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ABSTRACT. We develop the theory of the Fourier and Radon transforms of sections of equivariant vector bundles over symmetric spaces of the noncompact type. As an application, we show that wave propagation governed by the Maxwell and massless Dirac equations on the odd-dimensional hyperboloid is sharp. In particular, we prove Huygens' principle for these equations.

0. Introduction. Harmonic analysis, in its commutative and noncommutative forms, is currently one of the most important and useful areas in Mathematics. Harmonic analysis may be defined as the attempt to decompose function spaces over spaces with symmetry by taking spectral decompositions of differential operators which respect the symmetry; or in brief, as the spectral theory of invariant differential operators. The ability to find spectral decompositions is the ability to solve differential equations, and so one is led inevitably to the Fourier transform and its variants. Sufficient symmetry, i.e. the presence of a large enough transformation group, is extremely useful both in finding the "right" differential equations, and in solving them; it also seems to be the correct setting in which to define a Fourier transform. A look at the long history of harmonic analysis and of Lie theory helps explain why this happy convergence of goals and means is not entirely accidental. At the same time, it allows us to state the purpose of the present paper.

After the early investigations of Gauss and Riemann into the geometry of surfaces and of space, it became possible to put the study of the physical world and of symmetry on a geometric basis. When Sophus Lie began to work, the most sophisticated tools available for theoretical studies of the physical world were partial differential equations, for example, the Laplace, wave, Maxwell, and heat equations. Lie noticed that almost all properties of differential equations that were useful in their solution had to do with behavior under groups of transformations of the underlying space. He was led to the idea that one might be able to do for partial differential equations what Galois had done for algebraic equations: roughly speaking, to reduce their solution to group theory. This core idea has spread to become ubiquitous in science, sometimes in ways that Lie could not have imagined. In other ways, the ideas of Lie, Felix Klein, and others have succeeded, remarkably, much

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as planned. Lie groups permeate modern Physics; they operate not just on space or spacetime, but on phase and configuration spaces, on fibers of bundles, and on a variety of objects constructed from these. The parallel development of analysis on Lie groups and homogeneous spaces has made it possible to mount ever better direct and formalized attacks on differential equations, for example, the wave and Maxwell equations, through the exploitation of symmetry. This development has also allowed a change of perspective to take hold, not only in Mathematics, but also in the other sciences: the transformation group of a space has come to be seen as, in a sense, more fundamental than the space itself. Within Lie theory, this thinking is implemented by viewing a homogeneous space as a quotient G/H of a group G by a subgroup H; that is, by noticing that the space is already implicit in the group. In Physics, the study of partial differential equations with symmetry groups has led to the detailed study of representations of these groups. The idea is not just to describe known physical particles and fields in terms of representations (typically carried by the space of solutions of a differential equation, or by the quotient of some larger function space by this solution space), but rather to construct predictive theory based on classification results for representations. Slowly, the group representation aspect of a particle has come to be seen as fundamental, to the point that one often sees particles defined and labelled by group representations. This motivation has supplied much of the impetus for the central problem of group representation theory, that of classifying irreducible unitary representations of a given Lie group. First proposed by Bargmann and Wigner for the Lorentz group, this problem was then developed in more generality by Gelfand, Godement, Mackey, Mautner, Naimark, Segal, and others. For *semisimple* Lie groups, the study of representations and the related problem of determining the *Plancherel formula* was taken up by Harish-Chandra, and this brings us back to harmonic analysis.

On curved spaces, the notions of *systems* of fields and of differential equations give way to those of vector bundles and of differential operators on vector bundle sections. The Maxwell equations are an example of a system that, in the curved space setting, can be properly understood only in bundle terms (in this case, bundles of differential forms). The same is true of the Dirac equation, with the added restriction that now, Lie theory is a prerequisite even for the construction of the bundle involved. In harmonic analysis, the theory of bundle valued objects is somewhat underdeveloped relative to that of scalar valued (i.e., trivial bundle valued) objects; the same is true to a lesser extent in representation theory. For example, various classification problems for invariant differential operators have long been completely understood in the scalar case, but remain elusive in the bundle case.

Our purpose here is to develop the theory of the Fourier and Radon transform of vector bundle sections over symmetric spaces of the noncompact type, to show how such tools can be used to solve invariant differential equations, and to deduce important properties of solutions. Specifically, we work with the Maxwell and (massless) Dirac equations, with a special view toward properties that imply *sharp propagation* of information; that is, propagation at characteristic speed (the "speed of light"), without dispersion. The best-known such property is *Huygens' principle*; this is also the most elementary in the sense of being directly expressable in terms of support properties of solutions (as opposed to functional analytic constructs or conservation laws). We also consider the somewhat weaker property of *equipartition* of energy or of charge. In earlier work, we considered similar questions in the scalar case [**3, 17**]. The direct inspiration for those papers and for this one is Helgason's direct (i.e. non-transform) treatment [**12**] of Huygens' principle for the wave equation on a symmetric space. Our work can also be seen as a further development of fundamental work of Harish-Chandra and of Helgason on the Fourier and Radon transforms.

The organization of our paper is as follows. Secs. 1 and 4 relate objects from differential geometry, for example connections and Laplacians, to objects from Lie theory, for example differentiation from the left and right, and the *Casimir operator*. These relations are almost trivial in the case of scalar valued functions on homogeneous spaces, but require a certain degree of care in the case of bundles. Our central result here is Proposition 4.1, which relates the geometer's *Bochner Laplacian* to the Casimir operator of G acting in bundles over a (suitably reductive) homogeneous space G/H. The Bochner Laplacian is easily related to, for example, differential form and spin Laplacians, and it is straightforward to follow the effect of the Casimir operator as Fourier and Radon transforms are applied, so Proposition 4.1 is a "bridge" sufficient for our purposes. In Sec. 2, we develop the theory of the bundle-valued Fourier transform on symmetric spaces G/K of the noncompact type, for semisimple groups G with one conjugacy class of Cartan subgroup. The main result is Theorem 2.2, which gives the Fourier inversion and Plancherel formulas in the bundle setting. Here the analytic power derives from Harish-Chandra's theory of the *operator valued* Fourier transform. The most convenient tool for the study of support properties of solutions of differential equations is the *Radon transform*, which we develop in the bundle setting in Sec. 3. The main result here is a support lemma of Paley-Wiener type, Lemma 3.3, which relates the support of a vector bundle section, the support of its Radon transform, and an exponential type estimate on its Fourier transform. The analytic power is supplied by Delorme's Paley-Wiener theorem for functions on G.

In Sec. 5, we specialize some of our results to the case of the odd-dimensional hyperboloid $\mathrm{SO}_0(2k+1,1)/\mathrm{SO}_0(2k+1) = \mathrm{Spin}_0(2k+1,1)/\mathrm{Spin}_0(2k+1)$, the setting in which we shall apply the Radon transform to questions about the Dirac and Maxwell equations. In particular, we make contact with weight arithmetic for $\mathrm{Spin}_0(2k+1,1)$ and for the groups $\mathrm{Spin}(m)$, and express our Laplacians in these terms. Sec. 6 treats the Dirac equation and a spinor wave equation. The main results are Theorem 6.8 (Huygens' principle and equipartition of charge for the Dirac equation), and Corollary 6.9 (equipartition of energy for the spinor wave equation). In Sec. 7, we treat the Maxwell equations. The main results are Theorem 7.6 (Huygens' principle and equipartition of energy for Maxwell's equations), and Theorem 7.8 (equipartition of energy for a differential form wave equation with side condition). Huygens' principle for the Dirac and Maxwell systems on the odd-dimensional hyperboloid H^{2k+1} can also be derived from Ørsted's results in [**18**], which are obtained in the somewhat different setting of intrinsically Lorentzian, locally conformally flat spaces. Huygens' principle for Maxwell's equations on H^{2k+1} was also proved by Strichartz [**20**] using different methods. Our approach to sharp wave propagation in the bundle valued case seems to indicate, as indeed all other approaches do, that the first-order Dirac and Maxwell systems are extremely natural: our arguments go through only because of special characteristics of the representations defining the appropriate bundles, and of the equations; it is not possible simply to constuct first-order systems in arbitrary equivariant vector bundles which behave in this way.

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1. Preliminary remarks. In this section, we assume that $\mathcal{M} = G/H$ is a homogeneous space of a connected, semisimple Lie group G with closed isotropy subgroup H. We adopt the usual convention of denoting the Lie algebra of a Lie group by the corresponding small German letter; in particular, we have $\mathfrak{h} \subset \mathfrak{g}$. We assume further that $\mathcal{M}$ is *strongly reductive* in the sense that there is there is a vector subspace $\mathfrak{s}$ of $\mathfrak{g}$ with

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s} \text{ (as vector spaces)}, \tag{1.1}$$

$$\mathrm{Ad}(H)\mathfrak{s} \subset \mathfrak{s}, \qquad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{h}. \tag{1.2}$$

For example, $\mathcal{M}$ could be a symmetric space like the hyperboloid $H^n = \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ or sphere $\mathrm{SO}(n+1)/\mathrm{SO}(n)$.

Remark 1.1. Let $B_{\mathfrak{g}}(X,Y) = \mathrm{tr}\,\mathrm{ad}\,X\,\mathrm{ad}\,Y$ be the Killing form of $\mathfrak{g}$. Under the above assumptions, the restriction of $B_{\mathfrak{g}}$ to $\mathfrak{s}$ is nondegenerate, and thus defines a nondegenerate pseudo-Riemannian metric on $\mathcal{M}$ as follows. The splitting (1.1) gives rise to an identification of $\mathfrak{s}$ with the tangent space $T_o\mathcal{M}$ at the identity coset, or *origin* $o = eH$ of $\mathcal{M}$, and thus to a nondegenerate bilinear form g_o on $T_o\mathcal{M}$. By the $\mathrm{Ad}(G)$ invariance of the Killing form, g_o can be pulled back to a nondegenerate bilinear form g_x on $T_x\mathcal{M}$ for each $x \in \mathcal{M}$; the desired metric is then $g : x \mapsto g_x$. In general, g is not positive or negative definite. In special cases, we shall choose normalizations of the Killing form distinguished by the desire for a certain normalized curvature (for example, constant sectional curvature ∓ 1 on the hyperboloid and sphere respectively), or by the desire (when relevant and possible) to have a restricted Killing form that agrees with an intrinsic Killing form. Such renormalizations will, of course, have an effect on the computation of the *Casimir operator* of $\mathfrak{g}$ or one of its Lie subalgebras. Note that without an assumption of positive definiteness, when we speak of "orthonormal" bases and local frames $\{X_i\}$ in this section and in Sec. 4, the sense is that the inner product of X_i and X_j is $\pm\delta_{ij}$. The definition of the Casimir operator of a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ can be given in these terms as follows: if b is some chosen nondegenerate bilinear form on $\mathfrak{q}$ (usually a normalization of the restriction of $B_{\mathfrak{g}}$), and if $X_1, \ldots, X_n$ is a basis of $\mathfrak{q}$ with $b(X_i, X_j) = \varepsilon_i\delta_{ij}$, $\varepsilon_i = \pm 1$, then $\mathrm{Cas}_{\mathfrak{q}} = -\sum_{i=1}^n \varepsilon_i X_i^2 \in U(\mathfrak{g})$. Since we are mainly interested here in Riemannian symmetric spaces, indefinite inner products will appear only in auxiliary propositions which we wish to prove in reasonable generality.

Remark 1.2. The splitting (1.1) defines a natural left-invariant connection ∇ on the principal bundle $H \to G \to \mathcal{M}$ (take $\mathfrak{h}$ to be vertical and $\mathfrak{s}$ to be horizontal), and thus on the vector bundle $\mathbf{V}_\tau = G \times_\tau V_\tau$ associated to a finite-dimensional representation (τ, V_τ) of H. We call this the *canonical* connection on $H \to G \to \mathcal{M}$ or on $\mathbf{V}_\tau$. By [15, X.3.3], ∇ agrees with the Levi-Civita (pseudo-Riemannian) connection ∇^{LC} on the tangent bundle $T\mathcal{M} = G \times_{\mathrm{Ad}} \mathfrak{s}$ in our setting. We fix this choice of connection throughout this paper.

Remark 1.3. There is a standard identification of the space $C^\infty(\mathcal{M}, \mathbf{V}_\tau)$ of C^∞ sections f of $\mathbf{V}_\tau$ with the space $C^\infty(G;\tau)$ of C^∞ functions $f^\natural : G \to V_\tau$ satisfying $f^\natural(gh) = \tau(h^{-1})f^\natural(g)$ for all $g \in G$, $h \in H$. (In fact, when it causes no difficulty, we shall sometimes blur the distinction between f and $f^\natural$.) We can use this identification to state the standard relation between the connection and its covariant derivative: if $g \in G$, $X \in C^\infty(T\mathcal{M})$,

$$(1.3) \qquad (\nabla_X f)^\natural(g) = (X^\natural f^\natural)(g),$$

where $X^\natural$ is the horizontal lift of X to G via ∇. We would also like a formula for the canonical connection that is more adapted to Lie-theoretic calculations. Since ∇ is left-invariant and the expression $\nabla_X f$ is $C^\infty(\mathcal{M})$-linear in the X argument, all information will be contained in a formula for $(\nabla_X f)^\natural(e)$ in terms of $(X^\natural)_e$ and $f^\natural$. Let $\mathcal{X} \in \mathfrak{g}$ be the image of $(X^\natural)_e$ under the usual identification of T_eG with $\mathfrak{g}$; since $(X^\natural)_e$ is horizontal, $\mathcal{X} \in \mathfrak{s}$. Since $\mathcal{X}_e = (X^\natural)_e$, it is immediate from (1.3) that

$$(\nabla_X f)^\natural(e) = \frac{d}{dt}\bigg|_{t=0} f^\natural(\exp(t\mathcal{X})).$$

By the left invariance of ∇, if $x = gH$ is arbitrary in G/H, then

$$(\nabla_X f)^\natural(g) = \frac{d}{dt}\bigg|_{t=0} f^\natural(g\exp(t\mathcal{X})),$$

where $\mathcal{X} \in \mathfrak{s}$ is determined by

$$\mathcal{X}_g = (X^\natural)_g\,.$$

Remark 1.4. Choose an orthonormal basis $X_1, \ldots, X_n$ for $\mathfrak{s}$ in some normalization $b_\mathfrak{s} = B_\mathfrak{g}|_{\mathfrak{s}\times\mathfrak{s}}$ of the Killing form, $b_\mathfrak{s}(X_i, X_j) = \varepsilon_i\delta_{ij}$, $\varepsilon_i = \pm 1$. Then it is immediate from the last remark that

$$(z_1, \ldots, z_n) \leftrightarrow \exp(\sum_{i=1}^n z_i X_i)H$$

gives a normal coordinate system at at $o \in \mathcal{M}$.

2. The bundle valued Fourier transform. Let G be a connected semisimple Lie group and K a maximal compact subgroup. Then $X = G/K$ is a Riemannian symmetric space of the noncompact type. Suppose that G has one conjugacy class of Cartan subgroups (occC). (See [**22**, Sec. 7.9] or [**11**, Theorem IX.6.1].) In this section, we would like to define a Fourier transform $\tilde{}$ on sections of K-bundles $\mathbf{V}_\tau = G \times_\tau V$ over X, (V, τ) an irreducible representation of K, and use Harish-Chandra's theory of the operator-valued Fourier transform $\mathcal{F}$ on the space $C_c^\infty(G)$ to write down Fourier inversion and Plancherel formulas for $\tilde{}$. Here and below, C_c^∞ means C^∞ with compact support.

To introduce the Fourier transform, we shall need some basic definitions from semisimple structure theory. Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, fix a positive open Weyl chamber $\mathfrak{a}_+^*$ in $\mathfrak{a}^*$, and let $G = KAN$, $\mathfrak{g} = \mathfrak{k}+\mathfrak{a}+\mathfrak{n}$

be the corresponding Iwasawa decomposition. Let ρ be half the sum of the positive $(\mathfrak{g},\mathfrak{a})$ roots:

$$\rho(H)=\tfrac{1}{2}\mathrm{tr}\,(\mathrm{ad}\,H)\Big|_{\mathfrak{n}},\quad H\in\mathfrak{a}.$$

The Weyl group of $(\mathfrak{g},\mathfrak{a})$ is $W=M'/M$, where M and M' are respectively the centralizer and normalizer of $\mathfrak{a}$ in K. Note that there are natural actions of W on the set $\hat{M}$ of equivalence classes (σ,U_σ) of irreducible representations of M, and on $\mathfrak{a}^*$. The exponential map is a diffeomorphism of $\mathfrak{a}$ onto A. If $a\in A$ and $\nu\in\mathfrak{a}^*_{\mathbb{C}}$, let

$$a^\nu=e^{\nu(\log a)}.$$

Consider the *minimal parabolic subgroup* MAN corresponding to our choices. *Principal series* representations are parameterized by $(\sigma,\nu)\in\hat{M}\times\mathfrak{a}^*_{\mathbb{C}}$. The representation $\pi_{\sigma,\nu}$ acts by left translation in the Hilbert space $\mathcal{H}_{\sigma,\nu}$ obtained by completion of the space

$$\{\phi\in C(G,U_\sigma)\mid \phi(gman)=a^{-\nu-\rho}\sigma(m)^{-1}\phi(g),\ g\in G\}$$

in the norm

$$\|\phi\|^2=\int_K\|\phi(k)\|^2dk. \tag{2.1}$$

$\pi_{\sigma,\nu}$ is unitary for ν purely imaginary on $\mathfrak{a}$. As a K-module, $(\pi_{\sigma,\nu}|_K,\mathcal{H}_{\sigma,\nu})$ is independent of ν, because restriction to K is an isometry of $\mathcal{H}_{\sigma,\nu}$ onto the K-module $\mathcal{H}_\sigma$ obtained by completion of

$$\{\phi\in C(K,U_\sigma)\mid\phi(km)=\sigma(m)^{-1}\phi(k),\ k\in K\}$$

in the norm (2.1), for all ν. In the following we identify $\mathcal{H}_{\sigma,\nu}$ and $\mathcal{H}_\sigma$ whenever it is convenient.

For $\tau\in\hat{K}$, $\sigma\in\hat{M}$, we write $\tau\downarrow\sigma$ or $\sigma\uparrow\tau$ if the multiplicity $m_\sigma(\tau)$ of σ in the restriction of τ to M is nonzero. *Frobenius reciprocity* sets up a natural identification of $\mathrm{Hom}_K(\mathcal{H}_\sigma,V_\tau)$ with $\mathrm{Hom}_M(U_\sigma,V_\tau)$.

The operator-valued Fourier transform of $F\in C_c^\infty(G)$ is

$$(\mathcal{F}F)(\sigma,\nu)=\int_G F(g)\pi_{\sigma,\nu}(g)dg\in\mathrm{HS}(\mathcal{H}_{\sigma,\nu}), \tag{2.2}$$

where "HS" stands for "Hilbert-Schmidt". The corresponding Plancherel decomposition is as follows: let $\mathcal{L}$ and $\mathcal{R}$ be the left and right regular representations of G in $L^2(G)$. Then

$$L^2(G)\cong_{G\times G}\bigoplus_{\sigma\in\hat{M}}\int^{\oplus}_{\nu\in\sqrt{-1}\mathfrak{a}^*_+}\mathcal{H}_{\sigma,\nu}\otimes\mathcal{H}^*_{\sigma,\nu}d\nu, \tag{2.3}$$

where $d\nu$ is a choice of Lebesgue measure on $\sqrt{-1}\mathfrak{a}^*$, and the representation acting on the left-hand side is $\mathcal{L}\otimes\mathcal{R}$.

