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# ON THE TOPOLOGY
# OF
# SPACES OF HOLOMORPHIC MAPS

by

Jens Gravesen

Thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Oxford.
Trinity Term, 1987.

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## Abstract

In the thesis the space of based holomorphic maps, between a Riemann surface and a generalized flag manifold or a loop group, is compared with the corresponding space of continuous maps.

Both spaces have connected components labeled by a (multi-) degree, and if the Riemann surface is the sphere, then it is shown that the two spaces have the same homology in the limit where the degree tends to infinity.

The main idea is a generalization of the concept of a principal part of a meromorphic functions to a principal part of a holomorphic map into a flag manifold or a loop group. Then the space of holomorphic maps can be replaced with a configuration space of principal parts, which makes it possible to apply standard techniques in the proofs.

The space of holomorphic maps from $\mathbf{CP}^1$ to a loop group $\Omega G$ can be identified with a moduli space of holomorphic $G_{\mathbf{C}}$-bundles over $\mathbf{CP}^1 \times \mathbf{CP}^1$. When $\mathbf{CP}^1$ is replaced by a general Riemann surface $X$, then the space of holomorphic maps from $X$ to $\Omega G$ is only a subset of the moduli space of bundles over $X \times \mathbf{CP}^1$, but the space of configurations of principal parts can be identified with the full moduli space. Again it is shown that, when the degree tends to infinity, the homology of the moduli space tends to the homology of the space of continuous maps from $X$ to $\Omega G$.

Geometrical Methods In Quantum Field Theory

# ON THE TOPOLOGY OF SPACES OF HOLOMORPHIC MAPS

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## Contents



## 1. Introduction

Let $X$ and $Y$ be two complex manifolds and form the two spaces $\mathrm{Hol}(X,Y)$ of holomorphic maps $X \to Y$ and $\mathrm{Map}(X,Y)$ of continuous maps $X \to Y$, both equipped with the compact-open topology.

We will study the inclusion

$$\mathrm{Hol}(X,Y) \hookrightarrow \mathrm{Map}(X,Y)$$

in the case, where $X$ is a Riemann surface and $Y$ is a generalized flag manifold or a loop group.

Let $\mathrm{Hol}_n^*(X,Y)$ and $\mathrm{Map}_n^*(X,Y)$ denote the spaces of based maps of degree $n$. In [**18**] G. Segal showed that the inclusion

$$\mathrm{Hol}_n^*(X,\mathbf{CP}^m) \hookrightarrow \mathrm{Map}_n^*(X,\mathbf{CP}^m)$$

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is a homology equivalence up to dimension $(n-2g)(2m-1)$, where $g$ is the genus of $X$. If $X = \mathbf{CP}^1$, it is even a homotopy equivalence up to dimension $n(2m-1)$.

Segal conjectured that a similar statement holds, if $\mathbf{CP}^m$ is replaced by a flag manifold or a Grassmannian. By using Segal's result and induction arguments this was confirmed in the case of a flag manifold by M. A. Guest in [6] and in the case of a Grassmannian by F. C. Kirwan in [8].

If $G$ is a compact Lie group, then the loop group $\Omega G$ of smooth based maps $S^1 \to G$ has many properties similar to a Grassmannian, see [13], [14] and [15]. So it is natural to try to extend Segal's result to the inclusion

$$\mathrm{Hol}_n^*(X, \Omega G) \hookrightarrow \mathrm{Map}_n^*(X, \Omega G),$$

and this is indeed the purpose of this work.

In [1] M. F. Atiyah describes how a holomorphic map $X \to \Omega G$ gives rise to a holomorphic $G_{\mathbf{C}}$-bundle over $X \times \mathbf{CP}^1$, where $G_{\mathbf{C}}$ is the complexification of $G$. To be more precise, let $\mathcal{M}_n(X \times \mathbf{CP}^1, X \vee \mathbf{CP}^1, G_{\mathbf{C}})$ be the space of based isomorphism classes of holomorphic $G_{\mathbf{C}}$-bundles over $X \times \mathbf{CP}^1$, trivial over the axis $X \vee \mathbf{CP}^1$ and with characteristic class $n$. Then there is an imbedding

$$\mathrm{Hol}_n^*(X, \Omega G) \hookrightarrow \mathcal{M}_n(X \times \mathbf{CP}^1, X \vee \mathbf{CP}^1, G_{\mathbf{C}}),$$

and if $X = \mathbf{CP}^1$, then it is a diffeomorphism onto an open subset and it is homotopic to a diffeomorphism onto all of $\mathcal{M}_n(\mathbf{CP}^1 \times \mathbf{CP}^1, \mathbf{CP}^1 \vee \mathbf{CP}^1, G_{\mathbf{C}})$.

The main result (theorem 7.8) is that

$$\lim_{n\to\infty} H_*(\mathcal{M}_n(X \times \mathbf{CP}^1, X \vee \mathbf{CP}^1, G_{\mathbf{C}})) = H_*(\mathrm{Map}_0^*(X, \Omega G)).$$

If $X = \mathbf{CP}^1$, then $\mathrm{Hol}_n^*(\mathbf{CP}^1, \Omega G) \hookrightarrow \mathcal{M}_n(\mathbf{CP}^1 \times \mathbf{CP}^1, \mathbf{CP}^1 \vee \mathbf{CP}^1, G_{\mathbf{C}})$ is a homotopy equivalence and as the methods work equally well for a generalized flag manifold,

$$\lim_{n\to\infty} H_*(\mathrm{Hol}_n^*(\mathbf{CP}^1, Y)) = H_*(\mathrm{Map}_0^*(\mathbf{CP}^1, Y))$$

with $Y$ a generalized flag manifold or a loop group. The degree $n$ might be a multi-index $n = (n_1, \ldots, n_r)$ and then $n \to \infty$ means $n_i \to \infty$ all $i = 1, \ldots, r$.

Segal's results on projective spaces are stronger. In particular, in each dimension $q$ the limit $\lim_{n\to\infty} H_q(\mathrm{Hol}_n^*(X, \mathbf{CP}^1))$ is obtained after a finite number of steps. If this result on projective spaces could be proved in the framework of this paper, then the analogous result for loop groups would probably hold.

There is one result in this direction as the induced map

$$\pi_0(\mathrm{Hol}^*(\mathbf{CP}^1, Y)) \to \pi_0(\mathrm{Map}(\mathbf{CP}^1, Y))$$

is an injection. By the connection with vector bundles this gives yet a proof of the connectivety of certain moduli spaces in algebraic geometry, see [3] and it's references.

If $D$ is the open unit disk in C, then the inclusion $\mathrm{Hol}(D,Y) \hookrightarrow \mathrm{Map}(D,Y)$ is a homotopy equivalence. In fact, if $f_t(z) = f(tz)$, then $(t,z) \mapsto f_t$, $t \in [0,1]$, defines a contraction of both spaces onto $Y$ considered as the space of constant maps. As a surface $X$ can be made by gluing disks together, one could hope to prove that the inclusion $\mathrm{Hol}(X,Y) \hookrightarrow \mathrm{Map}(X,Y)$ is a homotopy equivalence by an induction argument. The principle is very simple. If $X$ is the union of two subsets $X_1$ and $X_2$, then the restrictions give cartesian diagrams

$$\begin{array}{ccccccc}
\mathrm{Hol}(X_1 \cup X_2, Y) & \longrightarrow & \mathrm{Hol}(X_1, Y) & & \mathrm{Map}(X_1 \cup X_2, Y) & \longrightarrow & \mathrm{Map}(X_1, Y) \\
\downarrow & & \downarrow & \hookrightarrow & \downarrow & & \downarrow \\
\mathrm{Hol}(X_2, Y) & \longrightarrow & \mathrm{Hol}(X_1 \cap X_2, Y) & & \mathrm{Map}(X_2, Y) & \longrightarrow & \mathrm{Map}(X_1 \cap X_2, Y).
\end{array}$$

Suppose the result is true for $X_1$, $X_2$ and $X_1 \cap X_2$. If the diagrams are homotopy cartesian, then we can conclude that the result is true for $X_1 \cup X_2$. This is the case if the vertical maps are fibrations. Unfortunately, the maps in the left-hand diagram are not fibrations so we have to be more clever.

A based holomorphic map $X \to \mathbf{CP}^1$ is uniquely determined by its zeros and poles and Segal uses this fact to replace the study of holomorphic maps with the study of configurations of zeros and poles. Similarly, we will use that a based holomorphic map $X \to \mathbf{CP}^1$ is uniquely determined by its principal parts, and replace the study of holomorphic maps with the study of configurations of principal parts.

As the diffeomorphism group does not act on such configurations, we have to enlarge the space. The 'configuration' space we consider consists of pairs of a complex structure on the underlying real manifold $M$ and a configuration of principal parts in this complex structure. Now the diffeomorphism group acts on the space, but it is no longer a true configuration space, since a global quantity, namely the complex structure, is introduced.

In section 2 we study complex structures on two dimensional manifolds. Much of the material is standard and is only included to establish the notation. For that reason there will be statements without proof or specific references. The missing details can be found in [**4**] , [**5**] and [**19**]. The main result is lemma 2.10, which roughly states that a meromorphic function depends continuously on it's principal parts and the complex structure.

In section 3 and 4 the necessary features of flag manifolds and loop groups are described. Once again most of the material is standard. The main results are lemma 3.1 and 4.6, which are generalizations of lemma 2.10.

We need to vary the complex structure and we only want to consider maps which have principal parts, so in section 5 we introduce the space $\mathcal{H}(M,Y)$ of pairs $(f,J)$ where $J$ is a complex structure on $M$ and $f$ is a map $M \to Y$, which is holomorphic in this complex structure and do not map entirely into infinity. If $\overline{D}$ is the closed unit disk, then we show that $\mathcal{H}(\overline{D},Y)$ is weak homotopy equivalent to $\mathrm{Map}(\overline{D},Y)$.

In section 6, a principal part of a holomorphic map into $Y$ is defined and the space $\mathcal{P}(M,Y)$ of pairs $(\xi,J)$, where $J$ is a complex structure on $M$ and $\xi$ is a configuration of principal parts in this structure, is introduced. There is a natural map $\mathcal{H}(M,Y) \to \mathcal{P}(M,Y)$, and if $\partial M \neq \emptyset$, then the map is surjective and a weak homotopy equivalence. The most important property of the space $\mathcal{P}(M,Y)$ is that, under certain conditions on an inclusion $M_1 \subseteq M_2$, the restriction map

$$\mathcal{P}(M_2,Y) \longrightarrow \mathcal{P}(M_1,Y)$$

is a quasifibration. It enables us to get the desired result for a union $M_1 \cup M_2$ if it is known for $M_1$, $M_2$ and $M_1 \cap M_2$.

This is used in section 7, where the results are proved. Starting with the result for $\overline{D}$, we follow the methods of [10] using induction on the number of handles in a handle decomposition of $M$. As long as $M$ is not closed, the relevant restriction maps are quasifibrations, so $\mathcal{H}(M,Y)$ is weak homotopy equivalent to $\mathrm{Map}(M,Y)$, if $\partial M \neq \emptyset$. When the manifold is closed, it is necessary to introduce a stabilized space $\widehat{\mathcal{P}}$.

By adding a principal part near infinity, we get a map $\mathcal{P} \to \mathcal{P}$ which increases the degree and $\widehat{\mathcal{P}}$ is the telescope of the sequence $\mathcal{P} \to \mathcal{P} \to \mathcal{P} \to \cdots$. Now the relevant restriction maps become homology fibrations and we can conclude that $\widehat{\mathcal{P}}$ and $\mathrm{Map}^*(M,Y)$ have the same homology type. The next step is to show that if $\mathcal{P}_J$ is the space of configurations of principal parts in a fixed complex structure $J$, then the inclusion $\mathcal{P}_J \hookrightarrow \mathcal{P}$ is a homotopy equivalence. Finally we show that $\mathcal{P}_{J,n}$ can be identified with $\mathcal{M}_n(X \times \mathbf{CP}^1, X \vee \mathbf{CP}^1, G_{\mathbf{C}})$, where $X$ is $M$ equipped with the complex structure $J$.

At the end is an appendix describing some topological concepts : homotopy theoretical fiber product, quasifibration, homology fibration and the telescope construction.

I would like to end the introduction by acknowledge my dept to M. F. Atiyah for proposing the problem and giving valuable suggestions and above all to my supervisor G. Segal for encouragement and for teaching me all I know about loop groups and configuration spaces.

Finally, I would like to express my gratitude to my Danish supervisor V. Lundsgaard Hansen. He was the first teacher I met at the university and ever since his enthusiasm, encouragement and support has been indispensable for me.

My stay at Oxford was made possible by a 'kandidatstipendium' from the University of Copenhagen, and the thesis has been completed in Denmark, while working first at the Mathematical Institute, the Technical University of Denmark, and now at IMFUFA, Roskilde University Center.

## 2. Complex Structures on Two Dimensional Manifolds

Let $M$ be a compact, connected, oriented two dimensional $C^\infty$-manifold without boundary. The topology of $M$ is completely described by a single number, the genus $g \in \mathbf{N} \cup \{0\}$.

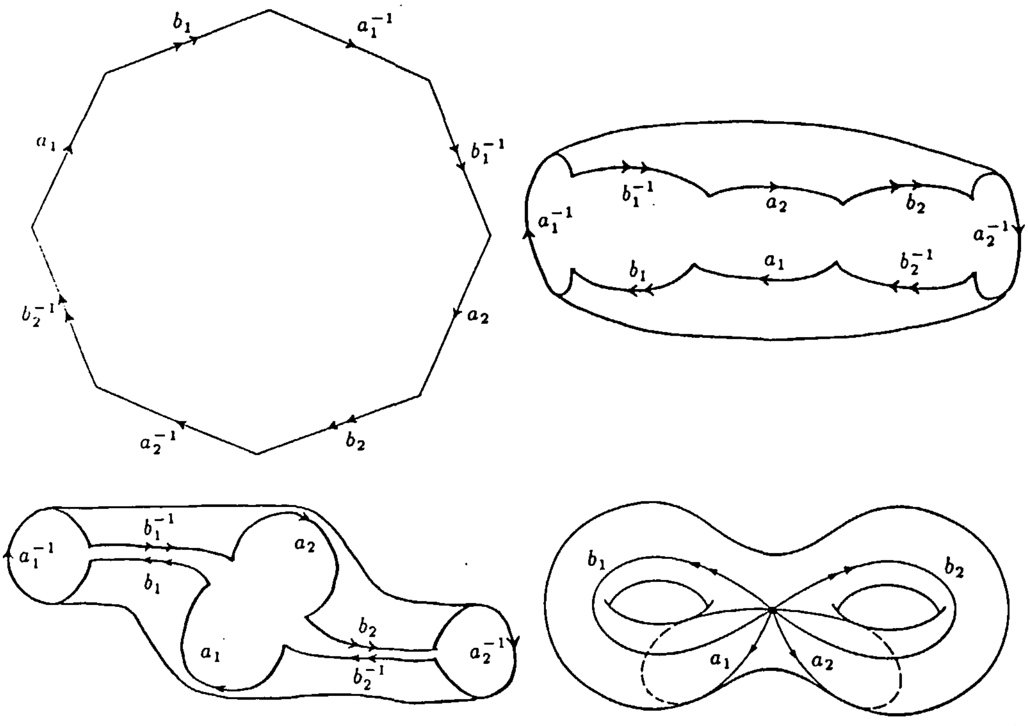



figure 2.1

If $g \geq 1$, then $M$ is homeomorphic to the space obtained from a polygon with $4g$ edges labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots; a_g, b_g, a_g^{-1}, b_g^{-1}$, by identifying $a_i$ with $a_i^{-1}$ and $b_i$ with $b_i^{-1}$, $i = 1, \ldots, g$, see figure 2.1.

After this identification, $a_1, b_1, \ldots, a_g, b_g$ become closed curves on $M$ and these curves generate the fundamental group $\pi_1(M)$. We have in fact that $\pi_1(M)$ is the free group generated by $a_1, b_1, \ldots, a_g, b_g$ divided by the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

The expression $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ is called the *symbol* of the surface. The first homology group $H_1(M, \mathbf{Z})$ is a free $\mathbf{Z}$-module with basis $a_1, b_1, \ldots, a_g, b_g$ and the intersection form $H_1(M, \mathbf{Z}) \times H_1(M, \mathbf{Z}) \to \mathbf{Z}$ is determined by

$$a_i a_j = b_i b_j = 0 \qquad \text{and} \qquad a_i b_j = -b_j a_i = \delta_{ij} \qquad i, j = 1, \ldots, g.$$

So in the basis $(a_1, \ldots, a_g, b_1 \ldots, b_g)$ the intersection form has the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where 0 is the $g \times g$ zero matrix and $I$ is the $g \times g$ identity matrix. Any basis for $H_1(M, \mathbf{Z})$ with such an intersection matrix is called a *canonical homology basis* .

A *Riemann surface* is a connected two-dimensional $C^\infty$-manifold $M$ equipped with an atlas $(U_\alpha, z_\alpha)$, where the chart $z_\alpha \colon U_\alpha \to \mathbf{C}$ is a diffeomorphism onto an open subset of the complex plane, such that the transition function $z_\alpha \circ z_\beta^{-1}$ is holomorphic for all $\alpha, \beta$.

A chart $z\colon U \to \mathbf{C}$ is also called a *local parameter* or a *local coordinate.* If $M$ is compact, then $M$ is called a *closed* Riemann surface, otherwise it is called an *open* Riemann surface.

A continuous map $f\colon M \to M'$ between Riemann surfaces, or more general between complex manifolds, is called *holomorphic*, if for every local coordinate $(U, z)$ on $M$ and every local coordinate $(U', z')$ on $M'$, with $U \cap f^{-1}(U') \neq \emptyset$, the map $z' \circ f \circ z^{-1}\colon z(U \cap f^{-1}(U')) \to \mathbf{C}$ is holomorphic. The map is called *conformal*, if it also is bijective. In this case the inverse map $f^{-1}\colon M' \to M$ is conformal too. A holomorphic map into $\mathbf{C}$ is called a *holomorphic function* and a holomorphic map into $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$, which is not identical $\infty$, is called a *meromorphic function.*

The *sheaf of holomorphic functions* is denoted $\mathcal{O}$, and the *sheaf of meromorphic functions* is denoted $\mathcal{M}$, i.e. if $U \subseteq M$ is open, then

$$\mathcal{O}(U) = \{f\colon U \to \mathbf{C} \mid f \text{ is holomorphic }\}$$

and

$$\mathcal{M}(U) = \{f\colon U \to \mathbf{CP} \mid f \text{ is holomorphic and } f(U) \cap \mathbf{C} \neq \emptyset\}.$$

If $z = x+iy$ and $\tilde{z} = \tilde{x}+i\tilde{y}$ are local parameters on $M$, then the transition function is holomorphic, so by the Cauchy-Riemann equations

$$dx \wedge dy = \det \begin{pmatrix} \frac{\partial x}{\partial \tilde{x}} & \frac{\partial x}{\partial \tilde{y}} \\ \frac{\partial y}{\partial \tilde{x}} & \frac{\partial y}{\partial \tilde{y}} \end{pmatrix} d\tilde{x} \wedge d\tilde{y} = \det \begin{pmatrix} \frac{\partial x}{\partial \tilde{x}} & -\frac{\partial y}{\partial \tilde{x}} \\ \frac{\partial y}{\partial \tilde{x}} & \frac{\partial x}{\partial \tilde{x}} \end{pmatrix} d\tilde{x} \wedge d\tilde{y}$$

$$= \left( \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 + \left(\frac{\partial y}{\partial \tilde{y}}\right)^2 \right) d\tilde{x} \wedge d\tilde{y} = \left|\frac{dz}{d\tilde{z}}\right|^2 d\tilde{x} \wedge d\tilde{y}$$

and a Riemann surface is orientable. In the following it is assumed that the orientation is compatible with the complex structure, i.e. if $\Omega$ is the *volume form* on $M$, then $\Omega = f dx \wedge dy$ with $f > 0$ for any local parameter $z = x + iy$.

If $z\colon U \to \mathbf{C}$ is a local parameter on $M$, then multiplication by $i$ in $\mathbf{C}$ induces a real isomorphism $J_x\colon T_xM \to T_xM$ with $J_x^2 = -1$ for all $x \in U$, defined by $J_x(v) = (dz)^{-1}(i\, dz(v))$ for $v \in T_xM$. As the transition functions are holomorphic, the endomorphism $J_x$ is independent of the choice of parameter $z$. Hence there is a well-defined smooth section $J$ in the endomorphism bundle $\operatorname{End}(TM)$, which satisfies $J^2 = -1$.

The complexification of the tangent bundle is $TM_{\mathbf{C}} = TM \otimes_{\mathbf{R}} \mathbf{C}$ and the complexification of the cotangent bundle is $T^*M_{\mathbf{C}} = T^*M \otimes_{\mathbf{R}} \mathbf{C}$. We can obviously identify $T^*M_{\mathbf{C}}$ with the complex dual bundle of $TM_{\mathbf{C}}$, i.e. $T^*M_{\mathbf{C}} \cong (TM_{\mathbf{C}})^*$. The bundle of complex valued forms on $M$ is

$$\bigwedge M_{\mathbf{C}} = \bigwedge(T^*M_{\mathbf{C}}) \cong (\bigwedge(T^*M)) \otimes_{\mathbf{R}} \mathbf{C}.$$

We have

$$\bigwedge M_{\mathbf{C}} = \bigwedge^0 M_{\mathbf{C}} \oplus \bigwedge^1 M_{\mathbf{C}} \oplus \bigwedge^2 M_{\mathbf{C}},$$

where

$$\bigwedge^i M_{\mathbf{C}} = \bigwedge^i (T^* M_{\mathbf{C}}) \cong (\bigwedge^i (T^* M_{\mathbf{C}})) \otimes_{\mathbf{R}} \mathbf{C} \qquad i = 0,1,2$$

is the bundle of complex valued $i$-forms on $M$. In particular, $\bigwedge^0 M_{\mathbf{C}} = M \times \mathbf{C}$ and $\bigwedge^1 M_{\mathbf{C}} = T^* M_{\mathbf{C}}$. The real isomorphism $J\colon TM \to TM$ induces a complex isomorphism $J\colon TM_{\mathbf{C}} \to TM_{\mathbf{C}}$ and the dual isomorphism $J^*\colon T^*M_{\mathbf{C}} \to T^*M_{\mathbf{C}}$, which both have eigenvalues $\pm i$. Let $TM^{1,0}$ and $\bigwedge^{1,0} M$ denote the two bundles of $(+i)$-eigenspaces and $TM^{0,1}$ and $\bigwedge^{0,1} M$ denote the two bundles of $(-i)$-eigenspaces. They are smooth subbundles of $TM_{\mathbf{C}}$ and $\bigwedge^1 M_{\mathbf{C}}$ respectively. Furthermore, $\bigwedge^{1,0} M$ is the complex dual of $TM^{1,0}$ and $\bigwedge^{0,1} M$ is the complex dual of $TM^{0,1}$.

If $z = x + iy$ is a local parameter on $M$, then $(dx, dy)$ is a local basis for $\bigwedge^1 M$ and hence for $\bigwedge^1 M_{\mathbf{C}}$. We have

$$J^* dx = -dy \qquad \text{and} \qquad J^* dy = dx,$$

so if

$$dz = dx + i dy \qquad \text{and} \qquad d\bar{z} = dx - i dy,$$

then $(dz, d\bar{z})$ is a basis for $\bigwedge^1 M_{\mathbf{C}}$, $dz$ is a basis for $\bigwedge^{1,0} M$ and $d\bar{z}$ is a basis for $\bigwedge^{0,1} M$.

Note that $-J^*$ is the Hodge star-operator restricted to one-forms for any metric, compatible with the complex structure, i.e. any metric, where $J$ is a positive rotation through $\pi/2$. Finally $dz \wedge d\bar{z} = -2i dx \wedge dy$, so the volume form $\Omega$ is pointwise a positive multiple of $i dz \wedge d\bar{z}$ for any local parameter $z$.

The *conjugation* $Q\colon TM_{\mathbf{C}} \to TM_{\mathbf{C}}$ is a conjugated complex linear isomorphism defined by $Q(V \otimes \alpha) = v \otimes \overline{\alpha}$. It satisfies $Q = Q^{-1}$ and gives a conjugated linear isomorphism $Q\colon TM^{1,0} \to TM^{0,1}$. Similarly, for $i = 0,1,2$, there are conjugations, also denoted $Q$, from $\bigwedge^i M_{\mathbf{C}}$ to itself. Again $Q = Q^{-1}$ and gives a conjugated linear isomorphism $Q\colon \bigwedge^{1,0} M \to \bigwedge^{0,1} M$. We also use the notation $Q(a) = \overline{a}$, and if $z$ is a local parameter, then $Q(dz) = \overline{dz} = d\bar{z}$ and $Q(d\bar{z}) = \overline{d\bar{z}} = dz$.

As $M$ is a complex manifold, the tangent bundle $TM$ is a holomorphic complex vector bundle over $M$, and as such we will denote it $TM_J$. If $v \in TM$ is a tangent vector and $\alpha + i\beta$ is a complex number with $\alpha, \beta \in \mathbf{R}$, then $(\alpha + i\beta)v = \alpha v + \beta J v$, and if $z = x + iy$ is a local parameter on $M$, then the vector field $\partial/\partial x$ gives a local holomorphic trivialization of $TM_J$. The complex vector bundles $T^{1,0}M$ and $\bigwedge^{1,0} M$ are canonical isomorphic to $TM_J$ and $(TM_J)^*$ respectively, so they are, in particular, holomorphic vector bundles. The isomorphism $TM_J \tilde{\to} TM^{1,0}$ is given by $v \mapsto \pi_{1,0}(v \otimes 1)$, where $\pi_{i,j}$ is the projection $TM_{\mathbf{C}} = T^{1,0}M \oplus T^{0,1}M \to T^{i,j}M$ and the isomorphism $\bigwedge^{1,0} M \tilde{\to} (TM_J)^*$ is the complex dual.

The exterior differential

$$\Omega^0(M) \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} 0,$$

where $\Omega^i M$ is the space of smooth sections in $\bigwedge^i M$, extends by complex linearity to act on complex valued forms and we get the sequence

$$\Omega^0 M_{\mathbf{C}} \xrightarrow{d} \Omega^1 M_{\mathbf{C}} \xrightarrow{d} \Omega^2 M_{\mathbf{C}} \xrightarrow{d} 0,$$

where $\Omega^i M_{\mathbf{C}}$ is the space of smooth sections in $\bigwedge^i M_{\mathbf{C}}$. Let $\Omega^{i,j} M$ denote the space of smooth sections in $\bigwedge^{i,j} M$ and define

$$\partial = \pi_{1,0} \circ d \colon \Omega^0 M_{\mathbf{C}} \longrightarrow \Omega^{1,0} M_{\mathbf{C}} \qquad \text{and} \qquad \overline{\partial} = \pi_{0,1} \circ d \colon \Omega^0 M_{\mathbf{C}} \longrightarrow \Omega^{0,1} M_{\mathbf{C}}.$$

We also denote $d|_{\Omega^{1,0}M} = \overline{\partial}$ and $d|_{\Omega^{0,1}M} = \partial$. Then $d = \partial + \overline{\partial}$ and using the complex conjugation defined above, we have

$$\partial = Q \circ \overline{\partial} \circ Q \quad \text{and} \quad \overline{\partial} = Q \circ \partial \circ Q.$$

If $z = x + iy$ is a local parameter on $M$ and $f, g \colon \mathbf{C} \to \mathbf{C}$ are smooth complex functions then

$$d(f \circ z) = \partial(f \circ z) + \overline{\partial}(f \circ z) = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \overline{z}}\, d\overline{z}$$

and

$$\begin{aligned} d\big((f \circ z)\, dz + (g \circ z)\, d\overline{z}\big) &= \overline{\partial}(f \circ z) \wedge dz + \partial(g \circ z) \wedge d\overline{z} \\ &= \frac{\partial f}{\partial \overline{z}}\, d\overline{z} \wedge dz + \frac{\partial g}{\partial z}\, dz \wedge d\overline{z} \\ &= \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \overline{z}}\right) dz \wedge d\overline{z}, \end{aligned}$$

with

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A complex one-form $\omega \in \Omega^1 M_{\mathbf{C}}$ is called

*exact*, if $\omega = df$ for some $f \in \Omega^0 M_{\mathbf{C}}$,
*co-exact*, if $J^*\omega$ is exact, i.e. if $\omega = J^* df$ for some $f \in \Omega^0 M_{\mathbf{C}}$,
*closed*, if $d\omega = 0$,
*co-closed*, if $J^*\omega$ is closed, i.e. if $dJ^*\omega = 0$,
*harmonic*, if $\omega$ is closed and co-closed,
*holomorphic*, if one and hence all of the following five equivalent conditions are satisfied

i) $\omega$ is a holomorphic section in $\bigwedge^{1,0} M$,
ii) locally $\omega = df$ with $f$ holomorphic,
iii) locally $\omega = f\, dz$ with $f$ holomorphic,
iv) $\omega \in \Omega^{1,0} M$ and is closed, i.e. $J^*\omega = i\omega$ and $d\omega = 0$,
v) $\omega = \alpha - iJ^*\alpha$ for some harmonic $\alpha$.

We also call a harmonic one-form a *harmonic differential* and a holomorphic one-form a *holomorphic differential.*

By deRham's and Hodge's theorems

$$H^1(M,\mathbf{C}) \cong \{\text{closed one-forms}\}/\{\text{exact one-forms}\} \cong \{\text{harmonic one-forms}\}$$

If $c$ is a one-cycle and $\omega$ is a closed one-form, the pairing

$$H_1(M,\mathbf{C}) \times H^1(M,\mathbf{C}) \longrightarrow \mathbf{C}\colon \big([c],[\omega]\big) \longmapsto \langle c,\omega\rangle$$

is given by

$$\langle c,\omega\rangle = \int_c \omega.$$

In particular, the space of harmonic one forms is a $2g$-dimensional complex vector space where $g$ is the genus of $M$. If $c_1 = a_1, \ldots, c_g = a_g, c_{g+1} = b_1, \ldots, c_{2g} = b_g$ is a canonical homology basis, then we can determine a basis $\alpha_1, \ldots, \alpha_{2g}$ for the harmonic differentials by demanding that

$$\langle c_i, \alpha_j\rangle = \delta_{ij}.$$

If the space of holomorphic differentials is denoted $\mathcal{H}$, and $\overline{\mathcal{H}}$ denotes the space of anti-holomorphic differentials, i.e. differentials of the form $\overline{\omega}$ with $\omega$ holomorphic, then there is a direct sum decomposition

$$\{\text{harmonic differentials}\} = \mathcal{H} \oplus \overline{\mathcal{H}}.$$

So the space of holomorphic differentials is $g$-dimensional, and if

$$\omega_i = \alpha_i - iJ^*\alpha_i, \qquad i = 1, \ldots, g,$$

then $\omega_1, \ldots, \omega_g$ is a basis for $\mathcal{H}$.

A *meromorphic differential* $\omega$ on a Riemann surface $M$ is a holomorphic differential on $M \setminus \{p_1, \ldots, p_n\}$ such that for each local parameter $z$ on $M$, $\omega = f\,dz$ with $f$ a meromorphic function.

If $\omega$ is a meromorphic differential, $z$ is a local parameter vanishing at a point $p \in M$ and $\omega = f(z)\,dz$ with $f(z) = \sum_{k=n}^{\infty} a_k z^k$ and $a_n \neq 0$, then the *order* of $\omega$ at $p$ is

$$\operatorname{ord}_p \omega = \operatorname{ord}_0 f = n.$$

If $\operatorname{ord}_p \omega < 0$, then $p$ is called a *pole* and the *residue* of $\omega$ at $p$ is then

$$\operatorname{res}_p \omega = a_{-1} = \int_c \omega$$

with $c$ a small closed curve around $p$.

