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Separability and the Jones Polynomial

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Abstract

The Jones polynomial $V_L(t)$ of oriented links and knots is defined as the trace of a representation of the braid groups in a tower of finite separable T-extensions. Finite separable extensions are the split separable extensions of IMFUFA text 210 (1991), just renamed in honor of example 4.3. T stands for trace on the smallest algebra, which, when composed with conditional expectation, defines trace elsewhere on the tower of algebras. The tower of algebras is obtained through iteration of a basic construction for finite separable extensions.

Separability and the Jones Polynomial

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1 Introduction

Jones' index theory of type II_1 von Neumann algebra subfactors was published in 1983, and led in the spring of the following year to a new polynomial invariant of knots and links. Subsequently, the Jones polynomial was generalized in different directions and several old problems of Tait's in knot theory were solved. Certain key ingredients of Jones' theory may be reduced to algebra in different ways. For example, the Jones polynomial may be defined from certain traces of Ocneanu's on a sequence of finite dimensional algebras named after Hecke. A second example: the semi-discrete index spectrum of II_1 subfactors may be obtained from the classification of matrix norms of the 0-1 matrices - accomplished long ago - the 0-1 matrices arising as the inclusion matrices of the multimatrix e_i-algebras $\mathcal{A}_{\beta,n} \subseteq \mathcal{A}_{\beta,n+1}$. Another algebraic direction to Jones' theory was started by M. Pimsner and S. Popa in [16] in which they find an "orthonormal basis" that shows a II_1 factor M to be a finitely generated projective module over a finite index subfactor N, the index being the Hattori-Stallings rank of the module [10].

In pursuing the algebraic direction of Pimsner and Popa, the author and D. Kastler in [10] showed that $N \subseteq M$ is a separable extension of rings. The reader will recall the notion of a separable field extension [11] and separable algebra over a field [15], which received various insightful generalizations in Hochschild's homological algebra [7, 1945], and in Auslander and Goldman's theory of the Brauer group of a commutative ring [1, 1960]. In this paper we build from a small system of axioms, one being relative separability and another its module-theoretic dual notion of split extension, the Jones theory leading up to the V_L polynomial for a link L. We define finite separable extensions of k-algebras, and show that these possess the key elements of

Jones' theory: basic construction, index, and, upon iteration of the basic construction to produce a tower of algebras, a countable family of idempotents satisfying braid-like relations. Including trace in this picture, one trace T on the tower of algebras extending the previous by composing it with the conditional expectation, the Jones polynomial V_L of a link L can then be defined by normalizing the trace of a representation of the braid group in a finite separable T-extension.

The structure theory of finite separable extensions of algebras, its relations with representation theory of groups, and the duality of separable and split extension is treated in my paper [9]. In the present paper we further discuss a dimension question and what relation finite separability has with quasi-Frobenius extensions.

2 Finite separable extensions

Let k be a commutative ring, and k° its group of units. Let S be a subalgebra of a faithful k-algebra A such that $1_A \in S$. We identify k with $k1_A$. We consider only natural module and bimodule structure coming from inclusion $S \hookrightarrow A$ and tensor product over S. Let μ denote the multiplication map $A \otimes_S A \to A$, an A-A bimodule morphism defined by $a_0 \otimes a_1 \mapsto a_0 a_1$.

Definition 2.1 A is called a finite separable extension of S if there exists an element $f \in A \otimes_S A$, an S-S bimodule homomorphism $E : A \to S$, and $\tau \in k^\circ$ such that

1. $af = fa \ (\forall a \in A)$ and $\mu(f) = 1$;
2. $E(1) = 1$;
3. $\mu(1 \otimes_S E)f = \mu(E \otimes_S 1)f = \tau$.

An element f satisfying axiom 1 is called a separating element, or a separability element, and its existence alone defines a separable extension of rings, a theory generalizing separable algebras and developed by Sugano and several others [3].