Remark 2.1. By a theorem of Bruhat [4, Theorem 7.2], the $\pi_{\sigma,\nu}$ are irreducible for almost all $\nu \in \sqrt{-1}\mathfrak{a}^*$ when G is semisimple. In our occC setting, they are irreducible for *all* $\nu \in \sqrt{-1}\mathfrak{a}^*$ by [7, Theorem 41.1]. Moreover, for $\nu, \nu' \in \sqrt{-1}\mathfrak{a}^*$, $\pi_{\sigma,\nu}$ is equivalent to $\pi_{\sigma',\nu'}$ if and only if there is a $w \in W$ with $(\sigma', \nu') = (w\sigma, w\nu)$ [4, Theorem 7.2].

To invert $\mathcal{F}$, it suffices to recover $F(e)$ from $\mathcal{F}F$, since we can then apply the result to any left translate of F. By Harish-Chandra's inversion formula, [22, Theorem 8.15.4], there exists a positive normalization of $d\nu$ such that

$$F(e) = \sum_{\sigma \in \hat{M}} \int_{\nu \in \sqrt{-1}\mathfrak{a}^*_+} (\mathrm{tr}(\mathcal{F}F)(\sigma,\nu))m(\sigma,\nu)d\nu, \tag{2.4}$$

where $m(\sigma, \nu)$ is the *Plancherel density*. A formula for $-m(\sigma,\nu)$ is given in [22, p. 294] (see also [7, Theorem 24.1]). It follows from this formula that $m(\sigma,\nu)$ can be written as $m(\sigma,\nu) = |\eta(\sigma,\nu)|^2$, where $\eta(\sigma,\cdot)$ is a complex polynomial on $\mathfrak{a}^*_c$ which is real on $\mathfrak{a}^*$. (η is unique up to multiplication by ± 1.) When σ is the trivial M-type then $\eta(\sigma,\nu)$ is (plus or minus) the inverse of Harish-Chandra's c-function. The corresponding Plancherel formula is

$$\int_G |F(g)|^2 dg = \sum_{\sigma \in \hat{M}} \int_{\nu \in \sqrt{-1}\mathfrak{a}^*_+} \|(\mathcal{F}F)(\sigma,\nu)\|^2_{\mathrm{HS}} m(\sigma,\nu)d\nu. \tag{2.5}$$

Let (τ, V_τ) be an irreducible representation of K, and consider the vector bundle $\mathbb{V}_\tau = G \times_\tau V_\tau$ associated to τ and the principal fibration $K \to G \to G/K$. We identify the section space $C^\infty(G/K, \mathbb{V}_\tau)$ with $C^\infty(G;\tau)$ as in Remark 1.3. Because K is compact, this also identifies $C^\infty_c(G/K, \mathbb{V}_\tau)$ with $C^\infty_c(G;\tau)$. Similarly, we denote by $L^2(G;\tau)$ the space of V_τ-valued L^2-functions on G satisfying the above transformation rule, and by $\mathcal{L}$ the natural representation of G on this space.

The Plancherel decomposition of $\mathcal{L}$ on $L^2(G;\tau)$ follows from (2.3) above. Indeed,

$$L^2(G, V_\tau) \cong_{G\times G} \bigoplus_\sigma \int^\oplus_{\nu \in \sqrt{-1}\mathfrak{a}^*_+} \pi_{\sigma,\nu} \otimes \pi^*_{\sigma,\nu} \otimes V_\tau \, d\nu,$$

so the right transformation rule defining $L^2(G;\tau)$ gives

$$L^2(G;\tau) \cong_G \bigoplus_\sigma \int^\oplus_{\nu \in \sqrt{-1}\mathfrak{a}^*_+} \pi_{\sigma,\nu} \otimes (\pi^*_{\sigma,\nu} \otimes V_\tau)^K d\nu,$$

where $\mathcal{L}$ is the representation acting on the left-hand side. But $(\pi^*_{\sigma,\nu} \otimes V_\tau)^K = \mathrm{Hom}_K(\mathcal{H}_\sigma, V_\tau)$ is naturally identified with $\mathrm{Hom}_M(U_\sigma, V_\tau)$, so

$$\mathcal{L} \cong_G \bigoplus_{\sigma \uparrow \tau} \int^\oplus_{\nu \in \sqrt{-1}\mathfrak{a}^*_+} \pi_{\sigma,\nu} \otimes 1_{\mathrm{Hom}_M(U_\sigma, V_\tau)} d\nu,$$

where 1 denotes the trivial representation.

This decomposition of $\mathcal{L}$ is implemented by the following Fourier transform. If $\alpha \in \mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)$, we define

$$\tilde{f}(\sigma,\nu)(\alpha) = \int_{G/K} \pi_{\sigma,\nu}(g)\alpha(f(g))d(gK) \in \mathcal{H}_\sigma, \quad \sigma \in \hat{M},\ \nu \in \mathfrak{a}_{\mathbb{C}}^*$$

for $f \in C_c^\infty(G;\tau)$. In this way, $\tilde{f}(\sigma,\nu)$ can be viewed as an element of

$$\mathcal{H}_\sigma \otimes \mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)^* = \mathcal{H}_\sigma \otimes \mathrm{Hom}_K(\mathcal{H}_\sigma, V_\tau) \cong \mathcal{H}_\sigma \otimes \mathrm{Hom}_M(U_\sigma, V_\tau).$$

Here we employ the natural identification of $\mathrm{Hom}_K(\mathcal{H}_\sigma, V_\tau)$ with $\mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)^*$, and use Frobenius reciprocity to identify $\mathrm{Hom}_K(\mathcal{H}_\sigma, V_\tau)$ with $\mathrm{Hom}_M(U_\sigma, V_\tau)$. In particular, only finitely many σ can contribute: $\tilde{f}(\sigma,\nu)(\alpha) = 0$ unless $\sigma \uparrow \tau$. Notice that $f \mapsto \tilde{f}$ maps $C_c^\infty(G;\tau)$ equivariantly into $\mathcal{H}_{\sigma,\nu} \otimes \mathrm{Hom}_M(U_\sigma, V_\tau)$.

We can now state:

Theorem 2.2. *Suppose τ is an irreducible representation of K, and let $n = \dim\tau$. Let $f \in C_c^\infty(G;\tau)$.*
(a) (Fourier inversion formula.) *Let $\Phi_\sigma : \mathcal{H}_\sigma \otimes \mathrm{Hom}_K(\mathcal{H}_\sigma, V_\tau) \to V_\tau$ be the contraction $\Phi_\sigma(h \otimes \varphi) = \varphi(h)$. Then*

$$f(g) = \frac{1}{n}\sum_{\sigma\uparrow\tau}\int_{\nu\in\sqrt{-1}\mathfrak{a}_+^*} \Phi_\sigma((\pi_{\sigma,\nu}(g^{-1})\otimes 1)\tilde{f}(\sigma,\nu))m(\sigma,\nu)d\nu.$$

(b) (Plancherel formula.) *We have*

$$\|f\|^2 = \frac{1}{n}\sum_{\sigma\uparrow\tau}\int_{\nu\in\sqrt{-1}\mathfrak{a}_+^*} \|\tilde{f}(\sigma,\nu)\|^2 m(\sigma,\nu)d\nu.$$

Proof. It suffices to prove (a) for $g = e$, since we can then apply this result to left translates of f. Thus the claim is that

$$f(e) \stackrel{?}{=} \frac{1}{n}\sum_\sigma \int_\nu \Phi_\sigma(\tilde{f}(\sigma,\nu))m(\sigma,\nu)d\nu. \tag{2.6}$$

(Here and in the rest of the proof, the sum is over σ with $\sigma \uparrow \tau$, and the ν-integral is over $\sqrt{-1}\mathfrak{a}_+^*$.) We apply (2.4) to the function $F(g) = \langle f(g), v\rangle$ where $v \in V_\tau$. It follows from (2.2) that $(\mathcal{F}F)(\sigma,\nu) = (\mathcal{F}F)(\sigma,\nu)P_\tau$, where P_τ is the orthogonal projection of $\mathcal{H}_\sigma$ onto its τ-isotypic component. (2.4) becomes

$$F(e) = \sum_\sigma \int_\nu \mathrm{tr}(P_\tau(\mathcal{F}F)(\sigma,\nu)P_\tau)m(\sigma,\nu)d\nu. \tag{2.7}$$

Now pick orthonormal bases $v_1, \dots, v_n$ and $\varphi_1, \dots, \varphi_m$ for V_τ and $\mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)$ respectively, with respect to K-invariant inner products on V_τ and $\mathcal{H}_\sigma$. The $\varphi_j v_i$ are an orthonormal basis for $P_\tau \mathcal{H}_\sigma$, and we get from (2.7) that

$$F(e) = \sum_\sigma \int_\nu \sum_{i,j} \langle (\mathcal{F}F)(\sigma,\nu)\varphi_j v_i, \varphi_j v_i \rangle m(\sigma,\nu)d\nu.$$

By (2.2),

$$(2.8) \qquad \begin{aligned}(\mathcal{F}F)(\sigma,\nu)\varphi_j v_i &= \int_G \langle f(g), v\rangle \pi_{\sigma,\nu}(g)\varphi_j v_i dg \\ &= \int_{G/K}\int_K \langle \tau(k^{-1})f(g), v\rangle \pi_{\sigma,\nu}(g)\varphi_j \tau(k) v_i dk\, d(gK).\end{aligned}$$

For any endomorphism A of V_τ we have $\int_K \tau(k)A\tau(k^{-1})dk = (\mathrm{tr}\, A)I/n$ by Schur's Lemma. With $Au = \langle u, v\rangle w$, we get

$$\int_K \langle \tau(k^{-1})u, v\rangle \tau(k) w\, dk = \frac{1}{n}\langle w, v\rangle u$$

for any three vectors $u, v, w \in V_\tau$. Applied to (2.8), this gives

$$(2.9) \qquad (\mathcal{F}F)(\sigma,\nu)\varphi_j v_i = \frac{1}{n}\langle v_i, v\rangle \int_{G/K} \pi_{\sigma,\nu}(g)\varphi_j(f(g))d(gK) = \frac{1}{n}\langle v_i, v\rangle \tilde{f}(\sigma,\nu)(\varphi_j).$$

Inserting this into (2.7), we get

$$\langle f(e), v\rangle = \frac{1}{n}\sum_\sigma \int_\nu \sum_{i,j} \langle v_i, v\rangle \langle \tilde{f}(\sigma,\nu)(\varphi_j), \varphi_j v_i\rangle m(\sigma,\nu)d\nu.$$

Since

$$\sum_{i,j} \langle v_i, v\rangle \langle \tilde{f}(\sigma,\nu)(\varphi_j), \varphi_j v_i\rangle = \sum_i \langle \tilde{f}(\sigma,\nu)(\varphi_j), \varphi_j v\rangle = \langle \Phi_\sigma(\tilde{f}(\sigma,\nu)), v\rangle$$

and v was arbitrary, (2.6) and hence (a) is established.

By definition,

$$(2.10) \qquad \|f\|^2 = \int_{G/K} \|f(g)\|^2 d(gK) = \sum_{l=1}^n \int_G |\langle f(g), v_l\rangle|^2 dg.$$

We want to apply the operator-valued Plancherel formula (2.5) to $F(g) = \langle f(g), v\rangle$. By (2.9),

$$(2.11) \qquad \|(\mathcal{F}F)(\sigma,\nu)\|_{\mathrm{HS}}^2 = \sum_{i,j} \|(\mathcal{F}F)(\sigma,\nu)\varphi_j v_i\|^2 = \frac{1}{n^2}\sum_{i,j} |\langle v_i, v\rangle|^2 \|\tilde{f}(\sigma,\nu)(\varphi_j)\|^2.$$

Let $F_l(g) = \langle f(g), v_l \rangle$ and apply (2.5) and (2.11) in (2.10):

$$\begin{aligned}
\|f\|^2 &= \sum_{l=1}^{n} \sum_{\sigma} \int_{\nu} \|\mathcal{F}F_l(\sigma,\nu)\|_{\mathrm{HS}}^2 m(\sigma,\nu)d\nu \\
&= \frac{1}{n^2} \sum_{\sigma} \int_{\nu} \sum_{l,i,j} |\langle v_i, v_l\rangle|^2 \|\tilde{f}(\sigma,\nu)(\varphi_j)\|^2 m(\sigma,\nu)d\nu \\
&= \frac{1}{n} \sum_{\sigma} \int_{\nu} \sum_{j} \|\tilde{f}(\sigma,\nu)(\varphi_j)\|^2 m(\sigma,\nu)d\nu \\
&= \frac{1}{n} \sum_{\sigma} \int_{\nu} \|\tilde{f}(\sigma,\nu)\|^2 m(\sigma,\nu)d\nu,
\end{aligned}$$

proving (b). □

Remark 2.3. Notice that though the Hilbert space $L^2(G;\tau)$ decomposes as a finite direct sum over $\sigma \uparrow \tau$ of invariant subspaces, this decomposition is in general not inherited by the subspace $C_c^\infty(G;\tau)$. Indeed, any continuous intertwining operator A from a $\pi_{\sigma,\nu}$ to a $\pi_{\sigma',\nu'}$ with complex valued ν and ν' will give rise to the relation

$$\begin{aligned}
A\tilde{f}(\sigma,\nu)(\alpha) &= \int_{G/K} A\pi_{\sigma,\nu}(g)\alpha(f(g))d(gK) \\
&= \int_{G/K} \pi_{\sigma',\nu'}(g)A\alpha(f(g))d(gK) \\
&= \tilde{f}(\sigma',\nu')(A\alpha).
\end{aligned}$$

Since $\tilde{f}$ is holomorphic, it follows that the Fourier transforms $\tilde{f}(\sigma,\cdot)$ and $\tilde{f}(\sigma',\cdot)$ are not independent.

3. The bundle valued Radon transform. For $f \in C_c^\infty(G;\tau)$ we define the *Radon transform* as the V_τ valued function

$$\hat{f}(g) = a(g)^\rho \int_N f(gn)dn$$

on G. Here $a(g) \in A$ is defined by the Iwasawa decomposition: $g \in Ka(g)N$, and again, $a^\nu = e^{\nu(\log a)}$ for $a \in A, \nu \in \mathfrak{a}_{\mathbb{C}}^*$. The defining integral of $\hat{f}$ converges locally uniformly in g since N is closed and f has compact support. Hence $\hat{f}$ is smooth.

Let $\Xi = G/MN$ be the space of horocycles in G/K [9]. Since $\hat{f}(gmn) = \tau(m)^{-1}\hat{f}(g)$ for $g \in G$, $m \in M$, and $n \in N$, we may view $\hat{f}$ as a section of the vector bundle $G \times_{MN} V_\tau$ over Ξ, where M acts on V_τ by $\tau|_M$ and N acts trivially.

Notice that if τ is the trivial representation then $\hat{f}$ is the Radon transform of f in the sense of [8], except for the factor a^ρ. In this case there is a simple relation between the Radon and Fourier transforms of functions on X: essentially $\tilde{f}$ is obtained from $\hat{f}$

by a (Euclidean) Fourier transform on A [10, p. 458, equation 7]. This relation can be generalized to the present situation, where we work in bundles over X and Ξ, as follows.

For each $\sigma \in \hat{M}$ we define the σ-Radon transform of f by

$$\hat{f}_\sigma(\beta) = \beta \circ \hat{f}, \qquad \beta \in \mathrm{Hom}_M(V_\tau, U_\sigma).$$

$\hat{f}_\sigma(\beta)$ may be viewed as a section of the vector bundle $G \times_{MN} U_\sigma$ over Ξ, where N as before acts trivially. For $\alpha \in \mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)$ let $\dot{\alpha}$ denote the element of $\mathrm{Hom}_M(V_\tau, U_\sigma)$ given by $\dot{\alpha}(v) = \alpha(v)(e)$. Then $\alpha \mapsto \dot{\alpha}$ sets up the isomorphism of $\mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)$ with $\mathrm{Hom}_M(V_\tau, U_\sigma)$ implied by Frobenius reciprocity.

Lemma 3.1. *Let $f \in C_c^\infty(G;\tau)$. Then*

$$\tilde{f}(\sigma,\nu)(\alpha)(k) = \int_A a^\nu \hat{f}_\sigma(\dot{\alpha})(ka)da$$

for all $\sigma \in \hat{M}$, $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, $\alpha \in \mathrm{Hom}_K(V_\tau, \mathcal{H}_\sigma)$, $k \in K$.

Proof. By definition of $\tilde{f}$ and invariance of the measure on G/K we have

$$\begin{aligned}\tilde{f}(\sigma,\nu)(\alpha)(k) &= \int_{G/K} \pi_{\sigma,\nu}(g)\alpha(f(g))(k)d(gK)\\ &= \int_{G/K} \pi_{\sigma,\nu}(g)\alpha(f(kg))(e)d(gK)\end{aligned}$$

which by the Iwasawa decomposition $G = ANK$ can be written as an integral over $A \times N$:

$$\begin{aligned}&= \int_A \int_N \pi_{\sigma,\nu}(an)\alpha(f(kan))(e)dn\,da\\ &= \int_A \int_N a^{\nu+\rho}\alpha(f(kan))(e)dn\,da.\end{aligned}$$

The latter identity follows from the fact that by definition of the representation $(\pi_{\sigma,\nu}, \mathcal{H}_\sigma)$ we have $\pi_{\sigma,\nu}(an)h(e) = a^{\nu+\rho}h(e)$ for any element $h \in \mathcal{H}_\sigma$.

The lemma now follows immediately from the definition of $\hat{f}_\sigma$. □

Let (τ_i, V_i), $i = 1, 2$ be finite dimensional representations of K. Notice that the elements of $S_d(\mathfrak{a}) \otimes \mathrm{Hom}_M(V_1, V_2)$ naturally define invariant differential operators of order $\leq d$ from the vector bundle $G \times_{MN} \tau_1|_M$ to the vector bundle $G \times_{MN} \tau_2|_M$. Here $S_d(\mathfrak{a})$ denotes the set of elements in the symmetric algebra $S(\mathfrak{a})$ of degree $\leq d$.

