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THE FUNCTIONAL DETERMINANT OF A FOUR-DIMENSIONAL BOUNDARY VALUE PROBLEM

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Abstract

Working on four-dimensional manifolds with boundary, we consider formally self-adjoint, elliptic boundary value problems (A, B), A being the interior and B the boundary operator. These problems (A, B) should be valued in a tensor-spinor bundle; should depend in a universal way on a Riemannian metric g and be formally self-adjoint; should behave in an appropriate way under conformal change $g \to \Omega^2 g$, Ω a smooth positive function; and the leading symbol of A should be positive definite. We view the functional determinant $\det A_B$ of such a problem as a functional on a conformal class $\{\Omega^2 g\}$, and develop a formula for the quotient of the determinant at $\Omega^2 g$ by that at g. (Analogous formulas are known to be intimately related to physical string theories in dimension two, and to sharp inequalities of borderline Sobolev imbedding and Moser-Trudinger types for the boundariless case in even dimensions.) When the determinant in a background metric g_0 is explicitly computable, the result is a formula for the determinant at each metric $\Omega^2 g_0$ (not just a quotient of determinants). For example, we compute the functional determinants of the Dirichlet and Robin (conformally covariant Neumann) problems for the Laplacian in the ball B^4, using our general quotient formulas in the case of the conformal Laplacian, together with an explicit computation on the hemisphere H^4.

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ABSTRACT. Working on four-dimensional manifolds with boundary, we consider formally self-adjoint, elliptic boundary value problems (A, B), A being the interior and B the boundary operator. These problems (A, B) should be valued in a tensor-spinor bundle; should depend in a universal way on a Riemannian metric g and be formally self-adjoint; should behave in an appropriate way under conformal change $g \to \Omega^2 g$, Ω a smooth positive function; and the leading symbol of A should be positive definite. We view the functional determinant $\det A_B$ of such a problem as a functional on a conformal class $\{\Omega^2 g\}$, and develop a formula for the quotient of the determinant at $\Omega^2 g$ by that at g. (Analogous formulas are known to be intimately related to physical string theories in dimension two, and to sharp inequalities of borderline Sobolev imbedding and Moser-Trudinger types for the boundariless case in even dimensions.) When the determinant in a background metric g_0 is explicitly computable, the result is a formula for the determinant at each metric $\Omega^2 g_0$ (not just a quotient of determinants). For example, we compute the functional determinants of the Dirichlet and Robin (conformally covariant Neumann) problems for the Laplacian in the ball B^4, using our general quotient formulas in the case of the conformal Laplacian, together with an explicit computation on the hemisphere H^4.

0. INTRODUCTION

The zeta function determinant $\det A$ of an elliptic differential operator A is important in Quantum Field Theory because it provides a regularization of the functional integral, formally identical to a functional determinant. (The adjective "functional" indicates that the integral, or determinant, is taken over an infinite-dimensional function space.) Originally of interest on four-dimensional manifolds, these objects have recently been intensively studied by physicists and mathematicians in two dimensions, in connection with String Theory, the isospectral problem, and uniformization problems. In each of these applications, the operator $A = A_g$ should be built naturally from a Riemannian metric g on a compact manifold M (and possibly some related extra information, like spin structure), and one is concerned with $\det A_g$ as a functional on the cone $\{g\}$ of Riemannian metrics on M, or more precisely, the quotient of $\{g\}$ by the action of the diffeomorphism group $\mathrm{Diffeo}(M)$. A key point has been the behavior of the determinant under *conformal* change of g; that is, replacement of g by $\Omega^2 g$, where Ω is a smooth positive function. The idea is that if A has reasonable conformal behavior, then the behavior of $\det A$ should be predictable,

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much as the behavior of the fundamental solution of A is. In the two-dimensional case, this thinking gets one quite far, as the quotient of $\{g\}$ by the groups Diffeo(M) and $C_+^\infty(M)$ (the positive functions Ω acting by $g \mapsto \Omega^2 g$) is a finite-parameter object; see, e.g. [**O**, **OPS1-2**]. Since diffeomorphisms act on conformal factors Ω, this quotient has the form $\mathcal{G} = \{g\}/(\text{Diffeo}(M) \ltimes C_+^\infty(M))$; i.e., the total group is a semidirect product. In dimensions three and higher, $\mathcal{G}$ is much larger, and in particular is in no sense a finite-parameter object. Even though it is not clear how one would go about tracking the behavior of the functional determinant as the metric g cuts across conformal classes, it seems timely to return to four dimensions, and, inspired by two-dimensional successes, at least handle the behavior of $\det A$ as a functional on a conformal class $C_+^\infty(M) \cdot g$.

For compact manifolds without boundary, some results are already in place. For a computation in connection with Yang-Mills theory on four-manifolds, see [**CT**]. In [**BØ3**], Branson and Ørsted derived a formula for the functional determinant of a strongly elliptic differential operator, with reasonable conformal properties, over a Riemannian four-manifold without boundary; this is analogous to the much-studied *Polyakov formulas* on two-manifolds. Branson, Chang, and Yang [**BCY**] used these formulas to study the isospectral and extremal (uniformization) problems in four dimensions, trying to get analogues of the two-dimensional results of Onofri [**O**] and of Osgood, Phillips, and Sarnak [**OPS1-2**]. The conformal behavior of the functional determinant in dimension two is intimately related to the Moser-Trudinger inequality, which expresses the continuity (and, in its sharp form, is the norm calculation for) the embedding

$$L_1^2 \hookrightarrow e^L \tag{0.1}$$

of the Sobolev class L_1^2 in the Orlicz class e^L. (0.1) may be regarded as a limit of borderline Sobolev inequalities $L_\nu^2 \hookrightarrow L^{2/(1-\nu)}$ (where L^p is the usual Lebesgue class) as $\nu \uparrow 1$, or as $m \downarrow 2$. Roughly speaking, the log-determinant (logarithm of the functional determinant) is the quantity that (0.1) asserts to be nonnegative. In general dimension m, the borderline Sobolev inequalities correspond to the imbeddings $L_\nu^2 \hookrightarrow L^{2m/(m-2\nu)}$, and the limiting case is an inequality of Moser-Trudinger type, corresponding to the imbedding $L_{m/2}^2 \hookrightarrow e^L$; this has been studied by Adams [**A**] and by Beckner [**Be**]. In dimension four, [**BØ3**] and [**BCY**] show that the logarithm of the functional determinant is a linear combination of two terms, one of which describes the embedding $L_2^2 \hookrightarrow e^L$, and the other of which describes the "ordinary" borderline Sobolev embedding $L_1^2 \hookrightarrow L^4$. Up to normalization, the $L_2^2 \hookrightarrow e^L$ and $L_1^2 \hookrightarrow L^4$ terms are connected by one *coupling constant*, say a. The two inequalities "work together" (the quantities asserted to be nonnegative do not appear with opposite signs) if and only if $a \geq 0$. $a = a[A]$ depends on the elliptic operator A whose functional determinant we are studying; for example, A could be the conformal Laplacian Y or the square $\nabla\!\!\!/^2$ of the Dirac operator $\nabla\!\!\!/$. But $a[A]$ is universal in the sense of being independent of the particular manifold and Riemannian metric; indeed, the number $a[A]$ can be computed from a knowledge of the heat invariants, which are similarly universal. Fortunately, $a[Y]$ and $a[\nabla\!\!\!/^2]$ are positive; this makes possible, among other things, the extremal results of [**BCY**, Sec. 5] for the log-determinant on S^4.

In this paper, we begin the extension of this program to four-dimensional manifolds with boundary, inspired by the quite complete two-dimensional treatment of [**OPS1-2**]. As will become clear, this is qualitatively harder than the boundariless case, but still tractable on on a conceptual as well as computational level. (Incidentally, the three-dimensional case is not interesting for manifolds without boundary, as the functional determinant is very rigid conformally in odd dimensions; see Sec. 2 below. The boundary value version of this three-dimensional problem is, however, interesting, though not as rich as the four-dimensional theory. Our results on the three-dimensional problem will appear separately.) It would seem that an effective treatment of isospectral and extremal problems in the boundary-value case would have to await a theory of boundary-value inequalities of Moser-Trudinger type; we note that an excellent theory of sharp borderline Sobolev inequalities is already in place [**E1-2**].

We shall need to be precise about three types of assumptions on the elliptic operator A and the boundary operator B which define our problem: (1) *analytic assumptions*, i.e. the strength of the ellipticity needed; (2) *naturality assumptions*; and (3) *conformal assumptions*. Since we wish to invoke invariant-theoretic properties of local spectral invariants associated to (A, B), specifically the heat invariants, we need to know that (A, B) enjoys suitable invariance properties; this is the rationale behind (2). (3) makes precise the "conformally reasonable behavior" mentioned above. We work out two examples in detail: the conformal Laplacian $Y = \Delta + \tau/6$ (τ = scalar curvature) with Dirichlet conditions, and Y with conditions of Neumann type called *Robin conditions* by physicists; specifically, the boundary operator here is $N - H/3$, where N is the inward unit normal derivative, and H is the trace of the boundary embedding's second fundamental form.

This paper is organized as follows. In Sec. 1, we summarize the invariant-theoretic background needed to extract information from the heat asymptotics on manifolds with boundary. Sec. 1 also describes a natural fourth-order differential operator P, originally introduced by Paneitz ([**P**]; see also [**Br2**, **ES**]) in connection with the interaction of the gauge and conformal groups on Maxwell fields; P seems to be absolutely central to four-dimensional functional determinant problems. In Sec. 1, we also make precise statements of the above-mentioned analytic and naturality assumptions. In Sec. 2, we define the functional determinant and prove a formula of Polyakov type for its conformal variation. (See also [**R**, **BØ2**] in the boundariless case.) Though the functional determinant is a *nonlocal* invariant of the spectrum of (A, B) (i.e., it is not the integral of a local expression), its conformal variation is local, and in fact is a heat invariant. In Secs. 3 and 4, we apply this variational formula in concert with invariant-theoretic and conformal geometric knowledge of the heat invariants in dimension four to get explicit local formulas for the quotient of functional determinants. At this point, the operators A and B have not been pinned down, apart from their naturality and conformal behavior; thus our formulas at this point depend on (exactly 13) parameters. In Sec. 6 we compute these parameters for the two choices of (A, B) mentioned above: the conformal Laplacian with Dirichlet and Robin conditions. In Sec. 5, still in the abstract (parameter-dependent) setting, we compute determinant quotients on special manifolds; specifically the unit four-hemisphere H^4, the unit four-ball B^4, the spherical shell $\mathcal{A}^4_s = \{x \in \mathbb{R}^m \mid 1 \leq |x| \leq s\}$ for $s > 1$, and the cylinder $\mathcal{C}^4_h = [0, h] \times S^3$ for $h > 0$, all with their standard metrics. H^4 is conformally equivalent to B^4 and $\mathcal{A}^4_s$ to $\mathcal{C}^4_h$ (when $h = \log s$); this provides checks on our calculations. It also provides a value for the functional determinant on the ball since, as we show in Sec. 7, everything is explicitly computable on H^4. For example, we find that if Y_- (resp. Y_+) is the conformal Laplacian with Dirichlet (resp. Robin) conditions, then

$$\begin{aligned}
-\log\det Y_\pm &= \tfrac{1}{3}\zeta_R'(-3)+\tfrac{1}{6}\zeta_R'(-1)+\tfrac{1}{288}\pm\tfrac{1}{2}\zeta_R'(-2) && \text{on } H^4,\\
-\log\det Y_- &= \tfrac{1}{3}\zeta_R'(-3)+\tfrac{1}{6}\zeta_R'(-1)+\tfrac{1}{288}-\tfrac{1}{2}\zeta_R'(-2)+(4\log 2+\tfrac{17}{21})/360 && \text{on } B^4,\\
-\log\det Y_+ &= \tfrac{1}{3}\zeta_R'(-3)+\tfrac{1}{6}\zeta_R'(-1)+\tfrac{1}{288}+\tfrac{1}{2}\zeta_R'(-2)+(4\log 2-\tfrac{1}{3})/360 && \text{on } B^4,
\end{aligned}$$

where ζ_R is the Riemann zeta function. This shows, in particular, that the minimality result of [**BCY**, Sec. 5] for $\det Y$ in the conformal class of the standard metric on S^4 does not readily extend to the hemisphere: passage from the round H^4 metric to the flat B^4 metric "improves" (lowers) both functionals. In an appendix (Sec. 8), we collect in one place the definitions of local invariants used in developing the determinant quotient formulas, and prove some facts (used in Sec. 7) about zeta functions associated to spheres.

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1. Local invariants, natural differential operators and boundary value problems, and the heat invariants

Let M be a smooth, compact, m-dimensional Riemannian manifold with smooth boundary ∂M. Denote by g the metric tensor on M; the pullback of g under the inclusion $\partial M \hookrightarrow M$ is a Riemannian metric on ∂M. Let R be the Riemann curvature tensor of g, with the sign convention that makes $R^1{}_{212}$ positive on standard spheres. We adopt the convention that letters $i, j, \ldots$ run from 1 to m, and index a local coordinate frame and coframe on M. We raise and lower indices using the metric tensor, and sum over repeated indices. The *Ricci tensor* ρ of M has $\rho_{ij} = R^k{}_{ikj}$, and the *scalar curvature* of M is $\tau = \rho^i{}_i$.

Additional invariants describe the embedding of ∂M, and are defined as tensor fields over ∂M (as opposed to M). Let N be the inward unit geodesic normal in a collar for ∂M in M, and consider local coordinates (x^i) in a neighborhood of a point of ∂M for which $\partial/\partial x^m = N$, and for which the x^a, $a = 1, \ldots m-1$ are local coordinates on ∂M. Letters $a, b, \ldots$ will run from 1 to $m-1$, and index coordinate frames and coframes of this type on ∂M. The subscript N will be interchangeable with m in this setting, and will serve to indicate that we are working in such a coordinate system. We denote the coordinate coframe element dx^m by $N_\flat$. The (*second*) *fundamental form* L of the boundary embedding is a symmetric 2-tensor defined by

$$L_{ab} := -\tfrac{1}{2} N g_{ab}\,.$$

The trace $H := L^a{}_a$ of L is a multiple of the *mean curvature*. Here we have used $g|_{\partial M}$, the pullback of g to ∂M under the inclusion, to raise the boundary index; we shall always use $g|_{\partial M}$ as the metric on ∂M. Repeated boundary indices are, of course, summed from 1 to $m-1$. L measures the deviation of the boundary embedding from total geodesy; that is, it is the obstruction to the possibility of finding coordinates x^i which are normal on both M and ∂M.

A symmetric 2-tensor G is defined by

$$G^a{}_b := R^a{}_{NbN}\,,$$

and we let $F := G^a{}_a$. The symmetric 2-tensor T is defined by

$$T_{ab} := R^c{}_{acb}\,.$$

Note that $(T+G)_{ab} = \rho_{ab}$, and that $T^a{}_a = R^{ca}{}_{ca} = \tau - 2F$. We use g and its pullback $g|_{\partial M}$ to define quantities like $|\rho|^2 = \rho^{ij}\rho_{ij}$, $|L|^2 = L^{ab}L_{ab}$, $\langle L, G\rangle = L^{ab}G_{ab}$, etc. Intrinsic objects on ∂M which are analogous to objects on M will usually be denoted with a tilde; for example, $\tilde{g} = g|_{\partial M}$; ∇, $\tilde{\nabla}$ are the Levi-Civita connections on M and ∂M respectively; and Δ, $\tilde{\Delta}$ are the Laplacians on functions. The Riemannian measure on (M, g) will be denoted by dx, and the Riemannian measure on $(\partial M, \tilde{g})$ by dy. Our sign convention for the Laplacian gives $\Delta = -d^2/dx^2$ on $\mathbb{R}^1$. We shall sometimes use a standard abbreviation in which indices after a bar indicate covariant differentiations with respect to ∇, for example $\varphi_{ij|kl} = \nabla_l\nabla_k\varphi_{ij} := (\nabla\nabla\varphi)_{lkij}$; and indices after a colon similarly indicate covariant diffentiations with respect to $\tilde{\nabla}$.

Let

$$\begin{aligned} J &= \tau/2(m-1), \\ V &= (\rho - Jg)/(m-2), \qquad (1.1) \\ C^i{}_{jkl} &= R^i{}_{jkl} + V_{jk}\delta^i{}_l - V_{jl}\delta^i{}_k + V^i{}_l g_{jk} - V^i{}_k g_{jl}\,. \end{aligned}$$

C is the *Weyl conformal curvature tensor*. C, V, J carry the information in R in a way which is better adapted to conformal variational calculations than are R, ρ, τ. Specifically, let the metric run through a conformal curve $g[\varepsilon\omega] = e^{2\varepsilon\omega}g[0]$ for $\omega \in C^\infty(M)$ and ε a real parameter. Then $(d/d\varepsilon)|_{\varepsilon=0}(g[\varepsilon\omega]) = 2\omega g[0]$ and

$$(d/d\varepsilon)|_{\varepsilon=0}C[\varepsilon\omega] = 0, \qquad (1.2)$$

$$(d/d\varepsilon)|_{\varepsilon=0}J[\varepsilon\omega] + 2\omega J[0] = \Delta[0]\omega, \qquad (1.3)$$

$$(d/d\varepsilon)|_{\varepsilon=0}V[\varepsilon\omega] = -(\nabla\nabla)[0]\omega. \qquad (1.4)$$

Here we have used the following convention, which will be maintained throughout this paper: given a conformal class of metrics

$$\langle g[0]\rangle := \{e^{2\omega}g[0] \mid \omega \in C^\infty(M)\},$$

and a metric-dependent quantity T, we indicate that T should be evaluated in $g[\omega] := e^{2\omega}g[0]$ by writing $T[\omega]$. For example, the conformal invariance of $|C|^2dx$ on four-manifolds can be expressed as

$$(|C|^2dx)[\omega] = (|C|^2dx)[0], \qquad m = 4, \quad \omega \in C^\infty(M). \qquad (1.5)$$

We shall need the *Paneitz quantity*

$$Q = -2|V|^2 + \tfrac{m}{2}J^2 + \Delta J \qquad (1.6)$$

and *Paneitz operator*

$$P = \Delta^2 + \delta\{(m-2)J - 4V\cdot\}d + \frac{m-4}{2}Q.$$

Here d is the exterior derivative, δ is the formal adjoint of d, and $V\cdot$ is the bundle endomorphism $\varphi = (\varphi_i) \mapsto (V_i{}^j\varphi_j)$ on the cotangent bundle T^*M. By [**P**], [**Br2**, Theorem 1.21], [**ES**], P is conformally covariant: given a conformal class $\langle g_0\rangle$,

$$e^{\frac{m+4}{2}\omega}P[\omega] = P[0]\mu(e^{\frac{m-4}{2}\omega}), \quad \text{all } \omega \in C^\infty(M), \qquad m \neq 1,2, \tag{1.7}$$

where for any $u \in C^\infty(M)$, $\mu(u)$ denotes multiplication by u. The infinitesimal form of (1.7) is

$$(d/d\varepsilon)|_{\varepsilon=0}P[\varepsilon\omega] = -4\omega P[0] + \frac{m-4}{2}[P[0],\mu(\omega)]. \tag{1.8}$$

A conformal variational formula for the local scalar invariant Q in dimension $m = 4$ will be important for us. To get this, let $m \geq 3$ be arbitrary for the moment, and let

$$P_0 = P - \frac{m-4}{2}Q = \Delta^2 + \delta\{(m-2)J - 4V\cdot\}d.$$

Applying the conformal covariance relation (1.7) to the function 1, we get

$$\begin{aligned}\frac{m-4}{2}Q[\omega]e^{\frac{m+4}{2}\omega} &= \left(P_0[0] + \frac{m-4}{2}Q[0]\right)e^{\frac{m-4}{2}\omega}\\ &= P_0[0]\left(e^{\frac{m-4}{2}\omega} - 1\right) + \frac{m-4}{2}Q[0]e^{\frac{m-4}{2}\omega},\end{aligned}$$

since P_0 annihilates constants. This leads to the identity

$$\begin{aligned}Q[\omega]e^{\frac{m+4}{2}\omega} &= \frac{2}{m-4}P_0[0]\left(e^{\frac{m-4}{2}\omega} - 1\right) + Q[0]e^{\frac{m-4}{2}\omega}\\ &= P[0](\omega + (m-4)\omega^2a((m-4)\omega)) + Q[0] + (m-4)\omega b((m-4)\omega),\end{aligned} \tag{1.9}$$

where a and b are entire functions. This identity holds for $m \neq 4$, but since all conformal variational calculations can be done within spaces of polynomial invariants with rational-in-m coeffients, analytic continuation in m is justified, and we get

$$P[0]\omega + Q[0] = Q[\omega]e^{4\omega}, \qquad m = 4, \tag{1.10}$$

Taking the variation of (1.10), we have

$$P[0]\omega = (d/d\varepsilon)|_{\varepsilon=0}Q[\varepsilon\omega] + 4\omega Q[0], \qquad m = 4. \tag{1.11}$$

1.1 Remark. We shall work with differential operators on bundles of *tensor-spinors* over (M, g). One way to describe these is as follows. A tensor-spinor bundle $\mathbb{V}$ is a vector bundle associated to the principal $\mathrm{O}(m)$-bundle of orthonormal frames, the $\mathrm{SO}(m)$-bundle of oriented orthonormal frames, or the $\mathrm{Spin}(m)$-bundle of spin frames. (The groups $H = \mathrm{O}(m)$, $\mathrm{SO}(m)$, and $\mathrm{Spin}(m)$ are the natural structure groups of Riemannian, oriented Riemannian, and Riemannian spin geometry respectively.) That is, $\mathbb{V}$ has the form $\mathcal{F}_H \times_\rho V$, where (V, ρ) is a finite-dimensional representation of H, and $\mathcal{F}_H$ is the appropriate frame bundle. Since the defining representation T of $\mathrm{SO}(m)$ and the spin representation Σ of $\mathrm{Spin}(m)$ are faithful, any irreducible tensor-spinor bundle can be realized as a direct summand of an iterated tensor product

$$\mathcal{T}^{\alpha,\sigma}_{\beta,\tau} = (TM)^{\otimes\alpha} \otimes (T^*M)^{\otimes\beta} \otimes (\Sigma M)^{\otimes\sigma} \otimes (\Sigma^* M)^{\otimes\tau}, \quad \alpha, \beta, \sigma, \tau \in \mathbb{N}.$$

We shall need a quick review of some basic results on the small time asymptotic expansion of the trace of the heat operator. Details can be found in [**G2**], especially Sec. 1.9.

1.2 Analytic Assumptions. *Let A be a differential operator of positive order on sections of a tensor-spinor bundle $\mathbb{V}$ over M. Suppose that A is formally self-adjoint and has positive definite leading symbol $\sigma_{\mathrm{lead}}(A)$; that is, $\sigma_{\mathrm{lead}}(A)(x, \xi)$ is positive definite in $\mathrm{End}\,\mathbb{V}_x$ for all $x \in M$ and $0 \neq \xi \in T^*_x M$. Let B be an operator on the bundle of Cauchy data for A on ∂M with the property that the pair (A, B) is elliptic.*

1.3 Remark. Formal self-adjointness and the assumption on the leading symbol make sense because tensor-spinor bundles over a Riemannian manifold come equipped with Riemannian vector bundle structures. Since $\sigma_{\mathrm{lead}}(A)(x, -\xi) = (-1)^{\mathrm{ord}(A)}\sigma_{\mathrm{lead}}(A)(x, \xi)$, the assumption of positive definite leading symbol forces the order of A to be even. We shall always denote $\mathrm{ord}(A)$ by $2\ell > 0$, so that $\sigma_{\mathrm{lead}}(A) = \sigma_{2\ell}(A)$. We do not give the definition of ellipticity for a boundary value problem here as it is somewhat technical and distracting; see [**G2**, Sec. 1.9] for this.

