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DISTRIBUTED PARAMETER SYSTEMS:

Dirichlet feedback control problems.

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PSEUDO-DIFFERENTIAL PERTURBATIONS AND STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS: Dirichlet feedback control problems.

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ORIENTATION.

This paper is concerned with boundary feedback stabilization problems for parabolic and hyperbolic evolution equations associated with elliptic differential operators. Boundary feedback systems have been studied intensively during the last ten years, with a monumental contribution due to I. Lasiecka & R. Triggiani; we refer here only to [7], [8] and [9] . Lasiecka & Triggiani employ a semigroup-rooted method, as related to work of Washburn [15] and Balakrishnan [1]. This semigroup-integral representation method is based on the theory of fractional power spaces and second order operators are considered. We describe here a pseudo-differential operator method that allow us to work directly in $L^2(\Omega)$ and we construct in a "classical" manner the resolvent of the 2m-order operator of the evolution equation. The explicit resolvent construction is then used to derive stabilization results for the boundary feedback semigroup of the problem.

More important, however, is the application of the pseudo-differential method to characterize various types of feedback systems appearing in the litterature. Some of the first results on the stabilization of feedback systems are due to Nambu [11] and Triggiani [12], where the systems are stabilized by a suitable manipulation (control) in the operator equation. Such manipulations are covered by what we denote by perturbations of the first kind. More complicated than manipulating with the operator equation, is manipulation with the boundary condition. Stabilization of a system by change of boundary condition is treated in [7], [8] and [9]. It is this kind of manipulation with the boundary condition that is usually understood as a boundary control, or as a boundary feedback for systems where the control is a feedback of the state. We denote this kind of control a perturbation of the second kind. For both types of perturbations it is most realistic to assume that the feedback controls are of finite rank.

One of the main results of this paper is that we can, in general, transform a perturbation of the second kind into the more simple and well understood perturbation of the first kind, where a "classical" approach can be taken for obtaining stabilization results. It turns out that the transformation can be regarded as a generalized change of coordinates in the space $H^m(\Omega)$, a feature that is also of interest when considering optimal control of systems, but this is not elaborated on in this paper.

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For the sake of generality, we also introduce a mixing of the perturbations. We manipulate <u>both</u> the operator equation <u>and</u> the boundary condition and call this a <u>perturbation of the third kind.</u> For such systems it is possible to construct a system operator that is <u>variational</u> (i.e. associated with a suitable sesquilinear form), and from calculation of this operator the linking of "interior terms" and "boundary terms" is revealed. The stabilization procedure suggested for such systems with mixed feedbacks is probably not optimal in any sense, but it is simple, and the theory elucidates the nature of the boundary feedback systems.

The pseudo-differential approach thus allows us, in a unified setting, to obtain stabilization results for all three kinds of perturbations and for parabolic as well as hyperbolic equations. In all cases we conclude that it is possible to construct finite rank feedback mechanisms that give exponential decrease of the L^2-norm of the state. For perturbations of the first and second kind this is achieved by means of the pole-assignment theorem, while for perturbations of the third kind interior and boundary terms are balanced.

The results on perturbations of first and second kind are contained in § 5 of the paper, while § 6 deals with the perturbations of the third kind. § 1 is an introduction to the notation used throughout the paper, § 2 introduces the specific form of the perturbations considered. § 3 and § 4 deals with some results of pseudo-differential calculus, most importantly the transformation techniques employed.

§ 1. INTRODUCTION AND NOTATION.

We consider stabilization of parabolic and hyperbolic differential equations of the form

$$(1,1) \quad \begin{cases} \partial_t u + Au = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \gamma u = 0 & \text{on } \Gamma \text{ , for } t > 0 \\ u = u_0 & \text{in } \Omega \text{ , for } t = 0 \end{cases}$$

and

$$(1,2) \quad \begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \gamma u = 0 & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega \text{ , at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases}$$

Here A is a formally selfadjoint, uniformly strongly elliptic differential operator of order 2m, of the form

$$(1,3) \quad A = \sum_{|\alpha|, |\beta| \leq m} D^\beta a_{\alpha\beta}(x) D^\alpha \ .$$

with $C^\infty(\overline{\Omega})$-coefficients $a_{\alpha\beta}$ on a bounded, open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega = \Gamma$. γ is the <u>Dirichlet trace operator</u>

$$(1,4) \quad \gamma = \{\gamma_j\}_{0 \leq j < m} \ , \text{ where}$$

$$(1,5) \quad \gamma_j u = \left(\frac{1}{i}\frac{\partial}{\partial n}\right)^j u \, \Big|_\Gamma$$

(n is the normal, directed inward.)

We denote similarly

$$(1,6) \quad \upsilon = \{\gamma_j\}_{m \leq j < 2m}$$

the <u>Neumann trace operator</u>, and we define the <u>Cauchy-data</u> ρu as

$$(1,7) \quad \rho u = \{\gamma u, \ \upsilon u\},$$

moreover we use the multi-indexs notation

$$(1.8) \qquad \begin{cases} D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_n^{\alpha_n}, \\ D_j^{\alpha_j} = \left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)^{\alpha_j} \end{cases}$$

The <u>Dirichlet realization</u> A_γ of A is the operator acting like A in $L^2(\Omega)$, and with domain

$$(1.9) \qquad D(A_\gamma) = \{u \in H^{2m}(\Omega) \mid \gamma u = 0\} = H^{2m}(\Omega) \cap H_o^m(\Omega)$$

where $H^s(\Omega)$ is the Sobolev space of $L^2(\Omega)$-functions with $L^2(\Omega)$-derivatives up to order s. It is well known that A_γ is an unbounded, selfadjoint operator in $L^2(\Omega)$, and since the embedding $H^s(\Omega) \to H^t(\Omega)$ is compact for $s > t$, the resolvent $R(\lambda, A_\gamma)$ of A_γ is a compact operator in $L^2(\Omega)$ for all λ outside the spectrum, $sp(A_\gamma)$, of A_γ. Hence A_γ has a sequence of real eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ converging to infinity. Wee see that (1.1) and (1.2) are the time dependent evolution problems associated with A_γ, generalizing the heat equation, resp. the wave equation. When $\lambda_1 > 0$, all solutions $u(t,x)$ of (1.1) are exponentially decreasing for $t \to \infty$, and all solutions of (1.2) are bounded; we will call this the <u>stable</u> case. However, if some eigenvalues are negative, there are solutions both of (1.1) and (1.2) that blow up in an exponential manner for $t \to \infty$. It is therefore of interest to investigate how one can change the systems to obtain the stable case, and this is the aim of this paper.

§ 2. PERTURBATIONS OF THE BOUNDARY VALUE PROBLEMS.

By a __perturbation of the first kind__ of the system (1,1), we will understand a system of the form

$$(2,1) \quad \begin{cases} \partial_t u + Au + Gu = 0 \quad \text{in } \Omega \text{ , for } t > 0 \\[2mm] \gamma u = 0 \quad \text{on } \Gamma \text{ , for } t > 0 \\[2mm] u = u_0 \text{ in } \Omega \text{ , at } t = 0 \end{cases}$$

Here the interior operator A is replaced by $A + G$ where G has finite rank and is of the special form $G = KT$, where

(i) T is a __trace operator__ that maps functions on Ω into functions on Γ, of the form as a column vector

(ii) K is a __Poisson operator__ that maps functions on Γ into functions on Ω, of the form as a row vector

An operator G of this special form is denoted a __singular Green operator__, this kind of operator is carefully explained, as well as the trace- and Poisson- operators, in Grubb [5]. They are all operators entering in the "Boutet de Monvel-calculus" (c.f. Boutet de Monvel [2]).

Stabilization of the a priori unstable system (1,1) by a perturbation of the first kind has been studied e.g. in Nambu [11] and Triggiani [14], and it is shown there that it is possible to choose G of finite rank, such that (2,1) is stable.

By a __perturbation of the second kind__ of the system (1,1), we will understand a system of the form

$$(2,2) \quad \begin{cases} \partial_t u + Au = 0 \quad \text{in } \Omega \text{ , for } t > 0 \\[2mm] \gamma u = T'u \text{ on } \Gamma \text{ , for } t > 0 \\[2mm] u = u_0 \quad \text{in } \Omega \text{ , at } t = 0 \end{cases}$$

where the boundary operator γ is replaced by $\gamma - T'$, T' being a trace operator of finite rank.

Both (2.1) and (2.2) are so called <u>boundary feedback systems</u>, and we will especially be interested in the case where T' is of the special form

$$(2.3) \qquad T'u = \sum_{j=1}^{N} (u|w_j)g_j$$

(here $(\cdot|\cdot)$ is the usual $L^2(\Omega)$ - inner product, $w_j \in C^{\infty}(\overline{\Omega})$, $g_j \in C^{\infty}(\Gamma)^m$, $j = 1, 2, \ldots, N$).

For applications, w_j can be thought of as sensor functions and g_j as boundary actuaters.

T' defined in (2.3) is called a <u>finite dimensional feedback operator</u>, and in contrast to γ it is of a <u>non-local</u> nature. (One of the major differences between the perturbations (2.1) and (2.2) is that in (2.1) the boundary condition is <u>local</u>, whereas in (2.2) it is non-local).

Boundary feedback systems have been studied in a number of papers by I. Lasiecka and R. Triggiani (see e.g. Lasiecka & Triggiani [7], [8] and [9]). One of the main results is that, under suitable hypotheses, it is possible to choose the functions w_j and g_j appearing in (2.3), such that the system is stable. Lasiecka and Triggiani take a semigroup approach to investigate the system (2.2), using developments on the semigroup approach presented in Washburn [15] and Balakrishnan [1]. (The basic idea of a semigroup model is presented in Fattorini [4], where ordinary differential equations are considered.)

By a <u>perturbation of the third kind</u> of the system (1.1), we will understand a system of the form

$$(2.4) \qquad \begin{cases} \partial_t u + Au + Gu = 0 \quad \text{in } \Omega, \text{ for } t > 0 \\ \quad \gamma u = T'u \text{ on } \Gamma, \text{ for } t > 0 \\ \quad u = u_0 \quad \text{in } \Omega, \text{ at } t = 0 \end{cases}$$

where the operators G and T' are of the types considered above.

We define perturbations of the system (1.2) analogously.

In the following we present a pseudo-differential operator method to investigate the systems above. this gives us in an easy way many of the results of Lasiecka & Triggiani, [7], [8] and [9],as well as similar results for the hyperbolic problem

$$(2.5) \quad \begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \gamma u = T'u & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega \text{ , at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases}$$

(a perturbation of the second kind of (1.2)).

Here we would like to point out that there now exists a quite general theory that includes all the above perturbations when they have "smooth coefficients", namely the theory of pseudo-differential boundary problems. For these, the solvability of parabolic problems like (2.2) (and far more general cases) has been discussed in great detail in Grubb [5]. However, in the work that follows, we use only some basic results of the pseudo-differential methods, but the techniques are crucial for the simplicity of the proofs, and the theory is very helpful for the understanding of the underlying problems.

Our main result is that we can, in general, transform a perturbation of the second kind into a perturbation of the first kind, whenever the boundary condition is normal, in the sense described in Grubb [5]. This includes all "classical" normal boundary conditions, as well as the feedback boundary condition $\gamma u - T'u = 0$, with T' given by (2.3). In this case, however, a special transformation of the systems (2.2) and (2.5) proves to be very useful. It turns out that the transformation can be regarded as a generalized change of coordinates, and the resulting, transformed system operator $A + G$ is merely a finite dimensional, A-bounded perturbation of A. In this case, stabilization theory for perturbations of the first kind is straightforward, as we can apply the well known "pole assignment theorem" due to Wonham (see Wonham [16]).

§ 3. NORMAL OPERATOR REALIZATIONS.

Assume that $Tu = 0$ is a <u>normal boundary condition</u> in the sense of Grubb [5]. We will then define a <u>normal realization</u> of the operator A (1,3) the following way: Let A_T be the operator acting like A in $L^2(\Omega)$. · with domain

$$(3.1) \qquad D(A_T) = \{u \in H^{2m}(\Omega) \mid Tu = 0\}$$

Then the realization A_T of A is called a normal realization.

It is shown in Grubb [5] Lemma 1.6.8 that normal realizations have dense domains in $L^2(\Omega)$. According to Grubb, there exists an operator Λ , (see (4.4)) which is a homeomorphism in $H^s(\Omega)$ for any $s \geq 0$, such that Λ defines a bijection

$$(3.2) \qquad \Lambda : D(A_T) \overset{\sim}{\to} D(A_\gamma) = H^{2m}(\Omega) \bigcap H^m_0(\Omega)$$

where A_γ is the Dirichlet realization from § 1.
Moreover, we have

LEMMA 3.1
Let T' be given by (2,3) and define

$$(3.3) \qquad T = \gamma - T'$$

Then $Tu = 0$ is a normal boundary condition, and the operator realization A_T of A is a closed, densely defined operator in $L^2(\Omega)$.

Proof:
We only have to show that A_T is closed.

Let (u_n) be a sequence in $D(A_T)$ converging to $u \in L^2(\Omega)$, and assume that (Au_n) converges to $v \in L^2(\Omega)$.

We must show that $u \in D(A_T)$ with $Au = v$.

Since $u_n \to u$ in $L^2(\Omega)$, we have that $Au_n \to Au$ in $\mathbb{D}'(\Omega)$ (the dual space of $C_0^\infty(\Omega)$), so $v = Au \in L^2(\Omega)$ and $u_n \to u$ in the space $\{v \in L^2(\Omega) \mid Av \in L^2(\Omega)\}$. Now $\tau u_n \to \tau u$ in $\prod H^{-\frac{1}{2}-k}(\Gamma)$ (see Lions & Magenes [10]) and since

$$\tau u_n = \sum_{j=1}^{N} (u_n|w_j)g_j \quad \text{where} \quad (u_n|w_j) \to (u|w_j) , \; j = 1, 2, \dots, N,$$

we see that $\tau u_n \to \sum_{j=1}^{N} (u|w_j)g_j$, so $\tau u = \sum_{j=1}^{N} (u|w_j)g_j$ as an element

of $\prod_{0 \leq k < m} H^{-\frac{1}{2}-k}(\Gamma)$. Now, since

$$g_j \in C^\infty(\Gamma)^m \subseteq \prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma) , \; j = 1, 2, \dots, N , \quad \text{we have that}$$

$$\tau u \in \prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma) .$$

But then by the regularity of the Dirichlet problem for A, $Au \in L^2(\Omega)$ and $\tau u \in \prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma)$ imply that $u \in H^{2m}(\Omega)$.

Altogether, $u \in H^{2m}(\Omega)$ with $\tau u - \sum_{j=1}^{N} (u|w_j)g_j = 0$, so $u \in D(A_T)$. $\square$

§ 4: THE PSEUDO-DIFFERENTIAL TRANSFORMATIONS

Consider for $\ell = 1, 2$ the parabolic, resp. hyperbolic perturbation of the second kind

$$(4.1) \qquad \partial_t^\ell u + A_T u = 0 , \; u \in D(A_T)$$

with $T = \gamma - T'$, T' given by (2.3).