Lemma 3.2. *Let $D : C^\infty(G;\tau_1) \to C^\infty(G;\tau_2)$ be an invariant differential operator of order $d \in \mathbb{N}$. Then there exists an element $\hat{D} \in S_d(\mathfrak{a}) \otimes \mathrm{Hom}_M(V_1, V_2)$ such that $(Df)\hat{} = \hat{D}\hat{f}$ for all $f \in C_c^\infty(G;\tau_1)$.*

Proof. By [22, 5.4.11], D is given by an element u of $(U_d(\mathfrak{g}) \otimes \mathrm{Hom}(V_1, V_2))^K$. (In general, if V is a K-module, V^K denotes the vector space of K-invariant elements of V.) By the Poincare-Birkhoff-Witt Theorem, there exist finitely many elements $v_i \in U_d(\mathfrak{a}), w_i \in U_d(\mathfrak{k}), z_i \in \mathrm{Hom}(V_1, V_2)$ such that $u = \sum_i v_i \otimes w_i \otimes z_i$ modulo $\mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{Hom}(V_1, V_2)$. Moreover, the K invariance of u implies that $\sum_i v_i \otimes w_i \otimes z_i$ is M-invariant, because M normalizes $\mathfrak{n}$. As in Sec. 2, we let $\mathcal{R}$ denote the right regular action of G, and use the same notation for the corresponding action of $\mathfrak{g}$, and the extension of this latter action to $U(\mathfrak{g})$. Then

$$(Df)\hat{}(x) = \sum_i a(x)^\rho \int_N z_i \mathcal{R}(v_i w_i) f(xn)dn = \sum_i a(x)^\rho \int_N z_i \tau_1(w_i^\vee)\mathcal{R}(v_i) f(xn)dn,$$

where $w \mapsto w^\vee$ is the anti-automorphism of $U(\mathfrak{g})$ generated by $X \mapsto -X$ for $X \in \mathfrak{g}$, and hence

$$(Df)\hat{}(x) = \sum_i z_i' a(x)^\rho \int_N (\mathcal{R}(v_i)f)(xn)dn$$

with $z_i' = z_i \tau_1(w_i^\vee) \in \mathrm{Hom}(V_1, V_2)$. By a change of variables it is easily seen that

$$a(x)^\rho \int_N (\mathcal{R}(X)\phi)(xn)dn = \mathcal{R}(X + \rho(X)) \left(a(\cdot)^\rho \int_N \phi(\cdot\, n)dn \right)(x)$$

for $X \in \mathfrak{a}, \phi \in C_c(G)$. Hence we obtain

$$(Df)\hat{} = \sum_i z_i' \mathcal{R}(v_i')\hat{f},$$

where v_i' is a ρ-shift of v_i. Since we have

$$\sum v_i' \otimes z_i' \in (U_d(\mathfrak{a}) \otimes \mathrm{Hom}(V_1, V_2))^M = U_d(\mathfrak{a}) \otimes \mathrm{Hom}_M(V_1, V_2))$$

the lemma is proved. □

Let D be as above, let $\hat{D} = \sum_i v_i \otimes z_i \in S_d(\mathfrak{a}) \otimes \mathrm{Hom}_M(V_1, V_2)$, and let $\sigma \in \hat{M}$. Then

$$(Df)\hat{}_\sigma(\beta) = \sum_i (\mathcal{R}(v_i)\hat{f}_\sigma)(\beta \circ z_i) \tag{3.1}$$

for $\beta \in \mathrm{Hom}_M(V_1, U_\sigma)$, and hence by Lemma 3.1 we obtain that

$$(Df)\tilde{}(\sigma, \nu)(\alpha) = \sum_i v_i(-\nu)\tilde{f}(\sigma, \nu)(\alpha_i) \tag{3.2}$$

for $\alpha \in \mathrm{Hom}_K(V_1, \mathcal{H}_\sigma)$. Here $\alpha_i \in \mathrm{Hom}_K(V_2, \mathcal{H}_\sigma)$ is the element determined by $\dot{\alpha}_i = \dot{\alpha} \circ z_i \in \mathrm{Hom}_M(V_2, U_\sigma)$.

For the Fourier and Radon transforms we have the following support theorem which generalizes results of Helgason in the case where τ is the trivial representation [**10**, Lemma 8.1].

Let B_r denote the ball of radius $r > 0$ around the origin in X. Since $G = KAK$ and the distance function is K-invariant, we have

$$B_r = \{kaK \in X \mid k \in K,\ a \in A,\ \|\log a\| \le r\}.$$

Similarly we define

$$\beta_r = \{kaMN \in \Xi \mid k \in K,\ a \in A,\ \|\log a\| \le r\}.$$

Lemma 3.3. *Let $f \in C_c^\infty(G;\tau)$ and let $r > 0$. The following conditions are equivalent:*

(1) $\operatorname{supp} f$ *is contained in* B_r.
(2) $\operatorname{supp} \hat{f}$ *is contained in* β_r.
(3) $\sup_{\nu \in \mathfrak{a}_\mathbb{C}^*}(1 + \|\nu\|)^N e^{-r\|\operatorname{Re}\nu\|}\|\tilde{f}(\sigma,\nu)\| < \infty$ *for all* $N \in \mathbb{R}$, $\sigma \in \hat{M}$.

Proof. Let "dist" be the Riemannian distance function on G/K. From [**11**, p. 278, Exercise B.2(iv)], we have that $\operatorname{dist}(anK, o) \geq \operatorname{dist}(aK, o)$ for all $a \in A$, $n \in N$, where $o = eK$ is the origin. It follows from this and the K-invariance of the distance function that $\operatorname{dist}(kanK, o) \geq \operatorname{dist}(aK, o)$ for all $k \in K, a \in A$, $n \in N$. Hence $\|\log a\| \geq r$ implies $kanK \notin B_r$ for all $n \in N$, and we get that (1) implies (2).

Notice that by the Paley-Wiener theorem for $\mathbb{R}^n$, (3) is the condition for the map $\nu \mapsto \tilde{f}(\sigma,\nu)$ to be the (Euclidean) Fourier transform of a function on A, supported on the set where $\|\log a\| \leq r$, for each σ. Hence (2) is equivalent to (3) by Lemma 3.1. It remains to prove that (2) implies (1).

We shall use the left K-finite expansions $f = \sum_{\delta \in \hat{K}} f^\delta$ and $\hat{f} = \sum_{\delta\in\hat{K}}(\hat{f})^\delta$ of f and $\hat{f}$. If $\delta \in \hat{K}$, f^δ is the component of f that transforms according to the representation δ from the left. We have that $f^\delta(g) = (\dim\delta)\int_K \chi_\delta(k) f(k^{-1}g)dk$ and $\hat{f}^\delta(g) = (\dim\delta)\int_K \chi_\delta(k)\hat{f}(k^{-1}g)dk$, where χ_δ is the character of δ, and hence we see that $\operatorname{supp} f \subset B_r$ (resp. $\operatorname{supp}\hat{f} \subset \beta_r$) if and only if $\operatorname{supp} f^\delta \subset B_r$ (resp. $\operatorname{supp}\hat{f}^\delta \subset \beta_r$) for all δ, and that $\hat{f}^\delta = \widehat{f^\delta}$ (the order of the integrals over K and N can be interchanged).

Assume that (2) holds. To obtain (1) we may (and hence do) assume f to be left K-finite, by the remarks in the previous paragraph.

We now apply the Paley-Wiener theorem of Delorme [**5**], which shows that $F \in C_c^\infty(G)$ has support in B_r if and only if for all $u, u' \in U(\mathfrak{k})$ and all natural numbers N,

$$\sup_{\sigma\in\hat{M},\nu\in\mathfrak{a}_\mathbb{C}^*} (1 + \|\sigma\| + \|\nu\|)^N e^{-r\|\operatorname{Re}\nu\|}\|\pi_\sigma(u)(\mathcal{F}F)(\sigma,\nu)\pi_\sigma(u')\| < \infty.$$

Here the operator norm is used on the operator $T = \pi_\sigma(u)(\mathcal{F}F)(\sigma,\nu)\pi_\sigma(u')$, which is defined on $\mathcal{H}_\sigma^\infty$, the space of smooth functions in $\mathcal{H}_\sigma$:

$$\|T\| = \sup_{\psi\in\mathcal{H}_\sigma^\infty,\|\psi\|\leq 1} \|T\psi\|. \tag{3.3}$$

As in the proof of Theorem 2.2, we apply to $F(g) = \langle f(g), v\rangle$, $v \in V_\tau$. Since f is left K-finite and transforms according to the trivial representation of K on the right, F is K-finite from both sides. Hence the applications of $\pi_\sigma(u)$ and $\pi_\sigma(u')$ are superfluous. Furthermore, as we know from before, only finitely many σ (those for which $\sigma \uparrow \tau$) contribute. Hence $\operatorname{supp} F \subset B_r$ if and only if for each $\sigma \uparrow \tau$ and $N \in \mathbb{N}$,

$$\sup_{\nu\in\mathfrak{a}_\mathbb{C}^*}(1 + \|\nu\|)^N e^{-r\|\operatorname{Re}\nu\|}\|\mathcal{F}F(\sigma,\nu)\| < \infty. \tag{3.4}$$

By (3.3) and (2.9),

$$\begin{aligned}\|\mathcal{F}F(\sigma,\nu)\| &\doteq \sup_{i,j} \|\mathcal{F}F(\sigma,\nu)\varphi_j v_i\| \\ &\doteq \sup_{i,j} |\langle v_i, v\rangle| \|\tilde{f}(\sigma,\nu)(\varphi_j)\| \\ &\doteq \|v\|\,\|\tilde{f}(\sigma,\nu)\|,\end{aligned}$$

where "$\doteq$" means equal up to equivalent norms with computable bounds that depend only on σ. Since we already saw that (2) implies (3), it follows that (3.4) holds for all v, and hence $\operatorname{supp} f \subset B_r$. Hence (2) implies (1). □

In the proof above, we applied Delorme's Paley-Wiener theorem for $\mathcal{F}$ to prove the equivalence of a support assumption on f (or $\hat{f}$) and an exponential type assumption on $\tilde{f}$, under the assumption that $f \in C_c^\infty(G;\tau)$. In fact a much stronger result, a Paley-Wiener theorem for the bundle valued Fourier transform (where in particular we do *not* start with the assumption that f is compactly supported), can be obtained this way. We omit the details.

It is convenient to work with a modified version of the Radon transform. Recall the Plancherel density $m(\sigma,\nu) = |\eta(\sigma,\nu)|^2$ from Sec. 2. Since $\eta(\sigma,\nu)$ is a polynomial in ν, we can define a differential operator J_σ on A with constant real coefficients by

$$\int_A a^\nu (J_\sigma\varphi)(a)da = \eta(\sigma,\nu)\int_A a^\nu \varphi(a)da \tag{3.5}$$

for all $\varphi \in C_c^\infty(A)$. Since A commutes with M and normalizes N, the operator J_σ acts naturally on sections of the bundle $G\times_{MN} U_\sigma$. We define

$$\mathfrak{R}_\sigma f = J_\sigma \hat{f}_\sigma.$$

$\mathfrak{R}_\sigma f(\beta)$ is a section of $G\times_{MN} U_\sigma$ for each $\beta \in \operatorname{Hom}_M(V_\tau, U_\sigma)$. It follows from Lemma 3.1 above that

$$\eta(\sigma,\nu)\tilde{f}(\sigma,\nu)(\alpha)(k) = \int_A a^\nu \mathfrak{R}_\sigma f(\dot{\alpha})(ka)da$$

for $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, $\alpha \in \operatorname{Hom}_K(V_\tau, \mathcal{H}_\sigma)$ and $k \in K$.

Since $\eta(\sigma,\nu)$ is a polynomial, $\tilde{f}(\sigma,\nu)$ will satisfy condition (3) of Lemma 3.3 if and only if $\eta(\sigma,\nu)\tilde{f}(\sigma,\nu)$ satisfies it. Hence we conclude from this lemma and the Euclidean Paley-Wiener theorem that if $f \in C_c^\infty(G;\tau)$ then

$$\operatorname{supp} f \subset B_r \iff \forall\sigma \in \hat{M}:\ \operatorname{supp}\mathfrak{R}_\sigma f \subset \beta_r. \tag{3.6}$$

From Theorem 2.2 and the Euclidean Plancherel theorem we easily obtain the following Plancherel formula for the Radon transform. Assume that τ is irreducible. Then

$$(\dim\tau)\|f\|^2 = \sum_{\sigma\uparrow\tau} \|\mathfrak{R}_\sigma f\|^2, \qquad f \in C_c^\infty(G;\tau). \tag{3.7}$$

By polarization, it follows that

$$(3.8)\qquad (\dim\tau)\langle f,g\rangle=\sum_{\sigma\uparrow\tau}\langle\mathfrak{R}_\sigma f,\mathfrak{R}_\sigma g\rangle,\qquad f,g\in C_c^\infty(G;\tau).$$

In analogy with [**17**, Proposition 1], the modified Radon transform has the following property.

Lemma 3.4. *Let $f\in C_c^\infty(G;\tau)$ and let $r\geq 0$. Assume that*

$$\operatorname{supp}\mathfrak{R}_\sigma f\subset\{kaMN\in\Xi:k\in K,a\in A,\|\log a\|\geq r\}$$

for all $\sigma\uparrow\tau$. Then

$$\operatorname{supp}f\subset\{kaK\in X:k\in K,a\in A,\|\log a\|\geq r\}.$$

Proof. It suffices to prove that $f(x_0)=0$ for any $x_0\in G/K$ with $\operatorname{dist}(x_0,o)<r$. Fix such an x_0, and choose a cutoff function $\varphi\in C_c^\infty(G/K)$ with $\varphi\geq 0$, $\varphi(x_0)>0$, and $\operatorname{supp}\varphi\subset B_s$ for some $s<r$. Regarding φ as a right-K-invariant function on G, multiplication by φ is an operator that preserves $C_c^\infty(G;\tau)$. By the Plancherel formula (3.8),

$$\int_{G/K}\|\varphi f\|^2d(gK)=\int_{G/K}\langle f,\varphi^2f\rangle d(gK)=\text{const}\sum_{\sigma\uparrow\tau}\int_{G/MN}\langle\mathfrak{R}_\sigma f,\mathfrak{R}_\sigma(\varphi^2f)\rangle d(gMN).$$

By (3.6) the support of $\mathfrak{R}_\sigma(\varphi^2f)$ is contained in β_s, which is disjoint with the assumed support of $\mathfrak{R}_\sigma f$, so the latter integral is zero. Hence φf vanishes identically, and we conclude that $f(x_0)=0$. □

Finally, though we do not need it in the sequel, we note that there is also an inversion formula for the bundle valued Radon transform. For any smooth section φ of $G\times_{MN}V_\tau$ we define

$$(3.9)\qquad \check\varphi(g)=\int_{K/M}a(gk)^{-\rho}\tau(k)\varphi(gk)d(kM),\quad g\in G.$$

Then $\check\varphi\in C^\infty(G;\tau)$. It is easily seen that

$$(3.10)\qquad \int_{G/MN}\langle\hat f(g),\varphi(g)\rangle d(gMN)=\int_{G/K}\langle f(g),\check\varphi(g)\rangle d(gK)$$

for all $f\in C_c^\infty(G;\tau)$; for this reason $\varphi\mapsto\check\varphi$ is called the *dual Radon transform.* Notice however that compact support for φ in general does not imply compact support for $\check\varphi$. Let J_σ^* be the constant coefficient differential operator on A defined in analogy with (3.5), but with $\eta(\sigma,-\nu)$ in place of $\eta(\sigma,\nu)$. Then J_σ^* is the formal adjoint of J_σ (as suggested by the notation). Let $C^\infty(G;\sigma)$ denote the space of smooth sections of $G\times_{MN}U_\sigma$. For $\varphi\in C^\infty(G;\sigma)$ and for $\gamma\in\operatorname{Hom}_M(U_\sigma,V_\tau)$ we now define

$$\mathfrak{R}_\sigma^*\varphi(\gamma)=(\gamma\circ J_\sigma^*\varphi)^\vee\in C^\infty(G;\tau).$$

We view $\mathfrak{R}_\sigma^*$ as an operator from $C^\infty(G;\sigma) \otimes \mathrm{Hom}_M(U_\sigma, V_\tau)$ to $C^\infty(G;\tau)$. For $f \in C_c^\infty(G;\tau)$, we may view $\mathfrak{R}_\sigma f$ as an element of $C^\infty(G;\sigma) \otimes \mathrm{Hom}_M(U_\sigma, V_\tau)$, and then it follows from (3.10) that $\mathfrak{R}_\sigma^*$ is the adjoint of $\mathfrak{R}_\sigma$, as suggested by the notation. We now get from (3.8) that

$$(\dim \tau) f = \sum_{\sigma \uparrow \tau} \mathfrak{R}_\sigma^* \mathfrak{R}_\sigma f$$

for $f \in C_c^\infty(G;\tau)$ and τ irreducible.

Notice that an alternate proof of Lemma 3.4 is obtained from this formula: If φ vanishes on β_τ then it follows from (3.9) that $\check{\varphi}$ and hence also $\mathfrak{R}_\sigma^* \varphi$ vanishes on B_τ.

4. The Bochner Laplacian. We would now like to display the power and utility of the bundle valued Radon transform by applying it to wave propagation problems that are essentially bundle valued. Though we are mostly interested in the Dirac and Maxwell equations, which live in spinor and form bundles, it costs nothing extra to work in a general K-bundle setting for the time being. To make contact with various bundle-valued differential operators from Geometry and Physics, it is necessary to understand the connection between the group-theoretic Laplacian, i.e. the Casimir operator, and the various Laplacians which are definable in a more general differential geometric setting, for example, the form, spinor, Bochner, and Lichnerowicz Laplacians. These technical problems are not present in the scalar valued setting, but confront us immediately in the bundle valued case. For example, sharp propagative properties of wave motion are extremely sensitive even to constant shifts in the appropriate wave equation; and in the L^2 index theory of a symmetric space or a quotient of such, where null spaces of Laplacians contain all the information, constant shifts have a highly nontrivial effect. Thus even in those cases where it is possible to guarantee that two Laplacians differ by a constant multiple of the identity, it is important to know the constant.

