in some countries (like France, Germany, Hungary, the former Soviet Union, China, to name just a few examples) the cultural and ideological maintenance of society is, historically, a particularly important reason for providing substantial mathematics education to large segments of the population.

After having outlined - from a considerable 'cruising altitude', I have to admit - the fundamental reasons which seem to have determined societies' decision to give the majority some degree of mathematical education, we shall now deal with goals of mathematics teaching a little more closely. For simplicity, we shall structure the discussion according to educational levels: primary and lower secondary, and upper secondary and (parts of) tertiary.

Primary and lower secondary mathematics teaching

As was suggested in the previous section, since the early 19th century elementary mathematics education has been given to children for two reasons.

First of all, the technological and socio-economic development of society requires a competent work force, in all sectors, consisting of individuals who possess knowledge, skills, flexibility, and attitudes so as to allow them to obtain, manage and develop jobs in the present and in the future. In addition to a number of job specific qualifications, this requires a fair amount of general qualifications, and the more so the more change-over, innovation, reorganisation etc. the work force is likely to be faced with. Various kinds of mathematics (typically arithmetic with applications, but also aspects of elementary applied geometry, and mensuration) are included in the general qualifications, and often in the specific qualifications as well. In times prior to the advent and proliferation of electronic calculators and computers, it was not unusual that primary and lower secondary mathematics teaching was to contribute to the education of 'human calculators' for business and commerce. Against this background, it is not only in the interest of society, but also in the objective interest of individuals who want attractive jobs, that the education system provides a general preparation for various sorts of occupation. Moreover, as everyone in day-to-day private and social life outside education and occupation will frequently encounter phenomenae, problems and challenges with an applied mathematical component - especially in the economic sphere (like shopping, taxes, mortgages), cooking, design, do-it-yourself work, recreational activity, and so forth - this is also a strong reason to equip pupils in primary education with mathematical prerequisites which can help them to cope with everyday life. As mentioned in the previous section, in many democratic countries, today, it is further intended to empower pupils to enter society as competent, independent, active and critical individuals and citizens.

To what extent, and how, is this profile of reasons reflected in the goals of primary mathematics teaching in different periods?

Let us have a look at a few historical examples. In an address delivered to the American Institute of Boston in 1830, Warren Colburn (Colburn, 1830/1912 (1970)), a mathematics educator and text book author, stated the following purpose of the teaching of arithmetic:

> With regard to its application, there are very few persons, either male or female, arrived at years of discretion, who have not occasions daily to make use of arithmetic in some form or other in the ordinary routine of business. And the person most ready in calculation is much the most likely to succeed in business of any kind. [...]
>
> Arithmetic, when properly taught, is acknowledged by all to be very important as a discipline of the mind; so much so that even if it had no practical application which would render it valuable on its own account, it would still be worthwhile to bestow a considerable portion of time on it for this purpose alone. This is a very important consideration, though a secondary one compared with its practical utility.

This is just one of several examples which demonstrate that in the 19th century elementary mathematics, including arithmetic, was taught not only with the purpose of being useful in practical domains but also of fostering mental discipline (cf. Stanic, 1987). Thus, mental discipline was not attached solely to higher mathematics, like for instance Euclidean geometry. It could be developed by elementary arithmetic as well. We have another example of this in the official goals of primary mathematics teaching issued in 1900 by the Danish Ministry of Education in a general circular (Kirke- og Undervisningsministeriet, 1900) (my translation):

> The teaching of arithmetic should be aimed at developing the minds of children and at accustoming them to exert energy and perseverance in their thinking, while they should acquire, at the same time, the arithmetic skills so valuable in practical life. It has to be emphasised that children should not solely be enabled to deal with numbers as figures but also to acquire real proficiency in mastering numbers as quantities, magnitudes, and ratios. So, mental arithmetic should in principle be given priority over blackboard arithmetic, even if, for practical reasons, the latter has to occupy the majority of the lessons as regards the older children.

Generally speaking, it is a characteristic of 19th century mathematics education that explicit goals were almost absent in written presentations of curricula and syllabi. Reasons and goals are to be found in the theoretical and practical contributions by general educationists and mathematics educators to the shaping of curricula. This is particularly true with countries such as England and the USA in which there never used to be national curricula or central curriculum authorities. It was customary to jump directly from overall justifications of mathematics education - and in many cases these were lacking as well - to lists of specific topics, concepts, facts, results and skills to be included in the mathematical syllabus. If we combine this observation with reports on how teaching was usually conducted those days, and with specimens of written examination papers (see, for instance, Howson, 1982, Appendix), we may conclude that the goals mainly reduced to a limited number of objectives: Pupils should know, normally by rote learning, various mathematical facts, procedures and methods, and possess computational skills in solving standard exercise type problems in pure or applied contexts correctly, fast and accurately. The presence of these abilities was tested in written examination papers (in some countries sometimes also in oral interviews) in which understanding, reasoning, explanation, formulation, exploration, creativity, critical judgement and so forth were of secondary or no importance compared with the obtaining of sufficiently many correct 'bottom line answers'. Thus, even if the capability of elementary mathematics to further the development of general mental and intellectual capacities, in particular reasoning power, was praised in discussions and debates on mathematics education there is not much evidence that the obtaining of such capacities was actually pursued as a direct and explicit goal in real teaching. Rather, it seems to have been supposed that the development of pupils' intellectual apparatus would result automatically from the mere acquisition of mathematical knowledge and skills.

The early 20th century saw pragmatic and utility oriented movements in mathematics education in many countries. At the same time it began to be questioned in various quarters (cf. Niss, 1981, and Stanic, 1986 & 1987), in particular in the USA, whether mathematics education was in fact able to foster general mental abilities to the extent usually claimed. This gave rise to attempts to critically review the reasons and goals for primary and lower secondary mathematics education and to revise the corresponding curricula. Instead of expecting mathematics education to serve formative ends, emphasis was placed on its ability to serve practical ends much more specifically. In the 'Summary of Recommendations' of the Report by the Mathematical Association on 'The Teaching of Mathematics in Public and Secondary Schools' in 1919 (quoted in Howson, 1982, p 228-29; see also Niss, 1981), the following two goals were stated (together with twelve others) - but for boys only:

> (1) That a boy's educational course at school should fit him for citizenship in the broadest sense of the word: that, to this end, the moral, literary, scientific (including mathematical), physical and aesthetic sides of his nature must be developed. That in so far as mathematics is concerned, his education should enable him not only to apply his mathematics to practical affairs, but also to have some appreciation of those greater problems of the world, the solution of which depends on mathematics and science.
>
> (2) That the utilitarian aspect and application of mathematics should receive a due share of attention in the mathematical course.

Similar goals can be found in other countries at the same time, for instance in the national Swedish curriculum for primary and lower secondary schools 1919 ('Undervisningsplan för rikets folkskolor', see Undervisningsplan, 1919) which states (my translation)

> The teaching of arithmetic and geometry in 'folkskolan' [primary and lower secondary school (MN)] has, as its main purpose, to provide pupils, according to their age and stage of

> development, with knowledge and skills in arithmetic with particular regard to what is needed in day-to-day life, as well as some familiarity with the drawing, description, mensuration and computation of geometrical figures and quantities.