Using (3.2) this can be transformed into

$$(4.2) \qquad \partial_t^\ell \Lambda^{-1} v + A_T \Lambda^{-1} v = 0 , \; v \in D(A_\gamma)$$

where $v = \Lambda u$. Acting with Λ from the left in (4.2) we find

$$(4.3) \qquad \partial_t^\ell v + \Lambda A_T \Lambda^{-1} v = 0 , \; v \in D(A_\gamma) .$$

From Grubb [5], Lemma 1.6.8, the structure of Λ and Λ^{-1} are as follows

$$(4.4) \qquad \begin{cases} \Lambda &= 1 - K_\delta T' \\ \Lambda^{-1} &= 1 - K_\delta Q_\delta T' \end{cases}$$

where K_δ, $\delta \in \,]0, \delta_0]$ is a family of Poisson operators and Q_δ is a certain δ-dependent pseudo-differential operator, that is bijective and elliptic in $\prod_{0 \leq k < 2m} H^{s-k}(\Omega)$, $s \geq 0$.

Then

$$(4.5) \qquad \Lambda A_T \Lambda^{-1} = (1 - K_\delta T') A_T (1 - K_\delta Q_\delta T') = A_T + G$$

where

$$(4.6) \qquad G = K_\delta T' A_T K_\delta Q_\delta T' - A_T K_\delta Q_\delta T' - K_\delta T' A_T$$

is a singular Green operator of finite rank.

Hence (4.3) is a perturbation of the first kind.

However in the case, where T' is of the special form (2.3), we can use a more explicit transformation.

Assume, for the moment, that 0 is not an eigenvalue of A_γ. This is only a temporary assumption that will be removed later (see after remark 5.3). Let K_γ be the Poisson operator that solves the Dirichlet problem for A, i.e. $u = K_\gamma \varphi$ is the solution of

$$(4.7) \qquad \begin{cases} Au = 0 & \text{in } \Omega \\ \gamma u = \varphi & \text{on } \Gamma \end{cases}$$

Since $K_\gamma T'$ has finite rank, the bounded operator $1 - K_\gamma T'$ is a Fredholm operator with index 0 in $L^2(\Omega)$, and it maps $H^{2m}(\Omega)$ into $H^{2m}(\Omega)$. (cf. Hörmander [6], ch. 19.1 for a simple explanation of Fredholm operators like $1 - K_\gamma T'$)

Since

$$(4.8) \qquad \gamma(1 - K_\gamma T')u = \gamma u - T'u = Tu$$

we have that

$$(4.9) \qquad (1 - K_\gamma T')D(A_T) \subseteq D(A_\gamma)$$

and moreover, if $u \in D(A_T)$ and $v = (1 - K_\gamma T')u$, then

$$(4.10) \qquad Av = A(1 - K_\gamma T')u = Au$$

Now assume that T' can be chosen such that $1 - K_\gamma T'$ is a bijection in $H^{2m}(\Omega)$ (this will be done later on, se § 5). Then referring again to the theory of Fredholm operators, we have that $1 - K_\gamma T'$ defines a homeomorphism:

$$(4.11) \qquad 1 - K_\gamma T' : D(A_T) \overset{\sim}{\to} D(A_\gamma) \quad (= H^{2m}(\Omega) \cap H_0^m(\Omega)) .$$

just like the Λ-operator above.

We have then established the useful factorization

$$(4.12) \qquad A_T = A_\gamma(1 - K_\gamma T') .$$

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Now proceeding as above, the problems

(4.13) $\qquad \partial_t^\ell u + A_T u = 0$, $u \in D(A_T)$, $\ell = 1, 2$.

transforms into

(4.14) $\qquad \partial_t^\ell v + (1 - K_\gamma T')A_\gamma v = 0$, $v \in D(A_\gamma)$, $\ell = 1,2$,

where $v = (1 - K_\gamma T')u$.

Thus we have transformed the perturbation of the second kind (4.13) into a perturbation of the first kind (4.14), and we are able to calculate the system operator in an easy way.

We have

THEOREM 4.1

The boundary feedback systems

(4.15) $\qquad \begin{cases} \partial_t u + Au = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \quad \gamma u = T'u & \text{on } \Gamma \text{ , for } t > 0 \\ \quad u = u_0 & \text{in } \Omega \text{ , at } t = 0 \end{cases}$

and

(4.16) $\qquad \begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \quad \gamma u = T'u & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\ \quad u = u_0 & \text{in } \Omega \text{ , at } t = 0 \\ \quad \partial_t u = u_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases}$

with T' given by (2.3), transform into the systems

(4.15') $\qquad \begin{cases} \partial_t v + Av - K_\gamma T'Av = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \quad \gamma v = 0 & \text{on } \Gamma \text{ , for } t > 0 \\ \quad v = v_0 & \text{in } \Omega \text{ , for } t = 0 \end{cases}$

and

$$(4.16') \quad \begin{cases} \partial_t^2 v + Av - K_\gamma T'Av = 0 \quad \text{in } \Omega \text{ , for } \quad t \in \mathbb{R} \\ \qquad\qquad\qquad \gamma v = 0 \quad \text{on } \Gamma \text{ , for } \quad t \in \mathbb{R} \\ \qquad\qquad\qquad v = v_0 \quad \text{in } \Omega \text{ , for } \quad t = 0 \\ \qquad\qquad \partial_t v = v_1 \quad \text{in } \Omega \text{ , for } \quad t = 0 \end{cases}$$

$\square$

Since

$$(4.17) \qquad K_\gamma T'Av = \sum_{j=1}^{N} (Av|w_j)K_\gamma g_j$$

for $v \in H^{2m}(\Omega)$, $K_\gamma T'A$ has finite rank, and we see that

$$(4.18) \qquad \tilde{A} = A - K_\gamma T'A$$

can be regarded as a finite dimensional perturbation of A.

We obviously have ($||\cdot||_s$ is the $H^s(\Omega)$-norm):

$$(4.19) \qquad ||K_\gamma T'Av||_0 \leq c||Av||_0 \leq c||Av||_0 + ||v||_0$$

for $v \in D(A_\gamma)$, so $K_\gamma T'A$ is A-bounded. Since A_γ is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$, so is $\tilde{A}_\gamma$, form the perturbation result in Zabczyk [17], prop. 1.

We have

THEOREM 4.2

The realization $\tilde{A}_\gamma$ of the operator

$$(4.20) \qquad \tilde{A} = A - K_\gamma T'A$$

with domain

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$$(4.21) \qquad D(\tilde{A}_\gamma) = H^{2m}(\Omega) \bigcap H_0^m(\Omega) \quad (= D(A_\gamma))$$

is the infinitesimal generator of an analytic semigroup $e^{-\tilde{A}_\gamma t}$, $t \geq 0$ on $L^2(\Omega)$, giving the solution to (4.15') as

$$(4.22) \qquad v(t,x) = e^{-\tilde{A}_\gamma t} v_0(x) \ , \ x \in \Omega \ , \ t \geq 0 \ ,$$

when $v_0 \in L^2(\Omega)$. The solution to the original system (4.15) is then

$$(4.23) \quad u(t,x) = (1 - K_\gamma T')^{-1} e^{-\tilde{A}_\gamma t} (1 - K_\gamma T') u_0(x) \ , \ x \in \Omega, \ t \geq 0$$

when $u_0 \in L^2(\Omega)$. $\square$

Remark 4.3
All the semigroup theory used above is covered by [5].

§ 5. AN APPLICATION OF THE PSEUDO-DIFFERENTIAL TRANSFORMATION TO STA-BILIZATION.

We will now show how the transformation from § 4 can be used as a short-cut to some of the results of Lasiecka & Triggiani ([7],[8],[9]), which have been a great motivation to us.

The assumed instability of the systems (1,1) and (1,2) is caused by the negative eigenvalues in the pure point spectrum $sp(A_\gamma)$ of A_γ , and we will show that we can choose a finite dimensional feedback boundary condition

$$(5,1) \qquad\qquad \gamma u = T'u$$

where T' is defined by

$$(5,2) \qquad\qquad T'u = \sum_{j=1}^{N} (u|w_j)g_j$$

(see (2,4)), such that the systems

$$(5,3) \qquad \begin{cases} \partial_t u + Au = 0 & \text{in } \Omega \text{ , for } t > 0 \\ \gamma u = T'u \text{ in } \Gamma \text{ , for } t > 0 \\ u = u_0 \text{ in } \Omega \text{ , for } t = 0 \end{cases}$$

and

$$(5,4) \qquad \begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \gamma u = T'u \text{ on } \Gamma \text{ , for } t \in \mathbb{R} \\ u = u_0 \text{ in } \Omega \text{ , at } t = 0 \\ \partial_t u = u_1 \text{ in } \Omega \text{ , at } t = 0 \end{cases}$$

are stable systems (in the sense described in § 1). We will apply the pseudo-differential transformation to the perturbations of the second kind (5,3) and (5,4) and then apply Wonhams "pole assignment theorem" (Wonham [16]) on the resulting perturbations of the first kind. This, combined with a classical resolvent analysis, gives us the desired results.

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Let the eigenvalues of A_γ be arranged in a non-decreasing sequence

$$(5.5) \qquad \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{K-1} \leq 0 \leq \lambda_K \leq \ldots \; .$$

each eigenvalue repeated according to multiplicity, and let $\{\varphi_j\}_{j \geq 1}$ be a corresponding set of orthonormalized eigenfunctions of A_γ. Now define P_u and P_s as the orthogonal projekctions of $L^2(\Omega)$ on the orthogonal subspaces X_u, resp. X_s, defined by

$$(5.6) \qquad \begin{cases} X_u = \text{span } \{\varphi_j\}_{1 \leq j < K} \\ X_s = \overline{\text{span}} \; \{\varphi_j\}_{j \geq K} \end{cases}$$

<u>Remark 5.1</u>

The results in Lasiecka & Triggiani [7], [8] and [9] are formulated as if non-selfadjoint realizations are treated as well, but on the other hand the treatment is based heavily on the orthogonal projections on the eigenspaces X_u and X_s. Orthogonality of eigenspaces in general requires at least that A_γ is normal, i.e. $A_\gamma A_\gamma^* = A_\gamma^* A_\gamma$, but we know of no Dirichlet realization A_γ satisfying this without being selfadjoint.□

Since X_u and $X_s \cap D(A_\gamma)$ are invariant subspaces for A_γ, we can define the restrictions

$$(5.7) \qquad \begin{cases} A_u = A_\gamma \big|_{X_u} \\ A_s = A_\gamma \big|_{X_s} \cap D(A_\gamma) \end{cases}$$

Then A_u is a bounded operator on X_u and A_s is an unbounded operator with domain $D(A_s) = X_s \cap D(A_\gamma)$. Notice that P_u and P_s commute with A_γ on $D(A_\gamma)$. Now writing $f_u = P_u f$, $f_s = P_s f$ for $f \in L^2(\Omega)$, we have that when $u \in D(A_T)$, (see 3.1) and (4.11)), then $v = (1-K_\gamma T')u \in D(A_\gamma)$ satisfies

$$Av = Au$$

$$(5.8) \qquad v_u \in X_u$$

$$v_s \in D(A_s)$$

Now we use the factorization

(5.9)
$$A_T = A_\gamma(1-K_\gamma T')$$

in the discussion of the resolvent equation

(5.10)
$$(A_T-\lambda)u = f \ , \ f \in L^2(\Omega) \ .$$

First we consider the case where we are allowed to <u>decouple</u> the feedback by assuming that

(5.11)
$$P_s w_j = 0 \ , \ j = 1, \ 2, \ \ldots \ , \ N$$

(i.e. the w_j are in X_u ; the "unstable" eigenspace).

Then we write (5.10) in projected and factorized form

(5.12)
$$\begin{bmatrix} P_u \\ P_s \end{bmatrix} (A_\gamma(1-K_\gamma T'))(u_u + u_s) - \lambda(u_u + u_s) = \begin{bmatrix} f_u \\ f_s \end{bmatrix}$$

which because of (5.11) reduces to

(5.13)
$$A_u u_u - A_u P_u K_\gamma T' u_u - \lambda u_u = f_u$$

(5.14)
$$- A_s P_s K_\gamma T' u_u + A u_s - \lambda u_s = f_s$$

where we observe that (5.13) is a finite dimensional resolvent equation for the matrix operator

(5.15)
$$\overline{A}_u = A_u - A_u P_u K_\gamma T' \ .$$

To this we can apply Wonhams theorem to stabilize the unstable part of the system. We then have (for $m = 1$, one of Lasiecka and Triggiani's results) ;

THEOREM 5.2
Assume that the Neumann traces $\{v\varphi_j\}_{1 \leq j < K}$ are linearly independent, so that

(5.16)
$$\dim (vX_u) = \dim(X_u) \ (= K - 1)$$

and let $\{c_j\}_{1\leq j<K}$ be an arbitrary given set of $K-1$ distinct, real numbers.

Then there exists a number N and a set

(5.17)
$$\{w_j, g_j\}_{1\leq j\leq N}$$

where $w_j \in X_u$ and $g_j \in C^\infty(\Gamma)^m$, such that with

(5.18)
$$T'u = \sum_{j=1}^{N} (u|w_j)g_j .$$

the eigenvalues of the matrix operator

(5.19)
$$\overline{A}_u = A_u - A_u P_u K_\gamma T'$$

on X_u, are $\{c_j\}_{1\leq j<K}$.

The number N can be taken as the largest multiplicity of the unstable eigenvalues $\{\lambda_j\}_{1\leq j<K}$. In particular, $N = 1$ when the eigenvalues are simple.

Proof:
Assume first that <u>all</u> of the eigenvalues $\{\lambda_j\}_{1\leq j<K}$ are simple and take $N = 1$. Consider T' of the form

(5.20)
$$T'u = (u|w)g .$$

In the basis $\{\varphi_j\}_{1\leq j<K}$ of X_u, the matrix $\overline{A}_u$ has the form

(5.21)
$$\begin{bmatrix} \lambda_1 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & \ldots\ldots & & \lambda_{K-1} \end{bmatrix} - P(g)W^t$$

where

$$(5.22) \qquad P(g) = \begin{bmatrix} \lambda_1(K_\gamma g | \varphi_1) \\ \lambda_2(K_\gamma g | \varphi_2) \\ \vdots \\ \lambda_{K-1}(K_\gamma g | \varphi_{K-1}) \end{bmatrix}$$

and W^t is the transposed of the matrix given by

$$(5.23) \qquad W = \begin{bmatrix} (\varphi_1 | w) \\ (\varphi_2 | w) \\ \vdots \\ (\varphi_{K-1} | w) \end{bmatrix} .$$

Consider now the control matrix of the pair (A_u, W) :

$$(5.24) \quad [W \; A_u W \; \ldots \; A_u^{K-2} W] = \begin{bmatrix} (\varphi_1 | w) & \lambda_1(\varphi_1 | w) & \ldots & \lambda_1^{K-2}(\varphi_1 | w) \\ (\varphi_2 | w) & \lambda_2(\varphi_2 | w) & \ldots & \lambda_2^{K-2}(\varphi_2 | w) \\ \vdots & \vdots & & \vdots \\ (\varphi_{K-1} | w) & \lambda_{K-1}(\varphi_{K-1} | w) & \ldots & \lambda_{K-1}^{K-2}(\varphi_{K-1} | w) \end{bmatrix}$$

The determinant of the control matrix is computed to be

$$(5.25) \qquad \prod_{1 \leq j < K} (\varphi_j | w) \prod_{1 \leq \ell < k < K} (\lambda_k - \lambda_\ell)$$

by reduction to a Vandermonde determinant. Since the eigenvalues are assumed to be simple, we can choose $w \in X_u$, satisfying $(\varphi_j | w) \neq 0$ for $j = 1, 2, \ldots, K - 1$, such that the determinant is different from 0. This implies that the pair (A_u, W) is controllable, so by the pole assignment theorem there exists a matrix

$$(5.26) \qquad \tilde{P} = \begin{bmatrix} p_1 \\ p_2 \\ \\ p_{K-1} \end{bmatrix} , \quad p_j \in \mathbb{C} , \quad j = 1, 2, \ldots, K - 1$$

for wich the matrix $A_u - \tilde{P} W^t$ has the set $\{c_j\}_{1 \leq j < K}$ as eigenvalues. Now we will choose $g \in C^\infty(\Gamma)^m$ such that $\tilde{P} = P(g)$, i.e. such that

$$(5.27) \qquad \lambda_j(K_\gamma g | \varphi_j) = p_j$$

for $j = 1, 2, \ldots, K - 1$.