If $p$ is a point on a Riemann surface $M$, $n \geq 2$ and $z$ is a local parameter vanishing at $p$, then there exists a meromorphic differential $\tau_p^{(n)}$, which is holomorphic on $M \setminus \{p\}$ and has singularity $z^{-n}$ at $p$, i.e. locally around $p$ we have

$$\tau_p^{(n)} = (z^{-n} + f(z))\, dz$$

with $f$ holomorphic. The order of $\tau_p^{(n)}$ at $p$ is $-n$ and can be arbitrarily low. The order can not be arbitrarily high, in fact, if $p$ is a point on $M$, then there are precisely $g$ integers

$$0 = \mu_1 < \mu_2 < \cdots < \mu_g \leq 2g - 2$$

such that there exist holomorphic differentials $\xi_1, \ldots, \xi_g$ with

$$\operatorname{ord}_p \xi_i = \mu_i \qquad i = 1, \ldots, g.$$

The numbers $\mu_1 + 1, \mu_2 + 1, \ldots, \mu_g + 1$ are called the *gaps* at $p$. The differentials $\xi_1, \ldots, \xi_g$ are necessarily linearly independent, why they form a basis for the holomorphic differentials.

If $z$ is a local parameter vanishing at $p$ and the differentials above have the local expression $\xi_i = \phi_i(z)\, dz$ with $\phi_i(z) = \sum_{k=0}^{\infty} a_{ik} z^k$ and $a_{i\mu_j} = \delta_{ij}$ (which always can be achieved), then $\xi_1, \ldots, \xi_g$ is called the basis *adapted* to the point $p$. It is not uniquely determined by $p$, but depends on the choice of the parameter $z$.

The *weight* of $p$ is the number

$$\tau(p) = \sum_{i=1}^{g} (\mu_j + 1 - j),$$

and as $\mu_j \geq j - 1$, $j = 1, \ldots, g$, the weight $\tau(p) \geq 0$, and $\tau(p) = 0$ iff $\mu_j = j - 1$, $j = 1, \ldots, g$ iff $\mu_g = g - 1$. The points $p \in M$ with $\tau(p) \geq 1$ are called *Weierstrass points* and there are only finitely many of such points on $M$. In fact

$$\sum_{p \in M} \tau(p) = (g-1)g(g+1).$$

If $z$ is a local parameter and $\omega_1, \ldots, \omega_g$ is a basis for the holomorphic differentials, which locally is given by $\omega_i = \phi_i(z)\, dz$, then

$$\tau(p) = \operatorname{ord}_{z(p)} \det[\phi_1, \phi_2, \ldots, \phi_g]$$

with

$$[\phi_1(z), \phi_2(z), \ldots, \phi_g(z)] = \begin{pmatrix} \phi_1(z) & \phi_2(z) & \cdots & \phi_g(z) \\ \phi_1'(z) & \phi_2'(z) & \cdots & \phi_g'(z) \\ \vdots & \vdots & & \vdots \\ \phi_1^{(g-1)}(z) & \phi_2^{(g-1)}(z) & \cdots & \phi_g^{(g-1)}(z) \end{pmatrix}$$

so locally the Weierstrass points are given as the zeros of a holomorphic function.

Let $p \in M$ and let $z$ be a local parameter vanishing at $p$. A *principal part* at $p$ is a rational function of $z$ of the form $f(z) = \sum_{k=-n}^{-1} a_k z^k$. If $p_1, \dots, p_m$ are $m$ distinct points on $M$, $z_j$ is a local parameter vanishing at $p_j$ and $f_j(z_j) = \sum_{k=-n}^{-1} a_{jk} z_j^k$ is a principal part at $p_j$ then the collection $\{f_1, f_2, \dots, f_m\}$ will be called a *system* or a *configuration of principal parts*. The points $p_1, \dots, p_m$ are called the *poles* of the configuration.

As it stands, a configuration of principal parts depends on the choices of local parameters. Instead we can consider an open covering $(U_i)$ of $M$ and an assignment of a meromorphic function $f_i$ on $U_i$ for each $i$, such that the difference $f_i - f_j$ is a holomorphic function on $U_i \cap U_j$ all $i, j$. Such a gadget is called a *Mittag-Lefler distribution*, and if the sets $U_i$ are domains for local parameters $z_i \colon U_i \to \mathbf{C}$, then we obviously have a one to one correspondence between Mittag-Lefler distributions and configurations of principal parts.

As a third approach, we consider the quotient sheaf $\mathcal{P} = \mathcal{M}/\mathcal{O}$ called the *sheaf of principal parts*, i.e. $\mathcal{P}$ is the sheaf generated by the presheaf $U \mapsto \mathcal{M}(U)/\mathcal{O}(U)$, where $\mathcal{O}$ is the sheaf of holomorphic functions and $\mathcal{M}$ is the sheaf of meromorphic functions. A configuration of principal parts is now defined as an element of $\mathcal{P}(M)$, i.e. as a global section of the sheaf $\mathcal{P}$. This is by definition the same as a Mittag-Lefler distribution on $M$.

The short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow 0$$

of sheaves induces a long exact cohomology sequence

$$0 \longrightarrow \mathcal{O}(M) \longrightarrow \mathcal{M}(M) \longrightarrow \mathcal{P}(M) \xrightarrow{\delta} H^1(M, \mathcal{O}) \longrightarrow \dots .$$

A configuration $\xi \in \mathcal{P}(M)$ comes from a globally defined meromorphic function iff $\delta\xi = 0$. If $\xi$ is a configuration represented by meromorphic functions $f_i \colon U_i \to \mathcal{P}$ and $\omega$ is a holomorphic differential, then

$$\mathrm{res}_p(f_i\omega) = \mathrm{res}(f_j\omega) \qquad \text{all } p \in U_i \cap U_j,$$

so we may define

$$\mathrm{res}_p(\xi) = \mathrm{res}_p(f_i\omega), \qquad \text{if } p \in U_i.$$

By Dolbout's theorem, $H^1(M, \mathcal{O})$ is the dual of the holomorphic differentials and if $\omega$ is a holomorphic differential, then

$$\langle \omega, \delta\xi \rangle = \mathrm{res}(\xi\omega) = \sum_{p \in M} \mathrm{res}_p(\xi\omega).$$

This leads to

(2.1) LEMMA. *If $\xi \in \mathcal{P}$ is a configuration of principal parts on a closed Riemann surface $M$ of genus $g$, and $p \in M$ is a point, which is neither a pole of $\xi$ nor a Weierstrass point, then there exists a meromorphic function $f$ on $M$, which on $M \setminus \{p\}$ has $\xi$ as its principal parts and has $\operatorname{ord}_p f \geq -g$.*

PROOF: Let $z$ be a local parameter vanishing at $p$ and let $\omega_1, \ldots, \omega_g$ be a basis for the holomorphic differentials adapted to the point $p$, i.e. $\omega_i = (z^{i-1} + z^g)g_i(z)\,dz$ with $g_i(0) \neq 0$. If we put

$$f_0 = -\sum_{k=1}^{g} \operatorname{res}(\xi\omega_k)z^{-k},$$

then the configuration $\xi \cup \{f_0\}$ satisfies

$$\operatorname{res}\big((\xi \cup \{f_0\})\omega_i\big) = \operatorname{res}(\xi\omega_i) + \operatorname{res}_p(f_0\omega_i) = 0 \qquad \text{all } i = 1, \ldots, g.$$

Hence $\delta(\xi \cup \{f_0\}) = 0$, and there exists a meromorphic function $f\colon M \to \mathbf{CP}^1$, which has $\xi \cup \{f_0\}$ as its principal parts. This function clearly satisfies the conditions in the lemma. ∎

If $p$ is a Weierstrass point, we still have the same kind of result, but we may need a pole of higher order at $p$.

Earlier we saw that a complex structure induces a section $J$ in $\operatorname{End}(TM)$ with $J^2 = -1$. Such a section is called an *almost complex structure* and in the case of two dimensional manifolds, all almost complex structures come from a complex structure. So we may define the space of complex structures on an oriented two dimensional manifold $M$ with volume form $\Omega$ as

$$\mathcal{C}(M) = \{\text{smooth sections } J \text{ in } \operatorname{End}(TM) \mid J^2 = -1 \text{ and } \Omega(v, Jv) \geq 0 \text{ all } v \in TM\}$$

equipped with the $C^\infty$-topology. This definition also makes sense, if $M$ has boundary and corners and in the following this may be the case.

If $M$ is a two dimensional manifold and $J \in \mathcal{C}(M)$, then $M_J$ denotes $M$ equipped with the complex structure $J$. As the complex structure now can vary, we will speak of $J$-holomorphic and $J$-harmonic functions, maps, forms ect.

Up to isomorphism, there is only one complex structure on $S^2$. If $J_0 \in \mathcal{C}(M)$ denotes the standard complex structure on $S^2$, i.e. $S^2_{J_0} = \mathbf{CP}^1$, and $J \in \mathcal{C}(M)$ is any complex structure, then there exists a diffeomorphism $\phi_J$ of $S^2$, which is holomorphic considered as a map from $S^2_J$ to $S^2_{J_0}$. If furthermore $\phi_J(0) = 0$, $\phi_J(1) = 1$ and $\phi_J(\infty) = \infty$, then $\phi_J$ is unique. We equip the group $\operatorname{Diff}(S^2)$ of diffeomorphisms of $S^2$ with the $C^\infty$-topology and have

(2.2) PROPOSITION. *The map $\mathcal{C}(S^2) \to \operatorname{Diff}(S^2)\colon J \mapsto \phi_J$ is continuous.*

PROOF: $\operatorname{Diff}(S^2)$ acts on $\mathcal{C}(S^2)$ from the left by $\phi.J = \phi_* J \phi_*^{-1}$. Let $\operatorname{Diff}^k(S^2)$ denote diffeomorphisms of Sobolev class $k$ and let $\mathcal{C}^k(S^2)$ denote the space of complex structures of Sobolev class $k$, both equipped with the corresponding topologies. These spaces are smooth Hilbert manifolds, and the action of $\mathrm{Diff}(S^2)$ on $\mathcal{C}(S^2)$ extends to a smooth action

$$\mathrm{Diff}^{k+1}(S^2) \times \mathcal{C}^k(S^2) \longrightarrow \mathcal{C}^k(S^2).$$

We only have to show that the orbit map

$$\mathcal{O}\colon \mathrm{Diff}^{k+1}(S^2) \longrightarrow \mathcal{C}^k(S^2)\colon \phi \longmapsto \phi.J_0 = \phi_* J_o \phi_*^{-1}$$

is a submersion for all $k$. As right translation

$$\mathrm{Diff}^{k+1}(S^2) \longrightarrow \mathrm{Diff}^{k+1}(S^2)\colon \psi \longmapsto \psi \circ \phi$$

is smooth for all $\phi \in \mathrm{Diff}^{k+1}(S^2)$, it is enough to show that the differential at the identity

$$D = d\mathcal{O}_{id}\colon T_{id}\,\mathrm{Diff}^{k+1}(S^2) \longrightarrow T_{J_0}\mathcal{C}^k(S^2)$$

induces an isomorphism

$$T_{id}\,\mathrm{Diff}^{k+1}(S^2)/\ker(D) \xrightarrow{\simeq} T_{J_0}\mathcal{C}^k(S^2).$$

We have that

$$T_{id}\,\mathrm{Diff}^{k+1}(S^2) = \{\text{vector fields on } S^2 \text{ of Sobolev class } k+1\},$$

and if

$$F = \{A \in \mathrm{End}(TS^2) \mid AJ_0 + J_0 A = 0\},$$

then

$$T_{J_0}\mathcal{C}^k(S^2) = \{\text{sections in } F \text{ of Sobolev class } k\}.$$

The Riemann sphere $S^2_{J_0} = \mathbf{CP}^1$ can be covered by two $J_0$-holomorphic charts $\mathbf{C} \to S^2$ with the transition function $\mathbf{C}^* \to \mathbf{C}^*\colon z \mapsto z^{-1}$. The corresponding transition function for the tangent bundle is the map

$$\mathbf{C}^* \times \mathbf{C} \longrightarrow \mathbf{C}^* \times \mathbf{C}\colon (z,u) \longmapsto (z^{-1}, -z^{-2}u),$$

and for the cotangent bundle it is

$$\mathbf{C}^* \times \mathbf{C} \longrightarrow \mathbf{C}^* \times \mathbf{C}\colon (z,\omega) \longmapsto (z^{-1}, -\bar{z}^2\omega),$$

where $(z,\omega)$ acts on $(z,u)$ by $\langle (z,\omega),(z,u)\rangle = Re(\overline{\omega}u)$. The endomorphism bundle, restricted to the two charts, is just $\mathbf{C}\times M_{\mathbf{R}}(2,2)$, where $M_{\mathbf{R}}(2,2)$ is the space of real two by two matrices. The standard complex structure $J_0$ is in both trivializations given by the constant map

$$J_0\colon \mathbf{C} \longrightarrow M_{\mathbf{R}}(2,2)\colon z \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence the bundle $F$ is in both trivializations given by matrices of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, and the map $a+ib \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, identifies the space of such matrices with $\mathbf{C}$. Then $(z,f) \in F$ acts on $(z,u) \in \mathbf{C}\times\mathbf{C} \subseteq TM$ by $(z,f)(z,u) = (z, f\bar{u})$. So the transition function for $F$ is

$$\mathbf{C}^* \times \mathbf{C} \longrightarrow \mathbf{C}^* \times \mathbf{C}\colon (z,f) \longmapsto (z^{-1}, (\bar{z}z^{-1})^2 f)\,.$$

A priori, $F$ is just a real two-dimensional vector bundle, but the transition function is complex linear in each fiber, so $F$ can be considered as a complex line bundle over $S^2$.

Now we want to calculate the local expression for the operator $D = dO_{id}$ and its symbol. Let $v$ be a vector field on $S^2$ and let $\phi_t\colon S^2 \to S^2$ be the flow generated by $v$. Locally $\phi_t = (\phi_{1t}, \phi_{2t})$ and $\phi_0 = id = (x,y)$, so

$$\phi_{t*} = \begin{pmatrix} \dfrac{\partial \phi_{1t}}{\partial x} & \dfrac{\partial \phi_{1t}}{\partial y} \\ \dfrac{\partial \phi_{2t}}{\partial x} & \dfrac{\partial \phi_{2t}}{\partial y} \end{pmatrix},$$

$$\phi_{t*}^{-1} = \left(\frac{\partial \phi_{1t}}{\partial x}\frac{\partial \phi_{2t}}{\partial y} - \frac{\partial \phi_{2t}}{\partial x}\frac{\partial \phi_{1t}}{\partial y}\right) \begin{pmatrix} \dfrac{\partial \phi_{2t}}{\partial y} & -\dfrac{\partial \phi_{1t}}{\partial y} \\ -\dfrac{\partial \phi_{2t}}{\partial x} & \dfrac{\partial \phi_{1t}}{\partial x} \end{pmatrix}$$

and

$$\phi_{0*} = \phi_{0*}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Differentiation with respect to $t$, letting $t=0$ and using $\dot{\phi}_0 = v = (v_1, v_2)$ leads to

$$\dot{\phi}_{0*} = \begin{pmatrix} \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y} \\ \dfrac{\partial v_2}{\partial x} & \dfrac{\partial v_2}{\partial y} \end{pmatrix},$$

$$\dot{\phi}_{0*}^{-1} = -\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \dfrac{\partial v_2}{\partial y} & -\dfrac{\partial v_1}{\partial y} \\ -\dfrac{\partial v_2}{\partial x} & \dfrac{\partial v_1}{\partial x} \end{pmatrix},$$

and finally

$$\begin{aligned} Dv &= \frac{d}{dt}\left(\phi_{t*} \circ J_0 \circ \phi_{t*}^{-1}\right)\Big|_{t=0} = \dot{\phi}_{0*} \circ J_0 \circ \phi_{0*}^{-1} + \phi_{0*} \circ J_0 \circ \dot{\phi}_{0*}^{-1} \\ &= \begin{pmatrix} \dfrac{\partial v_1}{\partial y} & -\dfrac{\partial v_1}{\partial x} \\ \dfrac{\partial v_2}{\partial y} & -\dfrac{\partial v_2}{\partial x} \end{pmatrix} - \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{\partial v_2}{\partial x} & -\dfrac{\partial v_1}{\partial x} \\ \dfrac{\partial v_2}{\partial y} & -\dfrac{\partial v_1}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\partial v_1}{\partial y} + \dfrac{\partial v_2}{\partial x} & -\dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} \\ -\dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} & -\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y} \end{pmatrix}. \end{aligned}$$

In complex notation

$$D = \frac{\partial}{\partial y} - i\frac{\partial}{\partial x} = -2i\frac{\partial}{\partial \bar{z}},$$

and the symbol is the map

$$\sigma\colon \mathbf{C}\times\mathbf{C}\times\mathbf{C} \longrightarrow \mathbf{C}\times\mathbf{C}\times\mathbf{C}\colon (z,\omega,u) \longmapsto (z,\omega,-i\omega u).$$

So $D$ is an elliptic operator and the proposition is proved as soon as we prove that the cokernel of $D$ is zero. As the space of conformal maps from $\mathbf{CP}^1$ to itself is three-dimensional, so is the kernel of $D$. Thus we need only prove that the index $i(D)$ of $D$ is three. By the Atiyah-Singer index theorem

$$i(D) = \big(ch(D)\cdot \mathcal{T}(S^2)\big)\,[S^2],$$

where $ch(D)$ is the Chern character of $D$ and $\mathcal{T}(S^2)$ is the Todd class of $S^2$ (which is 1).

Considered as complex line bundles over $S^2$, we have $TS^2 = T^*S^2 = E$ and $F = E\otimes E = E^2$ because, when restricted to $S^1$, $-\bar{z}^2 = -z^{-2}$ and $(\bar{z}z^{-1})^2 = z^{-4}$.

Let $P$ be the orthonormal frame bundle of $S^2$ with the standard metric. Then $P$ is a principal $SO(2)$-bundle over $S^2$, and as a real bundle $E = P\times_{SO(2)}\mathbf{R}^2$ with the standard action of $SO(2)$ on $\mathbf{R}^2$. There is a group isomorphism $SO(2)\cong U(1)$ by which the standard action of $U(1)$ on $\mathbf{C}\cong\mathbf{R}^2$ corresponds to the standard action of $SO(2)$ on $\mathbf{R}^2$. As complex bundles $E = P\times_{U(1)}\mathbf{C}$ and $F = P\times_{U(1)}\mathbf{C}_2$, where $\mathbf{C}_2$ is $\mathbf{C}$ with the $U(1)$-action $(e^{i\theta},z)\mapsto e^{2i\theta}z$. If $\pi\colon T^*S^2\to S^2$ is the projection, then

$$\pi^*(E) = P\times_{U(1)}(\mathbf{C}\times\mathbf{C}), \qquad \pi^*(F) = P\times_{U(1)}(\mathbf{C}\times\mathbf{C}_2)$$

and the symbol $\sigma\colon \pi^*(E)\to\pi^*(F)$ is induced by the map

$$P\times\mathbf{C}\times\mathbf{C} \longrightarrow P\times\mathbf{C}\times\mathbf{C}_2\colon (u,\omega,z)\longmapsto (u,\omega,-i\omega z).$$

Let $B_G = \mathbf{CP}^\infty$ be the classifying space for $G = SO(2) = U(1)$, let $E_G$ be the universal $G$-bundle over $B_G$ and let $f\colon S^2\to B_G$ be the classifying map for the principal $G$-bundle $P$, i.e. $P = f^*(E_G)$. If

$$\widetilde{V}^* = \widetilde{V} = E_G\times_G\mathbf{R}^2, \qquad \widetilde{E} = E_G\times_G\mathbf{C} \qquad\text{and}\qquad \widetilde{F} = E_G\times_G\mathbf{C}_2,$$

then

$$T^*S^2 = f^*(\widetilde{V}), \qquad E = f^*(\widetilde{E}) \qquad\text{and}\qquad F = f^*(\widetilde{F}),$$

and if $\widetilde{\pi}\colon \widetilde{V}^*\to B_G$ is the projection, then

$$\widetilde{\pi}^*(\widetilde{E}) = E_G\times_G(\mathbf{R}^2\times\mathbf{C}) = E_G\times_G(\mathbf{C}\times\mathbf{C})$$
$$\widetilde{\pi}^*(\widetilde{F}) = E_G\times_G(\mathbf{R}^2\times\mathbf{C}_2) = E_G\times_G(\mathbf{C}\times\mathbf{C}_2).$$

The symbol $\sigma$ is the pullback of the map $\widetilde{\sigma}\colon \widetilde{\pi}^*(\widetilde{E}) \to \widetilde{\pi}^*(\widetilde{F})$ induced by the map

$$E_G \times \mathbf{C} \times \mathbf{C} \longrightarrow E_G \times \mathbf{C} \times \mathbf{C}\colon (u, \omega, z) \longmapsto (u, \omega, i\omega z).$$

By [**12**, chap. III theorem 1]

$$ch(D) = f^*\Big(-\frac{ch(\widetilde{E}) - ch(\widetilde{F})}{\chi(\widetilde{V}^*)}\Big),$$

and by [**7**, chap. V theorem 3.15 and 3.23]

$$ch(\widetilde{E}) = \exp(\chi(\widetilde{E})) \in H^*(\mathbf{CP}^\infty)$$

and

$$ch(\widetilde{F}) = ch(\widetilde{E}^2) = ch(\widetilde{E})^2 \in H^*(\mathbf{CP}^\infty).$$

Finally $\widetilde{E} = \widetilde{V}$ as real vector bundles, so $\chi(\widetilde{E}) = \chi(\widetilde{V})$ and

$$\begin{aligned} ch(D) &= f^*\Big(-\frac{ch(\widetilde{E}) - ch(\widetilde{F})}{\chi(\widetilde{E})}\Big) = f^*\Big(-\exp(\chi(\widetilde{E}))\frac{1 - \exp(\chi(\widetilde{E}))}{\chi(\widetilde{E})}\Big) \\ &= f^*\Big(-(1 + \chi(\widetilde{E}) + \ldots)\,\frac{1 - (1 + \chi(\widetilde{E}) + \frac{1}{2}\chi(\widetilde{E})^2 + \ldots)}{\chi(\widetilde{E})}\Big) \\ &= f^*\left((1 + \chi(\widetilde{E}) + \ldots)\,(1 + \tfrac{1}{2}\chi(\widetilde{E}) + \ldots)\right) \\ &= f^*\left(1 + \tfrac{3}{2}\chi(\widetilde{E}) + \ldots\right) = 1 + \tfrac{3}{2}\chi(f^*\widetilde{E}) \\ &= 1 + \tfrac{3}{2}\chi(E) = 1 + \tfrac{3}{2}\chi(TS^2), \end{aligned}$$

hence

$$\begin{aligned} i(D) &= \left(ch(D)\cdot \mathcal{T}(S^2)\right)[S^2] = \left((1 + \tfrac{3}{2}\chi(S^2))\cdot 1\right)[S^2] \\ &= \tfrac{3}{2}\chi(S^2) = \tfrac{3}{2}\cdot 2 = 3. \end{aligned}$$

and the proof is finished. ∎

Similarly, up to isomorphism, there is only one complex structure on the closed unit disk. To be precise, if $J_0$ is the standard complex structure on $\overline{D}$, then we have

(2.3) PROPOSITION. *Let $J \in \mathcal{C}(\overline{D})$ be a complex structure on the closed unit disk. Then there exists a homeomorphism $\phi_J\colon \overline{D} \to \overline{D}$, such that the restriction to the open disk $D$ is a holomorphic map $D_J \to D_{J_0}$. Furthermore $\phi_J$ can be chosen such that, if $J_n \to J$ in $\mathcal{C}(\overline{D})$, then $\phi_{J_n}|_K \to \phi_J|_K$ in the $C^\infty$-topology for all compact subsets $K$ of $D$.*

PROOF: By the results in [**17**] there exists a continuous map $f\colon \mathcal{C}(\overline{D}) \to \mathcal{C}(S^2)$ such that $f(J)|_{\overline{D}} = J$. By proposition 2.2 there exists a holomorphic diffeomorphism

$$\widetilde{\phi}_{f(J)}\colon S^2_{f(J)} \longrightarrow \mathbf{CP}^1,$$

which depends continuously on $f(J)$. By the Riemann mapping theorem, the domain $\widetilde{\phi}_{f(J)}(D)$ can be mapped conformally onto the unit disk $D$. As the boundary of $\widetilde{\phi}_{f(J)}(D)$ is $\widetilde{\phi}_{f(J)}(S^1)$, which is a Jordan curve, this conformal map extends, by [9, theorem 2.24], to a homeomorphism

$$\psi_J : \widetilde{\phi}_{f(J)}(\overline{D}) \longrightarrow \overline{D}.$$

If we choose $\psi_J$ such that $\psi_J(\widetilde{\phi}_{f(J)}(0)) = 0$ and $\psi'_J(\widetilde{\phi}_{f(J)}(0)) > 0$, then $\psi_J$ is uniquely determined. Finally put

$$\phi_J = \psi_J \circ \widetilde{\phi}_{f(J)}|_{\overline{D}}.$$

If $J_n \to J$ in $C(\overline{D})$, then $f(J_n) \to f(J)$ in $C(S^2)$ and $\widetilde{\phi}_{f(J_n)} \to \widetilde{\phi}_{f(J)}$ in the $C^\infty$-topology. Choose $z_0 \in \widetilde{\phi}_{f(J)}(D)$. Then $z_0 \in \widetilde{\phi}_{f(J_n)}(D)$ for $n$ sufficiently large and we can define

$$\widetilde{\psi}_n : \widetilde{\phi}_{f(J_n)}(\overline{D}) \longrightarrow \overline{D} \qquad \text{and} \qquad \widetilde{\psi} : \widetilde{\phi}_{f(J)}(\overline{D}) \to \overline{D}$$

to be the conformal map with $\widetilde{\psi}_n(z_0), \widetilde{\psi}(z_0) = 0$ and $\widetilde{\psi}'_n(z_0), \widetilde{\psi}'(z_0) > 0$. As $\widetilde{\phi}_{f(J_n)}(D)$ converges to $\widetilde{\phi}_{f(J)}(D)$ in the sense of [9, page 33], $\widetilde{\psi}_n{}^{-1}|_D \to \widetilde{\psi}^{-1}|_D$ uniformly on compact subsets of $D$, see [9, theorem 2.1] (Caratheodory's mapping theorem). As $\widetilde{\psi}_n^{-1}$ and $\widetilde{\psi}^{-1}$ are holomorphic maps, all derivatives converge uniformly on compact subsets of $D$.

Let $\theta_n = \widetilde{\psi}_n \circ \psi_{J_n}^{-1}$ and $\theta = \widetilde{\psi} \circ \psi_J^{-1}$. Then $\theta_n$ and $\theta$ are Möbius transformations of $D$ with

$$\theta_n(0) = \widetilde{\psi}_n(\widetilde{\phi}_{f(J_n)}(0)) \longrightarrow \widetilde{\psi}(\widetilde{\phi}_{f(J)}(0)) = \theta(0), \qquad \text{for } n \longrightarrow \infty,$$
$$\theta'_n(0) = \widetilde{\psi}'_n(\widetilde{\phi}_{f(J_n)}(0)) \Big/ \psi'_{J_n}(\widetilde{\phi}_{f(J_n)}(0)) > 0$$

and

$$\theta'(0) = \widetilde{\psi}'(\widetilde{\phi}_{f(J)}(0)) \Big/ \psi'_J(\widetilde{\phi}_{f(J)}(0)) > 0.$$

For any compact subset $K$ of $D$,

$$\theta_n|_K \longrightarrow \theta|_K \qquad \text{and} \qquad \psi_{J_n}^{-1}|_K = (\psi_n^{-1} \circ \theta_n)|_K \longrightarrow (\psi^{-1} \circ \theta)|_K = \psi_J^{-1}|_K$$

in the $C^\infty$-topology. This implies that $\phi_{J_n}|_K \to \phi_J|_K$ in the $C^\infty$-topology. ∎

If $J$ is a complex structure on a two dimensional manifold $M$, then we can define an inner product $(\ ,\ )_J$ on the bundle $\bigwedge^1 M_{\mathbb{C}}$ by

$$(\alpha, \beta)_J \Omega = \alpha \wedge -\overline{J^* \beta},$$

where $\Omega$ is the fixed volume form on $M$. On $\bigwedge^0 M_{\mathbf{C}}$ and $\bigwedge^2 M_{\mathbf{C}}$ there are inner products

$$(f,g)_J = (f\Omega, g\Omega)_J = f\bar{g},$$

which, despite the notation, are independent of $J$. If we define $J^* f = -f\Omega$ and $J^* f\Omega = -f$, then

$$(\phi,\psi)_J\Omega = \phi \wedge -\overline{J^*\psi} \qquad \text{all } \phi,\psi \in {\textstyle\bigwedge}^i M_{\mathbf{C}} \quad \text{and} \quad i = 0,1,2.$$

We define inner products on the space of sections $\Omega^i M_{\mathbf{C}}$ by

$$\langle \phi,\psi\rangle_J = \int_M (\phi,\psi)_J\Omega = \int_M \phi \wedge -\overline{J^*\psi}.$$

The complex structure $J$ also gives rise to operators

$$d, \bar{\partial}_J, \partial_J : \Omega^i M_{\mathbf{C}} \longrightarrow \Omega^{i+1} M_{\mathbf{C}} \qquad i = 0,1$$

and the adjoint operators

$$d_J^*, \bar{\partial}_J^*, \partial_J^* : \Omega^i M_{\mathbf{C}} \longrightarrow \Omega^{i-1} M_{\mathbf{C}} \qquad i = 1,2$$

defined by

$$\langle L\phi,\psi\rangle_J = \langle \phi, L_J^*\psi\rangle_J \qquad L = d, \bar{\partial}_J, \partial_J$$

We have the following expressions for the adjoint operators

(2.4) LEMMA. *If $M$ is compact and without boundary, then*

$$d_J^* = -J^* d J^*, \qquad \bar{\partial}_J^* = -J^*\partial_J J^* \qquad \text{and} \qquad \partial_J^* = -J^*\bar{\partial}_J J^*.$$

PROOF: Let $\phi \in \Omega^k M_{\mathbf{C}}$ and $\psi \in \Omega^{k+1} M_{\mathbf{C}}$. Then

$$\begin{aligned}
\langle d\phi,\psi\rangle_J &= \int_M d\phi \wedge -\overline{J^*\psi} = \int_M \left(d(\phi \wedge -\overline{J^*\psi}) - (-1)^k \phi \wedge d(-\overline{J^*\psi})\right) \\
&= \int_M (-1)^k \phi \wedge (-1)^k J^* J^* d\overline{J^*\psi} = -\int_M \phi \wedge -J^*\overline{J^* d J^*\psi} \\
&= \langle \phi, -J^* d J^*\psi\rangle_J
\end{aligned}$$