The existence of f is equivalent to μ being a split epimorphism of A-bimodules, which is in turn equivalent to the vanishing of relative Hochschild cohomology groups [1] with arbitrary coefficients ($n > 0$),

$$H^n(A, S; -) = 0.$$

The map $E : A \to S$ satisfying axiom 2 is a conditional expectation as in operator theory, and its existence for a subalgebra $S \subseteq A$ defines a split extension of rings. It is equivalent to requiring the subalgebra S be a direct summand in the bimodule ${}_S A_S$: since the inclusion map splits the kernel exact sequence of the S-S epimorphism $E : A \to S$, we note that

$$A = S \oplus \ker E.$$

Conditional expectations and separating elements are not unique, but axiom 3 demands the existence of a conditional expectation $E : A \to S$ and separating element in $A \otimes_S A$

$$f = \tau \sum_{i=1}^{n} x_i \otimes_S y_i$$

such that $\tau \in k^\circ$ and

$$\sum_{i=1}^{n} E(x_i) y_i = \sum_{i=1}^{n} x_i E(y_i) = 1. \tag{1}$$

Indeed, a short computation reveals that we may choose $x_1 = y_1 = 1$ and $x_i, y_i \in \ker E$ for $i = 2, \ldots, n$. We will say that E and f are compatible in case they satisfy axiom 3. Now fix the notations $S \subseteq A$, f, E, x_i, y_i, μ, and τ for the rest of this paper.

Lemma 2.1 For every $a \in A$, we have

$$\sum_{i=1}^{n} E(ax_i) y_i = \sum_{i=1}^{n} x_i E(y_i a) = a. \tag{2}$$

Proof. One can define a linear map from $A \otimes_S A \otimes_k \hom_S(A, S) \to \hom_k(A, S)$ and make use of axiom 1 together with equation (1). The argument is repeatable on the right. □

Corollary 2.1 E is a nondegenerate S-valued bilinear form on A such that $\{E(-x_i)\}_{i=1}^n$ is a dual basis of $\{y_i\}_{i=1}^n$ for the projective module ${}_S A$ and $\{E(y_i -)\}_{i=1}^n$ is a dual basis of $\{x_i\}_{i=1}^n$ for A_S.

Proof. Nondegeneracy of E follows from assuming $E(ax) = 0$ for every $x \in A$, then $a = \sum E(ax_i) y_i = 0$. □

[1] defined in [8]

3 The basic construction

Define a k-algebra structure on the k-module, $A \otimes_S A$, on which we place a k-algebra structure with multiplication given by

$$(a_0 \otimes_S a_1)(a_2 \otimes_S a_3) = a_0 E(a_1 a_2) \otimes_S a_3.$$

The unity element of A_1 is $1 = \sum_{i=1}^{n} x_i \otimes_S y_i$.

Define index of S in A, $[A : S]_E = \tau^{-1}$, an invertible element in k. This definition is independent of f since $\mu(1) = \tau^{-1}$.

Proposition 3.1 A_1 is a finite separable extension of A with index $[A : S]_E$.

Proof. A separating element

$$f = \sum_{i=1}^{n} x_i \otimes_S 1 \otimes_S y_i$$

is compatible with the conditional expectation

$$E_1 = \tau\mu : A_1 \to A \quad \square$$

Remark 3.1 Note that the element $e_1 = 1 \otimes_S 1$ in A_1 is an idempotent, and a cyclic generator of A_1 as an A-A bimodule. Also note the identities,

$$e_1 a e_1 = E(a) e_1, \tag{3}$$

for every $a \in A$ and

$$E_1(e_1) = \tau 1. \tag{4}$$

Proposition 3.2 If B is a finite separable extension of A, which in turn is a finite separable extension of S, with conditional expectations E_1 and E_2, resp., then B is a finite separable extension of S with index satisfying Lagrange's equation,

$$[B : S]_{E_2 \circ E_1} = [B : A]_{E_1} [A : S]_{E_2}.$$

Proof. Let $(B, A, , E_1, f_1 = \tau_1 \sum_{i=1}^m u_i \otimes_A v_i)$, $(A, S, E_2, f_2 = \tau_2 \sum_{i=1}^n x_i \otimes_S y_j)$ be the data for finite separable extension. Note that $E = E_2 \circ E_1 : B \to S$ is a conditional expectation, and

$$f = \tau_1 \tau_2 \sum_{i=1}^m \sum_{j=1}^n u_i x_j \otimes_S y_j v_i$$

is a compatible separating element in $B \otimes_S B$. □

As a corollary we note that finite separable extension is closed under tensor product with index behaving multiplicatively. Since every algebra is a finite separable extension of itself with index 1, we are permitted a change of ring k on a finite separable extension of k-algebras - with no alteration to the index.