Now from the formula (A.10) in the appendix we have that

$$(5.28) \qquad (K_\gamma g | \varphi_j) = \frac{-1}{\lambda_j} (g | \mathscr{A}^{10^*} \upsilon \varphi_j)_\Gamma .$$

Here $\mathscr{A}^{10}$ is (since A is elliptic) an invertible $m \times m$ matrix of dif-
ferential operators over Γ, that maps $\prod\limits_{m \leq k < 2m} H^{2m-k-\frac{1}{2}}(\Gamma)$ onto $\prod\limits_{0 \leq k < m} H^{m+k+\frac{1}{2}}(\Gamma)$,
and $(\cdot | \cdot)_\Gamma$ is the $L^2(\Gamma)^m$ – inner product.

Since the set

$$(5.29) \qquad \{ \upsilon \varphi_1, \upsilon \varphi_2, \ldots, \upsilon \varphi_{K-1} \}$$

is linearly independent, so is also the set

$$(5.30) \qquad \{ \mathscr{A}^{10^*} \upsilon \varphi_1, \mathscr{A}^{10^*} \upsilon \varphi_2, \ldots, \mathscr{A}^{10^*} \upsilon \varphi_{K-1} \} .$$

Hence it is possible to choose $g \in C^\infty(\Gamma)^m$, satisfying

$$(5.31) \qquad (g | \mathscr{A}^{10^*} \upsilon \varphi_j) = -p_j , \quad j = 1, 2, \ldots, K - 1$$

and this choice of g provides us with the desired $P(g)$, in wiew of
(5.28). This ends the proof in the case of simple eigenvalues.

Now assume that one or more of the eigenvalues $\{\lambda_j\}_{1 \leq j < K}$ have multi-
plicity larger than 1, and let us take σ to be the largest occuring
multiplicity. Take $N = \sigma$ and consider T' of the form

$$(5.32) \qquad T'u = \sum_{j=1}^{\sigma} (u | w_j) g_j .$$

Now $A_u - A_u P_u K_\gamma T'$ can be written

$$
(5.33) \qquad \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & & & & \vdots \\ \vdots & & . & & & \vdots \\ 0 & . & . & . & . & \lambda_{K-1} \end{bmatrix} - P(\{g_i\}_{1 \leq i \leq \sigma}) \, W_\sigma^t
$$

where

$$
(5.34) \quad P(\{g_i\}_{1 \leq i \leq \sigma}) = \begin{bmatrix} \lambda_1(K_\gamma g_1 | \varphi_1) & \lambda_1(K_\gamma g_2 | \varphi_2) \ldots \lambda_1(K_\gamma g_\sigma | \varphi_1) \\ \lambda_2(K_\gamma g_1 | \varphi_2) & \lambda_2(K_\gamma g_2 | \varphi_2) \ldots \lambda_2(K_\gamma g_\sigma | \varphi_2) \\ \vdots & \\ \lambda_{K-1}(K_\gamma g_1 | \varphi_{K-1}) & \lambda_{K-1}(K_\gamma g_2 | \varphi_{K-1}) \; \lambda_{K-1}(K_\gamma g_\sigma | \varphi_{K-1}) \end{bmatrix}
$$

and

$$
(5.35) \qquad W_\sigma = \begin{bmatrix} (\varphi_1 | w_1) & (\varphi_1 | w_2) & . & . & . & (\varphi_1 | w_\sigma) \\ (\varphi_2 | w_1) & (\varphi_2 | w_2) & . & . & . & (\varphi_2 | w_\sigma) \\ \vdots & & \\ (\varphi_{K-1} | w_1) & (\varphi_{K-1} | w_2) & . & . & (\varphi_{K-1} | w_\sigma) \end{bmatrix}
$$

Considering the form of the control matrix

$$
(5.36) \qquad \begin{bmatrix} W_\sigma & A_u W_\sigma & . & . & . & A_u^{K-2} W_\sigma \end{bmatrix}
$$

we see that if $w_1, w_2, \ldots, w_\sigma$ are chosen in X_u such that

$$
(5.37) \qquad \text{rank } W_\sigma = \sigma
$$

then the rank of the control matrix (5.36) is $K - 1$ (because a regular $(K-1) \times (K-1)$ – submatrix can be extracted, after suitable row-column operations). Then according to Wonhams theorem there exists a complex matrix

$$
(5.38) \qquad \tilde{P}_\sigma = \begin{bmatrix} p_{11} & p_{12} & . & . & . & . & p_{1\sigma} \\ p_{21} & p_{22} & . & . & . & . & p_{2\sigma} \\ \vdots & \vdots & & & & & \vdots \\ p_{K-1,1} & p_{K-1,2} & . & . & . & p_{K-1,\sigma} \end{bmatrix}
$$

such that the eigenvalues of the matrix $A_u - \tilde{P}_\sigma W_\sigma^t$ are $\{c_j\}_{1 \leq j < K}$.

Now (5.28) takes the form

$$(5.39) \qquad (K_\gamma g_i | \varphi_j) = \frac{-1}{\lambda_j} (g_i | \mathscr{A}^{10*} \upsilon \varphi_j)_\Gamma \; .$$

In wiew of (5.16) we can choose $g_i \in C^\infty(\Gamma)^m$, $i = 1, 2, \ldots, \sigma$ satisfying

$$(5.40) \quad (g_i | \mathscr{A}^{10*} \upsilon \varphi_j)_\Gamma = - P_{ji} \; , \quad j = 1, 2, \ldots, K-1 \; ; \; i = 1, 2, \ldots, \sigma \; ,$$

and this choice of $\{g_i\}_{1 \leq i \leq \sigma}$ provides us with the desired $P(\{g_i\}_{1 \leq i \leq \sigma})$. □

Remark 5.3

For the application of the Wonham theorem here, it is important that the range of $P_u K_\gamma$ fills out all of X_u ; this is reformulated as the question whether the Neumann traces of the Dirichlet eigenfunctions in X_u are linearly independent. In that case the results are easy to formulate and allow N to be very low, otherwise the results become increasingly complicated and require (in general) higher N, the more linear dependence there is. Lasiecka & Triggiani have in [8] an upper and lower bound on the number N of feedback terms necessary in (5.18), once the number of linearly independent Neumann traces are given, but the discussion of the size of $\dim(\upsilon X_u)$ for differential operators on general domains in $\mathbb{R}^n$ in the litterature is far from complete, as far as we know. □

Let us now dispose of the temporary assumption $0 \notin \mathrm{sp}(A_\gamma)$. Everything done this far can without changes be done with the translated operator $A_\gamma - \delta$, $\delta > 0$, so if $0 \in \mathrm{sp}(A_\gamma)$, we take a $\delta > 0$ such that $0 \notin \mathrm{sp}(A_\gamma - \delta)$. Then after determining the new realization $A_T - \delta$ that moves the eigenvalues $\lambda_1 - \delta, \lambda_2 - \delta, \ldots, \lambda_{K-1} - \delta$, and leaves $\lambda_j - \delta$, $j \geq K$ unaffected, we just add δ to $A_T - \delta$. Then the moved eigenvalues will be increased with δ , and so will the unaffected eigenvalues, i.e. they will return to be the eigenvalues of A_s .

Let us now choose the set $\{c_j\}_{1 \leq j < K}$ occuring in Theorem 5.2 such that $c_j \geq \lambda_K \, (> 0)$, $j = 1, 2, \ldots, K - 1$.
With T' chosen according to the theorem, the operator $P_u(1 - K_\gamma T')$ is injective, hence bijective, on X_u , and since $w_j \in X_u$, $1 - K_\gamma T'$ is

the identity on X_s . Then, as promised in § 4 , $1 - K_\gamma T'$ is bijective from $H^{2m}(\Omega)$ to $H^{2m}(\Omega)$ ($K_\gamma T'$ has C^∞-range) and maps $D(A_T)$ onto $D(A_\gamma)$. This justifies the factorization (4.12).

Define $R(\lambda, \overline{A}_u)$ as the resolvent of $\overline{A}_u$ in X_u , for all $\lambda \notin \{c_j\}_{1 \leq j < K}$ and let $R(\lambda, A_s)$ be the resolvent of A_s in X_s , defined for all $\lambda \notin \{\lambda_j\}_{j \geq K}$.

We write the solution to (5.13) as

$$(5.41) \qquad u_u = R(\lambda, \overline{A}_u)f_u \ , \ \lambda \notin \{c_j\}_{1 \leq j < K}$$

Using that if $u \in D(A_T)$, then $v = (1 - K_\gamma T')u \in D(A_\gamma)$ satisfies

$$(5.42) \qquad v_s = P_s(1 - K_\gamma T')u = P_s(1 - K_\gamma T')(u_u + u_s)$$

$$= u_s - P_s K_\gamma T' u_u \in D(A_s)$$

and we write (5.14) as

$$(5.43) \qquad (A_s - \lambda)(u_s - P_s K_\gamma T' u_u) = f_s + \lambda P_s K_\gamma T' u_u \ .$$

Then, for all $\lambda \notin \{\lambda_j\}_{j \geq K}$:

$$(5.44) \qquad u_s - P_s K_\gamma T' u_u = R(\lambda, A_s)(f_s + \lambda P_s K_\gamma T' u_u) \ .$$

Inserting (5.41), we find for all $\lambda \notin (\{c_j\}_{1 \leq j < K} \cup \{\lambda_j\}_{j \geq K})$:

$$(5.45) \quad u_s = P_s K_\gamma T' R(\lambda, \overline{A}_u)f_u + R(\lambda, A_s)(f_s + \lambda P_s K_\gamma T' R(\lambda, \overline{A}_u)f_u)$$

so that

$$(5.46) \quad u = u_u + u_s$$
$$= R(\lambda, \overline{A}_u)f_u + P_s K_\gamma T' R(\lambda, \overline{A}_u)f_u + R(\lambda, A_s)f_s + \lambda R(\lambda, A_s)P_s K_\gamma T' R(\lambda, \overline{A}_u)f_u$$

$$= (1 + P_s K_\gamma T' + \lambda R(\lambda, A_s)P_s K_\gamma T')R(\lambda, \overline{A}_u)f_u + R(\lambda, A_s)f_s \ .$$

We see that we have

LEMMA 5.3

The resolvent $R(\lambda, A_T)$ solving (5.13) – (5.14) is

$$(5.47) \qquad R(\lambda, A_T)f = (R_{11} \ \ R_{12}) \begin{bmatrix} f_u \\ f_s \end{bmatrix}$$

where

$$(5.48) \qquad R_{11} = (1 + P_s K_\gamma T' + \lambda R(\lambda, A_s) P_s K_\gamma T') \ R(\lambda, \overline{A}_u)$$

$$R_{12} = R(\lambda, A_s)$$

so $R(\lambda, A_T)$ is well defined for all λ outside the spectrum $\{c_j\}_{1 \le j < K} \cup \{\lambda_j\}_{j \ge K}$ of A_T, and maps $L^2(\Omega)$ into $H^{2m}(\Omega)$. □

LEMMA 5.4

The resolvent $R(\lambda, A_T)$ satisfies the inequality

$$(5.49) \qquad ||R(\lambda, A_T)||_{L^2, L^2} \le \frac{c}{\text{dist}(\lambda, \text{co}(sp(A_T)))}$$

as an operator in $L^2(\Omega)$. Here $c > 0$ is a constant independent of λ, and $\text{co}(sp(A_T))$ is the convex hull of the spectrum of A_T.

Proof:

Note that $\text{co}(sp(A_T)) = [\lambda_K, \infty[$.

A_s is a selfadjoint, positive operator on $X_s \cap D(A_\gamma)$, satisfying:

$$||(A_s - \lambda)u||_o ||u||_o \geq |((A_s - \lambda)u|u)|$$

$$= |((A_s - \text{Re } \lambda)u|u) - i \text{ Im } \lambda ||u||_o^2|$$

$$= (((A_s - \text{Re } \lambda)u|u)^2 + (\text{Im } \lambda ||u||_o)^2)^{1/2}$$

$$\geq \begin{cases} |\text{Im } \lambda| \ ||u||_o^2 & \text{if } \text{Re } \lambda \geq \lambda_K \\ ((\lambda_K - \text{Re } \lambda)^2 + (\text{Im } \lambda)^2)^{1/2} \ ||u||_o^2 & \text{if } \text{Re } \lambda \leq \lambda_K \end{cases}$$

$$\geq \text{dist } (\lambda, \text{co}(\text{sp}(A_T))) \ ||u||_o^2 \quad \text{hence}$$

(5.50) $$||R(\lambda, A_s)||_{L^2, L^2} \leq (\text{dist } (\lambda, \text{co}(\text{sp}(A_T))))^{-1} \ .$$

$R(\lambda, \overline{A}_u)$ is a $(K - 1) \times (K - 1)$ matrix of the form

(5.51) $$R(\lambda, \overline{A}_u) = (\det(\overline{A}_u - \lambda)^{-1} p(\lambda, \overline{A}_u)$$

where $p(\lambda, \overline{A}_u)$ is a polynomium in λ (this follows easily from the inversion formula for matrices), and

(5.52) $$\det(\overline{A}_u - \lambda) = \text{constant} \cdot \prod_{1 \leq j < K} (\lambda - c_j) \ .$$

Therefore $||R(\lambda, \overline{A}_u)||_{L^2, L^2}$ is $O(|\lambda - c_j|^{-1})$ in a neighbourhood of $c_k \in \{c_j\}_{1 \leq j < K}$. For $|\lambda| \to \infty$ we write

(5.53) $$\overline{A}_u - \lambda = -\lambda(1 - \lambda^{-1}\overline{A}_u)$$

and we see that $R(\lambda, \overline{A}_u)$ is $O(|\lambda|^{-1})$ for $|\lambda| \to \infty$, since $(1 - \lambda^{-1}\overline{A}_u)$ can be inverted by a Neumann series for $|\lambda| > ||\overline{A}_u||_{L^2, L^2}$. We can then conclude that

(5.54) $$||R(\lambda, \overline{A}_u)||_{L^2, L^2} \leq M(\text{dist}(\lambda, \text{co}(\text{sp}(\overline{A}_u))))^{-1}$$

where M is a constant, independent of λ .