On $\Omega^0$, $\bar{\partial}_J = \pi_{0,1} \circ d$, so on $\Omega^1$ :

$$\bar{\partial}_J^* = d_J^* \circ \pi_{0,1}^* = -J^* d J^* \pi_{0,1} = -J^*(\partial_J + \bar{\partial}_J)\pi_{0,1} J^* = -J^*\partial_J J^*$$

and on $\Omega^1$, $\bar{\partial}_J = d \circ \pi_{1,0}$, so on $\Omega^2$ :

$$\bar{\partial}_J^* = \pi_{1,0}^* \circ d_J^* = \pi_{1,0}(-J^* d J^*) = -J^*\partial_J J^*$$

and similarly for $\partial_J$. ∎

We inductively define Sobolev inner products on $\Omega^i M_{\mathbf{C}}$ for $i = 0, 1, 2$ by

$$\begin{aligned}
\langle \phi, \psi \rangle_{J,0} &= \langle \phi, \psi \rangle_J, & \phi, \psi \in \Omega^i M_{\mathbf{C}},\\
\langle f, g \rangle_{J,k} &= \langle f, g \rangle_J + \langle df, dg \rangle_{J,k-1}, & f, g \in \Omega^0 M_{\mathbf{C}},\\
\langle \alpha, \beta \rangle_{J,k} &= \langle \alpha, \beta \rangle_J + \langle d\alpha, d\beta \rangle_{J,k-1} & \\
&\quad + \langle d_J^* \alpha, d_J^* \beta \rangle_{J,k-1}, & \alpha, \beta \in \Omega^1 M_{\mathbf{C}}
\end{aligned}$$

and

$$\langle \omega, \mu \rangle_{J,k} = \langle \omega, \mu \rangle_J + \langle d_J^* \omega, d_J^* \mu \rangle_{J,k-1} \qquad \omega, \mu \in \Omega^1 M_{\mathbf{C}}.$$

It is easily seen that $J^*$ is an isometry with respect to these inner products. As usual the corresponding Sobolev norms are defined by

$$\|\psi\|_{J,k} = \sqrt{\langle \psi, \psi \rangle_{J,k}}, \qquad \psi \in \Omega^i M_{\mathbf{C}} \qquad i = 0, 1, 2.$$

Fix a complex structure $J_0$ on $M$ and define for $T \in \operatorname{End}(T^* M_{\mathbf{C}}) = \operatorname{End}(\bigwedge^1 M_{\mathbf{C}})$ and $k \in \mathbf{N}$ a norm

$$\|T\|_k = \sup \left\{ \|T\alpha\|_{J_0,l} \;\middle|\; \|\alpha\|_{J_0,l} \leq 1 \text{ and } l \leq k \right\}.$$

Then for all $k$ :

$$\begin{aligned}
\|T\|_l &\leq \|T\|_k, & \text{if } l \leq k,\\
\|TS\|_k &\leq \|T\|_k \|S\|_k, & \\
\|T\alpha\|_k &\leq \|T\|_k \|\alpha\|_k & \text{and}\\
\|J_0^*\|_k &= \|1\|_k = 1. &
\end{aligned}$$

If $\alpha, \beta \in \bigwedge^1 M_{\mathbf{C}}$, then $\alpha \wedge J^* \beta = \alpha \wedge -J_0^* J_0^* J^* \beta$, so $\langle \alpha, \beta \rangle_J = -\langle \alpha, J_0^* J^* \beta \rangle_{J_0}$ and hence

$$\begin{aligned}
\left| \langle \alpha, \beta \rangle_J - \langle \alpha, \beta \rangle_{J_0} \right| &= \left| \langle \alpha, (1 + J_0^* J^*) \beta \rangle_{J_0} \right| \leq \|\alpha\|_{J_0} \, \|(1 + J_0^* J^*) \beta\|_{J_0}\\
&\leq \|1 + J_0^* J^*\|_0 \, \|\alpha\|_{J_0} \, \|\beta\|_{J_0} = \|J_0^*(J^* - J_0^*)\|_0 \, \|\alpha\|_{J_0} \, \|\beta\|_{J_0}\\
&\leq \|J_0^*\|_0 \, \|J^* - J_0^*\|_0 \, \|\alpha\|_{J_0} \, \|\beta\|_{J_0} = \|J^* - J_0^*\|_0 \, \|\alpha\|_{J_0} \, \|\beta\|_{J_0}
\end{aligned}$$

especially $\left| \|\alpha\|_J^2 - \|\alpha\|_{J_0}^2 \right| \leq \|J^* - J_0^*\|_0 \, \|\alpha\|_{J_0}^2$. This inequality generalizes to

(2.5) LEMMA. *If* $\|J^* - J_0^*\|_k \leq 1$, *then*

$$\left| \|f\|_{J,k}^2 - \|f\|_{J_0,k}^2 \right| \leq 4^{k-1} \, \|J^* - J_0^*\|_{k-1} \, \|f\|_{J_0,k}^2, \qquad \textit{all } f \in \Omega^0 M_{\mathbf{C}}$$

*and*

$$\big|\|\alpha\|^2_{J,k} - \|\alpha\|^2_{J_0,k}\big| \le 4^k \, \|J^* - J_0^*\|_k \, \|f\|^2_{J_0,k}, \qquad \textit{all } \alpha \in \Omega^1 M_{\mathbf{C}}$$

PROOF: By induction on $k$. The case $k = 0$ is trivial for $f$ and shown above for $\alpha$. If $k > 0$ and the lemma is true for $k - 1$, then

$$\begin{aligned}\big|\|f\|^2_{J,k} - \|f\|^2_{J_0,k}\big| &= \big|\|df\|^2_{J,k-1} - \|df\|^2_{J_0,k-1}\big| \\ &\le 4^{k-1}\|J^* - J_0^*\|_{k-1} \, \|df\|^2_{J_0,k-1} \\ &\le 4^{k-1}\|J^* - J_0^*\|_{k-1} \, \|f\|^2_{J_0,k}\end{aligned}$$

If $k = 1$, then

$$\begin{aligned}\big|\|\alpha\|^2_{J,1} - \|\alpha\|^2_{J_0,1}\big| &\le \big|\|\alpha\|^2_J - \|\alpha\|^2_{J_0}\big| + \big|\|d_J^*\alpha\|^2 - \|d_{J_0}^*\alpha\|^2\big| \\ &\le \|J^* - J_0^*\|_0 \, \|\alpha\|^2_{J_0} + \big(\|d_J^*\alpha\| + \|d_{J_0}^*\alpha\|\big) \, \big\|(d_J^* - d_{J_0}^*)\alpha\big\| .\end{aligned}$$

As

$$\big\|(d_J^* - d_{J_0}^*)\alpha\big\| = \|d(J^* - J_0^*)\alpha\| \le \|(J^* - J_0^*)\alpha\|_{J_0,1} \le \|J^* - J_0^*\|_1 \, \|\alpha\|_{J_0,1}$$

and

$$\begin{aligned}\|d_J^*\alpha\| + \|d_{J_0}^*\alpha\| &\le 2\|d_{J_0}^*\alpha\| + \|d(J^* - J_0^*)\alpha\| \\ &\le 2\|\alpha\|_{J_0,1} + \|(J^* - J_0^*)\alpha\|_{J_0,1} \\ &\le (2 + \|J^* - J_0^*\|_1) \, \|\alpha\|_{J_0,1} \le 3\|\alpha\|_{J_0,1},\end{aligned}$$

we see that

$$\begin{aligned}\big|\|\alpha\|^2_{J,1} - \|\alpha\|^2_{J_0,1}\big| &\le \|J^* - J_0^*\|_0 \, \|\alpha\|^2_{J_0} + 3\|J^* - J_0^*\|_0 \, \|\alpha\|^2_{J_0,1} \\ &\le 4\|J^* - J_0^*\|_0 \, \|\alpha\|^2_{J_0,1}.\end{aligned}$$

If $k > 1$ and the lemma is true for $k - 1$, then

$$\begin{aligned}\big|\|\alpha\|^2_{J,k} - \|\alpha\|^2_{J_0,k}\big| &\le \big|\|\alpha\|^2_J - \|\alpha\|^2_{J_0}\big| + \big|\|d\alpha\|^2_{J,k-1} - \|d\alpha\|^2_{J_0,k-1}\big| \\ &\quad + \big|\|d_J^*\alpha\|^2_{J,k-1} - \|d_{J_0}^*\alpha\|^2_{J_0,k-1}\big| \\ &\le \big|\|\alpha\|^2_J - \|\alpha\|^2_{J_0}\big| + \big|\|d\alpha\|^2_{J,k-1} - \|d\alpha\|^2_{J_0,k-1}\big| \\ &\quad + \big|\|d_J^*\alpha\|^2_{J,k-1} - \|d_J^*\alpha\|^2_{J_0,k-1}\big| \\ &\quad + \big|\|d_J^*\alpha\|^2_{J_0,k-1} - \|d_{J_0}^*\alpha\|^2_{J_0,k-1}\big| .\end{aligned}$$

As $J^*d\alpha = J_0^*d\alpha$, the first two terms gives

$$\begin{aligned}&\big|\|\alpha\|^2_J - \|\alpha\|^2_{J_0}\big| + \big|\|d\alpha\|^2_{J,k-1} - \|d\alpha\|^2_{J_0,k-1}\big| \\ &\qquad \le \|J^* - J_0^*\|_0 \, \|\alpha\|^2_{J_0} + 4^{k-2}\|J^* - J_0^*\|_{k-2} \, \|d\alpha\|^2_{J_0,k-1} \\ &\qquad \le 4^{k-2}\|J^* - J_0^*\|_k \, \|d\alpha\|^2_{J_0,k}.\end{aligned}$$

The third term gives

$$\left| \|d_J^*\alpha\|_{J,k-1}^2 - \|d_J^*\alpha\|_{J_0,k-1}^2 \right| \leq 4^{k-2} \|J^* - J_0^*\|_{k-2} \|d_J^*\alpha\|_{J_0,k-1}^2,$$

and as

$$\begin{aligned} \|d_J^*\alpha\|_{J_0,k-1}^2 &= \|dJ_0^* J_0^* J^* \alpha\|_{J_0,k-1}^2 \leq \|J_0^* J^* \alpha\|_{J_0,k}^2 \\ &\leq \|J_0^*\|_k^2 \|J^*\|_k^2 \|\alpha\|_{J_0,k}^2 \leq \left(\|J^* - J_0^*\|_k + \|J_0^*\|_k\right)^2 \|\alpha\|_{J_0,k}^2 \\ &\leq 4\|\alpha\|_{J_0,k}^2, \end{aligned}$$

we get

$$\left| \|d_J^*\alpha\|_{J,k-1}^2 - \|d_J^*\alpha\|_{J_0,k-1}^2 \right| \leq 4^{k-1} \|J^* - J_0^*\|_{k-2} \|\alpha\|_{J_0,k}^2.$$

We saw that $\|d_J^*\alpha\|_{J_0,k-1}^2 \leq 2\|\alpha\|_{J_0,k}^2$, so the fourth term gives

$$\begin{aligned} &\left| \|d_J^*\alpha\|_{J_0,k-1}^2 - \|d_{J_0}^*\alpha\|_{J_0,k-1}^2 \right| \\ &\qquad \leq \left(\|d_J^*\alpha\|_{J_0,k-1} + \|d_{J_0}^*\alpha\|_{J_0,k-1}\right) \left\|(d_J^* d_{J_0}^*)\alpha\right\|_{J_0,k-1}^2 \\ &\qquad \leq 3\|\alpha\|_{J_0,k} \|d(J^* - J_0^*)\alpha\|_{J_0,k-1} \\ &\qquad \leq 3\|\alpha\|_{J_0,k} \|(J^* - J_0^*)\alpha\|_{J_0,k} \\ &\qquad \leq 3\|J^* - J_0^*\| \|\alpha\|_{J_0,k}^2. \end{aligned}$$

All in all

$$\begin{aligned} \left| \|\alpha\|_{J,k}^2 - \|\alpha\|_{J_0,k}^2 \right| &\leq \left(4^{k-2} + 4^{k-1} + 3\right) \|J^* - J_0^*\|_k \|\alpha\|_{J_0,k}^2 \\ &\leq 4^k \|J^* - J_0^*\|_k \|\alpha\|_{J_0,k}^2, \end{aligned}$$

which finishes the proof. ∎

Let $\lambda$ be the first positive eigenvalue for the Laplacian

$$\Delta_J = d_J^* d = 2\overline{\partial}_J^* \overline{\partial}_J,$$

acting on functions. If $f \perp \ker \Delta_J$, i.e. $\int_M f\,\Omega = 0$, then

$$\langle \overline{\partial}_J f, \overline{\partial}_J f \rangle_J = \langle f, \overline{\partial}_J^* \overline{\partial}_J f \rangle_J = \tfrac{1}{2} \langle f, \Delta_J f \rangle_J \geq \tfrac{\lambda}{2} \langle f, f \rangle_J.$$

So $\|f\|^2 \leq \frac{2}{\lambda} \|\overline{\partial}_J f\|_J^2$. Furthermore

$$\begin{aligned} \langle df, df \rangle_J &= \langle f, d_J^* df \rangle_J = \langle f, 2\overline{\partial}_J^* \overline{\partial}_J f \rangle_J = 2\langle \overline{\partial}_J f, \overline{\partial}_J f \rangle_J, \\ \|d_J^* df\|_{J,k} &= 2\|\overline{\partial}_J^* \overline{\partial}_J f\|_{J,k} = 2\|J^* \partial_J J^* \overline{\partial}_J f\|_{J,k} = 2\|d_J^* \overline{\partial}_J f\|_{J,k}, \end{aligned}$$

and hence

$$\|df\|_{J,k}^2 = \|df\|_J^2 + \|d_J^* df\|_{J,k-1}^2 = 2\|\overline{\partial}_J f\|_J^2 + 4\|d_J^* \overline{\partial}_J f\|_{J,k-1}^2 \leq 4\|\overline{\partial}_J f\|_{J,k}^2.$$

We can now show

(2.6) PROPOSITION. *Let $f_n$, $f \in \Omega^0 M_{\mathbf{C}}$ and let $J_n$, $J \in C(M)$. Suppose that $\int_M f_n \Omega = \int_M f \Omega = 0$ all $n \in \mathbf{N}$, $J_n \to J$ and $\overline{\partial}_{J_n} f_n \to \overline{\partial}_J f$ in the $C^\infty$-topology. Then $f_n \to f$ in the $C^\infty$-topology.*

PROOF: First observe that

$$\|f_n - f\|_{J,k}^2 \leq (1 + \tfrac{2}{\lambda})\|d(f_n - f)\|_{J,k-1}^2 \leq 4(1 + \tfrac{2}{\lambda})\|\overline{\partial}_J(f_n - f)\|_{J,k-1}^2$$

and

$$\|\overline{\partial}_J(f - f_n)\|_{J,k-1} \leq \|\overline{\partial}_J f - \overline{\partial}_{J_n} f_n\|_{J,k-1} + \|(\overline{\partial}_{J_n} - \overline{\partial}_J) f_n\|_{J,k-1}.$$

As $\overline{\partial}_{J_n} \to \overline{\partial}_J$ we only need to show that $\|f_n\|_{J,k}$ is bounded. If we in lemma 2.5 replace $J_0$ with $J$, then

$$\begin{aligned}
\|f_n\|_{J,k}^2 &\leq (1 + \tfrac{2}{\lambda})\|df_n\|_{J,k-1}^2 \\
&\leq (1 + \tfrac{2}{\lambda})\left(1 + 4^{k-1}\|J^* - J_n^*\|\right)\|df_n\|_{J_n,k-1}^2 \\
&\leq (1 + \tfrac{2}{\lambda})\left(1 + 4^{k-1}\|J^* - J_n^*\|\right)\|\overline{\partial}_{J_n} f_n\|_{J_n,k-1}^2 \\
&\leq (1 + \tfrac{2}{\lambda})\left(1 + 4^{k-1}\|J^* - J_n^*\|\right)^2\|\overline{\partial}_{J_n} f_n\|_{J,k-1}^2,
\end{aligned}$$

which is bounded, because $J_n \to J$ and $\overline{\partial}_{J_n} f_n \to \overline{\partial}_J f$. ∎

The $J$-harmonic one-forms are characterized by being closed and orthogonal to the exact one-forms with respect to $\langle\ ,\ \rangle_J$. We fix a basis $(\alpha_1(J), \ldots, \alpha_{2g}(J))$ for the $J$-harmonic one-forms by demanding that

$$\int_{c_i} \alpha_j(J) = \delta_{i,j} \qquad i,j = 1,2,\ldots,2g,$$

where $(c_1, c_2, \ldots, c_{2g})$ is a fixed canonical homology basis.

(2.7) PROPOSITION. *Let $(\alpha_1(J), \ldots, \alpha_{2g}(J))$ be a basis for the $J$-harmonic one-forms as above. If $J_n \to J$ in the $C^\infty$-topology, then $\alpha_i(J_n) \to \alpha_i(J)$ in the $C^\infty$-topology, all $i = 1,2,\ldots,2g$.*

PROOF: Let $i \in \{1,2,\ldots,2g\}$ be given. To ease the notation put $\alpha_n = \alpha_i(J_n)$ and $\alpha = \alpha_i(J)$. As $\alpha$ and $\alpha_n$ represent the same cohomology class, $\alpha_n = \alpha + d\phi_n$, where $d\phi_n$ is uniquely determined by

$$\langle d\phi_n, d\psi\rangle_{J_n} = \langle \alpha, d\psi\rangle_{J_n} \qquad \text{all } d\psi.$$

We shall show that $d\phi_n \to 0$. Given $k \in \mathbf{N}$, we may assume that $\|J_n^* - J^*\|_k \leq 1$, and then

$$\|d\phi_n\|_{J,k}^2 \leq (1 + 4^k)\|d\phi_n\|_{J_n,k}^2,$$

so we only need to show that

$$\|d\phi_n\|_{J_n,k} \longrightarrow 0 \qquad \text{all } k.$$

First consider the case $k = 0$. As

$$\begin{aligned}\|d\phi_n\|^2_{J_n} &= \langle d\phi_n, d\phi_n\rangle_{J_n} = \langle \alpha, d\phi_n\rangle_{J_n} - \langle \alpha, d\phi_n\rangle_J \\ &= -\langle \alpha, (1 + J^* J_n^*) d\phi_n\rangle_J \le \|\alpha\|_J \, \|1 + J^* J_n^*\| \, \|d\phi_n\|_J \\ &\le \|\alpha\|_J \, \|J^* - J_n^*\| \, 2\|d\phi_n\|_{J_n},\end{aligned}$$

we have

$$\|d\phi_n\|_{J_n} \le 2\|\alpha\|_J \, \|J^* - J_n^*\| \longrightarrow 0.$$

If $k > 0$, then we put

$$L_n = \underbrace{\ldots d^*_{J_n} d\, d^*_{J_n}}_{k \text{ terms}}.$$

The adjoint with respect to $\langle \ , \ \rangle_{J_n}$ is

$$L_n^* = \underbrace{d\, d^*_{J_n} d \ldots}_{k \text{ terms}}.$$

Similarly we put

$$L = \underbrace{\ldots d^*_J d\, d^*_J}_{k \text{ terms}} \qquad \text{and} \qquad L^* = \underbrace{d\, d^*_J d \ldots}_{k \text{ terms}}.$$

As $\|d\phi_n\|^2_{J_n,k} = \|d\phi_n\|^2_{J_n,k-1} + \|L_n d\phi_n\|^2_{J_n}$, an induction argument gives that we only need to consider the last term.

$$\begin{aligned}\|L_n d\phi_n\|^2_{J_n} &= \langle d\phi_n, L_n^* L_n d\phi_n\rangle_{J_n} \\ &= \langle \alpha, L_n^* L_n d\phi_n\rangle_{J_n} - \langle \alpha, L^* L d\phi_n\rangle_J \\ &= \langle L_n\alpha, L_n d\phi_n\rangle_{J_n} - \langle L\alpha, L d\phi_n\rangle_J \\ &= -\langle L_n\alpha, J^* J_n^* L_n d\phi_n\rangle_J - \langle L\alpha, L d\phi_n\rangle_J \\ &= \langle (L - L_n)\alpha, J^* J_n^* L_n d\phi_n\rangle_J - \langle L\alpha, (J^* J_n^* L_n + L) d\phi_n\rangle_J \\ &\le \|(L - L_n)\alpha\|_J \, \|J^* J_n^* L_n d\phi_n\|_J + \|L\alpha\|_J \, \|(J^* J_n^* L_n + L) d\phi_n\|_J.\end{aligned}$$

As $L_n \to L$ and $J^* J_n^* \to -1$, we only need to show that $\|d\phi_n\|_{J,k}$ is bounded, and as $\|d\phi_n\|^2_{J,k} \le (1 + 4^k)\|d\phi_n\|^2_{J_n,k}$, we can show instead that $\|d\phi_n\|^2_{J_n,k}$ is bounded. The case $k = 0$ is already shown, and if $k > 0$, then as above we only need to consider $\|L_n d\phi_n\|_{J_n}$. We have

$$\begin{aligned}\|L_n d\phi_n\|^2_{J_n} &= \langle d\phi_n, L_n^* L_n d\phi_n\rangle_{J_n} = \langle \alpha, L_n^* L_n d\phi_n\rangle_{J_n} \\ &= \langle L_n\alpha, L_n D\phi_n\rangle \le \|L_n\alpha\|_{J_n} \, \|L_n d\phi_n\|_{J_n},\end{aligned}$$

so

$$\|L_n d\phi_n\|_{J_n} \leq \|L_n\alpha\|_{J_n} \leq \|\alpha\|_{J_n,k} \leq (1+4^k)\|\alpha\|_{J,k},$$

and the proof is complete. ∎

We get a basis $(\omega_1(J), \omega_2(J), \ldots, \omega_g(J))$ for the $J$-holomorphic differentials by putting

$$\omega_j(J) = \alpha_j(J) - iJ^*\alpha_j(J) \qquad j = 1, 2, \ldots, g.$$

Hence the holomorphic differentials depend continuously on the complex structure. Similarly we have

(2.8) PROPOSITION. *The Weierstrass points depend continuously on the complex structure.*

PROOF: It is a local question, so consider the Weierstrass points in some disk $D' \subseteq M$. Choose, continuously dependent on $J$, a $J$-holomorphic homeomorphism $\phi_J \colon D \to D'$. Let $(\omega_1(J), \omega_2(J), \ldots, \omega_g(J))$ be a basis for the $J$-holomorphic differentials as above and define holomorphic functions $f_{J,j} \colon D \to \mathbf{C}$ by

$$f_{J,j}\, dz = \phi_J^*(\omega_j(J)) \qquad j = 1, 2, \ldots, g.$$

These functions depend continuously on $J$ as does the matrix

$$[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)] = \begin{pmatrix} f_{J,1} & f_{J,2} & \cdots & f_{J,g} \\ f'_{J,1} & f'_{J,2} & \cdots & f'_{J,g} \\ \vdots & \vdots & & \vdots \\ f_{J,1}^{(g-1)} & f_{J,2}^{(g-1)} & \cdots & f_{J,g}^{(g-1)} \end{pmatrix}$$

Now we only have to observe that the $J$-Weierstrass points in $D'$ is the image by $\phi_J$ of the zeros of $\det[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)]$. ∎

With the same notation as above, assume that $\phi_J(0)$ is the same point $p$ for all $J \in \mathcal{C}(M)$ and that $p$ is a non-Weierstrass point in the complex structure $J_0$. For $J$ in a neighbourhood of $J_0$, $\det[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)] \neq 0$. So the inverse matrix $[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)](0)^{-1}$ exists, and it depends continuously on $J$. If

$$(\xi_1(J), \xi_2(J), \ldots, \xi_g(J)) = \\ (\omega_1(J), \omega_2(J), \ldots, \omega_g(J))[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)](0)^{-1},$$

then $(\xi_1(J), \xi_2(J), \ldots, \xi_g(J))$ is a basis for the $J$-holomorphic differentials adapted to the point $p$, and we have shown

(2.9) LEMMA. *If $p$ is a non $J_0$-Weierstrass point, then for $J$ in a neighbourhood of $J_0$, we can find a basis for the $J$-holomorphic differentials adapted to the point $p$, which depends continuously on $J$.*

If $U$ is a domain in $\mathbf{C}$ and $f \colon U \to \mathbf{CP}^1$ is a meromorphic function with a finite number of poles, then we can write $f = \frac{p}{q} + h$, where $p$ and $q$ are polynomials and $h \colon U \to \mathbf{C}$ is holomorphic. The following lemma is a generalization of this result.

(2.10) LEMMA. *Let $M$ be a closed surface and let $D_0$, $D_1$ and $D_2$ be open disks in $M$ such that $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ and $\overline{D}_0 \subseteq D_1$. Put $T = D_1 \setminus \overline{D}_0$, let $J$ be a complex structure on $M$ and let $Q \in D_2$ be a non $J$-Weierstrass point. If*

$$f\colon T \longrightarrow \mathbf{C} \text{ is } J\text{-holomorphic},$$

*then there exist unique $J$-holomorphic functions*

$$F_1\colon D_1 \longrightarrow \mathbf{C} \qquad \text{and} \qquad F_2\colon M \setminus (\overline{D}_0 \cap \{Q\}) \longrightarrow \mathbf{C},$$

*such that if $z$ is a $J$-parameter vanishing at $Q$, then*

$$f = F_1|_T + F_2|_T \qquad \text{and} \qquad F_2(z) = \sum_{n=-g}^{\infty} d_n z^n \text{ with } d_0 = 0.$$

*Furthermore, if $z$ depends continuously on $J$ (which we may assume), then $F_1$ and $F_2$ depend continuously on $f$ and $J$ in the compact-open topology, as long as $Q$ is a non-Weierstrass point.*

PROOF: Uniqueness is clear. To prove the existence, we first consider the case $f(w) = \sum_{n=-n}^{N} c_n w^n$, where $w$ is a parameter on $D_1$, vanishing at $P \in D_0$. From lemma 2.1 we know that there exists a meromorphic function $F_2$ on $M$, which at $P$ has the same principal part as $f$, has no poles outside $\{P, Q\}$ and at $Q$ has the expression $F_2(z) = \sum_{n=-g}^{\infty} d_n z^n$. We may of course assume that $d_0 = 0$. If we put $F_1 = f - F_2|_T$, then $F_1$ extends to a holomorphic function $D_1 \to \mathbf{C}$.

The next step is to show that $F_1$ and $F_2$ depend continuously on $f$ and $J$. For that purpose we will determine the principal part of $F_2$ at $Q$.

Let $c_1$ be a circle in $T$ around $P$ and let $c_2$ be a circle in $D_2$ around $Q$. Let $(\xi_1, \xi_2, \ldots, \xi_g)$ is a basis for the $J$-holomorphic differentials adapted to the point $Q$, i.e.

$$\xi_j = \big(z^j + (\text{order} \geq g)\big)\, dz.$$

The principal part of $F_2$ at $Q$ is

$$f' = \sum_{n=-g}^{-1} d_n z^n,$$

and the coefficients $d_{-1}, d_{-2}, \ldots, d_{-g}$ can be determined by

$$d_{-k} = \int_{c_2} f'\xi_k = \int_{c_2} F_2 \xi_k = \pm \int_{c_1} F_2 \xi_k = \pm \int_{c_1} f \xi_k.$$

If we choose $z$ to depend continuously on $J$, then $(\xi_1, \xi_2, \ldots, \xi_g)$ and the numbers $d_{-1}, d_{-2}, \ldots, d_{-g}$ depends continuously on $J$. Hence if we consider $f'$ as a function $D_2 \setminus \{Q\} \to \mathbf{C}$, then $f'$ depend continuously on $J$ and $f$. If $\widetilde{D}_1 \subseteq D_1$ and $\widetilde{D}_2 \subseteq D_2$ are closed disks containing $\overline{D}_0$ and $Q$ respectively in theirs interior, then we can extend $f|_{\tilde{D}_1 \setminus D_0}$ and $f'|_{\tilde{D}_2 \setminus \{Q\}}$ to one smooth function $G\colon M \setminus (D_0 \cup \{Q\}) \to \mathbf{C}$ such that $G$ depends continuously on $f$ and $f'$ and hence on $f$ and $J$.

We define a differential $g$ on $M$ by

$$g = \begin{cases} -\bar{\partial}_J G, & \text{on } M \setminus (\tilde{D}_1 \cup \tilde{D}_2) \\ 0, & \text{on } \tilde{D}_1 \cup \tilde{D}_2 \end{cases} .$$

This complex one-form on $M$ depends continuously on $f$ and $J$. Consider the equation

$$\bar{\partial}_J u = g \qquad \text{and} \qquad u(Q) = 0$$

on $M$. As

(1) $F_2 - G$ is a solution on $M \setminus (\tilde{D}_1 \cup \tilde{D}_2)$,
(2) $(F_2 - G)|_T = F_2|_T - f = F_1|_T$ extends $J$-holomorphicly to $D_1$ and
(3) $(F_2 - G)|_{D_2 \setminus \{Q\}} = F_2|_{D_2 \setminus \{Q\}} - f'$ extends $J$-holomorphicly to $D_2$,

the equation does have a solution. By proposition 2.6 the solution depends continuously on $J$ and $g$, and hence on $J$ and $f$. This implies that $F_1$ and $F_2$ depend continuously on $f$ and $J$.

If $f\colon T \to \mathbf{C}$ is an arbitrarily $J$-holomorphic function, then we can write

$$f = \lim_{n \to \infty} f_n \qquad \text{with} \qquad f_n = \sum_{k=-n}^{n} c_k w^k.$$

As $f_n \mapsto (F_{1,n}, F_{2,n})$ is linear and continuous, we can put $F_1 = \lim_{n \to \infty} F_{1,n}$ and $F_2 = \lim_{n \to \infty} F_{2,n}$. ∎

## 3. Flagmanifolds

Let $\mathbf{k} = (k_1, k_2, \ldots, k_r)$ be an ordered set of positive integers and put $n = \sum k_i$. The *(generalized) flag manifold* $Fl_{\mathbf{k}}$ is the space of subspaces $(E_1, E_2, \ldots, E_r)$ of $\mathbf{C}^n$, such that $\dim(E_i) = k_1 + k_2 + \cdots + k_i$ and $E_1 \subsetneq E_2 \subsetneq \ldots \subsetneq E_r = \mathbf{C}^n$. If $\mathbf{k} = (1, 1, \ldots, 1)$, then we have an ordinary flag manifold and write $Fl(\mathbf{C}^n)$ for $Fl_{\mathbf{k}}$. If $\mathbf{k} = (k, n-k)$, then $Fl_{\mathbf{k}}$ is the *Grassmannian* $Gr_k(\mathbf{C}^n)$ of all k-dimensional subspaces of $\mathbf{C}^n$, and if $\mathbf{k} = (1, n)$, then $Fl_{\mathbf{k}} = Gr_1(\mathbf{C}^{n+1})$ is the projective space $\mathbf{CP}^n$.

The flag manifold $Fl_{\mathbf{k}}$ is a homogeneous space under the action of both the unitary group $U_n$ and the general linear group $Gl_n(\mathbf{C})$. The isotropy groups of the natural basepoint $(\mathbf{C}^{k_1}, \mathbf{C}^{k_1+k_2}, \ldots, \mathbf{C}^n)$ are $U_{\mathbf{k}} = U_{k_1} \times U_{k_2} \times \ldots \times U_{k_r} \subseteq U_n$ respectively the group $P_{\mathbf{k}} \subseteq Gl_n(\mathbf{C})$ of upper echelon matrices of type $\mathbf{k}$, i.e. matrices of the form

$$\begin{pmatrix} A_1 & * & \ldots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & A_r \end{pmatrix}$$

with $A_i \in Gl_{k_i}(\mathbf{C})$. Thus

$$Fl_{\mathbf{k}} = U_n/U_{\mathbf{k}} = Gl_n(\mathbf{C})/P_{\mathbf{k}}.$$

The flag manifold $Fl_{\mathbf{k}}$ is a subspace of a product of Grassmanians

$$Fl_{\mathbf{k}} \subseteq Gr_{k_1}(\mathbf{C}^n) \times Gr_{k_1+k_2}(\mathbf{C}^n) \times \ldots \times Gr_{k_1+\cdots+k_{r-1}}(\mathbf{C}^n),$$

and, as the Grassmanian $Gr_k(\mathbf{C}^n)$ can be imbedded in a projective space of dimension $\binom{n}{k} - 1$ by the map $\mathbf{C}^n \supseteq E \mapsto \bigwedge^k E \subseteq \bigwedge^k \mathbf{C}^n$, a flag manifold is a complex projective variety.

