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THE MASLOV INDEX: A FUNCTIONAL ANALYTICAL DEFINITION AND THE SPECTRAL FLOW FORMULA

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ABSTRACT. We give a functional analytical definition of the Maslov index for continuous paths in the Fredholm-Lagrangian Grassmannian without any assumptions at the endpoints and crossings. Our definition naturally appears when studying families of Dirac operators on a closed manifold partitioned by a hypersurface and the families of their Cauchy data spaces. We also show the relation between our definition and the definition given by J. Robbin and D. Salamon. We show by purely geometrical and functional analytical means that the abstract Cauchy data spaces of a closed symmetric operator are closed and Lagrangian and depend continuously on the operator. We define the spectral flow for continuous paths of self-adjoint unbounded Fredholm operators and prove our main theorem: its coincidence with the Maslov index of the corresponding family of abstract Cauchy data spaces.

CONTENTS

Introduction

In this paper we give a purely functional analytical proof of how the spectral flow of a continuous curve of self-adjoint (unbounded) Fredholm operators coincides with the Maslov index of the corresponding curve of abstract Cauchy data. This is one instance of a widely exploited procedure in differential geometry and mathematical physics, namely of identifying two quantities as being the same invariants, although they are defined in a completely different way.

We have taken as model the coincidence of the (analytical) Fredholm index of an elliptic operator on a closed manifold, solely defined in terms of the proper operator, with its topological index, solely defined in terms of the principal symbol. But instead of the Fredholm index, we consider the spectral flow of a family of Dirac operators over a closed manifold, and instead of the topological index, we consider the Maslov index. It is obtained by partitioning the manifold into two submanifolds with a common boundary hypersurface Σ and then counting the intersections of the corresponding family of Lagrangian Fredholm pairs of Cauchy data spaces along Σ.

This type of phenomena was first observed by Floer who examined in [10] a concrete and interesting example arising from Lagrangian intersection. Then Yoshida, [31] established a formula for the coincidence of the spectral flow with the Maslov index of a family of Dirac operators on a 3-dimensional closed manifold and applied it to the study of 3-dimensional manifold topology. More recently Nicolaescu, [22], [23], [25] announced a generalization of the *Spectral Flow Formula* (i.e. spectral flow = Maslov index) to higher dimensions for families of Dirac operators under certain assumptions and derived so-called splitting formulas for these quantities. Now there are many papers discussing this kind of topic, see e.g. Bunke [7], Furutani and Otsuki, [26], [11], Kirk and Klassen, [16], [17], [18], [19].

The purpose of this paper is to contribute to the understanding of the functional analytical aspect of cutting a manifold by a hypersurface and to give an elementary, purely functional analytical framework for the Spectral Flow Formula. To us, the functional analytical meaning of the geometric situation is to attain a closed symmetric operator with suitable properties from a self-adjoint operator by the splitting process; i.e. an inverse procedure of the classical von Neumann approach of seeking self-adjoint extensions. One of our aims of ignoring all features solely connected with the geometric formulation and keeping the arguments on the functional analytical level is to select only those assumptions which are essential for proving the Spectral Flow Formula.

Our functional analytical approach requires a new definition of the Maslov index. Ever since the pioneering work by Maslov and Arnold, [3], Leray, [20], et al. and for the infinite-dimensional case by Swanson, [30] and Nicolaescu, [22], [23], the Maslov index of cycles has been well studied. Recently Robbin and Salamon, [29] (see also Cappell, Lee, and Miller, [8]) have achieved a generalization for paths by deforming a continuous path into a smooth curve with only regular crossings with the Maslov cycle. We give a new definition not only for the finite-dimensional case but also for the infinite-dimensional case, which is meaningful for any continuous path without requiring any deformation and without any assumptions at the endpoints and crossings.

Our definition was inspired by a recent reformulation of the spectral flow in Phillips, [28] and is, in particular, based on the early study of the Fredholm-Lagrangian Grassmannian in Swanson, [30]. For cycles, our definition gives the usual Maslov index. For our definition, the additivity of the Maslov index under catenation follows immediately; this is the most important property for proving our Spectral Flow Formula (Theorem 5.1).

As mentioned already, special forms of the Spectral Flow Formula were obtained before, but in a way which

- requires differentiable curves of symmetric operators and Lagrangian spaces; invertibility at the endpoints; and regular crossings for the original curve, whereas our approach needs no assumptions other than continuity of the original curve. We simply exploit the differentiability and regular properties of the curve pieces linked up in a two-parameter construction.
- exploits special properties of Dirac operators on odd-dimensional manifolds, partly under strong assumptions about the metrics of the underlying manifold and bundles close to the boundary ('cylindrical ends'), whereas we keep the calculations on the Hilbert space level and only assume symmetry of the operator A; the existence of an extension with compact resolvent; and the non-existence of inner solutions, i.e. properties which are automatically fulfilled for all Dirac operators and operators of Dirac type.
- reduces the calculations to the case of one-dimensional intersections of Lagrangian subspaces which are trivial in symplectic algebra (there are no problems with sign changes) in return for a deformation of the curve, whereas we embed the original curve in a simple two-parameter family and then make the calculations for arbitrary finite- (not necessarily one-) dimensional intersections.
- requires calculation of the spectral flow and the Maslov index for the whole unit interval in order to prove the coincidence, whereas for our approach it suffices to prove the identity in arbitrarily small intervals, but for all kinds of situations at the endpoints, and then to apply a catenation argument.

The essence of the present approach is the following: To avoid the delicacies of the symplectic reduction to normal crossings and one-dimensional intersections, we concentrate on small intervals of the parameter space and small intervals of the spectrum.

We proceed as follows:

In Section 1 we explain all the technical notions of our use of the symplectic functional analysis. They are not new except for the definition of the Maslov index itself.

In Section 2 we review the definition by Robbin and Salamon and describe the relation between their definition and ours.

In Section 3 we give an example inspired by the Krein-Višik-Birman theory of self-adjoint extensions. We equip the space of abstract boundary values with a symplectic form and show that the (abstract) Cauchy data spaces are Lagrangians and vary continuously for symmetric, closed operators which have a self-adjoint Fredholm extension and no inner solutions. The example provides a general framework for dealing with families of self-adjoint elliptic operators on closed partitioned manifolds. The example also illustrates the need of establishing a purely functional analytical definition of the Maslov index for infinite-dimensional settings.

In Section 4 we review the definition of the spectral flow of a continuous family of bounded self-adjoint Fredholm operators by Phillips and adapt it to continuous families of unbounded Fredholm operators. To that purpose we discuss two different ways of assigning a continuous family of bounded Fredholm operators to the original family and show that they yield the same spectral flow.

In Section 5 we prove our main theorem, the coincidence of the spectral flow with the Maslov index of the corresponding family of (abstract) Cauchy data. In the proof we make use of both definitions of the Maslov index, ours and that of Robbin and Salamon.

In a subsequent paper we shall work out the application of our main theorem to the concrete geometrical situation of a partitioned manifold and the splitting formulas of spectral invariants.

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1. The Grassmannian of Lagrangian Fredholm Pairs

In this section, we develop the real and complex functional analysis of infinite-dimensional Lagrangians and obtain a well-defined complex symmetric generator W_Λ. The definition might seem artificial at first, and so the exact mathematical meaning of the operator W_Λ will be worked out, namely its role as a spectral counter for a straight functional analytical definition of the Maslov index.

1.1. **Symplectic Functional Analysis.** The basics of finite-dimensional symplectic linear algebra can easily be transferred to infinite dimensions. We fix the following notation:

Let $(H, \langle .,. \rangle, \omega)$ be a fixed symplectic, separable real Hilbert space and let $J : H \to H$ denote the corresponding almost complex structure defined by $\omega(x,y) = \langle Jx, y\rangle$ with $J^2 = -\mathrm{Id}$, ${}^tJ = -J$, and $\langle Jx, Jy\rangle = \langle x, y\rangle$. Here ${}^tJ$ denotes the transpose of J with regard to the (real) inner product $\langle .,. \rangle$. Let $\mathcal{L} = \mathcal{L}(H)$ denote the set of all Lagrangian subspaces of H (i.e. $\Lambda = (J\Lambda)^\perp$). It is naturally identified with the space $\mathcal{C}$ of self-adjoint involutions of H which anti-commute with J, the correspondence being given by

$$\mathcal{L} \ni \Lambda \mapsto C := 2P_\Lambda - \mathrm{Id} \in \mathcal{C} \quad \text{and} \quad \mathcal{C} \ni C \mapsto \Lambda := \{x \in H \mid Cx = x\},$$

where P_Λ denotes the orthogonal projection onto Λ. We topologize $\mathcal{L}$ by the topology of $\mathcal{C}$ as a subset in the space $\mathcal{B}(H)$ of bounded operators on H.

Topologically the space $\mathcal{L}$ itself is not interesting, since it is contractible. To get something topologically meaningful we fix one Lagrangian subspace Λ_0 and investigate

- the *Fredholm-Lagrangian Grassmannian* of H at Λ_0
$$\mathcal{FL}_{\Lambda_0} := \{\Lambda \in \mathcal{L} \mid (\Lambda, \Lambda_0) \text{ Fredholm pair}\}$$
(i.e. $\dim \Lambda \cap \Lambda_0 < \infty$ and $\operatorname{codim} \Lambda + \Lambda_0 < \infty$, hence $\Lambda + \Lambda_0$ closed) and
- the *Maslov cycle*
$$\mathcal{M}_{\Lambda_0} := \mathcal{FL}_{\Lambda_0} \setminus \mathcal{FL}^{(0)}_{\Lambda_0},$$
where $\mathcal{FL}^{(0)}_{\Lambda_0}$ denotes the subset of Lagrangians intersecting Λ_0 transversally, i.e. $\Lambda \cap \Lambda_0 = \{0\}$.

Arnold, [3] pointed out the interesting topological and geometric properties of the Maslov cycle in finite-dimensional symplectic vector space. These properties are widely exploited for establishing the Maslov index by topological means, but we proceed to the relevant operator spaces in an infinite-dimensional symplectic Hilbert space. The underlying Fredholm-Lagrangian Grassmannian was first considered in Swanson, [30].

To express the dimension of the intersection of an arbitrary $\Lambda \in \mathcal{FL}_{\Lambda_0}$ with Λ_0 fixed by the kernel of a suitably associated operator, we embed our Lagrangians in the naturally associated *complex* Fredholm-Lagrangian Grassmannian and apply suitable groups of unitary operators.

Using the almost complex structure J we first consider the space H as a *complex Hilbert space.* We denote it by H. The complex inner product of H is given by

$$\langle x, y\rangle_{\mathbf{C}} := \langle x, y\rangle - \sqrt{-1}\,\omega(x,y).$$

In the following, however, we shall stick to the inner product $\langle .,. \rangle$ of the real Hilbert space whenever we take an orthogonal complement.

Let $\mathcal{U}(\mathsf{H})$ denote the group of all unitary operators on H and $\mathcal{U}_c(\mathsf{H})$ the subgroup which preserves the Fredholm property, i.e. the operators of $\mathcal{U}(\mathsf{H})$ of the form $\mathrm{Id} + K$, where K is a compact operator. Then $\mathcal{U}_c(\mathsf{H})$ acts transitively on $\mathcal{FL}_{\Lambda_0}$ in a natural way. Let

$$\begin{array}{rccc} \rho: & \mathcal{U}_c(\mathsf{H}) & \longrightarrow & \mathcal{FL}_{\Lambda_0} \\ & U & \mapsto & U(\Lambda_0^\perp) \end{array}$$

denote the mapping defined by this action. It is the projection of the principal fibre bundle $\mathcal{U}_c(\mathsf{H})$ onto its base space $\mathcal{FL}_{\Lambda_0}$ (see Swanson, [30], Lemma 3). This ρ-correspondence between unitary operators and Fredholm Lagrangians ensures that any curve $\Lambda(t)$ can be lifted to a curve U_t. But this is not very suitable for defining the Maslov index, since the representation is not unique and, as a consequence, the spectrum of U_t contains redundant information on the intersection behaviour of the curve with the Maslov cycle. We show how to get around that problem by introducing a new conjugate ${}^TU_t$ and then forming the operator $U_t\,{}^TU_t$.

Since H can be considered the complexification $\Lambda_0 \otimes \mathbf{C} \cong \Lambda_0 + J\Lambda_0 = H$ of Λ_0, we attain a *complex conjugation* by

$$\begin{array}{ccccc} z = & x\otimes 1 + y\otimes\sqrt{-1} & \mapsto & x\otimes 1 - y\otimes\sqrt{-1} & =: \bar{z} \\ & \| & & \| & \\ & x + Jy & & x - Jy & \end{array}$$

where $x, y \in \Lambda_0$. Correspondingly, we denote by $\bar{A}$ and ${}^TA$ the bounded operators on H given by $\bar{A}(z) := \overline{A(\bar{z})}$ resp. ${}^TA := \overline{A^*}$ for $A \in \mathcal{B}(\mathsf{H})$. Notice that in difference to the real transpose ${}^tA$, the new conjugate ${}^TA$ is taken in the category of complex operators and with respect to the fixed Λ_0.

We must distinguish the *complexification* $H\otimes\mathbf{C}$ from H. The space $H\otimes\mathbf{C}$ splits into two subspaces $H\otimes\mathbf{C} = \mathsf{E}_- \oplus \mathsf{E}_+$ where $\mathsf{E}_\pm$ are the eigenspaces of $J\otimes\mathrm{Id}$ for the eigenvalues $\pm\sqrt{-1}$. We define the set $\mathcal{L}^{\mathbf{C}}$ of *complex Lagrangian subspaces* of $H\otimes\mathbf{C}$ by

$$L + (J\otimes\mathrm{Id})L = H\otimes\mathbf{C} \quad\text{and}\quad \langle L, (J\otimes\mathrm{Id})L\rangle^{\mathbf{C}} = 0,$$

where $\langle\cdot,\cdot\rangle^{\mathbf{C}}$ denotes the inner product in $H\otimes\mathbf{C}$, and obtain a natural embedding of $\mathcal{L}$ in $\mathcal{L}^{\mathbf{C}}$ given by

$$\mathcal{L} \ni \Lambda \mapsto \tau(\Lambda) := \Lambda\otimes\mathbf{C} \in \mathcal{L}^{\mathbf{C}}.$$

On $\mathcal{L}^{\mathbf{C}}$ acts the group $\mathcal{G}$ of unitary operators which commute with $J\otimes\mathrm{Id}$ or, equivalently, keep $\mathsf{E}_\pm$ invariant, so that $\mathcal{G}$ is isomorphic to $\mathcal{U}(\mathsf{E}_-)\times\mathcal{U}(\mathsf{E}_+)$. Correspondingly, the subgroup $\mathcal{G}_c$ of $\mathcal{G}$, consisting of operators of the form $\mathrm{Id} + K$ with K compact operator on $H\otimes\mathbf{C}$, splits into

$$\mathcal{G}_c \cong \mathcal{U}_c(\mathsf{E}_-)\times\mathcal{U}_c(\mathsf{E}_+).$$

It acts transitively on the *complex Fredholm-Lagrangian Grassmannian* $\mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$ which consists of all $L\in\mathcal{L}^{\mathbf{C}}$ forming a Fredholm pair with $\Lambda_0\otimes\mathbf{C}$. Let

$$\begin{array}{rccc} \rho^{\mathbf{C}}: & \mathcal{G}_c & \longrightarrow & \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}} \\ & g & \mapsto & g(\Lambda_0^\perp\otimes\mathbf{C}) \end{array}$$

denote the map defined by this action. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_c(\mathsf{H}) & \xrightarrow{\tilde{\tau}} & \mathcal{G}_c \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho^{\mathbf{C}}} \\ \mathcal{FL}_{\Lambda_0} & \xrightarrow{\tau} & \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}, \end{array}$$

where $\tilde{\tau}$ denotes the complexification $U \mapsto U \otimes \mathrm{Id} = \begin{pmatrix} \overline{U} & 0 \\ 0 & U \end{pmatrix}$. Here we identify $\mathcal{U}_c(\mathsf{H})$ with $\mathcal{U}_c(\mathsf{E}_\pm)$ by the (complex and anti-linear) isomorphisms

$$\begin{array}{ccc} \mathsf{H} & \xrightarrow{\cong} & \mathsf{E}_\pm \\ z & \mapsto & \frac{z\otimes 1 \mp Jz\otimes\sqrt{-1}}{\sqrt{2}} \end{array},$$

so that $\overline{U}$ operates on E_- and U on E_+ in the preceding matrix.