1.4 Remark. The bundle W of order 2ℓ Cauchy data for sections of $\mathbb{V}$ has a natural grading by subbundles

$$W = W_0 \oplus \ldots \oplus W_{2\ell-1}$$

where W_j holds the $j^{\underline{\mathrm{th}}}$ Cauchy datum. The boundary operator B for an elliptic boundary value problem is valued in an auxiliary bundle W' which admits a similar grading

$$W' = W'_0 \oplus \ldots \oplus W'_{2\ell-1}$$

but which has $\dim W' = \frac{1}{2}\dim W$. (See the examples below.)

Let A_B be the restriction of A to the subspace

$$C^\infty(M, \mathbb{V})_B = \{F \in C^\infty(M, \mathbb{V}) \mid B(\mathrm{CD}_{2\ell} F) = 0\}.$$

Here $\mathrm{CD}_{2\ell} : C^\infty(M, \mathbb{V}) \to C^\infty(\partial M, W)$ is the operator which assigns the order 2ℓ Cauchy data. If $f \in C^\infty(M)$, there is an asymptotic expansion

$$\text{(1.12)} \qquad \mathrm{Tr}_{L^2} f \exp(-tA_B) \sim \sum_{n=0}^{\infty} a_n(f, A, B) t^{(n-m)/2\ell}, \quad t \downarrow 0,$$

where

$$\text{(1.13)} \qquad a_n(f, A, B) = \int_M f a_n(x, A) dx + \sum_{\nu=0}^{n-1} \int_{\partial M} (N^\nu f) a_{n,\nu}(y, A, B) dy.$$

The $a_n(x, A)$ and $a_{n,\nu}(y, A, B)$ are locally computable from the total symbols of A and B in local coordinates.

1.5 Remark. The auxiliary function f is a device which allows us to observe the distributional behavior of the heat kernel at the boundary. We are forced to deal with this extra information because, as we shall see below, conformal deformation of the asymptotics of $\mathrm{Tr} \exp(-tA_B)$ and of the functional determinant naturally lead to the asymptotics of $\mathrm{Tr}\, \omega \exp(-tA_B)$, where ω is the infinitesimal conformal factor as above. Here and below, we write simply "Tr" for Tr_{L^2}, and use the notation "tr" for traces over vector bundle fibers.

1.6 Naturality Assumptions. *Suppose that A and B are given locally by universal, polynomial formulas in the jets of a Riemannian metric g; the inverse $g^\sharp$ of g; plus (if orientation is involved), a volume form E; plus (if spin structure is involved) the fundamental tensor-spinor γ. Suppose that, with respect to uniform dilations of the metric, A has homogeneity degree $-\operatorname{ord} A$, and the boundary condition does not change:*

$$\text{(1.14)} \quad \bar{g} = \alpha^2 g \;\; (\bar{E} = \alpha^m E, \; \bar{\gamma} = \alpha^{-1}\gamma) \Rightarrow \bar{A} = \alpha^{-2\ell} A, \;\; \mathcal{N}(\bar{B} \circ \overline{\mathrm{CD}}_{2\ell}) = \mathcal{N}(B \circ \mathrm{CD}_{2\ell}),$$

where $\mathcal{N}$ is the null space. Suppose further that A satisfies the analytic assumptions 1.2 categorically; that is, the realization of (A, B) on any Riemannian manifold $(M, \partial M)$ with boundary satisfies the analytic assumptions.

1.7 Remark. If M has spin structure, the *fundamental tensor-spinor*, or *Clifford section* γ is a section of $TM \otimes \operatorname{End} \Sigma M \cong_{\mathrm{Spin}(m)} TM \otimes \Sigma M \otimes \Sigma^* M$, where TM is the tangent bundle and ΣM the spinor bundle. γ satisfies the *Clifford relations*

$$\gamma^i \gamma^j + \gamma^j \gamma^i = -2 g^{ij} \mathrm{Id}_\Sigma,$$

where γ^i is the local section of $\operatorname{End} \Sigma$ gotten by tensoring with dx^i and contracting the TM argument. The scalings of E and γ posited in (1.14) are those which are consistent with the scaling of the metric; the scaling of γ being forced by the Clifford relations. The Levi-Civita connection on TM is lifted to the spinor bundle (if any), and extended to iterated tensor products of TM, ΣM, and their duals, so that we may take covariant derivatives of tensor-spinor fields. ∇g, ∇E, and $\nabla \gamma$ all vanish when defined.

1.8 Remark. Since a scaling $\bar{g} = \alpha^2 g$ of the metric induces a scaling $\bar{N} = \alpha^{-1}N$ of the inward unit normal, the operator $\mathrm{CD}_{2\ell}$ is sensitive to uniform dilation; thus we had to speak of $\overline{\mathrm{CD}}_{2\ell}$ in (1.14).

1.9 Remark. By Weyl's invariant theory, A is built polynomially (using tensor product and contraction) from g, its inverse $g^\sharp$, ∇, and iterated covariant derivatives of R; plus (if orientation is involved) E; plus (if spin structure is involved) γ. B is similarly built from $\tilde{g}$, $\tilde{g}^\sharp$, N, $N_\flat$, ∇_N, $\tilde{\nabla}$, and tangential covariant derivatives of R and L, plus the restrictions to ∂M of E and/or γ if applicable. As a result, the $a_n(x, A)$ are polynomial in g, $g^\sharp$, and iterated covariant derivatives of R; plus E and/or γ if applicable. The $a_{n,\nu}(y, A, B)$ are polynomial in $\tilde{g}$, $\tilde{g}^\sharp$, N, $N_\flat$, and iterated tangential covariant derivatives ($\tilde{\nabla}$) of $R_{\partial M}$ and of L, plus the restrictions to ∂M of E and/or γ if applicable.

1.10 Remark. We shall say that a local scalar invariant on M or ∂M, or a natural differential operator $\mathcal{A}$ on some $C^\infty(M, \mathbb{V})$, has *level* n if it scales according to $\bar{\mathcal{A}} = \alpha^{-n}\mathcal{A}$ under uniform dilation $\bar{g} = \alpha^2 g$ of the metric, $0 < \alpha \in \mathbb{R}$ (with the compatible scalings $\bar{E} = \alpha^m E$, $\bar{\gamma} = \alpha^{-1}\gamma$ if applicable). For example, it is part of the naturality assumptions 1.6 that A has level 2ℓ (equal to its order). It is straightforward to show that we may measure the level as follows. If $\mathcal{A}$ is a level n monomial local invariant or monomial natural differential operator on M, of degree (k_R, k_∇) in (R, ∇), then

$$2k_R + k_\nabla = n.$$

If $\mathcal{A}$ is a level n monomial local invariant or monomial natural differential operator on ∂M, of degree $(k_R, k_L, k_{\tilde{\nabla}}, k_N)$ in $(R, L, \tilde{\nabla}, \nabla_N)$, then

$$2k_R + k_L + k_{\tilde{\nabla}} + k_N = n.$$

In the study of the index, analytic torsion, and functional determinant, a special role is played by quantities of level m, the dimension. Thus in this paper, we shall be especially interested in level 4 objects on M, for example the Paneitz quantity Q and operator P; and level 3 objects on ∂M.

1.11 Remark. By the last two remarks, the assumption of *categorically* positive definite leading symbol implies that $\sigma_{2\ell}(A)$ is polynomial in g and $g^\sharp$; plus, if applicable, E and/or γ; that is, no higher jets of these objects are involved.

1.12 Remark. Parity considerations force $a_{\mathrm{odd}}(x, A) = 0$, but the $a_{\mathrm{odd},\nu}(y, A, B)$ are generally nonzero. Homogeneity considerations (i.e., comparison of the behavior of the two sides of (1.13) under uniform dilation of the metric) imply that $a_n(x, A)$ has level n. Similarly, the $a_{n,\nu}(y, A, B)$ must have level $n - 1 - \nu$.

2. The Functional Determinant and its Conformal Variation

We retain the notation of Sec. 1, and assume that our boundary value problem (A, B) satisfies the analytic and naturality assumptions 1.2, 1.6. The analytic assumptions guarantee that (A, B) has real eigenvalue spectrum $\lambda_0 \leq \lambda_1 \leq \dots \uparrow +\infty$, with corresponding eigensections in $C^\infty(M,\mathbb{V})_B$. We define the *zeta function* of the problem (A,B) by

$$\zeta_{A,B}(s)=\sum_{\lambda_j\neq 0}|\lambda_j|^{-s}.$$

There exist $\varepsilon>0$ and $j_0\in\mathbb{N}$ such that $\lambda_j\geq j^\varepsilon$ whenever $j\geq j_0$, so $\zeta_{A,B}(s)$ is manifestly well-defined and holomorphic for large $\operatorname{Re}s$. Since there are only finitely many nonpositive λ_j, the heat expansion (1.12) gives

$$\begin{aligned}\sum_{\lambda_j\neq 0}e^{-t|\lambda_j|}&=-q(A,B)+2\sum_{\lambda_j<0}\sinh t\lambda_j+\sum_{n=0}^{N}a_n(A,B)t^{(n-m)/2\ell}+O\left(t^{\frac{N-m+1}{2\ell}}\right)\\&=\sum_{n=0}^{N}\underline{a_n(A,B)}t^{(n-m)/2\ell}+O\left(t^{\frac{N-m+1}{2\ell}}\right),\end{aligned}\tag{2.1}$$

where $a_n(A,B)=a_n(1,A,B)$, $q(A,B)$ is the multiplicity of 0 as an eigenvalue of (A,B), and

$$\underline{a_n(A,B)}=\begin{cases}a_n(A,B)-q(A,B), & n=m,\\ a_n(A,B)+2\sum_{\lambda_j<0}\lambda_j^k/k!, & n=m+2\ell(1+2k),\ k\in\mathbb{N},\\ a_n(A,B) & \text{otherwise.}\end{cases}\tag{2.2}$$

Applying the Mellin transform, we get a meromorphic continuation of $\zeta_{A,B}(s)$ to $\mathbb{C}$:

$$\begin{aligned}\zeta_{A,B}(s)=\frac{1}{\Gamma(s)}\Bigg(&\sum_{n=0}^{N}\left(s-\frac{m-n}{2\ell}\right)^{-1}\underline{a_n(A,B)}\\&+\int_0^1 t^{s-1}O\left(t^{\frac{N-m+1}{2\ell}}\right)dt+\int_1^\infty t^{s-1}\sum_{\lambda_j\neq 0}e^{-t|\lambda_j|}dt\Bigg),\end{aligned}$$

where $O(t^{(N-m+1)/2\ell})$ is the error term from (2.1). In particular, $\zeta_{A,B}(s)$ is regular at $s=0$, and we define the *functional determinant* of the problem (A,B) by

$$\det A_B=(-1)^{\#\{\lambda_j<0\}}\exp(-\zeta'_{A,B}(0)).$$

2.1 Remark. It is important to note that the functional determinant is not invariant under uniform dilation of the metric. Suppose, as before, that $\bar{g}=\alpha^2g$, and if applicable, $\bar{E}=\alpha^mE$, $\bar{\gamma}=\alpha^{-1}\gamma$. Then

$$\begin{aligned}\zeta_{\bar{A},\bar{B}}(0)&=\zeta_{A,B}(0),\\ \det\overline{A_B}&=\alpha^{-2\ell\zeta_{A,B}(0)}\det A_B\,.\end{aligned}\tag{2.3}$$

That is, the quantity $\zeta_{A,B}(0)$ is scale-invariant, and the functional determinant has a scale homogeneity which depends on $\zeta_{A,B}(0)$. Thus the functional

$$\mathcal{P}(A,B,g) = \mathrm{vol}(g)^{2\ell\zeta_{A,B}(0)/m} \det A_B$$

is a scale-invariant "version" of the determinant. An added advantage of $\mathcal{P}(A,B,g)$ is that, like the determinant, it is a spectral invariant, since

$$a_0(A,B) = C\,\mathrm{vol}(g),$$

where C is a constant depending only on $\sigma_{2\ell}(A)$. (The number $2\ell/m$ can be recovered from the spectrum because $-m/2\ell$ is the leading exponent in the heat asymptotics (1.12).) We emphasize that there is no reason to expect $\det A_B$ or $\mathcal{P}(A,B,g)$ to be the integral of a local expression, as is $a_n(A,B)$.

2.2 Remark. If $m > 1$ and $\partial M \neq \emptyset$, the functional

$$\mathcal{P}_\lambda(A,B,g) = \mathrm{vol}(g)^{2\ell\lambda/m}\,\mathrm{vol}(\tilde{g})^{2\ell(\zeta_{A,B}(0)-\lambda)/(m-1)} \det A_B\,, \qquad \lambda \in \mathbb{R}, \tag{2.4}$$

is also scale-invariant, and this raises the interesting prospect of interaction with the isoperimetric problem, especially in connection with extremal problems. The new ingredient, $\mathrm{vol}(\tilde{g})$, is often a spectral invariant: $a_1(A,B)$ has the form $C\,\mathrm{vol}(\tilde{g})$ for some constant C which depends on (A,B) but not on M. Thus $\mathrm{vol}(\tilde{g})$ is determined by the spectrum when $C \neq 0$. To preserve the spirit of the endeavor, and with a view toward the isospectral problem, one would like to choose exponents in (2.4) which are spectral invariants, perhaps by choosing $\lambda = 0$ or $\lambda = \zeta_{A,B}(0)$. (See Theorem 4.10 below.) If M and/or ∂M is disconnected, there is also the possibility of giving different components different weights in making scale corrections. Specifically, if M_u and $(\partial M)_v$ are the (finitely many) connected components of M and ∂M respectively, and $g_u = g|_{M_u}$, $\tilde{g}_v = g|_{(\partial M)_v}$, we can consider

$$\left(\prod_u \mathrm{vol}(g_u)^{2\ell\lambda_u/m}\right)\left(\prod_v \mathrm{vol}(\tilde{g}_v)^{2\ell\tilde{\lambda}_v/(m-1)}\right)\det A_B\,,$$
$$\text{where } \sum_u \lambda_u + \sum_v \tilde{\lambda}_v = \zeta_{A,B}(0).$$

The problem is that this may move us outside the realm of spectral invariants.

2.3 Remark. Suppose we are given an elliptic boundary value problem (D,b) in which D is formally self-adjoint, but does not necessarily have positive definite leading symbol. Let d be the order of D. If $r \in \mathbb{Z}^+$, we can form a new elliptic problem $(D^r, b^{(r)})$ by taking the $r^{\underline{\text{th}}}$ power: the boundary condition determining $b^{(r)}$ is

$$b \circ \mathrm{CD}_d\,\varphi = b \circ \mathrm{CD}_d(D\varphi) = \ldots = b \circ \mathrm{CD}_d(D^{r-1}\varphi) = 0.$$

The operator corresponding to the problem $(D^r, b^{(r)})$ will be called $(D_b)^r$. If (D,b) is natural, then so is $(D^r, b^{(r)})$, and if r is even, D^r has positive definite leading symbol.

We shall now impose some additional *conformal* assumptions.

2.4 Conformal Assumptions. *Suppose that A is a positive integer power of a natural differential operator D, $A = D^h$, which is conformally covariant in the sense that given a conformal class $\langle g[0]\rangle$,*

$$e^{a+2\ell/h}D[\omega] = D[0]\mu(e^{a\omega}), \qquad \omega \in C^\infty(M), \tag{2.5}$$

for some $a \in \mathbb{R}$. Here, in case orientation and/or spin structure are used, $E[\omega] := e^{m\omega}E[0]$, $\gamma[\omega] := e^{-\omega}\gamma[0]$. for some $a \in \mathbb{R}$. Suppose that B arises as $b^{(h)}$ as in Remark 2.3, where (D, b) is an elliptic boundary value problem, and that the conformal behavior of b is compatible with (2.5) in the sense that $\mathcal{N}((b_\circ \mathrm{CD}_{2\ell/h})[\omega]) = \mathcal{N}((b_\circ \mathrm{CD}_{2\ell/h})[0]\mu(e^{a\omega}))$, or equivalently,

$$\mathcal{N}((b_\circ \mathrm{CD}_{2\ell/h})[\omega]) = e^{-a\omega}\mathcal{N}((b_\circ \mathrm{CD}_{2\ell/h})[0]). \tag{2.6}$$

2.5 Remark. Our conformal assumptions are weaker than the assertion that (A, B) is conformally covariant; this is the special case $h = 1$. When we work in this generality, we can handle, for example, the conformal Laplacian D on middle-forms ($m/2$-forms for m even) with a suitable boundary operator B. By [**Br1**], D has the form

$$\delta d - d\delta + (\text{Ricci term}),$$

where d is the exterior derivative, and δ is the formal adjoint of d. If M is oriented, D interchanges the two eigenbundles $\Lambda^{m/2}_{\pm}M$ of the Hodge $\star$ operator, unlike the form Laplacian $\Delta = \delta d + d\delta$, which preserves both $\Lambda^{m/2}_{+}M$ and $\Lambda^{m/2}_{-}M$. There do, in fact exist boundary conditions which are suitable in the sense of ellipticity and conformal covariance of the right weight. On the leading symbol level, the resulting boundary conditions are absolute or relative conditions [**BG**, Sec. 7], and the necessary lower-order corrections are given by actions of the fundamental form L. (These results will appear separately.) Note that since

$$D^2 = \Delta^2 + (\text{lower order}),$$

D is elliptic. On the other hand, the Dirichlet problem for the spin Laplacian, i.e., for the square of the Dirac operator $\nabla\!\!\!/$ on the spinor bundle ΣM, is outside the framework we have described, even though $\nabla\!\!\!/$ is conformally covariant. The reason is that Dirichlet boundary conditions for $\nabla\!\!\!/^2$ do not arise from the iteration process of Remark 2.3. In fact, there are no local boundary conditions for $\nabla\!\!\!/$ which are elliptic in the sense we need; this is, of course, what leads to the eta invariant of Atiyah-Patodi-Singer. $(\nabla\!\!\!/, \mathcal{D})$ is not elliptic, even though this problem satisfies our conformal assumptions ($\nabla\!\!\!/$ is conformally covariant).

2.6 Remark. The infinitesimal form of the conformal covariance relation (2.5) is

$$(d/d\varepsilon)|_{\varepsilon=0}D[\varepsilon\omega] = -(2\ell/h)D[0] + a[D[0], \mu(\omega)]. \tag{2.7}$$

The finite and infinitesimal forms of the conformal covariance relation are, in fact, equivalent: an application of (2.7) with $g[\varepsilon_0\omega]$ in place of $g[0]$ gives

$$(d/d\varepsilon)|_{\varepsilon=\varepsilon_0}\{e^{(a+2\ell/h)\varepsilon\omega}D[\varepsilon\omega]\mu(e^{-a\varepsilon\omega})\} = 0 \tag{2.8}$$

for any $\varepsilon_0 \in \mathbb{R}$, so that (2.5) is obtained. In practice, the way in which we shall enforce (2.6) is to show that

$$(b\circ \mathrm{CD}_{2\ell/h})[\omega] = \mathcal{A}[\omega](b\circ \mathrm{CD}_{2\ell/h})[0]\mu(e^{a\omega}),$$

where $\mathcal{A}[\omega]$ is a smooth, functorial, ω-dependent section of $\operatorname{Aut} W'$, with the curves $\mathcal{A}[\varepsilon\omega]$ smooth and $\mathcal{E}[0] = \mathrm{Id}_{W'}$. An argument like (2.8) shows that this is in turn enforced by its infinitesimal form

$$(d/d\varepsilon)|_{\varepsilon=0}(b\circ \mathrm{CD}_{2\ell/h})[\varepsilon\omega] = a(b\circ \mathrm{CD}_{2\ell/h})[0]\mu(\omega) + \mathcal{E}[\omega](b\circ \mathrm{CD}_{2\ell/h})[0],$$

where $\mathcal{E}[\omega] := (d/d\varepsilon)|_{\varepsilon=0}\mathcal{A}[\varepsilon\omega] \in C^\infty(\partial M, \operatorname{End} W')$. In fact, in our examples, the entries of $\mathcal{A}[\omega]$ in the block decomposition corresponding to the grading of Remark 1.4 have the form $\mu(e^{c\omega})$ for various powers c.

2.7 Example. Let A be the *conformal Laplacian*, or *Yamabe operator*

$$Y = \Delta + \frac{m-2}{4(m-1)}\tau.$$

Y is conformally covariant *of bidegree* $((m-2)/2, (m+2)/2)$:

$$Y[\omega] = e^{-\frac{m+2}{2}\omega}Y[0]\mu(e^{\frac{m-2}{2}\omega}).$$

Though Y can be viewed as a conformally *invariant* operator between density bundles, we choose not to do so, and instead view it as acting on sections of a trivial line bundle over M. Accordingly, Dirichlet conditions for Y are obtained by letting W_0 be a trivial line bundle over ∂M, setting $W_1 = 0$, and setting

$$B_{0,0} = \mathrm{Id}, \qquad B_{0,1} = B_{1,0} = B_{1,1} = 0$$

in the block decomposition of Remark 1.4. Dirichlet conditions are, of course, conformally compatible.

2.8 Example. There is also a conformally compatible Neumann condition, sometimes called the *Robin condition* by physicists. This is obtained by "playing off" the conformal variation of the mean curvature against that of the normal vector field N, just as the variation of the scalar curvature τ compensates that of the Laplacian Δ to form the conformal Laplacian. By [**BG**, Appendix],

$$(d/d\varepsilon)|_{\varepsilon=0}N[\varepsilon\omega] = -\omega N[0], \qquad (d/d\varepsilon)|_{\varepsilon=0}H[\omega] + \omega H[0] = -(m-1)\omega_{|N}H[0].$$

Thus for all $a, \alpha \in \mathbb{R}$,

$$(d/d\varepsilon)|_{\varepsilon=0}(N+aH)[\varepsilon\omega] + \omega(N+aH)[0] - \alpha[(N+aH)[0], \mu(\omega)] = (-a(m-1)-\alpha)\mu(\omega_{|N}).$$

As a result, there is an infinitesimal conformal covariance law for $\nabla + aH$ for each $a \in \mathbb{R}$:

$$(d/d\varepsilon)|_{\varepsilon=0}(N+aH)[\varepsilon\omega] = -\omega(N+aH)[0] - a(m-1)[(N+aH)[0], \mu(\omega)].$$

In particular, the boundary operator

$$N - \frac{m-2}{2(m-1)}H$$

is conformally compatible with Y. More precisely, to set up the Robin condition, we let W_1' be a trivial line bundle, $W_0' = 0$, $B_{1,1} = \mathrm{Id}$, $B_{0,1} = -(m-2)H/2(m-1)$, and $B_{0,0} = B_{1,0} = 0$. The Robin condition is important in the study of the Yamabe problem on manifolds with boundary; see [**E2**].

2.9 Remark. If (A,B) satisfies 1.2, 1.6, and 2.4, then so does $(A^r, B^{(r)})$ for each $r \in \mathbb{Z}^+$. This is sometimes useful in that it allows us to get rid of the (finite multiplicity) negative spectrum of (A_B) by passing to $(A_B)^2$.

An extremely important property from our point of view is a generalization of the scale-invariance property (2.3) to *pointwise* (conformal) scalings under the conformal assumptions 2.4. Following [**BØ1**], we call this a *conformal index* property.