In this section, we return to the general setting and notation of Sec. 1. Suppose that (τ, V_τ) is a finite-dimensional representation of H, and consider the associated vector bundle $\mathbf{V}_\tau$. Recall that the canonical connection ∇ on the principal bundle $H \to G \to \mathcal{M}$ gives rise to a canonical connection, also called ∇, on each such $\mathbf{V}_\tau$. $T^*\mathcal{M} \otimes \mathbf{V}_\tau = G \times_H (\mathfrak{s} \otimes V_\tau)$ is also an associated bundle; thus it carries the connection ∇. Let $g^\sharp$ be the metric on $T^*\mathcal{M}$ determined by the metric g on $T\mathcal{M}$, which is in turn constructed from the Killing form of $\mathfrak{g}$ as in Remark 1.1. The *Bochner Laplacian* $\mathcal{B} = \mathcal{B}_\tau$ is obtained by (1) applying $-\nabla\nabla : C^\infty(\mathbf{V}_\tau) \to C^\infty(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes \mathbf{V}_\tau)$, and (2) contracting $g^\sharp$ with the $T^*\mathcal{M} \otimes T^*\mathcal{M}$ argument to get a differential operator on $\mathbf{V}_\tau$. In detail, suppose $Z_1, \dots, Z_n$ is a local frame for $\mathcal{M}$ and $\omega_1, \dots, \omega_n$ the dual coframe. If $f \in C^\infty(\mathbf{V}_\tau)$,

$$\nabla f = \sum_j \omega_j \otimes \nabla_{Z_j} f,$$

$$\nabla\nabla f = \sum_{i,j} \omega_i \otimes (\omega_j \otimes \nabla_{Z_i} \nabla_{Z_j} f + \nabla_{Z_i} \omega_j \otimes \nabla_{Z_j} f). \tag{4.1}$$

Recall that the Levi-Civita connection ∇^{LC} of g agrees with the canonical connection on $T\mathcal{M} = G \times_\tau \mathfrak{s}$. If Γ_{ijk} are the Christoffel symbols of ∇^{LC} in the given frame, that is, $\nabla_{Z_i} Z_j = \sum_k \Gamma_{ijk} Z_k$, then $\nabla_{Z_i}\omega_j = -\sum_k \Gamma_{ikj}\omega_k$. Choose an orthonormal basis $X_1, \dots, X_n$ for $\mathfrak{s}$, $-B_{\mathfrak{g}}(X_i, X_j) = \varepsilon_i\delta_{ij}$, $\varepsilon_i = \pm 1$, and let the Z_i be partial derivatives with respect to the corresponding normal coordinates of Remark 1.4. Then $g^\sharp(\omega_i, \omega_j)(o) = \delta_{ij}$ and $\Gamma_{ijk}(o) = 0$. Thus for these choices, (4.1) implies

$$(\mathcal{B}f)_o = -\sum_i \varepsilon_i (\nabla^2_{Z_i} f)_o. \tag{4.2}$$

The horizontal lift $(Z_i)^\natural$ is invariant under left translations by the one-parameter group $\exp(tX_i)$ [**15**, X.2.4], and so agrees with X_i along this curve. Let $\mathcal{L}$ and $\mathcal{R}$ be the left and right regular actions of $U(\mathfrak{g})$ on $C^\infty(G)$. Restriction of $\mathcal{L}$ yields a left action on $C^\infty(G;\tau)$, and thus on $C^\infty(\mathbf{V}_\tau)$, with which it is naturally identified (Remark 1.3). $\mathcal{R}$ does not leave $C^\infty(G;\tau)$ invariant. Nevertheless, working at the identity in G, we can make the following computation. By (4.2) and Remark 1.3,

$$(\mathcal{B}f)^\natural(e) = -\sum_i \varepsilon_i (\mathcal{R}(X_i)^2 f^\natural)(e) = (\mathcal{R}(\mathrm{Cas}_{\mathfrak{g}} - \mathrm{Cas}_{\mathfrak{h}}) f^\natural)(e), \quad f \in C^\infty(\mathbf{V}_\tau). \tag{4.3}$$

Here the inner product used to compute $\mathrm{Cas}_{\mathfrak{h}}$ is the restriction of that used to compute $\mathrm{Cas}_{\mathfrak{g}}$; see Remark 1.1. If τ is irreducible, $\mathcal{R}(\mathrm{Cas}_{\mathfrak{h}})$ takes a constant value $\tau(\mathrm{Cas}_{\mathfrak{h}}) = C_\tau$ on $C^\infty(\mathbf{V}_\tau)$. If $X, Y \in \mathfrak{g}$ and $F \in C^\infty(G)$, then $(\mathcal{R}(X)F)(e) = -(\mathcal{L}(X)F)(e)$; thus

$$(\mathcal{R}(X)\mathcal{R}(Y)F)(e) = -(\mathcal{L}(X)\mathcal{R}(Y)F)(e) = -(\mathcal{R}(Y)\mathcal{L}(X)F)(e) = (\mathcal{L}(Y)\mathcal{L}(X)F)(e).$$

(4.3) therefore implies that

$$(\mathcal{B}f)_o = ((\mathcal{L}(\mathrm{Cas}_{\mathfrak{g}}) - C_\tau)f)_o.$$

Since $\mathcal{B}$ and $\mathcal{L}(\mathrm{Cas}_{\mathfrak{g}})$ are left invariant differential operators, $\mathcal{B}f = \mathcal{L}(\mathrm{Cas}_{\mathfrak{g}}) - C_\tau$. We have proved:

Proposition 4.1. *Let $\mathcal{M} = G/H$ be a strongly reductive homogeneous space of a connected, semisimple Lie group with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. Let ∇ be the canonical connection on the principal bundle $H \to G \to \mathcal{M}$, and $\mathcal{B}$ the Bochner Laplacian determined by ∇ and the metric g on $\mathcal{M}$. Let (τ, V_τ) be an irreducible finite-dimensional representation of H, and let $\mathbf{V}_\tau$ be the associated vector bundle over $\mathcal{M}$. Then*

$$\mathcal{B}\big|_{C^\infty(\mathbf{V}_\tau)} = \mathcal{L}(\mathrm{Cas}_{\mathfrak{g}})\big|_{C^\infty(\mathbf{V}_\tau)} - C_\tau,$$

where C_τ is the constant $\tau(\mathrm{Cas}_{\mathfrak{h}})$. $\square$

Remark 4.2. One sometimes sees the Bochner Laplacian defined as $\nabla^*\nabla$, i.e. by a formal adjoint construction. For this, we need a nondegenerate Hermitian structure h (not necessarily positive definite) on the bundle $\mathbf{V}_\tau$; this defines a nondegenerate Hermitian structure $\underline{h}$ on $T^*\mathcal{M} \otimes \mathbf{V}_\tau$ via $\underline{h}(\omega \otimes f, \omega' \otimes f') = g^\sharp(\omega, \omega')h(f, f')$, where $\omega, \omega' \in C^\infty(T^*\mathcal{M})$ and

$f, f' \in C^\infty(\mathbf{V}_\tau)$. The formal adjoint $\nabla^* : C^\infty(T^*\mathcal{M} \otimes \mathbf{V}_\tau) \to C^\infty(\mathbf{V}_\tau)$ of $\nabla : C^\infty(\mathbf{V}_\tau) \to C^\infty(T^*\mathcal{M} \otimes \mathbf{V}_\tau)$ is uniquely determined by the relation

$$\int_{\mathcal{M}} h(f, \nabla^*\varphi) d\mathrm{vol}_g = \int_{\mathcal{M}} \underline{h}(\nabla f, \varphi) d\mathrm{vol}_g,$$

where f and φ are C^∞ sections of $\mathbf{V}_\tau$ and $T^*\mathcal{M} \otimes \mathbf{V}_\tau$ respectively, either f or φ has compact support, and $d\mathrm{vol}_g$ is the Riemannian measure. Note that h is a section of the associated bundle $\mathbf{V}_\tau^* \otimes \mathbf{V}_\tau^*$, so the expression ∇h makes sense. Under the assumption that $\nabla h = 0$, it is straightforward to prove that $\mathcal{B}$ and $\nabla^*\nabla$ agree; we omit the details. Note that the condition $\nabla h = 0$ can be enforced without any reference to connections, by assuming that h is left invariant. Indeed, given any $X \in C^\infty(T\mathcal{M})$, we compute that

$$(\nabla_X h)^\natural(e) = (\mathcal{R}(\mathcal{X})h^\natural)(e) = -(\mathcal{L}(\mathcal{X})h^\natural)(e), \tag{4.4}$$

where $\mathcal{X} \in \mathfrak{s}$ has $\mathcal{X}_e = (X^\natural)_e$. If h is left-invariant, the expression in (4.4) vanishes, so $(\nabla h)_o = 0$, but since ∇ and h are left invariant, ∇h must vanish on all of $\mathcal{M}$.

5. Vector bundles over the hyperboloid and sphere. Now let $G = \mathrm{Spin}_0(n, 1)$, $n \geq 1$, and consider the symmetric space $H^n = G/K$, where $K = \mathrm{Spin}(n)$ is the maximal compact subgroup. H^n can also be written as $\mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$. G is semisimple, and H^n is strongly reductive as a homogeneous space in the sense of Sec. 1. If n is odd, then furthermore G has one conjugacy class of Cartan subgroup. M is a copy of $\mathrm{Spin}(n-1)$ (or $\mathrm{SO}(n-1)$ if G is taken to be $\mathrm{SO}_0(n, 1)$), and the inclusions $\mathfrak{m} \to \mathfrak{k} \to \mathfrak{g}$ are *standard*; that is, they arise from block stabilization in the defining representation of $\mathfrak{so}(n, 1)$. We shall always use a normalization of the Killing form on any $\mathfrak{so}(p, q)$ obtained from its defining representation ℓ via $b(X, Y) = \operatorname{tr} \ell(X)\ell(Y)$; we shall sometimes call b the *reduced Killing form*. This has the advantage that for any standard inclusion $\mathfrak{so}(p', q') \subset \mathfrak{so}(p, q)$, the restricted and intrinsic forms b agree. The relation to the usual Killing form $B = B_{\mathfrak{so}(p,q)}$ defined using the adjoint representation is $b = B/2(p+q-2)$. Note that our normalization affects the computation of the metric on H^n, the Casimir operators of $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{m}$, and the Bochner Laplacian. To study the sphere $S^n = \mathrm{Spin}(n+1)/\mathrm{Spin}(n)$, we reverse the sign of the reduced Killing form to get a positive definite metric. An added advantage of our normalizations is that we now have the *standard* sphere and hyperboloid; that is, the sphere with constant sectional curvature and radius 1, and the hyperboloid with constant sectional curvature -1. Indeed, according to a fairly general homogeneous space computation [**1**, 7.39], the scalar curvature of S^n or H^n in the Killing form metric is $-n/2$; for the sign-corrected metric on the sphere, $n/2$. By the above, our normalization divides the Killing form of $\mathfrak{so}(n, 1)$ by $2(n-1)$, thus divides the sphere and hyperboloid metrics by $2(n-1)$, and thus *multiplies* the scalar curvature by $2(n-1)$. The result is scalar curvature $\pm n(n-1)$ for S^n and H^n respectively. On a space of constant curvature, the scalar curvature is $n(n-1)$ times the sectional curvature, so we get the desired result.

We shall need to do some arithmetic with the highest weights of $\mathrm{Spin}(m)$-modules for $m = n, n-1$. Assume first that $n = 2k+1$ is odd. Recall that the dual $\mathrm{Spin}(2k+1)^\wedge$ of $\mathrm{Spin}(2k+1)$, $k \geq 1$, is parameterized by k-tuples of integers or proper half-integers $\tau \in \mathbb{Z}^k \cup (\frac{1}{2} + \mathbb{Z})^k$ with

$$\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k \geq 0.$$

That is, each such τ is the highest weight of a unique (up to equivariant isomorphism) irreducible Spin$(2k+1)$-module V_τ. The irreducible representations which factor through $SO(2k+1)$ are exactly those with $\tau \in \mathbb{Z}^k$.

When dealing with connected compact groups H, we shall abuse notation by identifying an irreducible representation and its highest weight. As before, we shall also use the same notation for a representation, the corresponding representation of $\mathfrak{h}$, and the extension of the latter to the universal enveloping algebra $U(\mathfrak{h})$. When writing weights, we omit terminal strings of zeroes, and write, e.g., a string of p ones as 1_p. For example, the exterior representations of $SO(2k+1)$ are $\Lambda^p \cong_{SO(2k+1)} \Lambda^{2k+1-p} \cong_{SO(2k+1)} = (1_p)$ for $p \leq k$; the spin representation of Spin$(2k+1)$ is $\Sigma \cong_{\text{Spin}(2k+1)} ((\frac{1}{2})_k)$.

The dual Spin$(2k)^\frown$ of Spin$(2k)$, $k \geq 1$, is parameterized by k-tuples σ of integers or proper half-integers satisfying

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{k-1} \geq |\sigma_k|;$$

the representations with $\sigma \in \mathbb{Z}^k$ are exactly those which factor through $SO(2k)$. Let U_σ be the irreducible $SO(2k)$-module with highest weight σ. We shall need the *branching rule* describing the restriction of a Spin$(2k+1)$-module to Spin$(2k)$. The branching has multiplicity one in the sense that $m_\sigma(\tau) = \dim \text{Hom}_{\text{Spin}(2k)}(U_\sigma, V_\tau|_{\text{Spin}(2k)})$ is either 0 or 1. As before, we say that $\tau \downarrow \sigma$ if $m_\sigma(\tau) \neq 0$. The branching rule reads:

$$\tau \downarrow \sigma \iff \tau_1 - \sigma_1 \in \mathbb{Z} \text{ and } \tau_1 \geq \sigma_1 \geq \tau_2 \geq \ldots \geq \tau_k \geq |\sigma_k|. \tag{5.1}$$

The exterior representations of $SO(2k)$ are $\Lambda^p \cong_{\text{Spin}(2k)} \Lambda^{2k-p} = (1_p)$ for $p < k$, and $\Lambda^k = \Lambda^k_+ \oplus \Lambda^k_-$, $\Lambda^k_\pm \cong_{SO(2k)} (1_{k-1}, \pm 1)$, and the spin representation of Spin$(2k)$ is $\Sigma = \Sigma_+ \oplus \Sigma_-$, $\Sigma_\pm \cong_{\text{Spin}(2k)} ((\frac{1}{2})_{k-1}, \pm\frac{1}{2})$.

Remark 5.1. With our normalizations, The Casimir operator of the upper left $\mathfrak{so}(n)$ subalgebra of $\mathfrak{so}(n,1)$ or $\mathfrak{so}(n+1)$ takes the value $\lambda(\text{Cas}_{\mathfrak{so}(n)}) = -\langle\lambda + 2\rho, \lambda\rangle$ in the irreducible Spin(n)-module with highest weight λ, where

$$2\rho = (n-2, n-4, \ldots, n-2[n/2])$$

is the sum of the positive roots of $\mathfrak{so}(n)$, and $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^{[n/2]}$. Indeed, there is some λ-independent constant a_n for which $\lambda(\text{Cas}_{\mathfrak{so}(n)}) = -a_n\langle\lambda + 2\rho, \lambda\rangle$, and the case of the defining representation (where $\lambda = (1)$ and $\lambda(\text{Cas}_{\mathfrak{so}(n)}) = 1-n$) identifies a_n as 1. On the sphere $S^n = \text{Spin}(n+1)/\text{Spin}(n)$, where we have reversed the sign of the reduced Killing form to get a positive definite metric; this makes the conclusion of Proposition 4.1

$$(\mathcal{B} - \text{Cas}_{\mathfrak{so}(n+1)})|_{C^\infty(S^n, \mathbb{V}_\lambda)} = -\langle\lambda + 2\rho, \lambda\rangle. \tag{5.2}$$

On the hyperboloid $H^n = \text{Spin}_0(n,1)/\text{Spin}(n)$, the reduced Killing form provides a positive definite metric, and Proposition 4.1 says

$$(\mathcal{B} - \text{Cas}_{\mathfrak{so}(n,1)})|_{C^\infty(H^n, \mathbb{V}_\lambda)} = \langle\lambda + 2\rho, \lambda\rangle. \tag{5.3}$$

In differential form bundles, one has the exterior derivative $d : C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p+1})$ and its formal adjoint, the *coderivative* $\delta : C^\infty(\Lambda^{p+1}) \to C^\infty(\Lambda^p)$. The *form Laplacian* is $\Delta = \delta d + d\delta$. The difference $W = \Delta - \mathcal{B}$ in general Riemannian manifolds is a much-studied object called the *Weitzenböck operator*; on spaces of constant sectional curvature, it is just a constant (depending on the order of form).

For the moment, let us work in the setting of an arbitrary n-dimensional smooth manifold with pseudo-Riemannian metric g. Choose an orthonormal frame $Z_1 \dots, Z_n$ and dual coframe $\omega_1, \dots, \omega_n$. Let u be a p-form and v a $(p+1)$-form, and recall the classical formulas

$$(du)(Z_{i_0}, \dots, Z_{i_p}) = \sum_{s=0}^{p} (-1)^s (\nabla_{Z_{i_s}} u)(Z_{i_0}, \dots, \hat{Z}_{i_s}, \dots, Z_{i_p}), \tag{5.4}$$

$$(\delta v)(Z_{i_1}, \dots, Z_{i_p}) = -\sum_{j=1}^{n} (\nabla_{Z_j} v)(Z_j, Z_{i_1}, \dots, Z_{i_p}). \tag{5.5}$$

Here ∇ is induced by the Levi-Civita connection, and the hat indicates absence. Of course, d does not depend on the Levi-Civita connection or the Riemannian metric, and in fact (5.4) holds in any symmetric connection on the tangent bundle. Let R be the Riemann curvature tensor: if X, Y, Z are vector fields and ω a one-form, then

$$R(\omega, Z, X, Y) = \langle R(X, Y)Z, \omega \rangle,$$

where

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

and $\langle\ ,\ \rangle$ is the dual pairing. It is almost immediate from (5.4, 5.5) that if ε denotes exterior multiplication by a one-form and ι interior multiplication by a vector field, then

$$W = -\sum_{i,j,k,l=1}^{n} R(\omega_i, Z_j, Z_k, Z_l)\varepsilon(\omega_i)\iota(Z_j)\varepsilon(\omega_k)\iota(Z_l).$$

Now suppose our manifold has constant sectional curvature s, so that

$$R(\omega_i, Z_j, Z_k, Z_l) = s(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}).$$

By the identities

$$\begin{aligned}
&\iota(X)\varepsilon(\omega) + \varepsilon(\omega)\iota(X) = \langle X, \omega \rangle, \\
&\varepsilon(\omega)^2 = 0, \\
&\sum_i \iota(Z_i)\varepsilon(\omega_i) = n - p \text{ on } \Lambda^p, \\
&\sum_i \varepsilon(\omega_j)\iota(Z_j) = p \text{ on } \Lambda^p,
\end{aligned}$$

we have

$$W|_{\Lambda^p} = sp(n-p).$$

We apply this now to H^n and S^n. Since $\langle (1_p) + 2\rho, (1_p) \rangle = p(n-p)$, and for $n = 2k$ even, $\langle (1_{k-1}, -1) + 2\rho, (1_{k-1}, -1) \rangle = k^2$, (5.2) and (5.3) show:

Lemma 5.2. *On H^n with the reduced Killing form metric of $\mathfrak{g} = \mathfrak{so}(n,1)$, or S^n with the sign-reversed reduced Killing form metric of $\mathfrak{g} = \mathfrak{so}(n+1)$, the form Laplacian $\Delta = \delta d + d\delta$ agrees with $\mathcal{L}(\mathrm{Cas}_\mathfrak{g})$ on the differential form bundles.* □

See [**13**] for a different proof of this fact in the case of S^n.