To get an isomorphism between $\mathcal{U}_c(\mathsf{H})$ and $\mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$, we use instead of $\tilde{\tau}$ the mapping

$$\begin{array}{cccc} \Phi : & \mathcal{U}_c(\mathsf{H}) & \longrightarrow & \mathcal{G}_c \\ & U & \mapsto & \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & U \end{pmatrix} \end{array},$$

where Id operates on E_- and U is considered an operator $\mathsf{E}_+ \to \mathsf{E}_+$. Splitting

$$\mathcal{G}_c = \left\{ \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & U \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \right\} = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right\},$$

we see that the range of Φ and the second factor of $\mathcal{G}_c$ intersect only in the identity and that the right inverse of Φ is given by $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \mapsto VU^{-1}$, where U^{-1} is considered an operator from E_+ to E_+ by successively identifying $\mathsf{E}_- \cong H_{\mathbf{C}} \cong \mathsf{E}_+$.

The preceding facts prove

Proposition 1.1. *For any real symplectic, separable Hilbert space H with fixed Lagrangian Λ_0, we have a homeomorphism*

$$\rho^{\mathbf{C}} \circ \Phi : \mathcal{U}_c(\mathsf{H}) \xrightarrow{\sim} \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}\,.$$

Note that the inverse $\beta : \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}} \to \mathcal{U}_c(\mathsf{H})$ is given on the range of τ by $\Lambda\otimes\mathbf{C} \mapsto U\overline{U}^{-1} = U^TU$, where $U \in \mathcal{U}_c(\mathsf{H})$ is chosen to express Λ in the form $U(\Lambda_0^\perp)$. That permits us to introduce our key operator for the direct functional analytical definition of the Maslov index:

Definition 1.2. For any $\Lambda \in \mathcal{FL}_{\Lambda_0}$ we define *the complex symmetric generator* of Λ (with regard to Λ_0) by

$$W_\Lambda := \beta(\Lambda\otimes\mathbf{C}) \overset{\text{Prop.1.1}}{=} U^TU \in \mathcal{U}_c(\mathsf{H})\,.$$

Note . As emphasized before, the operator U is not invariantly defined by Λ, but $W_\Lambda = U^TU$ is. This was already found by Leray, [20], Lemma 2.1, by direct calculation in the *real* Grassmannian. In our context, the invariance of W_Λ is just a geometric property of the *complex* Grassmannian, namely that the principal fibre bundle given by the action of the group $\mathcal{G}_c$ on $\mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$ has a global section provided by β. Moreover, the set of all such W_Λ is exactly the subset of all $W \in \mathcal{U}_c(\mathsf{H})$ with $W = {}^TW$.

Indeed, the operator W_Λ provides a basic counting device for defining the Maslov index, since we have also in the infinite-dimensional case

$$\ker W_\Lambda + \mathrm{Id} \cong \Lambda \cap \Lambda_0 + J(\Lambda \cap \Lambda_0) \cong (\Lambda \cap \Lambda_0) \otimes \mathbf{C} \subset \Lambda_0 \otimes \mathbf{C} \cong \mathsf{H}\,,$$

as noticed before in Arnold, [3], and Leray, [20] in the finite-dimensional case. We emphasize:

Lemma 1.3. *For any* $\Lambda \in \mathcal{FL}_{\Lambda_0}$ *we have*

$$\dim_{\mathbf{R}}(\Lambda \cap \Lambda_0) = \dim_{\mathbf{C}} \ker(W_\Lambda + \mathrm{Id}).$$

1.2. **A Proper Functional Analytical Definition of the Maslov Index for Arbitrary Paths.** Now let $\{\Lambda(t)\}_{t\in I}, I = [0,1]$ be a continuous path in $\mathcal{FL}_{\Lambda_0}$. Then the family $\{W_{\Lambda(t)}\}_{t\in I}$ of unitary operators on H is also a continuous family in the operator norm. To define the Maslov index we now adapt an idea from Phillips, [28]. He applied it for defining the spectral flow of a continuous path of self-adjoint, bounded Fredholm operators directly without deforming the path in a 'general' position, i.e. without demanding the operators at the endpoints to be invertible but admitting singular operators, and without demanding strictly increasing or strictly decreasing movement of the eigenvalues when passing zero but admitting arbitrary movements for varying t.

Let an operator family be given, which is parametrized by the unit interval $[0,1]$, and assume that we want to count the net number of eigenvalues, counted with multiplicities, which pass through a fixed gauge in the positive direction. Phillips' spectral oscillations (entering into the definition of the spectral flow) are on the real line around zero; ours (entering into the definition of the Maslov index) on the unit circle around $e^{i\pi}$. *If* one could hedge the oscillations into an interval $[-a,a]$ (or an arc $e^{i(\pi\pm a)}$) so that no eigenvalues could leak through the boundary $\pm a$ when the parameter runs from 0 to 1, then a reasonable definition of the number of crossings would be the difference between the number of eigenvalues, counted with their multiplicities, lying between 0 and a at the right end of the interval, minus the corresponding number of eigenvalues between 0 and a at the left end of the interval. That definition does not require any assumptions about regularity at the ends of the interval or of the crossings, and hence no deformations.

In general, such hedgings will not be possible, since ever new eigenvalues will cross in or leak out no matter how large or small a otherwise is chosen. But Phillips observed that the strategy works locally and can be patched together in the following way (adapted to our situation): We choose a partition $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ of the interval and positive numbers $0 < \varepsilon_j < \pi,\ j = 1, \ldots, N$ such that

$$\ker(W_{\Lambda(t)} - e^{i(\pi\pm\varepsilon_j)}) = \{0\} \tag{1.1}$$

for $t_{j-1} \le t \le t_j$ (see Figure 1). Here we use the fact that $W_{\Lambda(t)} - e^{i\pi}$ is a Fredholm operator (since $W_{\Lambda(t)}$ is unitary with eigenvalues on the unit

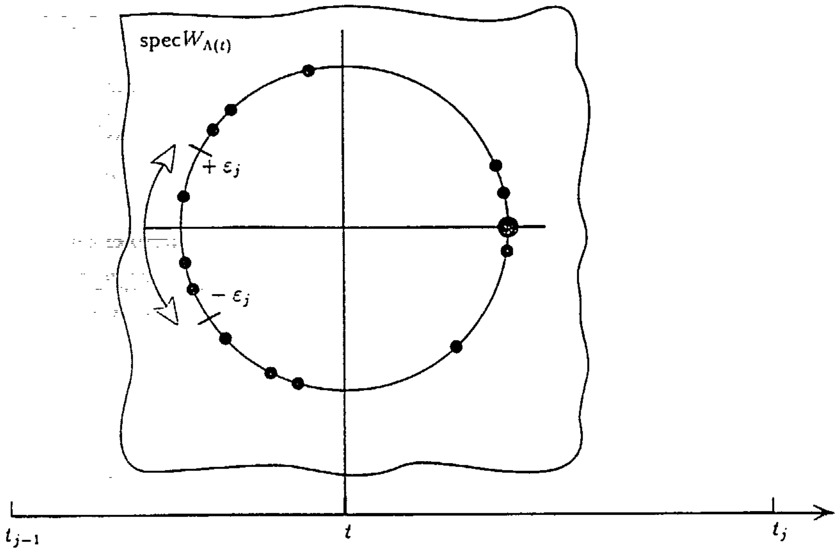

FIGURE 1. Horizontal and vertical spacing of the spectrum of the complex symmetric generator $W_{\Lambda(t)}$

circle and $W_{\Lambda(t)} - \mathrm{Id}$ is compact with discrete eigenvalues and the only accumulation point 0). Moreover, we use the continuity of the eigenvalues so that one ε_j satisfying (1.1) at a point t will also satisfy the equation in close neighbouring points and thus lead to the claimed finite interval $[t_{i-1}, t_i]$.

We can now define the Maslov index of a continuous path as the intersection number with the Maslov cycle $\mathcal{M}_{\Lambda_0}$ in the following way:

Definition 1.4. For any arbitrary continuous path $\Lambda = \{\Lambda(t)\}_{t\in I}$ in the Fredholm-Lagrangian Grassmannian $\mathcal{FL}_{\Lambda_0}$ of a real symplectic Hilbert space H at a fixed Lagrangian Λ_0, we define the *Maslov index* by

$$\mu(\Lambda) = \mu(\Lambda; \Lambda_0) := \sum_{j=1}^{N} k(t_j, \varepsilon_j) - k(t_{j-1}, \varepsilon_j)$$

with

$$k(t, \varepsilon_j) := \sum_{0\leq\theta<\varepsilon_j} \dim\ker(W_{\Lambda(t)} - e^{i(\pi+\theta)}) \quad \text{for } t_{j-1} \leq t \leq t_j\,,$$

where the horizontal and vertical spacing $(t_0, \ldots, t_N), (\varepsilon_1, \ldots, \varepsilon_N)$ is chosen as in (1.1).

By its very construction, our definition of the Maslov index does not depend on the choice of the horizontal or vertical spacing. To see this, it does not suffice to point to the continuity of the eigenvalues alone. We must also explain the role of ε by repeating the point of Phillips' argument: the function of ε_j chosen as in (1.1) is to lock the eigenvalues for $t \in [t_{j-1}, t_j]$ in the interval $[-\varepsilon, \varepsilon]$. Choosing any other such ε' will also lock the eigenvalues between ε and ε', see Figure 2.

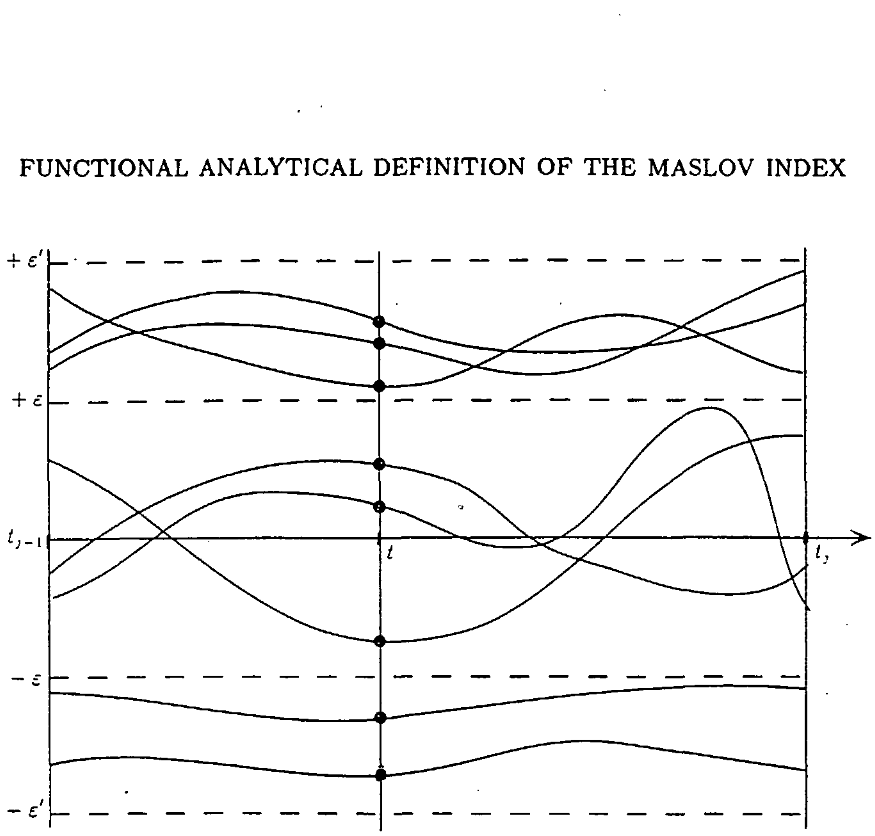

FIGURE 2. The locking of the eigenvalues between ε and ε'

All the 'axioms' for the Maslov index derived in Robbin and Salamon, [29], Theorem 2.3 (and given in similar form in Cappell, Lee, and Miller, [8]) follow at once from the construction of our Maslov index. We emphasize the following properties (bypassing the 'localization' and 'product' properties which go without saying for our functional analytical definition):

Theorem 1.5.

(I) *The Maslov index is well defined for* homotopy *classes of paths with fixed endpoints and distinguishes the homotopy classes. In particular, the Maslov index is invariant under* re-parametrization *of paths.*

(II) *The Maslov index is additive under* catenation, *i.e.*

$$\mu(\Lambda_1 * \Lambda_2) = \mu(\Lambda_1) + \mu(\Lambda_2),$$

where $\{\Lambda_1(t)\}, \{\Lambda_2(t)\}$ *are continuous paths with* $\Lambda_1(1) = \Lambda_2(0)$ *and*

$$(\Lambda_1 * \Lambda_2)(t) := \begin{cases} \Lambda_1(2t) & 0 \le t \le \frac{1}{2} \\ \Lambda_2(2t-1) & \frac{1}{2} < t \le 1 . \end{cases}$$

(III) *The Maslov index is* natural *under the action of the group* $\mathrm{Sp}(H)$ *of symplectic automorphisms of* H:

$$\mu(\Psi\Lambda; \Psi\Lambda_0) = \mu(\Lambda; \Lambda_0) \quad \text{for } \Lambda \in \mathcal{FL}_{\Lambda_0} .$$

(IV) *The Maslov index vanishes for paths which stay in one* stratum $\mathcal{FL}^{(k)}_{\Lambda_0}$ *of the stratified space* $\mathcal{FL}_{\Lambda_0} = \cup_{k=0}^{\infty} \mathcal{FL}^{(k)}_{\Lambda_0}$, *i.e. if* $\dim \Lambda(t) \cap \Lambda_0 = k$ *for one* $k \ge 0$ *and all* $t \in I$.