Although the utility and application oriented goals of mathematics education were in the centre of attention in the first quarter of the century, and the belief that mathematical training develops general mental capacities of relevance to extra-mathematical domains lost some of its former momentum, the assumption of a potential of mathematics education to serving formative goals was never completely abandoned. Rather, the notion of specific mental capacities was embedded in a much broader notion of 'the personality' as containing also beliefs, convictions, working habits, attitudes. For instance, the national Swedish curriculum in 1955 (Undervisningsplan, 1955) required that (my translation)

> 'The education of pupils' personality should be furthered by their experiencing the importance of conscientious and very precise work as well as the necessity of exerting thought and will power in order to solve given tasks.'

The German mathematics educator Wilhelm Birkemeier discussed, in 1923, the formative value of mathematics in a book of approximately the same title (Birkemeier, 1923 (1973)). Firstly, he invokes a (classical) distinction between the 'material' and the 'formal' formative value of an educational subject. The 'material' formative value is the inherent value obtained by acquiring knowledge of the very substance of the subject at issue. By definition such values cannot be acquired by the study of other subjects. As far as mathematics is concerned, the formative value is attached to knowing and understanding theorems and formulae, deduction and proof, methods and principles of construction. In contrast, the 'formal' formative value of an educational subject, in this case mathematics, is the value which can be derived from studies of mathematics to general application in contexts outside of mathematics itself. Birkemeier then goes on to speak about three different ways in which mathematics education can contribute to the formal formation of the personality (pp 80 - 107): (i) Development and training of the individual's fundamental mental dispositions and actions, i.e. all the mental capacities involved in performing intellectual activity and in gaining valuable experience. (ii) Development of the personality through 'co-activation' and 'co-training' of general psychological and intellectual functions such as, 'representing symbolically', 'denoting', 'making sense', 'ordering', 'distinguishing', 'connecting', 'formalising', all of which may, according to Birkemeier, be furthered by the teaching of arithmetic, while geometry contributes to developing 'spatial perception and imagination', 'comparison of magnitude and shape', 'functional thinking', 'abstracting', 'logical reasoning' etc.; (iii) Education in method, i.e. the ability and mental power to accomplish each step in a complex process.

Because of the economic and political recession that swept across the world in the 1930s, and because of the Second World War, there was a return to utility oriented and pragmatic goals of mathematics teaching during the 1930-40s. However, mathematics educators attempted to widen the perception of utility to encompass also goals related to the individual's private and social life. Leo J. Brueckner (Brueckner, 1939 (1970)) suggested four major functions of arithmetic '(a) the computational function, which deals with the development of essential computational skills; (b) the informational function, which deals with the development of an understanding of the history, evolution, and present status of institutions, such as banks, insurance, and taxation, that have been created by society to deal with the social uses of number; (c) the sociological function, which deals with the development of an awareness of the problems faced by these institutions and of the means, current and proposed, for solving those problems; and (d) the psychological function, which deals with the development of the power to do quantitative thinking and of an appreciation of the value and the significance of quantative data and methods in dealing with the affairs of life'. The overall purpose of such social arithmetic 'should be to give the pupils a rich social insight and an understanding of the function of number in daily life and to enable them to participate more effectively in the affairs of a changing industrial democratic society.'

At the same time as utilitarian aims were being re-emphasised, mathematics educators in several countries to an increasing extent began to criticise utilitarian mathematics teaching for promoting rote learning, unreflective use of procedures, lack of understanding of rules and processes, and so forth. For instance, a report, published in 1935 (Betænkning, 1935) by an official commission appointed by the Ministry of Education in Denmark, proposed the following goals of mathematics teaching at the lower secondary level (grades 6-9) (my translation):

> The goals of the teaching are:
>
> In *arithmetic*, to teach pupils to perform arithmetical computations with certainty and proficiency, on paper as well as mentally, and to apply these skills to not too complex tasks which, as much as possible, should bear relations to practical life. In addition, pupils should be able to explain the rules of arithmetic.
>
> In *algebra*, to clarify the general nature of the rules of arithmetic, as the knowledge of these rules, which pupils have acquired through the study of arithmetic, is to be extended to include symbolic algebraic expressions. Thus, there should be no attempt to establish a rigorous theory.
>
> In *geometry*, to develop pupils' visual awareness and ability through consideration and description of simple geometrical figures, and to provide insight into the properties and interrelations of these figures. Moreover, to teach them to accurately construct certain figures from given components. While emphasising empirical experience, the teaching should strive to awaken pupils' need for proof and proving so that they will realise the general validity of geometrical theorems.

As was the case in other countries, the above suggestions were not adopted as an official component in the curriculum. In the shadow of the black clouds of the thirties, society was not prepared to broaden the perspectives of mathematics education.

The development in mathematics education after the Second World War led to the advent and the dissemination of the so-called 'new mathematics curriculum' in a multitude of countries. As this spectacular development is well known and has been made subject of several studies, as well as of extensive and intensive discussions, I shall restrict myself to offering a few remarks before going on to deal with more recent developments and trends. In its immediate appearance, the 'new math' movement had its focus on intra-curricular matters of 'what?' and 'how?' rather than on 'why?'. The same was true with the subsequent scholarly and not so scholarly discussions about it. None the less, to a considerable extent the advocacy for the 'new math' was actually concerned with the justification and goals of mathematics education, though not in an univocal or unambiguous manner. In fact, different quarters involved in the multi-faceted movement emphasised different goals (cf., e.g., Cooper, 1985, and further considerations later in this section). Still, the 'new math' was supported by reference partly to new types of 'formative' aims, partly to new sorts of utilitarian goals related to the nature and rôle of mathematics in culture and society. The 'new math' educators underlined that modern society required individuals, citizens and work force members ('mathematical manpower' (College Entrance Examination Board, 1959 (1970), p 674)) who possessed a wide variety of general personal capacities of a formative nature, both as regards the character formation (concentration, observation, exactitude, perseverance) and the development of intellectual capacity (power of abstraction, generalisation, specialisation, precision in the use of words, logical thinking, analytical and research attitude, judgement).

Such abilities were considered significant not only as regards the formation of the personality, but also in relation to the flexible maintenance of jobs, the conditions and nature of which were seen as changing more rapidly than ever before. Thus, *understanding* was emphasised as the predominant goal of mathematics teaching at all levels, from kindergarten to university. In a wide variety of different contexts students - and citizens - would be faced with the need to sort, classify, structure, abstract, generalise, specialise, represent and interpret symbolically and graphically, justify and prove, encode and decode messages, formulate, communicate with others and so on and so forth. In other words, in spite of what subsequently happened in practice in many places - such as the kind of formalistic game-like plays in and with structures defined in terms of sets and logic, often devoid of sense-making relations to matters outside the structures themselves, which one could find in quite a few, say, Danish classrooms in the late 1960's and early 1970's - many of those involved in the 'new maths' reform, in particular in its first wave, saw the reform as a means also to utilitarian ends. This point of view was eloquently reflected in a report from a Unesco project on school mathematics in the Arab countries (Unesco, 1969, pp 13ff) which reads

> It is quite evident that the primary reason why mathematics and science have achieved more importance and wider public interest today lies in their uses and the way in which they have changed modern society. The main virtue of mathematics in modern society is the fact that it aids the non-mathematician, the applier, to do his job with greater efficiency and deeper insight. [...]
>
> Mathematics plays a fundamental role in the economic development of a country because it is a basic factor for all scientific and technological research and technical training. [...] It is equally important that mathematics of the more elementary variety be taught to large numbers of technicians and skilled workers, without whom the discoveries of the scientists and technologists cannot be effectively used in industry, agriculture, fisheries, mining, and practically all economically productive enterprises.