4 Examples

1. **Matrices.** The full matrix algebra $M_n(A)$ over any k-algebra A is a finite separable extension of A (embedded in the constant diagonals) so long as n is invertible in k. Of the n separability elements, $(j = 1, 2, \ldots, n)$

   $$f_j = \sum_{i=1}^n E_{ij} \otimes_A E_{ji},$$

   (E_{ij} is the (i,j)-matrix unit) we average to obtain $f = \frac{1}{n} \sum_{j=1}^n f_j$ as our separating element. The conditional expectation

   $$E(X) = \frac{1}{n} \sum_{i=1}^n X_{ii} \quad \text{where} \quad X = (X_{ij}) \in M_n(A).$$

   is easily computed to be compatible with f, having index reciprocal $\tau = \frac{1}{n^2}$.

   Taking different weighted averages of the elements f_j we can find separating elements and compatible expectations with index other than n^2.

2. **Subfactors.** Let N be a subfactor of M with Jones index $n \leq [M : N] < n + 1$. If $\{m_j\}_{j=1}^{n+1}$ is the Pimsner-Popa orthonormal basis [16] with respect to

the trace-preserving conditional expectation $E : M \to N$, a separating element compatible with E is then given by

$$\frac{1}{[M:N]} \sum_{j=1}^{n+1} m_j \otimes m_j^*$$

This example is observed and proven by D. Kastler and the author in [10]. The proof of these assertions follows from the relations in [16, p. 65], the isomorphism in [4, p. 189], the relations $e_1 x e_1 = E(x)e_1$ and the implication $e_1 x = e_1 y \Rightarrow x = y$, where e_1 denotes the projection of $L^2(M, \mathrm{tr})$ onto $L^2(N, \mathrm{tr})$.

3. **Finite Separable Extensions of Fields** F_2/F_1 with characteristic coprime to the degree n. Let α be a primitive element, $F_2 = F_1(\alpha)$, with minimal polynomial

$$p(x) = x^n - \sum_{i=0}^{n-1} c_i x^i$$

Let

$$E = \frac{1}{n}\mathrm{trace} : F_2 \to F_1,$$

the normalized trace, where trace is a nondegenerate bilinear form on the F_1-vector space F_2 with dual bases [12, cf. p. 213] $\{\alpha^i\}_{i=0}^{n-1}$ and

$$\{\frac{\sum_{j=0}^{i} c_j \alpha^j}{p'(\alpha)\alpha^{i+1}}\}_{i=0}^{n-1}.$$

A separating element is given [14] by

$$f = \sum_{i=0}^{n-1} \alpha^i \otimes_{F_1} \frac{\sum_{j=0}^{i} c_j \alpha^j}{p'(\alpha)\alpha^{i+1}}.$$

Denoting f by $\sum_{i=0}^{n-1} u_i \otimes v_i$ where $E(u_i v_j) = \frac{1}{n}\delta_{i,j}$, we easily compute $\sum u_i E(v_i) = \sum u_i E(u_0 v_i) = \frac{1}{n}$, since $u_0 = 1$. Letting $1 = \sum b_i v_i$, we get $\sum E(u_i) v_i = \sum b_j E(u_i v_j) v_i = \frac{1}{n} \sum b_j v_j = \frac{1}{n}$. Hence, f and E are compatible with index n. In characteristic p the index is $n \pmod{p}$.