To define the Maslov index, we embedded the real Fredholm Grassmannian $\mathcal{FL}_{\Lambda_0}$ in the complex Grassmannian $\mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$ with the inclusion given by complexification. Clearly, there are many more Lagrangian curves in $\mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$ than those coming from $\mathcal{FL}_{\Lambda_0}$. Also the *complex Maslov cycle* defined by

$$\mathcal{M}_{\Lambda_0\otimes\mathbf{C}} := \{L \in \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}} \mid L \cap (\Lambda_0 \otimes \mathbf{C}) \neq \{0\}\}$$

is substantially larger than the real Maslov cycle as defined before. Generalizing Lemma 1.3 we can characterize the complex Maslov cycle by the property that for all its L, the operator $\beta(L)$ has eigenvalue -1. This leads to a genuinely *complex* version of Definition 1.4:

Definition 1.6. For any continuous family $\{L(t)\} \in \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$ we define the *complex Maslov index* by

$$\mu^{\mathbf{C}}(\{L(t)\}) := \sum_{j=1}^{N} k(t_j, \varepsilon_j) - k(t_{j-1}, \varepsilon_j)\,,$$

where we replace the operator $W_{\Lambda(t)}$ by the operator $\beta(L(t))$ in the definition of the multiplicities $k(t_., \varepsilon_.)$.

Notice that Theorem 1.5 remains valid in the complex case. Furthermore, we see at once that $\mu^{\mathbf{C}}$ is the intersection number of the family $L(t)$ with the complex Maslov cycle, if $\{L(t)\}$ is in 'general position'. It is not difficult to derive the following formula:

Proposition 1.7. *Let $\{\Lambda(t)\} \in \mathcal{FL}_{\Lambda_0}$ and $\{L(t)\} \in \mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$ be two continuous families which have the same endpoints. They are homotopic in $\mathcal{FL}^{\mathbf{C}}_{\Lambda_0\otimes\mathbf{C}}$, if and only if*

$$\mu(\{\Lambda(t)\}) = \mu^{\mathbf{C}}(\{L(t)\}).$$

Remark 1.8. For loops, i.e. for $\Lambda(0) = \Lambda(1)$, and for finite-dimensional H, we notice that $\mu(\{\Lambda(t)\})$ is the winding number of the closed curve $\{\det W_{\Lambda(t)}\}_{t\in S^1}$. This is the original definition of the Maslov index as explained in Arnold, [3]. In a trivial way it can be transferred to infinite-dimensional inductive limits and was then generalized by Swanson, [30] to cycles in the full Fredholm-Lagrangian Grassmannian exploiting a homotopy argument by Palais, [27]. Similarly, we get for genuinely complex Lagrangian loops that $\mu^{\mathbf{C}}(\{L(t)\})$ is the winding number of $\{\det \beta(L(t))\}_{t\in S^1}$.

2. The Relation Between the Differential and Functional Analytical Definition of the Maslov Index

In this section we first recall the definition of the Maslov index for paths given in Robbin and Salamon, [29]; then we show the relation with our definition given in the preceding section.

2.1. Review of the Differential Definition. Here we assume that the symplectic vector space H is finite-dimensional. Notice though, that the definition of the Maslov index given by Robbin and Salamon can be extended immediately to the infinite-dimensional case following Swanson [30], Theorem 1.2, where the differentiable structure for $\mathcal{FL}_{\Lambda_0}$ was achieved (see also Nicolaescu [23]). In our proof of the Spectral Flow Formula (in Section 5) we shall apply their definition in the infinite-dimensional form to a trivial analytic family. Restricting ourselves to the finite-dimensional case makes the presentation more easy, since we can then identify $\mathcal{FL}_{\Lambda_0}$ with $\mathcal{L}$ and $\mathcal{U}_c(\mathsf{H})$ with $\mathcal{U}(\mathsf{H})$. We still fix one Lagrangian subspace Λ_0. Then, roughly speaking, Robbin and Salamon's *differential* approach for defining the Maslov index is based on three observations:

(i) Tangent vectors $(\Lambda, \dot\Lambda) \in T_\Lambda(\mathcal{L})$ can be regarded as symmetric bilinear forms $Q_{(\Lambda,\dot\Lambda)}$ on Λ in a natural way.

(ii) Any continuous path of Lagrangian subspaces is homotopic to a C^1 (or smooth) path $\{\Lambda(t)\}_{t\in I}$ which has only regular crossings with the Maslov cycle $\mathcal{M}_{\Lambda_0}$. Here *regular crossing* at $t \in I$ means that the symmetric bilinear form

$$Q_{(\Lambda(t),\dot\Lambda(t))}|_{\Lambda(t)\cap\Lambda_0} : \Lambda(t) \cap \Lambda_0 \times \Lambda(t) \cap \Lambda_0 \longrightarrow \mathbf{R}$$

is non-singular.

(iii) Since the regular crossings are isolated, adding the signatures and possible corrections at the ends of the path defines a number, the Robbin-Salamon (differential) Maslov index, which does not depend on the chosen homotopic deformation.

To explain observation (i), we set for a $(\Lambda, \dot\Lambda) \in T_\Lambda(\mathcal{L})$ and $x, y \in \Lambda$:

$$Q_{(\Lambda,\dot\Lambda)}(x,y) := \frac{d}{ds}\omega(x, B_s y)|_{s=0}\,, \tag{2.1}$$

where the family $\{B_s : \Lambda \to J\Lambda\}_{|s|\ll 1}$ of linear maps is chosen in such a way that its graph $\Lambda(s) := \{x + B_s x \mid x \in \Lambda\}$ becomes a C^1-curve through Λ at $s = 0$ with $\frac{d}{ds}\Lambda(s)|_{s=0} = \dot\Lambda$.

We shall not explain the deformation process underlying observation (ii), but now assume that $\{\Lambda_t\}_{t\in I}$ is a C^1-curve with only regular crossings with the Maslov cycle $\mathcal{M}_{\Lambda_0}$. We recall from [29] Robbin and Salamon's definition of the (differential) Maslov index by

$$\begin{aligned} \mu^{RS}(\Lambda;\Lambda_0) := {} & \frac{1}{2}\operatorname{sign} Q_{(\Lambda(0),\dot\Lambda(0))|\Lambda(0)\cap\Lambda_0} \\ & + \sum_{0<t<1} \operatorname{sign} Q_{(\Lambda(t),\dot\Lambda(t))|\Lambda(t)\cap\Lambda_0} + \frac{1}{2}\operatorname{sign} Q_{(\Lambda(1),\dot\Lambda(1))|\Lambda(1)\cap\Lambda_0}\,. \end{aligned} \tag{2.2}$$

The independence of the various choices claimed in observation (iii) follows immediately from the relation between Robbin and Salamon's differential and our functional analytical definition of the Maslov index for smooth paths.

2.2. **The Relation between the Differential and Functional Analytical Definition for Smooth Paths.** Let us assume that the curve $\{\Lambda(t)\}_{t\in I}$ is of C^2–class and has only regular crossings with the Maslov cycle $\mathcal{M}_{\Lambda_0}$. We show that our functional analytical definition of the Maslov index coincides with Robbin and Salamon's differential definition except for the corrections at the endpoints of the path, which by Robbin and Salamon sometimes cause the Maslov index to become a half-integer whereas, our definition always provides an integer (as wanted for **Z**-valued homotopy invariants). More precisely we have:

Theorem 2.1. *Under the preceding conditions of C^2– differentiability and regular crossings we have*

$$\mu = \mu^{RS} - \frac{k_{t=0}}{2} + \frac{k_{t=1}}{2}, \tag{2.3}$$

where $k(t)$ denotes the crossing dimension $\dim \Lambda(t) \cap \Lambda_0$.

Proof. First we show that our definition and that of Robbin and Salamon coincide in suitable small intervals by relating the eigenvalues of our symmetric unitary generator $W_{\Lambda(t)}$ on both sides of $e^{i\pi}$ with the positive and negative eigenvalues of the quadratic form $Q_{(\Lambda(t),\dot\Lambda(t))}|_{\Lambda(t)\cap\Lambda_0}$. Later we shall add over all small intervals and compare the eigenvalues at the endpoints of the interval I.

1. step: We consider a small neighbourhood of a point t_0, $0 < t_0 < 1$, where $\Lambda(t_0)\cap\Lambda_0 \neq \{0\}$. Let $U_t \in \mathcal{U}(\mathsf{H})$ be a curve of unitary transformations such that $U_t(J\Lambda_0) = \Lambda(t)$ for $|t - t_0| \ll 1$. If we write $U_t = X_t + \sqrt{-1}\,Y_t$, we can express the quadratic form on the variable space $\Lambda(t)$ of (2.1) as a quadratic form on the fixed space Λ_0 by substituting $x = U_{t_0}Ju$ and $y = U_{t_0}Jv$ with $u, v \in \Lambda_0$. As observed already by Robbin and Salamon, the coordinate change yields

$$Q_{(\Lambda(t_0),\dot\Lambda(t_0))}(U_{t_0}Ju, U_{t_0}Jv) = \langle \dot Y_{t_0}(u), X_{t_0}(v)\rangle - \langle \dot X_{t_0}(u), Y_{t_0}(v)\rangle\,. \tag{2.4}$$

2. step: Note that $\beta(\Lambda(t)\otimes\mathbf{C}) = U_t{}^T U_t =: W_t$ is our $W_{\Lambda(t)}$ of Definition 1.2. Writing $U_t = U_{t_0}e^{iA_t}$ and $W_t = W_{t_0}e^{iS_t}$ with self-adjoint A_t and S_t and $S_{t_0} = 0$ yields

$$Q_{(\Lambda(t_0),\dot\Lambda(t_0))}(-Y_{t_0}u + JX_{t_0}u, -Y_{t_0}v + JX_{t_0}v) = \langle \dot a_{t_0}(u), v\rangle\,, \tag{2.5}$$

where $A_t = a_t + ib_t$. Also we have the unitary equivalence

$${}^TU_{t_0}\dot S_{t_0} = 2\dot a_{t_0}{}^TU_{t_0}\,. \tag{2.6}$$

Equations (2.5) and (2.6) imply that $\operatorname{sign} Q_{(\Lambda(t_0),\dot\Lambda(t_0))|\Lambda(t_0)\cap\Lambda_0}$ coincides with the signatures of $\dot a_{t_0}$ and $\dot S_{t_0}$ on corresponding subspaces of Λ_0. The reason that b_{t_0} disappears in the signature formula is the non-uniqueness of picking U_t.

3. step: Now we relate the signature of the quadratic form at t_0 with the curve of eigenvalues of W_t for $|t - t_0| \ll 1$. We assume that

$$\dim_{\mathbf{R}}(\Lambda(t_0)\cap\Lambda_0) = \dim\ker(W_{t_0} - e^{i\pi}) = k > 0.$$

Now we lock the eigenvalues of W_t at $t = t_0$ ('vertically') by $\varepsilon > 0$ such that

$$\ker(W_{t_0} - e^{i(\pi+\theta)}) = \{0\} \quad \text{for } 0 < |\theta| \le \varepsilon$$

and ('horizontally') by $\delta > 0$ such that $\ker(W_t - e^{i(\pi\pm\varepsilon)})$ remains equal $\{0\}$, hence

$$\sum_{|\theta|\le\varepsilon} \dim\ker(W_t - e^{i(\pi+\theta)}) = k$$

for $|t - t_0| < \delta$.

Let

$$0 < \lambda_1 \le \cdots \le \lambda_p \quad \text{and} \quad 0 > \mu_1 \ge \cdots \ge \mu_q$$

denote the eigenvalues of $\dot{S}_{t_0}|_{\Lambda(t_0)\cap\Lambda_0}$. Since the crossing is assumed to be regular we have no vanishing eigenvalues, hence $p + q = k$. By the assumption that $\Lambda(t)$ is of C^2–class, the transformation W_t has (for t sufficiently close to t_0, say in the interval $[t_0 - \delta, t_0 + \delta]$) eigenvalues $\{\lambda_\ell(t)\}$ and $\{\mu_j(t)\}$ which bifurcate from -1 at t_0 in the following form

$$\lambda_\ell(t) = e^{i(\pi+\lambda_\ell t+O(t^2))}, \qquad \ell = 1, \ldots p,$$
$$\mu_j(t) = e^{i(\pi+\mu_j t+O(t^2))}, \qquad j = 1, \ldots q.$$

It follows that the point t_0, at which $\dim_{\mathbf{R}}(\Lambda(t_0) \cap \Lambda_0) > 0$, is isolated and that

$$\sum_{0\le\theta\le\varepsilon} \dim\ker(W_t - e^{i(\pi+\theta)}) = p \quad \text{and} \quad \sum_{-\varepsilon\le\theta<0} \dim\ker(W_t - e^{i(\pi+\theta)}) = q$$

for $t_0 < t \le t_0 + \delta$ (and vice versa for $t_0 - \delta \le t < t_0$), hence

$$\begin{aligned} (2.7) \quad \mu(\{\Lambda(t)\}_{t_0-\delta\le t\le t_0+\delta}; \Lambda_0) &= k(t_0 + \delta, \varepsilon) - k(t_0 - \delta, \varepsilon) = p - q \\ &= \operatorname{sign} Q_{(\Lambda(t_0),\dot{\Lambda}(t_0))}|_{\Lambda(t_0)\cap\Lambda_0} = \mu^{RS}(\{\Lambda(t)\}_{t_0-\delta\le t\le t_0+\delta}; \Lambda_0). \end{aligned}$$

4. step: We still have to compare the counting at the endpoints if the crossings are not transversal. At the left endpoint $t_0 = 0$ we have $k(0 + \delta, \varepsilon) = p$ and $k(0, \varepsilon) = \dim\Lambda(0) \cap \Lambda_0 = k = p + q$; hence our definition of the Maslov index contributes with $p - (p + q) = -q$, whereas Robbin and Salamon's definition contributes with $\frac{p-q}{2} = \frac{k}{2} - q$. Similarly, at the right endpoint $t_0 = 1$ we get $k(1, \varepsilon') - k(1 - \delta', \varepsilon') = k' - q' = p'$, whereas Robbin and Salamon get $\frac{p'-q'}{2} = p' - \frac{k'}{2}$. That explains the error terms in the formula (2.3) and we see our assertion by the additivity under catenation of paths.

□

3. An Example: The Fredholm Lagrangian of Abstract Cauchy Data Spaces

In this section we give an example of a continuous curve of Fredholm pairs of Lagrangians. We introduce the space B of abstract boundary values of a fixed closed, symmetric operator A in a real Hilbert space and endow B

with a symplectic structure. We show that the naturally defined Cauchy data space is closed and in fact Lagrangian; that it depends continuously on A; and that it forms a Fredholm pair with a fixed Lagrangian. We obtain the results under only two assumptions: that there exists a self-adjoint extension with compact resolvent and that there are no inner solutions.