We shall conclude this section of the Chapter by considering the development and current trends in the goals of primary and lower secondary mathematics teaching during the last couple of decades. If we were to sum up recent developments in one sentence, it could be done in this way: The goals of primary and lower secondary mathematics education have been *broadened* considerably so as to encompass the essential aspects of numeracy and 'mathematical literacy' in society. Many older goals have not been disposed of but have become embedded or simply absorbed in more general goals. Let us briefly review some of the most important changes (cf. International Commission on Mathematical Instruction, 1979; Morris, 1981; Department of Education and Science/Welsh Office, 1982; Curriculum and Evaluation Standards, 1989). The following are general aims which it seems that the vast majority of countries in the world wish to pursue:

Exterior aims:

* to provide substantial *mathematics education for all,* and not only to the future members of society's intellectual or social élite, while emphasising that mathematical competence, in some form or other, is available to everyone;

* to provide opportunities for *differentiated* teaching and learning to the individual learner, while paying attention to his or her personal background;

* to emphasise *participation and co-operation* amongst learners in dealing with collective tasks related to mathematics;

* to *assess* pupils' mathematical potential, achievement and performance in ways which are in accordance with the higher order goals of mathematics teaching and learning.

Interior aims:

* to focus on the needs and interests of *the individual learner,* in order to prepare him or her for active participation in all aspects of *private and social life,* including active and concerned *citizenship* in democratic society;

* to develop *pupils' personalities* by engendering or enriching self respect and self-confidence, independent and autonomous thinking (including logical thinking), the development of explorative and research attitudes, linguistic capacities, aesthetic experience and pleasure, etc.;

* to emphasise pupils' *mathematical activity* rather than their passive acquisition of knowledge;

* to emphasise *mathematical processes* (such as exploration, investigation, conjecturing, problem posing/formulation/solving, representing, proving, modelling) and not only products (concepts, results, methods, skills);

* to foster *mathematical thinking and creativity*, while emphasising that mathematics is a living subject resulting from human activity and from the continuing efforts of humankind over five millenia;

* to enable pupils to identify, pose, formulate and solve mathematical *problems*, whether pure or applied, whether closed or open;

* to enable pupils to understand and appreciate the special *nature of mathematics;*

* to enable pupils to apply mathematics to extra-mathematical situations by means of *models and modelling;*

* to enable pupils to critically *analyse and judge uses of mathematics* (their own as well as others') in extra-mathematical contexts;

* to provide students with an impression of and insight into the rôle of *mathematics in society and culture;*

* to make pupils familiar with current *information technology* in relation to mathematics;

However, one thing is to propose and establish general goals; it is something completely different to carry them out. This may be one of the reasons why in some countries one has returned to classical positions and placed more emphasis on providing descriptions of the sorts of performance and attainment to be expected or desired from pupils (cf. Brown, 1993). In other places the focus rather is on the categories or dimensions of content with which pupils should acquire experience, or on the sorts of activities and study forms which teachers may orchestrate in the classroom or with smaller groups of children. Moreover, in a great many places there is a marked gap between society's acknowledged goals of mathematics education and the reality of teaching and learning in the same society.

As appears from the above list of aims, they are not really specific to the primary and lower secondary levels. They might equally well be invoked vis-à-vis the upper secondary and tertiary levels. Of course, the interpretation and concretisation of the aims will be materialised differently at different levels, but - today - the goals are basically the same. (An illuminating and interesting exposition of the relationship between lower secondary and upper secondary goals in Japanese school mathematics is included in (Kawaguchi, 1980, p. 42f and p. 47)). This is a fairly recent trend. In former times, there was a clearer demarcation line between the goals of primary/lower secondary and upper secondary/tertiary teaching, respectively. Although it is true, as shown in previous sections of this Chapter, that the ultimate ends were often stated to be the same, the goals of primary/lower secondary teaching were much more centred around rote learning and specific procedural (especially computational) skills with an emphasis on mechanical drill (cf., e.g., the Spens report (1938), quoted in Howson, 1982, and in Niss, 1981, p. 16). Higher order complex goals were left to be pursued at the upper secondary/tertiary levels. There is hardly any doubt that the change, if at all, is due to (i) the spreading and increasing significance of mathematics to more and more domains of society and culture, (ii) the expansion of post-elementary general education to encompass a growing majority of an age cohort, an expansion which is now being continued to include post-secondary education as well.

These considerations take us to our next section.

Upper secondary and lower tertiary

Let us begin with the general observation that the overall oscillation (of a period of 2-3 decades, on average) between formative and utilitarian goals which was a characteristic trait of the development of elementary mathematics education over the past couple of centuries is no less manifest as far as upper secondary and tertiary education is concerned. On the contrary, the average amplitude is even greater than is the case with elementary mathematics teaching.

It was not until the second quarter of this century that post-elementary mathematics education began to be provided to more that a small minority of an age cohort, and in quite a few countries this did not even happen until the massive expansion of general education that took place from the 1960s and 1970s. In the last century and in the first decades of this century, upper secondary and tertiary education was reserved for the élite which was to occupy leading positions in government, administration, science, and culture. This determined the place and rôle of mathematics in upper secondary and tertiary curricula. Irrespective of the future careers students were likely to pursue, mathematics was present in upper secondary education mainly for the kinds of formative purposes which were discussed in previous sections of this Chapter, i.e. development of the personality, primarily as regards reasoning power in general and logical thinking in particular. Hosts of quotations could be offered as documentation of this claim (e.g. Charles Davies (1850) from whose book 'The Logic and Utility of Mathematics' interesting excerpts are quoted in Bidwell & Clason, 1970, pp 55ff, as is true with Edward Brooks (1883) quoted in Bidwell & Clason, 1970, pp 78ff. See also Schubring, 1983, especially Chapter 6, and Jahnke, 1990, especially Chapters II and V for exciting accounts of the rôle of mathematics in the German tradition of general humanistic education ('Allgemeinbildung')).

If, in the past, the proportion of a youth cohort which received post-elementary mathematical education was very small indeed, the proportion of those who were to continue with further studies or careers in science, engineering, surveying, as military officers, in certain specialised occupations in administration, business and commerce (e.g. public statistics and insurance) and a few other areas, was even smaller. With respect to these areas, mathematics either formed part of their very foundation (as is the case with astronomy and physics) or was an important tool in dealing with them. Thus, to students of such areas, mathematics was a service subject of varying degrees of importance. The purpose of the corresponding teaching of mathematics was to provide what we might call 'local utility'. The basic goal of the teaching of mathematics was simply inherited directly from that purpose: to teach the students the concepts, results, methods, and processes pertaining to the mathematical topics that happened to have been chosen for inclusion in the given syllabus.