4. **Crossed product algebras.** Let H be a subgroup of G with finite index $[G:H] \in k^{\circ}$, B a k-algebra with action $\alpha : G \to \text{Aut } B$. Then $A = B \times_{\alpha} G$ is a finite separable extension of $S = B \times_{\alpha} H$. For if $\{g_i\}_{i=1}^{n}$ is a left transversal of H in G, then

$$f = \frac{1}{[G:H]} \sum_{i=1}^{n} g_i \otimes_S g_i^{-1}$$

is a separating element compatible with the natural projection

$$\pi_H : B \times_{\alpha} G \to B \times_{\alpha} H;$$

whence $\tau = \frac{1}{[G:H]}$. Note that group algebras, and specifically those generated by Sylow p-subgroups of finite groups over characteristic p are included in this example. The proof of these assertions is elementary.

Remark 4.1 Galois extensions of commutative rings, multimatrix extensions $M_{n_1}(S) \times \cdots \times M_{n_t}(S)$, are finite separable extensions as are separable algebras over a local or global field (if dimension is coprime to the characteristic).

5 Tower of algebras

We have seen in section 2 that the basic construction is itself a finite separable extension with canonical expectation and same index. Let A_{i+1} be defined inductively for i = 1,2,3,... as the basic construction of the finite separable extension $A_{i-1} \subseteq A_i$. We make use of the natural notation $A_0 = A$, multiplication map $\mu_{i+1} : A_i \otimes_{A_{i-1}} A_i \to A_i$, and conditional expectation $E_{i+1} = \tau\mu_{i+1}$. By this iteration of the basic construction a tower of algebras over S is generated:

$$S \subseteq A \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

Theorem 5.1 If the ground ring k of a finite separable extension $S \subset A$ possesses an invertible solution t to the quadratic equation $(t+1)^2\tau = t$, then for every n there exists a nontrivial homomorphism of the braid group B_n into the group of units in A_{n-1}. Under the same hypothesis, there exists a nontrivial homomorphism of the Hecke algebra $H(t,n)$ into A_{n-1}.

Proof. We have the idempotent $e_{i+1} = 1 \otimes_{A_{i-1}} 1$ in A_{i+1}; the family of idempotents $\{e_i\}_{i=1}^{\infty}$ in the tower of algebras $S \subseteq A \subseteq A_1 \subseteq \ldots$ satisfies the braid-like relations ($|i-j| \geq 2$):

$$e_i e_{i+1} e_i = \tau e_i, \tag{5}$$

$$e_{i+1} e_i e_{i+1} = \tau e_{i+1}, \tag{6}$$

$$e_i e_j = e_j e_i. \tag{7}$$

Equation (6) follows from equations (3) and (4). Equation (7) is a simple consequence of noting that S is the centralizer subalgebra of e_1 in the basic construction. Equation (5) is the tedious computation that $e_1 e_2 e_1 = \tau e_1$.

Map the Artin generators $\{\sigma_i : i = 1, \ldots, n-1\}$ of B_n as follows:

$$\Phi_n : \quad \sigma_i \longmapsto w_i = (t+1)e_i - 1. \tag{8}$$

One can readily check that the w_i are units of A_{n-1} and satisfy the Artin relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i$$

Hence, the map Φ_n extends multiplicatively to a homomorphism of B_n into the group of units in A_{n-1}.

The Hecke algebras $H(t,n)$ have the standard presentation

$$\langle g_1, \ldots, g_{n-1} | g_i^2 = (t-1)g_i + t, \; g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, g_i g_j = g_j g_i, \; \forall i,j : |i-j| \geq 2 \rangle.$$

Since the w_i also satisfy $w_i^2 = (t-1)w_i + t$, an algebra homomorphism is obtained by sending each g_i onto w_i. □

6 Markov traces

Recall that a <u>trace</u> on a k-algebra B is a k-linear map $T : B \to k$ satisfying for each $x, y \in B$, $T(xy) = T(yx)$. In addition, we assume a trace to be normalized: $T(1) = 1$.