Our presentation is inspired by the Krein-Višik-Birman theory of self-adjoint extensions for closed, symmetric semi-bounded operators developed in the late 40s and early 50s for characterizing the ellipticity of boundary value problems for strongly elliptic differential operators. We refer to an expository article of 1980 by Alonso and Simon, [1], [2], and the detailed study by Grubb, in [12] and a series of subsequent papers where she worked out the theory also for general elliptic systems without semi-boundedness assumption. It is worth mentioning that one of our results, namely the closedness of the abstract Cauchy data space in the space of abstract boundary values, was already contained in the Krein-Višik-Birman theory, namely where it showed the 'soft extension' to be self-adjoint. New in our presentation is that we treat the space of abstract boundary values explicitly as factor space and equip it with a symplectic structure.

The usual way of introducing the Cauchy data space for a differential operator over a compact manifold with boundary is to take the L_2-closure for traces at the boundary of sufficiently differentiable solutions. This is theoretically not completely satisfactory. Instead, for elliptic operators one can use delicate trace theorems and Poisson type arguments to establish the Cauchy data space directly as a closed subspace of the L_2-sections space over the boundary, namely as the range of the Calderón projector, see e.g. [6], Chapter 13 where all the necessary tools for such a proof are provided and also the Lagrangian property is achieved for Dirac operators. For an alternative, direct proof see Grubb, [13], where - equivalent to the closedness of the Cauchy data space - the self-adjointness of the 'soft extension' is established for elliptic differential operators of first order. Our argument is more algebraic by deriving the Lagrangian property of the Cauchy data spaces (which implies the closedness) and the continuity immediately from the symplectic structure associated with any symmetric closed operator in (real) Hilbert space.

3.1. The Symplectic Space of Abstract Boundary Values.

We assume that H is a real Hilbert space with inner product $\langle .,. \rangle$; that D_m is a dense subspace of H; and that A is a closed symmetric operator in H defined on D_m. Let D_M denote the domain of the adjoint operator A^* of A. We have

$$D_M \supset D_m \quad \text{and} \quad A^*|_{D_m} = A,$$

i.e. A^* is a (closed) extension of A. Let $D_M^{\mathcal{G}}$ and $D_m^{\mathcal{G}}$ denote the corresponding Hilbert spaces equipped with the inner product coming from the graph norm

$$\langle x, y\rangle_{\mathcal{G}} := \langle x, y\rangle + \langle A^*x, A^*y\rangle \quad \text{for } x,\, y \in D_M.$$

Then $D_m^{\mathcal{G}}$ is a closed subspace of $D_M^{\mathcal{G}}$. We denote by $\|x\|_{\mathcal{G}} := \sqrt{\langle x, x\rangle_{\mathcal{G}}} = \sqrt{\langle x, x\rangle + \langle A^*x, A^*x\rangle}$. Set $\mathsf{B} := D_M^{\mathcal{G}}/D_m^{\mathcal{G}}$ with the canonical projection $\gamma : D_M^{\mathcal{G}} \to \mathsf{B}$ and with the quotient norm

$$\|\gamma(x)\|_{\mathsf{B}} := \inf_{a \in D_m} \|x + a\|_{\mathcal{G}} \quad \text{for } \gamma(x) \in \mathsf{B}.$$

We call B the space of *abstract boundary values* and γ the *abstract trace map*. We have a short exact sequence of Hilbert spaces

$$0 \longrightarrow D_m^{\mathcal{G}} \hookrightarrow D_M^{\mathcal{G}} \xrightarrow{\gamma} \mathsf{B} \longrightarrow 0 \tag{3.1}$$

which splits with a right inverse j of γ. Then $\widetilde{\mathsf{B}} := j(\mathsf{B})$ is a closed subspace of $D_M^{\mathcal{G}}$ characterized by

$$\widetilde{\mathsf{B}} \cong \mathsf{B}, \quad D_M^{\mathcal{G}} = D_m^{\mathcal{G}} \oplus \widetilde{\mathsf{B}} \quad \text{and} \quad D_m^{\mathcal{G}} \perp \widetilde{\mathsf{B}}. \tag{3.2}$$

More precisely, we have

Lemma 3.1. *(a) The realization $\widetilde{\mathsf{B}} := j(\mathsf{B})$ of B in $D_M^{\mathcal{G}}$ can be determined as*

$$\widetilde{\mathsf{B}} = \{y \in D_M \mid A^*y \in D_M \text{ and } A^{*2}y = -y\}. \tag{3.3}$$

(b) Let $D \subset D_M^{\mathcal{G}}$ be a subspace including D_m. Then D is closed in $D_M^{\mathcal{G}}$, if and only if $\gamma(D)$ is closed in B.

Proof. (a) Let $b \in \mathsf{B}$, say $b = \gamma(z)$ with $z \in D_M^{\mathcal{G}}$. We split $z = x + y$, where $x \in D_m^{\mathcal{G}}$ and $y \in D_m^{\mathcal{G}\perp}$. Then $y \perp D_m^{\mathcal{G}}$ means that $\langle x, y\rangle + \langle A^*x, A^*y\rangle = 0$. By definition

$$\langle A^*x, A^*y\rangle = \langle x, A^*A^*y\rangle,$$

hence $y \perp D_m^{\mathcal{G}}$, if and only if $\langle x, A^*A^*y\rangle = \langle x, -y\rangle$ for all $x \in D_m^{\mathcal{G}}$, i.e. $(A^*)^2y = -y$.
(b) If D is closed, the factor space $D/D_m^{\mathcal{G}} = \gamma(D)$ is a complete space, hence it is closed in B. □

We introduce a symplectic structure on B by setting

$$\omega([x],[y]) := \langle A^*x, y\rangle - \langle x, A^*y\rangle \quad \text{for } [x],\, [y] \in \mathsf{B}. \tag{3.4}$$

Note . To abbreviate our notation, we shall from time to time write $[x]$ for $\gamma(x)$ for denoting elements of B.

Proposition 3.2. *The form ω is a well-defined skew-symmetric bilinear form on $\mathsf{B} \times \mathsf{B}$ with the following properties:*

(i) *ω is bounded;*
(ii) *ω is non-degenerate.*

Proof. (i) is proved as follows using the elementary algebraic inequality $\sqrt{Bc}+\sqrt{bC} \le \sqrt{b+B}\sqrt{c+C}$ for non-negative reals:

$$
\begin{aligned}
(3.5)\quad |\omega([x],[y])| &\le |\langle A^*x,y\rangle| + |\langle x,A^*y\rangle| \\
&\le \|A^*x\|\,\|y\| + \|x\|\,\|A^*y\| \le \sqrt{\|x\|^2+\|A^*x\|^2}\,\sqrt{\|y\|^2+\|A^*y\|^2} \\
&= \|x\|_{\mathcal{G}}\,\|y\|_{\mathcal{G}}\,,
\end{aligned}
$$

where $x,y \in D_M (= D_M^{\mathcal{G}})$. Hence

$$|\omega([x],[y])| \le \|[x]\|_{\mathsf{B}}\|[y]\|_{\mathsf{B}}.$$

To prove (ii) we lift ω to the realization $\widetilde{\mathsf{B}}$ of B in $D_M^{\mathcal{G}}$. So, let $\widetilde{\omega}$ denote the form $\widetilde{\omega}(x,y) := \langle A^*x,y\rangle - \langle x,A^*y\rangle$ restricted to $\widetilde{\mathsf{B}}$, i.e.

$$\widetilde{\omega}(j([x]),j([y])) = \omega([x],[y]) \qquad \text{for all } [x],[y] \in \mathsf{B}.$$

Notice that

$$(3.6)\qquad A^*(\widetilde{\mathsf{B}}) \subset \widetilde{\mathsf{B}} \quad \text{and} \quad (A^*)^2 = -\mathrm{Id} \quad \text{on } \widetilde{\mathsf{B}}, \quad \text{and}$$

$$(3.7)\qquad \widetilde{\omega}(x,y) = \langle A^*x,y\rangle_{\mathcal{G}}\,.$$

From (3.6) and (3.7) we obtain that the mapping

$$
\begin{array}{rccc}
\tau: & \mathsf{B} & \longrightarrow & \mathsf{B}^* \\
& [x] & \mapsto & \tau_{[x]}([y]) := \omega([x],[y])
\end{array}
$$

is an isomorphism of the Hilbert space B onto its dual B^* = the space of bounded linear functionals; hence (ii) is proved. □

We can characterize various types of extensions of the fixed symmetric, closed operator A by corresponding properties of the domains and the abstract boundary values:

Lemma 3.3. *Let D be a subspace of D_M which contains D_m. Then the extension $A_D := A^*|_D$*
(a) is closed (as an operator in H), if and only if $\gamma(D)$ is closed (in B);
(b) the extension is self-adjoint, if and only if $\gamma(D)$ is a Lagrangian subspace of B; and
(c) it has compact resolvent, if and only if the inclusion $D^{\mathcal{G}} \hookrightarrow H$ is compact, where $D^{\mathcal{G}}$ denotes the domain D equipped with the graph norm.

Proof. (a) is just a reformulation of Lemma 3.1b; (b) and (c) are immediate from the definition. □

3.2. **Lagrangian Property of Cauchy Data Spaces.** We define the Cauchy data space $\gamma(\mathsf{S})$ and show that it is closed and isotropic and in fact Lagrangian for fixed real Hilbert space H; closed, densely defined symmetric A with domain D_m and adjoint A^* with domain D_M

As first suggested by Bojarski, [5] the concept of Cauchy data spaces is a fundamental tenet of any systematic study of splitting formulas for spectral invariants. This motivates the following definition in our abstract setting:

Definition 3.4. Let $\mathsf{S} := \ker A^*$ denote the solution space of A^*. The space S is closed in the graph norm in $D_M^{\mathcal{G}}$ and also in H. We call $\gamma(\mathsf{S})$ the *Cauchy data space* of A^*.

All the arguments in this section will assume the following:

Assumption 1. *There exists a* self-adjoint Fredholm *extension*

$$A_D := A^*|_D$$

defined on a domain D with $D_m \subset D \subset D_M$. So in particular $\gamma(D)$ is a Lagrangian subspace in B.

Assuming the existence of a Fredholm extension in our abstract setting corresponds to the ellipticity condition in the concrete setting. We shall exploit the following list of Fredholm properties:

<1> By definition $\ker A_D = D \cap \mathsf{S}$ is finite-dimensional;

<2> moreover, we have a short exact sequence

$$0 \longrightarrow D_m \cap \mathsf{S} \hookrightarrow D \cap \mathsf{S} \stackrel{\gamma|D\cap\mathsf{S}}{\longrightarrow} \gamma(D \cap \mathsf{S}) \longrightarrow 0,$$

which yields $D \cap \mathsf{S} \cong D_m \cap \mathsf{S} \oplus \gamma(D \cap \mathsf{S})$;

<3> clearly $\gamma(D \cap \mathsf{S}) \subset \gamma(D) \cap \gamma(\mathsf{S})$; in fact the spaces are equal since $\gamma(x) = \gamma(s)$ for $x \in D$ and $s \in \mathsf{S}$ implies $x - s \in D_m$, hence $s \in D$;

<4> $\operatorname{range} A_D = A^*(D)$ is closed in H and $\dim H/\operatorname{range} A_D < +\infty$, so $A^*(D_M)$ is also closed in H;

<5> $\ker A = D_m \cap \mathsf{S} = A^*(D_M)^\perp$ (the orthogonal complement taken in H); and

<6> $\ker A_D = (\operatorname{range} A_D)^\perp$ (the orthogonal complement taken in H).

Then Assumption 1 leads to the following proposition which is the main result of this section.

Proposition 3.5. *Under the preceding assumption (A closed symmetric with fixed self-adjoint Fredholm extension A_D) the Cauchy data space $\gamma(\mathsf{S})$ is a closed, Lagrangian subspace of* B *and belongs to the Fredholm-Lagrangian Grassmannian $\mathcal{FL}_{\Lambda_0}$ at $\Lambda_0 := \gamma(D)$.*

An astonishing aspect of symplectic functional analysis is that the proof of the preceding proposition can be kept completely elementary due to the following geometric comparison lemma, which says that any isotropic space intersecting a Lagrangian space transversally inherits the Lagrangian property.

Lemma 3.6. *Let (V, ω) be a real symplectic Hilbert space with a fixed Lagrangian subspace Λ_0 and an isotropic subspace Λ, i.e. $\Lambda \subset \Lambda^\circ$. Then Λ is Lagrangian (i.e. $\Lambda^\circ = \Lambda$) if*

$$\Lambda_0 \cap \overline{\Lambda} = \{0\} \text{ and } \Lambda_0 + \Lambda = V.$$

Proof. Let $x \in \overline{\Lambda}$, say $x = x_0 + x_1$ with $x_0 \in \Lambda_0$ and $x_1 \in \Lambda \subset \overline{\Lambda}$, hence $x_0 = x - x_1 \in \Lambda_0 \cap \overline{\Lambda}$, which must vanish, so $x = x_1 \in \Lambda$. This proves $\overline{\Lambda} = \Lambda$.

To prove the Lagrangian property of Λ, we take $x \in \Lambda^\circ$ and write it as $x = x_0 + x_1$ as before. Since $\Lambda \subset \Lambda^\circ$, we get

$$x_0 = x - x_1 \in \Lambda_0 \cap \Lambda^\circ = \Lambda_0^\circ \cap \Lambda^\circ = (\Lambda_0 + \Lambda)^\circ = V^\circ = \{0\},$$

hence $x = x_1 \in \Lambda$. □

Proof of Proposition 3.5. Step 1: Let $[x], [y] \in \gamma(\mathsf{S})$. Then

$$\omega([x],[y]) = \langle A^* x, y\rangle - \langle x, A^* y\rangle = 0,$$

hence $\gamma(\mathsf{S})$ isotropic.