This does not determine, however, the more specific aims and objectives of mathematics teaching. If we discern 'knowledge of content matter', 'skills', 'understanding', 'attitudes', and 'degree of independence of study', as basic kinds of goals, we cannot derive a distribution of relative weights for these kinds of goals from the fact that mathematics was meant to be a service subject. To that end, several other factors have to be taken into account, like, for instance, general views and doctrines of what it is to educate, teach and learn - and, not unimportantly, how knowledge and insight is assessed; the ways in which jobs are placed, divided and organised in society; society's views of individuals and their relations to others and to the community; and so forth. A closer inspection of the manner in which mathematics teaching was conducted, in classroom *practice*, in the past shows that also post-elementary mathematics teaching tended to emphasise the acquisition of 'knowledge of facts', 'procedural skills', and 'passive attitudes' through one-way communication, reproduction and drill, whereas 'understanding and insight' and 'method and strategy' gained through students' independent, creative, active and inter-active exploration and investigation in two-way communication with teachers and peers only seldom received attention worth mentioning. Ironically enough, this profile of goals was not restricted to service teaching only - which would not, perhaps, have been very surprising - but prevailed with the mathematics taught for the development of mental ability as well. This indicates that general educational tradition, environment and ideology are more important factors in the establishing of goals of mathematics teaching than one might tend to suppose in the first place.

The dominant development of upper secondary mathematics education at the turn of the century was a reform of the mathematical content. Although there is no simple relationship between the general aims of mathematics teaching and its content, it is normally the case that fundamental changes of content entail corresponding changes of objectives (sometimes the converse is also true, but not always). The second half of the 19th century saw a strongly increasing unification of mathematics as a research discipline - a process that continued throughout the 20th century. The classical division of mathematics into separate and fairly independent compartments: number theory, algebra, geometry, function theory and calculus, theoretical mechanics, applied mathematics, was being dissolved. An important contribution to and manifestation of this process was Felix Klein's so-called *Erlanger Programm* according to which it was an essential task of mathematics to identify and study structures that are left invariant under the action of different classes of functions. This placed the concept of function in the centre of attention, not only of mathematical research but also of upper secondary mathematics education. Several countries began to reform their upper secondary curricula to encompass various aspects of functions, including infinitesimal calculus or even analysis. In fact, the establishment of the International Commission on Mathematical Instruction (ICMI) in 1908 was essentially a result of the endeavour of the international community of mathematicians to promote and monitor reform in this direction. It took almost a quarter of a century until this reform process was accomplished in the majority of developed countries (cf. van der Blij et al., 1981).

In accordance with this movement, a new objective of upper secondary mathematics teaching was put forward in many places: That students should realise and appreciate the unity of mathematics and the rôle played by the concept of function (not to be confused with the theory of functions) for the emergence of this unity. This is splendidly reflected in Chapters II (see, for instance, pp 395-96) and VII (pp 442-449) of the remarkably thorough and eloquent report published by the National Committee on Mathematical Requirements under the auspices of the Mathematical Association of America (The Reorganization of Mathematics in Secondary Education, 1923 (1970)). In fact, this report provides a comprehensive and brilliant account of the aims of secondary mathematical instruction as perceived, at that time, by mathematics educators oriented towards the international reform movement. A summary of its considerations may therefore serve to outline and condense the state of thinking, as far as goals at the secondary level are concerned, at the forefront of mathematics education in the first half of this century.

The following formulation in the report (p. 395) deserves to be quoted:

> The primary purpose of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

As was customary at that time the report distinguishes between three classes of aims, which are not, however, considered mutually exclusive by the report: (1) practical or utilitarian, (2) disciplinary, and (3) cultural aims (p. 391). Each of these classes of aims are discussed in some detail.

Practical aims include (pp 391-92): (a) 'The immediate and undisputed utility' of the *fundamental processes of arithmetic* in the life of every individual. Here, the report emphasises students' understanding of the nature of the fundamental operations and their power to apply them in new situations; common sense and judgement in computing from approximate data; the development of self-reliance in the handling of numerical problems. (b) An *understanding of the language of algebra* and the ability to use this language intelligently and readily, including appreciation of the significance of formulae and setting up and solving equations in order to deal with simple problems. (c) An ability to understand and use the *fundamental laws of algebra* as being general in contradistinction to the particular rules of arithmetic. (d) The ability to understand and interpret correctly *graphical representations* of various kinds. (e) Familiarity with *geometrical forms* common in nature, industry, and life; the elementary properties and relations of these forms, including their *mensuration*; the development of *space perception* and *space imagination*.

Disciplinary aims include (pp 392-394): (a) 'The acquisition, in precise form, of those *ideas or concepts in terms of which quantitative thinking in the world is done*' (e.g. ratio, measurement, proportionality, similarity, positive and negative numbers, dependence between quantities). (b) 'The development of *ability to think clearly in* *terms of such ideas and concepts'*, including analysis of complex situations; recognition and expression of logical relations between interdependent factors; generalisation. (c) 'The acquisition of *mental habits and attitudes'*, including, among others, seeking for relations; an attitude of enquiry; a desire to understand; concentration and persistence; 'a love for precision, accuracy, throroughness, and clearness, and a distaste for vagueness and incompleteness'; 'a desire for orderly and logical organization as an aid to understanding and memory. (d) 'Many of these disciplinary aims are included in the broad sense of *the idea of relationship or dependence* - in what the mathematician in his technical vocabulary refers to as a "function" of one or more variables. Training in "functional thinking", that is thinking in terms of and about relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics' (p. 394). It is interesting, in this context, to note that the report explicitly abstains from entering the debate on the transferability of mental training gained in mathematics to other types of situations. (pp 392-94).

Cultural aims (p. 394) are 'those somewhat less tangible but none the less real and important intellectual, ethical, esthetic or spiritual aims that are involved in the development of appreciation and insight and the formation of ideals of perfection'. Such aims include (a) *appreciation of beauty* in the geometrical forms of nature, art, and industry; (b) *Ideals of perfection* as to logical structure, precision of statement of thought, logical reasoning, discrimination between the true and the false; (c) *Appreciation of the power of mathematics* - 'and the rôle that mathematics and abstract thinking, in general, have played in the development of civilization; in particular in science, in industry, and in philosophy. In this connection mention should be made of the religious effect, in the broad sense, which the study of the infinite and of the permanence of laws in mathematics tends to establish'.

When going on to discuss 'the point of view governing instruction', the report insists (p. 395) that 'Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes and principles in the solution of concrete problems rather than on the acquisition of mere facility and skill in manipulation.'

The genesis and the dissemination of the 'new mathematics' after the Second World War affected secondary and tertiary mathematics teaching to an even greater extent than was the case at the primary and lower secondary levels. Although much of the stimulus for the 'new maths' reform was actually found in utilitarian, and sometimes even industrial, aims (e.g., see Cooper, 1985, Chapters 5-8), other aims were put on the agenda as well. The authoritative and often quoted OECD report 'New Thinking in School Mathematics' (1961) of a seminar held in 1959 in Royaumont, France, proved instrumental in promoting the reform process. The seminar was much influenced by the 'bourbakist' perception of mathematics as being construed as a complex edifice of abstract structures founded on a few 'mother structures' (a set endowed with a relation, a set endowed with a composition, a set endowed with a topology (e.g. defined by a system of subsets called open sets)) all of which rely on the concept of set.