2.10 Conformal Index Theorem. *If (A,B) satisfies 1.2, 1.6, and 2.4, and $g[0]$ is a Riemannian metric on M, then the quantities $q(A,B)$, $\#\{\lambda_j < 0\}$, $a_m(A,B)$, and $\zeta_{A,B}(0)$ are constant on the conformal class $\langle g[0]\rangle$.*

Proof. Let D be as in 2.4. The spectral invariants of $A[\omega]$ on $\mathcal{N}((B \circ \mathrm{CD}_{2\ell})[\omega])$ are the same as those of $(\mu(e^{-(a+2\ell/h)\omega})D[0]\mu(e^{a\omega}))^h$ on $\mathcal{N}((B \circ \mathrm{CD}_{2\ell})[0]\mu(e^{a\omega}))$. The spectral invariants of the latter problem are the same as those of

$$\underline{A[\omega]} := e^{a\omega}(\mu(e^{-(a+2\ell/h)\omega})D[0]\mu(e^{a\omega}))^h \mu(e^{-a\omega}) = (\mu(e^{-2\ell\omega/h})D[0])^h$$

on $\mathcal{N}((B \circ \mathrm{CD}_{2\ell})[0])$. Here we have applied a "global gauge transformation" in conjugating by $\mu(e^{a\omega})$; this does not affect spectral data, and has the advantage of transforming the original problem into one in which the boundary condition is fixed. Note that all of the boundary value problems mentioned are elliptic because the original one is. Because A_B has pure eigenvalue spectrum, the null spaces $\mathcal{N}(A)$ and $\mathcal{N}(D)$ in $C^\infty(M,\mathbb{V})_B$ agree. But by the conformal covariance relations, the dimension of $\mathcal{N}(D)$ in $C^\infty(M,\mathbb{V})_B$ is conformally invariant; thus $q(A,B)$ is conformally invariant. By a straightforward extension of an argument in [**Bl**, Proposition 1], the number of negative eigenvalues of $(\mu(e^{-2\ell\omega/h})D[0])^h$ on $\mathcal{N}((B \circ \mathrm{CD}_{2\ell})[0])$ is independent of ε; this uses the fact that the number of zero eigenvalues is independent of ε. Since $\zeta_{A,B}(0) = a_m(A,B) - q(A,B)$, we just need to show that $a_m(A,B)$ is conformally invariant.

For this, fix $\omega \in C^\infty(M)$, and consider the conformal curve of metrics $g[\varepsilon\omega] = e^{2\varepsilon\omega}g[0]$. If we can show that the variation operator $(d/d\varepsilon)|_{\varepsilon=0}$ annihilates the functional $a_m(A,B)$, we are done, since this result may then be applied with any $g[\varepsilon_0\omega]$ in place of $g[0]$, and ω is arbitrary. By the preceding paragraph, it is sufficient to show that $(d/d\varepsilon)|_{\varepsilon=0}$ annihilates $a_m(\underline{A[\varepsilon\omega]}, B[0])$. The estimates in [**GS**] justify the following formal computation:

$$\begin{aligned}
\sum_{n=0}^{\infty}(d/d\varepsilon)|_{\varepsilon=0}a_n(A[\varepsilon\omega],B[\varepsilon\omega])t^{(n-m)/2\ell} &= \sum_{n=0}^{\infty}(d/d\varepsilon)|_{\varepsilon=0}a_n(\underline{A[\varepsilon\omega]},B[0])t^{(n-m)/2\ell} \\
&\sim (d/d\varepsilon)|_{\varepsilon=0}\operatorname{Tr}\exp(-t(\underline{A[\varepsilon\omega]})_B[0]) \\
&= -t\operatorname{Tr}\{((d/d\varepsilon)|_{\varepsilon=0}(\underline{A[\varepsilon\omega]})_B[0])\exp(-t(A_B)[0])\} \\
&= -t\operatorname{Tr}\{((d/d\varepsilon)|_{\varepsilon=0}(\mu(e^{-2\ell\varepsilon\omega/h})D[0])^h)_{B[0]}\exp(-t(A_B)[0])\} \\
&= 2\ell t\operatorname{Tr}\{\omega(A_B\exp(-tA_B))[0]\} \\
&= -2\ell t(d/dt)\operatorname{Tr}\{\omega\exp(-t(A_B)[0])\} \\
&\sim \sum_{n=0}^{\infty}(m-n)a_n(\omega,A[0],B[0])t^{(n-m)/2\ell},
\end{aligned} \tag{2.9}$$

where the asymptotics are for $t \downarrow 0$. Here we have used the fact that that $\exp(-t(A_B)[0])$ is a smoothing operator for $t > 0$, with the consequence that

$$\mathrm{Tr}(UVe^{-tA_B}) = \mathrm{Tr}(VUe^{-tA_B})$$

as long as U and V are finite-order pseudo-differential operators, and V commutes with A_B, Comparing coefficients for $n = m$, we get the result. □

In the course of the proof, we have actually computed the conformal variation of $a_n(A,B)$ for every n:

2.11 Corollary. *Under the assumptions of Theorem 2.10,*

$$(d/d\varepsilon)|_{\varepsilon=0} a_n(A[\varepsilon\omega], B[\varepsilon\omega]) = (m-n)a_n(\omega, A[0], B[0]). \qquad \square$$

The corollary shows that $a_n(A,B)$ is a conformal *primitive*, or integral, for $a_n(\omega, A, B)$ provided $n \neq m$. The following variational formula, which will be fundamental to our computations, shows that the functional determinant supplies the "missing" primitive for $a_m(\omega, A, B)$, at least when the conformal invariant $q(A,B)$ vanishes.

2.12 Theorem. *Suppose (A,B) satisfies 1.2, 1.6, and 2.4. Let $(M, g[0])$ be a particular manifold with boundary together with a conformal class on which $\mathcal{N}(A_B) = 0$, and let $\omega \in C^\infty(M)$. Then*

$$(d/d\varepsilon)|_{\varepsilon=0} \zeta'_{A[\varepsilon\omega],B[\varepsilon\omega]}(0) = 2\ell a_m(\omega, A[0], B[0]).$$

Proof. First assume that $(A_B)[0]$ is positive. By the conformal invariance of $q(A,B)$ and of $\#\{\lambda_j < 0\}$ (Theorem 2.10), $(A_B)[\varepsilon\omega]$ is positive for all $\varepsilon \in \mathbb{R}$, so that the Mellin transform relates the zeta function to $\mathrm{Tr}\exp(-tA_B)$, without the modifications of (2.2). The estimates in [**GS**] allow us to conclude that $(d/d\varepsilon)|_{\varepsilon=0}\zeta_{A,B}(s)$ is meromorphic, and that we can interchange the order of conformal variation and analytic continuation. For $\mathrm{Re}\, s$ large,

$$\begin{aligned} (d/d\varepsilon)|_{\varepsilon=0}\zeta'_{A[\varepsilon\omega],B[\varepsilon\omega]}(s) &= (d/ds)(d/d\varepsilon)|_{\varepsilon=0}\zeta_{A[\varepsilon\omega],B[\varepsilon\omega]}(s) \\ &= (d/ds)\left\{\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\{(d/d\varepsilon)|_{\varepsilon=0}\,\mathrm{Tr}\exp(-t(A_B)[\varepsilon\omega])\}dt\right\} \\ &= -(d/ds)\left\{\frac{2\ell}{\Gamma(s)}\int_0^\infty t^s (d/dt)\,\mathrm{Tr}\{\omega\exp(-t(A_B)[0])\}dt\right\} \\ &= (d/ds)\left\{\frac{2\ell s}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}\{\omega\exp(-t(A_B)[0])\}dt\right\}. \end{aligned} \tag{2.10}$$

Here we have integrated by parts in t, and used the computations in (2.9). Analytically continuing this formula, the value at $s = 0$ is the same as that of

$$\frac{2\ell}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}\{\omega\exp(-t(A_B)[0])\}dt,$$

that being $2\ell a_m(\omega, A[0], B[0])$.

To dispense with the positivity assumption on A_B, note that we have proved the result for the positive operator $(A_B)^2$. (Recall Remark 2.9.) But $\zeta_{A^2,B^{(2)}}(s) = \zeta_{A,B}(2s)$, so $\zeta'_{A^2,B^{(2)}}(0) = 2\zeta'_{A,B}(0)$. But by a straightforward extension of [**FG**, Theorem 2.4] to boundary value problems, $a_m(\omega, A^2, B^{(2)}) = a_m(\omega, A, B)$. □

2.13 Remark. The effect of zero spectrum on this argument is as follows. Since $q(A,B)$ is conformally invariant, nothing in (2.10) changes until we apply the ε-derivative; the trace on the third line becomes

$$\operatorname{Tr}(\omega\exp(-t(A_B)[0])-\mathcal{P}),\qquad \mathcal{P}=\operatorname{Proj}_{\mathcal{N}(A[0])\cap\mathcal{N}((B\circ\mathrm{CD}_{2\ell})[0])}\,.$$

The kernel function of $\omega\{\exp(-t(A_B)[0])-\mathcal{P}\}$ is

$$\omega(x)\left\{H(t,x,y)-\sum_{\lambda_j=0}\varphi_j(x)\otimes\varphi_j^*(y)\right\},$$

where $H(t,x,y)$ is the kernel function of $\exp(-t(A_B)[0])$, and $\{\varphi_j\}$ is an orthonormal basis of eigensections, $A[0]\varphi_j=\lambda_j\varphi_j$, $(B\circ\mathrm{CD}_{2\ell})[0]\varphi_j=0$. The conclusion is that

$$(d/d\varepsilon)|_{\varepsilon=0}\zeta'_{A[\varepsilon\omega],B[\varepsilon\omega]}(0)=2\ell\left(a_m(\omega,A[0],B[0])-\omega(x)\sum_{\lambda_j=0}|\varphi_j(x)|^2\right).$$

Thus an explicit formula for the local heat invariant $a_m(\omega,A,B)$, or such a formula together with an explicit knowledge of the null space $\mathcal{N}(A_B)$ when this null space is nonzero, is sufficient for an understanding of the conformal behavior of the functional determinant. Note that an explicit knowledge of $\mathcal{N}(A_B)$ is not an unreasonable expectation: if the scalar curvature of the background metric has positive scalar curvature, there can be no null space for an elliptic boundary problem Y_B based on the conformal Laplacian Y. For more general (A,B) satisfying 2.4, if M and ∂M are locally flat (for example, if M is a standard flat half-torus), $\mathcal{N}(A_B)$ can be given explicitly in the background metric, and thus in conformal metrics by the conformal covariance law.

The strategy for computing the functional determinant within a conformal class will be to integrate the variational formula along a one-parameter family $g[\varepsilon\omega]=e^{2\varepsilon\omega}g[0]$. The result will be a formula for the difference $\zeta'_{A[\omega],B[\omega]}(0)-\zeta'_{A[0],B[0]}(0)$; that is, for the quotient of determinants

$$(\det(A_B)[\omega])/(\det(A_B)[0]).$$

The formulas involve integrals of differential polynomials in ω, but such quantities cannot necessarily be re-expressed as integrals of scalar local invariants in the sense of Remark 1.9. For example, the quantity $\int\omega P\omega$, where P is the Paneitz operator, appears in our formulas; it cannot, in general, be expressed as the integral of a local scalar invariant of $g[\omega]$. This phenomenon is one conformal manifestation of the nonlocal nature of the functional determinant. To express everything in terms of differential polynomials, at least via the current methods, it is very important that we stay within a conformal class.

The problem of computing $(\det(A_B)[0])$, so that we have formulas for functional determinants instead of just quotients of such, may be approached separately; see Sec. 7 below.

3. Variational formulas and consequences of the conformal index property

In this section, fix $\omega \in C^\infty(M)$, and again consider the variation $(d/d\varepsilon)|_{\varepsilon=0}$ as the metric g runs through the conformal curve $g[\varepsilon\omega] = e^{2\varepsilon\omega}g[0]$ for a fixed (but arbitrary) $\omega \in C^\infty(M)$. We extend the definition of local scalar invariant to f*-augmented* local scalar invariants, $f \in C^\infty(M)$, by adding df (or $\tilde{d}f$ and Nf for boundary invariants) to the list of ingredients in Remark 1.9. (Note that suitable derivatives are also ingredients, so it is only the $0^{\underline{\text{th}}}$ derivative of f that does not come into play.) When there is no chance of confusion as to the choice of manifold or measure, or when these choices are arbitrary, we shall sometimes abbreviate $\int_M \cdot\, dx$ by $\int \cdot$, and $\int_{\partial M} \cdot\, dy$ by $\oint \cdot$.

We begin by choosing a nonstandard basis of the interior invariants.

3.1 Lemma. *With notation as in Sec. 1, the 4 quantities $|C|^2$, Q, J^2, ΔJ span the space of level 4 local scalar $O(m)$ invariants on M for $m \geq 3$; for $m \geq 4$ they are a basis. If $m \geq 5$, these 4 quantities are also a basis of the level 4 local scalar $SO(m)$-invariants. If $m = 4$ and $C_\pm$ are the self- and anti-self-dual parts of C, the 5 quantities $|C_+|^2$, $|C_-|^2$, Q, J^2, ΔJ are a basis of the level 4 local scalar $SO(4)$ invariants on M.*

Proof. Let $m \geq 3$. By (1.1), $\Delta\tau$ is a scalar multiple of ΔJ, $|\rho|^2$ is a linear combination of J^2 and $|V|^2$, and $|R|^2$ is a linear combination of J^2, $|V|^2$, and $|C|^2$. $|V|^2$ is a linear combination of Q, J^2, and ΔJ. By, e.g., **[G1]**, the 4 quantities $|R|^2$, $|\rho|^2$, τ^2, $\Delta\tau$ span the O(m) invariants, and are a basis for $m \geq 4$. For $m \geq 5$, all O(m)-irreducible summands of the vector bundle of which R is a section are also SO(m)-irreducible. For $m = 4$, this is true except for the O(4)-bundle of which C is a section; this splits into two irreducible summands under SO(4) **[S]**, and this induces the splitting $C = C_+ + C_-$. □

Using the invariant theory of **[BG]**, we can write down all the invariants that can appear in $a_4(f, A, B)$. In Table 3.1, we introduce abbreviations for some level 3 local scalar invariants on ∂M. $\operatorname{tr} L^3$ is an abbreviation for the local scalar invariant $L^a{}_b L^b{}_c L^c{}_a$. For convenience, all indices are written as subscripts, the convention being that one index in each pair is raised before summing.

Abbreviation	Invariant	Index expression
X_1	$N\tau$	$R_{ijij\|N}$
X_2	τH	$R_{ijij}L_{aa}$
X_3	FH	$R_{aNaN}L_{bb}$
X_4	$\langle G, L\rangle$	$R_{aNbN}L_{ab}$
X_5	$\langle T, L\rangle$	$R_{cacb}L_{ab}$
X_6	H^3	$L_{aa}L_{bb}L_{cc}$
X_7	$H\|L\|^2$	$L_{aa}L_{bc}L_{bc}$
X_8	$\operatorname{tr} L^3$	$L_{ab}L_{bc}L_{ca}$

Table 3.1

Let $f \in C^\infty(M)$ be an auxiliary function. In Table 3.2, we introduce abbreviations for some f-augmented level 3 local scalar invariants. Note that because the inward unit normal is extended to a collar as the tangent to a unit speed geodesic, the iterated partial derivatives $N \ldots Nf$ agree with the iterated covariant derivatives $f_{|N\ldots N} = (\nabla \ldots \nabla f)_{N\ldots N}$.

Abbreviation	Invariant	Index expression
$Y_1(f)$	$(Nf)\tau$	$f_{\vert N}R_{ijij}$
$Y_2(f)$	$(N^2 f)H$	$f_{\vert NN}L_{aa}$
$Y_3(f)$	$(-\tilde{\Delta} f)H$	$f_{:aa}L_{bb}$
$Y_4(f)$	$(Nf)H^2$	$f_{\vert N}L_{aa}L_{bb}$
$Y_5(f)$	$(Nf)F$	$f_{\vert N}R_{aNaN}$
$Y_6(f)$	$\langle\tilde{\nabla}\tilde{\nabla} f, L\rangle$	$f_{:ab}L_{ab}$
$Y_7(f)$	$(Nf)\vert L\vert^2$	$f_{\vert N}L_{ab}L_{ab}$
$Y_8(f)$	$N(-\Delta)f$	$f_{\vert iiN}$

Table 3.2

From [**BG**, Lemma 2.3] and the above, we get:

3.2 Lemma. *Suppose that either (A, B) is not orientation-sensitive or $m > 4$. Under the analytic and naturality assumptions 1.2, 1.6, $a_4(f, A, B)$ has the form*

$$a_4(f, A, B) = \int f\{\alpha_{1,1}|C|^2 + \alpha_{1,2}Q + \alpha_{1,3}J^2 + \alpha_{1,4}\Delta J\}$$
$$+ \oint \left(f\sum_{\mu=1}^{8} \alpha_{2,\mu}X_\mu + \sum_{\nu=1}^{8} \alpha_{3,\nu}Y_\nu(f) \right)$$

for some constants $\alpha_{\sigma,\mu}$ which depend only on the formal functorial expression for (A, B), and on m. (In particular, they do not depend on the particular manifold or metric.) If $m = 4$ and (A, B) is orientation-sensitive, the same is true with $\alpha^+_{1,1}|C_+|^2 + \alpha^-_{1,1}|C_-|^2$ in place of $\alpha_{1,1}|C|^2$ for for constants $\alpha^\pm_{1,1}$. $\square$

When $f = 1$, the invariants $Y_\nu(f)$ vanish, and integration by parts gives

$$\int \Delta J = \left(\oint X_1\right) \Big/ 2(m-1). \tag{3.1}$$

Thus $a_4(A, B)$ has the form

$$a_4(A, B) = \int \{\alpha_{1,1}|C|^2 + \alpha_{1,2}Q + \alpha_{1,3}J^2\} + \oint \sum_{\mu=1}^{8} \tilde{\alpha}_{2,\mu}X_\mu\,, \tag{3.2}$$

where the $\tilde{\alpha}_{2,1}$ are constants with the same dependence as above. (We make the obvious adjustment if $m = 4$ and (A, B) is orientation-sensitive.) Under the conformal assumptions 2.4, the number of undetermined coefficients in Lemma 3.2 and (3.2) is cut down considerably; the "axe" is the Conformal Index Theorem 2.10. To apply this, we need to know the conformal variations of the quantities involved.

3.3 Lemma. *Let $m = 4$. For the conformal variation above,*
(a) $(d/d\varepsilon)|_{\varepsilon=0}(|C|^2dx)[\varepsilon\omega] = 0$. *If M is oriented,* $(d/d\varepsilon)|_{\varepsilon=0}(|C_\pm|^2dx)[\varepsilon\omega] = 0$.
(b) $(d/d\varepsilon)|_{\varepsilon=0}\int(J^2dx)[\varepsilon\omega] = 2\int\omega((\Delta J)dx)[0] + \frac{1}{3}\oint(\{-\omega X_1 + Y_1(\omega)\}dy)[0]$.
(c) $(d/d\varepsilon)|_{\varepsilon=0}\int(|V|^2dx)[\varepsilon\omega] = 2\int\omega((\Delta J)dx)[0] + \oint(\{-\frac{1}{3}\omega X_1 - \frac{1}{6}Y_1(\omega) - Y_3(\omega) + Y_5(\omega) + Y_6(\omega)\}dy)[0]$.
(d) $(d/d\varepsilon)|_{\varepsilon=0}\int(\Delta Jdx)[\varepsilon\omega] = (d/d\varepsilon)|_{\varepsilon=0}\oint(J_{|N}dy)[\varepsilon\omega] = \oint(\{-\frac{1}{3}Y_1(\omega) - Y_8(\omega)\}dy)[0]$.
(e) $\frac{1}{2}(d/d\varepsilon)|_{\varepsilon=0}\int(Qdx)[\varepsilon\omega] = \oint(S(\omega)dy)[0]$, *where* $S(\omega) = \frac{1}{3}Y_1(\omega) + Y_3(\omega) - Y_5(\omega) - Y_6(\omega) - \frac{1}{2}Y_8(\omega)$.

Proof. (a) was already remarked as (1.5). (The statements about $C_\pm$ follow from the fact that the splitting into C_+ and C_- is conformally invariant.) For (b), we use (1.3) and integrate by parts to get:

$$\begin{aligned}(d/d\varepsilon)|_{\varepsilon=0}\int(J^2dx)[\varepsilon\omega] &= 2\int J(\Delta\omega)dx = 2\int\langle dJ, d\omega\rangle dx + 2\oint J\omega_{|N}dy\\ &= 2\int\omega(\Delta J)dx - 2\oint J_{|N}\omega dy + 2\oint J\omega_{|N}dy,\end{aligned}$$

where everything after the first "=" sign has an implicit [0] (is evaluated in $g[0]$).

For convenience in the rest of the proof, we write all indices as subscripts; one copy of each repeated index should be raised before summing. For (c), we use (1.4) and the Bianchi identity $V_{ij|i} = J_{|j}$, and integrate by parts:

$$\begin{aligned}(d/d\varepsilon)|_{\varepsilon=0}\int(|V|^2dx)[\varepsilon\omega] &= -2\int\langle V, \nabla\nabla\omega\rangle dx = 2\int\langle dJ, d\omega\rangle dx + 2\oint V_{iN}\omega_{|i}dy\\ &= 2\int\omega\Delta Jdx - 2\oint J_{|N}\omega dy + 2\oint V_{iN}\omega_{|i}dy.\end{aligned}$$

Again, everything after the first "=" sign is evaluated in $g[0]$. But

$$V_{Ni}\omega_{|i} = V_{NN}\omega_{|N} + V_{Na}\omega_{:a}\,,$$

and

$$\oint V_{Na}\omega_{:a} = -\oint V_{aN:a}\omega = -\tfrac{1}{2}\oint\rho_{aN:a}\omega.$$

By [**BG**, Lemma A.1(b)],

$$\rho_{aN:a} = H_{:aa} - L_{ab:ab}\,.$$

Since $V_{NN} = \frac{1}{2}F - \frac{1}{12}\tau$, integration by parts over ∂M (which has no boundary) gives

$$\oint V_{Ni}\omega_{|i} = \oint\left(\tfrac{1}{2}Y_5(\omega) - \tfrac{1}{12}Y_1(\omega) - \tfrac{1}{2}Y_3(\omega) + \tfrac{1}{2}Y_6(\omega)\right). \tag{3.3}$$

This and $J = \tau/6$ give (c).

The first equality in (d) is obtained by integrating by parts; the second is Lemma 3.4(a) below. For the proof of (e), we use (1.10):

$$\begin{aligned}\tfrac{1}{2}(d/d\varepsilon)|_{\varepsilon=0}\int(Qdx)[\varepsilon\omega] &= \tfrac{1}{2}\int(P\omega)dx = \tfrac{1}{2}\int\omega_{|iijj}dx - \int[\{J-2V\cdot\}d\omega]_{j|j}dx\\ &= -\tfrac{1}{2}\oint\omega_{|iiN}dy + \oint[\{J-2V\cdot\}d\omega]_N dy\\ &= -\tfrac{1}{2}\oint\omega_{|iiN}dy + \oint(J\omega_{|N} - 2V_{Ni}\omega_{|i})dy.\end{aligned}$$

(3.3) now finishes the proof. Alternatively, we could use the definition (1.6) of Q together with parts (b–d) to derive (e). □

We shall also need the conformal variations of the boundary invariants that do not automatically vanish for $\omega = 1$. The following can be read off from the variational formulas in the appendix to [**BG**]. (Note the differences in sign conventions.)