An element $(E_1, E_2, \ldots, E_r)$ of $Fl_{\mathbf{k}}$ can be represented by a $(n \times n)$-matrix $(a_{ij})$ in $Gl_n(\mathbf{C})$, such that $E_i$ is the span of the first $k_1 + k_2 + \cdots + k_i$ columns, where we regard the columns as elements of $\mathbf{C}^n$. The map

$$Fl_{\mathbf{k}} \longrightarrow \mathbf{CP}^{\binom{n}{k_1+\cdots+k_l}-1} : (E_1, E_2, \ldots, E_r) \longmapsto \bigwedge^{k_1+\cdots+k_l} E_l$$

is then given by

$$[a_{ij}] \longmapsto [\det(a_{ij})_{i,j \le k_1+\cdots+k_l}, \ldots],$$

where the determinant is taken of all possible $(k_1 + \cdots + k_l) \times (k_1 + \cdots + k_l)$-submatrices of the matrix formed by the first $k_1 + \cdots + k_l$ columns.

The part of $Fl_{\mathbf{k}}$, which lies in

$$\mathbf{C}^{\binom{n}{k_1}-1} \times \ldots \times \mathbf{C}^{\binom{n}{k_1+\cdots+k_{r-1}}-1} \subseteq \mathbf{CP}^{\binom{n}{k_1}-1} \times \ldots \times \mathbf{CP}^{\binom{n}{k_1+\cdots+k_{r-1}}-1}$$

is called the *affine part* of $Fl_{\mathbf{k}}$ and is denoted $(Fl_{\mathbf{k}})_a$. Clearly

$$(Fl_{\mathbf{k}})_a = \left\{ [a_{ij}] \;\middle|\; \det(a_{ij})_{i,j \le k_1+\cdots+k_l} \neq 0 \text{ all } l \right\},$$

and any element of $(Fl_{\mathbf{k}})_a$ can uniquely be represented by an $(n \times n)$-matrix of the form

$$\begin{pmatrix} E_1 & 0 & \ldots & 0 \\ * & E_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \ldots & * & E_r \end{pmatrix}$$

where $E_i$ is the identity $(k_i \times k_i)$-matrix. So as an algebraic variety, $(Fl_{\mathbf{k}})_a$ is a vector space. Furthermore, such matrices form a subgroup $N_{\mathbf{k}}$ of $Gl_n(\mathbf{C})$, which acts on $Fl_{\mathbf{k}}$ from the left and acts transitively and freely on $(Fl_{\mathbf{k}})_a$.

The *infinite part* $(Fl_{\mathbf{k}})_\infty$ of $Fl_{\mathbf{k}}$ is defined as

$$(Fl_{\mathbf{k}})_\infty = Fl_{\mathbf{k}} \setminus (Fl_{\mathbf{k}})_a$$

and is a subvariety of $Fl_{\mathbf{k}}$ given by the equation

$$\prod_{l=1}^{r-1} \det(a_{ij})_{i,j \le k_1+\cdots+k_l} = 0.$$

Unless $r = 2$ and we are considering a Grassmanian, $(Fl_{\mathbf{k}})_\infty$ is reducible with irreducible components $Y_1, Y_2, \ldots, Y_{r-1}$, where $Y_l$ is given by the equation

$$\det(a_{ij})_{i,j \leq k_1+\cdots+k_l} = 0.$$

From the fibration $U_{\mathbf{k}} \to U_n \to Fl_{\mathbf{k}}$ we see that $\pi_1(Fl_{\mathbf{k}}) = 0$, $\pi_2(Fl_{\mathbf{k}}) = \mathbf{Z}^{r-1}$ and hence

$$H_1(Fl_{\mathbf{k}}) = 0, \qquad H_2(Fl_{\mathbf{k}}) = \mathbf{Z}^{r-1}$$

and

$$H_{\dim(Fl_{\mathbf{k}})-2}(Fl_{\mathbf{k}}) \cong H^2(Fl_{\mathbf{k}}) \cong H_2(Fl_{\mathbf{k}})^* = \mathbf{Z}^{r-1}.$$

For $i = 1, \ldots, r-1$, there is an imbedding $\phi_i \colon \mathbf{CP}^1 \to Fl_{\mathbf{k}}$ given by

$$\phi_i(E) = (\mathbf{C}^{k_1}, \ldots, \mathbf{C}^{k_1+\cdots+k_{i-1}}, \mathbf{C}^{k_1+\cdots+k_i-1} \oplus E, \mathbf{C}^{k_1+\cdots+k_{i+1}}, \ldots, \mathbf{C}^n),$$

and we have the intersection indices

$$\phi_i(\mathbf{CP}^1) \cdot Y_j = \delta_{ij}.$$

Thus $(\phi_1(\mathbf{CP}^1), \ldots, \phi_{r-1}(\mathbf{CP}^1))$ is a basis for $H_2$ and $(Y_1, \ldots, Y_{r-1})$ is the dual basis for $H_{\dim(Fl_{\mathbf{k}})-2} = H_2^*$.

If $X$ is a closed Riemann surface, then the space of maps $f \colon X \to Fl_{\mathbf{k}}$ has components labelled by a multi-degree $(\deg_1 f, \ldots, \deg_{r-1} f) \in \mathbf{Z}^{r-1}$ where

$$\deg_i f = f(X).Y_i$$

is the intersection index between $f(X)$ and $Y_i$. We define a single degree by

$$\deg f = \deg_1 f + \cdots + \deg_{r-1} f = f(X).Y_\infty.$$

If $U$ is an open subset of a Riemann surface and $f \colon U \to Fl_{\mathbf{k}}$ is a holomorphic map with $f(U) \cap (Fl_{\mathbf{k}})_a \neq \emptyset$, then we can consider $f$ as a meromorphic map into $(Fl_{\mathbf{k}})_a \cong N_{\mathbf{k}} \cong \mathbf{C}^{\dim(N_{\mathbf{k}})}$. The set of poles, $f^{-1}((Fl_{\mathbf{k}})_\infty)$, is discrete, and hence each point $\alpha \in U$ has a neighbourhood $V$ of $\alpha$ such that $f^{-1}((Fl_{\mathbf{k}})_\infty) \cap V \subseteq \{\alpha\}$. For $i = 1, \ldots, r-1$, the $i$'th order of $f$ at $\alpha$ is defined as

$$\mathrm{ord}_{i,\alpha} f = \text{order of contact between } f(V) \text{ and } Y_i$$

and the total order of $f$ at $\alpha$ as

$$\begin{aligned} \mathrm{ord}_\alpha f &= \mathrm{ord}_{1,\alpha} f + \mathrm{ord}_{2,\alpha} f + \cdots + \mathrm{ord}_{r-1,\alpha} f \\ &= \text{order of contact between } f(V) \text{ and } (Fl_{\mathbf{k}})_\infty. \end{aligned}$$

If $U = X$, then

$$\deg_i f = \sum_{\alpha \in X} \mathrm{ord}_{i,\alpha} f = \sum_{\alpha \in f^{-1}(F_\infty)} \mathrm{ord}_{i,\alpha} f$$

and

$$\deg f = \sum_{\alpha \in X} \operatorname{ord}_\alpha f = \sum_{\alpha \in f^{-1}(F_\infty)} \operatorname{ord}_\alpha f.$$

Recall that the group $N_{\mathbf{k}} \subseteq GL_n(\mathbf{C})$ consists of matrices of the form

$$A = \begin{pmatrix} E_1 & 0 & \dots & 0 \\ A_{2,1} & E_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{r,1} & \dots & A_{r,r-1} & E_r \end{pmatrix},$$

where $E_i$ is the identity $(k_i \times k_i)$-matrix and $A_{i,j}$ is an arbitrary $(k_i \times k_j)$-matrix. For $l = 1, 2, \dots, r-1$ we put

$$N_l = \{A \mid i - j \neq k \implies A_{i,j} = 0\}$$

and

$$N_{\geq l} = N_l \oplus \cdots \oplus N_r.$$

Let $\pi_l$ denote the projection $N_{\mathbf{k}} = N_1 \oplus \cdots \oplus N_{r-1} \to N_l$ and $\pi_{\leq l}$ denote the projection $N_{\mathbf{k}} \to N_{\geq l}$. The composition in $N_{\mathbf{k}}$ is given by

$$(AB)_{i,j} = A_{i,j} + \sum_{l=j+1}^{i-1} A_{i,l} B_{l,j} + B_{i,j},$$

and if $N_{\mathbf{k}}$ is considered as a vector space, then

$$(A+B)_{i,j} = A_{i,j} + B_{i,j},$$

so if $A \in N_{\geq l}$, then

$$\pi_l(AB) = \pi_l(BA) = \pi_l(A+B) = \pi_l(A) + \pi_l(B).$$

On an open Riemann surface, any Mittag-Lefler distribution comes from a globally defined meromorphic function. If $\mathbf{CP}^1$ is replaced by a flag manifold $Fl_{\mathbf{k}}$, this generalizes to :

(3.1) LEMMA. *Let $\overline{M}$ be a compact surface with $\partial M \neq \emptyset$, and let $\overline{D}_1$, $\overline{D}_2$ be disjoint closed disks in $M = \overline{M} \setminus \partial M$. Put $c_i = \partial D_i$ and let $J \in \mathcal{C}(\overline{M})$. If, for $i = 1, 2$,*

$$f_i \colon \overline{D}_i \to Fl_{\mathbf{k}} \text{ is } J\text{-holomorphic with } f_i(c_i) \subseteq (Fl_{\mathbf{k}})_a,$$

*then there exist $J$-holomorphic maps*

$$f \colon \overline{M} \to Fl_{\mathbf{k}} \quad \text{and} \quad g_i \colon \overline{D}_i \to N_{\mathbf{k}} \quad i = 1, 2,$$

*such that*

$$f^{-1}((Fl_{\mathbf{k}})_\infty) \subseteq D_1 \cup D_2 \qquad \text{and} \qquad f_i = g_i\, f|_{\overline{D}_i} \qquad i = 1, 2.$$

*Furthermore, for small variations of $J$, the map $f$ can be chosen such that it depends continuously on $f_1, f_2$ and $J$.*

PROOF: First choose a closed surface $\widetilde{M}$ with $\overline{M} \subseteq \widetilde{M}$ and a continuous extension map $\mathcal{C}(\overline{M}) \to \mathcal{C}(\widetilde{M})$. Then any complex structure $J$ on $\overline{M}$ can be considered as a complex structure on $\widetilde{M}$. Next, choose a point $Q$ in $\widetilde{M} \setminus \overline{M}$, which is a non-Weierstrass point in the given complex structure.

We can find open disks $D_1'$ and $D_2'$, such that $\overline{D}_i' \subseteq D_i$ and $f_i^{-1}((Fl_{\mathbf{k}})_\infty) \subseteq D_i'$. Let $T_i = D_i \setminus D_i'$ and consider $f_i|_{T_i}$ as a map $T_i \to N_{\mathbf{k}}$. By lemma 2.10 we can write

$$f_i|_{T_i} = F_{i,1}|_{T_i} + F_{i,2}|_{T_i},$$

where $F_{i,1}\colon D_i \to N_{\mathbf{k}}$ and $F_{i,2}\colon \widetilde{M} \setminus (\overline{D}_i' \cup \{Q\}) \to N_{\mathbf{k}}$ are $J$-holomorphic and depend continuously on $f_1$, $f_2$ and $J$. If we put

$$F^1 = F_{1,2}|_{\widetilde{M}\setminus(\overline{D}_1'\cup\overline{D}_2'\cup\{Q\})} + F_{2,2}|_{\widetilde{M}\setminus(\overline{D}_1'\cup\overline{D}_2'\cup\{Q\})},$$

$$G_1^1 = F_{1,2}|_{D_1} - F_{1,1} \qquad \text{and} \qquad G_2^1 = F_{2,2}|_{D_2} - F_{2,1},$$

then $f_i|_{T_i} = F^1|_{T_i} + G_i^1|_{T_i}$. We put

$$\begin{aligned} f_i^2 &= \pi_{\geq 2}\,(G_i^1 f_i)|_{T_i} = (G_i^1 f_i)|_{T_i} - \pi_1\,(G_i^1 f_i)|_{T_i} \\ &= (G_i^1 f_i)|_{T_i} - \pi_1\,(G_i^1 + f_i)|_{T_i} = (G_i^1 f_i)|_{T_i} - \pi_1\,(F^1)|_{T_i} \end{aligned}$$

The same procedure as above shows that there exist $J$-holomorphic maps

$$F^2\colon \widetilde{M} \setminus (\overline{D}_1' \cup \overline{D}_2' \cup \{Q\}) \to N_{\geq 2} \qquad \text{and} \qquad G_i^2\colon D_i \to N_{\geq 2} \qquad i = 1, 2,$$

which depend continuously on $f_1$, $f_2$ and $J$, such that $f_i^2 = F^2|_{T_i} - G_i^2|_{T_i}$. We put

$$f_i^3 = \pi_{\geq 3}\,(G_i^2 G_i^1 f_i)|_{T_i} = (G_i^2 G_i^1 f_i)|_{T_i} - \pi_1\,(G_i^2 G_i^1 f_i)|_{T_i} - \pi_2\,(G_i^2 G_i^1 f_i)|_{T_i}.$$

As

$$\pi_1\,(G_i^2 G_i^1 f_i)|_{T_i} = \pi_1\,(G_i^1 f_i)|_{T_i} = \pi_1\,(F^1)|_{T_i}$$

and

$$\pi_2\,(G_i^2 G_i^1 f_i)|_{T_i} = \pi_2\,(G_i^2 + G_i^1 f_i)|_{T_i} = \pi_2\,(G_i^2)|_{T_i} + \pi_{\geq 2}\,(G_i^1 f_i)|_{T_i} = \pi_2\,(F^2)|_{T_i},$$

we have

$$f_i^3 = (G_i^2 G_i^1 f_i)|_{T_i} - \pi_1\,(F^1)|_{T_i} - \pi_2\,(F^2)|_{T_i}.$$

Continuing this way we end with

$$\begin{aligned} f_i^{r-1} &= (G_i^{r-2}G_i^{r-3}\dots G_i^2G_i^1 f_i)\big|_{T_i} - \pi_1\, F^1\big|_{T_i} - \dots - \pi_{r-2}\, F^{r-2}\big|_{T_i} \\ &= F^{r-1}\big|_{T_i} - G_i^{r-1}\big|_{T_i}, \end{aligned}$$

where $F^l\colon \widetilde{M}\setminus(\overline{D}_1'\cup\overline{D}_2'\cup\{Q\})\to N_{\geq l}$ and $G_i^l\colon D_i\to N_{\geq l}$. As

$$\begin{aligned} &F^{r-1}\big|_{T_i} = \\ &\quad G_i^{r-1}\big|_{T_i} + (G_i^{r-2}G_i^{r-3}\dots G_i^2G_i^1 f_i)\big|_{T_i} - \pi_1\, F^1\big|_{T_i} - \dots - \pi_{r-2}\, F^{r-2}\big|_{T_i} \\ &\qquad = (G_i^{r-1}G_i^{r-2}\dots G_i^2G_i^1 f_i)\big|_{T_i} - (\pi_1F^1 + \pi_2F^2\dots + \pi_{r-2}F^{r-2})\big|_{T_i}, \end{aligned}$$

we have

$$(G_i^{r-1}G_i^{r-2}\dots G_i^2G_i^1 f_i)\big|_{T_i} = (\pi_1F^1 + \pi_2F^2\dots + \pi_{r-2}F^{r-2} + F^{r-1})\big|_{T_i}.$$

So if we put $\widetilde{f} = \pi_1F^1 + \pi_2F^2 + \dots + \pi_{r-2}F^{r-2}$ and $\widetilde{g}_i = (G_i^{r-1}G_i^{r-2}\dots G_i^2G_i^1)^{-1}$, then $f_i|_{T_i} = \widetilde{g}_i|_{T_i}\,\widetilde{f}|_{T_i}$. As $\widetilde{f}|_{T_i} = \widetilde{g}_i|_{T_i}^{-1}\, f_i|_{T_i}$, $\widetilde{f}$ extends, as a $J$-holomorphic map into $Fl_{\mathbf{k}}$, to the interior of $D_i$. Likewise $\widetilde{g}_i|_{T_i} = f_i|_{T_i}^{-1}\,\widetilde{f}|_{T_i}$, so we can extend $\widetilde{g}_i$ to the boundary of $D_i$. We get $J$-holomorphic maps $f\colon \overline{M}\to Fl_{\mathbf{k}}$ and $g_i\colon \overline{D}_i\to N_{\mathbf{k}}$, such that $f_i = g_i\, f|_{\overline{D}_i}$ and $f^{-1}((Fl_{\mathbf{k}})_\infty)\subseteq D_1\cup D_2$. From the construction we see that $f$, $g_1$, and $g_2$ depend continuously on $f_1$, $f_2$ and $J$, as long as $Q$ is a non-Weierstrass point. ∎

REMARK. *If $\overline{M}\subseteq S^2$ then we do not need the assumption $\partial M\neq\emptyset$, i.e. the lemma holds for $\overline{M} = S^2$.*

## 4. Loop Groups

Let $G$ be a compact connected Lie group and consider the space of based loops in $G$, i.e. the space of smooth maps $\gamma\colon S^1\to G$ with $\gamma(1) = 1$. It is an infinite dimensional Lie group, and we let the *loop group* $\Omega G$ be the identity component†. If we consider Sobelev maps of some degree instead of smooth maps, then we even get a Hilbert Lie group.

The complexification of $G$ is denoted $G_{\mathbf{C}}$, and $LG_{\mathbf{C}}$ denotes the identity component of all loops in $G_{\mathbf{C}}$, i.e. all smooth maps $\gamma\colon S^1\to G_{\mathbf{C}}$. It is an infinite dimensional Lie group too, and we may consider $\Omega G$ as a subgroup of $LG_{\mathbf{C}}$. We define other subgroups of $LG_{\mathbf{C}}$ by

$$\begin{aligned} &L^+G_{\mathbf{C}} = \\ &\qquad \{\gamma\in LG_{\mathbf{C}} \mid \gamma \text{ is the boundary value of a holomorphic map } D\to G_{\mathbf{C}}\} \end{aligned}$$

†Normally all components is considered, but as we later will consider based maps into $\Omega G$, we will only need the identity component.

and

$L^-G_{\mathbf{C}} =$

$\{\gamma \in LG_{\mathbf{C}} \mid \gamma$ is the boundary value of a holomorphic map $D_\infty \to G_{\mathbf{C}}\}$,

where $D$ is the open unit disk in $\mathbf{C}$ and $D_\infty = \mathbf{CP}^1 \setminus \overline{D}$. Again we have Hilbert Lie groups, if we consider Sobolev maps.

If $LG$ denotes the identity component of all loops in $G$, then we may consider $G$ as the subgroup of constant loops in $LG$, and obviously $\Omega G = LG/G$. The loop group is also a homogeneous space of $LG_{\mathbf{C}}$, see [**15**] chapter 8. We state it as

(4.1) THEOREM. *Any loop $\gamma \in LG_{\mathbf{C}}$ can be factorized uniquely*

$$\gamma = \gamma_u \gamma_+$$

*with $\gamma_u \in \Omega G$ and $\gamma_+ \in L^+G_{\mathbf{C}}$, and the multiplication map*

$$\Omega G \times L^+G_{\mathbf{C}} \longrightarrow LG_{\mathbf{C}}$$

*is a diffeomorphism.*

In particular $\Omega G \cong LG_{\mathbf{C}}/L^+G_{\mathbf{C}}$. By Birkhoff's theorem, loops in $G_{\mathbf{C}}$ can be factorized in an other way, see [**15**] chapter 8,

(4.2) THEOREM. *Any loop $\gamma \in LG_{\mathbf{C}}$ can be factorized uniquely*

$$\gamma = \gamma_- \lambda \gamma_+$$

*with $\gamma_- \in L^-G_{\mathbf{C}}$, $\gamma_+ \in L^+G_{\mathbf{C}}$ and $\lambda \colon S^1 \to G$ a homeomorphism, which is uniquely determined up to conjugation by a constant loop. Loops with $\lambda = 1$ form a dense open subset of $LG_{\mathbf{C}}$, and the multiplication map*

$$L_1^-G_{\mathbf{C}} \times L^+G_{\mathbf{C}} \longrightarrow LG_{\mathbf{C}},$$

*with $L_1^-G_{\mathbf{C}} = \{\gamma \in L^-G_{\mathbf{C}} \mid \gamma(\infty) = 1\}$, is a diffeomorphism onto this subset.*

In the case of $G = S^1$ the only homeomorphism $\lambda \colon S^1 \to S^1$, which is homotopic to the constant $\lambda = 1$, is $\lambda = 1$, so $L\mathbf{C}^* = L_1^-\mathbf{C}^* \times L^+\mathbf{C}^*$ and $\Omega G \cong L_1^-\mathbf{C}^* \cong L_0^-\mathbf{C}$, where $L_0^-\mathbf{C}$ is the Lie algebra of $L_1^-\mathbf{C}^*$. In general, if $\mathcal{G}$ is the Lie algebra of $G$, then $\mathcal{G}_{\mathbf{C}} = \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ is the Lie algebra of $G_{\mathbf{C}}$ and

$$L_0^-\mathcal{G}_{\mathbf{C}} = \{\text{smooth maps } \gamma \longrightarrow \mathcal{G}_{\mathbf{C}} \mid \gamma \text{ is the boundary value of a holomorphic map } D_\infty \to \mathcal{G}_{\mathbf{C}} \text{ with } \gamma(\infty) = 0\}$$

is the Lie algebra of $L_1^-G_{\mathbf{C}}$.

It is easily seen that $LG_{\mathbf{C}}$ is a complex Lie group and that $L^+G_{\mathbf{C}}$ and $L_1^-G_{\mathbf{C}}$ are complex subgroups. From the identification $\Omega G = G_{\mathbf{C}}/L^+G_{\mathbf{C}}$, we have that $\Omega G$ is a complex manifold, but it is not a complex Lie group and the inclusion $\Omega G \hookrightarrow LG_{\mathbf{C}}$ is not holomorphic.

By the factorizations theorems (4.1) and (4.2), the group $L_1^-G_{\mathbf{C}}$ can be considered as an open dense subset of $\Omega G$, and the inclusion $L_1^-G_{\mathbf{C}} \hookrightarrow \Omega G$ is holomorphic. Moreover, left-translation in $LG_{\mathbf{C}}$ by an element $\gamma \in L_1^-G_{\mathbf{C}}$ induces a holomorphic map of $\Omega G$ to it self. We have in fact

(4.3) PROPOSITION. *The loopgroup $\Omega G$ is a complex manifold locally isomorphic to $L_1^- G_{\mathbf{C}}$, and left multiplication by a fixed element is holomorphic.*

*The composition $L_1^- G_{\mathbf{C}} \hookrightarrow LG_{\mathbf{C}} \to LG_{\mathbf{C}}/L^+G_{\mathbf{C}} \cong \Omega G$ is holomorphic and a diffeomorphism onto an open dense subset of $\Omega G$. Moreover the multiplication $L_1^- G_{\mathbf{C}} \times L_1^- G_{\mathbf{C}} \to L_1^- G_{\mathbf{C}}$ extends to a holomorphic left action $L_1^- G_{\mathbf{C}} \times \Omega G \to \Omega G$ of $L_1^- G_{\mathbf{C}}$ on $\Omega G$.*

The loop group $\Omega G$ can be considered as a kind of infinite dimensional Grassmannian, see [**15**]. If $\mathbf{P}(\ell^2(\mathcal{S}))$ is the projective space of the Hilbert space $\ell^2(\mathcal{S})$, where

$$\mathcal{S} = \{S \subseteq \mathbf{Z} \mid \mathrm{card}(S \setminus \mathbf{N}) = \mathrm{card}(\mathbf{N} \setminus S) \text{ is finite }\},$$

then there is a holomorphic imbedding $\Omega G \hookrightarrow \mathbf{P}(\ell^2(\mathcal{S}))$ by Plücker coordinates $\gamma \mapsto [\pi_S(\gamma)]_{S \in \mathcal{S}}$, and the subset $L_1^- G_{\mathbf{C}} \subseteq \Omega G$ is given by the equation $\pi_{\mathbf{N}}(\gamma) \neq 0$. So $\Omega G$ is not only a complex manifold, but even a complex projective variety and as such $L_1^- G_{\mathbf{C}}$ is the affine part of $\Omega G$.

This is very similar to the situation in the preceding section. The loop group $\Omega G$ corresponds to the flag manifold $Fl_{\mathbf{k}}$, and $L_1^- G_{\mathbf{C}}$ corresponds to the group $N_{\mathbf{k}} \cong (Fl_{\mathbf{k}})_a$. There is one difference between the groups $N_{\mathbf{k}}$ and $L_1^- G_{\mathbf{C}}$, namely the exponential map. It is an isomorphism in the case of $N_{\mathbf{k}}$, but this may not be so in the case of $L_1^- G_{\mathbf{C}}$, hence as a complex manifold $L_1^- G_{\mathbf{C}}$ need not be a vector space, but it is contractible by the homomorphisms $\gamma \mapsto \gamma_t$, $t \in [0,1]$, with $\gamma_t(z) = \gamma(t^{-1}z)$.

We will need the description of elements in $\Omega G$ as holomorphic bundles over $\mathbf{CP}^1$, see [**15**], section 8.10. The idea is simple. A loop $\gamma \in \Omega G$ is used to glue the trivial $G_{\mathbf{C}}$-bundle over $\overline{D}$ and $\overline{D}_\infty$ together and thus obtain a $G_{\mathbf{C}}$-bundle over $\mathbf{CP}^1$. The precise results are as follows

(4.4) PROPOSITION. *An element of $\Omega G$ is the same thing as an isomorphism class of pairs $(P, \tau)$, where $P$ is a holomorphic principal $G_{\mathbf{C}}$-bundle on $\mathbf{CP}^1$ and $\tau$ is a trivialization of $P$ over $\overline{D}_\infty$. The elements of $L_1^- G_{\mathbf{C}} \subseteq \Omega G$ correspond to pairs $(P, \tau)$, where $P$ is the trivial bundle, and the action of $L_1^- G_{\mathbf{C}}$ on $\Omega G$ corresponds to the map $(\gamma, (P, \tau)) \mapsto (P, \gamma\tau)$.*

A trivialization $\tau$ of $P$ over $\overline{D}_\infty$ is a smooth section of $P|_{\overline{D}_\infty}$, which is holomorphic over $D_\infty$.

If we consider holomorphic maps into $\Omega G$, then we have the following generalization

(4.5) PROPOSITION. *If $X$ is a complex manifold, then a holomorphic map from $X$ to $\Omega G$ is the same thing as an isomorphism class of pairs $(P, \tau)$, where $P$ is a holomorphic principal $G_{\mathbf{C}}$-bundle on $X \times \mathbf{CP}^1$ and $\tau$ is a trivialization of $P$ over $X \times \overline{D}_\infty$.*

As a manifold $G = (T_0 \times G_1 \times \ldots \times G_n)/K$, where $T_0$ is the identity component of the center of $G$, the $G_i$'s are compact simple simply-connected Lie groups and

$K$ is a finite subgroup of the center of $T_0 \times G_1 \times \ldots \times G_n$, see [16] theorem 6.4.5. Hence

$$\Omega G = \Omega T_0 \times \Omega G_1 \times \ldots \times \Omega G_n,$$

$$\pi_1(\Omega G) = \pi_2(G) = 0$$

and

$$\pi_2(\Omega G) = \pi_3(G_1) \times \ldots \times \pi_3(G_n) = \mathbf{Z}^n.$$

The Abelian component is a product of $S^1$'s, so $\Omega T_0 \cong L_1^-(\mathbf{C}^*)^d \cong L_0^-(\mathbf{C}^d)$, because in the case of $G_\mathbf{C} = \mathbf{C}^*$, the exponential map $L_0^-\mathbf{C} \to L_1^-\mathbf{C}^*$ is an isomorphism. As $L_0^-\mathbf{C}^d$ is a vector space, we will in the following assume that $G$ is a simple group. Then

$$\pi_1(\Omega G) = H_1(\Omega G) = 0 \qquad \text{and} \qquad \pi_2(\Omega G) = H_2(\Omega G) = \mathbf{Z},$$

and as in the case of flag manifolds, the space of maps $f\colon X \to \Omega G$, where $X$ is a closed Riemann surface, has components labeled by a degree $\deg f \in \mathbf{Z}$. The degree can be determined as the intersection index between $f(X)$ and the infinite part of $\Omega G$, which is $(\Omega G)_\infty = \Omega G \setminus L_1^- G_\mathbf{C}$.

If $U$ is an open subset of a Riemann surface $X$ and $f\colon U \to \Omega G$ is holomorphic with $f(U) \cap L_1^- G_\mathbf{C} \neq \emptyset$, then $f$ can be considered as a meromorphic map into $L_1^- G_\mathbf{C} = (\Omega G)_a$. The set of poles is $f^{-1}((\Omega G)_\infty)$, which is a discrete subset of $U$. If we use proposition 4.5 and identify $f$ with a pair $(P, \tau)$, where $P$ is a holomorphic $G_\mathbf{C}$-bundle over $U \times \mathbf{CP}^1$, then a point $\alpha \in U$ is a pole if and only if the line $\{\alpha\} \times \mathbf{CP}^1$ is a *jumping line*, i.e. the bundle $P|_{\{\alpha\}\times\mathbf{CP}^1}$ is non-trivial. To each point $\alpha$ exists a neighbourhood $V$ of $\alpha$, such that $f^{-1}((\Omega G)_\infty) \cap V \subseteq \{\alpha\}$ and we define the order of $f$ at $\alpha$ as

$$\operatorname{ord}_\alpha f = \text{ order of contact at } f(\alpha) \text{ between } f(V) \text{ and } (\Omega G)_\infty.$$

If $U = X$, then

$$\deg f = \sum_{\alpha \in X} \operatorname{ord}_\alpha f = \sum_{\text{poles } \alpha} \operatorname{ord}_\alpha f.$$

We end the chapter on loop groups with the equivalent of lemma 3.1

(4.6) LEMMA. *Let $\overline{M}$ be a compact surface with non-empty boundary, and let $D_1$, $D_2$ be disjoint closed disks in $M = \overline{M} \setminus \partial M$. Put $c_i = \partial D_i$ and let $J \in \mathcal{C}(\overline{M})$. If, for $i = 1, 2$,*

$$f_i\colon \overline{D}_i \longrightarrow \Omega G \text{ is } J\text{-holomorphic with } f_i(c_i) \subseteq L_1^- G_\mathbf{C},$$

*then there exist $J$-holomorphic maps*

$$f\colon \overline{M} \longrightarrow \Omega G \qquad \text{and} \qquad g_i\colon \overline{D}_i \longrightarrow L_1^- G_\mathbf{C} \qquad i = 1, 2,$$

*such that*

$$f^{-1}((\Omega G)_\infty) \subseteq D_1 \cup D_2 \qquad \text{and} \qquad f_i = g_i\, f|_{\overline{D}_i} \qquad i = 1,2.$$

*Furthermore, for small variations of $f_1$,$f_2$ and $J$, the map $f$ can be chosen such that it depends continuously on $f_1$,$f_2$ and $J$.*

PROOF: The two maps $f_1 \colon \overline{D}_1 \to \Omega G$ and $f_2 \colon \overline{D}_2 \to \Omega G$ correspond to two pairs $(P_i, \tau_i)$, where $P_i$ is a $J$-holomorphic $G_{\mathbf{C}}$-bundle over $\overline{D}_i \times \mathbf{CP}^1$ and $\tau_i$ is a trivialization of $P_i$ over $\overline{D}_i \times \overline{D}_\infty$. The bundle $P_i$ is trivial outside the jumping lines $f^{-1}((\Omega G)_\infty) \times \mathbf{CP}^1$, so, by gluing $P_1 \cup P_2$ to the trivial bundle bundle over $(\overline{M} \setminus (f_1^{-1}((\Omega G)_\infty) \cup f_2^{-1}((\Omega G)_\infty))) \times \mathbf{CP}^1$, we get a $J$-holomorphic $G_{\mathbf{C}}$-bundle $P$ over $\overline{M} \times \mathbf{CP}^1$.