Now let $\mathcal{M}$ be a smooth pseudo-Riemannian spin manifold of dimension m, with metric tensor g and fundamental tensor-spinor γ. γ is a section of $T\mathcal{M} \otimes \mathrm{End}\,\Sigma\mathcal{M}$, where the *spinor bundle* $\Sigma\mathcal{M}$ has fiber dimension $2^{[m/2]}$. If ω is a one-form, we contract in the first argument to get a section $\gamma(\omega)$ of $\mathrm{End}\,\Sigma\mathcal{M}$. This allows us to state the *Clifford relations*

$$\gamma(\omega)\gamma(\eta) + \gamma(\eta)\gamma(\omega) = -2g^\sharp(\omega,\eta)\,\mathrm{Id}_{\Sigma\mathcal{M}}, \quad \omega,\eta \in C^\infty(\mathcal{M}, T^*\mathcal{M}).$$

The *Dirac operator* on sections of $\Sigma\mathcal{M}$ is, in a local frame $\{X_i\}$ and dual coframe $\{\eta_i\}$

$$\nabla\!\!\!\!/ = \sum_i \gamma(\eta_i)\nabla_{X_i}.$$

To see that this is invariantly defined, note that ∇ carries $C^\infty(\mathcal{M},\Sigma\mathcal{M})$ to $C^\infty(\mathcal{M}, T^*\mathcal{M} \otimes \Sigma\mathcal{M})$. To get the Dirac operator, we just pair (contract) the $T^*\mathcal{M}$ argument with the $T\mathcal{M}$ argument from γ. The analogue of the Weitzenböck formula for spinors is the *Lichnerowicz formula* [**16**, (7)], which says that $\nabla\!\!\!\!/^{\,2} = \mathcal{B} + S/4$ on spinors, where S is the scalar curvature. In particular, $\nabla\!\!\!\!/^{\,2}$ is $\mathcal{B} + n(n-1)/4$ on spinors over S^n, and $\mathcal{B} - n(n-1)/4$ on spinors over H^n. The quantity $\langle \lambda + 2\rho, \lambda\rangle$ for $\lambda = ((\frac{1}{2})_{[n/2]})$, or for $n = 2k$ even and $\lambda = ((\frac{1}{2})_{k-1}, -\frac{1}{2})$, is $n(n-1)/8$. Thus by (5.2, 5.3), we have:

Lemma 5.3. *On H^n with the reduced Killing form metric of $\mathfrak{g} = \mathfrak{so}(n,1)$, on sections of the spinor bundle,*

$$\nabla\!\!\!\!/^{\,2} = \mathcal{L}(\mathrm{Cas}_\mathfrak{g}) - n(n-1)/8, \quad \mathcal{B} = \mathcal{L}(\mathrm{Cas}_\mathfrak{g}) + n(n-1)/8.$$

On S^n with the sign-reversed Killing form metric of $\mathfrak{g} = \mathfrak{so}(n+1)$,

$$\nabla\!\!\!\!/^{\,2} = \mathcal{L}(\mathrm{Cas}_\mathfrak{g}) + n(n-1)/8, \quad \mathcal{B} = \mathcal{L}(\mathrm{Cas}_\mathfrak{g}) - n(n-1)/8. \qquad \square$$

We can read off the effect of the center of $U(\mathfrak{g})$, and in particular, the Casimir operator, in the principal series representations from [**14**, 8.22 and 12.28]. In general this gives

$$\pi_{\sigma,\nu}(\mathrm{Cas}_\mathfrak{g}) = -\langle \nu,\nu\rangle + \langle \rho,\rho\rangle + \sigma(\mathrm{Cas}_\mathfrak{m}).$$

(Note that this formula is stable under renormalization of the Killing form.) In our special case, the positive $(\mathfrak{g},\mathfrak{a})$ root, which we shall denote by ν_0, has norm 1. Its multiplicity is $n-1$, so if we define a parameter $\lambda \in \mathbb{C}$ by $\nu = \sqrt{-1}\lambda\nu_0$, we get

$$\pi_{\sigma,\sqrt{-1}\lambda\nu_0}(\mathrm{Cas}_\mathfrak{g}) = \lambda^2 + \left(\frac{n-1}{2}\right)^2 + \sigma(\mathrm{Cas}_\mathfrak{m}).$$

Since the Fourier transform is a G-map, we can see the effect of our G/K Laplacians in the G/MAN picture. It follows from the above and Proposition 4.1 that:

Lemma 5.4. *Suppose $f \in C_c^\infty(G;\tau)$. Let $\nu_0 \in \mathfrak{a}^*$ be the positive $(\mathfrak{g},\mathfrak{a})$ root of $\mathfrak{so}(n,1)$, and define $\lambda = \lambda(\nu) \in \mathbf{C}$ by $\nu = \sqrt{-1}\lambda\nu_0$. Then*

$$(\mathcal{L}(\mathrm{Cas}_{\mathfrak{g}})f)^\sim(\sigma,\nu) = \left(\lambda^2 + \left(\frac{n-1}{2}\right)^2 + \sigma(\mathrm{Cas}_{\mathfrak{m}})\right)\tilde{f}(\sigma,\nu),$$

$$(\mathcal{B}f)^\sim(\sigma,\nu) = \left(\lambda^2 + \left(\frac{n-1}{2}\right)^2 - \tau(\mathrm{Cas}_{\mathfrak{k}}) + \sigma(\mathrm{Cas}_{\mathfrak{m}})\right)\tilde{f}(\sigma,\nu). \quad \square$$

This result points up the advantage of working with K-types for which $\mathrm{Cas}_{\mathfrak{m}}^\sigma$ is constant on $\{\sigma \mid \sigma \uparrow \tau\}$, or with systems of equations which imply the absence of certain "bad" M-types, on which $\mathrm{Cas}_{\mathfrak{m}}$ takes the "wrong" value. When we work with such special K-types and systems of equations, there emerge reasonable analogues of the Fourier and Radon transformation laws for the Laplacian of Euclidean space. This is exactly what happens in our situations: the spinor bundle, like the scalar bundle, gives a constant value for $\sigma(\mathrm{Cas}_{\mathfrak{m}})$, while (two of the four) Maxwell equations will enforce the annihilation of non-conforming M-types. In the following corollaries, we collect some of the above information in a form that will be useful for us.

Corollary 5.5. *Let $n = 2k+1 \geq 3$, and let $\tau = ((\frac{1}{2})_k)$. Then $\sigma \uparrow \tau$ if and only if $\sigma = \sigma_\pm = ((\frac{1}{2})_{k-1}, \pm\frac{1}{2})$. With the above normalizations,*

$$\sigma_\pm(\mathrm{Cas}_{\mathfrak{m}}) = -k(2k-1)/4.$$

Thus on smooth sections ψ of the spinor bundle of H^n,

$$(\nabla\!\!\!\!/^{\,2}\psi)^\sim(\sigma_\pm, \sqrt{-1}\lambda\nu_0) = \lambda^2\tilde{\psi}(\sigma_\pm, \sqrt{-1}\lambda\nu_0).$$

Proof. The first statement follows from the branching rule (5.1). The remarks at the beginning of the section about normalization of the $\mathfrak{so}(p,q)$ Killing forms, together with Remark 5.1, imply that for general $\sigma \in \hat{M}$,

$$\sigma(\mathrm{Cas}_{\mathfrak{m}}) = -\langle \sigma + 2\rho_{\mathfrak{m}}, \sigma\rangle,$$

where $2\rho_{\mathfrak{m}} = (2k-2,\ldots,2,0)$. This gives the formula for $\sigma_\pm(\mathrm{Cas}_{\mathfrak{m}})$. Lemmas 5.3 and 5.4 then give the formula for $\nabla\!\!\!\!/^{\,2}$ on the Fourier transformed side. $\square$

Corollary 5.6. *Let $n = 2k+1 \geq 3$, and let $\tau = \tau_p = (1_p)$ for $p \leq k-1$. Then $\sigma \uparrow \tau$ if and only if*

$$\sigma = (0), \quad p = 0;$$
$$\sigma = \sigma_{p,1} = (1_p) \text{ or } \sigma = \sigma_{p,0} = (1_{p-1}), \quad 1 \leq p \leq k-1;$$
$$\sigma = \sigma_\pm = (1_{k-1}, \pm 1) \text{ or } \sigma = \sigma_0 = (1_{k-1}), \quad p = k.$$

With the above normalizations,

$$(1_q)(\mathrm{Cas}_{\mathfrak{m}}) = -q(2k-q), \quad q \leq k;$$
$$(1_{k-1}, -1)(\mathrm{Cas}_{\mathfrak{m}}) = -k^2.$$

Thus on smooth p- or $(n-p)$-forms φ over H^n with $p < k$, if $\Delta = \delta d + d\delta$ is the form Laplacian,

$$(\Delta\varphi)^{\sim}(\sigma_{p,1}, \sqrt{-1}\lambda\nu_0) = (\lambda^2 + (k-p)^2)\tilde{\varphi}((1_p), \sqrt{-1}\lambda\nu_0),$$
$$(\Delta\varphi)^{\sim}(\sigma_{p,0}, \sqrt{-1}\lambda\nu_0) = (\lambda^2 + (k-p+1)^2)\tilde{\varphi}((1_{p-1}), \sqrt{-1}\lambda\nu_0).$$

On smooth k- or $(k+1)$-forms,

$$(\Delta\varphi)^{\sim}(\sigma_{\pm}, \sqrt{-1}\lambda\nu_0) = \lambda^2\tilde{\varphi}(\sigma_{\pm}, \sqrt{-1}\lambda\nu_0),$$
$$(\Delta\varphi)^{\sim}(\sigma_0, \sqrt{-1}\lambda\nu_0) = (\lambda^2 + 1)\tilde{\varphi}(\sigma_0, \sqrt{-1}\lambda\nu_0).$$

Proof. We refer to the branching rule (5.1), together with Lemmas 5.2 and 5.4. □

6. The Dirac equation and a spinor wave equation. Let $\mathcal{M}$ be a smooth oriented manifold of even dimension $n+1 = 2k+2$, equipped with a pseudo-Riemannian metric g. Assume that $\mathcal{M}$ is a spin manifold, and let Σ be the spinor bundle and D the Dirac operator. (Recall the definition of the Dirac operator from the last section.) The *Dirac equation* on $\mathcal{M}$ is the first order equation $D\beta = 0$ on a spinor field $\beta \in C^\infty(\mathcal{M}, \Sigma)$.

We are interested in the case in which $(\mathcal{M}, g)$ is a *factored* Lorentz manifold; that is, $\mathcal{M} = \mathbb{R} \times S$ and

$$g = -dt^2 + g_S, \tag{6.1}$$

where (S, g_S) is an $n = (2k+1)$-dimensional Riemannian manifold, and t is the standard parameter on $\mathbb{R}$. (6.1) is an abuse of notation; what it really means is that $g(\partial/\partial t, \partial/\partial t) = -1$, and that g_S is the pullback of g under the inclusion $S \cong \{t = t_0\} \to \mathcal{M}$.

Assume that S has spin structure, and let γ be the fundamental tensor-spinor of S. By [**16**, paragraph 2], we may (and do) assume that $\gamma(\omega)$ is fiberwise skew-adjoint on ΣS for each one-form ω on S. As a consequence of S having spin structure, the Lorentz manifold $\mathcal{M}$ also has spin structure, and there is a standard "concrete" construction of the spinor bundle $\Sigma\mathcal{M}$ and fundamental tensor-spinor α of $\mathcal{M}$, starting with the analogous objects ΣS and γ on S: the fiber $\Sigma_{(t,x)}\mathcal{M}$ is identified with $\Sigma_x S \oplus \Sigma_x S$,

$$\alpha(\partial_t) = \alpha_0 = \begin{pmatrix} 0 & \mathrm{Id}_{\Sigma S} \\ \mathrm{Id}_{\Sigma S} & 0 \end{pmatrix}$$

in block form, and $\alpha(\eta) = \gamma(\eta) \oplus \gamma(-\eta)$ if η is a one-form on $\mathcal{M}$ which is tangent to S (i.e., is annihilated by pairing with ∂_t). α satisfies the Clifford relations because γ does, and is annihilated by the Levi-Civita spin connection of $\mathcal{M}$ because γ enjoys the analogous property on S. Since n is odd, the bundle over $\mathcal{M}$ so obtained is an isomorphic copy of its spinor bundle. Let D and $\nabla\!\!\!/$ be the Dirac operators on $\mathcal{M}$ and S respectively. A spinor over $\mathcal{M}$ is then a pair (φ, ψ) of smooth t-dependent spinors over S, and the action of D is given by

$$D(\varphi, \psi) = -\alpha_0\partial_t(\varphi, \psi) + (\nabla\!\!\!/\varphi, -\nabla\!\!\!/\psi) = (-\partial_t\psi + \nabla\!\!\!/\varphi, -\partial_t\varphi - \nabla\!\!\!/\psi).$$

Hence the Dirac equation becomes the pair of first order equations

$$\partial_t\varphi + \nabla\!\!\!/\psi = 0, \quad \partial_t\psi - \nabla\!\!\!/\varphi = 0. \tag{6.2}$$

We shall call a pair (φ, ψ) satisfying (6.2) a *Dirac field* on $\mathcal{M}$.

Remark 6.1. The notation of the above construction is somewhat different from that customary in Physics: our γ endomorphisms are analogous to the physicist's σ (Pauli) matrices, our α endomorphisms to the physicist's γ matrices, and our $\alpha_0\alpha(X)$ to the physicist's α matrices.

Remark 6.2. Let V be a vector bundle over a factored Lorentz manifold ($\mathcal{M} = \mathbb{R} \times S, g = -dt^2 + g_S$), and let $P : C^\infty(\mathcal{M}, V) \to C^\infty(\mathcal{M}, V)$ be a differential operator with *metric leading symbol*: $\sigma_2(P)(\xi) = g^\sharp(\xi,\xi)\,\mathrm{Id}_V$ for all covector fields ξ. Then the hyperbolic equation $P\varphi = 0$ is called a *wave equation* and has *finite propagation speed*: if $\varphi \in C^\infty(\mathcal{M}, V)$, let $\mathrm{CD}_{t_0}(\varphi)$ be the Cauchy data $(\varphi, \partial_t\varphi)|_{t=t_0}$. Then if $P\varphi = 0$ and $\operatorname{supp}\mathrm{CD}_0(\varphi)$ is contained in the closed metric ball $B_r(x)$ of radius r about $x \in S$,

$$\operatorname{supp}\mathrm{CD}_t(\varphi) \subset B_{r+|t|}(x)$$

for all $t \in \mathbb{R}$.

The *spinor wave equation* is the second order equation

$$(\partial_t^2 + \not\nabla^2)\varphi = 0 \tag{6.3}$$

on a smooth t-dependent section φ of ΣS. Being a wave equation, (6.3) has finite propagation speed. Notice that if (φ, ψ) is a Dirac field, then both φ and ψ satisfy the spinor wave equation. In particular their Cauchy data propagate at finite speed.

Remark 6.3. Remaining in the general setting above, Dirac fields with compactly supported Cauchy data have a conserved *charge*. To explain this, we first claim that $\not\nabla$ is formally self-adjoint. Indeed, given $\varphi, \psi \in C^\infty(S, \Sigma S)$, with either φ or ψ compactly supported, consider the vector field X determined by φ, ψ, γ, and the fiber metric $h = \langle\cdot,\cdot\rangle$ via

$$(X, \omega) = \langle\varphi, \gamma(\omega)\psi\rangle, \quad \omega \in C^\infty(S, \Sigma S),$$

where $(\cdot,\cdot)$ is the dual pairing. It is immediate from $\nabla\gamma = 0$, $\nabla h = 0$, (5.5), and the fact that $\gamma(\omega)$ is skew-adjoint that

$$\delta X_\flat = -\langle\varphi, \not\nabla\psi\rangle + \langle\not\nabla\varphi, \psi\rangle,$$

where $X_\flat$ is the one-form associated to X by the metric g_S. Since $X_\flat$ is compactly supported, Stokes' Theorem gives $\int_S \delta X_\flat \, d\mathrm{vol}_{g_S} = 0$; the formal self-adjointness of $\not\nabla$ follows.

Now if $\varphi, \psi \in C_c^\infty(S, \Sigma S)$ and $D(\varphi, \psi) = 0$, then

$$\partial_t(|\varphi|^2 + |\psi|^2) = 2\,\mathrm{Re}(-\langle\varphi, \not\nabla\psi\rangle + \langle\not\nabla\varphi, \psi\rangle).$$

It follows from the above that the *charge*

$$Q = \tfrac{1}{2}\int_{t=t_0} (|\varphi|^2 + |\psi|^2)\, d\mathrm{vol}_{g_S}$$

is independent of t_0. Similarly, it is easily seen that if φ solves the spinor wave equation, then the *energy*

$$\mathcal{E} = \frac{1}{2}\int_{t=t_0} (|\partial_t\varphi|^2 + |\nabla\!\!\!/\,\varphi|^2) d\mathrm{vol}_{g_S}$$

is independent of t_0. For the reasoning behind the terms *charge* and *energy*, see [**23**, IV.B.6.I].

A Riemannian symmetric space $X = G/K$ of the noncompact type is always a spin manifold, since the spinorial obstruction is topological, and X is diffeomorphic to a Euclidean space, the Cartan complement $\mathfrak{p}$ of $\mathfrak{k}$. There arises a question, however, of whether the spinor bundle ΣX is an associated bundle $G \times_\tau V$ for some K-module (V, τ). Slebarski [**19**] gives a criterion for this: ΣX is an associated bundle if and only if the adjoint representation $\mathrm{ad}_{,}: \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ lifts to a homomorphism from K to $\mathrm{Spin}(\mathfrak{p})$. To take the most elementary example, it is important to view the hyperboloid H^n as $\mathrm{Spin}_0(n,1)/\mathrm{Spin}(n)$ rather than as $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ if we want to realize ΣH^n as an associated bundle.

We now specialize to the case of the hyperboloid $S = H^n = \mathrm{Spin}_0(n,1)/\mathrm{Spin}(n)$, $n = 2k+1 \geq 3$ odd. The spinor bundle $\Sigma = ((\frac{1}{2})_k)$ is similar to the scalar bundle in a sense that is very useful for us: recall from Corollary 5.5 that the K-module $\tau = ((\frac{1}{2})_k)$ restricts under M to $\sigma_+ \oplus \sigma_-$, where $\sigma_\pm = ((\frac{1}{2})_{k-1}, \pm\frac{1}{2})$, and that

$$(\nabla\!\!\!/^2\varphi)^\sim(\sigma_\pm, \sqrt{-1}\sigma\nu_0) = \lambda^2\tilde{\varphi}(\sigma_\pm, \sqrt{-1}\sigma\nu_0). \tag{6.4}$$

We shall show that as a result, the Dirac equation on the Lorentz manifold $\mathbb{R} \times H^{2k+1}$ enjoys properties of equipartition of charge and Huygens' principle, and the spinor wave equation has an equipartitioned energy.

Lemma 6.4. *Let φ be a C_c^∞ section of ΣH^n, $n = 2k+1 \geq 3$, and let ν_0 be the positive $(\mathfrak{g}, \mathfrak{a})$ root of $\mathfrak{so}(n,1)$. Then*

$$(\nabla\!\!\!/\,\varphi)^\sim(\sigma_\pm, \sqrt{-1}\lambda\nu_0) = u_\pm\lambda\tilde{\varphi}(\sigma_\pm, \sqrt{-1}\lambda\nu_0),$$

where $u_\pm$ is either 1 or -1, independently of φ (but as indicated, $u_\pm$ may depend on $\sigma_\pm$).

Proof. Because of (6.4) and the fact that $\sigma_\pm$ have multiplicity one, it follows from (3.2) that

$$(\nabla\!\!\!/\,\varphi)^\sim(\sigma_\pm, \sqrt{-1}\lambda\nu_0) = p_\pm(\lambda)\tilde{\varphi}(\sigma_\pm, \sqrt{-1}\lambda\nu_0)$$

for some first degree polynomials $p_\pm$ in λ. By (6.4) the square of each of these polynomials is λ^2, and the lemma follows. □

Let $H_0 \in \mathfrak{a}$ be the element determined by $\nu_0(H_0) = 1$, and put $a_y = \exp(yH_0) \in A$ for $y \in \mathbb{R}$. Let φ be as in the preceding lemma. Then it follows from Lemma 3.1 that we have

$$(\nabla\!\!\!/\,\varphi)^\wedge_{\sigma_\pm}(ka_y) = u_\pm\sqrt{-1}\partial_y\hat{\varphi}_{\sigma_\pm}(ka_y) \tag{6.5}$$

for $k \in K$.

Remark 6.5. A computation similar to the proof of Lemma 7.2 given in the next section shows that, more precisely, we have

$$(\nabla\!\!\!/ \varphi)\widehat{\ }_{\sigma\pm}(ka_y)(\beta) = \partial_y \hat{\varphi}_{\sigma\pm}(ka_y)(\beta\gamma(H_0))$$

where $\gamma(H_0) \in \mathrm{Hom}_M(V_\tau, V_\tau)$ is Clifford multiplication by H_0.