Definition 6.1 Suppose there exists a trace T_S on S. We say A is a <u>finite separable T-extension</u> of S if A is a finite separable extension of S with conditional expectation $E : A \to S$ such that $T_S \circ E$ defines a trace on A we denote by T_A.

Remark 6.1 Note that T_A restricted to S equals the trace T_S, We call T_A the Markov trace over S. Note the identity ($\forall s \in S, \quad a \in A$)

$$T_A(sa) = T_S(sE(a)) \tag{9}$$

Examples of finite separable T-extensions abound: each of the four main examples above of finite separable extension is also a T-extension with respect to the canonical trace on S.

Recall the notation $e_1 = 1 \otimes_S 1$ and $E_1 = \tau\mu$.

Theorem 6.1 Suppose A is a finite separable T-extension of S. Then the basic construction A_1 is a finite separable T-extension of A.

Proof. We must check that $T_{A_1} = T_A \circ E_1$ defines a trace. Note that the k-linear map $T_{A_1} : A_1 \to k$ is given by

$$T_{A_1}(\sum_{i=1}^{N} a_i e_1 b_i) = \tau \sum_{i=1}^{N} T_A(a_i b_i).$$

A computation with simple tensors and using equation (7) shows that T_{A_1} is a trace:

$$T_{A_1}(a_0 e_1 a_1 a_2 e_1 a_3) = T_{A_1}(a_0 E(a_1 a_2) e_1 a_3) =$$
$$\tau T_A(E(a_1 a_2) a_3 a_0) = \tau T_S(E(a_1 a_2) E(a_3 a_0)) =$$
$$T_{A_1}(a_2 e_1 a_3 a_0 e_1 a_1)$$

This together with proposition 1.1 completes the proof. □

We note the identity, which iterates up the tower of algebras over A ($\forall a \in A$):

$$T_{A_1}(a e_1) = \tau T_A(a) \tag{10}$$

Abbreviating the n'th Markov trace T_{A_n} to T_n, we get values for trace such as $T_5(e_4 e_3 e_2 e_3 e_5) = \tau^4$. In the next section we will need only know the values of trace on the e_i-algebra $\mathcal{A}_{\beta,n}$ generated by $1, e_1, \ldots, e_n$ within the tower ($\beta = [A : S]$).

7 The Jones polynomial

Jones associates in [5] a Laurent polynomial in $t^{\frac{1}{2}}$ to an oriented link L. This has been done in various ways via von Neumann algebras and Hecke algebras. In this section we show how it may be done with finite separable T-extensions.

Given a link L, we can find at least one braid α on n strings, which when closed up gives back the link L (a theorem of Alexander). Illustrations of closure are given below, and further reading on knot theory can be found in [2] and [6]. Now there is uncertainty in the correspondence $L \rightsquigarrow \alpha$. It turns out that for any other braid β on n strings, $\beta\alpha\beta^{-1}$ as well as the two braids $\sigma_n^{\pm 1}\alpha$ in B_{n+1} all [2] close up to give L: moreover, by a theorem of Markov the various finite compositions of conjugation and left multiplication of $\sigma_n^{\pm 1}, \sigma_{n+1}^{\pm 1}, \ldots$ are all the braids closing up to give L: these conjugation and left multiplication operations are called the first and second Markov moves, respectively.

Consider a finite separable T-extension A of S with index reciprocal $\tau = \frac{t}{(t+1)^2}$, such as may be obtained from a change of rings,

$$k \rightsquigarrow k[t, t^{-1}]/((t+1)^2\tau - t).$$

By theorem 5.1 the group homomorphism $\Phi_n : B_n \to A_{n-1}{}^{\circ}$ defined in equation (6). Making use of trace, we note that $T \circ \Phi_n$ is a character function on B_n (i.e., a scalar-valued function constant on conjugacy classes of the group). We need an adjustment or normalization for the other possible types of braids closing up to give the same link: the Jones polynomial $V_L(t)$ in $k[\sqrt{t}, \frac{1}{\sqrt{t}}]$ gets the job done.