3.4 Lemma. *Let* $X_i'(\omega) = (d/d\varepsilon)|_{\varepsilon=0}(e^{3\varepsilon\omega}X_i[\varepsilon\omega]) = (d/d\varepsilon)|_{\varepsilon=0}X_i[\varepsilon\omega] + 3\omega X_i[0]$. *Then:*
(a) $X_1'(\omega) = -2Y_1(\omega) - 2(m-1)Y_8(\omega)$.
(b) $X_2'(\omega) = -(m-1)Y_1(\omega) - 2(m-1)Y_2(\omega) - 2(m-1)Y_3(\omega) + 2(m-1)Y_4(\omega)$.
(c) $X_3'(\omega) = -(m-1)Y_2(\omega) - Y_3(\omega) + Y_4(\omega) - (m-1)Y_5(\omega)$.
(d) $X_4'(\omega) = -Y_2(\omega) - Y_5(\omega) - Y_6(\omega) + Y_7(\omega)$.
(e) $X_5'(\omega) = -Y_1(\omega) - Y_3(\omega) + Y_4(\omega) + 2Y_5(\omega) - (m-3)Y_6(\omega) + (m-3)Y_7(\omega)$.
(f) $X_6'(\omega) = -3(m-1)Y_4(\omega)$.
(g) $X_7'(\omega) = -2Y_4(\omega) - (m-1)Y_7(\omega)$.
(h) $X_8'(\omega) = -3Y_7(\omega)$. □

We retain the "prime" notation of Lemma 3.4 to derive some straightforward consequences in the next two lemmas.

3.5 Lemma. *Let*

$$\mathcal{L}_4 = -\frac{1}{m-1}X_2 + X_3 - (m-3)X_4 + X_5\,,$$
$$\mathcal{L}_5 = -\frac{2}{3(m-1)}X_6 + X_7 - \frac{m-1}{3}X_8\,.$$

Then $\mathcal{L}_s'(\omega) = 0$, $s = 4, 5$. □

3.6 Lemma. *Let* $m = 4$, *and let*

$$S = -\tfrac{1}{12}X_1 + \tfrac{1}{6}X_2 - X_4 + \tfrac{1}{9}X_6 - \tfrac{1}{3}X_8.$$

Then $S'(\omega) = -\mathcal{S}(\omega)$, *where* $\mathcal{S}(\omega)$ *is as in Lemma 3.3(e).* □

We can now harvest the consequences of the conformal index property, reducing the number of undetermined coefficients from the 20 in Lemma 3.2 to just 13 under the conformal assumptions.

3.7 Theorem. *Let $m = 4$, and suppose (A, B) satisfies 1.2, 1.6, and 2.4. Suppose that (A, B) is not orientation-sensitive. Then $a_4(f, A, B)$ has the form*

$$a_4(f,A,B) = \beta_1 \int f|C|^2 + \beta_2 \left\{ \tfrac{1}{2} \int fQ + \oint fS \right\}$$
$$+ \beta_3 \left\{ \int f\Delta J - \frac{1}{6} \oint fX_1 \right\} + \beta_4 \oint f\mathcal{L}_4 + \beta_5 \oint f\mathcal{L}_5 + \oint \sum_{\nu=1}^{8} \kappa_\nu Y_\nu(f),$$

where the constants β_μ, $\mu = 1, \ldots, 5$ and κ_ν, $\nu = 1, \ldots, 8$ depend only on the formal functorial expression for (A, B). In particular,

$$a_4(A,B) = \beta_1 \int |C|^2 + \beta_2 \left\{ \tfrac{1}{2} \int Q + \oint S \right\} + \beta_4 \oint \mathcal{L}_4 + \beta_5 \oint \mathcal{L}_5 . \tag{3.4}$$

If (A, B) is orientation-sensitive, the same is true with $\beta_{1,+}|C_+|^2 + \beta_{1,-}|C_-|^2$ in place of $\beta_1|C|^2$ in each formula.

Proof. Changing basis in the formula of Lemma 3.2, we may write $a_4(f, A, B)$ in the form

$$\beta_1 \int f|C|^2 + \beta_2 \left\{ \tfrac{1}{2} \int fQ + \oint fS \right\} + \beta_3 \left\{ \int f\Delta J - \frac{1}{6} \oint fX_1 \right\} + c \int fJ^2$$
$$+ \oint f\{\beta_4\mathcal{L}_4 + \beta_5\mathcal{L}_5 + \gamma_1X_1 + \gamma_2X_2 + \gamma_3X_3 + \gamma_4X_4 + \gamma_6X_6 + \gamma_8X_8\} + \oint \sum_{\nu=1}^{8} \kappa_\nu Y_\nu(f)$$

for some universal constants β_i, c, γ_j. In particular, by (3.1),

$$a_4(A,B) = \beta_1 \int |C|^2 + \beta_2 \left(\tfrac{1}{2} \int Q + \oint S \right) + c \int J^2 + \beta_4 \oint \mathcal{L}_4 + \beta_5 \oint \mathcal{L}_5 + \sum_{j=1,2,3,4,6,8} \oint \gamma_j X_j .$$

By Lemmas 3.3–3.6,

$$(d/d\varepsilon)|_{\varepsilon=0} a_4(A[\varepsilon\omega], B[\varepsilon\omega]) = 2c \int \omega(\Delta J)dx - \tfrac{1}{3}c \oint \omega X_1$$
$$+ (\tfrac{1}{3}c - 2\gamma_1 - 3\gamma_2) \oint Y_1(\omega) + (-6\gamma_2 - 3\gamma_3 - \gamma_4) \oint Y_2(\omega) + (-6\gamma_2 - \gamma_3) \oint Y_3(\omega)$$
$$+ (6\gamma_2 + \gamma_3 - 9\gamma_6) \oint Y_4(\omega) + (-3\gamma_3 - \gamma_4) \oint Y_5(\omega) - \gamma_4 \oint Y_6(\omega)$$
$$+ (\gamma_4 - 3\gamma_8) \oint Y_7(\omega) - 6\gamma_1 \oint Y_8(\omega)$$

By the linear independence of the invariants $\oint Y_j(\omega)$, we conclude that $c = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_6 = \gamma_8 = 0$. $\square$

It is now time to clarify the notion of conformal primitive.

3.8 Definition. *A real functional $\mathcal{P}$ on a conformal class $\langle g[0]\rangle$ on M is smooth if the functions*

$$\begin{aligned} &\mathbb{R}^N \to \mathbb{R}, \\ &\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \mapsto \mathcal{P}\left(g\left[2\sum_{i=1}^{N} \varepsilon_i \omega_i\right]\right) \end{aligned} \tag{3.5}$$

are C^∞ for all $N \in \mathbb{Z}^+$ and functions $\omega_i \in C^\infty(M)$. $\mathcal{P}$ has degree $n \in \mathbb{Z}^+$ if the maps (3.5) are polynomial in ε of degree n, and we use the terminology linear, quadratic, etc. for degree $1, 2, \dots$. A map $\mathcal{R}$ from $\langle g[0]\rangle$ to $C^\infty(M)$ is smooth (resp. has degree n) if $g \mapsto \mathcal{R}(g[\omega])(x)$ is smooth (resp. is polynomial of degree n in ε) for each $x \in M$, and similarly for maps to $C^\infty(\partial M)$.

3.9 Definition. *Let $\mathcal{P}$ and $\mathcal{T}$ be smooth functionals on a conformal class $\langle g[0]\rangle$ on M. $\mathcal{P}$ is a conformal primitive for $\mathcal{T}$ if*

$$(d/d\varepsilon)|_{\varepsilon=0}\mathcal{P}(g[\eta + \varepsilon\omega]) = \mathcal{T}(g[\eta]) \tag{3.6}$$

for all $\eta, \omega \in C^\infty(M)$. If in addition, a base metric g_0 is given and $\mathcal{P}(g[0]) = 0$, $\mathcal{P}$ is a base-pointed conformal primitive for $\mathcal{T}$.

3.10 Remark. A base-pointed conformal primitive $\mathcal{P}$ for $\mathcal{T}$, if it exists, is unique, since the curve $\alpha(\varepsilon) = \mathcal{P}(g[\varepsilon\omega])$ solves the initial value problem $(d/d\varepsilon)\alpha(\varepsilon) = \mathcal{T}(g[\varepsilon\omega])$, $\alpha(0) = 0$. If a functional $\mathcal{T}$ and a prospective conformal primitive $\mathcal{P}$ are given by universal formulas, it is sufficient to prove (3.6) at $\eta = 0$, since universality allows us to replace $g[0]$ by $g[\eta]$. If $\mathcal{T}(g[\omega])$ is a homogeneous polynomial functional of degree $n > 0$, then $\mathcal{T}(g[\omega])/n$ is a base-pointed conformal primitive for $\mathcal{T}(g[\omega])$. Thus a decomposition of a given functional into homogeneous polynomial functionals is sufficient information for the computation of a base-pointed conformal primitive. We shall sometimes use the abbreviation $\mathcal{T}[\omega]$ for $\mathcal{T}(g[\omega])$.

Motivated by Remark 3.10, we go on to compute the higher conformal variations of the local invariants in Theorem 3.7. The formulas for $\nabla[\omega]$, $R[\omega]$, $L[\omega]$, $\tilde{\nabla}[\omega]$, and $N[\omega]$ show that each term in that expression for $a_4(\omega, A[\omega], B[\omega])$ is polynomial of degree ≤ 4; this will also emerge from our calculations, so we omit the abstract proof. We first introduce abbreviations for some f-augmented local invariants which are quadratic and cubic in f.

Abbreviation	Invariant	Index expression
$Z_1(f,f)$	$(Nf)N^2f$	$f_{\vert N}f_{\vert NN}$
$Z_2(f,f)$	$(Nf)(-\tilde{\Delta})f$	$f_{\vert N}f_{:aa}$
$Z_3(f,f)$	$(Nf)^2H$	$f_{\vert N}f_{\vert N}L_{aa}$
$Z_4(f,f)$	$\vert\tilde{d}f\vert^2H$	$f_{:a}f_{:a}L_{bb}$
$Z_5(f,f)$	$\langle\tilde{d}f\otimes\tilde{d}f,L\rangle$	$f_{:a}f_{:b}L_{ab}$
$Z_6(f,f)$	$\langle\tilde{d}f,\tilde{d}(Nf)\rangle$	$f_{:a}(f_{\vert N})_{:a}$
$E_1(f,f,f)$	$(Nf)\vert\tilde{d}f\vert^2$	$f_{\vert N}f_{:a}f_{:a}$
$E_2(f,f,f)$	$(Nf)^3$	$f_{\vert N}f_{\vert N}f_{\vert N}$

Table 3.3

The notation $Z_i(f,f)$ indicates that Z_i can actually be thought of as a quadratic form $Z_i(f_1, f_2)$ after polarization; similarly for the cubic form determined by $E_j(f,f,f)$. Lemmas 3.11–3.18 immediately following are obtained by direct computation with the variational formulas and identities of [**BG**, Appendix], and integration by parts.

3.11 Lemma. *Let* $Y_i'(\omega,\omega) = (d/d\varepsilon)|_{\varepsilon=0}(e^{3\varepsilon\omega}Y_i(\omega)[\varepsilon\omega]) = (d/d\varepsilon)|_{\varepsilon=0}Y_i(\omega)[\varepsilon\omega]$ $+3\omega Y_i(\omega)[0]$. *Then:*
(a) $Y_1'(\omega,\omega) = -2(m-1)Z_1(\omega,\omega) - 2(m-1)Z_2(\omega,\omega) + 2(m-1)Z_3(\omega,\omega)$.
(b) $Y_2'(\omega,\omega) = -(m-1)Z_1(\omega,\omega) - Z_3(\omega,\omega) + Z_4(\omega,\omega)$.
(c) $Y_3'(\omega,\omega) = -(m-1)Z_2(\omega,\omega) + (m-3)Z_4(\omega,\omega)$.
(d) $Y_4'(\omega,\omega) = -2(m-1)Z_3(\omega,\omega)$.
(e) $Y_5'(\omega,\omega) = -(m-1)Z_1(\omega,\omega) - Z_2(\omega,\omega) + Z_3(\omega,\omega)$.
(f) $Y_6'(\omega,\omega) = -Z_2(\omega,\omega) + Z_4(\omega,\omega) - 2Z_5(\omega,\omega)$.
(g) $Y_7'(\omega,\omega) = -2Z_3(\omega,\omega)$.
(h) $Y_8'(\omega,\omega) = 2(m-3)Z_1(\omega,\omega) - 2Z_2(\omega,\omega) + 2Z_3(\omega,\omega) + 2(m-2)Z_5(\omega,\omega)$ $+2(m-2)Z_6(\omega,\omega)$. □

3.12 Lemma. *For all* $f,\omega \in C^\infty(M)$,
(a) $\oint Z_6(f,f) = -\oint Z_2(f,f)$.
(b) $\oint fZ_6(f,f) = -\oint fZ_2(f,f) - \oint E_1(f,f,f)$.
(c) $(d/d\varepsilon)|_{\varepsilon=0}\left(\oint(Y_8(\omega)dy)[\varepsilon\omega]\right) = (m-4)\oint \omega(Y_8(\omega)dy)[0] + \oint\{2(m-3)Z_1(\omega,\omega)$ $-2(m-1)Z_2(\omega,\omega) + 2Z_3(\omega,\omega) + 2(m-2)Z_5(\omega,\omega)\}$. □

3.13 Lemma. *Let* $Z_i'(\omega,\omega,\omega) = (d/d\varepsilon)|_{\varepsilon=0}(e^{3\varepsilon\omega}Z_i(\omega,\omega)[\varepsilon\omega]) = (d/d\varepsilon)|_{\varepsilon=0}Z_i(\omega,\omega)[\varepsilon\omega] +$ $3\omega Z_i(\omega,\omega)[0]$. *Then:*
(a) $Z_1'(\omega,\omega,\omega) = E_1(\omega,\omega,\omega) - E_2(\omega,\omega,\omega)$.
(b) $Z_2'(\omega,\omega,\omega) = (m-3)E_1(\omega,\omega,\omega)$.
(c) $Z_3'(\omega,\omega,\omega) = -(m-1)E_2(\omega,\omega,\omega)$.
(d) $Z_4'(\omega,\omega,\omega) = -(m-1)E_1(\omega,\omega,\omega)$.

(e) $Z_5'(\omega,\omega,\omega) = -E_1(\omega,\omega,\omega)$.
(f) $Z_6'(\omega,\omega,\omega) = -E_1(\omega,\omega,\omega)$. $\square$

3.14 Lemma. *The quantities* $E_i'(\omega,\omega,\omega,\omega) = (d/d\varepsilon)|_{\varepsilon=0}(e^{3\varepsilon\omega}E_i(\omega,\omega,\omega)[\varepsilon\omega])$ $= (d/d\varepsilon)|_{\varepsilon=0}Z_i(\omega,\omega,\omega)[\varepsilon\omega] + 3\omega Z_i(\omega,\omega,\omega)[0]$ *vanish identically.* $\square$

3.15 Lemma. *For* $m \geq 2$, *let*

$$\begin{aligned}
\ell_1(\omega) &= -Y_4(\omega) + (m-1)Y_7(\omega),\\
\ell_2(\omega) &= -Y_1(\omega) - (m-3)Y_2(\omega) + Y_3(\omega) - \frac{1}{m-1}Y_4(\omega) + (m-1)Y_5(\omega),\\
\ell_3(\omega) &= \frac{m^2-3m-2}{m-1}Y_2(\omega) - \frac{2m}{m-1}Y_3(\omega) - \frac{m^2-5m+2}{(m-1)^2}Y_4(\omega)\\
&\quad - (m-4)Y_5(\omega) + (m-2)Y_6(\omega) + Y_8(\omega).
\end{aligned}$$

Then $\ell_s'(\omega,\omega) = 0$ *for* $s = 1,2$, *and* $(d/d\varepsilon)|_{\varepsilon=0}\oint(\ell_3(\omega)dy)[\varepsilon\omega] = (m-4)\oint\omega(\ell_3(\omega)dy)[0]$. *In particular, if* $m = 4$, *then* $(d/d\varepsilon)|_{\varepsilon=0}\oint(\ell_s(\omega)dy)[\varepsilon\omega] = 0$, $s = 1,2,3$. $\square$

3.16 Lemma. *Let*

$$\mathcal{A} = X_6 - (m-1)^2 X_8 .$$

Then $\mathcal{A}''(\omega,\omega) = 0$. $\square$

3.17 Lemma. *Let*

$$\begin{aligned}
q_1(\omega) &= Y_3(\omega) - (m-1)Y_6(\omega),\\
q_2(\omega) &= (m-3)Y_1(\omega) + (m-3)(m-2)Y_2(\omega) - 2(m-2)Y_3(\omega),\\
q_3(\omega) &= (m-1)Y_1(\omega) - (m-1)(m-2)Y_2(\omega) + 2(m-2)Y_4(\omega).
\end{aligned}$$

Then $q_s''(\omega,\omega,\omega) = 0$, $s = 1,2,3$, *with the result that*

$$(d^2/d\varepsilon^2)|_{\varepsilon=0}\left(\oint(q_s dy)[\varepsilon\omega]\right) = 0, \qquad m = 4. \qquad \square$$

3.18 Lemma. *If* $f \in C^\infty(M)$, *then on* ∂M,

$$\begin{aligned}
f_{|i}{}^{i}{}_{N} &= f_{|NNN} + (f_{|N})_{:a}{}^{a} + 2\tilde\nabla^a(L_{ab}\tilde\nabla^b f)\\
&\quad - H_{:a}f_{:}{}^{a} - Ff_{|N} - |L|^2 f_{|N} - Hf_{|NN} . \qquad \square
\end{aligned}$$

The following lemma is a consequence of Lemmas 3.4, 3.11, 3.12, and 3.13.

3.19 Lemma. *If* $m = 4$ *and* S *is as in Lemma 3.6, then*

$$(d^2/d\varepsilon^2)|_{\varepsilon=0}\left(\oint\omega(Sdy)[\varepsilon\omega] - \frac{1}{3}\oint Y_3(\omega)[\varepsilon\omega]dy[\varepsilon\omega]\right) = 0. \qquad \square$$

4. Formulas for the quotient of functional determinants

Suppose we have a compact 4-manifold M with boundary ∂M, and a base-pointed conformal class $\langle g[0]\rangle$ is given. Suppose also that we have a boundary value problem (A, B) satisfying the assumptions of Theorem 2.12. A change of basis in the formula of Theorem 3.7 allows us to write

$$\begin{aligned} a_4(\omega, A, B)[\omega] = \beta_1 &\int \omega(|C|^2 dx)[\omega] \\ &+ \beta_2 \left\{ \tfrac{1}{2} \int \omega(Q dx)[\omega] + \oint ((\omega S - \tfrac{1}{3} Y_3(\omega)) dy)[\omega] \right\} \\ &+ \beta_3 \left\{ \int \omega((\Delta J) dx)[\omega] + \tfrac{1}{6} \oint ((Y_1(\omega) - \omega X_1) dy)[\omega] \right\} \\ &+ \beta_4 \oint \omega(\mathcal{L}_4 dy)[\omega] + \beta_5 \oint \omega(\mathcal{L}_5 dy)[\omega] \\ &+ \sum_{i=1}^{3} \lambda_i \oint (\ell_i(\omega) dy)[\omega] + \sum_{j=1}^{3} \sigma_j \oint (q_j(\omega) dy)[\omega] \\ &+ c_3 \oint (Y_3(\omega) dy)[\omega] + c_4 \oint (Y_4(\omega) dy)[\omega], \end{aligned} \tag{4.1}$$

where $\beta_\nu, \lambda_i, \sigma_j, c_3, c_4$ are universal constants depending on the universal polynomial expression for (A, B).

By Theorem 2.12, we can find a formula for

$$-\log|\det(A_B)|[\omega] + \log|\det(A_B)|[0]$$

by finding a base-pointed conformal primitive for each term in (4.1). By Theorem 2.10, the sign $(-1)^{\#\{\lambda_j < 0\}}$ of $\det A_B$ is conformally invariant, so this gives a formula for

$$\frac{\det(A_B)[\omega]}{\det(A_B)[0]}.$$

4.1 Lemma. *In (4.1), the β_ν terms for $\nu = 1, 4, 5$ and the λ_i terms for $i = 1, 2, 3$ have base-pointed conformal primitives*

$$\beta_1 \int \omega(|C|^2 dx)[0], \qquad \beta_4 \oint \omega(\mathcal{L}_4 dy)[0], \qquad \beta_5 \oint \omega(\mathcal{L}_5 dy)[0], \qquad \lambda_i \oint (\ell_i(\omega) dy)[0]$$

respectively.

Proof. By Lemmas 3.3(a) and 3.5, $|C|^2 dx$, $\mathcal{L}_4 dy$, and $\mathcal{L}_5 dy$ are conformal invariants in dimension 4; thus the relevant β_ν terms in (4.1) are linear on the conformal class $\langle g[0]\rangle$. By Lemma 3.15, the λ_i terms are also linear. We now apply Remark 3.10. □

4.2 Lemma. *A base-pointed conformal primitive for* $\int \omega(Qdx)[\omega]$ *is*

$$\mathcal{B}[\omega] := \tfrac{1}{2} \int \omega(P[0]\omega)dx[0] + \int \omega(Qdx)[0].$$

Proof. If $\eta, \omega \in C^\infty(M)$,

$$(d/d\varepsilon)|_{\varepsilon=0}\mathcal{B}[\eta + \varepsilon\omega] = \tfrac{1}{2} \int \{\eta(P[0]\omega) + \omega(P[0]\eta)\}dx[0] + \int \omega(Qdx)[0].$$

Evaluation at $\eta = \varepsilon_0\omega$ shows that the curve $\mathcal{B}[\varepsilon\omega]$ satisfies the initial value problem

$$(d/d\varepsilon)_{\varepsilon=\varepsilon_0}\mathcal{B}[\varepsilon\omega] = \int \{\varepsilon_0\omega(P[0]\omega) + \omega Q[0]\}dx[0], \qquad \mathcal{B}[0] = 0.$$

But by (1.10), the right side of this ordinary differential equation is

$$\int \omega(Qdx)[\varepsilon_0\omega],$$

as desired. □

4.3 Lemma. *A base-pointed conformal primitive for* $\mathcal{C}[\omega] := \oint((\omega S - \frac{1}{3}Y_3(\omega))dy)[\omega]$ *is*

$$\oint \omega(Sdy)[0] - \tfrac{1}{3} \oint (Y_3(\omega)dy)[0] - \tfrac{1}{2} \oint \omega(S(\omega)dy)[0]$$
$$+ \tfrac{1}{2} \oint (Z_2(\omega,\omega)dy)[0] - \tfrac{1}{6} \oint (Z_4(\omega,\omega)dy)[0].$$

Proof. By Lemma 3.19, $\mathcal{C}[\omega]$ is a quadratic functional. By Remark 3.10, its base-pointed conformal primitive is

$$\oint \omega(Sdy)[0] - \tfrac{1}{3} \oint (Y_3(\omega)dy)[0]$$
$$+ \tfrac{1}{2} \oint (\omega S'(\omega)[0] - \tfrac{1}{3}Y_3'(\omega,\omega)[0])dy[0].$$

The last line of this is computed using Lemmas 3.6 and 3.11. □

4.4 Lemma. *The* β_3 *term in (4.1) has base-pointed conformal primitive*

$$\tfrac{1}{2}\beta_3 \int \{(J^2dx)[\omega] - (J^2dx)[0]\}.$$

Proof. This is a restatement of Lemma 3.3(b). □

4.5 Lemma. *The sum of the σ_j, c_3, and c_4 terms in (4.1) has base-pointed conformal primitive*

$$\begin{aligned}
&\sigma_1\left\{\oint(q_1(\omega)dy)[0]+\oint((-Z_4+3Z_5)(\omega,\omega)dy)[0]\right\}\\
&+\sigma_2\left\{\oint(q_2(\omega)dy)[0]+\oint((-6Z_1+3Z_2+2Z_3-Z_4)(\omega,\omega)dy)[0]\right\}\\
&+\sigma_3\left\{\oint(q_3(\omega)dy)[0]+\oint((-9Z_2-3Z_4)(\omega,\omega)dy)[0]\right\}\\
&+c_3\left\{\oint(Y_3(\omega)dy)[0]+\oint((-\tfrac{3}{2}Z_2+\tfrac{1}{2}Z_4)(\omega,\omega)dy)[0]-\oint(E_1(\omega,\omega,\omega)dy)[0]\right\}\\
&+c_4\left\{\oint(Y_4(\omega)dy)[0]-3\oint(Z_3(\omega,\omega)dy)[0]+3\oint(E_2(\omega,\omega,\omega)dy)[0]\right\}.
\end{aligned}$$

Proof. By Lemma 3.17, the σ_j terms in (4.1) are quadratic on the conformal class $\langle g[0]\rangle$, and by Lemmas 3.11, 3.13, and 3.14, the c_4 and c_7 terms are cubic. We now apply Remark 3.10, using Lemmas 3.11 and 3.13 to compute expansions into homogeneous polynomial terms. □

We collect all this information in the following.