Obviosly $co(sp(\overline{A}_u)) \subset co(sp(A_T))$, and since $sp(\overline{A}_u)$ is bounded , $||\lambda R(\lambda,\overline{A}_u)||_{L^2,L^2}$ is $O(1)$ for $|\lambda| \to \infty$; moreover, $P_s K_\gamma T'$ is bounded and the decomposition $f \to f_u + f_s$ is bounded and fixed. Then, from the form of $R(\lambda,A_T)$ (5.47) - (5.48) , we find that

$$(5.55) \quad ||R(\lambda,A_T)||_{L^2,L^2} \leq \frac{M'}{\text{dist}(\lambda,co(sp(\overline{A}_u)))} + \frac{1}{\text{dist}(\lambda,co(sp(A_T)))}$$

$$\leq \frac{c}{\text{dist}(\lambda,co(sp(A_T)))} \quad . \quad \Box$$

Using Lemma 3.1, Theorem 5.2 and Lemma 5.4 we find

THEOREM 5.5

There exists a finite dimensional boundary condition

$$(5.56) \qquad\qquad \gamma u = T'u \quad \text{on } \Gamma$$

where

$$(5.57) \qquad\qquad T'u = \sum_{j=1}^{N} (u|w_j)g_j \quad .$$

$w_j \in X_u$, $g_j \in C^\infty(\Gamma)^m$, $j = 1, 2, \ldots, N$, such that the realization A_T of A , with domain

$$(5.58) \qquad D(A_T) = \{u \in H^{2m}(\Omega) | Tu = \gamma u - T'u = 0\}$$

is the infinitesimal generator of an analytic semigroup $e^{-A_T t}$, $t \geq 0$ on $L^2(\Omega)$, giving the solution to the Dirichlet boundary feedback parabolic system (5.3) as

$$(5.59) \qquad u(t,x) = e^{-A_T t} u_0(x) \quad , \quad x \in \Omega \quad , \quad t \geq 0 \quad ,$$

when $u_0 \in L^2(\Omega)$. The solution (5.59) satisfies the damping estimate

$$(5.60) \qquad ||u(t,\cdot)||_0 \leq M e^{-\lambda_K t} ||u_0||_0 , \ t \geq 0 , \ M > 0 ,$$

where λ_K is the first positive Dirichlet eigenvalue of A. Moreover, the operators

$$(5.61) \qquad \cos(A_T^{1/2}t) \quad \text{and} \quad \sin(A_T^{1/2}t)$$

are well defined, and we can write the solution to the hyperbolic problem (5.4) as

$$(5.62) \qquad u(t,x) = \cos(A_T^{1/2}t)u_0 + A_T^{-1/2} \sin(A_T^{1/2}t)u_1(x)$$

$x \in \Omega$, $t \in \mathbb{R}$, when $u_0, u_1 \in L^2(\Omega)$. $\square$

Remark 5.6

One of the slightly mysterious facts about the perturbation of the second kind, is that the operator A_T can <u>never</u> be semibounded (i.e. satisfy an inequality

$$(5.63) \qquad \text{Re } e^{i\theta}(A_T u | u) \geq c||u||_0^2$$

for some c and θ), when $T' \neq 0$. This is a consequence of Proposition 1.7.11 in Grubb [5], in particular, A_T can never be selfadjoint. Since A_T is never semibounded, the semigroup $e^{-A_T t}$, $t \geq 0$ is <u>never</u> a contraction semigroup, hence the constant M in (5.60) is always greater than 1. This is also noticed by Lasiecka & Triggiani, who consider the translated Laplacian in [8].

This phenomenon is not encountered in the Neumann feedback Problem. $\square$

Remark 5.7

Comparison of eqs. (5.59) and (4.23) show that for the semigroups we have

$$(5.64) \qquad e^{-A_T t} = (1 - K_\gamma T')^{-1} e^{-\tilde{A}_\gamma t} (1 - K_\gamma T').$$

This justifies the term "generalized change of coordinates" from § 2.□

Now it is straightforward to extend the theory to include more general cases where $P_s w_j \neq 0$. The operator T' considered above can be written

$$(5.65) \qquad T'u = \sum_{j=1}^{N} (u|P_u w_j)g_j \ .$$

so if we define the operator T'' as

$$(5.66) \qquad T''u = \sum_{j=1}^{N} (u|P_s w_j)g_j \ .$$

we see that the decoupled case considered above corresponds to the case where $T'' = 0$.

The operator T_1 , defined as

$$(5.67) \qquad T_1 u = \gamma u - T'u - T''u$$

defines a normal boundary condition $T_1 u = 0$ (in the sense of Grubb [5]), hence the operator realization A_{T_1} of A , with domain

$$(5.68) \qquad D(A_{T_1}) = \{u \in H^{2m}(\Omega) \,|\, T_1 u = 0\}$$

is a densely defined, closed operator in $L^2(\Omega)$. Here T_1 can be regarded as a perturbation of the trace operator $T = \gamma - T'$ since

$$(5.69) \qquad T_1 = \gamma - T' - T'' = T - T''$$

Let K_T be the Poisson solution operator defined by $u = K_T \varphi$, where u is the solution of

$$(5.70) \qquad \begin{cases} Au = 0 & \text{in } \Omega \\ Tu = \varphi & \text{on } \Gamma \end{cases}$$

We use here that A_T is made bijective, so K_T is well defined. Observe also the estimate

$$(5.71) \qquad ||K_T T''u||_0 = ||\sum_{j=1}^{N} (u|P_s w_j) K_T g_j||_0$$

$$\leq ||u||_0 \sum_{j=1}^{N} ||P_s w_j||_0 ||K_T g_j||_0 .$$

LEMMA 5.8

There exists a constant $r_1 > 0$, such that for $||P_s w_j||_0 < r_1$, $j = 1, 2, \ldots, N$ the operator

$$(5.72) \qquad 1 - K_T T''$$

is a homeomorphism in $H^{2m}(\Omega)$, and, in particular, defines a bijection

$$(5.73) \qquad 1 - K_T T'' : D(A_{T_1}) \overset{\sim}{\to} D(A_T) .$$

Proof:

Let $r_1 > 0$ be chosen, such that for $||P_s w_j||_0 < r_1$, $j = 1, 2, \ldots, N$, we have $||K_T T''||_{L^2,L^2} \leq \frac{1}{2}$. This is possible by (5.71). Now $1 - K_T T''$ is a bounded operator in $L^2(\Omega)$ and is inverted by a Neumann series

$$(5.74) \qquad (1 - K_T T'')^{-1} = \sum_{m=0}^{\infty} (K_T T'')^m .$$

converging in the operator norm in $L^2(\Omega)$.

Since K_T has range in $H^{2m}(\Omega)$, we see that $1 - K_T T''$ is a homeomorphism in $H^{2m}(\Omega)$, (see also § 4, where $1 - K_\gamma T$ was discussed) and, since

$$(5.75) \qquad T(1 - K_T T'')u = Tu - T''u = T_1 u ,$$

$1 - K_T T''$ defines a bijection from $D(A_{T_1})$ onto $D(A_T)$. $\square$

By Lemma 5.8 we have that if $u \in D(A_{T_1})$, then $v = (1 - K_T T'')u \in D(A_T)$, and $Au = Av$ in this case.

We wil now study the resolvent $R(\lambda, A_{T_1})$ of A_{T_1}, and we start out with the equation

$$(5.76) \qquad \begin{cases} (A - \lambda)u = f & \text{in } \Omega \\ T_1 u = 0 & \text{on } \Gamma \end{cases}$$

Using (5.73) with $v = (1 - K_T T'')u$ we get

$$(5.77) \qquad \begin{cases} (A - \lambda)(v + K_T T'' u) = f & \text{in } \Omega \\ Tv = 0 & \text{on } \Gamma \end{cases}$$

so that

$$(5.78) \qquad \begin{cases} (A - \lambda)v = f + \lambda K_T T'' u & \text{in } \Omega \\ Tv = 0 & \text{on } \Gamma \end{cases}$$

or

$$(5.79) \qquad \begin{cases} v = R(\lambda, A_T)(f + \lambda K_T T'' u) & \text{in } \Omega \\ Tv = 0 & \text{on } \Gamma \end{cases}$$

where the resolvent $R(\lambda, A_T)$ of A_T is used.

Then

$$(5.80) \qquad v = u - K_T T'' u = R(\lambda, A_T)(f + \lambda K_T T'' u)$$

so that

$$(5.81) \qquad (1 - (K_T T'' + \lambda R(\lambda, A_T) K_T T''))u = R(\lambda, A_T)f .$$

For $\epsilon > 0$, $0 < \theta < \frac{\pi}{2}$ we define the sector

$$W_{\lambda_K, \epsilon, \theta} = \{z \in \mathbb{C} | z = (\lambda_K - \epsilon) + re^{i\omega}, r \geq 0, \frac{\pi}{2} - \theta < \omega < \frac{3\pi}{2} + \theta\}$$

and from the estimates (5.49) and (5.71) we see that for any $\epsilon > 0$,

any $\theta \in]0,\frac{\pi}{2}[$, there exists a constant $r > 0$ such that for $w_j \in L^2(\Omega)$, satisfying $||P_s w_j||_0 < r \leq r_1$, $j = 1, 2, \ldots, N$, we have

$$(5.82) \qquad ||K_T T^{\cdot\cdot} + \lambda R(\lambda, A_T) K_T T^{\cdot\cdot}||_{L^2, L^2} \leq \frac{1}{2}$$

for all $\lambda \in W_{\lambda_K, \epsilon, \theta}$.

With the w_j , $j = 1, 2, \ldots, N$, chosen this way, the resolvent $R(\lambda, A_{T_1})$ of A_{T_1} is a well defined, bounded operator in $L^2(\Omega)$, given by (see (5.73))

$$(5.83) \qquad R(\lambda, A_T) = (1 - (K_T T^{\cdot\cdot} + \lambda R(\lambda, A_T) K_T T^{\cdot\cdot}))^{-1} R(\lambda, A_T)$$

$$= \sum_{m=0}^{\infty} (K_T T^{\cdot\cdot} + \lambda R(\lambda, A_T) K_T T^{\cdot\cdot})^m R(\lambda, A_T) ,$$

satisfying the estimate

$$(5.84) \qquad ||R(\lambda, A_{T_1})||_{L^2, L^2} \leq \frac{c_1}{|\lambda - \lambda_K|} ,$$

for all $\lambda \in W_{\lambda_K, \epsilon, \theta}$, where $c_1 > 0$ is a constant, depending only on ϵ and θ .

We have now

THEOREM 5.9

Let $\epsilon > 0$ be given, and assume that $P_u w_j$ and g_j , $j = 1, 2, \ldots, N$ satisfy the hypothesis of Theorem 5.5.

Then there exists a constant $r > 0$, such that for arbitrary choises of $P_s w_j$ with $||P_s w_j||_0 < r$, the finite dimensional Dirichlet boundary feedback condition

$$(5.85) \qquad \tau u = T'u + T^{\cdot\cdot}u = \sum_{j=1}^{N} (u|w_j)g_j$$

defines a realization A_{T_1} of A , with domain

(5.86) $D(A_{T_1}) = \{u \in H^{2m}(\Omega) \,|\, T_1 u = \gamma u - T'u - T''u = 0\}$

that is the infinitesimal generator of an analytic semigroup $e^{-A_{T_1} t}$, $t \geq 0$, on $L^2(\Omega)$, giving the solution to the Dirichlet boundary feedback control system

(5.87) $\begin{cases} \partial_t u + Au = 0 & \text{in } \Omega \text{ , for } t > 0 \\[2mm] \gamma u = \displaystyle\sum_{j=1}^{N} (u|w_j)g_j & \text{on } \Gamma \text{ , for } t > 0 \\[2mm] u = u_0 & \text{in } \Omega \text{ , at } t = 0 \text{ .} \end{cases}$

as

(5.88) $u(t,x) = e^{-A_{T_1} t} u_0(x)$, $t \geq 0$, $x \in \Omega$, $u_0 \in L^2(\Omega)$.

The solution (5.88) satisfies

(5.89) $||u(t,\cdot)||_0 \leq M' e^{-(\lambda_K - \epsilon)t} ||u_0||_0$, $t \geq 0$

where λ_K is the first positive Dirichlet eigenvalue of A , and M' is a constant, depending only on ϵ and θ ; θ defining the sector of analycity $W_{\lambda_K, \epsilon, \theta}$. $\square$

Remark 5.10
According to the proof of Lemma 5.7 , we have a factorization

(5.90) $A_{T_1} = A_T(1 - K_T T'') \quad \square$

We will now use the factorization (5.90) in the investigation of the hyperbolic problem for A_{T_1} .

The boundary value problem

(5.91) $\partial_t^2 u + A_{T_1} u = 0$, $u \in D(A_{T_1})$

transforms by (5,90) into

(5.92) $\qquad (1 - K_T T'')^{-1} \partial_t^2 v + A_T v = 0$, $v \in D(A_T)$

where $v = (1 - K_T T'')u$.

Acting with $(1 - K_T T'')$ from the left in (5.92) we find

(5.93) $\qquad \partial_t^2 v + (1 - K_T T'')A_T v = 0$, $v \in D(A_T)$

If the w_j are chosen such that $P_s w_j \in D(A_T^*)$, $j = 1, 2, \ldots, N$, then for $v \in D(A_T)$:

$$|| K_T T'' A_T v ||_o = || \sum_{j=1}^{N} (A_T v | P_s w_j) K_T g_j ||_o$$

$$= || \sum_{j=1}^{N} (v | A_T^* P_s w_j) K_T g_j ||_o$$

$$\leq || v ||_o \sum_{j=1}^{N} || A_T^* P_s w_j ||_o \, || K_T g_j ||_o$$

i.e. $K_T T'' A_T$ acts like a bounded operator on $D(A_T)$, for $P_s w_j \in D(A_T^*)$, $j = 1, 2, \ldots, N$.