For a smooth t-dependent section φ of ΣS which for each fixed t has compact support on S, we denote by $\hat{\varphi} = \hat{\varphi}(t, g)$ its Radon transform in the S variable.

Lemma 6.6. *Let (φ, ψ) be a smooth Dirac field on $\mathbb{R} \times H^n$, $n = 2k+1 \geq 3$, and assume that $(\varphi, \psi)|_{t=0}$ has compact support. Then (φ, ψ) has compact support on S for each fixed t, and*

$$\partial_t \hat{\varphi}_{\sigma\pm}(t, ka_y) = -u_\pm \sqrt{-1}\, \partial_y \hat{\psi}_{\sigma\pm}(t, ka_y)$$
$$\partial_t \hat{\psi}_{\sigma\pm}(t, ka_y) = u_\pm \sqrt{-1}\, \partial_y \hat{\varphi}_{\sigma\pm}(t, ka_y),$$

for $k \in K$, where $u_\pm$ is as in Lemma 6.4.

Proof. The proof is immediate from finite propagation speed, (6.2), and (6.5). □

Obviously, the same relations also hold for the modified Radon transforms of φ and ψ:

$$\partial_t \mathfrak{R}_{\sigma\pm}\varphi(t, ka_y) = -u_\pm \sqrt{-1}\, \partial_y \mathfrak{R}_{\sigma\pm}\psi(t, ka_y)$$
$$\partial_t \mathfrak{R}_{\sigma\pm}\psi(t, ka_y) = u_\pm \sqrt{-1}\, \partial_y \mathfrak{R}_{\sigma\pm}\varphi(t, ka_y).$$

For fixed k these are essentially the so-called *para-Cauchy-Riemann*, or *para-CR*, equations

$$\partial_t v = \partial_y w, \qquad \partial_t w = \partial_y v \tag{6.6}$$

on functions $v(t, y)$ and $w(t, y)$ on $\mathbb{R} \times \mathbb{R}$. For these equations we have the following result.

Lemma 6.7. *Let $v(t, y)$ and $w(t, y)$ be C^1 functions on $\mathbb{R} \times \mathbb{R}$ with values in a finite dimensional Hilbert space V, and satisfying the para-CR equations (6.6). Assume moreover that $v(0, \cdot)$ and $w(0, \cdot)$ are supported in $[-r, r]$. Then for each $t \in \mathbb{R}$ we have*

$$\operatorname{supp} v \subset [-r - t; r - t] \cup [-r + t; r + t], \tag{6.7}$$

and similarly for w. Moreover, for $|t| \geq r$,

$$\int_{\mathbb{R}} |v(t, y)|^2 dy = \int_{\mathbb{R}} |w(t, y)|^2 = constant. \tag{6.8}$$

Proof. It is easy to see that we may assume the functions are valued in $\mathbb{R}$.

Let $z = v + w$ and $\zeta = v - w$. Then z and ζ satisfy the first order system

$$\partial_t z = \partial_y z, \quad \partial_t \zeta = -\partial_y \zeta,$$

with the following explicit solution:

$$z(t,y) = z(0,y+t), \quad \zeta(t,y) = \zeta(0,y-t).$$

Hence $\operatorname{supp} z(t,\cdot) \subset [-r-t, r-t]$ and $\operatorname{supp} \zeta(t,\cdot) \subset [-r+t, r+t]$. From this (6.7) follows immediately.

To prove (6.8), we first notice that the sum of the two integrals in question is independent of t: using (6.6) we get

$$\partial_t \int (v^2 + w^2) dy = 2 \int (v\partial_t v + w\partial_t w) dy = 2 \int \partial_y (vw) dy = 0.$$

It now suffices to prove the equality of the two integrals for $|t| \geq r$. This follows from

$$\int (v^2 - w^2) dy = \int z\zeta dy = 0,$$

since the supports of z and ζ are disjoint for $|t| > r$. □

Let (φ, ψ) be a smooth Dirac field on $\mathbb{R} \times H^n$, $n = 2k+1 \geq 3$, and assume that (φ, ψ) has Cauchy data at $t = 0$ which are supported in the ball $B_r(o)$, where o is the origin. Applying Lemma 6.7 to $v = \mathfrak{R}_{\sigma_\pm} \varphi$ and $w = -\sqrt{-1} u_\pm \mathfrak{R}_{\sigma_\pm} \psi$, pointwise in $k \in K$, we obtain that $\mathfrak{R}_{\sigma_\pm} \varphi$ and $\mathfrak{R}_{\sigma_\pm} \psi$ at a given time t are supported in the set

$$\{k a_y MN \in \Xi \mid k \in K, |t| + r \geq |y| \geq |t| - r\}$$

and that

$$\|\mathfrak{R}_{\sigma_\pm} \varphi(t,\cdot)\|^2_{G/MN} = \|\mathfrak{R}_{\sigma_\pm} \psi(t,\cdot)\|^2_{G/MN} = \text{constant}$$

for $|t| \geq r$. We now obtain:

Theorem 6.8. *Let (φ, ψ) be a smooth solution of the Dirac equation on the hyperboloid $H^n = \operatorname{Spin}_0(n,1)/\operatorname{Spin}(n)$, $n = 2k+1 \geq 3$, with $\operatorname{supp}(\varphi,\psi)|_{t=0} \subset B_r(o)$. Then*
(a) (Huygens' principle.)

$$\operatorname{supp}(\varphi,\psi)(t,\cdot) \subset \{x \in G/K \mid |t| + r \geq \operatorname{dist}(x,o) \geq |t| - r\},$$

for all t, where $\operatorname{dist}(\cdot,\cdot)$ is the Riemannian distance function in H^n.
(b) (Equipartition of charge.)

$$\tfrac{1}{2}\|\varphi(t,\cdot)\|^2 = \tfrac{1}{2}\|\psi(t,\cdot)\|^2 = \tfrac{1}{2}Q,$$

for $|t| \geq r$, where Q is the t-independent total charge of (φ,ψ) described in Remark 6.3, and the norms are in $L^2(S, \Sigma S)$.

Proof. Apply Lemma 3.4 and the Plancherel formula (3.7). □

For the spinor wave equation we obtain:

Corollary 6.9. (Equipartition of energy.) *Suppose that φ is a smooth time-dependent section of ΣH^n, $n = 2k+1 \geq 3$, satisfying the spinor wave equation (6.3), and with* $\operatorname{supp}(\varphi|_{t=0}, (\partial_t\varphi)|_{t=0}) \subset B_r(o)$. *Then for $|t| \geq r$,*

$$\tfrac{1}{2}\|(\partial_t\varphi)(t,\cdot)\|^2 = \tfrac{1}{2}\|(\nabla\!\!\!/\,\varphi)(t,\cdot)\|^2 = \tfrac{1}{2}\mathcal{E}.$$

Proof. It is easily seen that $(\partial_t\varphi, \nabla\!\!\!/\,\varphi)$ is a Dirac field. □

Remark 6.10. For the spinor wave equation, the above argument does ***not*** give Huygens' principle. From the proof above we find only that $\partial_t\varphi$ and $\nabla\!\!\!/\,\varphi$ vanish in the *lacuna* $\{x \in G/K \mid \operatorname{dist}(x,o) < |t| - r\}$, whereas Huygens' principle would imply that φ itself vanishes there. This is in contrast to the scalar wave equation on G/K, for which it was shown in [17] that reduction via the Radon transform *does* give Huygens' principle. The difference results from the fact that the ordinary differential operator J in the modified Radon transform supplies at least one derivative (has no constant term) in the scalar case [17, Corollary 2], whereas this is not the case for the spinor bundle.

7. Maxwell's equations. Let $\mathcal{M}$ be a manifold of even dimension $n+1 = 2k+2$, equipped with a pseudo-Riemannian metric g. *Maxwell's equations* are the conditions

$$d\omega = 0, \quad \delta\omega = 0$$

on a differential form $\omega \in C^\infty(\mathcal{M}, \Lambda^m)$ of the middle order $k+1$.

As before, we assume that $(\mathcal{M}, g)$ is a factored Lorentz manifold: $\mathcal{M} = \mathbb{R} \times S$ with (S, g_S) an $n = 2k+1$-dimensional Riemannian manifold. We may then decompose any differential form $\varphi \in C^\infty(\mathcal{M}, \Lambda^p)$ as $\varphi = dt \wedge \varphi_0 + \varphi_1$, where φ_1 and φ_0 are t-dependent forms on S of orders p and $p-1$ respectively. If we write d, δ for the exterior derivative and coderivative in (S, g_S), and $\underline{d}, \underline{\delta}$ for the similar objects in $\mathcal{M}$, the graded Leibniz rule for $\underline{d}$ and integration by parts give

$$\underline{d}\varphi = dt \wedge (\partial_t\varphi_1 - d\varphi_0) + d\varphi_1,$$
$$\underline{\delta}\varphi = dt \wedge \delta\varphi_0 - (\partial_t\varphi_0 + \delta\varphi_1).$$

If ω is a middle-form as above, and we use the notation $E = \omega_0$, $H = \omega_1$ of electromagnetic theory, Maxwell's equations become the first order system

$$\partial_t E + \delta H = 0, \quad \partial_t H - dE = 0, \quad \delta E = 0, \quad dH = 0 \tag{7.1}$$

on t-dependent differential forms $E \in C^\infty(S, \Lambda^k)$ and $H \in C^\infty(S, \Lambda^{k+1})$. A pair (E, H) satisfying (7.1) is called a *Maxwell field* on $\mathcal{M}$.

In the classical setting $S = \mathbb{R}^3$, Λ^3 is identified with the trivial scalar bundle, and both Λ^1 and Λ^2 are identified with the tangent bundle; under these identifications, the δ in δH and the d in dE are both $-\nabla\times$, the d in dH is $\nabla\cdot$, and the δ in δE is $-\nabla\cdot$.

Let $\Delta = \delta d + d\delta$ be the form Laplacian. We shall call the equation

$$\partial_t^2 u = -\Delta u \tag{7.2}$$

on a differential form $u \in C^\infty(S, \Lambda^p)$ the *p-form wave equation*. Its solutions have finite propagation speed (cf. Remark 6.2). If (E, H) is a Maxwell field, E and H in particular satisfy the k- and $(k+1)$-form wave equations respectively. The Maxwell system, however, is strictly stronger than the system $(\partial_t^2 + \Delta)E = 0$, $(\partial_t^2 + \Delta)H = 0$.

Remark 7.1. The *energy density* of a Maxwell field at $t = t_0$ is the function $\varepsilon = \frac{1}{2}(|E|^2 + |H|^2)|_{t=t_0}$ on S, where the pointwise norms $|\cdot|$ are induced by g_S. The Maxwell equations imply that

$$\partial_t \varepsilon = \mathrm{Re}(-\langle E, \delta H\rangle + \langle dE, H\rangle).$$

Since δ is the formal adjoint of d with respect to g, we have that

$$\int_S \langle E, \delta H\rangle \, d\mathrm{vol}_{g_S} = \int_S \langle dE, H\rangle \, d\mathrm{vol}_{g_S},$$

and hence the (total) *energy*

$$\mathcal{E} = \tfrac{1}{2} \int_S (|E|^2 + |H|^2) d\mathrm{vol}_{g_S}$$

is independent of t. The manipulations are justified by finite propagation speed: if (E, H) has Cauchy data of bounded support at some t, it has such at all t.

Similarly, for a solution u to the p-form wave equation (7.2), the energy

$$\mathcal{E} = \tfrac{1}{2} \int_S (|\partial_t u|^2 + |du|^2 + |\delta u|^2) d\mathrm{vol}_{g_S}$$

is independent of t.

Now specialize to the case $S = H^n = G/K = \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$, $n = 2k+1 \geq 3$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and let τ_p be the p^{th} exterior representation of K on $\Lambda^p\mathfrak{p}$. Then the form bundle Λ^p over S is exactly the vector bundle $\mathbf{V}_{\tau_p}$ over G/K associated with τ_p. For $p \leq k$ we have $\tau_p = (1_p)$, and for $p > k$ we have $\tau_p = (1_{n-p})$.

To find the equations satisfied by the Radon transform of a Maxwell field, we first need to determine the effects of d and δ on the Radon transformed side. By definition, if $\varphi \in C_c^\infty(G; \tau_p)$, then $\hat{\varphi}_\sigma$ vanishes unless $\sigma \uparrow \tau_p$. Recall the M-decomposition of τ_p from Corollary 5.6.

Since $\sigma_{p,1} = \sigma_{p+1,0}$ for $p < k$ and $\sigma_{p,0} = \sigma_{p+1,1}$ for $k+1 \leq p < n$, we see that $\mathrm{Hom}_M(\Lambda^p(\mathfrak{p}), \Lambda^{p+1}(\mathfrak{p}))$ is one dimensional for all $p \neq k$, $p < n$. An explicit generator of this Hom space is obtained as follows: Denote by $\varepsilon(X)$ and $\iota(X)$, respectively, the exterior product by $X \in \mathfrak{p}$ in $\Lambda^*\mathfrak{p}$ and its adjoint, the interior multiplication. Let $H_0 \in \mathfrak{a}$ be the element determined by $\nu_0(H_0) = 1$, where ν_0 as before is the positive root. Then $\varepsilon(H_0)$ and $\iota(H_0)$ intertwine the M-actions because M centralizes H_0. For $p \neq k, n$ we conclude that $\mathrm{Hom}_M(\Lambda^p, \Lambda^{p+1}) = \mathbb{C}\varepsilon(H_0)$ and $\mathrm{Hom}_M(\Lambda^{p+1}, \Lambda^p) = \mathbb{C}\iota(H_0)$.

For $p = k$, the kernel of $\varepsilon(H_0) : \Lambda^k \to \Lambda^{k+1}$ equals the image of $\varepsilon(H_0) : \Lambda^{k-1} \to \Lambda^k$, hence it is isomorphic to $\sigma_{k-1,1} = \sigma_0$. Thus $\varepsilon(H_0)$ is a nonzero M-intertwinor between the $\sigma_\pm$ subspaces in Λ^k and Λ^{k+1}. Similarly, $\iota(H_0) : \Lambda^{k+1} \to \Lambda^k$ intertwines these M-actions, but the arrow goes in the other direction. In fact, by the general identity $\iota(X)\varepsilon(Y) + \varepsilon(Y)\iota(X) = \langle X, Y\rangle$, it must hold that $\varepsilon(H_0)$ and $\iota(H_0)$ are inverse to each other when restricted to the σ_+ or σ_- modules.

For $y \in \mathbb{R}$ let $a_y = \exp(yH_0) \in A$.

Lemma 7.2. *Let φ be a C_c^∞ section of $\Lambda^k H^n$ (with $n = 2k+1 \geq 3$). Then*

$$(d\varphi)\hat{}_\sigma(ka_y)(\beta) = \partial_y \hat{\varphi}_\sigma(ka_y)(\beta\varepsilon(H_0))$$

for all $\beta \in \mathrm{Hom}_M(\Lambda^{k+1}, U_\sigma)$, $\sigma \uparrow \tau_{k+1}$. In particular

$$(d\varphi)\hat{}_{\sigma_0} = 0.$$

Moreover, let ψ be a C_c^∞ section of $\Lambda^{k+1}H^n$, then

$$(\delta\psi)\hat{}_\sigma(ka_y)(\beta) = -\partial_y \hat{\psi}_\sigma(ka_y)(\beta\iota(H_0))$$

for all $\beta \in \mathrm{Hom}_M(\Lambda^k, U_\sigma)$, $\sigma \uparrow \tau_k$. In particular

$$(\delta\psi)\hat{}_{\sigma_0} = 0.$$

Proof. In order to be as general as possible, assume for the moment that φ and ψ are forms of order p and $p+1$, respectively, with no restriction on p. $\varphi^\natural$ is a $\Lambda^p(\mathfrak{p})$-valued function on G satisfying $\varphi(gk) = \tau_p(k^{-1})\varphi(g)$, and we have by (5.4, 5.5) and remark 1.3:

$$(d\varphi)^\natural = \sum_i \varepsilon(Y_i)\mathcal{R}(Y_i)\varphi^\natural,$$

where $\{Y_i\}$ is any basis of $\mathfrak{p}$. Similarly,

$$(\delta\psi)^\natural = -\sum_i \iota(Y_i)\mathcal{R}(Y_i)\psi^\natural.$$

Note that the Y_i do not push down to vector fields on G/K (such a pushdown would have to be left invariant, and the only such thing is 0). But given any point $x = gK \in G/K$, we can move the computation to $g \in G$ by picking an orthonormal basis of the tangent space to x and canonically identifying it with an orthonormal basis of $\mathfrak{p}$. The right-hand sides in each formula above are well-defined and have the correct right K covariance because of the sums; the individual terms enjoy no such properties.

A particularly useful basis for $\mathfrak{p}$ is obtained as follows. Let X_i be a basis for $\mathfrak{n}$, the Lie algebra of the Iwasawa component N, which is orthonormal in the inner product $b(X, \theta Y)$, where b is the reduced Killing form and θ is the Cartan involution. Put $Y_i = (X_i - \theta X_i)/\sqrt{2}$. Then H_0 together with the Y_i gives an orthonormal basis for $\mathfrak{p}$.

From the definition of the Radon transform we now obtain

$$(d\varphi)\hat{}(g) = a(g)^\rho \varepsilon(H_0)\int_N (\mathcal{R}(H_0)\varphi)(gn)dn + \sum_i a(g)^\rho \varepsilon(Y_i)\int_N (\mathcal{R}(Y_i)\varphi)(gn)dn.$$

As in the proof of Lemma 3.2 we may compute the first integral by a change of varables (an interchange of the order of integration and differentiation is justified by compact support) and obtain

$$a(g)^\rho \varepsilon(H_0)\int_N (\mathcal{R}(H_0)\varphi)(gn)dn = \varepsilon(H_0)(\mathcal{R}(H_0) + \rho(H_0))\hat{\varphi}(g).$$

For the second term we notice that since $X_i \in \mathfrak{n}$, we have

$$\int_N (\mathcal{R}(Y_i)\varphi)(gn)dn = \int_N (\mathcal{R}(Y_i - \sqrt{2}X_i)\varphi)(gn)dn,$$

and since $Y_i - \sqrt{2}X_i = -(X_i + \theta X_i)/\sqrt{2} \in \mathfrak{k}$, the transformation rule satisfied by φ shows that this integral equals

$$-d\tau_p(Y_i - \sqrt{2}X_i)\int_N \varphi(gn)dn.$$

We conclude that

$$(d\varphi)\hat{} = (\mathcal{R}(H_0)\varepsilon(H_0) + \gamma)\hat{\varphi},$$

where $\gamma = \rho(H_0)\varepsilon(H_0) - \sum_i \varepsilon(Y_i)d\tau_p(Y_i - \sqrt{2}X_i) \in \mathrm{Hom}(\Lambda^p, \Lambda^{p+1})$ is easily seen to be independent of the chosen basis X_i for $\mathfrak{n}$. It follows that $\gamma \in \mathrm{Hom}_M(\Lambda^p, \Lambda^{p+1})$, and hence

$$(d\varphi)\hat{}_\sigma(ka_y)(\beta) = \partial_y\hat{\varphi}_\sigma(ka_y)(\beta\varepsilon(H_0)) + \hat{\varphi}_\sigma(ka_y)(\beta\gamma)$$

for $\beta \in \mathrm{Hom}_M(\Lambda^{p+1}, U_\sigma)$.