After having discussed the increasing gap between school mathematics and university mathematics, caused by the rapid development of research mathematics while school mathematics had remained unchanged for almost a century (pp 105-106), 'New Thinking in School Mathematics' summed up the case for reform in five points as follows (p. 107):

> (a) The new developments in graduate and research mathematics imply a shift in emphasis for secondary school mathematics. New topics such as abstact algebra, vector spaces, theory of sets, etc. will enter the school programme, bringing a changed point of view on what mathematics is today.
>
> (b) The new applications of mathematics suggest new problem material. Probability, statistical inference, finite mathematical structures, linear programming, numerical analysis - all indicate expansion in useful applications of mathematics.
>
> (c) The development of new standards of accuracy and clarity of statements and the emphasis on mathematical structures indicate a need for reconsideration of the concepts embedded in our classical treatment of mathematics.
>
> (d) The tremendous increase in knowledge in the various branches of mathematics demands a synthesis and broader base for teaching at the pre-university level. To learn mathematics today requires more efficient and more general approaches.
>
> (e) The changes in cultural, industrial and economic patterns of many nations call for a basic change in educational patterns. More people must be better trained in scientific knowledge. Even laymen must come to understand science; today, knowing mathematics is basis to understanding science.

Although these points are not formulated as goals, they do in fact refer to goals, albeit implicitly. Thus, items (b) and (e) suggest a utilitarian purpose of mathematics education: to prepare people, laymen included, to live and act in a changing society marked by the increasing significance of science and technology and the essential rôle of (new) applications of mathematics for both. Items (a), (c) and (d) indirectly state the aim of bringing pre-university mathematics up to date with 'the new developments in graduate and research mathematics'. These new developments are characterised by a synthesis of the various branches of mathematics founded on general concepts, structures and approaches, and new topics based on these. Therefore it is an objective of mathematics teaching to equip students with knowledge and skills concerning mathematics in its new and thoroughly innovated shape.

It is well known that the 'new maths' reform soon became an object of criticism from more and more quarters, not only from such that were opposed to it from the very beginning but also from quite a few of those who initially supported it. The main reason was that in many places the actual implementation of the ideas turned out to give rise to results and problems which were not anticipated, let alone desired, in the original concept. In particular, it became clear that the study of very abstract and general concepts and structures developed into a 'facts and drill' aim in itself, instead of being a means to greater ends. Besides, the 'new math' reform was implemented along with the expansion of the educational system, but there was hardly any concurrent pedagogy available which could bring about the abstract and general species of modern mathematics to "the masses" who began to enter the higher levels of the educational systems. So, most of the 'new math' ideas were abandoned, although some of them are still with us, even if in a modified form, perhaps. Thus, the introduction of probability and statistics in modern curricula was actively stimulated by 'new math' thinking. It is important to note that the dissolution of the 'new math' approach was not primarily due to the purposes and ends it pursued, but mainly to the fact that it did not work as expected. Furthermore, in retrospect the experiences with 'new math' may be seen to have contributed greatly to the attribution of a key rôle to the consideration of goals in subsequent developments in mathematics education.

At any rate, while the subsequent reform movements of the late 1970s and 1980s - which, after having digested the lesson of the 'new math' reform, were neither concerted, coordinated, univocal or outspoken but pluralistic - all of them had subject matter, content and exposition, and teaching/study forms as their direct focus, they did indeed make their various cases with reference to goals of mathematics education. It is safe to claim that a good deal of the curriculum reforms that have been taking place since the 1970s have been accompanied by discussions about goals.

It is in this vein that Ubiratan D'Ambrosio's chapter (IX) of 'New trends in mathematics teaching IV' (D'Ambrosio, 1979,) - with the title 'Overall goals and objectives for mathematical education', a modification of the working title 'Why teach mathematics?' - is an attempt to assign a prominent rôle to the discussion of the justification and goals of mathematics teaching and to place this discussion in a multi-dimensional space of mathematical, scientific, technological, social, cultural, philosophical, and psychological matters. In so doing, he suggests that goals should be given comprehensive, yet casual, formulations in common sense terms free of pendantry and routine (pp 183-184). D'Ambrosio surveys the situation by listing - with endorsement - four goals as 'generally agreed upon':

> Goal 1: To achieve for each individual the mathematical competence appropriate for him.
> Goal 2: To prepare each individual for adult life, recognising that some students will require more mathematical instruction than others.
> Goal 3: To foster an appreciation of the fundamental usefulness of mathematics in our society.
> Goal 4: To develop proficiency in using mathematical models to solve problems.

Goals put forward by the 'Association des Professeurs de l'Enseignement Public', presented on p. 192 are in the same vein. 'The student should be able, as an objective of his schooling, to: (1) analyse different components of a situation; (2) recognize analogous situations; (3) choose a strategy adapted to a situation; (4) make himself understood by others; (5) have a critical atitude; (6) construct simple deductions and construct a chain of deductions; (7) predict a result and generalize; (8) construct a simple model; (9) participate in a collective task. At closer inspection, these goals fall into three different categories, all of which were in focus in the 1970s: what could be called 'new formativist goals' ((4), (5), (6), (9)) in which the development of the individual should be seen in the context of a community; goals concerning the 'general treatment of problems' ((1), (2), (3), (7)); and, modelling problems ((8)). Another example of national goals of the same kinds were introduced by the Austrian Ministry of Education, quoted in the chapter 'Mathematics education at upper secondary school, college and university transition' by Douglas Quadling, also in 'New trends in mathematics teaching, IV', 1979, p. 48. It refers to the following goals in relation to academic courses:

> precise and critical thought, logical deduction, precise verbal expression, independent intellectual activity, creativity and intuition, insight into mathematical subject matter and method, knowledge and abilities required for other subjects and for various real-world applications,

and goes on to specify

> knowledge of the axiomatic method, training in proof, training in the capacity for abstraction, familiarity with language and symbolic expression in the subject, fluency with the most important manipulative procedures, mathematical treatment of problems from natural science, technology, economics and other areas, development of geometrical intuition.

These, and a lot of other goals put forward in the trade, contain three components: mathematical; sociological; psychological, including affective. D'Ambrosio finds that in the course of the 20th century there has been a shift towards the latter two, and recently towards the last one (p. 182). In considerations characteristic to the 1970s he states, as his own opinion, p. 190, that 'emphasis should be changed from what is taught, that is, from curricula, syllabi and contents, to methodology, and to a new classroom environment, both geared towards creativity in its fullest sense'. D'Ambrosio's own opinion is expressed in his formulation of the basic goals of mathematics education : 'The primary objective of mathematical education is not to perpetuate knowledge or to push existing knowledge a little further (this will go on or will fade away as a result), but to foster the creation of new knowledge. Teaching is not the essential goal of this creative or contemplative form of mathematical education - the fundamental goal is to prepare a favourable setting for the creation of new knowledge.' (p. 193)

Another characteristic feature of the 1970s and 1980s was the emerging realisation that different quarters of society may hold different, or even opposing, goals. Thus, Douglas Quadling in his chapter of 'New trends in mathematics teaching, IV', p. 49 observes that the goals of 'curriculum developers, teacher trainers, writers of syllabuses and authors of text books, are not identical to those of students and teachers who often work to more limited goals: 'namely to acquire the knowledge and skills set out in the syllabus and textbook and the ability to apply such knowledge and skills to certain specific situations'. This is but one instance of a more general schism 'between the goals stressed by mathematics educators and those valued by employers or teachers of other subjects. Professors of engineering or chemistry in the university often look to the upper secondary school to provide their students with the mathematical skills and processes which are used in their own disciplines, and show little interest in rigorous foundations or a structural overview of mathematics [...]'. Naturally, part of the reason why different people hold different goals is that they have different target groups in mind. Some tend to pay particular attention to the needs of future academics, whereas others rather focus their interest on the needs of those whose future positions in society reflect a minimum of formal schooling. While there may be overall agreement concerning the higher order end of the goals spectrum ('ends' and 'purposes') for various target groups, disagreement is likely to occur when it comes to the specification of more particular goals, especially in the 'objectives' end of the spectrum.