As $\partial M \neq \emptyset$, there exists a trivialization $\tau$ of $P$ over $\overline{M} \times \overline{D}_\infty$. The pair $(P,\tau)$ corresponds to a $J$-holomorphic map $f \colon \overline{M} \to \Omega G$, and the difference between the trivializations $\tau|_{\overline{D}_i \times \overline{D}_\infty}$ and $\tau_i$ is a $J$-holomorphic map $g_i \colon \overline{D}_i \times \overline{D}_\infty \to G_{\mathbf{C}}$. We may assume that $g_i(x,\infty) = 1$ for all $x \in \overline{D}_i$, so $g_i$ is a $J$-holomorphic map $\overline{D}_i \to L_1^- G_{\mathbf{C}}$.

The maps $f$, $g_1$ and $g_2$ have all the required properties, and we only need to show that this process can be made continuously.

Let $y_0 = (f_1^0, f_2^0, J^0)$ be given. Put $U = D_1 \cup D_2$ and choose an open subset $V \subseteq \overline{M}$, such that $U \cup V = \overline{M}$ and $f_i^0(\overline{V} \cap \overline{D}_1) \subseteq (\Omega G)_a$ for $i = 1,2$. Finally, choose a neighbourhood $W$ of $y_0$ in

$$\big\{(f_1, f_2, J) \in \operatorname{Map}(\overline{D}_1, \Omega G) \times \operatorname{Map}(\overline{D}_2, \Omega G) \times \mathcal{C}(\overline{M}) \;\big|\; f_i(\overline{V} \cap \overline{D}_1) \subseteq (\Omega G)_a \text{ and } f_i \text{ is } J\text{-holomorphic } i = 1,2\big\}.$$

The evaluation map $F \colon W \times \overline{U} \to \Omega G$, given by $F((f_1, f_2, J), x) = f_i(x)$ if $x \in \overline{D}_i$, defines a pair $(P_U, \tau_U)$, where $P_U$ is a $G_{\mathbf{C}}$-bundle over $W \times \overline{U} \times \mathbf{CP}^1$, and $\tau_U$ is a trivialization of $P_U$ over $W \times \overline{U} \times \overline{D}_\infty$. The bundle $P_U$ is $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times U \times \mathbf{CP}^1$, and the trivialization $\tau_U$ is $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times U \times D_\infty$. Furthermore, $P_U$ can be trivialized over $W \times (\overline{U} \cap \overline{V}) \times \mathbf{CP}^1$, and the trivialization can be chosen such that it is $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times (U \cap V) \times \mathbf{CP}^1$.

By gluing $P_U$ to the trivial bundle over $W \times \overline{V} \times \mathbf{CP}^1$, we get a $G_{\mathbf{C}}$-bundle $P$ over $W \times \overline{M} \times \mathbf{CP}^1$, which is $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times M \times \mathbf{CP}^1$ and is trivial over $W \times \overline{V} \times \mathbf{CP}^1$. We only need to find a trivialization $\tau$ of $P$ over $W \times \overline{M} \times \overline{D}_\infty$, which is $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times M \times D_\infty$, and is equal to $\tau_U$ on $W \times \overline{U} \times \{\infty\}$.

If $x \in \overline{U} \cap \overline{V}$, then $F(y,x) \in (\Omega G)_a \cong L_1^- G_{\mathbf{C}}$, and the transition function from the trivialization over $W \times \overline{U} \times \overline{D}_\infty$ to the trivialization over $W \times \overline{V} \times \overline{D}_\infty$ is exactly $F|_{W \times (\overline{U} \cap \overline{V})}$, considered as a map $W \times (\overline{U} \cap \overline{V}) \times \overline{D}_\infty \to G_{\mathbf{C}}$.

Let $t \colon \overline{M} \to [0,1]$ be a smooth map, such that $t(\overline{M} \setminus U) = 0$ and $t(\overline{U} \setminus V) = 1$. We define

$$\psi_V \colon W \times \overline{V} \longrightarrow L_1^- G_{\mathbf{C}}$$

by

$$\psi_V(y,x)(z) = \begin{cases} 1 & \text{for } x \in \overline{V} \setminus U \\ F(y,x)(t(x)z) & \text{for } x \in \overline{U} \cap \overline{V}, \end{cases}$$

and $\psi_U \colon W \times \overline{U} \to L_1^- G_{\mathbf{C}}$ by $\psi_U = 1$ on $\overline{U} \setminus V$ and $\psi_U = F^{-1}\psi_V$ on $\overline{U} \cap \overline{V}$. The map $\psi_V$ defines an isomorphism of the trivial bundle over $W \times \overline{V} \times \overline{D}_\infty$, and $\psi_U$ defines an isomorphism of the trivial bundle over $W \times \overline{U} \times \overline{D}_\infty$. As $\psi_U = F\psi_V$, when restricted to $W \times (\overline{U} \cap \overline{V}) \times \overline{D}_\infty$, we get a trivialization $\phi$ of $P$ over $W \times \overline{M} \times \overline{D}_\infty$. The trivialization $\phi$ is holomorphic, when restricted to $\{(f_1, f_2, J)\} \times M \times \{\infty\}$, and is equal to $\tau_U$, when restricted to $W \times \overline{U} \times \{\infty\}$.

For any map $\psi \colon W \times \overline{M} \to L_1^- G_{\mathbf{C}}$, the product $\psi\phi$ is a new trivialization of $P$ over $W \times \overline{M} \times \overline{D}_\infty$. We want to find a $\psi$, such that $\psi\phi$ is $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times M \times D_\infty$. As $P$ is $J^0$-holomorphicly trivial over $\{y_0\} \times M \times D_\infty$, we can find $\psi \colon \overline{M} \to L_1^- G_{\mathbf{C}}$, such that $\psi\phi$ is $J^0$-holomorphic, when restricted to $\{y_0\} \times M \times D_\infty$. To ease notation, we assume that $\phi$ is already $J^0$-holomorphic, when restricted to $\{y_0\} \times M \times D_\infty$. This corresponds to assuming that $\psi_U$ and $\psi_V$ are $J^0$-holomorphic, when restricted to $\{y_0\} \times U \times D_\infty$ and $\{y_0\} \times V \times D_\infty$ respectively.

We shall find a map $\psi \colon W \times \overline{M} \to L_1^- G_{\mathbf{C}}$, such that $\psi\psi_U$ and $\psi\psi_V$ are $J$-holomorphic, when restricted to $\{(f_1, f_2, J)\} \times U$ and $\{(f_1, f_2, J)\} \times V$ respectively. We define

$$h \colon W \longrightarrow \Omega^1(\overline{M}, L_0^- \mathcal{G}_{\mathbf{C}}),$$

where $\Omega^1(\overline{M}, L_0^- \mathcal{G}_{\mathbf{C}}) = \{\text{one forms on } \overline{M} \text{ with values in } L_0^- \mathcal{G}_{\mathbf{C}}\}$, by

$$h((f_1, f_2, J)) = \begin{cases} -(\overline{\partial}_J \psi_U)\psi_U^{-1} & \text{on } U \\ -(\overline{\partial}_J \psi_V)\psi_V^{-1} & \text{on } V. \end{cases}$$

This is well-defined, because the difference between $\psi_U$ and $\psi_V$ is $J$-holomorphic.

Our task is to find $\psi$, such that $\psi^{-1}\overline{\partial}_J \psi = h$. If we put

$$\Omega^{0,1}(W \times \overline{M}, L_0^- \mathcal{G}_{\mathbf{C}}) = \{((f_1, f_2, J), h) \in W \times \Omega^1(\overline{M}, L_0^- \mathcal{G}_{\mathbf{C}}) \mid h \in \Omega^{0,1}_{J_0}(\overline{M}, L_0^- \mathcal{G}_{\mathbf{C}})\},$$

then $(y, h(y)) \in \Omega^{0,1}(W \times \overline{M}, L_0^- \mathcal{G}_{\mathbf{C}})$ all $y \in W$, and as $\phi$ is $J^0$-holomorphic, when restricted to $\{y_0\} \times M \times D_\infty$, $h(y_0) = 0$. Now consider the map

$$\begin{aligned} H \colon W \times C^\infty(\overline{M}, L_1^- G_{\mathbf{C}}) &\longrightarrow \Omega^{0,1}(W \times \overline{M}, L_0^- \mathcal{G}_{\mathbf{C}}) \\ ((f_1, f_2, J), \psi) &\longmapsto ((f_1, f_2, J), \psi^{-1}\overline{\partial}_J \psi). \end{aligned}$$

If there existed an inverse $H^{-1}$, then $\psi \colon W \times \overline{M} \to L_1^- G_C$ could be determined by $H^{-1}(y, h(y))(z) = (y, \psi(y, z))$.

We will soon need the inverse function theorem, so we must complete our different spaces, such that they become Hilbert or Banach manifolds. We do it without changes of notation, and such that $H$ extends to a smooth map. In the following we will drop the $J_0$ subscript so $\Omega^{0,1}(\overline{M}, L_0^- \mathcal{G}_{\mathbf{C}}) = \Omega^{0,1}_{J_0}(\overline{M}, L_0^- \mathcal{G}_{\mathbf{C}})$, $\overline{\partial} = \overline{\partial}_{J_0}$ etc.

The first step is to find the differential of $H$ at $(y_0, 1)$. We have

$$T_{(y_0,1)}(W \times C^\infty(\overline{M}, L_1^- G_{\mathbb{C}})) = T_{y_0}W \times \Omega^0(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$$

and

$$T_{(y_0,0)}\Omega^{0,1}(W \times \overline{M}, L_0^- \mathcal{G}_{\mathbb{C}}) = T_{y_0}W \times \Omega^{0,1}(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$$

Let $B \in T_{y_0}W$ and $A \in \Omega^0(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$. As $\overline{\partial}_J 1 = 0$ all $J \in \mathcal{C}(\overline{M})$, we get

$$d_{(y_0,1)}H(B, A) = (B, \overline{\partial}A).$$

The kernel of $\overline{\partial}$ is $\mathrm{Hol}(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$, which is a closed subspace of $\Omega^0(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$. Let $\mathrm{Hol}^\perp$ denote the orthogonal complement of $\mathrm{Hol}(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$ in $\Omega^0(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}})$ and consider the restriction $\overline{\partial}|_{\mathrm{Hol}^\perp}$. It is injective, and we want to show that it is an isomorphism.

Let $\widetilde{M}$ be a closed surface containing $\overline{M}$. We extend $J_0$ to a complex structure on $\widetilde{M}$ and thereby get an extension of $\overline{\partial}$ to an operator

$$\overline{\partial}\colon \Omega^0(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}}) \longrightarrow \Omega^{0,1}(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}}).$$

It is an isomorphism considered as an operator from $(\ker\overline{\partial})^\perp$ to $\mathrm{Im}\,\overline{\partial} = (\ker\overline{\partial}^*)^\perp$.

As $\overline{\partial}^* = -J_0^* \partial J_0^*$

$$\begin{aligned}(\ker\overline{\partial}^*)^\perp &= \left\{h \in \Omega^{0,1}(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}}) \mid \textstyle\int_{\widetilde{M}} \langle h \wedge -\overline{J^*\phi}\rangle = 0 \text{ all } \phi \in \ker\overline{\partial}^*\right\} \\ &= \left\{h \in \Omega^{0,1}(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}}) \mid \textstyle\int_{\widetilde{M}} \langle h \wedge \phi\rangle = 0 \text{ all } \phi \in \ker\overline{\partial}\right\}.\end{aligned}$$

The kernel of $\overline{\partial}$ acting on $\Omega^{1,0}(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}})$ is the holomorphic differentials on $\widetilde{M}$ with values in $L_0^- \mathcal{G}_{\mathbb{C}}$. If $\mathcal{H}$ denotes the space of complex valued holomorphic differentials on $\widetilde{M}$, then $\ker\overline{\partial} = \mathcal{H} \otimes L_0^- \mathcal{G}_{\mathbb{C}}$, so if $(\omega_1, \ldots, \omega_g)$ is an orthonormal basis for $\mathcal{H}$, then

$$(\ker\overline{\partial}^*)^\perp = \left\{h \in \Omega^{0,1}(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}}) \mid \textstyle\int_{\widetilde{M}} h \wedge \omega_i = 0 \quad i = 1, \ldots, g\right\}.$$

There exists a continuous extension $\widetilde{\mathrm{ext}}\colon \Omega^{0,1}(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}}) \to \Omega^{0,1}(\widetilde{M}, L_0^- \mathcal{G}_{\mathbb{C}})$, and we want to show that there exists a continuous extension

$$\mathrm{ext}\colon \Omega^{0,1}(\overline{M}, L_0^- \mathcal{G}_{\mathbb{C}}) \longrightarrow (\ker\overline{\partial}^*)^\perp.$$

Choose $f_1, \ldots, f_g \in \Omega^{0,1}(\widetilde{M})$, such that $\mathrm{supp}\, f_i \cap \overline{M} = \emptyset$ and $\int f_i \wedge \omega_j = \delta_{i,j}$. The required extension can now be defined by

$$\mathrm{ext}(h) = \widetilde{\mathrm{ext}}(h) - \sum_{i=1}^{g} f_i \otimes \int_{\widetilde{M}} \widetilde{\mathrm{ext}}(h) \wedge \omega_i.$$

The following composition

$$\Omega^{0,1}(\overline{M}, L_0^-\mathcal{G}_\mathbb{C}) \xrightarrow{\text{ext}} (\ker\overline{\partial}^*)^\perp \xrightarrow{\overline{\partial}^{-1}} (\ker\overline{\partial})^\perp \xrightarrow{r} \Omega^0(\overline{M}, L_0^-\mathcal{G}_\mathbb{C}) \xrightarrow{\text{proj}} \text{Hol}^\perp,$$

where $r$ is the restriction from $\widetilde{M}$ to $\overline{M}$, and proj is the orthogonal projection, is an inverse to $\overline{\partial}\colon \text{Hol}^\perp \to \Omega^{0,1}(\overline{M}, L_0^-\mathcal{G}_\mathbb{C})$.

We have proved that $H$ is a submersion in a neighbourhood of $(y_0, 1)$. Hence there exists a neighbourhood $\widetilde{W}$ of $(y_0, 0)$ in $\Omega^{0,1}(W \times \overline{M}, L_0^-\mathcal{G}_\mathbb{C})$, and a map $H^{-1}\colon \widetilde{W} \to W \times C^\infty(\overline{M}, L_1^- G_\mathbb{C})$, such that $H \circ H^{-1} = id$. As $h(y_0) = 0$, we may assume that $(y, h(y)) \in \widetilde{W}$ for all $y \in W$.

Now $\psi\colon W \to C^\infty(\overline{M}, L_1^- G_\mathbb{C})$ is defined by $(y, \psi(y)) = H^{-1}(y, h(y))$. A priori we only know that $\psi(y)$ is in some 'Sobolev-completion' of $C^\infty(\overline{M} \times \overline{D}_\infty, G_\mathbb{C})$, but as the preceding arguments go through for all completions, $\psi(y)$ must be a smooth map. ∎

## 5. Spaces of Holomorphic Maps

In the following $Y$ denotes either a flag manifold $Fl_\mathbf{k}$ or a loop group $\Omega G$. It is a complex manifold and even a complex projective variety. We let $Y_a$ denote the affine part of $Y$ and let $Y_\infty = Y \setminus Y_a$ denote the infinite part of $Y$. The affine part is isomorphic to a contractible complex Lie group $N$, and the composition $N \times N \to N$ extends to a holomorphic left action $N \times Y \to Y$ of $N$ on $Y$. The infinite part is the union $Y_\infty = Y_1 \cup \cdots \cup Y_r$ of finitely many irreducible algebraic varieties $Y_1, \ldots, Y_r$.

If $X$ is a Riemann surface and $f\colon X \to Y$ is a holomorphic map with $f(X) \not\subseteq Y_\infty$, then the set of poles, i.e. $f^{-1}(Y_\infty)$, is a discrete subset of $X$. To each point $\alpha \in X$ and $i = 1, \ldots, r$, the $i$'th order of $f$ at $\alpha$ is defined as

$$\text{ord}_{i,\alpha} f = \text{order of contact between } f(U) \text{ and } Y_\infty \text{ at } f(\alpha),$$

where $U$ is a neighbourhood of $\alpha$, such that $f^{-1}(Y_\infty) \cap U \subseteq \{\alpha\}$. The total order of $f$ at $\alpha$ is

$$\text{ord}_\alpha f = \text{ord}_{1,\alpha} f + \cdots + \text{ord}_{r,\alpha} f,$$

and $\alpha$ is a pole if and only if $\text{ord}_\alpha f > 0$. The $i$'th degree of $f$ is

$$\deg_i f = \sum_{\alpha \in X} \text{ord}_{i,\alpha} f = \sum_{\alpha \in f^{-1}(Y_i)} \text{ord}_{i,\alpha} f,$$

and the total degree is

$$\deg f = \deg_1 f + \cdots + \deg_r f = \sum_{\alpha \in X} \text{ord}_\alpha f = \sum_{\alpha \in f^{-1}(Y_\infty)} \text{ord}_\alpha f$$

The degrees may be infinite, but if $X$ is closed, then the degree is finite, and the $r$-tubel $(\deg_1 f, \dots, \deg_r f)$ determines the component of $\mathrm{Map}(X,Y)$, which $f$ lies in.

Let $\overline{M}$ be a compact two-dimensional manifold, possibly with boundary and corners, and put $M = \overline{M} \setminus \partial M$. Equip the space $\mathrm{Map}(M,Y)$ of continuous maps from $M$ to $Y$ with the compact-open topology.

In section 2 the space $\mathcal{C}(\overline{M})$ of complex structures on $\overline{M}$ was introduced. It can be considered as a space of smooth sections in a bundle over $\overline{M}$ and is equipped with the $C^\infty$-topology. Given a complex structure $J \in \mathcal{C}(\overline{M})$, then $M$ can be considered as a Riemann surface $M_J$. A map $f: M \to Y$ is called $J$-holomorphic, if it is holomorphic considered as a map $f: M_J \to Y$. The space

$$\mathrm{Hol}_J(M,Y) = \{f \in \mathrm{Map}(M,Y) \mid f \text{ is } J\text{-holomorphic}\}$$

is a closed subspace of $\mathrm{Map}(M,Y)$. If $f \in \mathrm{Hol}_J(M,Y)$ and $f(M) \cap Y_a \neq \emptyset$, then we have the concepts of poles, orders and degrees of $f$. If $M'$ is any subset of $M$, then we define

$$\mathcal{H}_n(\overline{M},M') = \big\{(f,J) \in \mathrm{Map}(M,Y) \times \mathcal{C}(\overline{M}) \;\big|\; f \text{ is } J\text{-holomorphic}, \\ f(M) \not\subseteq Y_\infty, \deg f = n \text{ and } f(M') \subseteq Y_a\big\},$$

$$\mathcal{H}_{\leq n}(\overline{M},M') = \bigcup_{k=0}^{n} \mathcal{H}_k(\overline{M},M')$$

and

$$\mathcal{H}(\overline{M},M') = \lim_{n\to\infty} \mathcal{H}_{\leq n}(\overline{M},M').$$

We put

$$\mathcal{H}_n(\overline{M}) = \mathcal{H}_n(\overline{M},\emptyset), \\ \mathcal{H}_{\leq n}(\overline{M}) = \mathcal{H}_{\leq n}(\overline{M},\emptyset)$$

and

$$\mathcal{H}(\overline{M}) = \mathcal{H}(\overline{M},\emptyset).$$

If the complex structure is fixed, then we have the spaces

$$\mathcal{H}_{J,n}(M,M') = \{f \in \mathrm{Map}(M,Y) \mid (f,J) \in \mathcal{H}_n(\overline{M},M')\}, \\ \mathcal{H}_{J,\leq n}(M,M') = \{f \in \mathrm{Map}(M,Y) \mid (f,J) \in \mathcal{H}_{\leq n}(\overline{M},M')\}$$

and

$$\mathcal{H}_J(M, M') = \lim_{n\to\infty} \mathcal{H}_{J,\leq n}(\overline{M}, M').$$

As before

$$\mathcal{H}_{J,n}(M) = \mathcal{H}_{J,n}(M, \emptyset),$$
$$\mathcal{H}_{J,\leq n}(M) = \mathcal{H}_{J,\leq n}(M, \emptyset)$$

and

$$\mathcal{H}_J(M) = \mathcal{H}_J(M, \emptyset).$$

The restriction of the projection $\mathrm{Map}(M, Y) \times \mathcal{C}(\overline{M}) \to \mathrm{Map}(M, Y)$ to $\mathcal{H}(\overline{M})$ fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_J(M) & \longrightarrow & \mathcal{H}(\overline{M}) \\ \downarrow & & \downarrow \\ \mathrm{Hol}_J(M, Y) & \longrightarrow & \mathrm{Map}(M, Y). \end{array}$$

In this section we consider the case $\overline{M} = \overline{D} = \{z \in \mathbf{C} \mid |z| \leq 1\}$ and show that the maps in the diagram are homotopy equivalences.

(5.1) LEMMA. *Let $J_0$ be any complex structure on $\overline{D}$. There exists a map $\psi$ from $\mathcal{H}(\overline{D})$ to $\mathcal{H}_{J_0}(D)$, such that $\psi(f, J_0) = f$, and the map $\mathcal{H}(\overline{D}) \to \mathcal{H}_{J_0}(D) \times \mathcal{C}(M)$ given by $(f, J) \mapsto (\psi(f, J), J)$ is a homeomorphism.*

PROOF: Let $\phi_J\colon D_{J_0} \to D_J$ be a holomorphic homeomorphism, continuously depending on $J$ and with $\phi_{J_0} = id$. We define $\psi$ by $\psi(f, J) = f \circ \phi_J$. ∎

(5.2) LEMMA. *The inclusion* $\mathrm{Hol}_J(D, Y) \hookrightarrow \mathrm{Map}(D, Y)$ *is a homotopy equivalence.*

PROOF: Let $J_0$ be the standard complex structure on $\overline{D}$ and let $\phi\colon \overline{D}_{J_0} \to \overline{D}_J$ be a holomorphic homeomorphism with $\phi(0) = 0$. Define for $t \in [0,1]$, $\psi_t\colon \overline{D} \to \overline{D}$ by $\psi_t(z) = \phi(t\phi^{-1}(z))$. Then $\psi_t$ is $J$-holomorphic for all $t \in [0,1]$, and $\psi_0 = 0$ and $\psi_1 = id$.

We define a homotopy inverse $F\colon \mathrm{Map}(D, Y) \to \mathrm{Hol}_J(D, Y)$ to the inclusion by $F(f)(z) = f(0)$, and only have to observe that $F$ is homotopic to the identity on both $\mathrm{Hol}_J(M, Y)$ and $\mathrm{Map}(D, Y)$ by the homotopy $(t, f) \mapsto f \circ \psi_t$. ∎

(5.3) LEMMA. *The map $\mathcal{H}_J(D) \to \mathrm{Hol}_J(D, Y) \setminus \mathrm{Hol}_J(D, Y_\infty)$ is a homotopy equivalence.*

PROOF: Let $\psi_t\colon \overline{D} \to \overline{D}$ be the map defined in the proof above. We define a homotopy inverse to the map in the lemma by $f \mapsto f \circ \psi_{1/2}$. ∎

The next step is to show that the inclusion

$$\mathrm{Hol}_J(D,Y) \setminus \mathrm{Hol}_J(D,Y_\infty) \hookrightarrow \mathrm{Hol}_J(D,Y)$$

is a homotopy equivalence, but first we need to show that the spaces we are considering contract onto manifolds. Put

$$\widetilde{\mathrm{Hol}}_J(\overline{D},Y) = \{f \in \mathrm{Map}(\overline{D},Y) \mid f|_D \text{ is } J\text{-holomorphic and } f(\overline{D}) \text{ is contained in a chart}\}.$$

Then we have

(5,4) LEMMA. $\widetilde{\mathrm{Hol}}_J(\overline{D},Y)$ *is a complex manifold modelled on* $\mathrm{Hol}_J(\overline{D},LN)$, *where* $LN$ *is the Lie algebra of* $N$.

PROOF: Let $f \in \widetilde{\mathrm{Hol}}_J(\overline{D},Y)$. Choose a neighbourhood $U$ of $f(\overline{D})$ in $Y$ and a chart $\phi\colon U \to LN$. For $v$ in a neighbourhood $V$ of 0 in $\mathrm{Hol}_J(\overline{D},LN)$, the map $\psi\colon V \to \widetilde{\mathrm{Hol}}_J(\overline{D},Y)$ given by $\psi(v) = \phi^{-1}\circ(\phi\circ f + v)$ is a chart around $f$. ∎

(5.5) LEMMA. *The inclusion* $\mathrm{Hol}_J(D,Y) \setminus \mathrm{Hol}_J(D,Y_\infty) \hookrightarrow \mathrm{Hol}_J(D,Y)$ *is a homotopy equivalence.*

PROOF: Choose a metric on $Y$ and a $k > 0$, such that any subset of $Y$ with diameter less than $k$ is contained in a chart. Let $\psi_t$ be the $J$-holomorphic map defined in the proof of lemma 5.2. For $f \in \mathrm{Hol}_J(D,Y)$, we put

$$t(f) = \max\{t \in [0,1/2] \mid \mathrm{diam}(f\circ\psi_t(\overline{D})) \le k\}.$$

The number $t(f)$ depends continuously on $f$, so we can define a map

$$\phi\colon \mathrm{Hol}_J(D,Y) \longrightarrow \widetilde{\mathrm{Hol}}_J(D,Y) \quad \text{by} \quad \phi(f) = f\circ\psi_{t(f)}.$$

This is a homotopy inverse to the restriction $r\colon \widetilde{\mathrm{Hol}}_J(D,Y) \to \mathrm{Hol}_J(D,Y)$, because

$$r\circ\phi(f) = f\circ\psi_{t(f)} \sim f\circ\psi_1 = f \quad \text{and} \quad \phi\circ r(f) = f\circ\psi_{t(f)} \sim f\circ\psi_1 = f$$

by obvious homotopies.

Moreover, the subspaces $\mathrm{Hol}_J(D,Y) \setminus \mathrm{Hol}_J(D,Y_\infty)$ and $\mathrm{Hol}_J(D,Y_\infty)$ are preserved by the homotopies. So it is enough to show that the inclusion

$$\widetilde{\mathrm{Hol}}_J(\overline{D},Y) \setminus \mathrm{Hol}_J(\overline{D},Y_\infty) \hookrightarrow \widetilde{\mathrm{Hol}}_J(\overline{D},Y)$$

is a homotopy equivalence.

This is the case because $\widetilde{\mathrm{Hol}}_J(\overline{D},Y)$ is a manifold and $\widetilde{\mathrm{Hol}}_J(\overline{D},Y) \cap \widetilde{\mathrm{Hol}}_J(\overline{D},Y_\infty)$ has infinite codimension in the sense of the following lemma. ∎

(5.6) LEMMA. *If $f \in \widetilde{\mathrm{Hol}}_J(\overline{D}, Y) \cap \mathrm{Hol}_J(\overline{D}, Y_\infty)$, then there exists a neighbourhood $W$ of 0 in $\mathrm{Hol}_J(\overline{D}, \mathbf{C})$, such that the imbedding*

$$W \longrightarrow \widetilde{\mathrm{Hol}}_J(\overline{D}, Y) : h \longmapsto f + hg$$

*maps $W \setminus \{0\}$ into $\widetilde{\mathrm{Hol}}_J(\overline{D}, Y) \setminus \mathrm{Hol}_J(\overline{D}, Y_\infty)$.*

PROOF: We can consider $f$ as a map $\overline{D} \to LN$ and as $Y_\infty$ has codimension two in $Y$, there exists $g \in \mathrm{Hol}_J(\overline{D}, LN)$, such that $f(0) + zg(0) \in Y_a$, if $0 < z \leq 1$. Then the map $z \mapsto f(x) + zg(x)$ only belongs to $\mathrm{Hol}_J(\overline{D}, Y_\infty)$ for $x$ in a discrete subset of $D$, hence

$$x_0 = \min\left(\{|x| \mid f(x) + zg(x) \in Y_\infty \text{ all } z \in \overline{D}\} \cup \{\tfrac{1}{2}\}\right) > 0.$$

Suppose $h_n \in \mathrm{Hol}_J(\overline{D}, \mathbf{C})$ is a sequence such that

(1) $h_n \longrightarrow 0$,
(2) $|h_n(x)| < 1/2$ and
(3) $f(x) + h_n(x)g(x) \in Y_\infty$ all $x$ and $n$.

If $|x| < x_0$, then $\{h_n(x) \mid n \in \mathbf{N}\}$ is a finite set, and as $h_n(x) \to 0$, there exists a number $n(x) \in \mathbf{N}$, such that $h_n(x) = 0$, if $n \geq n(x)$. As $\{x \mid |x| < x_0\}$ is uncountable, there must exists a number $n_0 \in \mathbf{N}$, such that the set $\{x \mid n(x) = n_0\}$ is infinite. If $n > n_0$, then $h_n(x) = 0$ for infinitely many points in $\{x \mid |x| < 1/2\}$, and hence $h_n = 0$. ∎

We finally state

(5.7) LEMMA. *Let $\overline{D}$ be the closed unit disk in $\mathbf{C}$ and let $J$ be any complex structure on $\overline{D}$. Then the maps in the commutative diagram*

$$\begin{array}{ccc} \mathcal{H}_J(D) & \longrightarrow & \mathcal{H}(\overline{D}) \\ \downarrow & & \downarrow \\ \mathrm{Hol}_J(D, Y) & \longrightarrow & \mathrm{Map}(D, Y) \end{array}$$

*are homotopy equivalences.*

PROOF: The two horizontal maps are homotopy equivalences by lemma 5.1 and lemma 5.6, and the lefthand vertical map is a homotopy equivalence by lemma 5.3 and lemma 5.5. But then the last map is a homotopy equivalence too. ∎

## 6. Spaces of Principal Parts

Let $J \in \mathcal{C}(M)$ be a complex structure. In analogy with the definition of ordinary principal parts on the Riemann surface $M_J$, we let $\mathcal{O}_J$ and $\mathcal{M}_J$ denote the sheaves of respectively $J$-holomorphic and $J$-meromorphic maps into $N$. More precisely, for an open subset $U \in M$, we let

$$\mathcal{O}_J(U) = \mathrm{Hol}_J(U,N) \qquad \text{and} \qquad \mathcal{M}_J(U) = \mathrm{Hol}_J(U,Y) \setminus \mathrm{Hol}_J(U,Y_\infty).$$

The action of $N$ on $Y$ induces an action of $\mathrm{Hol}_J(U,N)$ on $\mathrm{Hol}_J(U,Y)\setminus\mathrm{Hol}_J(U,Y_\infty)$, which clearly preserves poles and their orders. So we can define the quotient sheave $\mathcal{P}_J = \mathcal{M}_J/\mathcal{O}_J$ called the *sheave of J-principal parts.* A *configuration of J-principal parts* is a global section of $\mathcal{P}_J$.