The analogous computation for ψ shows that

$$(\delta\psi)\hat{}_\sigma(ka_y)(\beta') = -\partial_y\hat{\psi}_\sigma(ka_y)(\beta'\iota(H_0)) + \hat{\psi}_\sigma(ka_y)(\beta'\gamma^*) \tag{7.3}$$

for $\beta' \in \mathrm{Hom}_M(\Lambda^p, U_\sigma)$, where $\gamma^* \in \mathrm{Hom}_M(\Lambda^{p+1}, \Lambda^p)$ is the adjoint of γ.

To prove the theorem we need to establish that $\gamma = 0$ in the middle order $p = k$. This can be done by an explicit computation, but we prefer the following shortcut.

Consider first $(d\varphi)\hat{}_{\sigma\pm}$. By multiplicity one, γ is a constant multiple of $\varepsilon(H_0)$ on the $\sigma_\pm$-subspace in Λ^k, say $\gamma = \gamma_\pm\varepsilon(H_0)$, $\gamma_\pm \in \mathbb{C}$. Thus

$$(d\varphi)\hat{}_{\sigma\pm}(ka_y)(\beta) = (\partial_y + \gamma_\pm)\hat{\varphi}_{\sigma\pm}(ka_y)(\beta\varepsilon(H_0)),$$
$$(\delta\psi)\hat{}_{\sigma\pm}(ka_y)(\beta') = (-\partial_y + \overline{\gamma_\pm})\hat{\psi}_{\sigma\pm}(ka_y)(\beta'\iota(H_0)),$$

where β and β' are as above with $p = k$. Combining these equations we get

$$(\delta d\varphi)\hat{}_{\sigma\pm}(ka_y) = (-\partial_y^2 - 2(\mathrm{Im}\,\gamma_\pm)\partial_y + |\gamma_\pm|^2)\hat{\varphi}_{\sigma\pm}(ka_y).$$

Since $\sigma_\pm$ do not occur in the target bundle τ_{k-1} for δ we have by (3.1) that $(\delta\varphi)\hat{}_{\sigma\pm} = 0$, and hence also

$$(d\delta\varphi)\hat{}_{\sigma\pm} = 0.$$

By Corollary 5.6,

$$(\Delta\varphi)\tilde{}\,(\sigma_\pm, \sqrt{-1}\lambda\nu_0) = \lambda^2\tilde{\varphi}(\sigma_\pm, \sqrt{-1}\lambda\nu_0),$$

and we obtain from Lemma 3.1 that

$$(\Delta\varphi)\hat{}_{\sigma\pm}(ka_y) = -\partial_y^2\hat{\varphi}_{\sigma\pm}(ka_y).$$

Now compare with the formulas for $\delta d\varphi$ and $d\delta\varphi$. Using the Iwasawa decomposition $G = ANK$, it is easily seen that the range of the map $\varphi \mapsto \hat{\varphi}_{\sigma_\pm}$ is large enough to infer that $\gamma_\pm = 0$.

It remains to prove that γ vanishes on σ_0. Since $\varepsilon(H_0)$ vanishes on σ_0, we have

$$(d\varphi)\hat{}_{\sigma_0}(ka_y)(\beta) = \hat{\varphi}_{\sigma_0}(ka_y)(\beta\gamma) \tag{7.4}$$

Let $f \in C_c^\infty(\Lambda^{k-1})$ with $df \neq 0$. Since the $\sigma_\pm$ do not occur in Λ^{k-1} we have $(df)\hat{}_{\sigma_\pm} = 0$, hence $(df)\hat{}_{\sigma_0}$ must be non-zero. Since $dd = 0$, we get from (7.4) that

$$0 = (ddf)\hat{}_{\sigma_0}(ka_y)(\beta) = (df)\hat{}_{\sigma_0}(ka_y)(\beta\gamma),$$

and hence $\gamma = 0$. □

Remark 7.3. For forms φ and ψ of order p and $p+1$ respectively, we have similarly

$$(d\varphi)\hat{}_{\sigma}(ka_y)(\beta) = (\partial_y + k - p)\hat{\varphi}_\sigma(ka_y)(\beta\varepsilon(H_0)),$$
$$(\delta\psi)\hat{}_{\sigma}(ka_y)(\beta) = (-\partial_y + k - p)\hat{\psi}_\sigma(ka_y)(\beta\iota(H_0)).$$

Except for an ambiguity in the sign of the constant shift $k - p$, this can be obtained from Corollary 5.6 by an argument similar to that above. In their precise form the formulas are obtained by explicit computation of the operator γ. We omit the details.

Corollary 7.4. *Let φ be as in the previous lemma, and assume that $\delta\varphi = 0$. Then $\hat{\varphi}_{\sigma_0} = 0$.*

Proof. This follows immediately from (7.3) with $p = k - 1$ and $\psi = \varphi$, since $\iota(H_0)$ is non-zero on the σ_0 subspace in Λ^k. □

Similarly, if ψ is as above and $d\psi = 0$, then $\hat{\psi}$ has vanishing σ_0 part. It follows that the Fourier transforms of Maxwell fields have vanishing σ_0 part as a result of the equations $\delta E = dH = 0$.

Corollary 7.5. *Let (E, H) be a Maxwell field with compact support in H^n at some t. Then E and H have compact support in H^n for all t, and their Radon transforms satisfy the equations*

$$\hat{E}_{\sigma_0} = \hat{H}_{\sigma_0} = 0$$

and

$$\partial_t \hat{E}_{\sigma_\pm}(t, ka_y)(\beta) = \partial_y \hat{H}_{\sigma_\pm}(t, ka_y)(\beta\iota(H_0)),$$
$$\partial_t \hat{H}_{\sigma_\pm}(t, ka_y)(\beta') = -\partial_y \hat{E}_{\sigma_\pm}(t, ka_y)(\beta'\varepsilon(H_0)),$$

for all $\beta \in \mathrm{Hom}_M(\Lambda^k, U_{\sigma_\pm})$, $\beta' \in \mathrm{Hom}_M(\Lambda^{k+1}, U_{\sigma_\pm})$. □

Arguing as for the Dirac equation, we can now reduce the Huygens' principle and energy equipartition problems for Maxwell fields on H^n to the same problems for the para-CR equations (6.6) on $\sqrt{-1}\mathfrak{a}^* \cong \mathbb{R}$, and we obtain the theorem below. Notice that the para-CR equations are in fact the Maxwell equations on $\mathbb{R}$ (that is, with $\mathbb{R} \times \mathbb{R}$ as the spacetime). Indeed, a Maxwell field on $\mathbb{R}$ consists of an electric field v and a magnetic field $w\,dx$, where $v, w \in C^\infty(\Lambda^0\mathbb{R})$ and dx is the standard volume form on $\mathbb{R}$. The static (divergence) Maxwell equations $\delta v = 0$ and $d(w\,dx) = 0$ are vacuous ($\Lambda^{-1}\mathbb{R} = \Lambda^2\mathbb{R} = 0$), and the other two Maxwell equations are just (6.6).

Theorem 7.6. *Let (E, H) be a smooth solution of the Maxwell equations on $H^n = \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$, $n = 2k+1 \geq 3$, with* $\mathrm{supp}(E,H)|_{t=0} \subset B_r(o)$. *Then*
(a) (Huygens' principle.)

$$\mathrm{supp}(E,H)(t,\cdot) \subset \{x \in H^n \mid |t| + r \geq \mathrm{dist}(x,o) \geq |t| - r\},$$

for all t.
(b) (Equipartition of energy.)

$$\tfrac{1}{2}\|E\|^2 = \tfrac{1}{2}\|H\|^2 = \tfrac{1}{2}\mathcal{E},$$

for $|t| \geq r$, where $\mathcal{E}$ is the t-independent total energy of (E,H). □

Remark 7.7. Both $\mathbb{R}$ and H^n are homogeneous with invariant metrics, so the analogous statement can be made with an arbitrary $(t_0, x) \in \mathbb{R} \times H^n$ in place of $(0, o)$.

For the k-form wave equation on H^{2k+1}, we get equipartition and Huygens' principle only if we impose the extra condition that the solution u is a cycle, that is $\delta u = 0$. Note that for solutions of the equation, the cyclicity condition is equivalent to the fixed-time contraint

$$\delta(u|_{t=0}) = \delta((\partial_t u)|_{t=0}) = 0. \tag{7.5}$$

Indeed, if u satisfies the k-form wave equation, δu satisfies the $(k-1)$-form wave equation. The Plancherel formula shows that solutions of the Cauchy problem are unique, so (7.5) implies that $\delta u = 0$ for all t. Our result is:

Theorem 7.8. *Let u be a smooth t-dependent k-form on H^n satisfying the k-form wave equation (7.2), and having Cauchy data satisfying the constraint (7.5). If* $\mathrm{supp}\, du|_{t=0} \subset B_r(o)$ *and* $\mathrm{supp}(\partial_t u)|_{t=0} \subset B_r(o)$, *then for $|t| \geq r$,*

$$\tfrac{1}{2}\|\partial_t u(t,\cdot)\|^2 = \tfrac{1}{2}\|du(t,\cdot)\|^2 = \tfrac{1}{2}\mathcal{E}$$

where $\mathcal{E}$ is the total energy. If $\mathrm{supp}\, u|_{t=0} \subset B_r(o)$ *and* $\mathrm{supp}(\partial_t u)|_{t=0} \subset B_r(o)$ *then*

$$\mathrm{supp}\, u \subset \{(x,t) \mid |t| - r \leq \mathrm{dist}(o,x) \leq |t| + r\}$$

for all t.

Proof. The equipartition result is an immediate corollary of Theorem 7.6, since $(\partial_t u, du)$ is easily seen to be a Maxwell field. However, as with the spinor wave equation, this argument only gives that du and $\partial_t u$, but not u itself, vanish in the lacuna $\{(t,x) \mid x \in B_{|t|-r}\}$.

Since the Radon transforms $\mathfrak{R}_{\sigma_\pm} u(t, ka_y)$ both satisfy the wave equation $\partial_t^2 v = \partial_y^2 v$, and since $\mathfrak{R}_{\sigma_0} u = 0$, the vanishing of u in the lacuna will follow from [**17**, Lemma 1] and Lemma 3.4 once we prove that the second Cauchy datum $\partial_t \mathfrak{R}_{\sigma_\pm} u(0, ka_y)$ is a derivative in y of a compactly supported function. Now, since $G/K \cong \mathfrak{p}$ as a manifold, the homology of the de Rham complex with compact support vanishes in all degrees except $p = 0$ [**2**, Corollary 4.7.1]. Hence the cycle condition on u implies that there exists a compactly supported smooth $(k+1)$-form v such that $u(0,\cdot) = \delta v$. From this the desired property of $\mathfrak{R}_{\sigma_\pm} u$ follows. □

A similar result holds in degree $k+1$ with the side condition $du = 0$.

Remark 7.9. Instead of employing algebraic topology in the proof above, we could also have employed representation theory: The principal series representation $\pi_{\sigma_\pm,0}$ is equivalent to a quotient of $\pi_{\sigma_0,-\nu_0}$, cf. [21], hence there are surjective intertwining operators $T_\pm$ from $\mathcal{H}_{\sigma_0,-\nu_0}$ onto $\mathcal{H}_{\sigma_\pm,0}$. In fact, it follows from [6] that the intertwining operators in question are exactly the two components of the exterior derivative d on sections of $\Lambda^{k-1}(K/M)$. To explain this, note that $K/M \cong S^{2k}$ is even-dimensional, and so the middle-form bundle $\Lambda^k(K/M)$ splits under the structure group $M \cong \mathrm{SO}(2k)$ into two irreducible summands, namely the bundles associated to $\sigma_\pm$ under the principal fibration $M \to K \to K/M$. Projection onto these after an application of d results in two differential intertwining operators $d_\pm$, which are identical with $T_\pm$ up to constant factors. (The surjectivity of $d_\pm$ is actually equivalent, via Hodge theory, to a topological property of K/M, that being the vanishing of the $k^{\underline{\text{th}}}$ de Rham cohomology.)

If u satisfies the cycle condition $\delta u = 0$, we have that $\tilde{u}(\sigma_0, \nu) = 0$ for all ν, in particular for $\nu = -\nu_0$. It follows that $\tilde{u}(\sigma_\pm, 0) = 0$ (see Remark 2.3 and Corollary 7.4). Hence $\hat{u}_{\sigma_\pm}(ka)$ has a vanishing integral over $a \in A$.

By similar methods one can obtain from Remark 7.3 that p-forms satisfying the *shifted* wave equation

$$\partial_t^2 u = (-\Delta + (k-p)^2)u$$

with the side condition $\delta u = 0$ has an equipartitioned energy and satisfies Huygens' principle for $p \neq k$ (cf. [3] and [12] for $p = 0$, where the side condition is vacuous).

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Vejleder: Bernhelm Booss.

80/84 "KURSUSMATERIALE TIL MATEMATIK B".
Af: Mogens Brun Heefelt.

81/84 "FREKVENSAFHÆNGIG LEDNINGSEVNE I AMORFT GERMANIUM".
Specialerapport af: Jørgen Wind Petersen og Jan Christensen.
Vejleder: Niels Boye Olsen.

82/84 "MATEMATIK - OG FYSIKUNDERVISNINGEN I DET AUTOMATISEREDE SAMFUND".
Rapport fra et seminar afholdt i Hvidovre 25-27 april 1983.
Red.: Jens Højgaard Jensen, Bent C. Jørgensen og Mogens Niss.

83/84 "ON THE QUANTIFICATION OF SECURITY":
PEACE RESEARCH SERIES NO. I
Af: Bent Sørensen
nr. 83 er p.t. udgået

84/84 "NOGLE ARTIKLER OM MATEMATIK, FYSIK OG ALMENDANNELSE".
Af: Jens Højgaard Jensen, Mogens Niss m. fl.

85/84 "CENTRIFUGALREGULATORER OG MATEMATIK".
Specialerapport af: Per Hedegård Andersen, Carsten Holst-Jensen, Else Marie Pedersen og Erling Møller Pedersen.
Vejleder: Stig Andur Pedersen.

86/84 "SECURITY IMPLICATIONS OF ALTERNATIVE DEFENSE OPTIONS FOR WESTERN EUROPE".
PEACE RESEARCH SERIES NO. 2
Af: Bent Sørensen.

87/84 "A SIMPLE MODEL OF AC HOPPING CONDUCTIVITY IN DISORDERED SOLIDS".
Af: Jeppe C. Dyre.

88/84 "RISE, FALL AND RESURRECTION OF INFINITESIMALS".
Af: Detlef Laugwitz.

89/84 "FJERNVARMEOPTIMERING".
Af: Bjarne Lillethorup og Jacob Mørch Pedersen.

90/84 "ENERGI I 1.G - EN TEORI FOR TILRETTELÆGGELSE".
Af: Albert Chr. Paulsen.

91/85 "KVANTETEORI FOR GYMNASIET".
1. Lærervejledning
Projektrapport af: Biger Lundgren, Henning Sten Hansen og John Johansson.
Vejleder: Torsten Meyer.

92/85 "KVANTETEORI FOR GYMNASIET".
2. Materiale
Projektrapport af: Biger Lundgren, Henning Sten Hansen og John Johansson.
Vejleder: Torsten Meyer.

93/85 "THE SEMIOTICS OF QUANTUM - NON - LOCALITY".
Af: Peder Voetmann Christiansen.

94/85 "TREENIGHEDEN BOURBAKI - generalen, matematikeren og ånden".
Projektrapport af: Morten Blomhøj, Klavs Frisdahl og Frank M. Olsen.
Vejleder: Mogens Niss.

95/85 "AN ALTERNATIV DEFENSE PLAN FOR WESTERN EUROPE".
PEACE RESEARCH SERIES NO. 3
Af: Bent Sørensen

96/85 "ASPEKTER VED KRAFTVARMEFORSYNING".
Af: Bjarne Lilletorup.
Vejleder: Bent Sørensen.

97/85 "ON THE PHYSICS OF A.C. HOPPING CONDUCTIVITY".
Af: Jeppe C. Dyre.

98/85 "VALGMULIGHEDER I INFORMATIONSALDEREN".
Af: Bent Sørensen.

99/85 "Der er langt fra Q til R".
Projektrapport af: Niels Jørgensen og Mikael Klintorp.
Vejleder: Stig Andur Pedersen.

100/85 "TALSYSTEMETS OPBYGNING".
Af: Mogens Niss.

101/85 "EXTENDED MOMENTUM THEORY FOR WINDMILLS IN PERTURBATIVE FORM".
Af: Ganesh Sengupta.

102/85 OPSTILLING OG ANALYSE AF MATEMATISKE MODELLER, BELYST VED MODELLER OVER KØERS FODEROPTACELSE OG - OMSÆTNING".
Projektrapport af: Lis Eilertzen, Kirsten Habekost, Lill Røn og Susanne Stender.
Vejleder: Klaus Grünbaum.

103/85 "ØDSLE KOLDKRIGERE OG VIDENSKABENS LYSE IDEER".
Projektrapport af: Niels Ole Dam og Kurt Jensen.
Vejleder: Bent Sørensen.

104/85 "ANALOGREGNEMASKINEN OG LORENZLIGNINGER".
Af: Jens Jæger.

105/85 "THE FREQUENCY DEPENDENCE OF THE SPECIFIC HEAT AF THE GLASS REANSITION".
Af: Tage Christensen.

"A SIMPLE MODEL AF AC HOPPING CONDUCTIVITY".
Af: Jeppe C. Dyre.
Contributions to the Third International Conference on the Structure of Non - Crystalline Materials held in Grenoble July 1985.

106/85 "QUANTUM THEORY OF EXTENDED PARTICLES".
Af: Bent Sørensen.

107/85 "EN MYG GØR INGEN EPIDEMI".
- flodblindhed som eksempel på matematisk modellering af et epidemiologisk problem.
Projektrapport af: Per Hedegård Andersen, Lars Boye, CarstenHolst Jensen, Else Marie Pedersen og Erling Møller Pedersen.
Vejleder: Jesper Larsen.

108/85 "APPLICATIONS AND MODELLING IN THE MATEMATICS CUR - RICULUM" - state and trends -
Af: Mogens Niss.

109/85 "COX I STUDIETIDEN" - Cox's regressionsmodel anvendt på studenteroplysninger fra RUC.
Projektrapport af: Mikael Wennerberg Johansen, Poul Katler og Torben J. Andreasen.
Vejleder: Jørgen Larsen.

110/85 "PLANNING FOR SECURITY".
Af: Bent Sørensen

111/85 JORDEN RUNDT PÅ FLADE KORT".
Projektrapport af: Birgit Andresen, Beatriz Quinones og Jimmy Staal.
Vejleder: Mogens Niss.

112/85 "VIDENSKABELIGGØRELSE AF DANSK TEKNOLOGISK INNOVATION FREM TIL 1950 - BELYST VED EKSEMPLER".
Projektrapport af: Erik Odgaard Gade, Hans Hedal, Frank C. Ludvigsen, Annette Post Nielsen og Finn Physant.
Vejleder: Claus Bryld og Bent C. Jørgensen.