Let us finish this section by considering the goals of upper secondary and lower tertiary mathematics education as they currently appear to be perceived in several quarters around the world. As stated already in the previous section, the goals of these levels are the same as the ones encountered for the primary and lower secondary levels (cf. pp 17-18), only the specification, the concretisation, and the relative priorities differ between the levels. Clearly, at upper secondary and non-specialist tertiary levels the mathematical issues, content, methods, theories with which the goals are meant to be pursued are not the same as at the lower levels. If we were to portray the set of goals which, in the 1990s, are receiving particular attention, the result might, for instance, look as follows (again, cf. pp 17-18):

Mathematics education at a higher level should be provided to essentially all in a youth cohort, in order to prepare them for active participation in all aspects of private and social life in society, including future education and occupation. As this purpose relies on the rôle mathematics is playing in society and culture, it is an end of mathematics education that students should realise, understand, judge, and relate to those properties of mathematics that constitute this rôle. Therefore, it is an aim of mathematics teaching and learning at this level that students be able to exercise representative mathematical thinking and creativity and to engage in a variety of characteristic mathematical activities and processes. More specifically, it is an aim that students be able to engage - in non-routine, open situations - in exploration; representation; conjecturing; problem posing/formulation/solving; reasoning and proving. In the same vein it is an aim that students be able to analyse mathematical models of extra-mathematical matters, and to construct, for themselves, mathematical models. It is further an aim that students should develop an impression of and insight into the scientific and philosophical nature and status of mathematics and of its position in society and culture, and into the history and development of mathematics as a subject which is a result of human activity. In order to make the pursuit of these aims meaningful and realistic, it is an objective that students acquire knowledge and skills concerning such specific mathematical concepts, methods, processes, facts, results and theories that are of significance to the pursuit of the above-mentioned goals at the educational levels at issue.

III. Concluding remarks

During the 1990s the justification and goals of mathematics education have not been given a terrible amount of explicit attention by researchers in mathematics education. This seems to be due to two factors. Firstly, a rather widespread skepticism in the mathematics education community towards the notion of thinking in terms of curricula, in particular if curricula are considered as top-down prescriptions. This skepticism is probably based on the fairly recent realisation of the discrepancies between 'the matter meant' (the intended curriculum), 'the matter taught' (the implemented curriculum), and 'the matter learnt' (the realised curriculum) (cf. Bauersfeld, 1979, and Robitaille, 1981). As specially regards 'goals', the general uneasiness with prescriptive curricula goes along with a disbelief in the value of classical 'goals-means' hierarachies. Secondly, during the last decade, or so, the research community has focussed much more on 'learning' than on 'teaching', partly because of the discrepancies just referred to, partly because of an increased interest in the situations and needs of the individual learners and the differences between them.

Nevertheless, even if issues of 'justification' and 'goals' have not occupied a predominant position on the agenda of researchers in mathematics education, it would be wrong to infer that these have been relegated to playing an inferior rôle in mathematics education as such. As a matter of fact, issues of justification and goals have manifested themselves strongly, albeit indirectly, in two items on the mathematics education agenda. Since the late 1970s, we have been witnessing a strong emphasis on the process aspects of mathematics teaching and learning, such as mathematical thinking and creativity, pure and applied problem posing and solving, exploration and heuristics, and modelling. Moreover, the development and proliferation of information technology has given rise to fundamental questions concerning the future character and place of mathematics education. Whether discussed explicitly in terms of justification and goals, these issues - which have actually been dealt with by both didacticians of mathematics and agents of curriculum reform - are in fact intimately related to the very purposes and ends of mathematical education, and not the least so when connected to the 'mathematics for all' movement. The other item on the agenda of mathematics educators which is closely related to the issue of goals is 'assessment'. Since the mid-1980s mathematics educators have devoted growing attention to the reality and the potential of assessment in mathematics education. Undoubtedly, one crucial reason for this is the marked and increasing gap between the reality of established assessment modes and the development of mathematics teaching and learning. Clearly, this development is strongly associated with (revised) goals of mathematics education.

So, although it may well be that the justification and goals of mathematics education are not in the limelight of mathematics education research in the mid-1990s, they are certainly essential, even indispensable, components in the theory and practice of mathematics teaching and learning. When this is the case, it is in the interest of intellectual honesty, clarity, and enlightenment to bring these issues into the open and deal with them, both from descriptive/analytic and from normative perspectives. Therefore, we continuingly need updated descriptive/analytic studies to uncover, in concrete and specific terms, the actual justification and the goals of mathematics education in different countries/cultures, and we continuingly need normative studies to discuss, on theoretically and empirically sound foundations, the justification and goals of mathematics education in a world marked by constant change - in terms of the positions and functions of mathematics, the socio-economic, technological and political development of our nations, ideological and cultural values, and the rôles of education in all these processes.

To conclude, as has been argued in several places in this Chapter, the justification and goals of mathematics education deserve to be granted substantial interest. Whatever we do in the research, development and practice of mathematics education, its justification and goals are of crucial importance, for better or for worse. None the less, this should not lead us to conclude that most other matters are of secondary or derived importance. Far from it: The rest is not silence.

Acknowledgements

The author wishes to thank David Wheeler, Vancouver, Canada, for his valuable comments and suggestions concerning a variety of general and specific points in the various drafts of this Chapter. The author further wishes to thank his Chapter reviewers Geoff Wain, Sawarak, Malaysia, and George Stanic, Athens (Georgia), USA, for their many stimulating remarks and suggestions.

References:

D'Ambrosio, Ubiratan, 1979: 'Overall goals and objectives for mathematics education', in International Commission (op. cit.), pp 180-198

Bauersfeld, Heinrich, 1979: 'Research related to the mathematical learning process', in International Commission (op. cit.), pp 199-213

Betænkning vedrørende Undervisningen i Mellem- og Realskoler og i Skoler, der forberedertil Pigeskoleeksamen og til Almindelig Forberedelseseksamen, 1935, Copenhagen, A/S J.H. Schultz Bogtrykkeri, p 40

Bidwell, J.K. & Clason, R.G. (eds.), 1970: *Readings in the History of Mathematics Education,* Washington, DC, NCTM,

Birkemeier, Wilhelm, 1923 (1973): *Über den Bildungswert der Mathematik. Ein Beitrag zur Philosophischen Pädagogik,* Leipzig und Berlin , B.G. Teubner, 1923 (photographical reprint at Walluf bei Wiesbaden, Dr. Martin Sändig oGH, 1973)

van der Blij, Frederic; Hilding, Sven, & Weinzweig, A.I, 1981: 'New Goals for Old: An Analysis of Reactions to Recent Reforms in Several Cuntries', in R. Morris (op. cit.), pp 105-118

Brooks, Edward, 1883 (1970): *Mental Science and Methods of Mental Culture,* Lancaster, PA, Normal Publishing Company, quoted in excerpts in Bidwell & Clason (op. cit.) pp 78-90

Brown, Margaret, 1993: 'Assessment in mathematics education: Developments in philosophy and practice in the United Kingdom', in Niss, M. (ed.): *Cases of Assessment in Mathematics Education,* pp 71-84

Brueckner, Leo J., 1939 (1970): 'The Development of Ability in Arithmetic', originally published as Chapter XV in 'The Thirty-Eighth Yearbook of the National Society for the Study of Education' (G. M Whipple, ed.), Bloomington, IL, Public School Publishing Company, here quoted from Bidwell & Clason (op. cit.), p 570 & p 575

Colburn, Warren, 1830/1912 (1970): *Teaching of Artihmetic,* in J.K. Bidwell & R.G. Clason (op. cit.), p 24

College Entrance Examinaton Board, 1959 (1970): *Program for college preparatory mathematics,* excerpts quoted in Bidwell & Clason (op. cit.)