Therefore, (5.93) and with it (5.91) is nothing but a <u>bounded</u> perturbation of the equation

(5.94) $\qquad \begin{cases} \partial_t^2 v + Av = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\ \quad\quad Tv = 0 & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\ \quad\quad\; v = v_0 & \text{in } \Omega \text{ , at } t = 0 \\ \quad \partial_t v = v_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases}$

treated in Theorem 5.5, in this case. Since the spectrum of A_T is assumed to be contained in $\mathbb{R}_+$, we find from standard bounded perturba-

tion theory (see e.g. Sova [12] and Fattorini [3]) that the operators

$$(5.95) \qquad \cos((A_T - K_T T''A_T)t)$$

$$(5.96) \qquad \sin((A_T - K_T T''A_T)t)$$

are well defined, bounded operators in $L^2(\Omega)$ for $P_s w_j \in D(A_T^*)$, $j = 1, 2, \ldots, N$, and $t \in \mathbb{R}$. From Theorem 1.6.11 in Grubb [5] we find that

$$(5.97) \qquad D(A_T^*) = \{u \in H^{2m}(\Omega) \mid \overset{\bullet}{I}{}^X \mathscr{A}^{01*} \gamma u = 0\}$$

where $\overset{\bullet}{I}{}^X$ is the "reflection" of the index set, replacing $\{k\}_{0 \leq k < m}$ by $\{2m - k - 1\}_{0 \leq k < m}$, and $\mathscr{A}^{01}$ is the $m \times m$ - matrix differential operator, appearing in Greens formula ((A.4) in the appendix) for A, it is invertible since A is elliptic. We see from ex. 1.6.12. in Grubb [5] that the action of the realization A_T^* is of the form $A + G$, for a certain singular Green operator G of finite rank. But the condition of being in $D(A_T^*)$ is then simply to be in $H^{2m}(\Omega) \cap H_0^m(\Omega)$, that is

$$(5.98) \qquad D(A_T^*) = D(A_\gamma) .$$

We have now

THEOREM 5.11

Let the set $\{w_j, g_j\}_{1 \leq j \leq N}$ be chosen according to Theorem 5.9, and assume furthermore that $P_s w_j$, $j = 1, 2, \ldots, N$, are chosen in $D(A_\gamma) = H^{2m}(\Omega) \cap H_0^m(\Omega)$.

Then the operators

$$(5.99) \qquad C(t) = \cos((A_T - K_T T''A_T)t) \quad \text{and}$$

$$(5.100) \qquad S(t) = (A_T - K_T T''A_T)^{-1/2} \sin((A_T - K_T T''A_T)^{1/2}t)$$

on $L^2(\Omega)$, are well defined for $t \in \mathbb{R}$, giving the solution to the system

$$(5.101) \begin{cases} \partial_t^2 v + (1 - K_T T'')Av = 0 & \text{, in } \Omega \text{ , for } t \in \mathbb{R} \\[2mm] \gamma v = \displaystyle\sum_{j=1}^{N} (v|P_u w_j)g_j & \text{, on } \Gamma \text{ , for } t \in \mathbb{R} \\[2mm] v = v_0 & \text{, in } \Omega \text{ , at } t = 0 \\[2mm] \partial_t v = v_1 & \text{, in } \Omega \text{ , at } t = 0 \end{cases}$$

as

$$(5.102) \quad v(t,x) = C(t)v_0(x) + S(t)v_1(x) \text{ , } x \in \Omega \text{ , } t \in \mathbb{R} \text{ , } v_0, v_1 \in L^2(\Omega) \text{ ,}$$

The solution to the original system

$$(5.103) \begin{cases} \partial_t^2 u + Au = 0 & \text{in } \Omega \text{ , for } t \in \mathbb{R} \\[2mm] \gamma u = \displaystyle\sum_{j=1}^{N} (u|w_j)g_j & \text{on } \Gamma \text{ , for } t \in \mathbb{R} \\[2mm] u = u_0 & \text{in } \Omega \text{ , at } t = 0 \\[2mm] \partial_t u = u_1 & \text{in } \Omega \text{ , at } t = 0 \end{cases}$$

is then by (5.90)

$$(5.104) \quad u(t,x) = (1 - K_T T'')^{-1} C(t)(1 - K_T T'')u_0(x) +$$

$$(1 - K_T T'')^{-1} S(t)(1 - K_T T'')u_1(x), \ x \in \Omega, \ t \in \mathbb{R}, \ u_0, u_1 \in L^2(\Omega) \ .\square$$

§ 6. STABILIZATION BY PERTURBATIONS OF THE THIRD KIND.

We want to stabilize the systems (1.1) and (1.2) by changing <u>both</u> the boundary condition and the system operator. The stabilized systems will then take the form

$$(6.1) \qquad \begin{cases} \partial_t u + Au + Gu = 0 & \text{in } \Omega, \text{ for } t \geq 0 \\ \gamma u = T'u & \text{on } \Gamma, \text{ for } t \geq 0 \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0 \end{cases}$$

and

$$(6.2) \qquad \begin{cases} \partial_t^2 u + Au + Gu = 0 & \text{in } \Omega, \text{ for } t \in \mathbb{R} \\ \gamma u = T'u & \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega, \text{ at } t = 0 \end{cases}$$

and we say that the systems (6.1) and (6.2) are associated with the realization B, where

$$(6.3) \qquad B = (A + G)_T$$

$$(6.4) \qquad D(B) = \{u \in H^{2m}(\Omega) \mid Tu = 0\}$$

$$(6.5) \qquad T = \gamma - T'.$$

We will now determine operators G and T', such that (6.1) and (6.2) are stable systems of feedback type. Moreover, we will make B a selfadjoint generator of an analytic <u>contraction</u> semigroup.

As in the preceeding paragraph, let $\{\varphi_j\}_{j \geq 1}$ be an orthonormalized set of eigenfunctions for A_γ, enumerated according to the ordering of the eigenvalues (5.5), but define now the boundary feedback operator T' as

$$(6.6) \qquad T'u = \sum_{j=1}^{K-1} (u|\varphi_j)h_j$$

where $K - 1$ is the number of negative eigenvalues of A_γ, repeated according to multiplicity, and

$$(6.7) \qquad\qquad h_j \in C^\infty(\Gamma)^m \ , \ j = 1, 2, \ldots, K - 1$$

are chosen linearly independent, with no other conditions on them. Moreover, define the operator G as

$$(6.8) \qquad\qquad G = K'\upsilon + G'$$

where υ is the Neumann trace operator (1.6) and

$$(6.9) \qquad\qquad \begin{cases} K' = -T'^* \mathscr{A}^{01} \\ G' = -T'^* (\mathscr{S} - c)\gamma \end{cases}$$

Here c is a positive constant to be determined later, $\mathscr{A}^{01}$ is the upper right corner in the coefficient matrix (A.3) in Greens formula (see the appendix), wheras $\mathscr{S}$ is the coefficient matrix appearing in the boundary term in the "halfways" Greens formula for a convenient sesquilinear form $a(u,v)$ associated with A (see (A.6)). Here we will point out that the following construction is not very "economic", as we typically use a large number of feedback terms, but the construction clarifies to a great extent the interaction System operator equation $\longleftrightarrow$ boundary equation.

Let $a(u,v)$ be chosen such that it is $H_0^m(\Omega)$-coercive, then it is well known how A_γ is the <u>variational operator</u> associated with the triple $(a, H_0^m(\Omega), L^2(\Omega))$ (see e.g. Grubb [5], § 1.7). Let us define the sesquilinear form $a_1(u,v)$ on $H^m(\Omega)$ by

$$(6.10) \qquad\qquad a_1(u,v) = a(u,v) + c(T'u|\gamma v)_\Gamma \ .$$

Let U be the space

$$(6.11) \qquad\qquad U = \{u \in H^m(\Omega) \,|\, \gamma u = T'u\} \ ,$$

and observe that U is dense in $L^2(\Omega)$, since $T = \gamma - T'$ defines a normal boundary condition $Tu = 0$.

Let B_1 be the operator associated with the triple $(a_1, U, L^2(\Omega))$, defined as follows

$$(6.12) \quad \begin{cases} D(B_1) = \{u \in U \mid \exists f \in L^2(\Omega) \text{ so that } a_1(u,v) = (f|v) \text{ for all } v \in U\} \\ B_1 u = f \end{cases}$$

We will show that $B = B_1$, where B is the realization defined in $(6.3) - (6.5)$.

LEMMA 6.1

$$(6.13) \qquad\qquad B \subseteq B_1$$

Proof:

For $u \in D(B)$, $v \in U$ we have (see appendix)

$$a_1(u,v) = a(u,v) + c(T'u|\gamma v)_\Gamma$$

$$= (Au|v) - (\mathscr{A}^{01}\nu u + \mathscr{S}\gamma u|\gamma v)_\Gamma + c(T'u|\gamma v)_\Gamma$$

$$= (Au|v) - (\mathscr{A}^{01}\nu u + \mathscr{S}\gamma u|T'v)_\Gamma + c(\gamma u|T'v)_\Gamma$$

$$= (Au - T'^*(\mathscr{S} - c)\gamma u - T'^* \mathscr{A}^{01}\nu u|v)$$

$$= ((A + G)u|v) \ .$$

This shows that $u \in D(B_1)$ with $B_1 u = (A + G)u$, so it follows that $D(B) \subset D(B_1)$ with $B_1 u = Bu$ there. □

For $\ell \in \mathbb{N}$ we define the subspace W_ℓ of $H_0^m(\Omega)$ by

$$(6.14) \qquad\qquad W_\ell = \overline{\operatorname{span}}^m \{\varphi_j\}_{j \geq \ell} \qquad (H^m(\Omega) - \text{closure})$$

LEMMA 6.2

There exists a linearly independent set of functions $\{v_j\}_{1 \leq j < K}$ in $U \backslash W_K$, such that

$$(6.15) \qquad\qquad U = \operatorname{span} \{v_j\}_{1 \leq j < K} \dotplus W_K \ .$$

Proof

According to Grubb [5], lemma 1.6.8 (see our § 3) we have

$$(6.16) \qquad U = \Lambda^{-1} H_0^m(\Omega)$$

where Λ and Λ^{-1} are bounded operators in $H^s(\Omega)$, for all $s \geq 0$.

Choose a linearly independent set $\{z_j\}_{1 \leq j < K}$ in $H_0^m(\Omega)$ such that the matrix

$$(6.17) \qquad C = ((z_i | (\Lambda^{-1})^* \varphi_j)_{i,j})_{1 \leq i, j < K}$$

is regular, and define

$$(6.18) \qquad v_j = \Lambda^{-1} z_j , \quad j = 1, 2, \ldots, K-1 ;$$

this is clearly a linearly independent set in U.

Moreover, for $1 \leq j < K$

$$(6.19) \qquad \gamma v_j = \gamma \Lambda^{-1} z_j = \sum_{k=1}^{K-1} (\Lambda^{-1} z_j | \varphi_k) h_k$$

$$= \sum_{k=1}^{K-1} (z_j | (\Lambda^{-1})^* \varphi_k) h_k \neq 0$$

since C is regular, so none of the v_j lie in $H_0^m(\Omega)$.

Since

$$(6.20) \qquad \gamma \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{K-1} \end{bmatrix} = C \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{K-1} \end{bmatrix}$$

the set $\{\gamma v_j\}_{1 \leq j < K}$ is linearly independent, so $\gamma(\sum_{j=1}^{K-1} \alpha_j v_j) = 0$ implies

that $\alpha_j = 0$, $j = 1, 2, \ldots, K-1$.

Hence $\operatorname{span}\{v_j\}_{1 \leq j < K} \bigcap W_K = \{0\}$.

Since $W_K \subseteq U$ and $\operatorname{span}\{v_j\}_{1 \leq j < K} \subseteq U$ the inclusion

$$(6.21) \qquad \operatorname{span}\{v_j\}_{1 \leq j < K} \dotplus W_K \subseteq U$$

is evident.

To show the inclusion the other way, let $u \in U$ and invert (6.20) to get

$$(6.22) \qquad \gamma u = \sum_{j=1}^{K-1} (u|\varphi_j) h_j = \sum_{j=1}^{K-1} ((u|\varphi_j) \sum_{k=1}^{K-1} \alpha_{jk} \gamma v_k) \ , \ \alpha_{jk} \in \mathbb{C} \ .$$

The last term equals $\sum_{k=1}^{K-1} \beta_k \gamma v_k$, where

$$(6.23) \qquad \beta_k = \sum_{j=1}^{K-1} (u|\varphi_j) \alpha_{jk} \ , \ k = 1, 2, \ldots, K-1 \ .$$

If we define

$$(6.24) \qquad w = u - \sum_{j=1}^{K-1} \beta_j v_j$$

then

$$(6.25) \qquad \gamma w = \gamma u - \sum_{j=1}^{K-1} \beta_j \gamma v_j = 0 \ .$$

hence

$$(6.26) \qquad 0 = \gamma w = \sum_{j=1}^{K-1} (w|\varphi_j) h_j \ .$$

But the set $\{h_j\}_{1 \le j < K}$ is linearly independent, so (6.26) implies that

$$(6.27) \qquad (w|\varphi_j) = 0 \ , \ j = 1, 2, \ldots, K-1 \ .$$

Then, from (6.24)

$$(6.28) \qquad u = \sum_{j=1}^{K-1} \beta_j v_j + w \in \text{span}\{v_j\}_{1 \le j < K} \overset{\circ}{+} W_K \ . \square$$

Since $\overline{W}_K^{\ o}$ (i.e. the $L^2(\Omega)$-closure) has a L^2-complement $(\overline{W}_K^{\ o})^\perp$ of dimension $K-1$, we can assume that the v_j are chosen outside $\overline{W}_K^{\ o}$. Then we have a decomposition of $u \in U$ as

$$(6.29) \qquad u = v + w$$

where $v \in \text{span}\{v_j\}_{1 \le j < K}$, $v_j \notin (\overline{W}_K^{\ o})$ $j = 1, 2, \ldots, K-1$, and $w \in W_K$, and the projections

$$(6.30) \qquad \begin{cases} u \to v \\ u \to w \end{cases}$$

are then continuous in $L^2(\Omega)$ so we have en inequality

$$(6.31) \qquad ||v||_o + ||w||_o \le C||u||_o$$

with a constant $C > 0$.

LEMMA 6.3

$$a(u,v) \text{ is } m - \text{coercive on } U$$

<u>Proof</u>:

Let $u \in U$ and write $u = v + w$ as above

Then

$$a(u,u) = a(w,w) + 2\text{Re } a(v,w) + a(v,v) .$$

For any $\epsilon > 0$ we can find constants $C_1, C_2 > 0$, such that

$$(6.32) \qquad 2|a(v,w)| \leq 2C_1 ||w||_m ||v||_m$$

$$\leq \epsilon^2 ||w||_m^2 + \frac{C_1^2}{\epsilon^2} ||v||_m^2$$

$$\leq \epsilon^2 ||w||_m^2 + \frac{C_2}{\epsilon^2} ||u||_0^2$$

because of (6.31) and the fact that the norms are equivalent on a finite dimensional space.

Since $a(u,v)$ is $H_0^m(\Omega)$-coercive, again using (6.31) we find

$$(6.33) \qquad a(w,w) \geq C_3 ||w||_m^2 - C_4 ||w||_0^2$$

$$\geq C_3 ||w||_m^2 - C_5 ||u||_0^2 .$$

with constants $C_3, C_4, C_5 > 0$.