As noted above a pole, the order of a point and the degree of a configuration $\xi$ of principal parts are well defined concepts. We are only interested in finite configurations, so we put

$$\begin{aligned} \mathcal{P}_J(M) &= \left\{\text{global sections } \xi \text{ of } \mathcal{P}_J \;\middle|\; \deg \xi < \infty\right\}, \\ \mathcal{P}_{J,\leq n}(M) &= \left\{\xi \in \mathcal{P}_J(M) \;\middle|\; \deg \xi \leq n\right\} \end{aligned}$$

and

$$\mathcal{P}_{J,n}(M) = \left\{\xi \in \mathcal{P}_J(M) \;\middle|\; \deg \xi = n\right\}.$$

If $M'$ and $M$ both are subset of a surface $\widetilde{M}$, then we put

$$\begin{aligned} \mathcal{P}_J(M,M') &= \left\{\xi \in \mathcal{P}_J(M) \;\middle|\; \xi|_{M\cap M'} = 0\right\}, \\ \mathcal{P}_{J,\leq n}(M,M') &= \mathcal{P}_{J,\leq n}(M) \cap \mathcal{P}_J(M,M') \end{aligned}$$

and

$$\mathcal{P}_{J,n}(M,M') = \mathcal{P}_{J,n}(M) \cap \mathcal{P}_J(M,M').$$

Finally the complex structure varies, and we get the spaces

$$\begin{aligned} \mathcal{P}(\overline{M}) &= \bigcup_{J\in\mathcal{C}(\overline{M})} \mathcal{P}_J(M) \times \{J\}, \\ \mathcal{P}_{\leq n}(\overline{M}) &= \left\{(\xi,J) \in \mathcal{P}(\overline{M}) \;\middle|\; \deg \xi \leq n\right\}, \\ \mathcal{P}_n(\overline{M}) &= \left\{(\xi,J) \in \mathcal{P}(\overline{M}) \;\middle|\; \deg \xi = n\right\}, \\ \mathcal{P}(M,M') &= \left\{(\xi,J) \in \mathcal{P}(M) \;\middle|\; \xi|_{M\cap M'} = 0\right\}, \\ \mathcal{P}_{\leq n}(M,M') &= \mathcal{P}_{\leq n}(M) \cap \mathcal{P}(M,M') \end{aligned}$$

and

$$\mathcal{P}_n(M,M') = \mathcal{P}_n(M) \cap \mathcal{P}(M,M').$$

Let $\mathcal{A}(\overline{M}, M')$ be the quotient of the free Abelian monoid, generated by points of $\overline{M} \setminus M'$ by the relation, which identifies points on $\partial M$ with zero, see [**18**, page 45], and define the pole map

$$\mathcal{P}(\overline{M}, M') \longrightarrow \mathcal{A}(\overline{M}, M') \qquad \text{by} \qquad (\xi, J) \longmapsto \sum_{\alpha \in M} \operatorname{ord}_\alpha \xi.\alpha.$$

A $J$-holomorphic map $f\colon M \to Y$ with $f(M) \cap Y_a \neq \emptyset$ and $\deg f < \infty$ defines a configuration $[f]$ of $J$-principal parts with $\deg_\alpha[f] = \deg_\alpha f$ all $\alpha \in M$, i.e. we have a degree preserving map

$$\mathcal{H}(\overline{M}, M') \longrightarrow \mathcal{P}(\overline{M}, M')\colon (f, J) \longmapsto ([f], J).$$

(6.1) LEMMA. *Let $f, f' \in \mathcal{H}_J(M)$. Then $[f] = [f']$ if and only if there exists a $J$-holomorphic map $g\colon M \to N$, such that $f' = gf$.*

PROOF: The 'if' part is clear, so assume $[f] = [f']$. Let $\alpha_1, \dots, \alpha_n$ be the poles of $f$ and $f'$ and put $V = M \setminus \{\alpha_1, \dots, \alpha_n\}$. There exist neighbourhoods $U_i$ of $\alpha_i$ and $J$-holomorphic maps $g_i\colon U_i \to N$, such that $f'|_{U_i} = g_i\, f|_{U_i}$ all $i = 1, \dots n$. On $V$ we can consider $f$ and $f'$ as maps into $N$. So on $V \cap U_i$ we must have $g_i|_{V\cap U_i} = f'|_{V\cap U_i}\, f|_{V\cap U_i}^{-1}$ and hence $g\colon M \to N$ can be defined by

$$g(x) = \begin{cases} f'(x) f^{-1}(x), & \text{for } x \in V \\ g_i(x), & \text{for } x \in U_i. \end{cases} \blacksquare$$

The lemma says that the fiber at $([f], J)$ of the map $\mathcal{H}(\overline{M}) \to \mathcal{P}(\overline{M})$ is $\operatorname{Hol}_J(M, N)$.

In the case of $Y = \Omega G$, proposition 4.4 and proposition 4.5 imply that $\mathcal{P}_J(M)$ is the set of holomorphic $G_{\mathbf{C}}$-bundles on $M_J \times \mathbf{CP}^1$ with only finitely many jumping lines.

Before we equip $\mathcal{P}(\overline{M})$ with a topology, we will study the action of $\operatorname{Hol}_J(M, N)$ on $\mathcal{H}_J(M)$ a little closer

(6.2) LEMMA. *$\operatorname{Hol}_J(M, N)$ acts freely on $\mathcal{H}_J(M)$.*

PROOF: Let $g \in \operatorname{Hol}_J(M, N)$ and $f \in \mathcal{H}_J(M)$ and assume that $gf = f$. As $N$ acts freely on $Y_a$, $g(x) = 1$ for $x \in f^{-1}(Y_a)$, but $f^{-1}(Y_a)$ is dense in $M$, and thus $g = 1$. $\blacksquare$

(6.3) LEMMA. *Let $U \subseteq M$, let $\alpha_1, \dots, \alpha_n \in U \setminus \partial U$ and put $V = U \setminus \{\alpha_1, \dots, \alpha_n\}$. Let there be given a sequence of complex structures $J_n \in \mathcal{C}(\overline{M})$ and a sequence of maps $g_n\colon U \to N$, such that $g_n$ is $J_n$-holomorphic. If $J_n \to J \in \mathcal{C}(\overline{M})$ and $g_n|_V \to g$, where $g\colon V \to N$ is $J$-holomorphic, then $g$ extends to a $J$-holomorphic map $g\colon U \to N$, and $g_n \to g$.*

PROOF: Let $\alpha \in U \setminus V$ and choose a disk $D_\alpha$ in $(V \setminus \partial U) \cup \{\alpha\}$ around $\alpha$. Choose, continuously depending on $J' \in \mathcal{C}(M)$, a $J'$-holomorphic homeomorphism

$\phi_{J'}\colon D_\alpha \to D$, such that $\phi_{J'}(\alpha) = 0$. Let $c = \{z \in \mathbf{C} \mid |z| = \frac{2}{3}\}$. We can imbed $N$ as a closed subset of a complex topological vector space $E$. In the case of a loop group $E$ is not a Banach space, but there do exist norms $\|\ \|_m$ on $E$, and a sequence in $E$ converges if and only if it converges in all these norms. We can write

$$g(x) = \sum_{k=-\infty}^{\infty} a_k \phi_J(x)^k \qquad \text{for } x \in D_\alpha \setminus \{\alpha\}$$

with

$$a_k = \frac{1}{2\pi i}\int_c \frac{g\circ\phi_J^{-1}(z)}{z^{k+1}}dz \in E.$$

As $g_n \to g$ and $\phi_{J_n}^{-1} \to \phi_J^{-1}$ uniformly on $c$, we have $a_n = 0$ if $n < 0$. Thus $g$ extends to a $J$-holomorphic map $g\colon V \cup \{\alpha\} \to N$. Let $K = \phi_J^{-1}(\{z \in \mathbf{C} \mid |z| \le \frac{1}{3}\})$. It is a compact neighbourhood of $\alpha$, and $\operatorname{dist}(\phi_J(K), c) = \frac{1}{3}$, hence $\operatorname{dist}(\phi_{J_n}(K), c) > \frac{1}{4}$, if $n$ is sufficiently large. For such an $n$, an $x_0 \in K$ and a norm $\|\ \|$ as above on $E$

$$\|g_n(x_0) - g(x_0)\| = \left\| \frac{1}{2\pi i}\int_c \frac{g_n\circ\phi_{J_n}^{-1}(z)}{z-\phi_{J_n}(x_0)}dz - \frac{1}{2\pi i}\int_c \frac{g\circ\phi_J^{-1}(z)}{z-\phi_J(x_0)}dz \right\|$$

$$\le \frac{1}{2\pi}\int_c \left\| \frac{(z-\phi_J(x_0))\left(g_n\circ\phi_{J_n}^{-1}(z) - g\circ\phi_J^{-1}(z)\right) + (\phi_{J_n}(x_0) - \phi_J(x_0))g\circ\phi_J^{-1}(z)}{(z-\phi_{J_n}(x_0))(z-\phi_J(x_0))} \right\| dz$$

$$\le 3\int_c \left( \|g_n\circ\phi_{J_n}^{-1}(z) - g\circ\phi_J^{-1}(z)\| + \|\phi_{J_n}(x_0) - \phi_J(x_0)\|\,\|g\circ\phi_J^{-1}(z)\| \right) dz.$$

As $g_n\circ\phi_{J_n}^{-1}(z) \to g\circ\phi_J^{-1}(z)$ uniformly on $c$, $\phi_{J_n}(x_0) \to \phi_J(x_0)$ and $\|g\circ\phi_J^{-1}(z)\|$ is bounded on $c$, we have $\|g_n - g\| \to 0$ uniformly on $K$. Hence $g_n \to g$ uniformly on compact subsets of $V \cup \{\alpha\}$ and induction on the number of points in $U \setminus V$ finishes the proof. ∎

(6.4) LEMMA. *Let $J_n$ be a sequence of complex structures on $\overline{M}$, let $g_n$ be a sequence of maps $M \to N$ and let $f_n$ be a sequence of maps $M \to Y$, such that $g_n \in \mathrm{Hol}_{J_n}(M,N)$ and $f_n \in \mathcal{H}_{J_n}(M)$. If $J_n \to J \in \mathcal{C}(\overline{M})$, $f_n \to f \in \mathcal{H}_J(M)$ and $g_n f_n \to \tilde{f} \in \mathcal{H}_J(M)$, then there exists a $g \in \mathrm{Hol}_J(M,N)$, such that $g_n \to g$ and $\tilde{f} = gf$.*

PROOF: Put $V = f^{-1}(Y_a) \cap \tilde{f}^{-1}(Y_a)$. Then $M \setminus V$ is finite, and we can consider $f|_V$ and $\tilde{f}|_V$ as maps into $N$. Define $g\colon V \to N$ by $g = \tilde{f}|_V\, f|_V^{-1}$. Let $K$ be a compact subset of $V$. As $Y_a$ is open and $f(K) \subseteq Y_a$, we have that $f_n(K) \subseteq Y_a \cong N$ if $n$ is sufficiently large. Then $g_n|_K = g_n|_K\, f_n|_K\, f_n|_K^{-1} \to \tilde{f}_n|_K\, f_n|_K^{-1} = g_n|_K$. By lemma 6.3, $g$ extends to a $J$-holomorphic map $g\colon M \to N$ and $g_n \to g$, which in turn implies that $g_n f_n \to gf$, and thus $\tilde{f} = gf$. ∎

(6.5) COROLLARY. *$\mathrm{Hol}_J(M,N)$ acts properly on $\mathcal{H}_J(M)$.*

There is the following generalization of lemma 3.1 and lemma 4.6

(6.6) LEMMA. *Let $\overline{M}$ be a two dimensional compact connected manifold with non-empty boundary and let $\overline{D}_1, \dots, \overline{D}_n$ be disjoint closed disks in $M$. Suppose we have $J$-holomorphic maps*

$$f_i \colon \overline{D}_i \longrightarrow Y \quad \text{with} \quad f_i(\partial D_i) \subseteq Y_a \quad i = 1, \dots, n,$$

*then there exist $J$-holomorphic maps*

$$f \colon \overline{M} \longrightarrow Y \quad \text{and} \quad g_i \colon \overline{D}_i \longrightarrow N \qquad i = 1, \dots, n,$$

*such that*

$$f^{-1}(Y_\infty) = f_1^{-1}(Y_\infty) \cup \cdots \cup f_n^{-1}(Y_\infty) \qquad \text{and} \qquad f_i = g_i\, f|_{\overline{D}_i} \quad i = 1, \dots, n.$$

*Furthermore, for small variations of $f_1, \dots, f_n$ and $J$, the choices can be made, such that $f$ and $g_1, \dots, g_n$ depend continuously on $f_1, \dots, f_n$ and $J$.*

PROOF: The case of $n = 2$ is lemma 3.1 or lemma 4.6, except for the last statement about $g_1$ and $g_2$, and $n = 1$ follows trivially from $n = 2$ because, if $f_1 \colon \overline{D}_1 \to Y$ is a $J$-holomorphic map with $f_1(\partial D_1) \subseteq Y_a$, then we just choose any disk $\overline{D}_2 \subseteq M \setminus \overline{D}_1$ and let $f_2$ be a constant map from $\overline{D}_2$ to $Y_a$. The general case is shown by induction on $n$.

Assume the lemma is true for $n - 1\,(\geq 2)$. Choose a closed disk $\overline{D}'_2 \subseteq M \setminus \overline{D}_1$, such that $\overline{D}_2 \cup \cdots \cup \overline{D}_n \subseteq D'_2$ and use the lemma with $\overline{D}'_2$ instead of $\overline{M}$. We get maps $f'_2 \colon \overline{D}' \to Y$ and $g'_i \colon \overline{D}_i \to N$ for $i = 2, \dots, n$, such that

(1) $f_2'^{-1}(Y_\infty) = f_2^{-1}(Y_\infty) \cup \cdots \cup f_n^{-1}(Y_\infty)$,
(2) $f_i = g'_i\, f'_2|_{\overline{D}_i}$ and
(3) $f'_2$ depends continuously on $f_2, \dots, f_n$ and $J$.

Using the lemma on $f_1, f'_2$, we get maps $f \colon \overline{M} \to Y$, $g_1 \colon \overline{D}_1 \to N$ and $g' \colon \overline{D}'_2 \to N$, such that

(1) $f^{-1}(Y_\infty) = f_1^{-1}(Y_\infty) \cup f_2'^{-1}(Y_\infty) = f_1^{-1}(Y_\infty) \cup \cdots \cup f_n^{-1}(Y_\infty)$,
(2) $f_1 = g_1\, f|_{\overline{D}_1}$,
(3) $f'_2 = g'\, f|_{\overline{D}'_2}$ and
(4) $f$ depends continuously on $f_1$, $f'_2$ and $J$ and hence on $f_1, \dots, f_n$ and $J$.

Finally, let $g_i = g'_i\, g'|_{\overline{D}_i}$ for $i = 2, \dots, n$. Then $f_i = g'_i\, f'_2|_{\overline{D}_i} = g'_i\, g'|_{\overline{D}_i}$ and $f|_{\overline{D}_i} = g_i\, f|_{\overline{D}_i}$. Outside the poles $g_i = f_i\, f|_{\overline{D}_i}^{-1}$ so by lemma 6.3, $g_i$ depends continuously on $f_1, \dots, f_n$ and $J$. ∎

We immediately get

(6.7) COROLLARY. *If $\overline{M}$ is a compact connected surface with $\partial M \neq \emptyset$, then the map $\mathcal{H}(\overline{M}) \to \mathcal{P}(\overline{M})$ is surjective, and as sets $\mathcal{P}_J(M) = \mathcal{H}_J(M)/\operatorname{Hol}_J(M, N)$.*

We are now ready to define the topology on $\mathcal{P}(\overline{M})$ in the case, where $M$ has a boundary. If $K$ is a compact subset of $M$, then we let $\mathcal{H}(K)$ denote the space of pairs $(f, J) \in \operatorname{Map}(K, Y) \times \mathcal{C}(\overline{M})$, where $f$ extends to an element of $\mathcal{H}_J(U)$ for some neighbourhood $U$ of $K$. We define an equivalence relation $\sim$ on $\mathcal{H}(K)$ by letting $(f_1, J_1) \sim (f_2, J_2)$, if $J_1 = J_2$ and there exist a neighbourhood $U$ of $K$ and a map $g \in \operatorname{Hol}_{J_1}(U, N)$, such that $f_1 = g|_K f_2$. Let $\mathcal{P}(K) = \mathcal{H}(K)/\sim$, equipped with the quotient topology. Put the weakest topology on $\mathcal{P}_{\leq n}(\overline{M})$, which makes the restriction map $\mathcal{P}_{\leq n}(\overline{M}) \to \mathcal{P}(K)$ continuous for all compact subsets $K$ of $M$. Finally let $\mathcal{P}(\overline{M}) = \lim\limits_{n \to \infty} \mathcal{P}_{\leq n}(\overline{M})$. Then we have

(6.8) LEMMA. *The maps $\mathcal{H}(\overline{M}) \to \mathcal{P}(\overline{M}) \to \mathcal{A}(\overline{M})$ are continuous.*

If $\overline{D}_1, \ldots, \overline{D}_k$ are disjoint disks in $M$, and, for $i = 1, \ldots, k$, $f_i \colon D_i \to Y$ is a $J$-holomorphic map with $f_i(D_i) \cap Y_a \neq \emptyset$ and $\deg f_i < \infty$, then we get a configuration of $J$-principal parts in $M$ denoted $[f_1] \cup \cdots \cup [f_k]$, and no matter what the boundary of $M$ is, every configuration of $J$-principal parts is of this form.

We will study $\mathcal{P}_n(\overline{M})$ a little closer. If $\partial M \neq \emptyset$, then $\mathcal{P}_n(\overline{M})$ is an open dense subset of $\mathcal{P}_{\leq n}(\overline{M})$, which is a closed subset of $\mathcal{P}(\overline{M})$. We equip $\overline{D}$ with the standard complex structure and put

$$H_n = \left\{ f \in \operatorname{Map}(\overline{D}, Y) \mid f(S^1) \subseteq Y_a,\ f|_D \text{ is } J\text{-holomorphic and } \deg f|_D = n \right\}.$$

Choose, for $i = 1, \ldots, k$ and $J \in \mathcal{C}(\overline{M})$, imbeddings $\phi_{iJ} \colon \overline{D} \to M$, such that

(1) $\phi_{iJ}|_D$ is$J$-holomorphic,
(2) $\phi_{iJ}$ depends continuously on $J$ and
(3) $\phi_{iJ}(\overline{D}) \cap \phi_{iJ}(\overline{D}) = \emptyset$, if $i \neq j$.

If $n = n_1 + \cdots + n_k$, then there is a map

$$H_{n_1} \times \ldots \times H_{n_k} \times \mathcal{C}\left(\overline{M}\right) \longrightarrow \mathcal{P}_n\left(\overline{M}\right)$$

defined by

$$(f_1, \ldots, f_k, J) \longmapsto \left([f_1 \circ \phi_{1J}^{-1}] \cup \cdots \cup [f_k \circ \phi_{kJ}^{-1}], J\right).$$

Two sets of maps $(f_1, \ldots, f_k)$ and $(f_1', \ldots, f_k')$ give the same configuration if and only if there for each $i = 1, \ldots, k$ exists a map $g_i \in \operatorname{Hol}(\overline{D}, N)$, such that $f_i' = g_i f_i$. If we put $H_n/\sim\, = H_n/\operatorname{Hol}(\overline{D}, N)$, then we have

(6.9) LEMMA. *If $\partial M \neq \emptyset$, then the map above induces a local homeomorphism*

$$(H_{n_1}/\sim) \times \ldots \times (H_{n_k}/\sim) \times \mathcal{C}\left(\overline{M}\right) \longrightarrow \mathcal{P}_n\left(\overline{M}\right),$$

*and every element of $\mathcal{P}_n\left(\overline{M}\right)$ has a neighbourhood, which is the image of such a homeomorphism.*

PROOF: Let $(f_1, \ldots, f_k, J) \in H_{n_1} \times \ldots \times H_{n_k} \times \mathcal{C}(\overline{M})$. Choose closed disks $\overline{D}_1, \ldots, \overline{D}_k$ in $M$, such that $\overline{D}_i \subseteq \phi_{iJ}(D)$ and $\phi_{iJ}(f_i^{-1}(Y_\infty)) \subseteq D_i$. These conditions are then satisfied for small variations of $(f_1, \ldots, f_k, J)$. Consider the maps

$f_i \circ \phi_{iJ}^{-1}: \overline{D}_i \to Y$, for $i = 1, \ldots, k$. They are $J$-holomorphic and, by lemma 6.6, the top map in the commutative diagram

$$\begin{array}{ccc} H_{n_1} \times \ldots \times H_{n_k} \times \mathcal{C}\left(\overline{M}\right) & \longrightarrow & \mathcal{H}_n\left(\overline{M}\right) \\ \downarrow & & \downarrow \\ (H_{n_1}/\sim) \times \ldots \times (H_{n_k}/\sim) \times \mathcal{C}\left(\overline{M}\right) & \longrightarrow & \mathcal{P}_n\left(\overline{M}\right) \end{array}$$

is defined locally. This implies that the bottom map is continuous. Let, on the other hand, $[f_1] \cup \cdots \cup [f_k]$ be a configuration of $J$-principal parts in $M$, where $f_i: D_i \to Y$ is a $J$-holomorphic map with $f_i(D_i) \cap Y_a \neq \emptyset$, $\deg f_i = n_i$ and $\overline{D}_i$ and $\overline{D}_j$ disjoint if $i \neq j$. Choose a compact set $K \subseteq D_1 \cup \cdots \cap D_k$ with interior $\overset{\circ}{K}$ and imbeddings $\phi_{iJ'}: \overline{D} \to \overset{\circ}{K}$, such that $\phi_{iJ'}$ depends continuously on $J'$, $\phi_{iJ'}(\overline{D}) \subseteq D_i$ and $f_i^{-1}(Y_\infty) \subseteq \phi_{iJ'}(D)$. Theses conditions are satisfied for small variations of $(f_1, \ldots, f_k, J)$, so locally there are maps

$$\begin{array}{ccccc} & & \mathcal{H}\left(K\right) & \longrightarrow & H_{n_1} \times \ldots \times H_{n_k} \times \mathcal{C}\left(\overline{M}\right) \\ & & \downarrow & & \downarrow \\ \mathcal{P}_n\left(\overline{M}\right) & \longrightarrow & \mathcal{P}\left(K\right) & \longrightarrow & (H_{n_1}/\sim) \times \ldots \times (H_{n_k}/\sim) \times \mathcal{C}\left(\overline{M}\right) \end{array}$$

and this finishes the proof. ∎

In particular, the transition functions between spaces of the form

$$(H_{n_1}/\sim) \times \ldots \times (H_{n_k}/\sim) \times \mathcal{C}\left(\overline{M}\right)$$

are homeomorphisms. This is even the case if $\partial M = \emptyset$, because we can always remove a disk from $M$ without disturbing a given configuration of principal parts. If $\partial M = \emptyset$, then the topology on $\mathcal{P}(M)$ is defined by declaring the inclusions

$$(H_{n_1}/\sim) \times \ldots \times (H_{n_k}/\sim) \times \mathcal{C}(M) \hookrightarrow \mathcal{P}(M)$$

to be local homeomorphisms. The subspace $\mathcal{P}_n(M)$ is then open and closed in $\mathcal{P}(M)$, and we still have

(6.10) LEMMA. *The maps* $\mathcal{H}\left(\overline{M}\right) \to \mathcal{P}\left(\overline{M}\right) \to \mathcal{A}\left(\overline{M}\right)$ *are continuous.*

Let $\widetilde{H}_n = \{f \in H_n \mid f(\overline{D}) \text{ is contained in a chart}\}$. Then $\widetilde{H}_n$ is an open subset of $\widetilde{\mathrm{Hol}}(\overline{D}, Y)$ and hence a complex manifold modelled on $\mathrm{Hol}(\overline{D}, LN)$, c.f. lemma 5.4. The following result is obvious

(6.11) LEMMA. *The restriction of the action* $F\colon \mathrm{Hol}\left(\overline{D}, Y\right) \times \widetilde{H}_n \to H_n$ *to* $F^{-1}(\widetilde{H}_n)$ *is holomorphic.*

As a corollary we have

(6.12) LEMMA. *$\widetilde{H}_n/\!\sim$ is a manifold, and the projection $\widetilde{H}_n \to \widetilde{H}_n/\!\sim$ has local sections.*

PROOF: $\widetilde{\mathrm{Hol}}(\overline{D}, Y)$ acts freely and properly on $H_n$, and a neighbourhood of the identity acts smoothly on $\widetilde{H}_n$. ∎

In lemma 6.9 we may replace $H_n$ with $\widetilde{H}_n$, i.e. we have

(6,13) LEMMA. *The maps*

$$\left(\widetilde{H}_{n_1}/\!\sim\right) \times \ldots \times \left(\widetilde{H}_{n_k}/\!\sim\right) \times \mathcal{C}\left(\overline{M}\right) \longrightarrow \mathcal{P}_n\left(\overline{M}\right)$$

*are local homeomorphisms and covers $\mathcal{P}_n\left(\overline{M}\right)$.*

PROOF: As $\widetilde{H}_n$ is an open subset of $H_n$, the first part of the lemma is trivial, and we only have to show that any configuration is hit by such a map. Let $\xi$ be a configuration of $J$-principal parts in $M$ with poles $\alpha_1, \ldots, \alpha_k$. Choose disjoint closed disks $\overline{D}_1, \ldots, \overline{D}_k$ in $M$ and $J$-holomorphic maps $f_i \colon D_i \to Y$, such that $\alpha_i \in D_i$, and $\xi = [f_1] \cup \cdots \cup [f_k]$. By restricting $f_i$ to a smaller disk, we may assume that $f_i(D_i)$ is contained in a chart. Choose, for any complex structure $J'$ on $\overline{M}$, a $J'$-holomorphic imbedding $\phi_{iJ'} \colon \overline{D} \to D_i$, such that $\phi_{iJ'}$ depends continuously on $J'$, and $\phi_{iJ'}(0) = \alpha_i$. Then $f_i \circ \phi_{iJ'} \in \widetilde{H}_{n_i}$ with $n_i = \deg f_i = \mathrm{ord}_{\alpha_i} \xi$, and clearly $(\xi, J)$ is the image of $(f_1 \circ \phi_{1J'}, \ldots, f_k \circ \phi_{kJ'}, J)$ by the map

$$\left(\widetilde{H}_{n_1}/\!\sim\right) \times \ldots \times \left(\widetilde{H}_{n_k}/\!\sim\right) \times \mathcal{C}\left(\overline{M}\right) \longrightarrow \mathcal{P}_n\left(\overline{M}\right)$$

given by

$$(f_1', \ldots, f_k', J') \mapsto \left([f_1' \circ \phi_{1J'}^{-1}] \cup \cdots \cup [f_1' \circ \phi_{1J'}^{-1}], J'\right),$$

and the proof is complete. ∎

As the fiber of $\mathcal{H}(\overline{M}) \to \mathcal{P}(\overline{M})$ is $\mathrm{Hol}_J(M, N)$, which is contractible, it is not surprising that the map is a weak homotopy equivalence, but before we prove it, we need to show that it is a quasifibration.

(6.14) LEMMA. *If $\partial M \neq \emptyset$, then the map $\pi \colon \mathcal{H}_n\left(\overline{M}\right) \to \mathcal{P}_n\left(\overline{M}\right)$ is a quasifibration over any open subset of $\mathcal{P}_n\left(\overline{M}\right)$.*

PROOF: By [**2**, Satz 2.2], it is enough to show that $\pi$ is a quasifibration over arbitrarily small open subsets. Locally we have a commutative diagram

$$\begin{array}{ccc}
\widetilde{H}_{n_1} \times \ldots \times \widetilde{H}_{n_k} \times \mathcal{C}\left(\overline{M}\right) & \longrightarrow & \mathcal{H}_n\left(\overline{M}\right) \\
\downarrow & & \downarrow \pi \\
\left(\widetilde{H}_{n_1}/\!\sim\right) \times \ldots \times \left(\widetilde{H}_{n_k}/\!\sim\right) \times \mathcal{C}\left(\overline{M}\right) & \longrightarrow & \mathcal{P}_n\left(\overline{M}\right).
\end{array}$$

As there are local sections of $\widetilde{H}_{n_i} \to \widetilde{H}_{n_i}/\sim$, there are local sections of $\pi$. Let $\sigma: W \to \mathcal{H}_n(\overline{M})$ be a section of $\pi$ over an open subset $W \subseteq \mathcal{P}_n(\overline{M})$. We only need to show that $\pi|_{\pi^{-1}(W)} : \pi^{-1}(W) \to W$ is a quasifibration. Let

$$\widetilde{W} = \left\{(g, (\xi, J)) \in \mathrm{Map}(M, N) \times W \mid g \text{ is } J\text{-holomorphic}\right\},$$

and consider the map

$$\widetilde{W} \longrightarrow \pi^{-1}(W): (g, (\xi, J)) \longmapsto g\sigma(\xi, J).$$

It is a homeomorphism, so we only have to show that the projection $\widetilde{W} \to W$ is a quasifibration. This is trivial, as a contraction of $N$ induces a fiber preserving deformation of $\widetilde{W}$ onto $\{0\} \times W$. ∎

We can now show

(6.15) LEMMA. *If $\partial M \neq \emptyset$, then the map $\pi: \mathcal{H}\left(\overline{M}\right) \to \mathcal{P}\left(\overline{M}\right)$ is a quasifibration.*

PROOF: As $\mathcal{P}(\overline{M}) = \lim_{n\to\infty} \mathcal{P}_{\leq n}(\overline{M})$, it is enough to show that $\pi$ is a quasifibration, when restricted to $\mathcal{H}_{\leq n}(\overline{M})$, see [**2**,Satz 2.15], which we do by induction on $n$. Assume that the restriction to $\mathcal{H}_{\leq n-1}(\overline{M})$ is a quasifibration. Choose a neighbourhood $B(\epsilon)$ of $\partial M$ in $\overline{M}$, homeomorphic to $\partial M \times [0, \epsilon)$ and put

$$W = \left\{(\xi, J) \in \mathcal{P}_{\leq n}(\overline{M}) \mid \deg \xi|_{M\setminus B(\epsilon)} \leq n-1\right\}.$$

Then $W$ is a neighbourhood of $\mathcal{P}_{\leq n-1}(\overline{M})$ in $\mathcal{P}_{\leq n}(\overline{M})$, and it is enough to show that $\pi$ is a quasifibration, when restricted to $\pi^{-1}(W)$, $\mathcal{H}_n(\overline{M})$ and $\mathcal{H}_{\leq n}(\overline{M}) \cap \pi^{-1}(W)$ respectively. By lemma 6.14 , the last two restrictions are quasifibrations, so we need only consider $\pi|_{\pi^{-1}(W)} : \pi^{-1}(W) \to W$. As the fibers of $\pi$ are contractible, it is by [**2**,Hilfsatz 2.10] enough to find a deformation $\psi_t: W \to W$, $t \in [0,1]$, such that

(1) $\psi_0 = id$,
(2) $\psi_t\left(\mathcal{P}_{\leq n-1}\left(\overline{M}\right)\right) \subseteq \mathcal{P}_{\leq n-1}\left(\overline{M}\right)$ all $t$,
(3) $\psi_1(W) = \mathcal{P}_{\leq n-1}\left(\overline{M}\right)$ and
(4) $\psi_t$ lifts to a deformation of $\pi^{-1}(W)$.