Cooper, Barry, 1985: *Renegotiating Secondary School Mathematics. A Study of Curriculum Change and Stability,* London and Philadelphia, the Falmer Press

Curriculum and Evaluation Standards for School Mathematics, 1989, Prepared by the Working Groups of the Commision on Standards for School Mathematics of the National Council of Teachers of Mathematics, Reston, VA, NCTM

Department of Education and Science/Welsh Office, 1982: *Mathematics Counts* ('The Cockcroft Report'), Committee of Inquiry into the Teaching of Mathematics in Schools, London, Her Majesty's Stationary Office

Howson, A. Geoffrey,, 1982: *A history of mathematics education in England,* Cambridge, Cambridge Unversity Press

International Commission on Mathematical Instruction, 1979: *New trends in mathematics teaching, Volume IV,* Paris, Unesco

Jahnke, Hans Niels, 1990: *Mathematik und Bildung in der Humboldtschen Reform,* Studien zur Wissenschafts-Sozial- un Bildungsgeschichte der Mathematik Band 8, Göttingen, Vendenhoeck & Ruprecht,

Kawaguchi, Tadsu, 1980: 'Secondary School Mathematics in Japan', in R. Morris (op. cit.), pp 39-59

Kirke- og Undervisningsministeriet, 1900: *Cirkulære af 6. April til samtlige Skoledirektioner uden for Kjøbenhavn,* Copenhagen, 1990

Kolyagin, Yu.M., Lukanin, G.L., Oganesyan, V.A., 1980: 'Ways of Improving Mathematics Teaching in Soviet General Secondary Schools', in R. Morris (op. cit.), pp 70-86

Morris, R. (ed.), 1980: *Studies in Mathematics Education* **1**, Paris, Unesco

Morris, R. (ed.), 1981: *Studies in Mathematics Education* **2**, Paris, Unesco

New Thinking in School Mathematics, 1961, Organization for Economic Cooperation and Development, Paris

Niss, Mogens, 1981: 'Goals as a Reflection of the Needs of Society', in R. Morris (op. cit.) pp 1-21

Niss, Mogens, 1994: 'Mathematics in Society', in R. Biehler et al. (eds.) *Didactics of Mathematics as a Scientific Discipline,* Kluwer Academic Publishers, Dordrecht, pp 367-78

Quadling, Douglas, 1979: 'Mathematics education at upper secondary school, college and university transition', in International Commission (op. cit.), pp 47-65

The Reorganization of Mathematics in Secondary Education. A Report by the National Committee on Mathematical Requirements under the Auspices of the Mathematical Association of America, Inc., 1923 (1970), quoted in excerpts in Bidwel & Clason (op. cit), pp 382-459

Robitaille, David F., 1981: 'Goals of the Mathematics Curriculum of British Columbia: Intended, Implemented, and Realized', in R. Morris (op. cit.) pp 149-158

Schubring, Gert, 1983: *Die Entstehung des Mathematiklehrerberufs im 19. Jahrhundert. Studien und Materialien zum Prozess der Professionalisierung in Preussen (1810-1870),* Weinheim und Basel, Beltz Verlag

Soedijarto, R. Ibrahim, & Khodir, A., 1980: 'The Indonesian Modular Instructional System for School Mathematics', in R. Morris (op. cit.), pp 21-38

Stanic, George M.A., 1986: 'The growing crisis in mathematics education in the early twentieth century', *Journal for Research in Mathematics Education*, **17**, No. 3, 190-205

Stanic, George M.A., 1987: 'Mathematics Education in the United States at the Beginning og the Twentieth Century', in T.S. Popkewitz (ed.): *The Formation of the School Subjects. A Struggle for Creating an American Institution*, The Falmer Press, New York, Philadephia & London, pp 145-175

Undervisningsplan för rikets folkskolor, 1919, Stockholm, P.A. Norstedt & Söners förlag, quoted from an unpublished working paper by G. Emanuelsson and B. Johansson

Undervisningspla för rikets folkskolor, 1955, Stockholm, Svenska Bokforlaget Norstedt, quoted from an unpublished working paper by G. Emanuelsson and B. Johansson

Unesco mathematics project for the Arab States, 1969: *School Mathematics in Arab Countries*, Paris, Unesco (SC/WS/201, January 1969)

Mogens Niss,
13 November 1995

Liste over tidligere udsendte tekster kan rekvireres
på IMFUFA's sekretariat, tlf. 4674 2263 eller
e-mail: bs@ruc.dk

332/97 ANOMAL SWELLING AF LIPIDE DOBBELTLAG
Specialerapport af:
Stine Sofia Korremann
Vejleder: Dorthe Posselt

333/97 Biodiversity Matters
an extension of methods found in the literature on monetisation of biodiversity
by: Bernd Kuemmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM
by: Bernd Kuemmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids
by: Jeppe C. Dyre

336/97 PROBLEM-ORIENTATED GROUP PROJECT WORK AT ROSKILDE UNIVERSITY
by: Kathrine Legge

337/97 Verdensbankens globale befolkningsprognose
- et projekt om matematisk modellering
af: Jørn Chr. Bendtsen, Kurt Jensen,
Per Pauli Petersen
Vejleder: Jørgen Larsen

338/97 Kvantisering af nanolederes elektriske ledningsevne
Første modul fysikprojekt
af: Søren Dam, Esben Danielsen, Martin Niss,
Esben Friis Pedersen, Frederik Resen Steenstrup
Vejleder: Tage Christensen

339/97 Defining Discipline
by: Wolfgang Coy

340/97 Prime ends revisited - a geometric point of view -
by: Carsten Lunde Petersen

341/97 Two chapters on the teaching, learning and assessment of geometry
by Mogens Niss

342/97 LONG-TERM SCENARIOS FOR GLOBAL ENERGY DEMAND AND SUPPLY
A global clean fossil scenario discussion paper
prepared by Bernd Kuemmel
Project leader: Bent Sørensen

343/97 IMPORT/EKSPORT-POLITIK SOM REDSKAB TIL OPTIMERET UDNYTTELSE AF EL PRODUCERET PÅ VE-ANLÆG
af: Peter Meibom, Torben Svendsen, Bent Sørensen

344/97 Puzzles and Siegel disks
by Carsten Lunde Petersen

345/98 Modeling the Arterial System with Reference to an Anestesia Simulator
Ph.D. Thesis
by: Mette Sofie Olufsen