Step 2: Now we consider the sequence of continuous mappings

$$D_M^{\mathcal{G}} \xrightarrow{A^*} \operatorname{range} A^* \xrightarrow{\pi} \operatorname{range} A^* / \operatorname{range} A_D\,,$$

hence

$$D + \mathsf{S} = \{x \in D_M^{\mathcal{G}} \mid A^* x \in \operatorname{range} A_D\} = \ker \pi \circ A^*$$

must be closed and we have a Hilbert space isomorphism

$$D_M^{\mathcal{G}} / (D + \mathsf{S}) \overset{\to}{\cong} \operatorname{range} A^* / \operatorname{range} A_D\,. \tag{3.8}$$

Moreover $\gamma(D+\mathsf{S})$ is closed in B by Lemma 3.1b and coincides with $\gamma(D) + \gamma(\mathsf{S})$. From the closedness of $\gamma(D) + \gamma(\mathsf{S})$ we get

$$(\gamma(D) + \gamma(\mathsf{S}))^\circ = (\gamma(D) + \overline{\gamma(\mathsf{S})})^\circ = \gamma(D)^\circ \cap \overline{\gamma(\mathsf{S})}^\circ\,. \tag{3.9}$$

Step 3: Since $\gamma(\mathsf{S})$ is isotropic, we have also $\overline{\gamma(\mathsf{S})}$ isotropic. Recall also that $\gamma(D)$ is Lagrangian. This yields

$$\gamma(D)^\circ \cap \overline{\gamma(\mathsf{S})}^\circ \supset \gamma(D) \cap \overline{\gamma(\mathsf{S})} \supset \gamma(D) \cap \gamma(\mathsf{S}),$$

hence, with (3.9)

$$(\gamma(D) + \gamma(\mathsf{S}))^\circ \supset \gamma(D) \cap \gamma(\mathsf{S}). \tag{3.10}$$

Step 4: Now we exploit the Fredholm properties and get

$$\begin{aligned} D_m \cap \mathsf{S} \oplus \gamma(D) \cap \gamma(\mathsf{S}) &\overset{<2>,<3>}{\cong} D \cap \mathsf{S} \overset{<1>}{=} \ker A_D \\ &\overset{<6>}{=} (\operatorname{range} A_D)^\perp = H/\operatorname{range} A_D \cong H/\operatorname{range} A^* \oplus \operatorname{range} A^*/\operatorname{range} A_D. \end{aligned} \tag{3.11}$$

Since $D_m \cap \mathsf{S} \overset{<5>}{\cong} H/\operatorname{range} A^*$, this yields

$$\begin{aligned} \gamma(D) \cap \gamma(\mathsf{S}) &\cong \operatorname{range} A^*/\operatorname{range} A_D \overset{(3.8)}{\cong} D_M/(D + \mathsf{S}) \\ &\cong \mathsf{B}/\gamma(D+\mathsf{S}) = \mathsf{B}/(\gamma(D) + \gamma(\mathsf{S})). \end{aligned} \tag{3.12}$$

Moreover, for any closed subspace L in B we have

$$\mathsf{B}/L^\circ \cong \mathsf{B}^*/\tau_\omega(L) \cong \mathsf{B}^*/\tau_E(L) \cong \mathsf{B}/L^\perp\,,$$

where the isomorphisms $\tau_E, \tau_\omega : \mathsf{B} \to \mathsf{B}^*$ are given by $\tau_E([x])[y] := \langle [x],[y]\rangle_{\mathsf{B}}$ and $\tau_\omega([x])[y] := \omega([x],[y])$, hence

$$\dim \mathsf{B} \,/\, \gamma(D) + \gamma(\mathsf{S}) \;=\; \dim(\gamma(D) + \gamma(\mathsf{S}))^\circ \,. \tag{3.13}$$

Combined with formulas (3.10) and (3.12) this yields

$$(\gamma(D) + \gamma(\mathsf{S}))^\circ = \gamma(D) \cap \overline{\gamma(\mathsf{S})} = \gamma(D) \cap \gamma(\mathsf{S}). \tag{3.14}$$

Step 5: Set $\mu := \Lambda \cap \gamma(\mathsf{S})$. Since μ is finite-dimensional, it is closed. (3.14) yields $\mu \subset \mu^\circ = \gamma(D) + \gamma(\mathsf{S})$, i.e. μ is isotropic. Hence, in the reduced symplectic vector space μ°/μ we have

$$\gamma(D)/\mu \,\cap\, \overline{\gamma(\mathsf{S})/\mu} = \{0\} \quad\text{and}\quad \gamma(D)/\mu + \gamma(\mathsf{S})/\mu = \mu^\circ/\mu \,. \tag{3.15}$$

Clearly $\gamma(D)/\mu$ is Lagrangian in the factor space; hence we can apply Lemma 3.6 and get that $\gamma(\mathsf{S})/\mu$ is Lagrangian in μ°/μ, hence $\gamma(\mathsf{S})$ Lagrangian in μ° and in B.

From Formula (3.13) we see that $\gamma(\mathsf{S})$ forms a Fredholm pair with $\gamma(D)$. □

Corollary 3.7. *Let Λ be a Lagrangian subspace in B. Then $(\Lambda, \gamma(\mathsf{S}))$ is a Fredholm pair, if and only if $A_{\gamma^{-1}(\Lambda)} := A^*|_{\gamma^{-1}(\Lambda)}$ is a (self-adjoint) Fredholm operator. We then have*

$$\operatorname{index} A_{\gamma^{-1}(\Lambda)} = \mathbf{i}(\Lambda, \gamma(\mathsf{S})) = \dim \Lambda \cap \gamma(\mathsf{S}) - \operatorname{codim}(\Lambda + \gamma(\mathsf{S})) = 0.$$

3.3. **The Continuity of the Cauchy Data Spaces.** We shall investigate the Cauchy data spaces of operator families of the form $\{A^* + C_t\}_{t\in I}$ where A is a closed symmetric, densely defined operator in a Hilbert space H satisfying suitable additional assumptions. We assume that $\{C_t\}_{t\in I}$ is a continuous family (with respect to the operator norm) of bounded self-adjoint operators. Here the parameter t runs in the standard interval $I = [0,1]$.

Remark 3.8. On a compact manifold with or without boundary with fixed Riemannian metric and a fixed bundle of Clifford modules, such families arise naturally when investigating families of operators of Dirac type distinguished only by the *connection* of the Clifford modules. The operators of such families have the same principal symbol and are distinguished only by their zero-order parts which are bundle homomorphisms and can be considered in particular as a continuous family of bounded operators from L_2 to L_2.

We define the space of abstract boundary values and the abstract trace map $\gamma : D_M \to D_M^{\mathcal{G}}/D_m^{\mathcal{G}} = \mathsf{B}$ as before. Notice that even in the family situation, the vector spaces B and the mapping γ are fixed; but given by the graph of $A^* + C_t$, the inner product $\langle .,.\rangle_t^{\mathcal{G}}$ for $D_M^{\mathcal{G}}$ and B varies with varying parameter t, hence the splitting $j_t : \mathsf{B} \to D_M^{\mathcal{G}}$ varies and the embedding of B as subspace $j_t(\mathsf{B}) = \widetilde{\mathsf{B}}_t$ in D_M; yet all norms are equivalent, uniformly with respect to $t \in [0,1]$, to the norm defined just by A^*. In the following we fix the inner product defined by A^*.

We sharpen our previous Assumption 1 by demanding the existence of a D with $D_m \subset D \subset D_M$ such that $A_D := A^*|_D$ has compact resolvent, hence, in particular, the operator $A_D + C_t$ is a Fredholm operator for that fixed D and all $t \in I$.

The solution spaces $\mathsf{S}_t := \ker A^* + C_t$ and the corresponding Cauchy data spaces $\gamma(\mathsf{S}_t)$ can vary considerably. For the ease of presentation we shall fix the dimension of the inner solution spaces $\ker A + C_t = D_m \cap \mathsf{S}_t$ to 0:

Assumption 2. *We shall assume the non-existence of inner solutions for all operators* $A^* + C_t$, *i.e.*

$$D_m \cap \mathsf{S}_t = \{0\} \quad \text{for all } t \in [0,1].$$

Note . The non-existence of inner solutions ('unique continuation property') is not generally valid for elliptic differential operators but established for Dirac operators (see e.g. [6], Chapter 8).

Theorem 3.9. *Under the preceding assumptions (existence of a self-adjoint extension A_D with compact resolvent and non-existence of inner solutions), the spaces $\gamma(\mathsf{S}_t)$ of Cauchy data of a continuous family $\{A^* + C_t\}_{t\in I}$ vary continuously.*

Note . As usual, we define the continuous dependence of a family of subspaces of a Hilbert space on a parameter by the continuity of the corresponding orthogonal projections.

Proof. To prove the continuity, we need only to consider the local situation at $t = 0$. We carry out the proof in two steps. First we show that $\{\mathsf{S}_t\}_{t\in I}$ is a continuous family of subspaces of $D_M^{\mathcal{G}}$; then we show that $\gamma(\mathsf{S}_t)$ is a continuous family in B.

Step 1: We consider the bounded operator

$$\begin{array}{rccc} F_t: & D_M^{\mathcal{G}} & \longrightarrow & H \oplus \mathsf{S}_0 \\ & x & \mapsto & ((A^* + C_t)(x), P_0 x) \end{array},$$

where $P_0 : H \to \mathsf{S}_0$ denotes the orthogonal projection of H onto the subspace S_0 which is closed in $D_M^{\mathcal{G}}$ and in H.

Clearly F_0 is injective: $F_0(x) = 0$ implies $x \in \mathsf{S}_0$ and $x = P_0 x = 0$. The operator F_0 is also surjective: Since $A^* + C_0$ has no inner solutions, we have $\ker A + C_0 = D_m \cap \mathsf{S}_0 \cong \operatorname{coker} A^* + C_0$ which shows that the operator $A^* + C_0$ is surjective. Let $y \in H$ and $x \in \mathsf{S}_0$ and choose z with $(A^* + C_0)z = y$. Let $w := P_0(z) - x \in \mathsf{S}_0$. Then $F_0(z - w) = ((A^* + C_0)(z - w) = y, P_0(z - w) = x)$. This proves that F_0 is an isomorphism.

Then all operators F_t are isomorphisms for $0 \leq t \ll 1$, since F_t is a continuous family of operators. We define

$$\varphi_t := F_t^{-1} \circ F_0 : D_M^{\mathcal{G}} \cong D_M^{\mathcal{G}} \qquad \text{for } t \text{ small}.$$

We see that

$$\varphi_t(\mathsf{S}_0) = \mathsf{S}_t\,, \tag{3.16}$$

since each $z \in \varphi_t(\mathsf{S}_0)$ implies $F_t(z) = 0 + P_0(z)$, hence $(A^* + C_t)z = 0$; vice versa, each $z \in \mathsf{S}_t$ can be written in the form $F_t^{-1}F_0(y)$ with $y := P_0(z)$.

From (3.16) we get that

$$\{P_t := \varphi_t P_0 \varphi_t^{-1} : D_M^{\mathcal{G}} \longrightarrow S_t\}$$

is a continuous family of projections onto the solution spaces S_t. The projections are not necessarily orthogonal, but can be orthogonalized and remain continuous in t like in [6], Lemma 12.8.

Step 2: Now we must show that $\{\gamma(\mathsf{S}_t)\}$ is a continuous family in B. This is not proved by the formula $\gamma(\mathsf{S}_t) = \gamma(\varphi_t(\mathsf{S}_0))$ alone. We must modify the endomorphism φ_t of $D_M^{\mathcal{G}}$ in such a way that it keeps the subspace D_m invariant.

To do that we notice that $D_m + \mathsf{S}_0$ is closed in $D_M^{\mathcal{G}}$. We define a continuous family of mappings by

$$\begin{array}{rcccccc} \psi_t : D_M^{\mathcal{G}} = & D_m + \mathsf{S}_0 & + & (D_m + \mathsf{S}_0)^{\perp} & \longrightarrow & D_M^{\mathcal{G}} \\ & x + s & + & y & \mapsto & x + \varphi_t(s) + y \end{array}$$

with $\psi_0 = \mathrm{Id}$, hence ψ_t isomorphism for $t \ll 1$, and $\psi_t(D_m) = D_m$ for such small t. Hence we obtain a continuous family of mappings $\{\widetilde{\psi}_t : \mathsf{B} \to \mathsf{B}\}$ with $\widetilde{\psi}_t(\gamma(\mathsf{S}_0)) = \gamma(\mathsf{S}_t)$. From that we obtain a continuous family of projections as above. □

Remark 3.10. From the preceding arguments it follows as well that the Cauchy data spaces form a differentiable family, if $\{C_t\}$ is a differentiable family.

4. The Spectral Flow for Families of Self-Adjoint (Unbounded) Fredholm Operators

In this section we deal with continuous curves of self-adjoint Fredholm operators in a real separable Hilbert space.

First we consider the space $\widehat{\mathcal{F}}$ of *bounded* self-adjoint Fredholm operators. We recall its decomposition into three connected components and define the spectral flow for any continuous path $t \mapsto A_t \in \widehat{\mathcal{F}}$ for $t \in [0,1]$. The decomposition is treated in [4] and at length (in the very similar complex case) in [6], Chapter 16. The non-trivial component $\widehat{\mathcal{F}}_*$ is a classifying space for the functor $K\mathbf{R}^{-7}$ in the real case and for the functor K^{-1} in the complex case.

The original definition of the spectral flow and its generalization for smooth families of elliptic self-adjoint operators of positive order over a closed Riemannian manifold was worked out in [6], Chapter 17, namely by bringing the graph of the spectrum of the family in 'general position' by deformation and counting intersection numbers with $y = 0$. That 'perturbation' approach is meaningful only for loops (or loop defining curves, e.g.

operator curves with periodic spectrum or operator curves with invertible endpoints). But still it is very useful when carrying out concrete calculations under the assumption of smoothness and regularity of the crossings.

To get a more general theoretical picture, we follow a recent paper by Phillips, [28] who gives a purely functional analytical definition of the spectral flow for continuous curves (not necessarily loops) in $\widehat{\mathcal{F}}$ without any assumptions about the zero eigenvalues and without requiring any perturbation of the family.

Then we consider continuous families of (unbounded) self-adjoint Fredholm operators of the form $t \mapsto A_t := A_D + C_t$, where A_D is a fixed self-adjoint operator with compact resolvent and $\{C_t\}_{t\in I}$ a continuous family of bounded self-adjoint operators (here 'continuous' refers to the operator norm). As explained above, such families of unbounded operators arise naturally in two situations. Firstly, the situation occurs when one considers a Dirac operator acting on sections in a fixed Clifford module bundle E over a closed manifold M with fixed Riemannian structure, but varying the connection in E. Then A_t has the Sobolev space $\mathcal{H}^1(M;E)$ as domain and is considered a densely defined, closed operator in $L_2(M;E)$; or, more precisely, as a bounded Fredholm operator from $\mathcal{H}^1(M;E)$ into $L_2(M;E)$. Secondly, the situation occurs when one considers the Dirac operator solely on 'half' the manifold, say a codimension 0 submanifold M_+ with boundary, and imposes globally elliptic boundary conditions specifying a domain $D \subset L_2(M_+;E|_{M_+})$ and then vary the connection.

In both cases the operator A_t is transformed into a bounded self-adjoint Fredholm operator from $L_2(M;E)$ into $L_2(M;E)$ (resp. from $L_2(M_+;E|_{M_+})$ into $L_2(M_+;E|_{M_+})$) by the transformation

$$A_t \mapsto \mathcal{R}(A_t) := A_t\sqrt{\mathrm{Id}+A_t^2}^{\,-1}\,. \tag{4.1}$$

Here $\sqrt{\mathrm{Id}+A_t^2}^{\,-1}$ denotes the unique positive definite square root of the positive definite operator $(\mathrm{Id}+A_t^2)^{-1}$.

Recall that self-adjoint operators with compact resolvent like self-adjoint elliptic differential operators of positive order acting on sections in a vector bundle E over a closed manifold M have a discrete spectrum $\{\lambda_k\}_{k\in\mathbf{Z}}$ of finite multiplicity with $\pm\infty$ as the only accumulation points. There exists no essential spectrum, and the eigensections span the whole $L_2(E)$. Clearly the spectrum will be compressed to $\{\lambda_k/\sqrt{1+\lambda_k^2}\}$ under the transformation $\mathcal{R}$, but the transformation does not change the signs of corresponding eigenvalues.