Let exponent $e(\alpha)$ equal the sum of powers in the expansion of α in Artin generators, σ_i - an invariant of word reduction in B_n! Define the Jones polynomial of the link L with corresponding braid α in B_n as follows:

$$V_L(t) = \left(-\frac{t+1}{\sqrt{t}}\right)^{n-1} (\sqrt{t})^{e(\alpha)} T(\Phi_n(\alpha)) \tag{11}$$

To prove this an invariant under the second Markov move, note that $T(\Phi_{n+1}(\sigma^{\pm 1}\alpha) = T((t^{\pm 1}+1)e_n\Phi_n(\alpha)) - T(\Phi_n(\alpha)) = -\frac{t^{0,1}}{t+1}T(\Phi_n(\alpha))$; plugging

[2]The Artin generators σ_i cross the i'th and i+1'st strings while fixing the other strings.

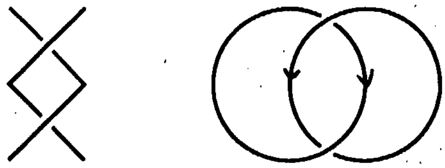

Figure 1. σ_1^2. The Hopf link

this in the right-hand side above, with appropriate changes made to the powers $n-1$ and $e(\alpha)$, one sees the polynomial does not change. It follows from Markov's and Alexander's theorems that $V_L(t)$ is an isotopy invariant of links.

Example 1: *The Hopf link.* The braid σ_1^2 in B_2 closes up to give the Hopf link, which is two interlocking circles oriented as shown below. The exponent of this braid is 2 and $n=2$. Hence,

$$V_L(t) = -\frac{t+1}{\sqrt{t}} tT((e_1(t+1)-1)^2) =$$

$$-(t+1)\sqrt{t}(t - \frac{2t}{t+1} + 1) = -t^{\frac{5}{2}} - t^{\frac{1}{2}}.$$

Example 2: *The left-handed trefoil knot.* The braid σ_1^3 in B_2 closes up to give the left-handed trefoil knot as shown below. The equality sign below should be understood as an isotopy in the ambient space. We compute:

$$V_L(t) = (-\frac{t+1}{\sqrt{t}})\sqrt{t}^3 T((t^3+1)e_1 - 1) =$$

$$-\frac{t}{t+1}(t^3+1)t - (t+1)^2) = -t^4 + t^3 + t.$$

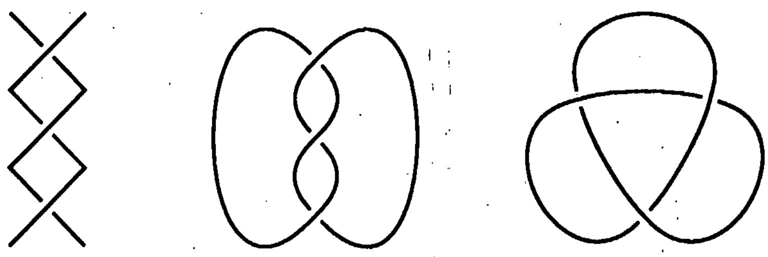

Figure 2. σ_1^3. Its closure: the left-handed trefoil knot.

The mirror image link. Suppose the braid $\alpha = \sigma_1^{n_1} \cdots \sigma_k^{n_k}$ closes up to give the oriented link L. Now we hold a mirror below the link as we look down on it, which turns overpasses to underpasses as we follow the arrows around the link (or knot): this is the mirror image knot denoted by L^-. Clearly, the braid $\beta = \sigma_1^{-n_1} \cdots \sigma_k^{-n_k}$ closes up to give L^-.

Observe the easy identities: $\frac{t+1}{\sqrt{t}} = \frac{t^{-1}+1}{\sqrt{t^{-1}}}$, $\Phi_n(\sigma_i^{-1}) = (t^{-1}+1)e_i - 1$ and $\frac{t}{(t+1)^2} = \frac{t^{-1}}{(t^{-1}+1)^2}$. It follows readily that the Jones polynomial of the mirror image link is $V_L(t^{-1})$. For example, the right-handed trefoil knot has polynomial $-t^{-4} + t^{-3} + t^{-1}$.