346/98 Klyngedannelse i en hulkatode-forstøvningsproces
af: Sebastian Horst
Vejledere: Jørn Borggren, NBI, Niels Boye Olsen

347/98 Verificering af Matematiske Modeller
- en analyse af Den Danske Eulerske Model
af: Jonas Blomqvist, Tom Pedersen, Karen Timmermann,
Lisbet Øhlenschlæger
Vejleder: Bernhelm Booss-Bavnbek

348/98 Case study of the environmental permission procedure and the environmental impact assessment for power plants in Denmark
by: Stefan Krüger Nielsen
Project leader: Bent Sørensen

349/98 Tre rapporter fra FAGMAT - et projekt om tal og faglig matematik i arbejdsmarkedsuddannelserne
af: Lena Lindenskov og Tine Wedege

350/98 OPGAVESAMLING - Bredde-Kursus i Fysik 1976 - 1998
Erstatter teksterne 3/78, 261/93 og 322/96

351/98 Aspects of the Nature and State of Research in Mathematics Education
by: Mogens Niss

352/98 The Herman-Swiatec Theorem with applications

by: Carsten Lunde Petersen

353/98 Problemløsning og modellering i en almendannende matematikundervisning

Specialerapport af: Per Gregersen og Tomas Højgaard Jensen

Vejleder: Morten Blomhøj

354/98 A GLOBAL RENEWABLE ENERGY SCENARIO

by: Bent Sørensen and Peter Meibom

355/98 Convergence of rational rays in parameter spaces

by: Carsten Lunde Petersen and Gustav Ryd

356/98 Terrænmodellering

Analyse af en matematisk model til konstruktion af terrænmodeller

Modelprojekt af: Thomas Frommelt, Hans Ravnkjær Larsen og Arnold Skimminge

Vejleder: Johnny Ottesen

357/98 *Cayleys Problem*

En historisk analyse af arbejdet med Cayley problem fra 1870 til 1918

Et matematisk videnskabsfagsprojekt af: Rikke Degn, Bo Jakobsen, Bjarke K.W. Hansen, Jesper S. Hansen, Jesper Udesen, Peter C. Wulff

Vejleder: Jesper Larsen

358/98 *Modeling of Feedback Mechanisms which Control the Heart Function in a View to an Implementation in Cardiovascular Models*

Ph.D. Thesis by: Michael Danielsen

359/99 *Long-Term Scenarios for Global Energy Demand and Supply Four Global Greenhouse Mitigation Scenarios*

by: Bent Sørensen

360/99 ***SYMMETRI I FYSIK***

En Meta-projektrapport af: Martin Niss, Bo Jakobsen & Tune Bjarke Bonné

Vejleder: Peder Voetmann Christiansen

361/99 *Symplectic Functional Analysis and Spectral Invariants*

by: Bernhelm Booss-Bavnbek, Kenro Furutani

362/99 *Er matematik en naturvidenskab? - en udspænding af diskussionen*

En videnskabsfagsprojekt-rapport af Martin Niss

Vejleder: Mogens Nørgaard Olesen

363/99 ***EMERGENCE AND DOWNWARD CAUSATION***

by: Donald T. Campbell, MArk H. Bickhard and Peder V. Christiansen

364/99 *Illustrationens kraft*

Visuel formidling af fysik

Integreret speciale i fysik og kommunikation

af: Sebastian Horst

Vejledere: Karin Beyer, Søren Kjørup

365/99 *To know - or not to know - mathematics, that is a question of context*

by: Tine Wedege

366/99 ***LATEX FOR FORFATTERE***

En introduktion til LATEX og IMFUFA-LATEX

af: Jørgen Larsen

367/99 **Boundary Reduction of Spectral invariants and Unique Continuation Property**

by Bernhelm Booss-Bavnbek

368/99 Kvartalsrapport for projektet
Scenarier for samlet udnyttelse af brint som energibærer i Danmarks fremtidige energisystem
Projektleder: Bent Sørensen
Opdateret til halvvejsrapport. Den nye udgave Tekst 368bis" kan hentes ned fra internettet fra adressen
http://mmf.ruc.dk/energy/report

369/99 Dynamics of Complex Quadratic Correspondences
by: Jacob Jalving

370/99 OPGAVESAMLING
Bredde-Kursus i Fysik 1976 – 1999
(erstatter tekst nr. 350/98)

371/99 Bevisets stilling -
beviser og bevisførelse i en gymnasial matematikundervisning
Matematikspeciale af: Maria Hermannsson
Vejleder: Mogens Niss

372/99 En kontekstualiseret matematikhistorisk analyse af ikke lineær programmering: Udviklingshistorie og multipel opdagelse
Ph.d.-afhandling af Tinne Hoff Kjeldsen

373/99 Criss-Cross Reduction of the Maslov Index and a Proof of the Yosida-Nicolaescu Theorem
by: Bernhelm Booss-Bavnbek, Kenro Furutani and Nobukazu Otsuki

374/99 Det hydrauliske spring
Et eksperimentelt studie af polygoner
Og hastighedsprofiler
Specialeafhandling af Anders Marcussen
Vejledere: Tomas Bohr, Clive Ellegaard og Bent C. Jørgensen

375/99 Begrundelser for Matematikundervisningen
i den lærde skole hhv. gymnasiet 1884-1914
Historie-speciale af: Henrik Andreassen

376/99 Universality of AC conduction in
disordered solids
by: Jeppe C. Dyre, Thomas B. Schrøder

377/99 The Kuhn-Tucker Theorem in
Nonlinear Programming:
A Multiple Discovery?
by: Tinne Hoff Kjeldsen

....

378/00 Solar energy preprints:
Renewable energy sources and thermal energy storage
Integration of photovoltaic cells into the global
Energy system
by: Bent Sørensen

379/00 EULERS DIFFERENTIALREGNING
Eulers indførelse af differentialregningen stillet
over for den moderne
En tredjesemesters projektrapport på den
naturvidenskabelige basisuddannelse
af: Uffe Thomas Volmer Jankvist, Rie Rose Møller
Pedersen, Maja Bagge Petersen
Vejleder: Jørgen Larsen

380/00 Matematisk Modellering af Hjertefunktionen
Isovolumetrisk ventrikulær kontraktion og
udpumpning til det cardiovaskulære system
Speciale/3.moduls-rapport
af: Gitte Andersen, Jakob Hilmer, Stine Weisbjerg
Vejleder: Johnny Ottesen

381/00 Matematikviden og teknologiske kompetencer hos
kortuddannede voksne
- Rekognosceringer og konstruktioner i
grænselandet mellem matematikkens
didaktik og forskning i voksenuddannelse
Ph.d.-afhandling af Tine Wedege

382/00 Den selvundvigende vandring
Et matematisk professionsprojekt
af: Martin Niss, Arnold Skimminge
Vejledere: John Villumsen, Viggo Andreassen

383/00 Beviser i matematik
af: Anne K.S.Jensen, Gitte M.Jensen,
Jesper Thrane, Karen L.A.W.Wille,
Peter Wulff
Vejleder: Mogens Niss

384/00 Hopping in Disordered Media:
A Model Glass Former and A Hopping Model
Ph.d. thesis by Thomas B. Schrøder
Supervisor: Jeppe C. Dyre

385/00 The Geometry of Cauchy Data Spaces
by: Bernhelm Booss-Bavnbek, K. Furutani,
K.P. Wojciechowski