4.6 Theorem. *Under the assumptions of Theorem 2.12, if (A,B) is not orientation-*

sensitive,

$$
\begin{aligned}
-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]} &= \beta_1\int\omega(|C|^2dx)[0]\\
&+\beta_2\Big\{\tfrac14\int\omega(P[0]\omega)dx[0]+\tfrac12\int\omega(Qdx)[0]+\oint\omega(Sdy)[0]-\tfrac13\oint(Y_3(\omega)dy)[0]\\
&\qquad-\tfrac12\oint\omega(S(\omega)dy)[0]+\oint((\tfrac12 Z_2-\tfrac16 Z_4)(\omega,\omega)dy)[0]\Big\}\\
&+\tfrac12\beta_3\int\{(J^2dx)[\omega]-(J^2dx)[0]\}+\beta_4\oint\omega(\mathcal{L}_4dy)[0]+\beta_5\oint\omega(\mathcal{L}_5dy)[0]\\
&+\sum_{i=1}^{3}\lambda_i\oint(\ell_i(\omega)dy)[0]\\
&+\sigma_1\Big\{\oint(q_1(\omega)dy)[0]+\oint((-Z_4+3Z_5)(\omega,\omega)dy)[0]\Big\}\\
&+\sigma_2\Big\{\oint(q_2(\omega)dy)[0]+\oint((-6Z_1+3Z_2+2Z_3-Z_4)(\omega,\omega)dy)[0]\Big\}\\
&+\sigma_3\Big\{\oint(q_3(\omega)dy)[0]+\oint((-9Z_2-3Z_4)(\omega,\omega)dy)[0]\Big\}\\
&+c_3\Big\{\oint(Y_3(\omega)dy)[0]+\oint((-\tfrac32 Z_2+\tfrac12 Z_4)(\omega,\omega)dy)[0]-\oint(E_1(\omega,\omega,\omega)dy)[0]\Big\}\\
&+c_4\Big\{\oint(Y_4(\omega)dy)[0]-3\oint(Z_3(\omega,\omega)dy)[0]+3\oint(E_2(\omega,\omega,\omega)dy)[0]\Big\}.
\end{aligned}
$$

If (A,B) is orientation-sensitive, $\beta_1\int\omega(|C|^2dx)[0]$ should be replaced by $\beta_{1,+}\int\omega(|C_+|^2dx)[0]+\beta_{1,-}\int\omega(|C_-|^2dx)[0]$. If $\mathcal{P}_\lambda$ is the functional of Remark 2.2, then

$$
\begin{aligned}
-(2\ell)^{-1}\log\frac{\mathcal{P}_\lambda(A,B,g[\omega])}{\mathcal{P}_\lambda(A,B,g[0])} &= -\frac{\lambda}{4}\log\frac{\int e^{4\omega}dx[0]}{v[0]}-\frac{a_4(A,B)-\lambda}{3}\log\frac{\oint e^{3\omega}dy[0]}{\tilde v[0]}\\
&\quad-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]},
\end{aligned}\tag{4.2}
$$

where $v[0]=\mathrm{vol}(g[0])$ and $\tilde v[0]=\mathrm{vol}(g[0])$. □

In the above, recall that $a_4(A,B)$ is conformally invariant. A more manageable version of this formula is obtained when we notice that several of its terms are also terms in a formula for $a_4(\omega,A,B)$. Indeed, if $\omega,\eta\in C^\infty(M)$, (4.1) can be modified to give a formula for $a_4(\omega,A,B)[\eta]$ just by replacing each $[\omega]$ by $[\eta]$. To make the formula even more easily applicable, we add the mild assumption that our chosen background metric on M has constant scalar curvature. We immediately have:

4.7 Corollary. *Suppose $(M,\langle g[0]\rangle)$ and (A,B) are as in Theorem 2.12, and that $g[0]$ has* *constant scalar curvature* τ_0. *Let* $\omega \in C^\infty(M)$. *Then*

$$
\begin{aligned}
-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]} &= a_4(\omega, A, B) \\
&+\beta_2\left\{\tfrac14\int\omega(P[0]\omega)dx[0] - \tfrac12\oint\omega(\mathcal{S}(\omega)dy)[0] + \oint((\tfrac12 Z_2 - \tfrac16 Z_4)(\omega,\omega)dy)[0]\right\} \\
&+\tfrac12\beta_3\left\{\int\{(J^2dx)[\omega] - (J^2dx)[0] - \tfrac13\tau_0\oint\omega_{|N}\right\} + \sigma_1\oint((-Z_4+3Z_5)(\omega,\omega)dy)[0] \\
&+\sigma_2\oint((-6Z_1+3Z_2+2Z_3-Z_4)(\omega,\omega)dy)[0] + \sigma_3\oint((-9Z_2-3Z_4)(\omega,\omega)dy)[0] \\
&+c_3\left\{\oint((-\tfrac32 Z_2+\tfrac12 Z_4)(\omega,\omega)dy)[0] - \oint(E_1(\omega,\omega,\omega)dy)[0]\right\} \\
&+c_4\left\{-3\oint(Z_3(\omega,\omega)dy)[0] + 3\oint(E_2(\omega,\omega,\omega)dy)[0]\right\}. \qquad \square
\end{aligned}
$$

Now recall the functionals $\mathcal{P}_\lambda$ of Remark 2.2, which involve the conformal index $a_4(A,B)$. It is sometimes useful to express the conformal index in terms of the Euler characteristic of M. Recall that $\chi(M) = \chi(\partial M) + \chi(M, \partial M)$; thus if m is even, $\chi(M) = \chi(M, \partial M)$. By the Chern-Gauss-Bonnet formula, if $m = 4$,

$$
\begin{aligned}
\chi(M) &= (32\pi^2)^{-1}\int_M(\tau^2 - 4|\rho|^2 + |R|^2)dx \\
&\quad + (24\pi^2)^{-1}\oint_{\partial M}(3\tau H - 6FH - 6\langle T, L\rangle + 2H^3 - 6H|L|^2 + 4\,\mathrm{tr}\,L^3)dy.
\end{aligned}
\tag{4.3}
$$

The interior integrand can be rewritten as $|C|^2 - 8|V|^2 + 8J^2 = |C|^2 + 4\{Q - \Delta J\}$, and thus the interior term in (4.3) equals

$$
(32\pi^2)^{-1}\left(\int_M(|C|^2 + 4Q)dx - \tfrac23\oint\tau_{|N}dy\right).
$$

Thus

$$
\begin{aligned}
\chi(M) &= (32\pi^2)^{-1}\int_M(|C|^2 + 4Q)dx \\
&\quad + (24\pi^2)^{-1}\oint_{\partial M}(-\tfrac12\tau_{|N} + 3\tau H - 6FH - 6\langle T, L\rangle + 2H^3 - 6H|L|^2 + 4\,\mathrm{tr}\,L^3)dy,
\end{aligned}
$$

or more compactly,

$$
\chi(M) = (32\pi^2)^{-1}\int_M(|C|^2 + 4Q)dx + (4\pi^2)^{-1}\oint_{\partial M}(S - \mathcal{L}_4 - \mathcal{L}_5)dy. \tag{4.4}
$$

At a background metric $g[0]$ with constant scalar curvature as in Theorem 4.7, the $\tau_{|N}$ contributions to the boundary integrals in the formulas for $\chi(M)$ and $a_4(A,B)$ disappear. It is now appropriate to distinguish two types of "model backgrounds":

4.8 Definition. *$(M, g[0])$ is a model background of type I if $(\nabla R)[0] = 0$, ∂M is totally geodesic, and M is connected. $(M, g[0])$ is a model background of type II if $g[0]$ is flat, $(\tilde{\nabla} L)[0] = 0$, and ∂M is connected.*

4.9 Lemma. *In a model background $(M, g[0])$ of type I, the boundary integrals in (3.4) and (4.4) vanish, and*

$$a_4(A[0], B[0]) = 4\pi^2 \beta_2 \chi(M) + (\beta_1 - \tfrac{1}{8}\beta_2)|C|^2[0]v[0].$$

In a model background $(M, g[0])$ of type II, the interior integrals in (3.4) and (4.4) vanish, and

$$a_4(A[0], B[0]) = 4\pi^2 \beta_2 \chi(M) + (\beta_4 + 6\beta_2)\mathcal{L}_4[0]\tilde{v}[0] + (\beta_5 + 6\beta_2)\mathcal{L}_5[0]\tilde{v}[0].$$

Proof. In the type I case, $\nabla R = 0 \Rightarrow \nabla C = 0$, so the connectedness of M guarantees that $|C|^2[0]$ is constant. In the type II case, $R = 0$, $\tilde{\nabla} L = 0 \Rightarrow \tilde{\nabla}\mathcal{L}_4 = \tilde{\nabla}\mathcal{L}_5 = 0$, so the connectedness of ∂M guarantees that $\mathcal{L}_4[0]$ and $\mathcal{L}_5[0]$ are constant. □

We combine these considerations to get a more natural form of (4.2):

4.10 Theorem. *Under the assumptions of Theorem 2.12, if $(M, g[0])$ is a background of type I and $\lambda = a_4(A, B)$, or if $(M, g[0])$ is a background of type II and $\lambda = 0$,*

$$\begin{aligned}
-(2\ell)^{-1} \log \frac{\mathcal{P}_\lambda(A, B, g[\omega])}{\mathcal{P}_\lambda(A, B, g[0])} &= \mathcal{E}_{(\text{I or II})} + \beta_2 \Big\{ \tfrac{1}{4} \int \omega(P[0]\omega) dx[0] - \tfrac{1}{3} \oint (Y_3(\omega) dy)[0] \\
&\quad - \tfrac{1}{2} \oint \omega(\mathcal{S}(\omega) dy)[0] + \oint ((\tfrac{1}{2} Z_2 - \tfrac{1}{6} Z_4)(\omega, \omega) dy)[0] \Big\} \\
&\quad + \tfrac{1}{2}\beta_3 \int \{(J^2 dx)[\omega] - (J^2 dx)[0]\} + \sum_{i=1}^{3} \lambda_i \oint (\ell_i(\omega) dy)[0] \\
&\quad + \sigma_1 \oint ((-Z_4 + 3Z_5)(\omega, \omega) dy)[0] \\
&\quad + \sigma_2 \Big\{ \oint (q_2(\omega) dy)[0] + \oint ((-6Z_1 + 3Z_2 + 2Z_3 - Z_4)(\omega, \omega) dy)[0] \Big\} \\
&\quad + \sigma_3 \Big\{ \oint (q_3(\omega) dy)[0] + \oint ((-9Z_2 - 3Z_4)(\omega, \omega) dy)[0] \Big\} \\
&\quad + c_3 \Big\{ \oint (Y_3(\omega) dy)[0] + \oint ((-\tfrac{3}{2} Z_2 + \tfrac{1}{2} Z_4)(\omega, \omega) dy)[0] - \oint (E_1(\omega, \omega, \omega) dy)[0] \Big\} \\
&\quad + c_4 \Big\{ \oint (Y_4(\omega) dy)[0] - 3 \oint (Z_3(\omega, \omega) dy)[0] + 3 \oint (E_2(\omega, \omega, \omega) dy)[0] \Big\},
\end{aligned}$$

where

$$\mathcal{E}_\text{I} = -\frac{a_4(A, B)}{4} \log \frac{\int e^{4(\omega - \bar{\omega})} dx[0]}{v[0]} = -\{\pi^2 \beta_2 \chi(M) + \tfrac{1}{4}(\beta_1 - \tfrac{1}{8}\beta_2)|C|^2[0]v[0]\},$$

and

$$\mathcal{E}_{\mathrm{II}} = -\frac{a_4(A,B)}{3}\log\frac{\oint e^{3(\omega-\tilde{\omega})}dy[0]}{\tilde{v}[0]}$$
$$= -\{\tfrac{4}{3}\pi^2\beta_2\chi(M) + \tfrac{1}{3}(\beta_4+6\beta_2)\mathcal{L}_4[0]\tilde{v}[0] + \tfrac{1}{3}(\beta_5+6\beta_2)\mathcal{L}_5[0]\tilde{v}[0]\}.$$

Here $\bar{\omega} := (\int \omega dx[0])/v[0]$ *is the mean value of* ω *over* M, *and* $\tilde{\omega} := (\oint \omega dy[0])/\tilde{v}[0]$ *is the mean value over* ∂M.

Proof. In the type I case, we absorb terms totalling $(\beta_1|C|^2[0] + \frac{1}{2}\beta_2 Q[0])\int \omega dx[0]$ (or $(\beta_{1,+}|C_+|^2[0] + \beta_{1,-}|C_-|^2[0] + \frac{1}{2}\beta_2 Q[0])\int \omega dx[0]$ if (A,B) is orientation-sensitive) into the first exponential term of (4.2); the coefficient of the second exponential term is 0. In the type II case, we absorb terms totalling $(\beta_2 S[0] + \beta_4\mathcal{L}_4[0] + \beta_5\mathcal{L}_5[0])\oint \omega dy[0]$ into the second exponential term of (4.2); the coefficient of the first exponential term is 0. We also make use of the fact that $\oint(q_1(\omega)dy)[0]$ vanishes, since $(\tilde{\nabla}L)[0] = 0$ in both the type I and type II cases. □

4.11 Remark. The choice $\lambda = a_4(A,B)$ or $\lambda = 0$ makes $\mathcal{P}_\lambda(A,B,g)$ a spectral invariant (recall Remark 2.2). The presence of the $|C|^2$ term in the case of a model background of type I is an indication that the analysis of the determinant functional will be heavily dependent on conformal geometry as well as on topology. See [**BCY**] for this analysis, and the effect of the $|C|^2$ term, in the boundariless case. Similarly, for a model background of type II, the $\mathcal{L}_4$ and $\mathcal{L}_5$ terms indicate a dependence on conformal geometry.

4.12 Remark. There need not be a model background of type I or II in a given conformal class, of course. It can happen, however, that there are model backgrounds of both types in the same conformal class. For example, the round metric on the closed hemisphere H^4 (type I) is conformal to the flat metric on the closed ball B^4 (type II). The standard metric on the cylinder $\mathcal{C}^4_h = [0,h]\times S^3$ of height h is conformal to the flat metric on the spherical shell $\mathcal{A}^4_s = \{x \in \mathbb{R}^4 \mid 1 \le |x| \le s\}$, $s = e^h$. Here the cylindrical geometry is type I; the shell geometry fails to be type II only because of its disconnected boundary.

5. Special manifolds

We would now like to do some computations in the special cases of the hemisphere, ball, cylinder, and spherical shell. Since the hemisphere and ball are conformally equivalent, and the cylinder of height h is conformally equivalent to the spherical shell of outer/inner radius ratio $s = e^h$, this will provide checks on our formulas, in that we can compute certain determinant quotients in two different ways. Moreover, since we can write down the spectra of the Dirichlet and Robin problems for the conformal Laplacian on the hemisphere, and compute the determinants of these problems explicitly (Sec. 7), we shall be able to compute the determinants of the similar problems on the ball. The following elementary observation will be useful.

5.1 Lemma. *If* ∂M *is totally geodesic and* κ *is the intrinsic scalar curvature of* ∂M, *then* $\tau = \kappa + 2F$ *on* ∂M.

Proof. We use total geodesy to pick coordinates at a point of ∂M which are normal for both g and $\tilde{g}$, then use the characterication of the Riemann tensor as the second-order part of the Taylor expansion of the metric, to show that $\kappa = R^{ab}{}_{ab}$. Since $\tau = R^{ij}{}_{ij} = R^{ab}{}_{ab} + 2R^{a}{}_{NaN}$, the result follows. $\square$

Now consider the upper hemisphere H^m in S^m, the boundary of which is the equator S^{m-1}, with the standard round metric $g[0]$ as background. Here the interior metric has $\nabla R = 0$ and the boundary embedding is totally geodesic, so $(H^m, g[0])$ is a model background of type I. In this background, $C = 0$, $V = \frac{1}{2}g$, and $J = m/2$. If $\Psi^2 := \Delta + (m-1)^2/4$, then the Paneitz operator and quantity are

$$P = (\Psi - \tfrac{3}{2})(\Psi - \tfrac{1}{2})(\Psi + \tfrac{1}{2})(\Psi + \tfrac{3}{2}), \qquad Q = m(m+2)(m-2)/8.$$

In particular, if $m = 4$, then $P = \Delta(\Delta + 2)$ and $Q = 6$. By (1.1),

$$R_{ijkl} = -g_{jk}g_{il} + g_{jl}g_{ik}.$$

Thus on the boundary $\partial H^m = S^{m-1}$ of H^m,

$$G = \tilde{g}, \qquad F = m - 1, \qquad T = (m-2)\tilde{g}.$$

L and H vanish, and by Lemma 3.18, if $f \in C^\infty(H^m)$,

$$Y_8(f) = N(-\Delta)f = N^3 f + (-\tilde{\Delta})(Nf) - (m-1)Nf.$$

We also have

$$X_i = 0,\ i = 1, \dots 8;$$
$$Y_2(f) = Y_3(f) = Y_4(f) = Y_6(f) = Y_7(f) = Z_3(f,f) = Z_4(f,f) = Z_5(f,f) = 0;$$

and

$$Y_1(f) = m(m-1)Nf, \qquad Y_5(f) = (m-1)Nf.$$

As a result, on ∂H^m,

$$\mathcal{L}_4 = \mathcal{L}_5 = 0,$$
$$\ell_1(f) = 0, \qquad \ell_2(f) = -(m-1)Nf,$$
$$\ell_3(f) = -(m-4)(m-1)Nf + N(-\Delta)f = N^3 f + (-\tilde{\Delta})(Nf) - (m-3)(m-1)Nf,$$
$$q_1(f) = 0, \qquad q_2(f) = m(m-1)(m-3)Nf, \qquad q_3(f) = m(m-1)^2 Nf.$$

On ∂H^4,

$$S = 0, \qquad \mathcal{S}(f) = Nf - \tfrac{1}{2}N(-\Delta)f = -\tfrac{1}{2}N^3 f - \tfrac{1}{2}(-\tilde{\Delta})(Nf) + \tfrac{5}{2}Nf.$$

Since $\mathrm{vol}(H^4) = 4\pi^2/3$,

$$a_4(A,B) = 2\pi^2\beta_2 Q[0]/3 = 4\pi^2\beta_2$$

for (A,B) satisfying 1.2, 1.6, and 2.4. (Alternatively, we can use the formula of Lemma 4.9 and the fact that $\chi(H^m) = 1$.) Specializing Theorem 4.10, the conclusion is:

5.2 Theorem. *Suppose that (A,B) satisfies 1.2, 1.6, and 2.4, and that $\mathcal{N}(A_B)=0$ on $(H^4,g[0])$, where $g[0]$ is the round metric. Let $\lambda=4\pi^2\beta_2$. Then for $\omega\in C^\infty(H^4)$,*

$$\begin{aligned}
-(2\ell)^{-1}\log\frac{\mathcal{P}_\lambda(A,B,g[\omega])}{\mathcal{P}_\lambda(A,B,g[0])} &= -\pi^2\beta_2\log\frac{\int_{H^4}e^{4(\omega-\bar\omega)}dx[0]}{4\pi^2/3}\\
&+\left(\tfrac{3}{4}\beta_2+3\sigma_2-9\sigma_3-\tfrac{3}{2}c_3\right)\oint_{\partial H^4}((N\omega)(-\tilde\Delta\omega)dy)[0]\\
&+\beta_2\left\{\tfrac14\int_{H^4}\omega((\Delta(\Delta+2)\omega)dx)[0]-\tfrac54\oint_{\partial H^4}\omega((N\omega)dy)[0]+\tfrac14\oint_{\partial H^4}\omega((N^3\omega)dy)[0]\right\}\\
&+\tfrac12\beta_3\left\{\int_{H^4}(J^2dx)[\omega]-16\pi^2/3\right\}\\
&+(-3\lambda_2-3\lambda_3+12\sigma_2+36\sigma_3)\oint_{\partial H^4}((N\omega)dy)[0]+\lambda_3\oint_{\partial H^4}((N^3\omega)dy)[0]\\
&-6\sigma_2\oint_{\partial H^4}((N\omega)(N^2\omega)dy)[0]-c_3\oint_{\partial H^4}((N\omega)|\tilde d\omega|^2dy)[0]+3c_4\oint_{\partial H^4}((N\omega)^3dy)[0].
\end{aligned}$$

A formula for the determinant functional

$$-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]}$$

is obtained by replacing

$$-\pi^2\beta_2\log\frac{\int_{H^4}e^{4(\omega-\bar\omega)}dx[0]}{4\pi^2/3}\qquad \textit{by}\qquad 4\pi^2\beta_2\bar\omega$$

in the formula for

$$-(2\ell)^{-1}\log\frac{\mathcal{P}_\lambda(A,B,g[\omega])}{\mathcal{P}_\lambda(A,B,g[0])}.\qquad\square$$

To set up the conformal diffeomorphism between the hemisphere and the ball, view S^m as the unit sphere of $\mathbb{R}^{m+1}$ with coordinate function $\xi=(u,s)\in\mathbb{R}^m\times\mathbb{R}$. Identify $\mathbb{R}^m$, whose coordinate will be called x, with the complement $S^m\setminus(0,-1)$ of the south pole via

$$x=\frac{u}{1+s},\qquad u=\frac{2x}{1+r^2},\qquad s=\frac{1-r^2}{1+r^2}=\cos p,\qquad \alpha=|u|=\sin p,$$

where $r=|x|$, and p is the *azimuthal* angle between the vector (u,s) and the ray emanating from the origin $(0,0)$ and passing through the north pole $(0,1)$. The standard metrics are related by

$$g_{\mathbb{R}^m}=\Phi^2g_{S^m},\qquad \Phi=\tfrac12(1+r^2)=\frac{1}{1+s}.$$

This version of the stereographic projection identifies the upper hemisphere in S^m with the unit ball in R^m; our two conformal metrics agree on the common boundary of H^m and B^m. The total interior volumes of our models differ, as

$$\mathrm{vol}(H^m)=\frac{(4\pi)^{m/2}\Gamma(m/2)}{2\Gamma(m)},\qquad \mathrm{vol}(B^m)=\frac{2\pi^{m/2}}{m\Gamma(m/2)},$$

so that

$$\frac{\mathrm{vol}(B^m)}{\mathrm{vol}(H^m)} = \frac{\Gamma(m)}{2^{m-2}m\Gamma(m/2)^2};$$

in particular,

$$\mathrm{vol}(B^4)/\mathrm{vol}(H^4) = 3/8.$$

The inward unit normal in H^m is $N = -\partial_p$. Here we use p, together with any coordinate system on the latitudes $s = \mathrm{const}$, to get local coordinates on H^m. Since $Ns = -\partial_p s = \alpha$ and $\partial_s \alpha = -s/\alpha$,

$$N = \alpha\partial_s, \qquad N^2 = \alpha^2\partial_s^2 - s\partial_s \qquad N^3 = \alpha^3\partial_s^3 - 3\alpha s\partial_s^2 - \alpha\partial_s$$

on $H^m \setminus \{(0,1)\}$. In particular,

$$N = \partial_s, \quad N^2 = \partial_s^2, \quad N^3 = \partial_s^3 - \partial_s \qquad \text{on } \partial H^m.$$