Choose a vector field on $\overline{M}$, such that the corresponding flow $\phi_t$ preserves $\overline{M}\setminus B(\epsilon)$ and has $\phi_1(\overline{M}) \subseteq \overline{M} \setminus B(\epsilon)$. We put $\widetilde{\psi}_t((f, J)) = (f \circ \phi_t, \phi_t(J))$. This defines a deformation $\widetilde{\psi}_t$ of $\pi^{-1}(W)$, which clearly descends to the wanted deformation $\psi_t$ of $W$. ∎

We have already noted that the fibers of $\mathcal{H}(\overline{M}) \to \mathcal{P}(\overline{M})$ are contractible, so we get

(6.16) LEMMA. *If $\partial M \neq \emptyset$, then the map $\mathcal{H}(\overline{M}) \to \mathcal{P}(\overline{M})$ is a weak homotopy equivalence.*

Two configurations $\xi_1$ and $\xi_2$ of $J$-principal parts without common poles give rise to a new configuration $\xi_1 \cup \xi_2$ of $J$-principal parts called the *union* or the *sum* of $\xi_1$ and $\xi_2$.

(6.17) LEMMA. *Addition of principal parts is a continuous map :*

$$\{((\xi_1, J), (\xi_2, J)) \in \mathcal{P}(\overline{M}) \times \mathcal{P}(\overline{M}) \mid \text{pole}\, \xi_1 \cap \text{pole}\, \xi_2 = \emptyset\} \longrightarrow \mathcal{P}(\overline{M}).$$

PROOF: Let $((\xi_{1n}, J_n), (\xi_{2n}, J_n)) \to ((\xi_1, J), (\xi_2, J))$ be a convergent sequence in the space above. Let $\alpha_1, \dots, \alpha_{k_1}$ be the poles of $\xi_1$ and let $\alpha_{k_1+1}, \dots, \alpha_k$ be the poles of $\xi_2$. Choose disjoint closed disks $\overline{D}_1, \dots, \overline{D}_k$ in $M$, with $\alpha_i \in D_i$ all $i = 1, \dots, k$. Let, for $j = 1, 2$, $\widetilde{\xi}_{jn}$ be the part of $\xi_{jn}$, which lies in $D_1 \cup \dots \cup D_k$. Then $(\widetilde{\xi}_{jn}, J_n) \to (\xi_j, J)$ and, for $n$ sufficiently large, $\deg \widetilde{\xi}_{jn} = \deg \xi_j = n_j$. We obviously have that $(\widetilde{\xi}_{1n} \cup \widetilde{\xi}_{2n}, J_n) \to (\xi_1 \cup \xi_2, J)$, and if $K$ is any compact subset of $M$, then $\widetilde{\xi}_{jn}|_K = \xi_{jn}|_K$, if $n$ is large. Hence $(\xi_{1n} \cup \xi_{2n}, J_n) \to (\xi_1 \cup \xi_2, J)$. ∎

(6.18) LEMMA. *The fiber of the pole map $\mathcal{P}_1(\overline{M}) \to \mathcal{A}_1(M)$, restricted to configurations with one simple pole, has $r$ connected components, one for each irreducible component $Y_i$ of $Y_\infty = Y_1 \cup \dots \cup Y_r$.*

PROOF: Let $\alpha \in M = \mathcal{A}_1(M)$ be given. Choose for $J \in \mathcal{C}(\overline{M})$, a $J$-holomorphic imbedding $\phi_J \colon \overline{D} \to M$, such that $\phi_J(0) = \alpha$ and $\phi_J$ depends continuously on $J$. The fiber over $\alpha$ of the pole map is homeomorphic to

$$\{([f], J) \in (\widetilde{H}_1/\!\sim) \times \mathcal{C}(\overline{M}) \mid f(0) \in Y_\infty\}.$$

As $\mathcal{C}(\overline{M})$ is contractible, it is enough to consider the space

$$\{[f] \in \widetilde{H}_1/\!\sim \mid f(0) \in Y_\infty\}.$$

Let $[f]$ be an element of this space. Then $f(\overline{D}) \cap Y_\infty = \{f(0)\}$, and the order of contact is one. Thus $f(0)$ is a simple point of $Y_\infty$, and as the sets $Y_i \cap Y_j$ consist of singular points for $i \neq j$, the fiber has at least $r$ connected components. On the other hand, the set of singular points in $Y_\infty$ is a proper subvariety of $Y_\infty$ and has at least complex codimension one. Hence the set $Y_i^s$ of points in $Y_i$, which is simple in $Y_\infty$, is connected. Around each point $y \in Y_i^s$, exist local coordinates $(u, v)$ on $Y$, such that $Y_i$ is given by the equation $u = 0$. In these coordinates, $f$ is given by a pair of maps

$$f(z) = (u(z), v(z)) \qquad \text{with} \qquad u(z) = \sum_{n=1}^{\infty} u_n z^n.$$

We put

$$f_t(z) = (u_t(z), v_t(z))$$

with

$$u_t(z) = z \sum_{n=1}^{\infty} u_n(tz)^{n-1} \qquad \text{and} \qquad v_t(z) = v(tz).$$

This gives us a curve $f_t$ from $f = f_1$ to $f_0$. The map $f_t$ has only one simple pole at 0 for all $t$, and $f_0(z) = (u_1 z, v(0))$. By covering a curve in $Y_i^s$ from $f_0(0) = (0, v(0))$ to a base point $y_i \in Y_i^s$ with a finite number of local coordinates, $f_0$ can be deformed such that the new $f_0$ has $f_0(0) = y_i$ and in local coordinates $f_0(z) = (u_1 z, 0)$. Finally we just have to deform $u_1$ into a base point. ∎

Higher order poles can be split continuously in the following sense

(6.19) LEMMA. *Given a J-principal part $\xi$ at $\alpha \in M$ and a neighbourhood $U$ of $\alpha$. Then $\xi$ can be deformed continuously into a configuration of principal parts in $U$, all with simple poles.*

PROOF: We use induction on the order $\operatorname{ord}_\alpha \xi$ of the principal part. If $\operatorname{ord}_\alpha \xi = 1$, there is nothing to show. So we need only to show that we continuously can split a principal part of order $m \geq 2$ into a configuration of two or more principal parts in $U$, which then necessarily have strictly lower orders.

We may assume that $U = D$, $\alpha = 0$ and $f\colon D \to Y$ is a representative for $\xi$, which maps $D$ into a chart. If $f(0) \in Y_\infty$ is a simple point, then there exist local coordinates $(u, v)$ on $Y$, such that $Y_\infty$ is given by the equation $u = 0$. The map $f$ is given by a pair of maps $f(z) = (u(z), v(z))$. Put $v_t = v$ and $u_t(z) = tz + u(z)$. Then $f_t(z) = (u_t(z), v_t(z))$ defines a curve $f_t$ starting at $f = f_0$. For $t \neq 0$, $f_t$ has a simple pole at 0 and hence some other pole in the vicinity of 0. If $f(0)$ is a singular point on $Y_\infty$, then it is obviously enough to find a curve $f_t$ with $f_0 = f$ such that $f_t(0)$ is a simple point on $Y_\infty$ for $t \neq 0$. Let $u$ be a local coordinate on $Y$ around $f(0)$, such that $f$ is given by $f(z) = u(z)$, with $u(0) = 0$. The singular points have at least complex codimension one in $Y_\infty$, so there exists a curve $\tilde{u}(t)$ such that $\tilde{u}(0) = 0$, which corresponds to the singular point $f(0)$, and $\tilde{u}(t)$ corresponds to simple point on $Y_\infty$ for $t \neq 0$. We define the curve $f_t$ by $f_t(z) = \tilde{u}(t) + u(z)$. ∎

REMARK. *If $Y_\infty$ is irreducible, then the last two results shows that the space $\mathcal{P}(\overline{M})$ is connected*

If $\overline{M'}$ is another compact surface and $\overline{M'} \subseteq \overline{M}$, then the restriction from $\overline{M}$ to $\overline{M'}$ is a continuous map $r\colon \mathcal{P}(\overline{M}) \to \mathcal{P}(\overline{M'})$ and the fiber $r^{-1}(\xi', J')$ is homeomorphic to $\{(\xi, J) \in \mathcal{P}(\overline{M}, M') \mid J|_{\overline{M'}} = J'\}$ by the map $(\xi, J) \mapsto (\xi \cup \xi', J)$. We shall show that $r$ is a quasifibration under certain conditions, but first we need a couple of lemmas.

We say that $\overline{M'} \subseteq \overline{M}$ is nicely imbedded, if $\partial M' \cap M$ only have finitely many connected components $\partial_1, \ldots, \partial_k$, and the closure $\overline{\partial}_i$ of each of these intersects $\partial M$ transversally and has a neighbourhood $B_i(\epsilon)$ in $\overline{M}$ homeomorphic to $\overline{\partial}_i \times (-\epsilon, \epsilon)$, such that $B_i(\epsilon) \cap B_j(\epsilon) = \emptyset$, if $i \neq j$. We put $B(\epsilon) = B_1(\epsilon) \cup \cdots \cup B_k(\epsilon)$. Then $B(\epsilon)$ is a neighbourhood of $\overline{\partial M' \cap M}$ homeomorphic to $\overline{\partial M' \cap M} \times (-\epsilon, \epsilon)$.

(6.20) LEMMA. *Let $\overline{M'} \subseteq \overline{M}$ be nicely imbedded and let $B(\epsilon)$ be a neighbourhood of $\overline{\partial M' \cap M}$ as above. If $K \subseteq \mathcal{C}(\overline{M})$ is compact, then we can find a map*

$$v: K \longrightarrow \{\text{vector fields on } \overline{M}\},$$

*such that*

(1) *$v(J)$ intersects $\partial M' \cap M$ transversally and points into $M'$,*
(2) *$v(J)$ do not point out of $\overline{M}$ and*
(3) *there exists a $\epsilon > 0$, such that $v(J)|_{B(\epsilon)}$ is $J$-holomorphic.*

PROOF: First we define $v(J)$ in a neighbourhood of each component of $\partial M' \cap M$, and then we just extend this family of vector fields to a family of vector fields on all of $\overline{M}$. So consider a component $\partial$ of $\partial M' \cap M$ and let $B(\epsilon)$ be a neighbourhood of $\overline{\partial}$ as above. Choose for $J \in K$, a $J$-holomorphic diffeomorphism $\phi_J: \overline{B(\epsilon)} \hookrightarrow \mathbf{C}$, continuously depending on $J$, in the $\mathbf{C}^\infty$-topology. Given $J \in K$, then we can choose a vector field $v$ on $\mathbf{C}$, such that

(1) $v$ is holomorphic in a neighbourhood $U$ of $\phi_J(\overline{\partial})$,
(2) $v$ intersects $\phi_J(\partial)$ transversally and points into $\phi_J(B(\epsilon) \cap M)$ and
(3) $v$ points into $\phi_J(B(\epsilon))$ at $\phi_J(\partial M \cap \overline{B(\epsilon)})$.

There exists a neighbourhood $W$ of $J$ in $K$, such that if $J' \in W$, then

(1) $\phi_{J'}(\overline{\partial}) \subseteq U$,
(2) $v$ intersects $\phi_{J'}(\partial)$ transversally and points into $\phi_{J'}(B(\epsilon) \cap M)$ and
(3) $v$ does not point out of $\phi_{J'}(B(\epsilon))$ at $\phi_{J'}(\partial M \cap \overline{B(\epsilon)})$.

As $K$ is compact, we can find finitely many vector fields $v_i$ on $\mathbf{C}$, open sets $U_i \subseteq \mathbf{C}$, and open sets $W_i \subseteq K$ (as above), such that $K \subseteq \bigcup W_i$. Let $(\rho_i)$ be a partition of unity on $K$, subordinated the cover $(W_i)$, and put $v(J) = \phi_{J*}^{-1}(\sum \rho_i(J) v_i)$. This is clearly a vector field with the desired properties. ∎

(6.21) LEMMA. *Let $\overline{M'} \subseteq \overline{M}$ be nicely imbedded and let $r: \mathcal{P}(\overline{M}) \to \mathcal{P}(\overline{M'})$ be the restriction map. If $W \subseteq \mathcal{P}_n(\overline{M'})$ is open, then $r|_{r^{-1}(W)} : r^{-1}(W) \to W$ is a quasifibration.*

PROOF: It is enough to show that $r$ has the following weak form of the homotopy lifting property:

If $P$ is compact, and $h: P \to r^{-1}(W)$ and $\overline{H}: P \times [0,1] \to W$ are maps, such that $\overline{H}(x,t) = r \circ h(x)$ all $x$, if $t \in [0,1/2]$, then there exists a lift of $\overline{H}$, i.e. a map $H: P \times [0,1] \to r^{-1}(W)$, such that $r \circ H = \overline{H}$ and $H(x,0) = h(x)$ all $x$.

Let $h$ and $\overline{H}$ be as above. If

$$\overline{H}(x,t) = (\xi'(x,t), J'(x,t)),$$

then

$$h(x) = (\xi'(x,0) \cup \xi(x), J(x)).$$

Choose an open set $U$, such that the poles of $\xi'(x,t))$ is contained in $U$ for all $(x,t)$, and $\overline{U} \subseteq M'$. Let $v(J(x))$ be the vector field in lemma 6.20, let $t \mapsto \phi(x,t)$ be the flow restricted to $\overline{M \setminus M'}$, let $B(\epsilon)$ be the neighbourhood of $\overline{\partial M' \cap M}$ in $\overline{M}$ and put $V = M \setminus \overline{B(\epsilon) \cup M'}$. We may assume that $\overline{B(\epsilon)} \cap \overline{U} = \emptyset$, $\phi(x,0) = id$ all $x \in P$, $M \setminus M' \subseteq \phi(x,1)(V)$ all $x \in P$ and that $\phi(x,t)$ is $J(x)$-holomorphic in a neighbourhood of $\overline{\partial M' \cap M}$, see figure 6.1.

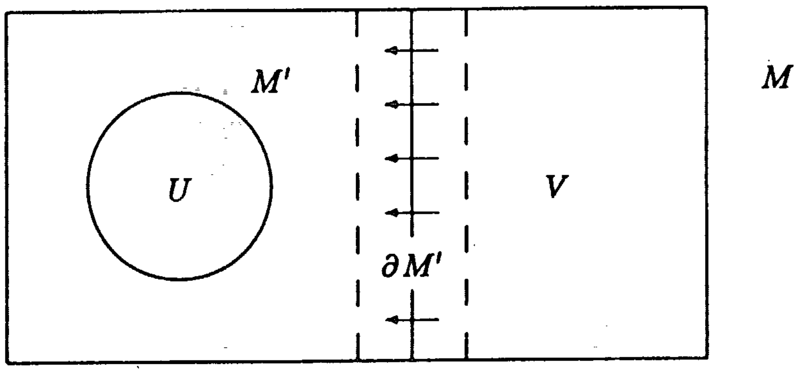


figure 6.1

As $\phi(x,t)$ is $J(x)$-holomorphic near $\overline{\partial M' \cap M}$, we can choose a continuous map $J\colon P \times [0,1] \to \mathcal{C}(\overline{M})$, such that

(1) $J(x,t)|_{\overline{M'}} = J'(x,t)$, all $t \in [0,1]$,
(2) $J(x,t)|_{\overline{M \setminus M'}} = \phi(x,2t)(J(x))|_{\overline{M \setminus M'}}$, for $t \in [0,1/2]$ and
(3) $J(x,t)|_{\overline{V}} = \phi(x,1)(J(x,t))|_{\overline{V}}$, for $t \in [1/2,1]$.

As the poles of $\xi(x) \circ \phi(x,1)$ lie in $V$, we can regard $\xi(x) \circ \phi(x,1)$ as a configuration of $J(x,t)$-principal parts for $t \in [1/2,1]$. Hence it is possible to define the homotopy $H\colon P \times [0,1] \to r^{-1}(W)$ by

$$H(x,t) = \begin{cases} (\xi'(x,t) \cup \xi(x) \circ \phi(x,2t), J(x,t)), & \text{for } 0 \le t \le 1/2 \\ (\xi'(x,t) \cup \xi(x) \circ \phi(x,1), J(x,t)), & \text{for } 1/2 \le t \le 1. \end{cases}$$

Obviously $r \circ H = \overline{H}$ and $H(x,0) = h(x)$ all $x$. ∎

We can now show

(6.22) PROPOSITION. *Let $\overline{M'} \subseteq \overline{M}$ be nicely imbedded and assume that every component of $\partial M'$ intersects $\partial M$. Then the restriction map $r\colon \mathcal{P}(\overline{M}) \to \mathcal{P}(\overline{M'})$ is a quasifibration.*

PROOF: As $\mathcal{P}(\overline{M'}) = \lim_{n\to\infty} \mathcal{P}_{\le n}(\overline{M'})$, it is enough to show that $r$ is a quasifibration over $\mathcal{P}_{\le n}(\overline{M'})$, which we do by induction on $n$. By lemma 6.21, $r$ is a quasifibration over $\mathcal{P}_{\le 0}(\overline{M'}) = \mathcal{P}_0(\overline{M'})$, i.e. the start of the induction is secured. Assume that $r$ is a quasifibration over $\mathcal{P}_{\le n-1}(\overline{M'})$.

Let $B'(\epsilon)$ be a neighbourhood of $\partial M'$ in $\overline{M'}$, homeomorphic to $\partial M' \times [0,\epsilon)$, and put

$$W = \{(\xi, J) \in \mathcal{P}_{\leq n}(\overline{M'}) \mid \deg \xi|_{M'\setminus B(\epsilon)} \leq n-1\}.$$

It is a neighbourhood of $\mathcal{P}_{\leq n-1}(\overline{M'})$ in $\mathcal{P}_{\leq n}(\overline{M'})$, and by lemma 6.21, $r$ is a quasifibration over $\mathcal{P}_n(\overline{M'})$ and $W \cap \mathcal{P}_n(\overline{M'})$. Thus, it is enough to show that $r$ is a quasifibration over $W$, see [**2**,Satz 2.2].

Choose a vector field on $\overline{M}$, such that the induced flow $\phi_t$ satisfies

(1) $\phi_t(M') \subseteq M'$ all $t$,
(2) $\phi_t(M' \setminus B'(\epsilon)) \subseteq M' \setminus B'(\epsilon)$ all $t$ and
(3) $\phi_1(M') \subseteq M' \setminus B'(\epsilon)$.

See figure 6.2.

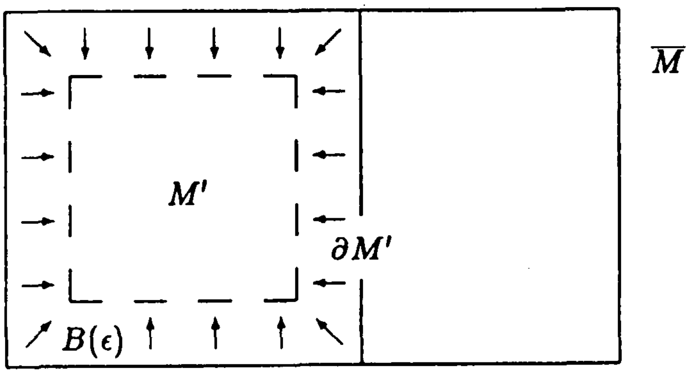



figure 6.2

We define deformations $d_t$ of $W$ and $D_t$ of $r^{-1}(W)$ by

$$d_t(\xi', J') = (\xi' \circ \phi_t, \phi_t(J')) \qquad \text{and} \qquad D_t(\xi, J) = (\xi \circ \phi_t, \phi_t(J)).$$

As $r \circ D_t = d_t \circ r$, $d_t(\mathcal{P}_{\leq n-1}(\overline{M'})) \subseteq \mathcal{P}_{\leq n-1}(\overline{M'})$ and $d_1(W) \subseteq \mathcal{P}_{\leq n-1}(\overline{M'})$, we only have to show that $D_1|_{r^{-1}((\xi',J'))} : r^{-1}((\xi', J')) \to r^{-1}(d_1(\xi', J'))$ is a weak homotopy equivalence, c.f. [**2**,Satz 2.10]. We have

$$r^{-1}((\xi', J')) \cong \{(\xi, J) \in \mathcal{P}(\overline{M}, \overline{M'}) \mid J|_{\overline{M'}} = J'\} = F_0,$$
$$r^{-1}(d_1(\xi', J')) \cong \{(\xi, J) \in \mathcal{P}(\overline{M}, \overline{M'}) \mid J|_{\overline{M'}} = \phi_1(J')\} = F_1,$$

and $D_1 \colon F_0 \to F_1$ is given by $D_1(\xi, J) = (\tilde{\xi} \cup \xi \circ \phi_1, \phi_1(J))$, where $\tilde{\xi}$ is a (possible empty) configuration of principal parts in $B'(\epsilon) \cap M'$, which by the flow $\phi_t$ is moved to $M \setminus M'$. The configuration $\xi \circ \phi_1$ is pushed away from $\partial M'$, and it is possible to move $\tilde{\xi}$ along $\partial M'$ to $\partial M$. Hence $D_1$ is homotopy equivalent to the map $D \colon F_0 \to F_1$, given by $D(\xi, J) = (\xi \circ \phi_1, \phi_1(J))$. We want to find a homotopy inverse $\tilde{D} \colon F_1 \to F_0$.

Let $B(\epsilon)$ be a neighbourhood of $\overline{\partial M' \cap M}$ in $\overline{M}$, which is homeomorphic to $\overline{\partial M' \cap M} \times (-\epsilon, \epsilon)$, and let $s \mapsto \psi(t, s)$ be the flow of a vector field on $\overline{M}$, such that

(1) $\psi(t, s)$ depends continuously on $(t, s)$,

(2) $\psi(t,s)$ is $\phi_t(J')$-holomorphic on $B(\epsilon) \cap \overline{M'}$ all $(t,s)$,
(3) $M \setminus (B(\epsilon) \cup M') \subseteq \psi(t,s)(M \setminus (B(\epsilon) \cup M'))$ all $(t,s)$,
(4) $M \setminus M' \subseteq \psi(t,1)(M \setminus (B(\epsilon) \cup M'))$ all $t$,
(5) there exists a $n \in \mathbf{N}$ such that
  (i) $\phi_{1/n}(M \setminus M') \subseteq \psi(t,s) \circ \phi_{1/n}(M \setminus M')$ all $(t,s)$,
  (ii) $\phi_{1/n}(M \setminus M') \subseteq \psi(t,1)(M \setminus M')$ all $t$.

If $J \in \mathcal{C}(\overline{M})$ and $J|_{\overline{M'}} = \phi_1(J')$, then we can define $\theta(J) \in \mathcal{C}(M')$ by

$$\theta(J) = \begin{cases} (\phi_{1/n}^{-1} \circ \psi(\frac{1}{n},1) \circ \phi_{1/n}^{-1} \circ \psi(\frac{2}{n},1) \circ \cdots \circ \phi_{1/n}^{-1} \circ \psi(1,1))(J) & \text{on } \overline{M} \setminus M' \\ J & \text{on } \overline{M'}. \end{cases}$$

Now $\widehat{D}\colon F_1 \to F_0$ is defined by

$$\widehat{D}(\xi, J) = \left(\xi \circ \phi_{1/n}^{-1} \circ \psi(1/n,1) \circ \phi_{1/n}^{-1} \circ \psi(2/n,1) \circ \cdots \circ \phi_{1/n}^{-1} \circ \psi(1,1), \theta(J)\right).$$

We shall show that $D \circ \widehat{D}$ and $\widehat{D} \circ D$ are homotopic to the identity. First we define $\theta_t\colon \overline{M} \to \overline{M}$ for $t \in [0,1]$ by

$$\theta_t = \phi_{k/n} \circ \psi(\tfrac{n-k}{n}, nt-k) \circ \phi_{1/n}^{-1} \circ \psi(\tfrac{n-k+1}{n}, 1) \circ \cdots \circ \phi_{1/n}^{-1} \circ \psi(1,1), \quad \text{if } \tfrac{k}{n} \le t \le \tfrac{k+1}{n}.$$

For a $J \in \mathcal{C}(\overline{M})$ with $J|_{\overline{M'}} = J'$, we define $\widetilde{\theta}_t(J) \in \mathcal{C}(\overline{M})$ by

$$\widetilde{\theta}_t(J)\Big|_{\overline{M'}} = J' \qquad \text{and} \qquad \widetilde{\theta}_t(J)\Big|_{\overline{M} \setminus M'} = \theta_t(J).$$

Finally $H_t\colon F_0 \to F_0$ is defined by $H_t(\xi, J) = (\xi \circ \theta_t, \widetilde{\theta}_t(J))$. Clearly $H_0 = id$ and $H_1 = \widehat{D} \circ D$. Next we define $\theta'_t\colon \overline{M} \to \overline{M}$ for $t \in [0,1]$ by

$$\theta'_t = \phi_{1/n}^{-1} \circ \psi(\tfrac{1}{n}, 1) \circ \cdots \circ \phi_{1/n}^{-1} \circ \psi(\tfrac{k}{n}, nt-k+1) \circ \phi_{k/n}, \quad \text{if } \tfrac{k-1}{n} \le t \le \tfrac{k}{n}.$$

For a $J \in \mathcal{C}(\overline{M})$ with $J|_{\overline{M'}} = \phi_1(J')$, we define $\widetilde{\theta}'_t(J) \in \mathcal{C}(\overline{M})$ by

$$\widetilde{\theta}'_t(J)\Big|_{\overline{M'}} = \phi_1(J') \qquad \text{and} \qquad \widetilde{\theta}'_t(J)\Big|_{\overline{M} \setminus M'} = \theta'_t(J).$$

Finally $H'_t\colon F_1 \to F_1$ is defined by $H'_t(\xi, J) = (\xi \circ \theta'_t, \widetilde{\theta}'_t(J))$. Clearly, $H'_0 = id$ and $H'_1 = D \circ \widehat{D}$. ∎

## 7. The Results

In this section we will show that the topology of the space of holomorphic maps resembles the topology of the space of continuous maps. First a non closed surface is considered.

(7.1) PROPOSITION. *Let $\overline{M}$ be a compact surface, and assume that every component of $\overline{M}$ has non empty boundary. Then the map $\mathcal{H}(\overline{M}) \to \mathrm{Map}(M,Y)$ is a weak homotopy equivalence.*

PROOF: We use induction on the number of handles in a handle decomposition of $\overline{M}$. The start of the induction is secured by lemma 5.7. So assume $\overline{M} = \overline{M}_1 \cup \overline{M}_2$, and that the proposition is true for $\overline{M}_1$, $\overline{M}_2$ and $\overline{M}_1 \cap \overline{M}_2$. We may assume that the inclusions $\overline{M}_1 \cap \overline{M}_2 \subseteq \overline{M}_1$ and $\overline{M}_2 \subseteq \overline{M}$ satisfy the conditions of proposition 6.22. Consider the diagram

$$
\begin{array}{ccccccccccc}
\mathcal{P}(\overline{M}) & \longrightarrow & \mathcal{P}(\overline{M}_1) & & \mathcal{H}(\overline{M}) & \longrightarrow & \mathcal{H}(\overline{M}_1) & & \mathrm{Map}(M,Y) & \longrightarrow & \mathrm{Map}(M_1,Y) \\
\downarrow & & \downarrow & \overset{\simeq}{\longleftarrow} & \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\
\mathcal{P}(\overline{M}_2) & \longrightarrow & \mathcal{P}(\overline{M}_1 \cap \overline{M}_2) & & \mathcal{H}(\overline{M}_2) & \longrightarrow & \mathcal{H}(\overline{M}_1 \cap \overline{M}_2) & & \mathrm{Map}(M_2,Y) & \longrightarrow & \mathrm{Map}(M_1 \cap M_2,Y)
\end{array}
$$

where the maps in the squares are restrictions. The maps between the squares are weak homotopy equivalences, except possibly, the map $\mathcal{H}(\overline{M}) \to \mathrm{Map}(M,Y)$. The right-hand square is homotopy cartesian, and if the middle square is weak homotopy cartesian, the proof is complete. The left-hand square is weak homotopy cartesian, because the vertical maps are quasifibrations, but then the middle square is weak homotopy cartesian too. ∎

If $M$ is an open surface, then by [**15**,Proposition 8.11.6], it can be shown that the inclusion $\mathcal{H}_J(M) \hookrightarrow \mathrm{Map}(M,Y)$ is a homotopy equivalence, but the result above is needed, when we close the surface.

It is unfortunately impossible to apply the proof of proposition 7.1 in the case, where $\partial M = \emptyset$, because the relevant restrictions are not quasifibrations. In order to overcome this difficulty, a new stabilized space is introduced.

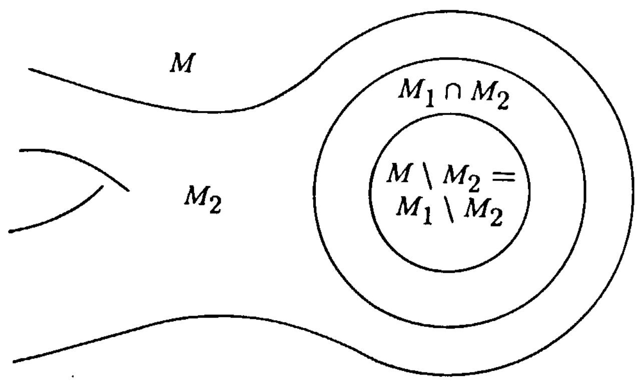



figure 7.1

Let $M$ be a closed surface. Choose open subsets $M_1, M_2 \subseteq M$, such that $\overline{M}_1$ and $\overline{M}_2$ are manifolds with boundaries, and $\overline{M}_1$ and $M \setminus M_2$ are closed disks with $M \setminus M_2 \subseteq M_1$. Then $M = M_1 \cup M_2$, and $M_1 \cap M_2$ is an annulus, see figure 7.1.

Choose a sequence of disks $D_1, D_2, \ldots$ in $M_1$ such that $\overline{D}_{k+1} \subseteq D_k$ all $k$, and $\overline{D}_\infty = \bigcap D_k$ is a disk with $\partial M_2 \cap D_\infty \neq \emptyset$. Choose for all $k$, a point

$\alpha_k \in D_k \setminus \overline{D}_{k+1}$, such that $\overline{\{\alpha_k \mid k \in \mathbf{N}\}} \cap \overline{M}_2 = \emptyset$. Choose continuously depending on $J \in \mathcal{C}(\overline{M}_1)$, a $J$-holomorphic imbedding

$$\phi_{Jk}\colon D \to D_k \setminus (\overline{D}_{k+1} \cup \overline{M}_2),$$

such that $\phi_{Jk}(0) = \alpha_k$. If $Y_1, \dots, Y_r$ are the irreducible components of $Y_\infty$, then, for each $i = 1, \dots, r$, we choose a holomorphic map $f_i\colon D \to Y$, such that 0 is the only pole and $\mathrm{ord}_{j,0} f_i = \delta_{ij}$, see figure 7.2.