113/85 "DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS 11".
Af: Bernhelm Booss og Krzysztof Wojciechowski.

114/85 "ANVENDELSE AF GRAFISKE METODER TIL ANALYSE AF KONTIGENSTABELLER".
Projektrapport af: Lone Biilmann, Ole R. Jensen og Arne-Lise von Moos.
Vejleder: Jørgen Larsen.

115/85 "MATEMATIKKENS UDVIKLING OP TIL RENÆSSANCEN".
Af: Mogens Niss.

116/85 "A PHENOMENOLOGICAL MODEL FOR THE MEYER-NELDEL RULE".
Af: Jeppe C. Dyre.

117/85 "KRAFT & FJERNVARMEOPTIMERING"
Af: Jacob Mørch Pedersen.
Vejleder: Bent Sørensen

118/85 "TILFÆLDIGHEDEN OG NØDVENDIGHEDEN IFØLGE PEIRCE OG FYSIKKEN".
Af: Peder Voetmann Christiansen

119/86 "DET ER GANSKE VIST - - EUKLIDS FEMTE POSTULAT KUNNE NOK SKABE RØRE I ANDEDAMMEN".
Af: Iben Maj Christiansen
Vejleder: Mogens Niss.

120/86 "ET ANTAL STATISTISKE STANDARDMODELLER".
Af: Jørgen Larsen

121/86 "SIMULATION I KONTINUERT TID".
Af: Peder Voetmann Christiansen.

122/86 "ON THE MECHANISM OF GLASS IONIC CONDUCTIVITY".
Af: Jeppe C. Dyre.

123/86 "GYMNASIEFYSIKKEN OG DEN STORE VERDEN".
Fysiklærerforeningen, IMFUFA, RUC.

124/86 "OPGAVESAMLING I MATEMATIK".
Samtlige opgaver stillet i tiden 1974-jan. 1986.

125/86 "UVBY,β - systemet - en effektiv fotometrisk spektralklassifikation af B-,A- og F-stjerner".
Projektrapport af: Birger Lundgren.

126/86 "OM UDVIKLINGEN AF DEN SPECIELLE RELATIVITETSTEORI".
Projektrapport af: Lise Odgaard & Linda Szkotak Jensen
Vejledere: Karin Beyer & Stig Andur Pedersen.

127/86 "GALOIS' BIDRAG TIL UDVIKLINGEN AF DEN ABSTRAKTE ALGEBRA".
Projektrapport af: Pernille Sand, Heine Larsen & Lars Frandsen.
Vejleder: Mogens Niss.

128/86 "SMÅKRYB" - om ikke-standard analyse.
Projektrapport af: Niels Jørgensen & Mikael Klintorp.
Vejleder: Jeppe Dyre.

129/86 "PHYSICS IN SOCIETY"
Lecture Notes 1983 (1986)
Af: Bent Sørensen

130/86 "Studies in Wind Power"
Af: Bent Sørensen

131/86 "FYSIK OG SAMFUND" - Et integreret fysik/historieprojekt om naturanskuelsens historiske udvikling og dens samfundsmæssige betingethed.
Projektrapport af: Jakob Heckscher, Søren Brønd, Andy Wierød.
Vejledere: Jens Høyrup, Jørgen Vogelius, Jens Højgaard Jensen.

132/86 "FYSIK OG DANNELSE"
Projektrapport af: Søren Brønd, Andy Wierød.
Vejledere: Karin Beyer, Jørgen Vogelius.

133/86 "CHERNOBYL ACCIDENT: ASSESSING THE DATA. ENERGY SERIES NO. 15.
AF: Bent Sørensen.

134/87 "THE D.C. AND THE A.C. ELECTRICAL TRANSPORT IN AsSeTe SYSTEM"
Authors: M.B.El-Den, N.B.Olsen, Ib Høst Pedersen, Petr Viščor

135/87 "INTUITIONISTISK MATEMATIKS METODER OG ERKENDELSESTEORETISKE FORUDSÆTNINGER"
MASTEMATIKSPECIALE: Claus Larsen
Vejledere: Anton Jensen og Stig Andur Pedersen

136/87 "Mystisk og naturlig filosofi: En skitse af kristendommens første og andet møde med græsk filosofi"
Projektrapport af Frank Colding Ludvigsen
Vejledere: Historie: Ib Thiersen
Fysik: Jens Højgaard Jensen

137/87 "HOPMODELLER FOR ELEKTRISK LEDNING I UORDNEDE FASTE STOFFER" - Resume af licentiatafhandling
Af: Jeppe Dyre
Vejledere: Niels Boye Olsen og Peder Voetmann Christiansen.

138/87 "JOSEPHSON EFFECT AND CIRCLE MAP."
Paper presented at The International Workshop on Teaching Nonlinear Phenomena at Universities and Schools, "Chaos in Education". Balaton, Hungary, 26 April-2 May 1987.

By: Peder Voetmann Christiansen

139/87 "Machbarkeit nichtbeherrschbarer Technik durch Fortschritte in der Erkennbarkeit der Natur"

Af: Bernhelm Booss-Bavnbek
Martin Bohle-Carbonell

140/87 "ON THE TOPOLOGY OF SPACES OF HOLOMORPHIC MAPS"

By: Jens Gravesen

141/87 "RADIOMETERS UDVIKLING AF BLODGASAPPARATUR - ET TEKNOLOGIHISTORISK PROJEKT"

Projektrapport af Finn C. Physant

Vejleder: Ib Thiersen

142/87 "The Calderón Projektor for Operators With Splitting Elliptic Symbols"

by: Bernhelm Booss-Bavnbek og
Krzysztof P. Wojciechowski

143/87 "Kursusmateriale til Matematik på NAT-BAS"

af: Mogens Brun Heefelt

144/87 "Context and Non-Locality - A Peircean Approach
Paper presented at the Symposium on the Foundations of Modern Physics The Copenhagen Interpretation 60 Years after the Como Lecture. Joensuu, Finland, 6 - 8 august 1987.

By: Peder Voetmann Christiansen

145/87 "AIMS AND SCOPE OF APPLICATIONS AND MODELLING IN MATHEMATICS CURRICULA"

Manuscript of a plenary lecture delivered at ICMTA 3, Kassel, FRG 8.-11.9.1987

By: Mogens Niss

146/87 "BESTEMMELSE AF BULKRESISTIVITETEN I SILICIUM"
- en ny frekvensbaseret målemetode.

Fysikspeciale af Jan Vedde

Vejledere: Niels Boye Olsen & Petr Viščor

147/87 "Rapport om BIS på NAT-BAS"

redigeret af: Mogens Brun Heefelt

148/87 "Naturvidenskabsundervisning med Samfundsperspektiv"

af: Peter Colding-Jørgensen DLH
Albert Chr. Paulsen

149/87 "In-Situ Measurements of the density of amorphous germanium prepared in ultra high vacuum"

by: Petr Viščor

150/87 "Structure and the Existence of the first sharp diffraction peak in amorphous germanium prepared in UHV and measured in-situ"

by: Petr Viščor

151/87 "DYNAMISK PROGRAMMERING"

Matematikprojekt af:
Birgit Andresen, Keld Nielsen og Jimmy Staal

Vejleder: Mogens Niss

152/87 "PSEUDO-DIFFERENTIAL PROJECTIONS AND THE TOPOLOGY OF CERTAIN SPACES OF ELLIPTIC BOUNDARY VALUE PROBLEMS"

by: Bernhelm Booss-Bavnbek
Krzysztof P. Wojciechowski

153/88 "HALVLEDERTEKNOLOGIENS UDVIKLING MELLEM MILITÆRE OG CIVILE KRÆFTER"

Et eksempel på humanistisk teknologihistorie
Historiespeciale

Af: Hans Hedal

Vejleder: Ib Thiersen

154/88 "MASTER EQUATION APPROACH TO VISCOUS LIQUIDS AND THE GLASS TRANSITION"

By: Jeppe Dyre

155/88 "A NOTE ON THE ACTION OF THE POISSON SOLUTION OPERATOR TO THE DIRICHLET PROBLEM FOR A FORMALLY SELFADJOINT DIFFERENTIAL OPERATOR"

by: Michael Pedersen

156/88 "THE RANDOM FREE ENERGY BARRIER MODEL FOR AC CONDUCTION IN DISORDERED SOLIDS"

by: Jeppe C. Dyre

157/88 " STABILIZATION OF PARTIAL DIFFERENTIAL EQUATIONS BY FINITE DIMENSIONAL BOUNDARY FEEDBACK CONTROL: A pseudo-differential approach."

by: Michael Pedersen

158/88 "UNIFIED FORMALISM FOR EXCESS CURRENT NOISE IN RANDOM WALK MODELS"

by: Jeppe Dyre

159/88 "STUDIES IN SOLAR ENERGY"

by: Bent Sørensen

160/88 "LOOP GROUPS AND INSTANTONS IN DIMENSION TWO"

by: Jens Gravesen

161/88 "PSEUDO-DIFFERENTIAL PERTURBATIONS AND STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS:
Dirichlet feedback control problems"

by: Michael Pedersen

162/88 "PIGER & FYSIK - OG MEGET MERE"

AF: Karin Beyer, Sussanne Blegaa, Birthe Olsen, Jette Reich , Mette Vedelsby

163/88 "EN MATEMATISK MODEL TIL BESTEMMELSE AF PERMEABILITETEN FOR BLOD-NETHINDE-BARRIEREN"

Af: Finn Langberg, Michael Jarden, Lars Frellesen

Vejleder: Jesper Larsen

164/88 "Vurdering af matematisk teknologi
Technology Assessment
Technikfolgenabschätzung"

Af: Bernhelm Booss-Bavnbek, Glen Pate med
Martin Bohle-Carbonell og Jens Højgaard Jensen

165/88 "COMPLEX STRUCTURES IN THE NASH-MOSER CATEGORY"

by: Jens Gravesen

166/88 "Grundbegreber i Sandsynligheds-regningen"

Af: Jørgen Larsen

167a/88 "BASISSTATISTIK 1. Diskrete modeller"

Af: Jørgen Larsen

167b/88 "BASISSTATISTIK 2. Kontinuerte modeller"

Af: Jørgen Larsen

168/88 "OVERFLADEN AF PLANETEN MARS"
Laboratorie-simulering og MARS-analoger undersøgt ved Mössbauerspektroskopi.

Fysikspeciale af:

Birger Lundgren

Vejleder: Jens Martin Knudsen
Fys.Lab./HCØ

169/88 "CHARLES S. PEIRCE: MURSTEN OG MØRTEL TIL EN METAFYSIK."

Fem artikler fra tidsskriftet "The Monist" 1891-93.

Introduktion og oversættelse:

Peder Voetmann Christiansen

170/88 "OPGAVESAMLING I MATEMATIK"

Samtlige opgaver stillet i tiden 1974 - juni 1988

171/88 "The Dirac Equation with Light-Cone Data"

af: Johnny Tom Ottesen

172/88 "FYSIK OG VIRKELIGHED"

Kvantemekanikkens grundlagsproblem i gymnasiet.

Fysikprojekt af:

Erik Lund og Kurt Jensen

Vejledere: Albert Chr. Paulsen og Peder Voetmann Christiansen

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173/89 "NUMERISKE ALGORITMER"

af: Mogens Brun Heefelt

174/89 " GRAFISK FREMSTILLING AF FRAKTALER OG KAOS"

af: Peder Voetmann Christiansen

175/89 " AN ELEMENTARY ANALYSIS OF THE TIME DEPENDENT SPECTRUM OF THE NON-STATONARY SOLUTION TO THE OPERATOR RICCATI EQUATION

af: Michael Pedersen

176/89 " A MAXIUM ENTROPY ANSATZ FOR NONLINEAR RESPONSE THEORY"

af : Jeppe Dyre

177/89 "HVAD SKAL ADAM STÅ MODEL TIL"

af: Morten Andersen, Ulla Engström, Thomas Gravesen, Nanna Lund, Pia Madsen, Dina Rawat, Peter Torstensen

Vejleder: Mogens Brun Heefelt

178/89 "BIOSYNTESEN AF PENICILLIN - en matematisk model"

af: Ulla Eghave Rasmussen, Hans Oxvang Mortensen, Michael Jarden

vejleder i matematik: Jesper Larsen
biologi: Erling Lauridsen

179a/89 "LÆRERVEJLEDNING M.M. til et eksperimentelt forløb om kaos"

af: Andy Wierød, Søren Brønd og Jimmy Staal

Vejledere: Peder Voetmann Christiansen
Karin Beyer

179b/89 "ELEVHÆFTE: Noter til et eksperimentelt kursus om kaos"

af: Andy Wierød, Søren Brønd og Jimmy Staal

Vejledere: Peder Voetmann Christiansen
Karin Beyer

180/89 "KAOS I FYSISKE SYSTEMER eksemplificeret ved torsions- og dobbeltpendul".

af: Andy Wierød, Søren Brønd og Jimmy Staal
Vejleder: Peder Voetmann Christiansen

181/89 "A ZERO-PARAMETER CONSTITUTIVE RELATION FOR PURE SHEAR VISCOELASTICITY"

by: Jeppe Dyre

183/89 "MATEMATICAL PROBLEM SOLVING, MODELLING. APPLICATIONS AND LINKS TO OTHER SUBJECTS - State. trends and issues in mathematics instruction

by: WERNER BLUM, Kassel (FRG) og MOGENS NISS, Roskilde (Denmark).

184/89 "En metode til bestemmelse af den frekvensafhængige varmefylde af en underafkølet væske ved glasovergangen"

af: Tage Emil Christensen

185/90 "EN NÆSTEN PERIODISK HISTORIE"

Et matematisk projekt

af: Steen Grode og Thomas Jessen

Vejleder: Jacob Jacobsen

186/90 "RITUAL OG RATIONALITET i videnskabers udvikling"

redigeret af Arne Jakobsen og Stig Andur Pedersen

187/90 "RSA - et kryptografisk system"

af: Annemette Sofie Olufsen, Lars Frellesen og Ole Møller Nielsen

Vejledere: Michael Pedersen og Finn Munk

188/90 "FERMICONDENSATION - AN ALMOST IDEAL GLASS TRANSITION"

by: Jeppe Dyre

189/90 "DATAMATER I MATEMATIKUNDERVISNINGEN PÅ GYMNASIET OG HØJERE LÆREANSTALTER

af: Finn Langberg

190/90 "FIVE REQUIREMENTS FOR AN APPROXIMATE NONLINEAR RESPONSE THEORY"

by: Jeppe Dyre

191/90 "MOORE COHOMOLOGY, PRINCIPAL BUNDLES AND ACTIONS OF GROUPS ON C*-ALGEBRAS"

by: Iain Raeburn and Dana P. Williams

192/90 "Age-dependent host mortality in the dynamics of endemic infectious diseases and SIR-models of the epidemiology and natural selection of co-circulating influenza virus with partial cross-immunity"

by: Viggo Andreasen

193/90 "Causal and Diagnostic Reasoning"

by: Stig Andur Pedersen

194a/90 "DETERMINISTISK KAOS"
Projektrapport af : Frank Olsen

194b/90 "DETERMINISTISK KAOS"
Kørselsrapport

Projektrapport af: Frank Olsen

195/90 "STADIER PÅ PARADIGMETS VEJ"
Et projekt om den videnskabelige udvikling der førte til dannelse af kvantemekanikken.

Projektrapport for 1. modul på fysikuddannelsen, skrevet af:

Anja Boisen, Thomas Hougård, Anders Gorm Larsen, Nicolai Ryge.

Vejleder: Peder Voetmann Christiansen

196/90 "ER KAOS NØDVENDIGT?"
– en projektrapport om kaos' paradigmatiske status i fysikken.

af: Johannes K. Nielsen, Jimmy Staal og Peter Bøggild

Vejleder: Peder Voetmann Christiansen

197/90 "Kontrafaktiske konditionaler i HOL

af: Jesper Voetmann, Hans Oxvang Mortensen og Aleksander Høst-Madsen

Vejleder: Stig Andur Pedersen

198/90 "Metal-Isolator-Metal systemer"
Speciale

af: Frank Olsen

199/90 "SPREDT FÆGTNING" Artikelsamling
af: Jens Højgaard Jensen

200/90 "LINEÆR ALGEBRA OG ANALYSE"
Noter til den naturvidenskabelige basisuddannelse.
af: Mogens Niss

201/90 "Undersøgelse af atomare korrelationer i amorfe stoffer ved røntgendiffraktion"

af: Karen Birkelund og Klaus Dahl Jensen

Vejledere: Petr Viščor, Ole Bakander

202/90 "TEGN OG KVANTER"
Foredrag og artikler, 1971-90.

af: Peder Voetmann Christiansen

203/90 "OPGAVESAMLING I MATEMATIK" 1974-1990

afløser tekst 170/88

204/91 "ERKENDELSE OG KVANTEMEKANIK"
et Breddemodul Fysik Projekt
af: Thomas Jessen
Vejleder: Petr Viščor

205/91 "PEIRCE'S LOGIC OF VAGUENESS"
by: Claudine Engel-Tiercelin
Department of Philosophy
Université de Paris-1
(Panthéon-Sorbonne)

206a+b/91 "GERMANIUMBEAMANALYSE SAMT A – GE TYNDFILMS ELEKTRISKE EGENSKABER"
Eksperimentelt Fysikspeciale
af: Jeanne Linda Mortensen
og Annette Post Nielsen
Vejleder: Petr Viščor

207/91 "SOME REMARKS ON AC CONDUCTION IN DISORDERED SOLIDS"

by: Jeppe C. Dyre

208/91 "LANGEVIN MODELS FOR SHEAR STRESS FLUCTUATIONS IN FLOWS OF VISCO-ELASTIC LIQUIDS"

by: Jeppe C. Dyre

209/91 "LORENZ GUIDE" Kompendium til den danske fysiker Ludvig Lorenz, 1829-91.

af: Helge Kragh

210/91 "Global Dimension, Tower of Algebras, and Jones Index of Split Seperable Subalgebras with Unitality Condition.

by: Lars Kadison

211/91 "I SANDHEDENS TJENESTE"
– historien bag teorien for de komplekse tal.
af: Lise Arleth, Charlotte Gjerrild, Jane Hansen, Linda Kyndlev, Anne Charlotte Nilsson, Kamma Tulinius.
Vejledere: Jesper Larsen og Bernhelm Booss-Bavnbek

212/91 "Cyclic Homology of Triangular Matrix Algebras"
by: Lars Kadison

213/91 "Disease-induced natural selection in a diploid host
by: Viggo Andreasen and Freddy B. Christiansen

214|91 "Halløj i æteren" - om elektromagnetisme. Oplæg til undervisningsmateriale i gymnasiet.

Af: Nils Kruse, Peter Gastrup, Kristian Hoppe, Jeppe Guldager

Vejledere: Petr Viscor, Hans Hedal

215|91 "Physics and Technology of Metal-Insulator-Metal thin film structures used as planar electron emitters

by: A.Delong, M.Drsticka, K.Hladil, V.Kolarik, F.Olsen, P.Pavelka and Petr Viscor.

216|91 "Kvantemekanik på PC'eren"

af: Thomas Jessen

217/92 "Two papers on APPLICATIONS AND MODELLING IN THE MATHEMATICS CURRICULUM"

by: Mogens Niss

218/92 "A Three-Square Theorem"

by: Lars Kadison