For loops (or families with periodic spectrum or with invertible ends) of unbounded operators of the form $\{A_t = A_D + C_t\}$, one can then proceed as in [6] and count the eigenvalue crossings with the line $y=0$, possibly after a small deformation making the crossings regular; or one can equivalently count the eigenvalue crossings of the transformed family, possibly also after a small deformation. Under these assumptions (smooth, periodic spectrum or invertible operators at the endpoints, and deformation into regular crossings) one need not look at the operators but only at the graph of the spectrum; and the only continuity argument required for concrete calculations in that case is the continuous dependence of the spectrum on deformations of the operator.

To obtain a general picture, one can drop these assumptions and define the spectral flow of the family $\{A_t\}$ directly on the operator level by the spectral flow of the transformed family. To this purpose we exploit an argument which R. Nest showed us and outline the seldom-mentioned proof that the transformation $\mathcal{R}$ of (4.1) transforms a continuous path $\{A_t = A_D + C_t\}$ into a continuous path $\{\mathcal{R}(A_t)\}$.

We shall also assign another continuous curve $\{t \mapsto \widetilde{A}_t\}$ in $\widehat{\mathcal{F}}_*$ to our curve $\{A_t\}$ by piecewise patching together spectral projections derived from $\{A_t\}$. The construction depends on horizontal and vertical spacing and other choices. We show that in our case it leads to the same spectral flow as the curve $\{\mathcal{R}(A_t)\}$.

4.1. **Phillips' Definition for Continuous Bounded Families.** Let H be a real separable Hilbert space and let $\widehat{\mathcal{F}}$ denote the space of bounded self-adjoint Fredholm operators from H to H. It is well known that $\widehat{\mathcal{F}}$ consists of three connected components (in the operator norm)

$$\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_- \cup \widehat{\mathcal{F}}_+ \cup \widehat{\mathcal{F}}_*\,,$$

namely the contractible spaces of essentially negative and essentially positive operators and the topologically non-trivial component of operators with essential spectrum on both sides of the real line.

Definition 4.1. [*J. Phillips, 1995.*] For any arbitrary continuous path $A : [0,1] \ni t \mapsto A_t \in \widehat{\mathcal{F}}$ we define the *spectral flow* by

$$\mathrm{sf}(A) := \sum_{j=1}^{N} k(t_j, \varepsilon_j) - k(t_{j-1}, \varepsilon_j)$$

with

$$k(t,\varepsilon_j) := \sum_{0 \le \theta < \varepsilon_j} \dim\ker(A_t - \theta) \quad \text{for } t_{j-1} \le t \le t_j\,,$$

where the horizontal and vertical spacing $(t_0, \ldots, t_N), (\varepsilon_1, \ldots, \varepsilon_N)$ is chosen such that

$$\ker(A_t - \varepsilon_j) = \{0\} \quad \text{and} \quad \dim\ker(A_t - \theta) < \infty \tag{4.2}$$

for $t_{j-1} \le t \le t_j$ and $0 \le |\theta| < \varepsilon_j$.

It is possible to choose vertical and horizontal spacing satisfying (4.2) since the spectrum of a self-adjoint bounded Fredholm operator changes continuously with the operator and the zero eigenvalue is discrete and of finite multiplicity. After Definition 1.4 we already referred Phillips' argument why the definition does not depend on the choice of the horizontal and vertical spacing.

We list the following properties of the spectral flow to emphasize formal similarities with the Maslov index (see Theorem 1.5):

Theorem 4.2.

(I') *The spectral flow is well defined for* homotopy *classes of paths with fixed endpoints and distinguishes the homotopy classes. In particular, it is invariant under* re-parametrization *of paths.*

(II') *The spectral flow is additive under* catenation, *i.e.*

$$\mathrm{sf}(A * B) = \mathrm{sf}(A) + \mathrm{sf}(B)\,,$$

where $\{A_t\}, \{B_t\}$ *are continuous paths with* $A_1 = B_0$ *and*

$$(A * B)_t := \begin{cases} A_{2t} & 0 \le t \le \frac{1}{2} \\ B_{2t-1} & \frac{1}{2} < t \le 1\,. \end{cases}$$

(III') *The spectral flow is* invariant *under the adjoint action of the full orthogonal group* $\mathcal{O}(H)$ *of* H.

(IV') *The spectral flow vanishes for paths which stay in one (connected)* stratum

$$\widehat{\mathcal{F}}_{\#}^{(k)} := \{F \in \widehat{\mathcal{F}}_{\#} \mid \dim\ker F = k\}, \quad \# \in \{-,+,*\}$$

of the stratified space $\widehat{\mathcal{F}}_{\#} = \cup_{k=0}^{\infty} \widehat{\mathcal{F}}_{\#}^{(k)}$, *i.e. if* $\dim\ker A_t = k$ *for one* $k \ge 0$ *and all* $t \in I$.

We can discuss the relations with the spectral flow $\mathrm{sf}^{\mathbf{C}}$ of the complex case in exactly the same way as we did in Section 1 for the Maslov index: First we embed the space $\mathcal{F} = \mathcal{F}(H)$ of Fredholm operators defined on the real Hilbert space H in the complex Fredholm operator space $\mathcal{F}(H \otimes \mathbf{C})$ with the inclusion given by complexification. Clearly, there are many more paths in $\widehat{\mathcal{F}}(H \otimes \mathbf{C})$ than those coming from $\widehat{\mathcal{F}}(H)$.

Theorem 4.2 remains valid in the complex case. It is not difficult to derive the following formula:

Proposition 4.3. *Let* $\{A_t\} \in \widehat{\mathcal{F}}(H)$ *and* $\{B_t\} \in \widehat{\mathcal{F}}(H \otimes \mathbf{C})$ *be two continuous paths which have the same endpoints. They are homotopic in* $\widehat{\mathcal{F}}(H \otimes \mathbf{C})$, *if and only if*

$$\mathrm{sf}(\{A_t\}) = \mathrm{sf}^{\mathbf{C}}(\{B_t\}).$$

In spite of formal similarities between the definition of the spectral flow and of the Maslov index, it must be noted that the spectral flow of a path in finite dimension depends only on the eigenvalues at the endpoints and consequently vanishes for loops (and this remains true for paths resp. loops in the components $\widehat{\mathcal{F}}_{\pm}$), whereas the Maslov index even in finite dimension depends on the path and not only on the endpoints: Hence the spectral flow (counting passages through 0 on the real line) is topologically only interesting when we have an infinite number of eigenvalues (or essential spectrum) on both sides of the real line. The Maslov index (counting passages through $e^{i\pi}$ on the circle) is always topologically interesting. Our understanding of the spectral flow as a quantum type invariant is nourished also by the observation that the spectral flow is defined directly by operators and their eigenvalues and not by 'classical' quantities; that it demands genuinely infinite-dimensional function spaces; and that in spite of its coincidence with the Maslov index (see below Section 5), it reflects the finer distinction between the components $\widehat{\mathcal{F}}_-, \widehat{\mathcal{F}}_+$, and $\widehat{\mathcal{F}}_*$.

4.2. **The Construction of a Continuous Curve of Bounded Operators by the Transformation $A \mapsto A(\mathrm{Id} + A^2)^{-1/2}$.** Now we consider a path $\{A_t = A_D + C_t\}_{t\in I}$ of (unbounded) self-adjoint Fredholm operators in H, where A_D is a fixed (unbounded) self-adjoint operator with compact resolvent and $\{C_t\}_{t\in I}$ is a continuous path of bounded self-adjoint operators on H.

To define the spectral flow of the family $\{A_t = A_D + C_t\}_{t\in I}$ we apply the transformation

$$
\begin{array}{llll} \mathcal{R}: & C\widehat{\mathcal{F}} & \longrightarrow & \widehat{\mathcal{F}} \\ & A & \mapsto & \mathcal{R}(A) := A\sqrt{\mathrm{Id} + A^2}^{\,-1}, \end{array} \tag{4.3}
$$

where $C\widehat{\mathcal{F}}$ denotes the space of (not necessarily bounded) self-adjoint Fredholm operators. We define convergence in $C\widehat{\mathcal{F}}$ by the *gap* metric, i.e. the convergence of the orthogonal projection operators onto the graphs of the Fredholm operators. It was shown by Cordes and Labrousse, [9], Addendum, Theorem 1, that on the subset of all bounded operators, the topology, induced by the gap metric for closed operators, is equivalent to that given by the operator norm.

Clearly $\mathcal{R}$ maps the connected component of $C\widehat{\mathcal{F}}$ which contains $\widehat{\mathcal{F}}_*$ into $\widehat{\mathcal{F}}_*$ (and the same holds for $\widehat{\mathcal{F}}_\pm$). From the Spectral Decomposition Theorem and the Weierstrass Approximation Theorem it follows that the mapping $\mathcal{R}$ restricted to $\widehat{\mathcal{F}}_*$ (or $\widehat{\mathcal{F}}_\pm$) is continuous and homotopic to the identity map of $\widehat{\mathcal{F}}_*$ (or $\widehat{\mathcal{F}}_\pm$), see Atiyah and Singer [4]. However, it seems unclear whether the mapping $\mathcal{R}$ is continuous on the whole space $C\widehat{\mathcal{F}}$ or on the subspace $\widehat{\mathcal{C}}$ of self-adjoint operators with compact resolvent.

Instead of discussing the continuity of the map $\mathcal{R}$, we shall show the continuity of the composed map

$$C \mapsto A_D + C \mapsto \mathcal{R}(A_D + C)$$

from $\widehat{\mathcal{B}}$ to $\widehat{\mathcal{F}}$, where $\widehat{\mathcal{B}}$ denotes the space of bounded self-adjoint operators on H.

Proposition 4.4. *Let S be a self-adjoint operator with spectral decomposition $S = \int_{-\infty}^{\infty} \lambda\, dE_\lambda$ and set $f_\varepsilon(z) := z\, e^{-(\frac{1}{2}+\varepsilon)\log(1+z^2)}$ for $\varepsilon > 0$.* [1] *Let $\Gamma = \Gamma_- \cup \Gamma_+$ denote the double cone around the x-axis with opening $(-\frac{\pi}{4}, \frac{\pi}{4})$*

[1] We fix the branch of $\log(1+z^2)$ for which $-\pi < \arg\log(1+z^2) < \pi$.

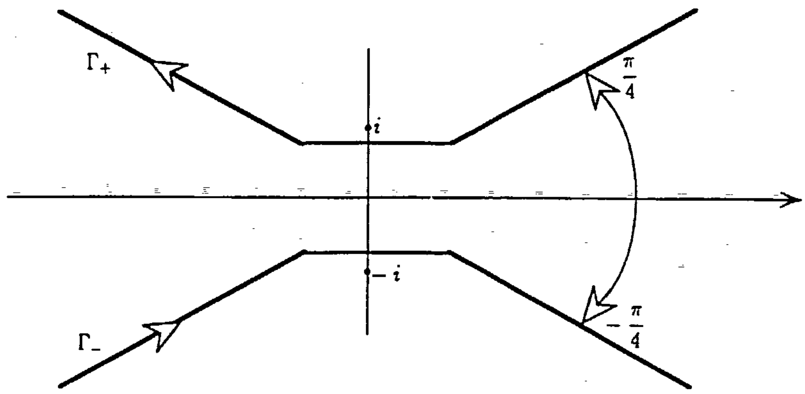

FIGURE 3. The integration path $\Gamma = \Gamma_+ \cup \Gamma_-$

turned off zero by passing inside $\pm i$ (see Figure 3). Then the 'Cauchy integral' converges and defines a bounded operator with

$$\frac{1}{2\pi i}\int_\Gamma f_\epsilon(\lambda)(\lambda - S)^{-1}d\lambda = \int_{-\infty}^{\infty} f_\epsilon(\theta)\,dE_\theta =: f_\epsilon(S)\,. \tag{4.4}$$

Proof. (By courtesy of R. Nest; see also [26] *and* [11].*)* Since the function f_ϵ is bounded, the integral on the right side of (4.4) exists and defines a bounded operator. From the estimate

$$|f_\epsilon(z)| \sim \frac{1}{|z|^{2\epsilon}} \qquad \text{as } |z| \to \infty \tag{4.5}$$

it follows that also the integral on the left side of (4.4) is well defined and defines a bounded operator. Therefore, to prove (4.4) it suffices to prove the coincidence of the two operators on the dense subspace $\cup_{N>0}P_N(H)$, where $P_N := \int_{-N}^{N} dE_\theta$.

So let $x \in H$ and $N > 0$. For $R \gg N$ we replace the infinite integration path Γ by the finite closed contour Γ_R as indicated in Figure 4. Then on the 'compact element' $P_N(x)$, the operator $f_\epsilon(S)$ takes the form

$$\begin{aligned} \int_{-N}^{N} f_\epsilon(\theta)\,dE_\theta &= \int_{-N}^{N} \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f_\epsilon(\lambda)}{\lambda - \theta}d\lambda\,dE_\theta \\ &= \int_{-N}^{N} \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f_\epsilon(\lambda)}{\lambda - \theta}d\lambda\,dE_\theta = \int_{-N}^{N} \frac{1}{2\pi i}\int_{\Gamma} \frac{f_\epsilon(\lambda)}{\lambda - \theta}d\lambda\,dE_\theta\,. \end{aligned} \tag{4.6}$$

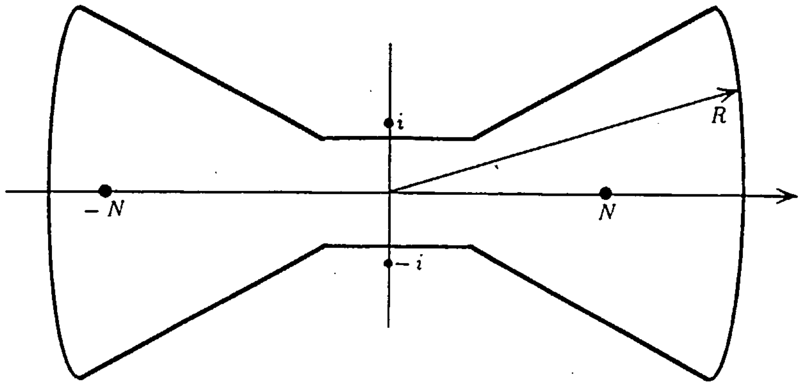

FIGURE 4. The integration path Γ_R for $R \gg N$

The last equality is proved by applying the estimation (4.5). Clearly, the last operator applied to $P_N(x)$ yields

$$
\begin{aligned}
(4.7)\quad &\frac{1}{2\pi i}\int_\Gamma f_\varepsilon(\lambda)\left(\int_{-N}^{N}\frac{1}{\lambda-\theta}dE_\theta\right)(P_N(x))\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma f_\varepsilon(\lambda)\left(\int_{-\infty}^{\infty}\frac{1}{\lambda-\theta}dE_\theta\right)(P_N(x))\,d\lambda \\
&= \frac{1}{2\pi i}\left(\int_\Gamma f_\varepsilon(\lambda)(\lambda-S)^{-1}d\lambda\right)(P_N(x))\,,
\end{aligned}
$$

which proves equation (4.4). □

Remark 4.5. If the Hilbert space H is real, the preceding proof must be carried out *after* complexifying H. Nevertheless, the resulting operator $f_\varepsilon(S)$ remains real, so that $f_\varepsilon(S)$ can be considered an operator on H.