Moreover, since

$$(6.34) \qquad a(v,v) \leq C_6 ||v||_m^2 \leq C_7 ||v||_0^2 \leq C_8 ||u||_0^2$$

with $C_6, C_7, C_8 > 0$, we have that

$$(6.35) \qquad a(u,u \geq (C_3 - \epsilon^2)||w||_m^2 - (\frac{C_2}{\epsilon^2} + C_5 + C_8)||u||_0^2$$

Since the norms of v are equivalent, we have positive constants C' and C'' , such that

(6.36)
$$C' \lVert v \rVert_0 \leq \lVert v \rVert_m \leq C'' \lVert v \rVert_0$$

so for all $\epsilon' > 0$:

(6.37)
$$\lVert w \rVert_m^2 \geq \lVert u \rVert_m^2 - 2 \lVert v \rVert_m \lVert u \rVert_m + \lVert v \rVert_m^2$$

$$\geq \lVert u \rVert_m^2 + C'^2 \lVert v \rVert_0^2 - \left(\frac{1}{\epsilon'^2} \lVert v \rVert_m^2 + \epsilon'^2 \lVert u \rVert_m^2 \right)$$

$$\geq (1 - \epsilon'^2) \lVert u \rVert_m^2 + \left(C'^2 - \frac{1}{\epsilon'^2} C''^2 \right) \lVert v \rVert_0^2$$

$$\geq (1 - \epsilon'^2) \lVert u \rVert_m^2 - \left(C'^2 - \frac{1}{\epsilon'^2} C''^2 \right) C \lVert u \rVert_0^2$$

Choosing $0 < \epsilon^2 < C_3$ and $0 < \epsilon' < 1$ and inserting (6.37) in (6.35), we get

(6.38)
$$a(u,u) \geq C_0 \lVert u \rVert_m^2 - k \lVert u \rVert_0^2$$

with constants C_0 and k , C_0 being positive.□

LEMMA 6.4

$$B_1 = B_1^*$$

Proof

Since for $u \in U$

(6.39) $a_1(u,u) = a(u,u) + c(Tu \mid \tau u)_\Gamma = a(u,u) + c(Tu \mid Tu)_\Gamma \geq a(u,u)$.

a_1 is m-coercive on U. Then B_1 is the variational operator associated with the triple $(a_1, U, L^2(\Omega))$, and since $a_1(u,v)$ is symmetric on U, $B_1 = B_1^*$. (See e.g. Grubb [5], § 1.7) □

Now we notice that the boundary value problem

(6.40)
$$\begin{cases} Au + Gu = f , & f \in L^2(\Omega) \\ Tu = 0 \end{cases}$$

is elliptic, since the Dirichlet problem for A

$$(6.41) \qquad \begin{cases} Au = f \ , \ f \in L^2(\Omega) \\ \gamma u = 0 \end{cases}$$

is elliptic and G has the form (6.8) – (6.9) where T'^* and T' obviously are integral operators with C^∞-kernels, hence of order $-\infty$. ((6.40) and (6.41) have the same principal symbols.) We can then use Theorem 1.6.11 and ex. 1.6.12 from Grubb [5] to show that $B = B^*$. It is shown there that when

$$(6.42) \qquad B = (A + G)_T$$

is an elliptic realization of a formally selfadjoint operator A, with

$$(6.43) \qquad \begin{cases} G = K'\upsilon + G' \\ T = \gamma - T' \end{cases}$$

then $B = B^*$ if

$$(6.44) \qquad \begin{cases} T' = -(\mathscr{A}^{01*})^{-1} \ K^{1*} \\ G' = G'^* + T'^* \ \mathscr{A}^{00*}\gamma \end{cases}$$

Using that $K' = -T'^* \ \mathscr{A}^{01}$ (see (6.9)) and that $\mathscr{A}^{00} = \mathscr{G} - \mathscr{G}^*$ (see appendix, Lemma (A.1)) we compute

$$(6.45) \qquad -(\mathscr{A}^{01*})^{-1} \ K'^* = (\mathscr{A}^{01*})^{-1} \ \mathscr{A}^{01*} \ T' = T'$$

and

$$(6.46) \qquad G'^* + T'^* \ \mathscr{A}^{00}\gamma = - \ T'^*(\mathscr{G}^*-c)\gamma + T'^*(\mathscr{G}^*-\mathscr{G})\gamma$$

$$= - \ T'^*(\mathscr{G}-c)\gamma$$

$$= G' \ ,$$

hence $B = B^*$, according to (6.44).

We have now

THEOREM 6.5

$$(6.47) \qquad\qquad B = B_1$$

Proof

$B \subseteq B_1$ implies that $B_1^* \subseteq B^*$. But $B = B^*$ and $B_1 = B_1^*$. $\square$

We have shown that the realization

$$(6.48) \qquad\qquad B = (A + G)_T$$

is the _variational_ operator associated with the triple $(a_1, U, L^2(\Omega))$.

THEOREM 6.6

Given $\zeta < 1$, there exists a constant $c > 0$ such that the sesquilinear form

$$(6.49) \qquad\qquad a_1(u,v) = a(u,v) + c(Tu|\tau v)_\Gamma$$

satisfies, for all $u \in U$:

$$(6.50) \qquad\qquad a_1(u,u) \geq \zeta \lambda_K ||u||_0^2 .$$

Here λ_K is the first positive Dirichlet eigenvalue of A.

Proof

We write $u \in U$ as $u = v + w$ as in (6.29).

Then

$$\frac{a_1(u,u)}{||u||_0^2} = \frac{a(v+w,v+w) + c(T(v+w)|\tau(v+w)_\Gamma}{(v+w|v+w)}$$

$$= \frac{a(v+w,v+w) + c(T(v+w)|T(v+w))_\Gamma}{(v+w|v+w)}$$

$$= \frac{a(w,w) + 2\mathrm{Re}\, a(v,w) + a(v,v) + c||Tv||_0^2}{||w||_0^2 + 2\mathrm{Re}\, (v|w) + ||v||_0^2}$$

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Now since

$$(6.51) \qquad a(w,w) \geq C_1 ||w||_m^2 - C_2 ||w||_0^2 \ , \ C_1, \ C_2 > 0 \ .$$

we have, for all $\epsilon > 0$:

$$(6.52) \qquad 2|a(v,w)| \leq \epsilon^2 ||w||_m^2 + \frac{C_3}{\epsilon^2} ||v||_m^2$$

$$\leq \epsilon^2 ||w||_m^2 + \frac{C_4}{\epsilon^2} ||v||_0^2$$

$$\leq \epsilon^2 (\frac{1}{C_1} a(w,w) + \frac{C_2}{C_1} ||w||_0^2) + \frac{C_4}{\epsilon^2} ||v||_0^2$$

$$\leq \epsilon^2 (\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1}) a(w,w) + \frac{C_4}{\epsilon^2} ||v||_0^2$$

with constants $C_3, \ C_4 > 0$.

Moreover, for all $\epsilon' > 0$:

$$(6.53) \qquad 2|(v|w)| \leq 2||v||_0 \ ||w||_0 \leq \epsilon'^2 ||w||_0^2 + \frac{1}{\epsilon'^2} ||v||_0^2$$

so we find that

$$(6.54) \qquad \frac{a_1(u,u)}{||u||_0^2} \geq \frac{(1 - \epsilon^2(\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1})) a(w,w) - (\frac{C_4}{\epsilon^2} + C_6) ||v||_0^2 + c ||Tv||_0^2}{(1 + \epsilon'^2) ||w||_0^2 + (1 + \frac{1}{\epsilon'^2}) ||v||_0^2}$$

where we also used that $|a(v,v)| \leq C_5 ||v||_m^2 \leq C_6 ||v||_0^2$. $C_5, \ C_6 > 0$.

Then

$$(6.55) \qquad \frac{a_1(u,u)}{||u||_0^2} \geq \frac{(1 - \epsilon^2(\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1})) \lambda_K ||w||_0^2 - (\frac{C_4}{\epsilon^2} + C_6) ||v||_0^2 + c ||Tv||_0^2}{(1 + \epsilon'^2) ||w||_0^2 + (1 + \frac{1}{\epsilon'^2}) ||v||_0^2}$$

T is injective from $\mathrm{span}\{v_j\}_{1 \leq j < K}$ to $\mathrm{span}\{h_j\}_{1 \leq j < K}$, so we have that

$$(6.56) \qquad ||Tv||_0^2 \geq C_7 ||v||_0^2 .$$

with a positive constant C_7.

Hence

$$(6.57) \qquad \frac{a_1(u,u)}{||u||_0^2} \geq \frac{\alpha\lambda_K ||w||_0^2 + \beta ||v||_0^2}{\mu ||w||_0^2 + \theta ||v||_0^2}$$

where

$$\alpha = 1 - \epsilon^2(\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1}) \;, \quad \beta = cC_7 - \frac{C_4}{\epsilon^2} - C_6$$

$$\mu = 1 + \epsilon'^2 \qquad\qquad , \quad \theta = 1 + \frac{1}{\epsilon'^2}$$

Considering the function

$$(6.58) \qquad f(s) = \frac{\alpha\lambda_K s + \beta}{\mu s + \theta} \;, \quad s > 0 .$$

which is decreasing for $\lambda_K \alpha\theta - \mu\beta < 0$, we see that

$$(6.59) \qquad \frac{a_1(u,u)}{||u||_0^2} \geq \frac{\alpha}{\mu} \lambda_K$$

if

$$(6.60) \qquad \beta > \frac{\alpha\theta}{\mu} \lambda_K .$$

so if

$$(6.61) \qquad c > \frac{(1 - \epsilon^2(\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1}))\lambda_K(1 + \frac{1}{\epsilon'^2})}{C_7(1 + \epsilon'^2)} + \frac{C_4}{C_7 \epsilon^2} + \frac{C_6}{C_7}$$

then

$$(6.62) \qquad a_1(u,u) \geq \frac{1 - \epsilon^2(\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1})}{1 + \epsilon'^2} \lambda_K \, ||u||_0^2 .$$

With ϵ and ϵ' chosen such that $\dfrac{1 - \epsilon^2(\frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1})}{1 + \epsilon'^2} = \zeta$ and c chosen such that (6.61) holds, we see that

$$(6.64) \qquad a_1(u,u) \geq \zeta\lambda_K ||u||_0^2$$

as claimed.□

We have now

THEOREM 6.7

Given any $\zeta < 1$, there exists a constant $c > 0$ such that the operator realization

$$(6.65) \qquad B = (A + G)_T \quad (\text{see } (6.3) - (6.5))$$

has its spectrum in the halfline $[\zeta\lambda_K, \infty[$, and is the infinitesimal gene-rator of an analytic semigroup e^{-Bt}, $t \geq 0$, on $L^2(\Omega)$, giving the solutions to the parabolic system

$$(6.66) \qquad \begin{cases} \partial_t u + Au + Gu = 0 & \text{in } \Omega, \text{ for } t > 0 \\ \gamma u = T'u & \text{on } \Gamma, \text{ for } t > 0 \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0 \end{cases}$$

as

$$(6.67) \qquad u(t,x) = e^{-Bt} u_0(x) , \quad x \in \Omega, \, t \geq 0 , \, u_0 \in L^2(\Omega) .$$

The solution satisfies

$$(6.68) \qquad ||u(t,\cdot)||_0 \leq e^{-\zeta\lambda_K t} ||u_0||_0 , \, t \geq 0 .$$

Moreover the operators

$$(6.69) \qquad \begin{cases} C(t) = \cos{(B^{1/2}t)} \\ S(t) = B^{-1/2} \sin{(B^{1/2}t)} \end{cases}$$

are well defined for $t \in \mathbb{R}$, giving the solution to the hyperbolic problem

$$(6.70) \qquad \begin{cases} \partial_t^2 u + Au + Gu = 0 & \text{in } \Omega, \text{ for } t \in \mathbb{R} \\ \gamma u = T'u & \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\ u = u_0 & \text{in } \Omega, \text{ at } t = 0 \\ \partial_t u = u_1 & \text{in } \Omega, \text{ at } t = 0 \end{cases}$$

as

$$(6.71) \qquad u(t,x) = C(t)u_0(x) + S(t)u_1(x), \quad x \in \Omega, \; t \in \mathbb{R},$$

when $u_0, u_1 \in L^2(\Omega)$. $\square$

Remark 6.8

In the light of Proposition 1.7.11 in Grubb [5], it is of interest to notice that the realization constructed above is weakly semibounded, because we compensate for the non-localness in the boundary condition by non-local terms in the system operator equation, satisfying

$$(6.72) \qquad K'^* = -\mathcal{A}^{01*} T'.$$

Moreover, we can notice that B is selfadjoint and is m-bounded, that is

$$(6.73) \qquad |(Bu|v)| \leq C_0 ||u||_m \, ||v||_m \quad \text{for } u, v \in D(B). \square$$

Example 6.9

Let us calculate the operator G in the case where $A = -\Delta$.

Since

$$(6.74) \qquad T'u = \sum_{j=1}^{K-1} (u|\varphi_j)h_j.$$

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we have that

$$(6.75) \qquad T'^* \psi = \sum_{j=1}^{K-1} (\psi | h_j)_\Gamma \varphi_j \ .$$

hence

$$(6.76) \qquad Gu = -T'^* \mathcal{A}^{01} \upsilon u - T'^*(\mathcal{G} - c)\gamma u$$

$$= - \sum_{j=1}^{K-1} (\mathcal{A}^{01} \upsilon u - (\mathcal{G} - c)\gamma u | h_j)_\Gamma \varphi_j \ .$$

Now the terms in Greens formula (see appendix (A.4)) are particularly simple since

$$(6.77) \qquad \mathcal{A} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ .$$

$$(6.78) \qquad (-\Delta u | v) - (u | -\Delta v) = i(\upsilon u | \gamma v)_\Gamma + i(\gamma u | \upsilon v)_\Gamma \ .$$

Then (6.76) reduces to

$$(6.79) \qquad Gu = - \sum_{j=1}^{K-1} (i\upsilon u + c\gamma u | h_j)_\Gamma \varphi_j \ . \square$$

APPENDIX: GREENS FORMULAS.

For the formally selfadjoint operator A (1,3) we have

$$(A.1) \qquad (Au|v) - (u|Av) = (\mathscr{A}\rho u|\rho v)_\Gamma$$

(see 1.7). Here $\mathscr{A}$ is a skew-triangular $2m \times 2m$ – matrix of differential operators over the boundary Γ, of the form

$$(A.2) \qquad \mathscr{A} = \mathscr{A}^0 + \mathscr{A}^1 = \begin{bmatrix} s_1^{\ 0} & \cdots & s_{2m-1}^{\ \ 0} & s_{2m}^{\ \ 0} \\ s_2^{\ 0} & \cdots & s_{2m}^{\ \ 0} & 0 \\ \vdots & & & \\ s_{2m}^{\ \ 0} & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} \text{lower} & & 0 \\ \text{order} & & \cdot & \cdot \\ & & \cdot & \cdot \\ 0 & \cdots & 0 \end{bmatrix}$$

where $s_k^{\ 0}$ is of order $2m - k$ (See Grubb [5], § 1.3).