8 Relation to ring extension theory

Bruno Müller published in [13, 1964-5] his theory of quasi-Frobenius extensions, in which various phenomena observable in this paper first appeared. We establish in the next theorem that the basic construction of a finite separable extension is none other than the endomorphism ring of the natural module associated to the extension. Then the inclusion of A in A_1 is simply the left regular represention of A in End A_S. For a quasi-Frobenius extension $A \supseteq B$, it was shown in [13] that the endomorphism ring End A_B is itself a quasi-Frobenius extension of A: then the converse question was taken up and settled by Müller. We explore what finite separability has to do with quasi-Frobenius extension in this section.

Proposition 8.1 A_1 is Morita equivalent to S.

Proof. Recall that $e_1 = 1 \otimes_S 1$, an idempotent in A_1. Define a ring homomorphism,

$$F: \quad A_1 \to \text{End } A_S$$

by

$$F(\sum_{i=1}^{m} a_i e_1 b_i) = \sum_{i=1}^{m} \lambda_{a_i} E \lambda_{b_i}$$

where $\lambda_x(y) = xy$ is the left multiplication map. It is easy to show that F is surjective, and slightly harder to show that it is injective. But A_S is a generator module since $E(1) = 1$ and finitely generated projective by an earlier result. It follows from basic Morita theory that A_1 and S are equivalent rings. □

A right quasi-Frobenius extension is a ring A and subring B containing 1_A subject to two axioms:

1. A_B is a finitely generated projective module;
2. A is isomorphic as $B-A$ modules to a direct summand of a direct sum of a finite number of copies of $\hom_B(A_B, B_B)$.

Theorem 8.1 A finite separable extension $A \supseteq S$ is a right quasi-Frobenius extension.

Proof. We have seen that the natural module A_S is a finitely generated projective.

Define an $S-A$ isomorphism $\Psi : A \longrightarrow \hom({}_A A_S, S_S)$ by

$$a \longmapsto E(a-)$$

where $E(a-)$ denotes the map $x \mapsto E(ax)$. Since E is nondegenerate (corollary, section 2), Ψ is injective. Ψ is surjective since, given a map f in the range, $\Psi : \sum_{i=1}^n f(x_i)y_i \mapsto f$. The $S-A$ action on $\hom(A_S, S_S)$ is given by the formula $(sga)(x) = s[g(ax)]$. Now it is clear that $\Psi(sab) = s\Psi(a)b$ □[3]

It is not true that every quasi-Frobenius extension is a finite separable extension. Indeed the next theorem can only be proven for finite separable extensions. A simple example of a Frobenius algebra (i.e., the paradigm of a quasi-Frobenius extension) not satisfying equality of global dimension of rings, $D(-)$, is a group algebra of a p-group in characteristic p.

Theorem 8.2 If A is a finite separable extension of S, then

$$D(A) = D(S).$$

Proof. D(-) may denote left, right, or weak global dimension of ring, though we give the argument only for left modules. Let pd denotes projective dimension. Recall the well-known inequality for any change of rings $S \longrightarrow A$ and an A-module M, pd $M_S \leq$ pd $M_A +$ pd A_S. Since the inclusion map of

[3]Left quasi-Frobenius extension may be defined oppositely. The inclusion in Theorem 8.1 is true for these as well.

$N \to N \otimes_S A$ is split by $Id \otimes_S E$ whenever A is a split extension of S, we easily see that for split extensions

$$D(S) \leq D(A) + \text{pd}\, A_S.$$

For finite separable extension we see moreover that $D(S) \leq D(A)$, since pd $A_S = 0$. Since A_1 is a finite separable extension of A we have $D(A) \leq D(A_1)$. But A_1 is Morita equivalent to S so that $D(A_1) = D(S)$. □

The last theorem is a generalization of Serre's extension theorem for group cohomology [17, the coprime index case]. For we recall that the cohomological dimension of a group G over a field k

$$\text{cd}_k(G) = \text{D}(k[G]).$$

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