In the notation of Theorem 5.2, if $g[0]$ is the hemisphere metric and $g[\omega]$ the ball metric, then $\omega = -\log(1+s)$. In particular,

$$\begin{aligned} \tilde{d}\omega = 0, \quad \tilde{\Delta}\omega = 0 \qquad &\text{on } H^m \setminus \{(0,1)\}, \\ \omega = 0, \quad -N\omega = N^2\omega = -N^3\omega = 1 \qquad &\text{on } \partial H^m. \end{aligned}$$

Specializing to the case $m = 4$ now, and looking at the formula of Theorem 5.2, our first need is for $(\Delta(\Delta+2))[0]\omega = P[0]\omega$. Applying our covariant setup in the form (1.10), we can immediately conclude that

$$P[0]\omega = Q[\omega]e^{4\omega} - Q[0] = -Q[0] = -6,$$

since all local scalar invariants vanish in the flat metric $g[\omega]$. $J^2[\omega]$ also vanishes for this reason. The surviving boundary terms in Theorem 5.2 all come from the expressions

$$\begin{aligned} \oint_{\partial H^4} ((N\omega)dy)[0] &= \oint_{\partial H^4} ((N^3\omega)dy)[0] = \oint_{\partial H^4} ((N\omega)(N^2\omega)dy)[0] \\ &= \oint_{\partial H^4} ((N\omega)^3 dy)[0] = -\mathrm{vol}(\partial H^4) = -\mathrm{vol}(S^3) = -2\pi^2. \end{aligned}$$

(Note especially that no β_2 boundary terms survive, since ω vanishes on the boundary.) Thus Theorem 5.2 specializes to

$$\begin{aligned} -(2\ell)^{-1}\log\frac{\mathcal{P}_\lambda(A,B,g[\omega])}{\mathcal{P}_\lambda(A,B,g[0])} &= -\pi^2\beta_2\left\{\log\frac{\mathrm{vol}(B^4)}{4\pi^2/3} - 4\bar{\omega}\right\} - \tfrac{3}{2}\beta_2\int_{H^4}\omega dx[0] \\ &\quad - \tfrac{8}{3}\pi^2\beta_3 - 2\pi^2(-3\lambda_2 - 2\lambda_3 + 6\sigma_2 + 36\sigma_3 + 3c_4). \end{aligned}$$

In this, the $\bar{\omega}$ and $\int\omega$ terms combine to give

$$\begin{aligned} -(2\ell)^{-1}\log\frac{\mathcal{P}_\lambda(A,B,g[\omega])}{\mathcal{P}_\lambda(A,B,g[0])} &= -\pi^2\beta_2\log\frac{\mathrm{vol}(B^4)}{4\pi^2/3} + \tfrac{3}{2}\beta_2\int_{H^4}\omega dx[0] \\ &\quad - \tfrac{8}{3}\beta_3\pi^2 - 2\pi^2(-3\lambda_2 - 2\lambda_3 + 6\sigma_2 + 36\sigma_3 + 3c_4). \end{aligned}$$

To compute the $\int \omega$ term, let $d\theta$ be a volume form on a standard (radius 1) S^3; then $\alpha^3 d\theta$ is a volume form on the latitude in H^4 with radius α, and

$$-\alpha^3 dp \wedge d\theta = \alpha^2 ds \wedge d\theta = (1 - s^2) ds \wedge d\theta$$

is a volume form on $H^4 \setminus \{(0,1)\}$. Since $\mathrm{vol}(S^3) = 2\pi^2$ and $\omega = \omega(s) = -\log(1+s)$ is constant on latitudes,

$$\int_{H^4} \omega dx[0] = 2\pi^2 \int_0^1 (1-s^2)\omega(s) ds = -2\pi^2 \int_1^2 t(2-t) \log t \, dt = \pi^2(\tfrac{13}{9} - \tfrac{8}{3}\log 2).$$

Here we have used the integral formula

$$\int t^n (\log t) dt = \frac{t^{n+1}}{n+1}\left(\log t - \frac{1}{n+1}\right) + \mathrm{const}$$

in the cases $n = 1, 2$. Since $\mathrm{vol}(B^4) = \pi^2/2$, we have:

5.3 Corollary. *With assumptions as in Theorem 5.2 and $g[\omega]$ the flat B^4 metric,*

$$-(2\ell)^{-1} \log \frac{\mathcal{P}_\lambda(A, B, g[\omega])}{\mathcal{P}_\lambda(A, B, g[0])} =$$
$$\pi^2\{(\tfrac{13}{6} - \log 6)\beta_2 - \tfrac{8}{3}\beta_3 + 6\lambda_2 + 4\lambda_3 - 12\sigma_2 - 72\sigma_3 - 6c_4\}. \qquad \square$$

In the expression for the quotient of determinants (as opposed to scale-invariant determinant functionals) is simply missing the $\mathrm{vol}(B^4)/\mathrm{vol}(H^4)$ contribution:

5.4 Corollary. *With assumptions as in Theorem 5.2 and $g[\omega]$ the flat B^4 metric,*

$$-(2\ell)^{-1} \log \frac{\det(A_B)[\omega]}{\det(A_B)[0]} =$$
$$\pi^2\{(\tfrac{13}{6} - 4\log 2)\beta_2 - \tfrac{8}{3}\beta_3 + 6\lambda_2 + 4\lambda_3 - 12\sigma_2 - 72\sigma_3 - 6c_4\}. \qquad \square$$

An interesting check on our calculations can be made by specializing Theorem 4.10 to the flat metric on the unit ball B^4, a model background of type II, and viewing the round H^4 metric as the perturbation rather than the background. This is not simply the same calculation in disguise; different terms from the determinant quotient formula contribute to the answer, which is, of course, the reciprocal of the determinant quotient just computed. To set up the calculation, let the dimension m be unrestricted for the moment. Since all interior invariants vanish in a flat metric,

$$|C|^2 = J^2 = Q = 0, \qquad P = \Delta^2.$$

The fundamental form and normalized mean curvature are

$$L = \tilde{g}, \qquad H = m - 1,$$

so

$$X_6 = (m-1)^3, \qquad X_7 = (m-1)^2, \qquad X_8 = m-1.$$

Because the Riemann tensor R vanishes,

$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

As a result of the formulas for the X_i,

$$\mathcal{L}_4 = \mathcal{L}_5 = 0.$$

By Lemma 3.18,

$$Y_8(f) = N^3 f + (-\tilde{\Delta})(Nf) + 2(-\tilde{\Delta})f - (m-1)N^2 f - (m-1)Nf.$$

Furthermore,

$$\begin{aligned}
&Y_1(f) = Y_5(f) = 0\\
&Y_2(f) = (m-1)N^2 f, \qquad Y_3(f) = (m-1)(-\tilde{\Delta})f, \qquad Y_4(f) = (m-1)^2 Nf,\\
&Y_6(f) = (-\tilde{\Delta})f, \qquad Y_7(f) = (m-1)Nf,
\end{aligned}$$

and

$$\begin{aligned}
&\ell_1(f) = 0, \qquad \ell_2(f) = -(m-3)(m-1)N^2 f + (m-1)(-\tilde{\Delta})f - (m-1)Nf,\\
&\ell_3(f) = N^3 f + (-\tilde{\Delta})(Nf) + (m^2-4m-1)N^2 f - m(-\tilde{\Delta})f - (m^2-4m+1)Nf,\\
&q_1(f) = 0, \qquad q_2(f) = (m-3)(m-2)\{(m-1)N^2 f - 2(-\tilde{\Delta})f\},\\
&q_3(f) = (m-1)^2(m-2)\{-N^2 f + 2Nf\},\\
&Z_3(f,f) = (m-1)(Nf)^2, \qquad Z_4(f,f) = (m-1)|\tilde{d}\omega|^2, \qquad Z_5(f,f) = |\tilde{d}\omega|^2.
\end{aligned}$$

If $m = 4$,

$$S = 2, \qquad \mathcal{S}(f) = -\tfrac{1}{2}N^3 f - \tfrac{1}{2}(-\tilde{\Delta})(Nf) + \tfrac{3}{2}N^2 f + (-\tilde{\Delta})f + \tfrac{3}{2}Nf.$$

The conformal index, being a conformal invariant, is that already computed in the H^4 background, viz. $4\pi^2\beta_2$. Specializing Theorem 4.10, we have:

5.5 Theorem. *Suppose that (A, B) satisfies 1.2, 1.6, and 2.4, and that $\mathcal{N}(A_B) = 0$ on*

$(B^4, g[0])$, where $g[0]$ is the standard flat metric. Then for $\omega \in C^\infty(B^4)$,

$$\begin{aligned}
-(2\ell)^{-1}\log\frac{\mathcal{P}_0(A,B,g[\omega])}{\mathcal{P}_0(A,B,g[0])} &= -\frac{4\pi^2\beta_2}{3}\log\frac{\oint_{\partial B^4} e^{3(\omega-\tilde{\omega})}dy[0]}{2\pi^2}\\
&+\left(\tfrac{3}{4}\beta_2+3\sigma_2-9\sigma_3-\tfrac{3}{2}c_3\right)\oint_{\partial B^4}((N\omega)(-\tilde{\Delta}\omega)dy)[0]\\
&+\beta_2\left\{\tfrac{1}{4}\int_{B^4}\omega\Delta^2[0]\omega dx[0]+\tfrac{1}{4}\oint_{\partial B^4}\omega((N^3\omega)dy)[0]\right.\\
&\qquad\left.-\tfrac{3}{4}\oint_{\partial B^4}\omega((N^2\omega)dy)[0]-\tfrac{3}{4}\oint_{\partial B^4}\omega((N\omega)dy)[0]\right\}\\
&+\tfrac{1}{2}\beta_3\int_{B^4}(J^2dx)[\omega]+(-3\lambda_2-\lambda_3+6\sigma_2-18\sigma_3)\oint_{\partial B^4}((N^2\omega)dy)[0]\\
&+(-3\lambda_2-\lambda_3+36\sigma_3+9c_4)\oint_{\partial B^4}((N\omega)dy)[0]+\lambda_3\oint_{\partial B^4}((N^3\omega)dy)[0]\\
&-6\sigma_2\oint_{\partial B^4}((N\omega)(N^2\omega)dy)[0]+(6\sigma_2-9c_4)\oint_{\partial B^4}((N\omega)^2dy)[0]\\
&+\left(-3\sigma_2-9\sigma_3+\tfrac{3}{2}c_3\right)\oint_{\partial B^4}(|\tilde{d}\omega|^2dy)[0]-c_3\oint_{\partial B^4}((N\omega)|\tilde{d}\omega|^2dy)[0]\\
&+3c_4\oint_{\partial B^4}((N\omega)^3dy)[0].
\end{aligned}$$

A formula for the determinant functional

$$-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]}$$

is obtained by replacing

$$-\frac{4\pi^2\beta_2}{3}\log\frac{\oint_{\partial B^4} e^{3(\omega-\tilde{\omega})}dy[0]}{2\pi^2} \qquad \textit{by} \qquad 4\pi^2\beta_2\tilde{\omega}$$

in the formula for

$$-(2\ell)^{-1}\log\frac{\mathcal{P}_0(A,B,g[\omega])}{\mathcal{P}_0(A,B,g[0])}. \qquad \square$$

Now specialize further to the case where $g[\omega]$ is the round hemisphere metric. In this case,

$$\omega = \log\frac{2}{1+r^2},$$

so

$$\omega = N^2\omega = 0, \qquad N\omega = -N^3\omega = 1.$$

By (1.10),

$$\Delta^2[0]\omega = Q[\omega]e^{4\omega} - Q[0] = 6e^{4\omega}.$$

To evaluate $\int_{B^4} \omega e^{4\omega} dx[0]$, note that $e^{4\omega}dx[0] = dx[\omega]$ is the measure on the round hemisphere, so our previous calculations give

$$\tfrac{1}{4}\int_{B^4} \omega(\Delta^2[0]\omega)dx[0] = \tfrac{3}{2}\int_{B^4} \omega e^{4\omega} dx[0] = \tfrac{3}{2}\pi^2(\tfrac{8}{3}\log 2 - \tfrac{13}{9}).$$

Thus the surviving terms in this special case are

$$\begin{aligned}
&-(2\ell)^{-1}\log\frac{\mathcal{P}_0(A,B,g[\omega])}{\mathcal{P}_0(A,B,g[0])} = -(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]} \\
&\quad = \tfrac{3}{2}\pi^2(\tfrac{8}{3}\log 2 - \tfrac{13}{9}) + 2\beta_3 \operatorname{vol}(H^4) + \operatorname{vol}(S^3)\{-3\lambda_2 - 2\lambda_3 + 6\sigma_2 + 36\sigma_3 + 3c_4\} \\
&\quad = \pi^2\{(4\log 2 - \tfrac{13}{6})\beta_2 + \tfrac{8}{3}\beta_3 - 6\lambda_2 - 4\lambda_3 + 12\sigma_2 + 72\sigma_3 + 6c_4\};
\end{aligned}$$

that is, we get the answer predicted by Corollary 5.4.

The case of the ball is helpful in getting a determinant quotient formula in the case of the spherical shell

$$\mathcal{A}_s^4 = \{x \in \mathbb{R}^m \mid 1 \leq |x| \leq s\},$$

where we assume $s > 1$. What we need to know is the case of the unit ball B_1^4 treated above, plus the case of the ball B_s^4 of radius s. Formulas for the larger ball can be obtained by scaling those for B_1^4 and keeping track of the effect of the unit normal's direction on the sign of each term. Note that our setup is in terms of Riemannian measures rather than volume elements, so dy is not signed. All level 3 local invariants on ∂M, including f-augmented ones, reverse sign when the direction of N is reversed. (Recall that L changes sign with N.) The result of this bookkeeping is the following:

5.6 Theorem. *Suppose that (A, B) satisfies 1.2, 1.6, and 2.4, and that $\mathcal{N}(A_B) = 0$ on $(\mathcal{A}_s^4, g[0])$, where $g[0]$ is the standard flat metric. Let $\mathcal{X} = -\partial_r$, where r is the radial* *spherical coordinate in* $\mathbb{R}^m$. *Then for* $\omega \in C^\infty(\mathcal{A}^4_s)$,

$$
\begin{aligned}
-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]} &= (\tfrac{3}{4}\beta_2 + 3\sigma_2 - 9\sigma_3 - \tfrac{3}{2}c_3)\left(\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}\omega)(-\tilde{\Delta}\omega)dy)[0] \\
&+\beta_2\Big\{4\pi^2(\tilde{\omega}(S^3_s) - \tilde{\omega}(S^3_1)) + \tfrac{1}{4}\int_{\mathcal{A}^4_s}\omega\Delta^2[0]\omega dx[0] + \tfrac{1}{4}\left(\oint_{S^3_s} - \oint_{S^3_1}\right)\omega((\mathcal{X}^3\omega)dy)[0] \\
&\quad - \tfrac{3}{4}\left(s^{-1}\oint_{S^3_s} - \oint_{S^3_1}\right)\omega((\mathcal{X}^2\omega)dy)[0] - \tfrac{3}{4}\left(s^{-2}\oint_{S^3_s} - \oint_{S^3_1}\right)\omega((\mathcal{X}\omega)dy)[0]\Big\} \\
&+\tfrac{1}{2}\beta_3\int_{\mathcal{A}^4_s}(J^2dx)[\omega] + (-3\lambda_2 - \lambda_3 + 6\sigma_2 - 18\sigma_3)\left(s^{-1}\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}^2\omega)dy)[0] \\
&+(-3\lambda_2 - \lambda_3 + 36\sigma_3 + 9c_4)\left(s^{-2}\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}\omega)dy)[0] \\
&+\lambda_3\left(\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}^3\omega)dy)[0] - 6\sigma_2\left(\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}\omega)(\mathcal{X}^2\omega)dy)[0] \\
&+(6\sigma_2 - 9c_4)\left(s^{-1}\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}\omega)^2dy)[0] \\
&+(-3\sigma_2 - 9\sigma_3 + \tfrac{3}{2}c_3)\left(s^{-1}\oint_{S^3_s} - \oint_{S^3_1}\right)(|\tilde{d}\omega|^2dy)[0] - c_3\left(\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}\omega)|\tilde{d}\omega|^2dy)[0] \\
&+3c_4\left(\oint_{S^3_s} - \oint_{S^3_1}\right)((\mathcal{X}\omega)^3dy)[0].
\end{aligned}
$$

Here S^3_r *is the sphere of radius* r *centered at the origin in* $\mathbb{R}^4$, *and* $\tilde{\omega}(S^3_r)$ *is the mean value of* ω *over this sphere.* □

We now specialize further to the case where the perturbed metric $g[\omega]$ is the standard metric on the cylinder $\mathcal{C}^4_h = [0,h]\times S^3$ with $h = \log s$, that being $dt^2 + d\theta^2$, where t is the parameter on $[0,h]$, and $d\theta^2$ is the standard metric on S^3. The shell $\mathcal{A}^4_s$ is diffeomorphically the cylinder $[1,s]\times S^3$; in these coordinates, its standard metric is $dr^2 + r^2d\theta^2$. Thus the diffeomorphism $(t,\theta)\mapsto(e^t,\theta)$ from $\mathcal{C}^4_h$ to $\mathcal{A}^4_s$ is conformal:

$$dr^2 + r^2d\theta^2 = r^2(dt^2 + d\theta^2).$$

That is, with $\mathcal{A}^4_s$ as the background, the $\mathcal{C}^4_h$ metric is $g[\omega]$ for $\omega = -\log r$; with $\mathcal{C}^4_h$ as the background, the $\mathcal{A}^4_s$ metric is $g[\omega]$ with $\omega = t$.

Let $g[0]$ be the flat $\mathcal{A}^4_s$ metric. We first note that the term in Theorem 5.6 which involves $\Delta^2[0]\omega$ vanishes by (1.10), since $J[\omega] = 1$ and $2V[\omega] = -dt^2 + d\theta^2$, so that $Q[\omega] = 0$. (Alternatively, we could use the fact that $\log r$ is a constant multiple of the fundamental solution of Δ^2 in $\mathbb{R}^4$.) The fact that $J[\omega] = 1$ also evaluates the β_3 term in Theorem 5.6 as $\frac{1}{2}\beta_3 h\,\mathrm{vol}(S^3)$. To evaluate the other terms, note that

$$\mathcal{X}\omega = r^{-1}, \qquad \mathcal{X}^2\omega = r^{-2}, \qquad \mathcal{X}^3\omega = 2r^{-3};$$

we need this at $r = s$ and $r = 1$. No boundary integrals except the mean value of ω over S^3_s survive the computation, and the result, using $\mathrm{vol}(S^3) = 2\pi^2$, is:

5.7 Corollary. *With assumptions as in Theorem 5.6 and $g[\omega]$ the standard $\mathcal{C}^4_h$ metric,*

$$-(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]} = 2\pi^2 h(-\beta_2 + \tfrac{1}{2}\beta_3). \qquad \square$$

Now let $g[0]$ be the cylinder metric $dt^2 + d\theta^2$; this is a model background of type I. Computing for the moment in a general dimension $m \geq 3$, we have

$$J = (m-2)/2, \quad V = \tfrac{1}{2}(-dt^2 + d\theta^2), \quad C = 0, \quad Q = m^2(m-4)/8.$$

The Paneitz operator is

$$P = \Delta^2 + \frac{m(m-4)}{2}\Delta + 4\partial_t^2 + \frac{m^2(m-4)^2}{16}.$$

Since $L = 0$ and τ is constant, $X_i = 0$ for $i = 1, \ldots 8$; in particular, $\mathcal{L}_4 = \mathcal{L}_5 = 0$, and if $m = 4$, $S = 0$. We have $F = 0$, and for all $f \in C^\infty(\mathcal{C}^m_h)$,

$$Y_2(f) = Y_3(f) = Y_4(f) = Y_5(f) = Y_6(f) = Y_7(f) = 0, \qquad Y_1(f) = (m-1)(m-2)Nf,$$
$$Y_8(f) = N^3 f + (-\tilde{\Delta})(Nf).$$

As a result,

$$\ell_1(f) = q_1(f) = 0, \qquad \ell_2(f) = -(m-1)(m-2)Nf, \qquad \ell_3(f) = N^3 f + (-\tilde{\Delta})(Nf),$$
$$q_2(f) = (m-1)(m-2)(m-3)Nf, \qquad q_3(f) = (m-1)^2(m-2)Nf,$$

and if $m = 4$,

$$\mathcal{S}(f) = 2Nf - \tfrac{1}{2}N^3 f - \tfrac{1}{2}(-\tilde{\Delta})(Nf).$$

Furthermore,

$$Z_3(f,f) = Z_4(f,f) = Z_5(f,f) = 0.$$

Now specialize to the case $m = 4$. The conformal index vanishes by the above and (3.4), or by Lemma 4.9, since $\chi(\mathcal{C}^m_h) = 0$; thus the scale-invariant functional involved in the specialization of Theorem 4.10 is $\mathcal{P}_0$. Collecting information, we have:

5.8 Theorem. *Suppose that (A, B) satisfies 1.2, 1.6, and 2.4, and that $\mathcal{N}(A_B) = 0$ on*

$(C_h^4, g[0])$, *where* $g[0]$ *is the standard cylinder metric. Let* $\mathcal{Y} = -\partial_t$. *Then for* $\omega \in C^\infty(C_h^4)$,

$$-(2\ell)^{-1}\log\frac{\mathcal{P}(A,B,g[\omega])}{\mathcal{P}(A,B,g[0])} = -(2\ell)^{-1}\log\frac{\det(A_B)[\omega]}{\det(A_B)[0]} =$$

$$\beta_2\left\{\tfrac{1}{4}\int_{C_h^4}\omega((\Delta^2[0]+4\partial_t^2))\omega dx[0] - \tfrac{1}{2}\oint^* \omega((2\mathcal{Y}\omega - \tfrac{1}{2}\mathcal{Y}^3\omega - \tfrac{1}{2}(-\tilde{\Delta})(\mathcal{Y}\omega))dy)[0]\right\}$$

$$+\tfrac{1}{2}\beta_3\left(\int_{C_h^4}(J^2dx)[\omega] - h\operatorname{vol}(S^3)\right) + (-6\lambda_2+6\sigma_2+18\sigma_3)\oint^*(\mathcal{Y}\omega)dy[0]$$

$$+\lambda_3\oint^*(\mathcal{Y}^3\omega)dy[0] + \lambda_3\oint^*((-\tilde{\Delta})(\mathcal{Y}\omega)dy)[0]$$

$$-6\sigma_2\oint^*(\mathcal{Y}\omega)(\mathcal{Y}^2\omega)dy[0] + (\tfrac{1}{2}\beta_2+3\sigma_2-9\sigma_3-\tfrac{3}{2}c_3)\oint^*(\mathcal{Y}\omega)((-\tilde{\Delta}\omega)dy)[0]$$

$$-c_3\oint^*(\mathcal{Y}\omega)(|\tilde{d}\omega|^2dy)[0] + 3c_4\oint^*(\mathcal{Y}\omega)^3dy[0],$$

where

$$\oint^* = \oint_{t=h} - \oint_{t=0}. \qquad \square$$

We specialize further to the perturbation which gives the shell $\mathcal{A}_s^4$ with $s = e^h$; that is, we set $\omega = t$. The surviving terms on the right in Theorem 5.8 are

$$\beta_2\left\{-\tfrac{1}{2}\oint_{t=h}\omega(2\mathcal{Y}\omega)dy[0]\right\} + \tfrac{1}{2}\beta_3(-h\operatorname{vol}(S^3)) = 2\pi^2h(\beta_2 - \tfrac{1}{2}\beta_3). \tag{5.1}$$

This checks with Corollary 5.7.