We usually write $\mathscr{A}$ in $m \times m$ – blocks as

$$(A.3) \qquad \mathscr{A} = \begin{bmatrix} \mathscr{A}^{00} & \mathscr{A}^{01} \\ \mathscr{A}^{10} & 0 \end{bmatrix}$$

and we have the following version of Greens formula

$$(A.4) \qquad (Au|v) - (u|Av) = (\mathscr{A}^{01}\nu u + \mathscr{A}^{00}\tau u|\tau v)_\Gamma + (\mathscr{A}^{10}\tau u|\nu v)_\Gamma$$

for $u, v \in H^{2m}(\Omega)$.

Now consider a symmetric sesquilinear form

$$(A.5) \qquad a(u,v) = \sum_{|\alpha|, |\beta| \le m} (a_{\alpha\beta} D^\beta u|D^\alpha v)$$

associated with A. For such a form we have a "halfways" Greens formula, for $u \in H^{2m}(\Omega)$ and $v \in H^m(\Omega)$:

$$(A.6) \qquad (Au|v) - a(u,v) = (\mathscr{A}^{01}\nu u + \mathscr{S}\tau u|\tau v)_\Gamma$$

where the operator $\mathscr{S}$ is of the same type as $\mathscr{A}^{00}$ in (A.3). Since $\mathscr{A}^{01*} = \mathscr{A}^{10}$ when A is formally selfadjoint, we have

LEMMA A.1.

$$(A.7) \qquad\qquad \mathscr{A}^{00} = \mathscr{S} - \mathscr{S}^{*}$$

Proof.

For $u,v \in H^{2m}(\Omega)$ we have:

$$(Au|v) = a(u,v) + (\mathscr{A}^{01}\upsilon u + \mathscr{S}\gamma u|\gamma v)_{\Gamma}$$

$$(u|Av) = \overline{(Av|u)} = a(u,v) + \overline{(\mathscr{A}^{01}\upsilon u + \mathscr{S}\gamma v|\gamma u)_{\Gamma}}$$

$$= a(u,v) + (\mathscr{A}^{01*}\gamma u|\upsilon v)_{\Gamma} + (\mathscr{S}^{*}\gamma u|\gamma v)_{\Gamma}$$

so that

$$(A.8) \qquad (Au|v) - (u|Av) = (\mathscr{A}^{01}\upsilon u + (\mathscr{S}-\mathscr{S}^{*})\,\gamma u|\gamma v)_{\Gamma} + (\mathscr{A}^{10}\,\gamma u|\upsilon v)_{\Gamma}$$

Comparing (A.8) with (A.4) gives us (A.7).$\square$

Now let K_{γ} be the Poisson solution operator to the Dirichlet problem for A, i.e. $u = K_{\gamma}\psi$, where

$$(A.9) \qquad\qquad \begin{cases} Au = 0 & \text{in } \Omega \\ \gamma u = \psi & \text{on } \Gamma \end{cases}$$

We can then specify the action of K_{γ} the following way:

PROPOSITION A.2.

The Poisson solution operator K_{γ} to the Dirichlet problem for A satisfies

$$(A.10) \qquad\qquad (K_{\gamma}\psi|\varphi_{j}) = \frac{-1}{\lambda_{j}}\,(\psi|\mathscr{A}^{10*}\upsilon\varphi_{j})_{\Gamma}\,. \quad j \geq 1\,.$$

Proof.

Just insert $u = K_{\gamma}\psi$ and $v = \varphi_{j}$ (the eigenfunctions of A_{γ}) in (A.4), and use that $\gamma v = 0$. Moreover, $\lambda_{j} = \overline{\lambda}_{j}$ since A_{γ} is selfadjoint.$\square$

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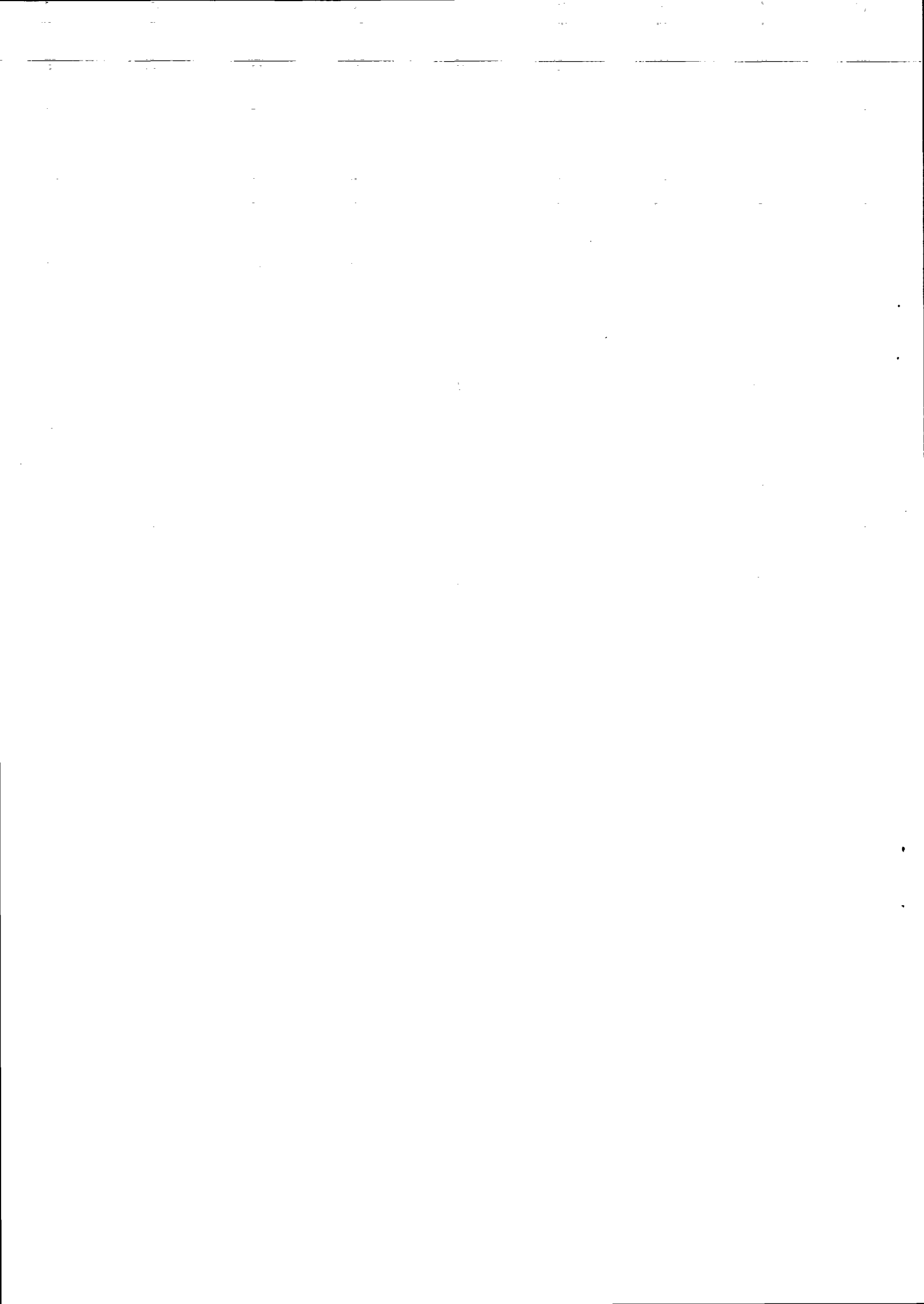
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1+11 ILLUSTRERET VED TO EKSEMPLER".
Projektrapport af: Torben O.Olsen, Lasse Rasmussen og
Niels Dreyer Sørensen.
Vejleder: Bent C. Jørgensen.

47/82 "BARSEBÄCK OG DET VÆRST OFFICIELT-TÆNKELIGE UHELD".
ENERGY SERIES NO. 5.
Af: Bent Sørensen.

48/82 "EN UNDERSØGELSE AF MATEMATIKUNDERVISNINGEN PÅ ADGANGS-
KURSUS TIL KØBENHAVNS TEKNIKUM".
Projektrapport af: Lis Eilertzen, Jørgen Karrebæk, Troels
Lange, Preben Nørregaard, Lissi Pedesen, Laust Rishøj,
Lill Røn og Isac Showiki.
Vejleder: Mogens Niss.

49/82 "ANALYSE AF MULTISPEKTRALE SATELLITBILLEDER".
Projektrapport af: Preben Nørregaard.
Vejledere: Jørgen Larsen og Rasmus Ole Rasmussen.

50/82 "HERSLEV - MULIGHEDER FOR VEDVARENDE ENERGI I EN
LANDSBY".
ENERGY SERIES NO. 6.
Rapport af: Bent Christensen, Bent Hove Jensen, Dennis
B. Møller, Bjarne Laursen, Bjarne Lillethorup og Jacob
Mørch Pedersen.
Vejleder: Bent Sørensen.

51/82 "HVAD KAN DER GØRES FOR AT AFHJÆLPE PIGERS BLOKERING
OVERFOR MATEMATIK ?"
Projektrapport af: Lis Eilertzen, Lissi Pedesen, Lill
Røn og Susanne Stender.

52/82 "DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS".
Af: Bernhelm Booss og Krzysztof Wojciechowski.

53/82 "THE CONSTITUTION OF SUBJECTS IN ENGINEERING
EDUCATION".
Af: Arne Jacobsen og Stig Andur Pedersen.

54/82 "FUTURES RESEARCH" - A Philosophical Analysis
of Its Subject-Matter and Methods.
Af: Stig Andur Pedersen og Johannes Witt-Hansen.

55/82 "MATEMATISKE MODELLER" - Litteratur på Roskilde
Universitetsbibliotek.
En biografi.
Af: Else Høyrup.

Vedr. tekst nr. 55/82 se også tekst nr. 62/83.

56/82 "EN - TO - MANGE" -
En undersøgelse af matematisk økologi.
Projektrapport af: Troels Lange.
Vejleder: Anders Madsen.

57/83 "ASPECT EKSPERIMENTET"-
Skjulte variable i kvantemekanikken?
Projektrapport af: Tom Juul Andersen.
Vejleder: Peder Voetmann Christiansen.
Nr. 57 er udgået.

58/83 "MATEMATISKE VANDRINGER" - Modelbetragtnin-
ger over spredning af dyr mellem småbiotoper
i agerlandet.
Projektrapport af: Per Hammershøj Jensen og
Lene Vagn Rasmussen.
Vejleder: Jørgen Larsen.

59/83 "THE METHODOLOGY OF ENERGY PLANNING".
ENERGY SERIES NO. 7.
Af: Bent Sørensen.

60/83 "MATEMATISK MODEKSPERTISE"- et eksempel.
Projektrapport af: Erik O. Gade, Jørgen Kar-
rebæk og Preben Nørregaard.
Vejleder: Anders Madsen.

61/83 "FYSIKS IDEOLOGISKE FUNKTION, SOM ET EKSEMPEL
PÅ EN NATURVIDENSKAB - HISTORISK SET".
Projektrapport af: Annette Post Nielsen.
Vejledere: Jens Høyrup, Jens Højgaard Jensen
og Jørgen Vogelius.

62/83 "MATEMATISKE MODELLER" - Litteratur på Roskilde
Universitetsbibliotek.
En biografi 2. rev. udgave.
Af: Else Høyrup.

63/83 "CREATING ENERGY FUTURES:A SHORT GUIDE TO ENER-
GY PLANNING".
ENERGY SERIES No. 8.
Af: David Crossley og Bent Sørensen.

64/83 "VON MATEMATIK UND KRIEG".
Af: Berhelm Booss og Jens Høyrup.

65/83 "ANVENDT MATEMATIK - TEORI ELLER PRAKSIS".
Projektrapport af: Per Hedegård Andersen, Kir-
sten Habekost, Carsten Holst-Jensen, Annelise
von Moos, Else Marie Pedersen og Erling Møller
Pedersen.
Vejledere: Bernhelm Booss og Klaus Grünbaum.

66/83 "MATEMATISKE MODELLER FOR PERIODISK SELEKTION
I ESCHERICHIA COLI".
Projektrapport af: Hanne Lisbet Andersen, Ole
Richard Jensen og Klavs Frisdahl.
Vejledere: Jørgen Larsen og Anders Hede Madsen.

67/83 "ELEPSOIDE METODEN - EN NY METODE TIL LINEÆR
PROGRAMMERING?"
Projektrapport af: Lone Biilmann og Lars Boye.
Vejleder: Mogens Brun Heefelt.

68/83 "STOKASTISKE MODELLER I POPULATIONSGENETIK"
- til kritikken af teoriladede modeller.
Projektrapport af: Lise Odgård Gade, Susanne
Hansen, Michael Hviid og Frank Mølgård Olsen.
Vejleder: Jørgen Larsen.

69/83 "ELEVFORUDSÆTNINGER I FYSIK"
  - en test i 1.g med kommentarer.

  Af: Albert C. Paulsen.

70/83 "INDLÆRINGS - OG FORMIDLINGSPROBLEMER I MATEMATIK
  PÅ VOKSENUNDERVISNINGSNIVEAU".
  Projektrapport af: Hanne Lisbet Andersen, Tor-
  ben J. Andreasen, Svend Åge Houmann, Helle Gle-
  rup Jensen, Keld Fl. Nielsen, Lene Vagn Ras-
  mussen.
  Vejleder: Klaus Grünbaum og Anders Hede Madsen.

71/83 "PIGER OG FYSIK"
  - et problem og en udfordring for skolen?
  Af: Karin Beyer, Sussanne Blegaa, Birthe Olsen,
  Jette Reich og Mette Vedelsby.

72/83 "VERDEN IFØLGE PEIRCE" - to metafysiske essays,
  om og af C.S Peirce.
  Af: Peder Voetmann Christiansen.

73/83 ""EN ENERGIANALYSE AF LANDBRUG"
  - økologisk contra traditionelt.
  ENERGY SERIES NO. 9
  Specialeopgave i fysik af: Bent Hove Jensen.
  Vejleder: Bent Sørensen.

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74/84 "MINIATURISERING AF MIKROELEKTRONIK" - om vi-
  denskabeliggjort teknologi og nytten af at lære
  fysik.
  Projektrapport af: Bodil Harder og Linda Szko-
  tak Jensen.
  Vejledere: Jens Højgaard Jensen og Bent C. Jørgensen.

75/84 "MATEMATIKUNDERVISNINGEN I FREMTIDENS GYMNASIUM"
  - Case: Lineær programmering.
  Projektrapport af: Morten Blomhøj, Klavs Frisdahl
  og Frank Mølgaard Olsen.
  Vejledere: Mogens Brun Heefelt og Jens Bjørneboe.

76/84 "KERNEKRAFT I DANMARK?" - Et høringssvar indkaldt
  af miljøministeriet, med kritik af miljøstyrelsens
  rapporter af 15. marts 1984.
  ENERGY SERIES No. 1o
  Af: Niels Boye Olsen og Bent Sørensen.

77/84 "POLITISKE INDEKS - FUP ELLER FAKTA?"
  Opinionsundersøgelser belyst ved statistiske
  modeller.
  Projektrapport af: Svend Åge Houmann, Keld Nielsen
  og Susanne Stender.
  Vejledere: Jørgen Larsen og Jens Bjørneboe.