A remarkable consequence of the preceding proposition is the following lemma.

Lemma 4.6. *Let S and C be self-adjoint operators with S unbounded and C bounded. Then we have in the operator norm*

$$\|f_\varepsilon(S+C) - f_\varepsilon(S)\| \le c\|C\|\,,$$

where the constant c neither depends on S nor on C nor on $\varepsilon > 0$.

Proof. By Proposition 4.4 we have

$$
\begin{aligned}
(4.8)\quad f_\varepsilon(S+C) - f_\varepsilon(S) &= \frac{1}{2\pi i}\int_\Gamma f_\varepsilon(\lambda)\left((\lambda-(S+C))^{-1} - (\lambda-S)^{-1}\right)d\lambda \\
&= \frac{-1}{2\pi i}\int_\Gamma f_\varepsilon(\lambda)(\lambda-(S+C))^{-1}\circ C\circ(\lambda-S)^{-1}d\lambda,
\end{aligned}
$$

hence

$$\|f_\varepsilon(S+C) - f_\varepsilon(S)\| \leq \frac{1}{2\pi}\int_\Gamma |f_\varepsilon(\lambda)|\frac{1}{|\Im(\lambda)|^2}\|C\|d\lambda = c\|C\|.$$

□

Since for any self-adjoint operator S with compact resolvent the limit $\lim_{\varepsilon\to 0} f_\varepsilon(S)$ exists in the strong sense and equals our transformed operator $\mathcal{R}(S) := S(\mathrm{Id} + S^2)^{-1/2}$ by Lebesgue's Convergence Theorem and the Resonance Theorem, we can conclude the main result of this subsection from Proposition 4.4 and Lemma 4.6.

Theorem 4.7. *Let S be a self-adjoint operator with compact resolvent in a real separable Hilbert space H. Then for any bounded self-adjoint operator C, the sum $S + C$ also has compact resolvent (hence is a closed Fredholm operator; and we have*

$$\|\mathcal{R}(S+C) - \mathcal{R}(S)\| \leq c\|C\|,$$

where the constant c neither depends on S nor on C.

We shall apply the preceding theorem in the following form:

Corollary 4.8. *Curves of self-adjoint (unbounded) Fredholm operators in separable real Hilbert space of the form $\{A_D + C_t\}_{t\in I}$ are mapped into continuous curves in $\widehat{\mathcal{F}}$ by the transformation $\mathcal{R}$ when A_D has compact resolvent and $\{C_t\}_{t\in I}$ is a continuous curve of bounded operators.*

Remark 4.9. Let

$$\begin{array}{cccc} \mathcal{T}_{A_D}: & \widehat{\mathcal{B}} & \longrightarrow & C\widehat{\mathcal{F}} \\ & C & \mapsto & A_D + C \end{array}$$

denote the translation by A_D, mapping bounded self-adjoint operators on H into closed self-adjoint Fredholm operators in H. On $C\widehat{\mathcal{F}}$, the gap topology is defined by the metric

$$g(A_1, A_2) := \sqrt{\|R_{A_1} - R_{A_2}\|^2 + \|A_1R_{A_1} - A_2R_{A_2}\|^2},$$

where $R_{A_.} := (\mathrm{Id}+A_.^2)^{-1}$ (see Cordes and Labrousse, [9] and also Kato, [14]). In Theorem 4.7 we proved that the composition $\mathcal{R}\circ\mathcal{T}_{A_D}$ is continuous.

$$\begin{array}{ccc} \widehat{\mathcal{B}} & \xrightarrow{\mathcal{T}_{A_D}} & C\widehat{\mathcal{F}} \\ & {\scriptstyle \mathcal{R}\circ\mathcal{T}_{A_D}} \searrow & \downarrow \mathcal{R} \\ & & \widehat{\mathcal{F}} \end{array}$$

Further, we can prove that the translation operator $\mathcal{T}_{A_D}$ is a continuous operator from $\widehat{\mathcal{B}}$ onto the subspace $\widehat{\mathcal{B}} + A_D \subset C\widehat{\mathcal{F}}$. The proof can be carried out along the same lines as the proof of Theorem 4.7 taking the functions $(1+\lambda^2)^{-1}$ and $\lambda(1+\lambda^2)^{-1}$ instead of $f_\varepsilon(\lambda)$. Thus it needs not take a limit $\varepsilon \to 0$. It is not clear, however, whether the inverse operator

$\mathcal{T}_{A_D}^{-1} : \widehat{\mathcal{B}} + A_D \to \widehat{\mathcal{B}}$ is continuous, and hence whether the transformation $\mathcal{R}$ is continuous on the space $C\widehat{\mathcal{F}}$.

Here we do not discuss the continuity of $\mathcal{T}_{A_D}^{-1}$ since the literature only examines examples of families of self-adjoint Fredholm operators of our form $\{A_D + C_t\}$ (see Floer, [10], Yoshida, [31], Nicolaescu, [22], [23], Kirk and Klassen, [15], [16], etc.).

We define:

Definition 4.10. Let $\{A_t\}_{t\in I}$ be a continuous curve of (unbounded) self-adjoint Fredholm operators of the form $A_t = A_D + C_t$ with the preceding assumptions. Then the *spectral flow* of the continuous, unbounded curve $\{A_t\}_{t\in I}$ is defined by the spectral flow of the continuous, bounded curve $\{\mathcal{R}(A_t)\}_{t\in I}$ in $\widehat{\mathcal{F}}$.

Remark 4.11. The properties listed in Theorem 4.2 for the spectral flow of families of bounded operators remain valid for our class of unbounded operators by the construction of the curve $\{\mathcal{R}(A_t)\}_{t\in I}$.

4.3. **An Alternative Construction of a Continuous Curve of Bounded Operators.** Phillips' idea was to define the spectral flow of a continuous curve of bounded self-adjoint Fredholm operators by piecewise hedging curves of eigenvalues. To construct a continuous curve $\{\widetilde{A}_t\}$ of bounded self-adjoint Fredholm operators solely from small intervals of the spectrum of our curve $\{A_t = A_D + C_t\}_{t\in I}$ and from the related eigenspaces, we sharpen Phillips' argument and hedge the branching of the zero eigenvalues only, quite in the same way as when determining the relations between the functional analytical and differential definition of the Maslov index in the proof of Theorem 2.1 above.

Proposition 4.12. *For any continuous family $\{A_t = A_D + C_t\}_{t\in I}$ of self-adjoint Fredholm operators there exists a partition $0 = t_0 < \cdots < t_N = 1$ of the interval and positive reals $a_1, \ldots, a_N$, such that we can construct continuous curves $A_t^{(j)}$ in $\widehat{\mathcal{F}}_*$ on each small interval $[t_j, t_{j+1}]$ with the following properties (for $t_j \le t \le t_{j+1}$):*

1. $\operatorname{spec}(A_t^{(j)}) = \{\operatorname{spec}(A_D + C_t) \cap (-a_j, a_j)\} \cup \{1, -1\}$ *with* $\operatorname{spec}_{ess}(A_t^{(j)}) = \{1, -1\}$, *and*
2. $\ker A_t^{(j)} = \ker(A_D + C_t)$.

Proof. Step 1: To construct the jump curve, we first consider the family A_s in a neighbourhood $t - \delta(t) \le s \le t + \delta(t)$ of a point $t \in I$, where $\ker A_t = \{0\}$. Then none of the A_s has eigenvalues at all in a small vertical interval. Hence there is no contribution to the spectral flow and we can define $A_s^{(t)} := T$ for s in the interval $[t - \delta(t), t + \delta(t)]$. Here $T : H \to H$ denotes an isomorphism which is equal Id on one infinite-dimensional subspace $L_{(t)}$ and equal $-\mathrm{Id}$ on the orthogonal, also infinite-dimensional subspace $L'_{(t)}$, the polarization $(L_{(t)}, L'_{(t)})$ chosen arbitrarily.

Step 2: Now we consider the family A_s close to a point t where we have $\dim\ker A_t > 0$. Let λ_1 denote the smallest positive eigenvalue and μ_1 the largest negative one. We choose a positive real number $a(t) < 1$ with $a(t) < \lambda_1$ and $a(t) < |\mu_1|$ and a $\delta(t) > 0$ such that $a(t), -a(t) \notin \mathrm{spec} A_s$ for $s \in [t - \delta(t), t + \delta(t)]$. We define for s in this interval

$$P_s^{(t)} := \frac{1}{2\pi i} \int_{|\lambda| = a(t)} (A_s - \lambda)^{-1}\, d\lambda\,,$$

hence $\mathrm{rank} P_s^{(t)} = \dim\ker A_t$. Then the operator

$$A_s P_s^{(t)} : H \longrightarrow H$$

is bounded because $P_s^{(t)}$ is of finite rank and A_s keeps range $P_s^{(t)}$ invariant for $t - \delta(t) \leq s \leq t + \delta(t)$.

Step 3: Now we can choose points $\widetilde{t}_0 = 0 < \widetilde{t}_1 < \cdots < \widetilde{t}_N = 1$ in such a way that $\widetilde{t}_{j+1} - \delta(\widetilde{t}_{j+1}) < \widetilde{t}_j + \delta(\widetilde{t}_j)$, and then choose points $t_j \in (\widetilde{t}_j - \delta(\widetilde{t}_j), \widetilde{t}_{j-1} + \delta(\widetilde{t}_{j-1}))$ with $0 = t_0 = \widetilde{t}_0 < \widetilde{t}_1 < t_1 < \widetilde{t}_2 < \cdots < \widetilde{t}_N < t_N = 1$. We set $a_j := a(\widetilde{t}_j)$ for $j = 0, \ldots, N$.

Next we choose polarizations of the infinite-dimensional

$$(\mathrm{range}\, P_{t_j}^{(\widetilde{t}_j)})^\perp = L_j \oplus L_j^\perp$$

with L_j and $L_j^\perp$ infinite-dimensional. We define an operator $\Pi_j : H \to H$ with $\mathrm{spec_{ess}} = \{-1, 1\}$ by

$$\Pi_j|_{\mathrm{range}\, P_{t_j}^{(\widetilde{t}_j)}} = 0, \quad \Pi_j|_{L_j} = \mathrm{Id}, \quad \text{and} \quad \Pi_j|_{L_j^\perp} = -\mathrm{Id}.$$

Step 4: Finally we define the jump curve

$$A_s^{(j)} := A_s P_s^{(\widetilde{t}_j)} + (O_s^j)^* \Pi_j O_s^j \quad \text{for } t_j \leq s \leq t_{j+1} \tag{4.9}$$

which satisfies the properties 1 and 2. Here the orthogonal projections are chosen in such a way that

$$P_s^{(\widetilde{t}_j)} = O_s^j P_{t_j}^{(\widetilde{t}_j)} (O_s^j)^*$$

for $t_j \leq s \leq t_{j+1}$. □

Since the dimensions of the kernels do not jump at the discontinuities of the curve and since the strata $\widehat{\mathcal{F}}_*^{(k)}$ are connected, we can insert continuous curve pieces at the discontinuities without changing the spectral flow. This yields a continuous curve $t \mapsto \widetilde{A}_t \in \widehat{\mathcal{F}}_*$ with the following property:

Corollary 4.13. *If each operator $A_t = A_D + C_t$ has an infinite number of eigenvalues on both sides of the real line, the curve $\{\widetilde{A}_t\}$ and the curve $\{\mathcal{R}(A_t)\}$ are homotopic in $\widehat{\mathcal{F}}_*$ in the sense of keeping the endpoints in two fixed strata of $\widehat{\mathcal{F}}_*$.*

Note . Whereas the preceding construction leads to a continuous curve $\{\widetilde{A}_t\}$ in $\widehat{\mathcal{F}}_*$ for any continuous curve of self-adjoint Fredholm operators of the form $A_t = A_D + C_t$, the transformation $\mathcal{R}$ leads only to a curve in $\widehat{\mathcal{F}}_*$ when the operators of the original curve have an infinite number of eigenvalues on both sides of the real line. Curves of positive or negative semi-bounded operators are mapped by $\mathcal{R}$ in $\widehat{\mathcal{F}}_+, \widehat{\mathcal{F}}_-$, but by the preceding construction invariably also in $\widehat{\mathcal{F}}_*$.

5. The Spectral Flow Formula

In this section we shall prove our main result, the equality of the spectral flow and the Maslov index:

Theorem 5.1. [Spectral Flow Formula]. *Let A be a closed symmetric operator in a real Hilbert space H with domain D_m and let $\{C_t\}_{t\in I}$ be a continuous family of bounded self-adjoint operators on H. We assume that*

1. *the operator A has a self-adjoint extension A_D with compact resolvent;*
2. *there exists a positive constant a such that*

$$D_m \cap \ker(A^* + C_t - s) = \{0\}$$

for any s with $|s| < a$ and any $t \in [0,1]$.

Then we have

$$\mathrm{sf}(\{A_D + C_t\}) = \mu(\{\gamma(\ker(A^* + C_t)\}, \gamma(D)), \tag{5.1}$$

where γ denotes the projection of the domain D_M of A^* onto the symplectic space $\mathsf{B} = D_M/D_m$ of abstract boundary values.

We notice that conditions 1 and 2 are naturally satisfied for operators of Dirac type (i.e. for first-order differential operators with principal symbol of A^2 defining the Riemannian metric) both over a closed manifold and over a manifold with boundary subject to global elliptic boundary conditions. Clearly, a perturbation by adding a real multiplum of the identity preserves the Dirac type and hence the non-existence of inner solutions ('unique continuation property'). This might, however, not be true for general first-order elliptic differential operators.

We recall that the left side of Formula (5.1) was defined in Definition 4.10 by associating a continuous curve of bounded self-adjoint Fredholm operators with the curve $\{A_D + C_t\}_{t\in I}$ (Theorem 4.7). For the right side of Formula (5.1) we recall that $\{\gamma(\ker(A^* + C_t))\}_{t\in I}$ is a continuous family of Lagrangian subspaces of B by the assumptions made and by Theorem 3.9, and that $(\gamma(\ker(A^* + C_t)), \gamma(D))$ is a Fredholm pair by Proposition 3.5. Hence we have a continuous curve $t \mapsto \Lambda(t) \in \mathcal{FL}_{\Lambda_0}$ in the Fredholm-Lagrangian Grassmannian with $\Lambda(t) := \gamma(\ker(A^* + C_t))$ and $\Lambda_0 := \gamma(D)$ and a family of unitary operators $\{W_t : \mathsf{H} \to \mathsf{H}\}$ defining the Maslov index of the curve (see Definition 1.4).