6. The Dirichlet and Robin problems for the conformal Laplacian

The determinant quotient formulas of Theorem 4.6 involve coefficients β_ν $(1 \le \nu \le 5)$, λ_i $(i = 1,2,3)$, σ_j $(j = 1,2,3)$, c_k $(k = 3,4)$ that depend only on the universal formula for (A, B), and not on the particular manifold M. In this section, we compute these constants for the two boundary value problems described in Examples 2.7–2.8.

The starting point is a formula of Branson and Gilkey [**BG**] for $a_4(A, B)$ for elliptic boundary value problems (A, B) in the case where: (1) A is a second-order differential operator with metric leading symbol on sections of a vector bundle V over M, i.e.

$$\sigma_2(A)(x,\xi) = |\xi|^2\operatorname{Id}_{V_x} = g^{ij}\xi_i\xi_j\operatorname{Id}_{V_x},$$

for all $(x, \xi) \in T^*M$, and (2) B gives either Dirichlet conditions, or Neumann conditions of the form

$$(\varphi_{|N} + S\varphi)|_{\partial M} = 0, \tag{6.1}$$

where S is a smooth section of $\operatorname{End} V|_{\partial M}$. For convenience, we state these results in the present notation. There is no restriction on the dimension m, and there no assumptions on naturality or the conformal behavior of (A, B).

6.1 Theorem [BG]. *Under the above assumptions on A, there is a unique connection ∇ on V such that $A = \Delta_V - \mathcal{E}$, where $\Delta_V = -g^{ij}\nabla_i\nabla_j$ is the Bochner Laplacian of ∇, and $\mathcal{E}$ is a smooth section of* $\operatorname{End} V$. *If B gives Dirichlet conditions, we write $a_n(f,A,\mathcal{D})$ for $a_n(f,A,B)$, and have*

$$\begin{aligned}360(4\pi)^{m/2}a_4(f,A,\mathcal{D}) = &\int \operatorname{tr}_V f\{-60\Delta_{\operatorname{End} V}\mathcal{E} + 60\tau\mathcal{E} + 180\mathcal{E}^2\\ &+30\Omega^{ij}\Omega_{ij} - 12\Delta\tau + 5\tau^2 - 2|\rho|^2 + 2|R|^2\}\\ +&\oint \operatorname{tr}_{V|_{\partial M}}\Big(f\{-120\mathcal{E}_{|N} - 18\tau_{|N} + 120\mathcal{E}H\\ &+20\tau H - 4FH + 12\langle G,L\rangle - 4\langle T,L\rangle - 24\tilde\Delta H\\ &+\tfrac{40}{21}H^3 - \tfrac{88}{7}H|L|^2 + \tfrac{320}{21}\operatorname{tr} L^3\}\\ &+f_{|N}\{-180\mathcal{E} - 30\tau - \tfrac{180}{7}H^2 + \tfrac{60}{7}|L|^2\}\\ &+24f_{|NN}H + 30(\Delta f)_{|N}\Big),\end{aligned}$$

where Ω is the curvature of ∇. □

The connection V determines a connection on $\operatorname{End} V$, and this is used to form $\Delta_{\operatorname{End} V}$. The invariants $\oint fL_{ab:}{}^{ab} = \oint\langle\tilde\nabla\tilde\nabla f, L\rangle$, $\oint f\Omega^a{}_{N:a}$, and $\oint f_{|N}F$, which could appear in the above formula, do so with coefficient 0. Note that we have not quite written things in the form (1.13); $(\Delta f)_{|N}$ has been used instead of $f_{|NNN}$ in our basis of invariants. This turns out to be convenient for most practical purposes; if desired, Lemma 3.18 can be used to switch to a basis consistent with (1.13).

6.2 Theorem [BG]. *Under the above assumptions, if B gives Neumann conditions of the form (6.1), we write $a_n(f,A,S)$ for $a_n(f,A,B)$, and have*

$$\begin{aligned}360(4\pi)^{m/2}a_4(f,A,S) = &\int \operatorname{tr}_V f\{-60\Delta_{\operatorname{End} V}\mathcal{E} + 60\tau\mathcal{E} + 180\mathcal{E}^2\\ &+30\Omega^{ij}\Omega_{ij} - 12\Delta\tau + 5\tau^2 - 2|\rho|^2 + 2|R|^2\}\\ +&\oint \operatorname{tr}_{V|_{\partial M}}\Big(f\{240\mathcal{E}_{|N} + 42\tau_{|N} + 120\mathcal{E}H\\ &+20\tau H - 4FH + 12\langle G,L\rangle - 4\langle T,L\rangle - 24\tilde\Delta H\\ &+\tfrac{40}{3}H^3 + 8H|L|^2 + \tfrac{32}{3}\operatorname{tr} L^3\\ &+720S\mathcal{E} + 120S\tau + 144SH^2 + 48S|L|^2\\ &+480S^2H + 480S^3 - 120\tilde\Delta_{\operatorname{End} V|_{\partial M}}S\}\\ &+f_{|N}\{180\mathcal{E} + 30\tau + 12H^2 + 12|L|^2 + 72SH + 240S^2\}\\ &+f_{|NN}\{24H + 120S\} - 30(\Delta f)_{|N}\Big).\end{aligned}$$ □

Again the invariants $\oint fL_{ab:}{}^{ab} = \oint\langle\tilde\nabla\tilde\nabla f, L\rangle$, $\oint f\Omega^a{}_{N:a}$, and $\oint f_{|N}F$ appear with coefficient 0, as does the new invariant $\oint fSF$.

For the conformal Laplacian Y with either Dirichlet or Robin conditions,

$$\Omega = 0, \qquad \mathcal{E} = -\frac{m-2}{2}J = -\frac{m-2}{4(m-1)}\tau.$$

To evaluate the interior terms of a_4 for either problem, we compute that

$$\begin{aligned}\tau^2 &= 4(m-1)^2J^2,\\ |\rho|^2 &= (m-2)^2|V|^2 + (3m-4)J^2,\\ |R|^2 &= |C|^2 + 4(m-2)|V|^2 + 4J^2.\end{aligned}$$

Recall the formula (1.6) for Q. Writing $(Y,\mathcal{D})$ and $(Y,\mathcal{R})$ for the Dirichlet and Robin problems, we have:

6.3 Lemma. *The interior terms of $360(4\pi)^{m/2}a_4(f,Y,\mathcal{D})$ or of $360(4\pi)^{m/2}a_4(f,Y,\mathcal{R})$ in the formula of Theorem 6.1 or 6.2 are*

$$\begin{aligned}&\int f\Big(2|C|^2 - 2(m-2)(m-6)|V|^2 + (5m-16)(m-6)J^2 + 6(m-6)\Delta J\Big)\\ &= \int f\Big(2|C|^2 + 2(m-6)Q - 2(m-4)(m-6)|V|^2\\ &\qquad + 4(m-4)(m-6)J^2 + 4(m-6)\Delta J\Big). \qquad \square\end{aligned}$$

In the last expression, we have used a highly linearly dependent list of local invariants; the terms that survive upon restriction to dimension $m = 4$ are linearly independent. The factor of $m-6$ in the terms that are not local conformal invariants is expected; see [**BG**, Lemma 3.1(c)]. Recalling the notation of Tables 3.1 and 3.2, we have:

6.4 Lemma. *The boundary terms of $360(4\pi)^{m/2}a_4(f,Y,\mathcal{D})$ in the formula of Theorem 6.1 are*

$$\begin{aligned}&\oint\Big(f\Big\{\frac{6(2m-7)}{m-1}X_1 - \frac{10(m-4)}{m-1}X_2 - 4X_3 + 12X_4 - 4X_5 + \frac{40}{21}X_6 - \frac{88}{7}X_7 + \frac{320}{21}X_8\Big\}\\ &\quad + \frac{15(m-4)}{m-1}Y_1(f) + 24Y_2(f) + 24Y_3(f) - \frac{180}{7}Y_4(f) + \frac{60}{7}Y_7(f) - 30Y_8(f)\Big).\end{aligned}$$

(The invariants $Y_5(f)$ and $Y_6(f)$ appear with coefficient 0.) $\square$

We now change the basis of invariants to that of Theorem 3.7, and check that the coefficients c and γ_i vanish as asserted there. For this, note that in dimension 4,

$$
\begin{aligned}
\mathcal{L}_4 &= -\tfrac{1}{3}X_2 + X_3 - X_4 + X_5\,, \\
\mathcal{L}_5 &= -\tfrac{2}{9}X_6 + X_7 - X_8\,, \\
\ell_1(f) &= (-Y_4 + 3Y_7)(f), \\
\ell_2(f) &= (-Y_1 - Y_2 + Y_3 - \tfrac{1}{3}Y_4 + 3Y_5)(f), \\
\ell_3(f) &= (\tfrac{2}{3}Y_2 - \tfrac{8}{3}Y_3 + \tfrac{2}{9}Y_4 + 2Y_6 + Y_8)(f), \\
q_1(f) &= (Y_3 - 3Y_6)(f), \\
q_2(f) &= (Y_1 + 2Y_2 - 4Y_3)(f), \\
q_3(f) &= (3Y_1 - 6Y_2 + 4Y_4)(f).
\end{aligned}
\qquad (m=4) \tag{6.2}
$$

We then change to the basis of (4.1) and compute the following.

6.5 Theorem. *If $\bar{\beta}_\nu = (4\pi)^2 \cdot 360\beta_\nu$ and similarly for λ_i, σ_j, and c_k, then for the problem $(Y, \mathcal{D})$ in dimension $m = 4$,*

$$
\begin{aligned}
&\bar{\beta}_1 = 2, \quad \bar{\beta}_2 = \bar{\beta}_3 = -8, \quad \bar{\beta}_4 = -4, \quad \bar{\beta}_5 = -\tfrac{88}{7}\,, \\
&\bar{\lambda}_1 = \tfrac{20}{7}\,, \quad \bar{\lambda}_2 = 0, \quad \bar{\lambda}_3 = -30, \\
&\bar{\sigma}_1 = -20, \quad \bar{\sigma}_2 = \tfrac{35}{3}\,, \quad \bar{\sigma}_3 = -\tfrac{31}{9}\,, \\
&\bar{c}_3 = 8, \quad \bar{c}_4 = -\tfrac{152}{63}\,. \qquad \square
\end{aligned}
$$

For the Robin problem,

$$S = -\frac{m-2}{2(m-1)}H\,.$$

The easiest way to compute is to find the difference between the Robin and Dirichlet heat invariants; this we do in dimension $m = 4$ only:

$$
\begin{aligned}
(4\pi)^2 \cdot 360(a_4(f,Y,\mathcal{R}) - a_4(f,Y,\mathcal{D})) &= \oint \Big(f\{-\tfrac{64}{63}H^3 + \tfrac{32}{7}H|L|^2 \\
&\quad - \tfrac{32}{7}\operatorname{tr} L^3 + 40\tilde{\Delta}H\} + f_{|N}\{\tfrac{848}{21}H^2 + \tfrac{24}{7}|L|^2\} - 40f_{|NN}H - 60(\Delta f)_{|N}\Big) \\
&= \oint \Big(f\{-\tfrac{64}{63}X_6 + \tfrac{32}{7}X_7 - \tfrac{32}{7}X_8\} - 40Y_2(f) - 40Y_3(f) \\
&\quad + \tfrac{848}{21}Y_4(f) + \tfrac{24}{7}Y_7(f) + 60Y_8(f)\Big)
\end{aligned}
\tag{6.3}
$$

Using (6.2), we write this in terms of the invariants $\mathcal{L}_\nu$, $\ell_i(f)$, $q_j(f)$, $Y_3(f)$, and $Y_4(f)$:

$$
\begin{aligned}
(4\pi)^2 \cdot 360(a_4(f,Y,\mathcal{R}) - a_4(f,Y,\mathcal{D})) &= \oint \Big(\tfrac{32}{7}f\mathcal{L}_5 + \tfrac{8}{7}\ell_1(f) + 60\ell_3(f) + 40q_1(f) \\
&\quad - 20q_2(f) + \tfrac{20}{3}q_3(f) + \tfrac{32}{21}Y_4(f)\Big), \qquad m = 4.
\end{aligned}
$$

This gives:

6.6 Theorem. *If* $\bar\beta_\nu = (4\pi)^2 \cdot 360\beta_\nu$ *and similarly for* λ_i, σ_j, *and* c_k, *then for the problem* $(Y, \mathcal{R})$ *in dimension* $m = 4$,

$$\begin{array}{llll}
\bar\beta_1 = 2, & \bar\beta_2 = \bar\beta_3 = -8, & \bar\beta_4 = -4, & \bar\beta_5 = -8, \\
\bar\lambda_1 = 4, & \bar\lambda_2 = 0, & \bar\lambda_3 = 30, & \\
\bar\sigma_1 = 20, & \bar\sigma_2 = -\frac{25}{3}, & \bar\sigma_3 = \frac{29}{9}, & \\
\bar c_3 = 8, & \bar c_4 = -\frac{8}{9}. & \square &
\end{array}$$

Apropos Theorems 5.2, 5.5, 5.6, and 5.8, we remark that

$$\begin{aligned}
&(\tfrac{3}{4}\bar\beta_2 + 3\bar\sigma_2 - 9\bar\sigma_3 - \tfrac{3}{2}\bar c_3)(Y, \mathcal{D}) = 48, \\
&(\tfrac{3}{4}\bar\beta_2 + 3\bar\sigma_2 - 9\bar\sigma_3 - \tfrac{3}{2}\bar c_3)(Y, \mathcal{R}) = -72, \\
&(-3\bar\lambda_2 - 3\bar\lambda_3 + 12\bar\sigma_2 + 36\bar\sigma_3)(Y, \mathcal{D}) = 106, \\
&(-3\bar\lambda_2 - 3\bar\lambda_3 + 12\bar\sigma_2 + 36\bar\sigma_3)(Y, \mathcal{R}) = -74, \\
&(-3\bar\lambda_2 - \bar\lambda_3 + 6\bar\sigma_2 - 18\bar\sigma_3)(Y, \mathcal{D}) = 162, \\
&(-3\bar\lambda_2 - \bar\lambda_3 + 6\bar\sigma_2 - 18\bar\sigma_3)(Y, \mathcal{R}) = -138, \\
&(-3\bar\lambda_2 - \bar\lambda_3 + 36\bar\sigma_3 + 9\bar c_4)(Y, \mathcal{D}) = -810/7, \\
&(-3\bar\lambda_2 - \bar\lambda_3 + 36\bar\sigma_3 + 9\bar c_4)(Y, \mathcal{R}) = 78, \\
&(6\bar\sigma_2 - 9\bar c_4)(Y, \mathcal{D}) = 642/7, \\
&(6\bar\sigma_2 - 9\bar c_4)(Y, \mathcal{R}) = -42, \\
&(-3\bar\sigma_2 - 9\bar\sigma_3 + \tfrac{3}{2}\bar c_3)(Y, \mathcal{D}) = (-3\bar\sigma_2 - 9\bar\sigma_3 + \tfrac{3}{2}\bar c_3)(Y, \mathcal{R}) = 8, \\
&(\tfrac{1}{2}\bar\beta_2 + 3\bar\sigma_2 - 9\bar\sigma_3 - \tfrac{3}{2}\bar c_3)(Y, \mathcal{D}) = 50, \\
&(\tfrac{1}{2}\bar\beta_2 + 3\bar\sigma_2 - 9\bar\sigma_3 - \tfrac{3}{2}\bar c_3)(Y, \mathcal{R}) = -70, \\
&(-6\bar\lambda_2 + 6\bar\sigma_2 + 18\bar\sigma_3)(Y, \mathcal{D}) = (-6\bar\lambda_2 + 6\bar\sigma_2 + 18\bar\sigma_3)(Y, \mathcal{R}) = 8.
\end{aligned}$$

By Corollaries 5.3 and 5.4 and equation (5.1), we have:

6.7 Corollary. *If* $g[0]$ *is the standard* H^4 *metric and* $g[\omega]$ *the standard* B^4 *metric, then* $a_4(Y, \mathcal{D}) = a_4(Y, \mathcal{R}) = -1/180$. *For* $\lambda = -1/180$,

$$\begin{aligned}
\log \frac{\mathcal{P}_\lambda(Y, \mathcal{D}, g[\omega])}{\mathcal{P}_\lambda(Y, \mathcal{D}, g[0])} &= -(\log 6 + \tfrac{17}{21})/360 < 0, \\
\log \frac{\det(Y_\mathcal{D})[\omega]}{\det(Y_\mathcal{D})[0]} &= -(4\log 2 + \tfrac{17}{21})/360 < 0, \\
\log \frac{\mathcal{P}_\lambda(Y, \mathcal{R}, g[\omega])}{\mathcal{P}_\lambda(Y, \mathcal{R}, g[0])} &= -(\log 6 - \tfrac{1}{3})/360 < 0, \\
\log \frac{\det(Y_\mathcal{R})[\omega]}{\det(Y_\mathcal{R})[0]} &= -(4\log 2 - \tfrac{1}{3})/360 < 0.
\end{aligned}$$

If $g[0]$ is the standard C_h^4 metric and $g[\omega]$ the standard $\mathcal{A}_s^4$ metric, $s = e^h$, then $a_4(Y, \mathcal{D}) = a_4(Y, \mathcal{R}) = 0$, and

$$\log \frac{\mathcal{P}_0(Y, \mathcal{D}, g[\omega])}{\mathcal{P}_0(Y, \mathcal{D}, g[0])} = \log \frac{\det(Y_{\mathcal{D}})[\omega]}{\det(Y_{\mathcal{D}})[0]} = h/360.$$

Proof. Aside from direct computation, what we need to verify is that the null spaces of the problems vanish on the spaces in question. But the lowest possible eigenvalue of either problem on H^4 or C_h^4 is 1/6 times the (positive) constant scalar curvature of $g[0]$. □

Branson, Chang, and Yang [**BCY**, Sec. 5] have shown that the scale invariant determinant functional for Y on the conformal class of the round metric $g[0]$ on S^4 is minimized exactly at $g[0]$, and at the metrics $h^* g[0]$ gotten by pulling $g[0]$ back under a conformal diffeomorphism h of $(S^4, g[0])$. In light of this, Corollary 6.7 can be interpreted as saying that passage from H^4 to B^4 has *improved* (i.e. lowered) the scale-invariant determinant functionals for both $(Y, \mathcal{D})$ and $(Y, \mathcal{R})$. Roughly speaking, round is "best" in the boundariless case, but flat is "better" when boundaries are allowed.

7. The value of the functional determinant on the hemisphere and ball

In this section, the index j will always run over the natural numbers $\mathbb{N}$.

The Hurwitz zeta functions are

$$\zeta_a(s) = \sum_j (j+a)^{-s}, \quad a > 0,$$

and the Riemann zeta function is $\zeta_R(s) = \zeta_1(s)$. Note that

$$\begin{aligned} (d/da)\zeta_a(s) &= -s\zeta_a(s+1), \\ \zeta_a(s) - \zeta_{a+1}(s) &= a^{-s}. \end{aligned} \tag{7.1}$$

Consider the *double* zeta functions

$$\begin{aligned} h_a(s) &= \sum_j [(j+a)(j+a+1)]^{-s}, \\ f_a(s) &= \sum_j (2j+2a+1)[(j+a)(j+a+1)]^{-s}. \end{aligned}$$

In analogy with (7.1), we have

$$(d/da)h_a(s) = -sf_a(s+1), \tag{7.2}$$

$$(d/da)f_a(s) = (2-4s)h_a(s) - sh_a(s+1), \tag{7.3}$$

$$h_a(s) - h_{a+1}(s) = [a(a+1)]^{-s}, \tag{7.4}$$

$$f_a(s) - f_{a+1}(s) = (2a+1)[a(a+1)]^{-s}. \tag{7.5}$$

All these zeta functions have isolated simple poles. All identities below are valid in their elementary form for large $\operatorname{Re} s$, and for all s in the sense of analytic continuation. In and thus

$$\begin{aligned}
\mathcal{I}_5 &= \int_0^\infty \left((t^{-2} - 6t^{-3} + 12t^{-4})e^{-at} - 12t^{-3}\frac{e^{-(a+1)t}}{1-e^{-t}}\right) dt \\
&= \frac{d}{ds}\bigg|_{s=0} \left\{\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-at}dt - \frac{6}{\Gamma(s)}\int_0^\infty t^{s-3}e^{-at}dt + \frac{12}{\Gamma(s)}\int_0^\infty t^{s-4}e^{-at}dt \right. \\
&\qquad \left. - \frac{12}{\Gamma(s)}\int_0^\infty t^{s-3}\frac{e^{-(a+1)t}}{1-e^{-t}}\,dt\right\} \\
&= \frac{d}{ds}\bigg|_{s=0} \left\{\frac{a^{1-s}}{s-1} - \frac{6a^{2-s}}{(s-1)(s-2)} + \frac{12a^{3-s}}{(s-1)(s-2)(s-3)} - \frac{12}{(s-1)(s-2)}\zeta_{a+1}(s-2)\right\} \\
&= a(a+1)(2a+1)\log a - a - \tfrac{9}{2}a^2 - \tfrac{11}{3}a^3 - 6\zeta'_{a+1}(-2) - 9\zeta_{a+1}(-2).
\end{aligned}$$

This gives

$$\begin{aligned}
f'_a(-1) = 4\zeta'_{a+1}(-3) &+ 2\zeta'_{a+1}(-1) + 11\zeta_{a+1}(-2) - 2\zeta_{a+1}(-1) \\
&+ \tfrac{11}{3}a^3 + \tfrac{9}{2}a^2 + \tfrac{5}{6}a - \tfrac{1}{8} - a(a+1)(2a+1)\log a.
\end{aligned}$$

By (8.2),

$$f'_a(-1) = 4\zeta'_{a+1}(-3) + 2\zeta'_{a+1}(-1) - a(a+1)(2a+1)\log a + \tfrac{1}{24},$$

as desired for (7.9).

Now consider

$$\begin{aligned}
h_a(s) &= \sum_j [(j+a)(j+a+1)]^{-s} \\
&= \frac{1}{\Gamma(s)^2}\int_0^\infty\int_0^\infty (uv)^{s-1}\sum_j e^{-(j+a)(u+v)-v}du\,dv \\
&= \frac{1}{\Gamma(s)^2}\int_0^1 [\theta(1-\theta)]^{s-1}\left(\int_0^\infty t^{2s-1}e^{\theta t}\frac{e^{-(a+1)t}}{1-e^{-t}}\,dt\right)d\theta \\
&= \frac{1}{\Gamma(s)^2}\sum_{k=0}^\infty c_k(s+1)I_{a,k}(s+\tfrac{1}{2}) \\
&= \frac{1}{\Gamma(s)}\sum_{k=0}^\infty e_k(s)\zeta_{a+1}(2s+k),
\end{aligned}$$

where

$$e_k(s) := \Gamma(s+k)/k!\,.$$

(7.7) is immediate from this; the first term on the right in (7.7) is produced by the singularity of ζ_{a+1} at $s = 1$. In calculating $h'_a(0)$, we encounter the singularity of ζ_{a+1} at the $k = 1$ term; in calculating $h'_a(-1)$, at the $k = 3$ term.