78/84 "JÆVNSTRØMSLEDNINGSEVNE OG GITTERSTRUKTUT I
  AMORFT GERMANIUM".
  Specialrapport af: Hans Hedal, Frank C. Ludvigsen
  og Finn C. Physant.
  Vejleder: Niels Boye Olsen.

79/84 "MATEMATIK OG ALMENDANNELSE".
  Projektrapport af: Henrik Coster, Mikael Wenner-
  berg Johansen, Povl Kattler, Birgitte Lydholm
  og Morten Overgaard Nielsen.
  Vejleder: Bernhelm Booss.

80/84 "KURSUSMATERIALE TIL MATEMATIK B".
  Af: Mogens Brun Heefelt.

81/84 "FREKVENSAFHÆNGIG LEDNINGSEVNE I AMORFT GERMANIUM".
  Specialerapport af: Jørgen Wind Petersen og Jan
  Christensen.
  Vejleder: Niels Boye Olsen.

82/84 "MATEMATIK - OG FYSIKUNDERVISNINGEN I DET AUTO -
  MATISEREDE SAMFUND".
  Rapport fra et seminar afholdt i Hvidovre
  25-27 april 1983.
  Red.: Jens Højgaard Jensen, Bent C. Jørgensen
  og Mogens Niss.

83/84 "ON THE QUANTIFICATION OF SECURITY":
  PEACE RESEARCH SERIES NO. 1
  Af: Bent Sørensen
  nr. 83 er p.t. udgået

84/84 "NOGLE ARTIKLER OM MATEMATIK, FYSIK OG ALMENDANNELSE".
  Af: Jens Højgaard Jensen, Mogens Niss m. fl.

85/84 "CENTRIFUGALREGULATORER OG MATEMATIK".
  Specialerapport af: Per Hedegård Andersen, Carsten Holst-
  Jensen, Else Marie Pedersen og Erling Møller Pedersen.
  Vejleder: Stig Andur Pedersen.

86/84 "SECURITY IMPLICATIONS OF ALTERNATIVE DEFENSE OPTIONS
  FOR WESTERN EUROPE".
  PEACE RESEARCH SERIES NO. 2
  Af: Bent Sørensen.

87/84 "A SIMPLE MODEL OF AC HOPPING CONDUCTIVITY IN DISORDERED
  SOLIDS".
  Af: Jeppe C. Dyre.

88/84 "RISE, FALL AND RESURRECTION OF INFINITESIMALS".
  Af: Detlef Laugwitz.

89/84 "FJERNVARMEOPTIMERING".
  Af: Bjarne Lillethorup og Jacob Mørch Pedersen.

90/84 "ENERGI I 1.G - EN TEORI FOR TILRETTELÆGGELSE".
  Af: Albert Chr. Paulsen.

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91/85 "KVANTETEORI FOR GYMNASIET".
  1. Lærervejledning
  Projektrapport af: Biger Lundgren, Henning Sten Hansen
  og John Johansson.
  Vejleder: Torsten Meyer.

92/85 "KVANTETEORI FOR GYMNASIET".
  2. Materiale
  Projektrapport af: Biger Lundgren, Henning Sten Hansen
  og John Johansson.
  Vejleder: Torsten Meyer.

93/85 "THE SEMIOTICS OF QUANTUM - NON - LOCALITY".
  Af: Peder Voetmann Christiansen.

94/85 "TREENIGHEDEN BOURBAKI - generalen, matematikeren
  og ånden".
  Projektrapport af: Morten Blomhøj, Klavs Frisdahl
  og Frank M. Olsen.
  Vejleder: Mogens Niss.

95/85 "AN ALTERNATIV DEFENSE PLAN FOR WESTERN EUROPE".
  PEACE RESEARCH SERIES NO. 3
  Af: Bent Sørensen

96/85 "ASPEKTER VED KRAFTVARMEFORSYNING".
  Af: Bjarne Lilletorup.
  Vejleder: Bent Sørensen.

97/85 "ON THE PHYSICS OF A.C. HOPPING CONDUCTIVITY".
  Af: Jeppe C. Dyre.

98/85 "VALGMULIGHEDER I INFORMATIONSALDEREN".
  Af: Bent Sørensen.

99/85 "Der er langt fra Q til R".
  Projektrapport af: Niels Jørgensen og Mikael Klintorp.
  Vejleder: Stig Andur Pedersen.

100/85 "TALSYSTEMETS OPBYGNING".
  Af: Mogens Niss.

101/85 "EXTENDED MOMENTUM THEORY FOR WINDMILLS IN
  PERTURBATIVE FORM".
  Af: Ganesh Sengupta.

102/85 OPSTILLING OG ANALYSE AF MATEMATISKE MODELLER, BELYST
  VED MODELLER OVER KØERS FODEROPTAGELSE OG - OMSÆTNING".
  Projektrapport af: Lis Eilertzen, Kirsten Habekost, Lill Røn
  og Susanne Stender.
  Vejleder: Klaus Grünbaum.

103/85 "ØDSLE KOLDKRIGERE OG VIDENSKABENS LYSE IDEER".
        Projektrapport af: Niels Ole Dam og Kurt Jensen.
        Vejleder: Bent Sørensen.

104/85 "ANALOGREGNEMASKINEN OG LORENZLIGNINGER".
        Af: Jens Jæger.

105/85 "THE FREQUENCY DEPENDENCE OF THE SPECIFIC HEAT AF THE
        GLASS REANSITION".
        Af: Tage Christensen.

        "A SIMPLE MODEL AF AC HOPPING CONDUCTIVITY".
        Af: Jeppe C. Dyre.
        Contributions to the Third International Conference
        on the Structure of Non - Crystalline Materials held
        in Grenoble July 1985.

106/85 "QUANTUM THEORY OF EXTENDED PARTICLES".
        Af: Bent Sørensen.

107/85 "EN MYG GØR INGEN EPIDEMI".
        - flodblindhed som eksempel på matematisk modelle-
        ring af et epidemiologisk problem.
        Projektrapport af: Per Hedegård Andersen, Lars Boye,
        CarstenHolst Jensen, Else Marie Pedersen og Erling
        Møller Pedersen.
        Vejleder: Jesper Larsen.

108/85 "APPLICATIONS AND MODELLING IN THE MATEMATICS CUR -
        RICULUM" - state and trends -
        Af: Mogens Niss.

109/85 "COX I STUDIETIDEN" - Cox's regressionsmodel anvendt på
        studenteroplysninger fra RUC.
        Projektrapport af: Mikael Wennerberg Johansen, Poul Kat-
        ler og Torben J. Andreasen.
        Vejleder: Jørgen Larsen.

110/85 "PLANNING FOR SECURITY".
        Af: Bent Sørensen

111/85 JORDEN RUNDT PÅ FLADE KORT".
        Projektrapport af: Birgit Andresen, Beatriz Quinones
        og Jimmy Staal.
        Vejleder: Mogens Niss.

112/85 "VIDENSKABELIGGØRELSE AF DANSK TEKNOLOGISK INNOVATION
        FREM TIL 1950 - BELYST VED EKSEMPLER".
        Projektrapport af: Erik Odgaard Gade, Hans Hedal,
        Frank C. Ludvigsen, Annette Post Nielsen og Finn
        Physant.
        Vejleder: Claus Bryld og Bent C. Jørgensen.

113/85 "DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS 11".
        Af: Bernhelm Booss og Krzysztof Wojciechowski.

114/85 "ANVENDELSE AF GRAFISKE METODER TIL ANALYSE
        AF KONTIGENSTABELLER".
        Projektrapport af: Lone Biilmann, Ole R. Jensen
        og Anne-Lise von Moos.
        Vejleder: Jørgen Larsen.

115/85 "MATEMATIKKENS UDVIKLING OP TIL RENÆSSANCEN".
        Af: Mogens Niss.

116/85 "A PHENOMENOLOGICAL MODEL FOR THE MEYER-
        NELDEL RULE".
        Af: Jeppe C. Dyre.

117/85 "KRAFT & FJERNVARMEOPTIMERING"
        Af: Jacob Mørch Pedersen.
        Vejleder: Bent Sørensen

118/85 "TILFÆLDIGHEDEN OG NØDVENDIGHEDEN IFØLGE
        PEIRCE OG FYSIKKEN".
        Af: Peder Voetmann Christiansen

119/86 "DET ER GANSKE VIST - - EUKLIDS FEMTE POSTULAT
        KUNNE NOK SKABE RØRE I ANDEDAMMEN".
        Af: Iben Maj Christiansen
        Vejleder: Mogens Niss.

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        Af: Jørgen Larsen

121/86 "SIMULATION I KONTINUERT TID".
        Af: Peder Voetmann Christiansen.

122/86 "ON THE MECHANISM OF GLASS IONIC CONDUCTIVITY".
        Af: Jeppe C. Dyre.

123/86 "GYMNASIEFYSIKKEN OG DEN STORE VERDEN".
        Fysiklærerforeningen, IMFUFA, RUC.

124/86 "OPGAVESAMLING I MATEMATIK".
        Samtlige opgaver stillet i tiden 1974-jan. 1986.

125/86 "UVBYβ - systemet - en effektiv fotometrisk spektral-
        klassifikation af B-,A- og F-stjerner".
        Projektrapport af: Birger Lundgren.

126/86 "OM UDVIKLINGEN AF DEN SPECIELLE RELATIVITETSTEORI".
        Projektrapport af: Lise Odgaard & Linda Szkotak Jensen
        Vejledere: Karin Beyer & Stig Andur Pedersen.

127/86 "GALOIS' BIDRAG TIL UDVIKLINGEN AF DEN ABSTRAKTE
        ALGEBRA".
        Projektrapport af: Pernille Sand, Heine Larsen &
        Lars Frandsen.
        Vejleder: Mogens Niss.

128/86 "SMÅKRYB" - om ikke-standard analyse.
        Projektrapport af: Niels Jørgensen & Mikael Klintorp.
        Vejleder: Jeppe Dyre.

129/86 "PHYSICS IN SOCIETY"
        Lecture Notes 1983 (1986)
        Af: Bent Sørensen

130/86 "Studies in Wind Power"
        Af: Bent Sørensen

131/86 "FYSIK OG SAMFUND" - Et integreret fysik/historie-
        projekt om naturanskuelsens historiske udvikling
        og dens samfundsmæssige betingethed.
        Projektrapport af: Jakob Heckscher, Søren Brønd,
        Andy Wierød.
        Vejledere: Jens Høyrup, Jørgen Vogelius,
        Jens Højgaard Jensen.

132/86 "FYSIK OG DANNELSE"
        Projektrapport af: Søren Brønd, Andy Wierød.
        Vejledere: Karin Beyer, Jørgen Vogelius.

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        ENERGY SERIES NO. 15.
        AF: Bent Sørensen.

134/87 "THE D.C. AND THE A.C. ELECTRICAL TRANSPORT IN AsSeTe SYSTEM"
        Authors: M.B.El-Den, N.B.Olsen, Ib Høst Pedersen,
                  Petr Viscör

135/87 "INTUITIONISTISK MATEMATIKS METODER OG ERKENDELSES-
        TEORETISKE FORUDSÆTNINGER"
        MASTEMATIKSPECIALE: Claus Larsen
        Vejledere: Anton Jensen og Stig Andur Pedersen

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        første og andet møde med græsk filosofi"
        Projektrapport af Frank Colding Ludvigsen
        Vejledere: Historie: Ib Thiersen
                   Fysik: Jens Højgaard Jensen

137/87 "HOPMODELLER FOR ELEKTRISK LEDNING I UORDNEDE
        FASTE STOFFER" - Resume af licentiatafhandling
        Af: Jeppe Dyre
        Vejledere: Niels Boye Olsen og
                   Peder Voetmann Christiansen.

138/87 "JOSEPHSON EFFECT AND CIRCLE MAP."
Paper presented at The International .
Workshop on Teaching Nonlinear Phenomena
at Universities and Schools, "Chaos in
Education". Balaton, Hungary, 26 April-2 May 1987.

By: Peder Voetmann Christiansen

13 9/87 "Machbarkeit nichtbeherrschbarer Technik
durch Fortschritte in der Erkennbarkeit
der Natur"

Af: Bernhelm Booss-Bavnbek
Martin Bohle-Carbonell

140/87 "ON THE TOPOLOGY OF SPACES OF HOLOMORPHIC MAPS"

By: Jens Gravesen

141/87 "RADIOMETERS UDVIKLING AF BLODGASAPPARATUR –
ET TEKNOLOGIHISTORISK PROJEKT"

Projektrapport af Finn C. Physant
Vejleder: Ib Thiersen

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Splitting Elliptic Symbols"

by: Bernhelm Booss-Bavnbek og
Krzysztof P. Wojciechowski

143/87 "Kursusmateriale til Matematik på NAT-BAS"

af: Mogens Brun Heefelt

144/87 "Context and Non-Locality – A Peircan Approach
Paper presented at the Symposium on the
Foundations of Modern Physics The Copenhagen
Interpretation 60 Years after the Como Lecture.
Joensuu, Finland, 6 – 8 august 1987.

By: Peder Voetmann Christiansen

145/87 "AIMS AND SCOPE OF APPLICATIONS AND
MODELLING IN MATHEMATICS CURRICULA"
Manuscript of a plenary lecture delivered at
ICMTA 3, Kassel, FRG 8.-11.9.1987
By: Mogens Niss

146/87 "BESTEMMELSE AF BULKRESISTIVITETEN I SILICIUM"
– en ny frekvensbaseret målemetode.
Fysikspeciale af Jan Vedde
Vejledere: Niels Boye Olsen & Petr Viščor

147/87 "Rapport om BIS på NAT-BAS"
redigeret af: Mogens Brun Heefelt

148/87 "Naturvidenskabsundervisning med
Samfundsperspektiv"
af: Peter Colding-Jørgensen DLH
Albert Chr. Paulsen

149/87 "In-Situ Measurements of the density of amorphous
germanium prepared in ultra high vacuum"
by: Petr Viščor

150/87 "Structure and the Existence of the first sharp
diffraction peak in amorphous germanium
prepared in UHV and measured in-situ"
by: Petr Viščor

151/87 "DYNAMISK PROGRAMMERING"
Matematikprojekt af:
Birgit Andresen, Keld Nielsen og Jimmy Staal
Vejleder: Mogens Niss

152/87 "PSEUDO-DIFFERENTIAL PROJECTIONS AND THE TOPOLOGY
OF CERTAIN SPACES OF ELLIPTIC BOUNDARY VALUE
PROBLEMS"

by: Bernhelm Booss-Bavnbek
Krzysztof P. Wojciechowski

153/88 "HALVLEDERTEKNOLOGIENS UDVIKLING MELLEM MILITÆRE
OG CIVILE KRÆFTER"
Et eksempel på humanistisk teknologihistorie
Historiespeciale
Af: Hans Hedal
Vejleder: Ib Thiersen

154/88 "MASTER EQUATION APPROACH TO VISCOUS LIQUIDS AND
THE GLASS TRANSITION"

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OPERATOR TO THE DIRICHLET PROBLEM FOR A FORMALLY
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BY FINITE DIMENSIONAL BOUNDARY FEEDBACK CONTROL:
A pseudo-differential approach."

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RANDOM WALK MODELS"
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