Proof. We prove the theorem in three steps. First we construct a suitably fine horizontal spacing $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ and a vertical spacing $\{a_1, \ldots, a_N\}$. Then we show

$$\operatorname{sf}\left(\{\mathcal{R}(A_D + C_{t_{i+1}}) - s\}_{0 \le s \le \frac{a_{i+1}}{\sqrt{1+a_{i+1}^2}}}\right) = \mu\left(\{\gamma(\ker(A^* + C_{t_{i+1}} - s)\}_{0 \le s \le a_{i+1}}, \gamma(D)\right) \tag{5.2}$$

for that spacing. This is the main part of the proof. It consists, so to speak, of establishing the coincidence of the spectral flow and the Maslov index for segments of analytic families. This will be done by explicit calculation identifying the two invariants with the integer $-\sum_{0 \le s \le a_{i+1}} \dim\ker(A_D + C_{t_{i+1}} - s)$. Finally, we show how the general case follows from the special case.

First step: To construct a suitable horizontal and vertical spacing, we choose a $t \in [0,1]$. We denote the smallest positive eigenvalue of $A_D + C_t$ by $\lambda_1(t)$ and the largest negative one by $\mu_1(t)$. We distinguish two cases:

(I.) $\ker(A_D + C_t) = \{0\}$. For the vertical spacing we choose a positive

$$b(t) < \min\{\lambda_1(t), |\mu_1(t)|, a\}.$$

For the horizontal spacing we take a $\delta(t) > 0$ such that the small box is kept free of eigenvalues (see Figure 5), namely

$$\operatorname{spec}(A_D + C_{t'}) \cap (-b(t), b(t)) = \emptyset \quad \text{for } t' \in (t - \delta(t), t + \delta(t)).$$

(II.) $\ker(A_D + C_t) \neq \{0\}$. In this case we choose a positive

$$b(t) < \min\left\{\frac{\lambda_1(t)}{3}, \frac{|\mu_1(t)|}{3}, a\right\}.$$

For the horizontal spacing we take a $\delta(t) > 0$ such that the eigenvalues in a small box are confined by two strips (see Figure 6a), namely

$$\operatorname{spec}(A_D + C_{t'}) \cap (b(t), 2b(t)) = \emptyset$$

and

$$\operatorname{spec}(A_D + C_{t'}) \cap (-2b(t), -b(t)) = \emptyset$$

for $t' \in (t - \delta(t), t + \delta(t))$.

Second step: In the first case, the regular case, we have for each $\tilde{\delta} \le \delta(t)$

$$\operatorname{sf}\left(\{\mathcal{R}(A_D + C_{t'})\}_{t - \tilde{\delta} \le t' \le t + \tilde{\delta}}\right) = 0,$$

since $A_D + C_{t'}$ is invertible for $t - \delta(t) \le t' \le t + \delta(t)$. We also have

$$\mu\left(\{\gamma(\ker(A^* + C_{t'})\}_{t - \tilde{\delta} \le t' \le t + \tilde{\delta}}, \gamma(D)\right) = 0,$$

since $\gamma(\ker(A^* + C_{t'}) \cap \gamma(D) = \{0\}$ for $t - \delta(t) \le t' \le t + \delta(t)$.

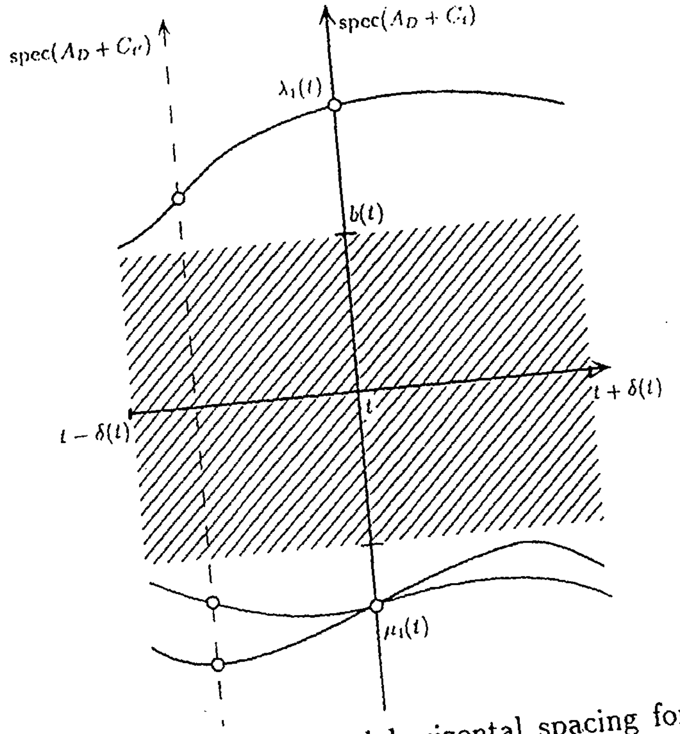

FIGURE 5. Vertical and horizontal spacing for $\ker(A_D + C_t) = \{0\}$

For the second, singular case, we recall that the eigenvalues of the operator $\mathcal{R}(A_D + C_{t'}) - s$ are of the form

$$\frac{\lambda}{\sqrt{1+\lambda^2}} - s,$$

where λ is an eigenvalue of the operator $A_D + C_{t'}$. Hence the spectral flow of the family $\{\mathcal{R}(A_D + C_{t'}) - s\}_{0 \le s \le \frac{b(t)}{\sqrt{1+b(t)^2}}}$ equals

$$-\sum_{0 \le s \le \frac{b(t)}{\sqrt{1+b(t)^2}}} \dim\ker(\mathcal{R}(A_D + C_{t'}) - s) = -\sum_{0 \le s \le b(t)} \dim\ker(A_D + C_{t'} - s)$$

(see also Figure 6b). We shall show that this integer equals the Maslov index of the family $\{\Lambda(s,t') := \gamma(\ker(A^* + C_{t'} - s))\}_{0 \le s \le b(t)}\}$ at the Lagrangian $\Lambda_0 := \gamma(D)$ for each fixed $t' \in (t - \delta(t), t + \delta(t))$. We have

$$\Lambda(s,t') \cap \Lambda_0 = \{0\},$$

if $s \notin \operatorname{spec}(A_D + C_{t'}) \cap [0, b(t)]$. By our assumption 1, the set $\operatorname{spec}(A_D + C_{t'}) \cap [0, b(t)]$ contains only a finite number of elements, hence the intersection of the curve $\{\Lambda(s,t')\}_{0 \le s \le b(t)}$ with $\gamma(D)$ is non-trivial only at finitely many points; and these points are the eigenvalues λ of the operator $A_D + C_{t'}$ with $0 \le \lambda \le b(t)$. Clearly, the family $\{\Lambda(s,t')\}_{0 \le s \le b(t)}$ is a smooth curve.

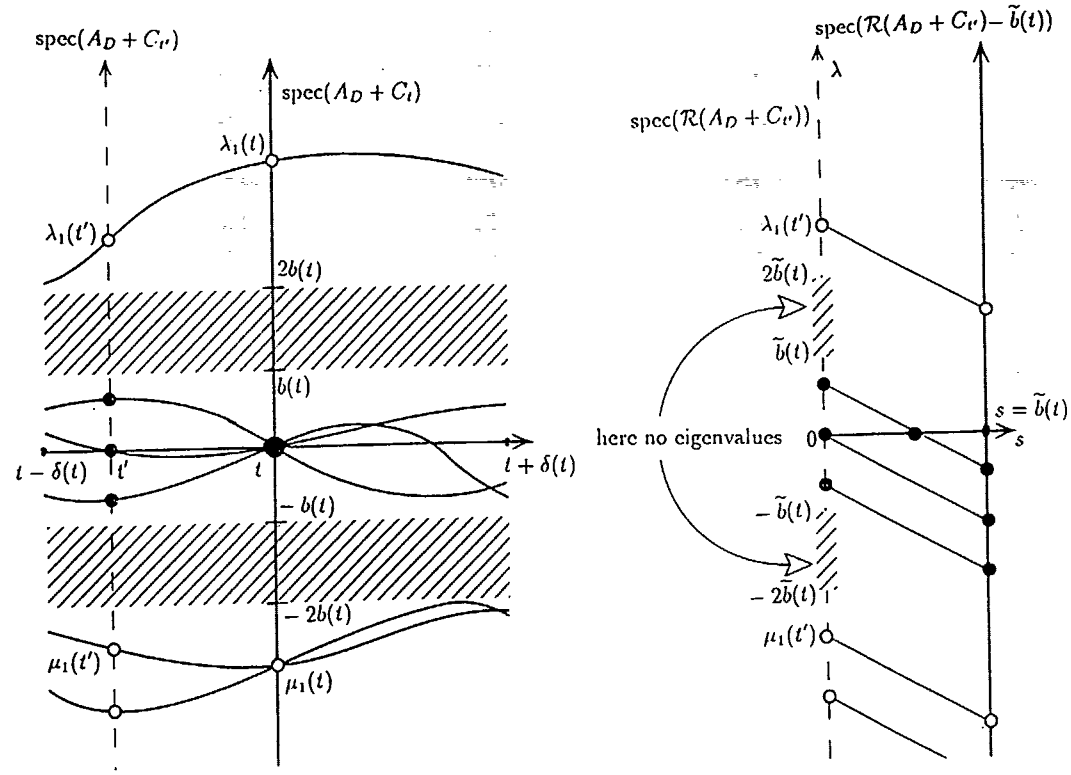

FIGURE 6. a) Vertical and horizontal spacing for $\ker(A_D + C_t) \neq \{0\}$ b) The spectral flow of the linear family $\{\mathcal{R}(A_D + C_{t'}) - s\}_{s\in[0,\tilde{b}(t)]}$, $\tilde{b}(t) = b(t)(1 + b(t)^2)^{-1/2}$

We determine the quadratic form $Q_{(\Lambda(\lambda,t'),\dot{\Lambda}(\lambda,t'))}$ for all such eigenvalues λ and fixed t':

$$Q_{(\Lambda(\lambda,t'),\dot{\Lambda}(\lambda,t'))}([x],[x]) := \frac{d}{d\theta}\omega([x], B_\theta[x])|_{\theta=0} \quad \text{for } [x] = \gamma(x) \in \mathsf{B}\,,$$

where $\mathsf{B} := D_M/D_m$ denotes our symplectic space of boundary values and $B_\theta : \Lambda(\lambda, t') \to \Lambda(\lambda, t')^\perp$ is chosen in such a way that $\{[x] + B_\theta[x] \mid [x] \in \Lambda(\lambda, t')\} = \Lambda(\lambda + \theta, t')$ for θ close to 0, hence $B_0 = 0$.

Let $x \in \ker(A_D + C_{t'} - \lambda)$ or equivalently $\gamma(x) \in \Lambda(\lambda, t') \cap \Lambda_0$, and θ sufficiently small. Then we can choose a smooth family $\{u_\theta \in \ker(A^* + C_{t'} - \lambda - \theta)\}$ such that

$$\gamma(x) + B_\theta(\gamma(x)) = \gamma(u_\theta) \quad \text{and} \quad u_0 = x,$$

hence

$$\begin{aligned}(5.3)\quad \omega\left(\gamma(x), B_\theta(\gamma(x))\right) &= \langle A^*x, u_\theta - x\rangle - \langle x, A^*(u_\theta - x)\rangle \\ &= \langle (A^* + C_{t'} - \lambda)x, u_\theta - x\rangle - \langle x, (A^* + C_{t'} - \lambda)(u_\theta - x)\rangle = -\langle x, \theta u_\theta\rangle.\end{aligned}$$

Differentiating yields

$$\frac{d}{d\theta} - \langle x, \theta u_\theta \rangle|_{\theta=0} = -\langle x, u_0 \rangle = -\langle x, x \rangle < 0$$

for $x \neq 0$; hence $Q_{(\Lambda(\lambda,t'),\dot\Lambda(\lambda,t'))}|_{\Lambda(\lambda,t')\cap\gamma(D)}$ is negative definite.

This implies (i) that the crossings are all regular at $s = \lambda$ eigenvalue of $A_D + C_{t'}$ so that we can apply Theorem 2.1 and determine the Maslov index by adding the signatures of the crossing forms; and (ii) that this signature is just the dimension of the kernel of $A_D + C_{t'} - \lambda$. Hence

$$(5.4)\quad \mu(\{\Lambda(s,t')\}_{0\le s\le b(t)}, \gamma(D)) = \sum_{0\le s\le b(t)} \operatorname{sign} Q_{(\Lambda(s,t'),\dot\Lambda(s,t'))}|_{\Lambda(s,t')\cap\Lambda_0}$$
$$= -\sum_{0\le s\le b(t)} \dim \Lambda(s,t')\cap\gamma(D) = -\sum_{0\le s\le b(t)} \dim\ker(A_D + C_{t'} - s).$$

Based on these considerations we can choose the desired horizontal spacing $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ and vertical spacing $\{a_1 := b(t_1), \ldots, a_N := b(t_N)\}$.

Third step: Now we have at each small interval $[t_i, t_{i+1}]$ a loop of the two-parameter family $\{\mathcal{R}(A_D + C_t) - s\}$ which is contractible in $\widehat{\mathcal{F}}$, hence

$$(5.5)$$
$$\operatorname{sf}\left(\{\mathcal{R}(A_D + C_t)\}_{t_i\le t\le t_{i+1}}\right) + \operatorname{sf}\left(\{\mathcal{R}(A_D + C_{t_{i+1}}) - s\}_{0\le s\le \frac{a_{i+1}}{\sqrt{1+a_{i+1}^2}}}\right)$$
$$- \operatorname{sf}\left(\{\mathcal{R}(A_D + C_{t_i}) - s\}_{0\le s\le \frac{a_{i+1}}{\sqrt{1+a_{i+1}^2}}}\right) = 0,$$

and

$$(5.6)\quad \mu\left(\{\gamma(\ker(A^* + C_t)\}_{t_i\le t\le t_{i+1}}, \gamma(D)\right)$$
$$+ \mu\left(\{\gamma(\ker(A^* + C_{t_{i+1}} - s)\}_{0\le s\le a_{i+1}}, \gamma(D)\right)$$
$$- \mu\left(\{\gamma(\ker(A^* + C_{t_i} - s)\}_{0\le s\le a_{i+1}}, \gamma(D)\right) = 0.$$

By the preceding step, the spectral flow and the Maslov index coincide for linear families, hence for each small interval $[t_i, t_{i+1}]$

$$\operatorname{sf}\left(\{\mathcal{R}(A_D + C_t)\}_{t_i\le t\le t_{i+1}}\right) = \mu\left(\{\gamma(\ker(A^* + C_t)\}_{t_i\le t\le t_{i+1}}, \gamma(D)\right),$$

which proves our theorem by additivity under catenation. □

Remark 5.2. Note that the left side of Formula (5.1) is also well defined for *complex operators*; the right side, however, is ambiguous in usual symplectic algebra for *complex spaces*, but well defined in our setting, see Definition 1.6; then our preceding theorem translates immediately to the complex case.

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af: Lena Lindenskov, Statens Humanistiske Forskningsråd, RUC, IMFUFA

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