For the $s = 0$ calculation, note that

$$e_0(s) = \Gamma(s), \qquad e_1(s) = s\Gamma(s),$$
$$e_k(0) = 1/k, \qquad k \geq 2.$$

Thus

$$h'_a(0) = 2\zeta'_{a+1}(0) + \alpha(a) + \mathcal{I}_2\,,$$

where $\alpha(a)$ is defined by

$$\zeta_{a+1}(1+s) = \frac{1}{s} + \alpha(a) + O(s),$$

and

$$\begin{aligned}
\mathcal{I}_2 :&= \sum_{k=2}^{\infty} \frac{1}{k}\zeta_{a+1}(k) = \sum_{k=2}^{\infty} \frac{1}{k!}\int_0^\infty t^{k-1}\frac{e^{-(a+1)t}}{1-e^{-t}}\,dt \\
&= \int_0^\infty (e^t - 1 - t)t^{-1}\frac{e^{-(a+1)t}}{1-e^{-t}}\,dt \\
&= \int_0^\infty \left(t^{-1}e^{-at} - \frac{e^{-(a+1)t}}{1-e^{-t}}\right)dt \\
&= \frac{d}{ds}\bigg|_{s=0}\left\{\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}dt - \frac{1}{\Gamma(s)}\int_0^\infty t^s\frac{e^{-(a+1)t}}{1-e^{-t}}dt\right\} \\
&= \frac{d}{ds}\bigg|_{s=0}\{a^{-s} - s\zeta_{a+1}(s+1)\} \\
&= -\log a - \alpha(a).
\end{aligned}$$

This gives

$$h'_a(0) = 2\zeta'_{a+1}(0) - \log a,$$

as desired for (7.9).

For the $s = -1$ calculation, note that

$$e_2(s) = \tfrac{1}{2}s(s+1)\Gamma(s), \qquad e_3(s) = \tfrac{1}{6}s(s+1)(s+2)\Gamma(s),$$
$$e_k(-1) = 1/k(k-1), \qquad k \geq 4.$$

Thus

$$h'_a(-1) = 2\zeta'_{a+1}(-2) - 2\zeta'_{a+1}(-1) + \zeta_{a+1}(-1) - \tfrac{1}{2}\zeta_{a+1}(0) - \tfrac{1}{6}\alpha(a) - \mathcal{I}_4\,,$$

where

$$\begin{aligned}
\mathcal{I}_4 :&= \sum_{k=4}^{\infty}\frac{k-2}{k!}\int_0^\infty t^{k-3}\frac{e^{-(a+1)t}}{1-e^{-t}}\,dt \\
&= \int_0^\infty \{(t^{-2} - 2t^{-3})(e^t - 1) + 2t^{-2} - \tfrac{1}{6}\}\frac{e^{-(a+1)t}}{1-e^{-t}}\,dt \\
&= \frac{d}{ds}\bigg|_{s=0}\left\{\frac{a^{1-s}}{s-1} - \frac{2a^{2-s}}{(s-1)(s-2)} + \frac{2}{s-1}\zeta_{a+1}(s-1) - \tfrac{1}{6}s\zeta_{a+1}(s+1)\right\} \\
&= -a + a\log a - \tfrac{3}{2}a^2 + a^2\log a - 2\zeta_{a+1}(-1) - 2\zeta'_{a+1}(-1) - \tfrac{1}{6}\alpha(a).
\end{aligned}$$

The total is

$$h'_a(-1) = 2\zeta'_{a+1}(-2) + 3\zeta_{a+1}(-1) - \tfrac{1}{2}\zeta_{a+1}(0) - (a^2 + a)\log a + a + \tfrac{3}{2}a^2;$$

by (8.2),

$$h'_a(-1) = 2\zeta'_{a+1}(-2) - (a^2 + a)\log a,$$

as desired for (7.9).

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Vejleder: Anders Madsen

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Projektrapport af: Tom J. Andersen, Tommy R. Andersen, Gert Krenøe og Peter H. Lassen
Vejleder: Bernhelm Boss.

3/78 "OPGAVESAMLING", breddekursus i fysik.
Af: Lasse Rasmussen, Aage Bonde Kræmmer og Jens Højgaard Jensen.

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Af: Mogens Niss
Nr. 4 er p.t. udgået.

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Af: Helge Kragh.
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Af: Karin Beyer, Jens Højgaard Jensen og Bent C. Jørgensen.

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Af: B.V. Gnedenko.
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Projektrapport af: Lasse Rasmussen.
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Projektrapport af: Jan Christensen og Jeanne Mortensen.
Vejledere: Karin Beyer og Peder Voetmann Christiansen.

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Af: Jørgen Larsen.

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18/79 "MATHEMATICS AND THE REAL WORLD", Proceedings af an International Workshop, Roskilde University Centre, Denmark, 1978. Preprint.
Af: Bernhelm Booss og Mogens Niss (eds.)

19/79 "GEOMETRI, SKOLE OG VIRKELIGHED".
Projektrapport af: Tom J. Andersen, Tommy R. Andersen og Per H.H. Larsen.
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Projektrapport af: Niels Jørgensen & Mikael Klintorp.
Vejleder: Jeppe Dyre.

129/86 "PHYSICS IN SOCIETY"
Lecture Notes 1983 (1986)
Af: Bent Sørensen

130/86 "Studies in Wind Power"
Af: Bent Sørensen

131/86 "FYSIK OG SAMFUND" - Et integreret fysik/historieprojekt om naturanskuelsens historiske udvikling og dens samfundsmæssige betingethed.
Projektrapport af: Jakob Heckscher, Søren Brønd, Andy Wierød.
Vejledere: Jens Høyrup, Jørgen Vogelius, Jens Højgaard Jensen.

132/86 "FYSIK OG DANNELSE"
Projektrapport af: Søren Brønd, Andy Wierød.
Vejledere: Karin Beyer, Jørgen Vogelius.

133/86 "CHERNOBYL ACCIDENT: ASSESSING THE DATA.
ENERGY SERIES NO. 15.
AF: Bent Sørensen.

134/87 "THE D.C. AND THE A.C. ELECTRICAL TRANSPORT IN AsSeTe SYSTEM"
Authors: M.B.El-Den, N.B.Olsen, Ib Høst Pedersen, Petr Visčor

135/87 "INTUITIONISTISK MATEMATIKS METODER OG ERKENDELSESTEORETISKE FORUDSÆTNINGER"
MASTEMATIKSPECIALE: Claus Larsen
Vejledere: Anton Jensen og Stig Andur Pedersen

136/87 "Mystisk og naturlig filosofi: En skitse af kristendommens første og andet møde med græsk filosofi"
Projektrapport af Frank Colding Ludvigsen
Vejledere: Historie: Ib Thiersen
Fysik: Jens Højgaard Jensen

137/87 "HOPMODELLER FOR ELEKTRISK LEDNING I UORDNEDE FASTE STOFFER" - Resume af licentiatafhandling
Af: Jeppe Dyre
Vejledere: Niels Boye Olsen og Peder Voetmann Christiansen.

138/87 "JOSEPHSON EFFECT AND CIRCLE MAP."
Paper presented at The International Workshop on Teaching Nonlinear Phenomena at Universities and Schools, "Chaos in Education". Balaton, Hungary, 26 April-2 May 1987.

By: Peder Voetmann Christiansen

139/87 "Machbarkeit nichtbeherrschbarer Technik durch Fortschritte in der Erkennbarkeit der Natur"

Af: Bernhelm Booss-Bavnbek
Martin Bohle-Carbonell

140/87 "ON THE TOPOLOGY OF SPACES OF HOLOMORPHIC MAPS"

By: Jens Gravesen

141/87 "RADIOMETERS UDVIKLING AF BLODGASAPPARATUR - ET TEKNOLOGIHISTORISK PROJEKT"

Projektrapport af Finn C. Physant

Vejleder: Ib Thiersen

142/87 "The Calderón Projektor for Operators With Splitting Elliptic Symbols"

by: Bernhelm Booss-Bavnbek og
Krzysztof P. Wojciechowski

143/87 "Kursusmateriale til Matematik på NAT-BAS"

af: Mogens Brun Heefelt

144/87 "Context and Non-Locality - A Peircean Approach
Paper presented at the Symposium on the Foundations of Modern Physics The Copenhagen Interpretation 60 Years after the Como Lecture. Joensuu, Finland, 6 - 8 august 1987.

By: Peder Voetmann Christiansen

145/87 "AIMS AND SCOPE OF APPLICATIONS AND MODELLING IN MATHEMATICS CURRICULA"

Manuscript of a plenary lecture delivered at ICMTA 3, Kassel, FRG 8.-11.9.1987

By: Mogens Niss

146/87 "BESTEMMELSE AF BULKRESISTIVITETEN I SILICIUM"
- en ny frekvensbaseret målemetode.

Fysikspeciale af Jan Vedde

Vejledere: Niels Boye Olsen & Petr Viščor

147/87 "Rapport om BIS på NAT-BAS"

redigeret af: Mogens Brun Heefelt

148/87 "Naturvidenskabsundervisning med Samfundsperspektiv"

af: Peter Colding-Jørgensen DLH
Albert Chr. Paulsen

149/87 "In-Situ Measurements of the density of amorphous germanium prepared in ultra high vacuum"

by: Petr Viščor

150/87 "Structure and the Existence of the first sharp diffraction peak in amorphous germanium prepared in UHV and measured in-situ"

by: Petr Viščor

151/87 "DYNAMISK PROGRAMMERING"

Matematikprojekt af:
Birgit Andresen, Keld Nielsen og Jimmy Staal

Vejleder: Mogens Niss

152/87 "PSEUDO-DIFFERENTIAL PROJECTIONS AND THE TOPOLOGY OF CERTAIN SPACES OF ELLIPTIC BOUNDARY VALUE PROBLEMS"

by: Bernhelm Booss-Bavnbek
Krzysztof P. Wojciechowski

153/88 "HALVLEDERTEKNOLOGIENS UDVIKLING MELLEM MILITÆRE OG CIVILE KRÆFTER"

Et eksempel på humanistisk teknologihistorie
Historiespeciale

Af: Hans Hedal

Vejleder: Ib Thiersen

154/88 "MASTER EQUATION APPROACH TO VISCOUS LIQUIDS AND THE GLASS TRANSITION"

By: Jeppe Dyre

155/88 "A NOTE ON THE ACTION OF THE POISSON SOLUTION OPERATOR TO THE DIRICHLET PROBLEM FOR A FORMALLY SELFADJOINT DIFFERENTIAL OPERATOR"

by: Michael Pedersen

156/88 "THE RANDOM FREE ENERGY BARRIER MODEL FOR AC CONDUCTION IN DISORDERED SOLIDS"

by: Jeppe C. Dyre

157/88 " STABILIZATION OF PARTIAL DIFFERENTIAL EQUATIONS BY FINITE DIMENSIONAL BOUNDARY FEEDBACK CONTROL: A pseudo-differential approach."

by: Michael Pedersen

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159/88 "STUDIES IN SOLAR ENERGY"

by: Bent Sørensen

160/88 "LOOP GROUPS AND INSTANTONS IN DIMENSION TWO"

by: Jens Gravesen

161/88 "PSEUDO-DIFFERENTIAL PERTURBATIONS AND STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS:
Dirichlet feedback control problems"

by: Michael Pedersen

162/88 "PIGER & FYSIK - OG MEGET MERE"

AF: Karin Beyer, Sussanne Blegaa, Birthe Olsen,
Jette Reich , Mette Vedelsby

163/88 "EN MATEMATISK MODEL TIL BESTEMMELSE AF PERMEABILITETEN FOR BLOD-NETHINDE-BARRIEREN"

Af: Finn Langberg, Michael Jarden, Lars Frellesen

Vejleder: Jesper Larsen

164/88 "Vurdering af matematisk teknologi
Technology Assessment
Technikfolgenabschätzung"

Af: Bernhelm Booss-Bavnbek, Glen Pate med
Martin Bohle-Carbonell og Jens Højgaard Jensen

165/88 "COMPLEX STRUCTURES IN THE NASH-MOSER CATEGORY"

by: Jens Gravesen

166/88 "Grundbegreber i Sandsynligheds-regningen"

Af: Jørgen Larsen

167a/88 "BASISSTATISTIK 1. Diskrete modeller"

Af: Jørgen Larsen

167b/88 "BASISSTATISTIK 2. Kontinuerte modeller"

Af: Jørgen Larsen

168/88 "OVERFLADEN AF PLANETEN MARS"
Laboratorie-simulering og MARS-analoger undersøgt ved Mössbauerspektroskopi.

Fysikspeciale af:

Birger Lundgren

Vejleder: Jens Martin Knudsen
Fys.Lab./HCØ

169/88 "CHARLES S. PEIRCE: MURSTEN OG MØRTEL TIL EN METAFYSIK."

Fem artikler fra tidsskriftet "The Monist" 1891-93.

Introduktion og oversættelse:

Peder Voetmann Christeansen

170/88 "OPGAVESAMLING I MATEMATIK"

Samtlige opgaver stillet i tiden 1974 - juni 1988

171/88 "The Dirac Equation with Light-Cone Data"

af: Johnny Tom Ottesen

172/88 "FYSIK OG VIRKELIGHED"

Kvantemekanikkens grundlagsproblem i gymnasiet.

Fysikprojekt af:

Erik Lund og Kurt Jensen

Vejledere: Albert Chr. Paulsen og
Peder Voetmann Christiansen

173/89 "NUMERISKE ALGORITMER"

af: Mogens Brun Heefelt

174/89 " GRAFISK FREMSTILLING AF FRAKTALER OG KAOS"

af: Peder Voetmann Christiansen

175/89 " AN ELEMENTARY ANALYSIS OF THE TIME DEPENDENT SPECTRUM OF THE NON-STATONARY SOLUTION TO THE OPERATOR RICCATI EQUATION

af: Michael Pedersen

176/89 " A MAXIUM ENTROPY ANSATZ FOR NONLINEAR RESPONSE THEORY"

af : Jeppe Dyre

177/89 "HVAD SKAL ADAM STÅ MODEL TIL"

af: Morten Andersen, Ulla Engström, Thomas Gravesen, Nanna Lund, Pia Madsen, Dina Rawat, Peter Torstensen

Vejleder: Mogens Brun Heefelt

178/89 "BIOSYNTESEN AF PENICILLIN - en matematisk model"

af: Ulla Eghave Rasmussen, Hans Oxvang Mortensen, Michael Jarden

vejleder i matematik: Jesper Larsen
biologi: Erling Lauridsen

179a/89 "LÆRERVEJLEDNING M.M. til et eksperimentelt forløb om kaos"

af: Andy Wierød, Søren Brønd og Jimmy Staal

Vejledere: Peder Voetmann Christiansen
Karin Beyer

179b/89 "ELEVHÆFTE: Noter til et eksperimentelt kursus om kaos"

af: Andy Wierød, Søren Brønd og Jimmy Staal

Vejledere: Peder Voetmann Christiansen
Karin Beyer

180/89 "KAOS I FYSISKE SYSTEMER eksemplificeret ved torsions- og dobbeltpendul".

af: Andy Wierød, Søren Brønd og Jimmy Staal
Vejleder: Peder Voetmann Christiansen

181/89 "A ZERO-PARAMETER CONSTITUTIVE RELATION FOR PURE SHEAR VISCOELASTICITY"

by: Jeppe Dyre

183/89 "MATEMATICAL PROBLEM SOLVING, MODELLING. APPLICATIONS AND LINKS TO OTHER SUBJECTS - State. trends and issues in mathematics instruction

by: WERNER BLUM, Kassel (FRG) og
MOGENS NISS, Roskilde (Denmark)

184/89 "En metode til bestemmelse af den frekvensafhængige varmefylde af en underafkølet væske ved glasovergangen"

af: Tage Emil Christensen

185/90 "EN NÆSTEN PERIODISK HISTORIE"

Et matematisk projekt

af: Steen Grode og Thomas Jessen

Vejleder: Jacob Jacobsen

186/90 "RITUAL OG RATIONALITET i videnskabers udvikling"

redigeret af Arne Jakobsen og Stig Andur Pedersen

187/90 "RSA - et kryptografisk system"

af: Annemette Sofie Olufsen, Lars Frellesen og Ole Møller Nielsen

Vejledere: Michael Pedersen og Finn Munk

188/90 "FERMICONDENSATION - AN ALMOST IDEAL GLASS TRANSITION"

by: Jeppe Dyre

189/90 "DATAMATER I MATEMATIKUNDERVISNINGEN PÅ GYMNASIET OG HØJERE LÆREANSTALTER

af: Finn Langberg

190/90 "FIVE REQUIREMENTS FOR AN APPROXIMATE NONLINEAR RESPONSE THEORY"

by: Jeppe Dyré

191/90 "MOORE COHOMOLOGY, PRINCIPAL BUNDLES AND ACTIONS OF GROUPS ON C*-ALGEBRAS"

by: Iain Raeburn and Dana P. Williams

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by: Viggo Andreasen

193/90 "Causal and Diagnostic Reasoning"

by: Stig Andur Pedersen

194a/90 "DETERMINISTISK KAOS"

Projektrapport af : Frank Olsen

194b/90 "DETERMINISTISK KAOS"
Kørselsrapport

Projektrapport af: Frank Olsen

195/90 "STADIER PÅ PARADIGMETS VEJ"
Et projekt om den videnskabelige udvikling der førte til dannelse af kvantemekanikken.

Projektrapport for 1. modul på fysikuddannelsen, skrevet af:

Anja Boisen, Thomas Hougård, Anders Gorm Larsen, Nicolai Ryge.

Vejleder: Peder Voetmann Christiansen

196/90 "ER KAOS NØDVENDIGT?"
- en projektrapport om kaos' paradigmatiske status i fysikken.

af: Johannes K. Nielsen, Jimmy Staal og Peter Bøggild

Vejleder: Peder Voetmann Christiansen

197/90 "Kontrafaktiske konditionaler i HOL

af: Jesper Voetmann, Hans Oxvang Mortensen og Aleksander Høst-Madsen

Vejleder: Stig Andur Pedersen

198/90 "Metal-Isolator-Metal systemer"
Speciale
af: Frank Olsen

199/90 "SPREDT FÆGTNING" Artikelsamling
af: Jens Højgaard Jensen

200/90 "LINEÆR ALGEBRA OG ANALYSE"
Noter til den naturvidenskabelige basisuddannelse.
af: Mogens Niss

201/90 **"Undersøgelse af atomare korrelationer i amorfe stoffer ved røntgendiffraktion"**

af: Karen Birkelund og Klaus Dahl Jensen

Vejledere: Petr Viščor, Ole Bakander

202/90 **"TEGN OG KVANTER"**
Foredrag og artikler, 1971-90.

af: Peder Voetmann Christiansen

203/90 **"OPGAVESAMLING I MATEMATIK" 1974-1990**

afløser tekst 170/88

204/91 **"ERKENDELSE OG KVANTEMEKANIK"**

et Breddemodul Fysik Projekt

af: Thomas Jessen

Vejleder: Petr Viščor

205/91 **"PEIRCE'S LOGIC OF VAGUENESS"**

by: Claudine Engel-Tiercelin
Department of Philosophy
Université de Paris-1
(Panthéon-Sorbonne)

206a+b/91 **"GERMANIUMBEAMANALYSE SAMT A - GE TYNDFILMS ELEKTRISKE EGENSKABER"**

Eksperimentelt Fysikspeciale

af: Jeanne Linda Mortensen

og Annette Post Nielsen

Vejleder: Petr Viščor

207/91 "SOME REMARKS ON AC CONDUCTION IN DISORDERED SOLIDS"

by: Jeppe C. Dyre

208/91 "LANGEVIN MODELS FOR SHEAR STRESS FLUCTUATIONS IN FLOWS OF VISCO-ELASTIC LIQUIDS"

by: Jeppe C. Dyre

209/91 "LORENZ GUIDE" Kompendium til den danske fysiker Ludvig Lorenz, 1829-91.

af: Helge Kragh

210/91 "Global Dimension, Tower of Algebras, and Jones Index of Split Seperable Subalgebras with Unitality Condition.

by: Lars Kadison

211/91 "I SANDHEDENS TJENESTE"

- historien bag teorien for de komplekse tal.

af: Lise Arleth, Charlotte Gjerrild, Jane Hansen, Linda Kyndlev, Anne Charlotte Nilsson, Kamma Tulinius.

Vejledere: Jesper Larsen og Bernhelm Booss-Bavnbek

212/91 "Cyclic Homology of Triangular Matrix Algebras"

by: Lars Kadison

213/91 "Disease-induced natural selection in a diploid host

by: Viggo Andreasen and Freddy B.Christiansen

214|91 "Halløj i æteren" - om elektromagnetisme. Oplæg til undervisningsmateriale i gymnasiet.

Af: Nils Kruse, Peter Gastrup, Kristian Hoppe, Jeppe Guldager

Vejledere: Petr Viscor, Hans Hedal

215|91 "Physics and Technology of Metal-Insulator-Metal thin film structures used as planar electron emitters

by: A.Delong, M.Drsticka, K.Hladil, V.Kolarik, F.Olsen, P.Pavelka and Petr Viscor.

216|91 "Kvantemekanik på PC'eren"

af: Thomas Jessen

217/92 "Two papers on APPLICATIONS AND MODELLING IN THE MATHEMATICS CURRICULUM"

by: Mogens Niss

218/92 "A Three-Square Theorem"

by: Lars Kadison

219/92 "RUPNOK - stationær strømning i elastiske rør"

af: Anja Boisen, Karen Birkelund, Mette Olufsen

Vejleder: Jesper Larsen

220/92 "Automatisk diagnosticering i digitale kredsløb"

af: Bjørn Christensen, Ole Møller Nielsen

Vejleder: Stig Andur Pedersen

221/92 "A BUNDLE VALUED RADON TRANSFORM, WITH APPLICATIONS TO INVARIANT WAVE EQUATIONS"

by: Thomas P. Branson, Gestur Olafsson and Henrik Schlichtkrull

222/92 On the Representations of some Infinite Dimensional Groups and Algebras Related to Quantum Physics

by: Johnny T. Ottesen

223/92 *THE FUNCTIONAL DETERMINANT*

by: Thomas P. Branson

224/92 *UNIVERSAL AC CONDUCTIVITY OF NON-METALLIC SOLIDS AT LOW TEMPERATURES*

by: Jeppe C. Dyre

225/92 *"HATMODELLEN" Impedansspektroskopi i ultrarent en-krystallinsk silicium*

af: Anja Boisen, Anders Gorm Larsen, Jesper Varmer, Johannes K. Nielsen, Kit R. Hansen, Peter Bøggild og Thomas Hougaard

Vejleder: Petr Viscor

226/92 *"METHODS AND MODELS FOR ESTIMATING THE GLOBAL CIRCULATION OF SELECTED EMISSIONS FROM ENERGY CONVERSION"*

by: Bent Sørensen

227/92 "Computersimulering og fysik"

af: Per M.Hansen, Steffen Holm, Peter Maibom, Mads K. Dall Petersen, Pernille Postgaard, Thomas B.Schrøder, Ivar P. Zeck

Vejleder: Peder Voetmann Christiansen

228/92 "Teknologi og historie"

Fire artikler af:

Mogens Niss, Jens Høyrup, Ib Thiersen, Hans Hedal

229/92 "Masser af information uden betydning"

En diskussion af informationsteorien i Tor Nørretranders' "Mærk Verden" og en skitse til et alternativ basseret på andenordens kybernetik og semiotik.

af: Søren Brier

230/92 "Vinklens tredeling - et klassisk problem"

et matematisk projekt af

Karen Birkelund, Bjørn Christensen

Vejleder: Johnny Ottesen

231A/92 "Elektrondiffusion i silicium - en matematisk model"

af: Jesper Voetmann, Karen Birkelund, Mette Olufsen, Ole Møller Nielsen

Vejledere: Johnny Ottesen, H.B.Hansen

231B/92 "Elektrondiffusion i silicium - en matematisk model" Kildetekster

af: Jesper Voetmann, Karen Birkelund, Mette Olufsen, Ole Møller Nielsen

Vejledere: Johnny Ottesen, H.B.Hansen

232/92 "Undersøgelse om den simultane opdagelse af energiens bevarelse og isærdeles om de af Mayer, Colding, Joule og Helmholtz udførte arbejder"

af: L.Arleth, G.I.Dybkjær, M.T.Østergård

Vejleder: Dorthe Posselt

233/92 "The effect of age-dependent host mortality on the dynamics of an endemic disease and Instability in an SIR-model with age-dependent susceptibility

by: Viggo Andreasen