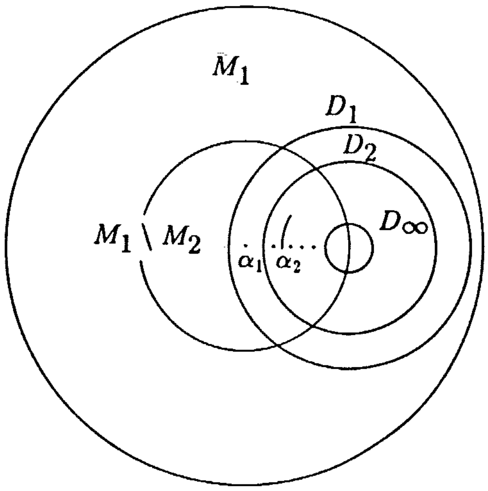



figure 7.2

We define a $J$-principal part $\xi_{Jk}$ at $\alpha_k$, with $\mathrm{ord}_j\, \xi_{Jk} = \delta_{ij}$ where $k \equiv i \pmod r$, by $\xi_{Jk} = [f_i \circ \phi_{Jk}^{-1}]$. Define imbeddings

$$\mathcal{P}(M, \overline{D}_k) \longrightarrow \mathcal{P}(M, \overline{D}_{k+1})\colon (\xi, J) \longmapsto (\xi \cup \xi_{J|_{\overline{M}_1}k}, J)$$

and

$$\mathcal{P}(M_1, \overline{D}_k) \longrightarrow \mathcal{P}(M_1, \overline{D}_{k+1})\colon (\xi, J) \longmapsto (\xi \cup \xi_{Jk}, J),$$

and form the telescopes

$$\widehat{\mathcal{P}}(M, \overline{D}_\infty) = \mathrm{Tel}\left(\mathcal{P}(M, \overline{D}_1) \longrightarrow \mathcal{P}(M, \overline{D}_2) \longrightarrow \dots\right)$$

and

$$\widehat{\mathcal{P}}(\overline{M}_1, \overline{D}_\infty) = \mathrm{Tel}\left(\mathcal{P}(\overline{M}_1, \overline{D}_1) \longrightarrow \mathcal{P}(\overline{M}_1, \overline{D}_2) \longrightarrow \dots\right).$$

The imbeddings, defining the telescopes, fit in as the horizontal maps in the following commutative diagrams,

$$\begin{array}{ccc} \mathcal{P}(M, \overline{D}_k) & \longrightarrow & \mathcal{P}(M, \overline{D}_{k+1}) \\ \downarrow & & \downarrow \\ \mathcal{P}(\overline{M}_2, \overline{D}_\infty) & = & \mathcal{P}(\overline{M}_2, \overline{D}_\infty), \end{array}$$

$$\begin{array}{ccc} \mathcal{P}(\overline{M}_1, \overline{D}_k) & \longrightarrow & \mathcal{P}(\overline{M}_1, \overline{D}_{k+1}) \\ \downarrow & & \downarrow \\ \mathcal{P}(\overline{M}_1 \cap \overline{M}_2, \overline{D}_\infty) & = & \mathcal{P}(\overline{M}_1 \cap \overline{M}_2, \overline{D}_\infty) \end{array}$$

and

$$\begin{array}{ccccccc} \dots \longrightarrow & \mathcal{P}(M, \overline{D}_k) & \longrightarrow & \mathcal{P}(M, \overline{D}_{k+1}) & \longrightarrow \dots \\ & \downarrow & & \downarrow & \\ \dots \longrightarrow & \mathcal{P}(\overline{M}_1, \overline{D}_k) & \longrightarrow & \mathcal{P}(\overline{M}_1, \overline{D}_{k+1}) & \longrightarrow \dots, \end{array}$$

where the vertical maps are the restrictions. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{P}}(M, \overline{D}_\infty) & \longrightarrow & \widehat{\mathcal{P}}(\overline{M}_1, \overline{D}_\infty) \\ \downarrow & & \downarrow \\ \mathcal{P}(\overline{M}_2, \overline{D}_\infty) & \longrightarrow & \mathcal{P}(\overline{M}_1 \cap \overline{M}_2, \overline{D}_\infty), \end{array}$$

which we will show is homology cartesian. By lemma 3 in the appendix, it is enough to show

(7.2) PROPOSITION. *The restriction maps*

$$r\colon \widehat{\mathcal{P}}(M, \overline{D}_\infty) \longrightarrow \mathcal{P}(\overline{M}_2, \overline{D}_\infty) \quad \text{and} \quad r\colon \widehat{\mathcal{P}}(\overline{M}_1, \overline{D}_\infty) \longrightarrow \mathcal{P}(\overline{M}_1 \cap \overline{M}_2, \overline{D}_\infty)$$

*are homology fibrations.*

PROOF: Let $\overline{M}'$ denote either $M$ or $\overline{M}_1$ and put $\overline{M}'_2 = \overline{M}_2 \cap \overline{M}'$. Let $(\xi_0, J_0)$ belong to $\mathcal{P}(\overline{M}', \overline{D}_\infty)$, let $\beta_1, \dots, \beta_k$ be the poles of $\xi_0$ and let $\nu_1, \dots, \nu_k$ be their orders. Let $B(\epsilon)$ be a neighbourhood of $\partial M'_2$ in $M$, which is homeomorphic to $\partial M'_2 \times (-\epsilon, \epsilon)$, and choose $\epsilon > 0$ such that

(1) $\alpha_i, \beta_j \notin \overline{B(2\epsilon)}$ for all $i = 1, 2, \dots$ and $j = 1, \dots, k$,
(2) $\overline{D}_\infty \subseteq B(2\epsilon) \cup M_1$, and
(3) $D_\infty \cap (M_2 \setminus B(2\epsilon)) \neq \emptyset$,

see figure 7.3. Choose for $i = 1, \dots, k$ an open disk $U_i$ around $\beta_i$, such that

(1) $\overline{U}_i \subseteq M'_2 \setminus (\overline{B(2\epsilon)} \cup \overline{D}_\infty)$ for all $i$ and
(2) $\overline{U}_i \cap \overline{U}_j = \emptyset$ if $i \neq j$.

The set

$$\left\{(\xi, J) \in \mathcal{P}\left(\overline{M}'_2, \overline{D}_\infty\right) \mid \deg \xi|_{U_i} = \nu_i \text{ and } \operatorname{pole}(\xi) \subseteq U_1 \cup \dots \cup U_k \cup B(\epsilon)\right\}$$

is a neighbourhood of $(\xi_0, J_0)$ in $\mathcal{P}(\overline{M}', \overline{D}_\infty)$. So by lemma 6.13, $(\xi_0, J_0)$ has a neighbourhood homeomorphic to

$$\left(\widetilde{H}_{\nu_1}/\sim\right) \times \dots \times \left(\widetilde{H}_{\nu_k}/\sim\right) \times \left\{(\xi, J) \in \mathcal{P}\left(\overline{M}'_2, \overline{D}_\infty\right) \mid \operatorname{pole}(\xi) \subseteq B(\epsilon)\right\}.$$

As $\widetilde{H}_{\nu_1}/\sim$ is a manifold, $(\xi_0, J_0)$ has a neighbourhood $W$ in $\mathcal{P}(\overline{M'_2}, \overline{D}_\infty)$, homeomorphic to $B_1 \times \ldots \times B_k \times B$, where $B_i$ is an open contractible subset of $\widetilde{H}_{\nu_i}/\sim$, and

$$B = \left\{(\xi, J) \in \mathcal{P}\left(\overline{M'_2}, \overline{D}_\infty\right) \mid \text{pole}(\xi) \subseteq B(\epsilon)\right\}.$$

From the diagram

$$\begin{array}{ccc} r^{-1}(W) & \xrightarrow{\ \sim\ } & B_1 \times \ldots \times B_k \times r^{-1}(B) \\ \Big\downarrow {\scriptstyle r|_{r^{-1}(W)}} & & \Big\downarrow {\scriptstyle id \times r|_{r^{-1}(B)}} \\ W & \xrightarrow{\ \sim\ } & B_1 \times \ldots \times B_k \times B, \end{array}$$

it is seen that we only have to show that $B$ is contractible, and that the inclusions of the fibers of $r$ in $r^{-1}(B)$ are homology equivalences.

Choose a vector field on $M$, which vanishes outside $B(2\epsilon)$, is tangent to $\partial D_i$ all $i$, is transversal to $\partial M'_2$ and points into $M'_2$, see figure 7.3.

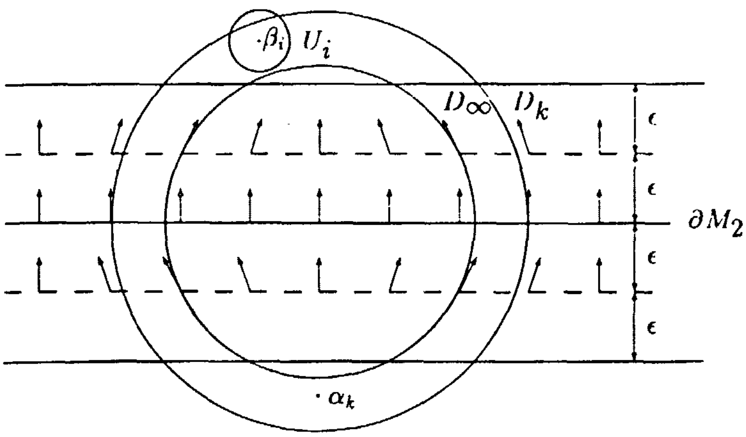



figure 7.3

Let $\phi_t$ be the flow on $M$, induced by this vector field. We may assume that the following conditions are satisfied.

(1) $\phi_t = id$ outside $B(2\epsilon)$ for all $t$,
(2) $\phi_t(D_k) = D_k$ for all $t$
(3) $\phi_t(M'_2 \cup B(\epsilon)) \subseteq M'_2 \cup B(\epsilon)$ for all $t$, and
(4) $\phi_1(M'_2 \cup B(\epsilon)) \subseteq M'_2 \setminus \overline{B(\epsilon)}$.

The flow $\phi_t$ induces a deformation $h_t$ of $B$, given by $h_t(\xi, J) = (\xi \circ \phi_t, \phi_t(J))$. As $h_1(B) \cong \mathcal{C}(\overline{M'_2})$, $B$ is contractible. Let $(\xi', J') \in B$. We shall show that the inclusion $r^{-1}(\xi', J') \hookrightarrow r^{-1}(B)$ is a homology equivalence. Define a deformation $H_t$ of $r^{-1}(B)$ by $H_t(\xi, J, s) = (\xi \circ \phi_t, \phi_t(J), s)$. Then

$$H_1\left(r^{-1}(B)\right) = \left\{(\xi, J, s) \in \widehat{\mathcal{P}}\left(\overline{M'}, \overline{D}_\infty\right) \mid \text{pole}(\xi) \subseteq M \setminus \overline{M_2 \cup B(\epsilon)}\right\}.$$

Put

$$F_1 = \left\{(\xi, J, s) \in H_1\left(r^{-1}(B)\right) \mid J|_{\overline{M'_2}} = \phi_1|_{\overline{M'_2}}(J')\right\},$$

and consider the diagram

$$\begin{array}{ccc} r^{-1}(\xi',J') & \longrightarrow & r^{-1}(B) \\ \big\downarrow{\scriptstyle H_1} & & \big\downarrow{\scriptstyle H_1} \\ F_1 & \longrightarrow & H_1(r^{-1}(B)). \end{array}$$

We will show that the two vertical maps and the lower horizontal map are homology equivalences, and hence that the top horizontal map is a homology equivalence.

First consider $H_1\colon r^{-1}(B) \to H_1(r^{-1}(B))$. If $i\colon H_1(r^{-1}(B)) \to r^{-1}(B)$ is the inclusion, then $i \circ H_1 = H_1 \sim H_0 = id$, and as $H_t(H_1(r^{-1}(B))) \subseteq H_1(r^{-1}(B))$ for all $t$, we also have $H_1 \circ i = H_1|_{H_1(r^{-1}(B))} \sim H_0|_{H_1(r^{-1}(B))} = id$. Next consider the inclusion $F_1 \hookrightarrow H_1(r^{-1}(B))$. Choose a deformation $D_t$ of $C(\overline{M}')$ such that :

(1) $D_0 = id$,
(2) $D_t(J)|_{M\setminus(M_2\cup B(\epsilon))} = J|_{M\setminus(M_2\cup B(\epsilon))}$ for all $J$,
(3) $D_t(J)|_{\overline{M}'_2} = J'$ if $J|_{\overline{M}'_2} = J'$ for all $t$, and
(4) $D_1(J)|_{\overline{M}'_2} = J'$ for all $J$.

Define a deformation $\widetilde{D}_t$ of $H_1(r^{-1}(B))$ by $\widetilde{D}_t(\xi, J, s) = (\xi, D_t(J), s)$. This deformation contracts $H_1(r^{-1}(B))$ onto $F_1$, hence the inclusion $F_1 \hookrightarrow H_1(r^{-1}(B))$ is a homotopy equivalence.

Only the map $H_1\colon r^{-1}(\xi', J') \to F_1$ remains. Let

$$F_0 = \left\{(\xi, J, t) \in \widehat{\mathcal{P}}\left(\overline{M}', \overline{D}_\infty\right) \mid \operatorname{pole}(\xi) \subseteq M \setminus \overline{M}_2 \text{ and } J|_{\overline{M}'_2} = J'\right\}.$$

This space is homeomorphic to $r^{-1}(\xi', J')$ by the map $F_0 \to r^{-1}(\xi', J')$, which maps $(\xi, J, t)$ to $(\xi \cup \xi', J, t)$. By this identification, $H_1$ corresponds to the map $H\colon F_0 \to F_1$ given by

$$(\xi, J, t) \longmapsto ((\xi \cup \xi') \circ \phi_1, \phi_1(J), t) = (\xi \circ \phi_1 \cup \xi' \circ \phi_1, \phi_1(J), t)\,.$$

By lemma 6.19 and 6.18, we can split $\xi' \circ \phi_1$ into simple principal parts, move these principal parts along $\phi_1^{-1}(\partial M_2)$ to the points $\alpha_k$, and finally deform them into the standard form $\xi_{kJ}$. The spaces $F_0$ and $F_1$ are the telescopes of the sequences $F_0^1 \to F_0^2 \to \ldots$ and $F_1^1 \to F_1^2 \to \ldots$ respectively, where

$$F_0^n = \left\{(\xi, J) \in \mathcal{P}\left(\overline{M}', \overline{D}_n\right) \mid \operatorname{pole}(\xi) \subseteq M \setminus \overline{M}_2 \text{ and } J|_{\overline{M}'_2} = J'\right\}$$

and

$$F_1^n = \left\{(\xi, J) \in \mathcal{P}\left(\overline{M}', \overline{D}_n\right) \mid \operatorname{pole}(\xi) \subseteq M \setminus \overline{M_2 \cup B(\epsilon)} \text{ and } J|_{\overline{M}'_2} = \phi_1(J')|_{\overline{M}'_2}\right\}.$$

We define $\widetilde{H}\colon F_0^n \to F_1^n$ by $\widetilde{H}(\xi, J) = (\xi \circ \phi_1, \phi_1(J))$. By lemma 5 in the appendix, we only have to show that $\widetilde{H}$ is a homotopy equivalence, and this can be proved by the same method as in the proof of proposition 6.22. ∎

We can now show that $\widehat{\mathcal{P}}(M,\overline{D}_\infty)$ and

$$\operatorname{Map}\left(M,\overline{D}_\infty;Y,Y_a\right)=\left\{f\in\operatorname{Map}(M,Y)\mid f(\overline{D}_\infty)\subseteq Y_a\right\}$$

have the same homology type. Let $H_1$ be the homotopy theoretical fiber product of

$$\begin{array}{ccc} & & \widehat{\mathcal{P}}\left(\overline{M}_1,\overline{D}_\infty\right) \\ & & \downarrow \\ \mathcal{P}\left(\overline{M}_2,\overline{D}_\infty\right) & \longrightarrow & \mathcal{P}\left(\overline{M}_1\cap\overline{M}_2,\overline{D}_\infty\right), \end{array}$$

let $H_2$ be the homotopy theoretical fiber product of

$$\begin{array}{ccc} & & \mathcal{H}\left(\overline{M}_1,\overline{D}_1\right) \\ & & \downarrow \\ \mathcal{H}\left(\overline{M}_2,\overline{D}_1\right) & \longrightarrow & \mathcal{H}\left(\overline{M}_1\cap\overline{M}_2,\overline{D}_1\right), \end{array}$$

and let $H_3$ be the homotopy theoretical fiber product of

$$\begin{array}{ccc} & & \operatorname{Map}\left(M_1,\overline{D}_\infty;Y,Y_a\right) \\ & & \downarrow \\ \operatorname{Map}\left(\overline{M}_2,\overline{D}_\infty;Y,Y_a\right) & \longrightarrow & \operatorname{Map}\left(\overline{M}_1\cap\overline{M}_2,\overline{D}_\infty;Y,Y_a\right). \end{array}$$

The inclusion $\operatorname{Map}(M,\overline{D}_\infty;Y,Y_a)\hookrightarrow H_3$ is a homotopy equivalence, and by proposition 7.2, the inclusion $\widehat{\mathcal{P}}(M,\overline{D}_\infty)\hookrightarrow H_1$ is a homology equivalence. All in all we have

(7.3) THEOREM. *In the commutative diagram*

$$\begin{array}{ccccc} \widehat{\mathcal{P}}\left(M,\overline{D}_\infty\right) & \longleftarrow & \mathcal{H}\left(M,\overline{D}_1\right) & \longrightarrow & \operatorname{Map}\left(M,\overline{D}_\infty;Y,Y_a\right) \\ \downarrow & & \downarrow & & \downarrow \\ H_1 & \longleftarrow & H_2 & \longrightarrow & H_3, \end{array}$$

*the bottom horizontal maps are weak homotopy equivalences, the left-hand vertical map is a homology equivalence and the right-hand vertical map is a homotopy equivalence.*

PROOF: We only have to show that

$$\mathcal{H}(\overline{M}_1,\overline{D}_1)\longrightarrow\widehat{\mathcal{P}}(M,\overline{D}_\infty)\qquad\text{and}\qquad\mathcal{H}(\overline{M}_1,\overline{D}_1)\longrightarrow\operatorname{Map}(M,\overline{D}_\infty;Y,Y_a)$$

are equivalences, but this is trivial, as all three spaces are contractible. ∎

The same conclusion holds, if the complex structure is fixed, but before we can show that, some terminology is needed.

Imbed $M$ in $\mathbf{R}^3$, and choose a tubular neighbourhood $U$ of $M$. The imbedding and $U$ can be chosen, such that any subset of $M$ with diameter less than 10, is contained in a disk in $M$ and has it's convex hull contained in $U$. Let $\alpha_1, \dots, \alpha_n \in M$ be points with weights $\nu_1, \dots, \nu_n$. If $\operatorname{diam}(\{\alpha_1, \dots, \alpha_n\}) \leq 10$, then the ordinary center of mass lies in $U$ and can be projected down to a point on $M$, which we will call the *center of mass*, and which depends continuously on the configuration $(\alpha_1^{\nu_1}, \dots, \alpha_n^{\nu_n})$ of points in $M$.

Choose a point $x_\infty \in M$, and put $M' = M \setminus \{x_\infty\}$. Blow the metric up at $x_\infty$, such that any subset of $M'$ with diameter less than 10 is contained in a disk in $M'$, and any configuration of points in $M \setminus \{x\}$ with diameter less than 10 has a well defined center of mass.

Let $r \in \mathbf{R}_+$. A configuration of points $\xi \in \mathcal{A}_{\leq n}(M')$ is called *r-small*, if

$$\operatorname{diam}(\xi) \leq r \cdot 4^{\deg \xi - n}.$$

(7.4) LEMMA. *If $\xi_1$ and $\xi_2$ are r-small and $\xi_1 \cap \xi_2 \neq \emptyset$, then $\xi_1 \cup \xi_2$ is r-small.*

PROOF: If $\xi_1 \subseteq \xi_2$ or $\xi_2 \subseteq \xi_1$ there is nothing to show, so we may assume that $\deg(\xi_1 \cup \xi_2) \geq \max\{\deg \xi_1, \deg \xi_2\} + 1$. Then

$$\begin{aligned} \operatorname{diam}(\xi_1 \cup \xi_2) &\leq \operatorname{diam} \xi_1 + \operatorname{diam} \xi_2 \leq r \cdot 4^{\deg \xi_1 - n} + r \cdot 4^{\deg \xi_2 - n} \\ &\leq 2r \cdot 4^{\max\{\deg \xi_1, \deg \xi_2\} - n} \leq r \cdot 4^{\deg(\xi_1 \cup \xi_2) - n}, \end{aligned}$$

i.e. $\xi_1 \cup \xi_2$ is $r$-small. ∎

Two configurations $\xi_1$ and $\xi_2$ are called *r-independent*, if any $r$-small subconfiguration of $\xi_1 \cup \xi_2$ is contained in either $\xi_1$ or $\xi_2$.

(7.5) LEMMA. *If $\xi$ is not r-small, then we can write $\xi = \xi_1 \cup \xi_2$ with $\xi_1 \cap \xi_2 = \emptyset$ and $\xi_1, \xi_2 \neq \emptyset$, such that any proper 2r-small subconfiguration is contained in either $\xi_1$ or $\xi_2$.*

REMARK. *Then the configurations $\xi_1$ and $\xi_2$ are r-independent, but they need not be 2r-independent, because $\xi$ may be 2r-small.*

PROOF: Choose $x, y \in \xi$, such that $\operatorname{dist}(x, y) = \operatorname{diam} \xi \geq r^{\deg \xi + n}$. Let $\xi_1$ be a maximal $2r$-small proper subconfiguration of $\xi$ containing $x$, and let $\xi_2 = \xi \setminus \xi_1$. Then $\operatorname{diam} \xi_1 < 2r \cdot 4^{\deg \xi - 1 - n}$, and hence

$$\operatorname{dist}(y, \xi) \geq \operatorname{dist}(x, y) - \operatorname{diam} \xi_1 > r \cdot 4^{\deg \xi + n} - 2r \cdot 4^{\deg \xi - 1 - n} = 2r \cdot 4^{\deg \xi - 1 - n}.$$

Assume $\xi' \subseteq \xi$ is $2r$-small, $\xi' \cap \xi_1 \neq \emptyset$ and $\xi' \cap \xi_2 \neq \emptyset$. We shall show that $\xi' = \xi$. As $\xi' \cap \xi_1 \neq \emptyset$, lemma 7.4 implies that $\xi' \cup \xi_1$ is $2r$-small, and as $\xi_1$ is maximal, we must have $\xi' \cup \xi_1 = \xi$. Especially $y \in \xi'$, and hence

$$\operatorname{diam} \xi' \geq \operatorname{dist}(y, \xi_1) > 2r \cdot 4^{\deg \xi - 1 - n}.$$

As $\xi'$ is $2r$-small, we have $\deg \xi' > \deg \xi - 1$ and thus $\xi' = \xi$. ∎

We can now show

(7.6) LEMMA. *Let $M$ be a closed surface with base point $x_\infty$ and let $J$ be any complex structure on $M$. Then the inclusion $\mathcal{P}_J(M,\{x_\infty\}) \hookrightarrow \mathcal{P}(M,\{x_\infty\})$ is a homotopy equivalence.*

PROOF: It is clearly enough to show that the inclusion

$$\mathcal{P}_{J,n}(M,\{x_\infty\}) \hookrightarrow \mathcal{P}_n(M,\{x_\infty\})$$

is a homotopy equivalence for all $n$.

We want to define a map $\mathcal{P}_{\leq n}(M,\{x_\infty\}) \times \mathcal{C}(M) \to \mathcal{P}_{\leq n}(M,\{x_\infty\})$ of the form $(\xi, J, J') \mapsto (\psi(\xi, J, J'), J')$, which preserves degree and satisfies

(1) $\psi(\xi, J, J) = \xi$ and
(2) $\psi(\xi_1 \cup \xi_2, J, J') = \psi(\xi_1, J, J') \cup \psi(\xi_2, J, J')$ if pole$(\xi_1)$ and pole$(\xi_2)$ are 2-independent, considered as elements of $\mathcal{A}_{\leq n}(M')$.

The map $\psi$ turns $J$-principal parts into $J'$-principal parts. We define $\psi$ inductively, but first we choose a vector field $v$ on $M$, which only vanishes at $x_\infty$.

If $(\xi, J, J') \in \mathcal{P}_1(M,\{x_\infty\}) \times \mathcal{C}(M)$ and $\alpha \in M'$, then we let $D_\alpha$ be the disk in $M$ with center $\alpha$ and radius one. Let $\phi_{J',J}\colon D_{\alpha J'} \to D_{\alpha J}$ be the unique holomorphic homeomorphism such that $\phi_{J',J}(\alpha) = \alpha$ and $d\phi_{J',J}(v(\alpha)) = c \cdot v(\alpha)$ with $c > 0$. Define $\psi$ by $\psi(\xi, J, J') = \xi \circ \phi$. As $\phi$ depends continuously on $\alpha$, $J$ and $J'$, the map $\psi$ depends continuously on $(\xi, J)$ and $J'$. Condition 2 is empty in this case, and as $\phi = id$, if $J = J'$, condition 1 is satisfied.

Assume that $\psi$ is defined on $\mathcal{P}_{\leq(k-1)}(M,\{x_\infty\}) \times \mathcal{C}(M)$ with $2 \leq k \leq n$, and put

$$\begin{aligned}\widetilde{\mathcal{P}} &= \{(\xi, J) \in \mathcal{P}_{\leq k}(M,\{x_\infty\}) \mid \text{pole}(\xi) \text{ is 1-small} \implies \deg\xi \leq k-1\} \\ &= \mathcal{P}_{\leq(k-1)}(M,\{x_\infty\}) \cup \{(\xi, J) \in \mathcal{P}_k(M,\{x_\infty\}) \mid \operatorname{diam}\text{pole}(\xi) > 4^{k-n}\}.\end{aligned}$$

If $\deg\xi = k$, and $\operatorname{diam}\text{pole}(\xi) > 4^{k-n}$, then we write $\xi = \xi_1 \cup \xi_2$ according to lemma 7.5. Define $\widetilde{\psi}$ on $\widetilde{\mathcal{P}} \times \mathcal{C}(M)$ by

$$\widetilde{\psi}(\xi, J, J') = \begin{cases} \psi(\xi, J, J'), & \text{if } \deg\xi \leq k-1, \\ \psi(\xi_1, J, J') \cup \psi(\xi_2, J, J'), & \text{if } \deg\xi = k. \end{cases}$$

As $\psi$ satisfies condition 2, $\widetilde{\psi}$ is well-defined, and clearly $\widetilde{\psi}$ is continuous and satisfies condition 1 and 2.

We now let

$$\begin{aligned}\overline{\mathcal{P}} &= \{(\xi, J) \in \mathcal{P}_{\leq k}(M,\{x_\infty\}) \mid \text{pole}(\xi) \text{ is 2-small} \implies \deg\xi \leq k-1\} \\ &= \mathcal{P}_{\leq(k-1)}(M,\{x_\infty\}) \cup \{(\xi, J) \in \mathcal{P}_k(M,\{x_\infty\}) \mid \operatorname{diam}\text{pole}(\xi) > 2\cdot 4^{k-n}\} \subseteq \widetilde{\mathcal{P}}\end{aligned}$$

If $\deg\xi = k$, and $\operatorname{diam}\text{pole}(\xi) \leq 2\cdot 4^{k-n}$, i.e. if $\xi \notin \overline{\mathcal{P}}$, then we let $\alpha$ be the center of mass of pole$(\xi)$ and put $D_\alpha = \{x \in M \mid \text{dist}(x, \alpha) < 5\}$. As $D_\alpha$ is a disk in $M$ containing pole$(\xi)$, we can define $\phi_{J',J}\colon D_{\alpha J'} \to D_{\alpha J}$ as above. Choose a homotopy $H\colon \mathcal{C}(M) \times \mathcal{C}(M) \times [0,1] \to \mathcal{C}(M)$, such that

(1) $H(J, J', 0) = J$ for all $J$ and $J'$,
(2) $H(J, J', 1) = J'$ for all $J$ and $J'$ and
(3) $H(J, J, t) = J$ for all $J$ and $t$.

Put $t(\xi) = 4^{n-k} \cdot \operatorname{diam}\operatorname{pole}(\xi) - 1$ and define $\psi$ on $\mathcal{P}_{\leq k}(M, \{x_\infty\})$ by

$$\psi(\xi, J, J') = \begin{cases} \widetilde{\psi}(\xi, J, J'), & \text{if } (\xi, J) \in \overline{\mathcal{P}}, \\ \widetilde{\psi}\left(\xi, J, H\left(J, J', t(\xi)\right)\right) \circ \phi_{J', H(J, J', t(\xi))}, & \text{if } \deg \xi = k \text{ and } \\ & 0 \leq t(\xi) \leq 1, \\ \xi \circ \phi_{J', J}, & \text{if } \deg \xi = k \text{ and } \\ & t(\xi) \leq 0. \end{cases}$$

It is easily checked that $\psi$ is well-defined, continuous and satisfies condition 1 and condition 2.

We define

$$\theta \colon \mathcal{P}_n\left(M, \{x_\infty\}\right) \longrightarrow \mathcal{P}_{J'}\left(M, \{x_\infty\}\right) \qquad \text{by} \qquad \theta(\xi, J) = \psi(\xi, J, J').$$

If

$$i \colon \mathcal{P}_{J', n}\left(M, \{x_\infty\}\right) \hookrightarrow \mathcal{P}_n\left(M, \{x_\infty\}\right)$$

is the inclusion $\xi \mapsto (\xi, J')$, then

$$\theta \circ i(\xi) = \psi(\xi, J', J') = \xi$$

and

$$i \circ \theta(\xi, J) = \left(\psi(\xi, J, J'), J'\right).$$

The first composition is the identity and the last composition is homotopic to the identity by the homotopy $(\xi, J, t) \mapsto (\psi(\xi, J, H(J, J', t), H(J, J', t))$, i.e. $\theta$ is a homotopy inverse to $i$. ∎

Put $\mathcal{P}^*(M) = \mathcal{P}(M, \{x_\infty\})$ and $\mathcal{P}^*_J(M) = \mathcal{P}_J(M, \{x_\infty\})$. Let $D'$ be any disk in $M$ containing $x_\infty$. By choosing a vector field, which pushes principal parts away from $x_\infty$, we see that the inclusion $\mathcal{P}(M, \overline{D'}) \hookrightarrow \mathcal{P}^*(M)$ is a homotopy equivalence. We put

$$\widehat{\mathcal{P}}_0\left(M, \overline{D}_\infty\right) = \operatorname{Tel}\left(\mathcal{P}_0(M, \overline{D}_1) \longrightarrow \mathcal{P}_0(M, \overline{D}_2) \longrightarrow \ldots\right) \subseteq \widehat{\mathcal{P}}\left(M, \overline{D}_\infty\right).$$

If $x_\infty \in D_\infty$, then by the remarks above and lemma 7.6 :

$$\begin{aligned} H_*\left(\widehat{\mathcal{P}}_0\left(M, \overline{D}_\infty\right)\right) &= \lim_{n \to \infty} H_*\left(\mathcal{P}_n\left(M, \overline{D}_{n+1}\right)\right) = \lim_{n \to \infty} H_*\left(\mathcal{P}^*_n(M)\right) \\ &= \lim_{n \to \infty} H_*\left(\mathcal{P}^*_{J,n}(M)\right), \end{aligned}$$

for all $J \in \mathcal{C}(M)$. Similarly we let

$$\operatorname{Map}^*_0(M, Y) = \left\{f \colon M \to Y \mid f(x_\infty) = 1 \in N \cong Y_a \text{ and } \deg_1 f = \cdots = \deg_k f = 0\right\}$$

and

$$\mathrm{Map}_0\left(M,\overline{D}_\infty;Y,Y_a\right) = \\ \left\{f\colon M\to Y \mid f(\overline{D}_\infty)\subseteq Y_a \text{ and } \deg_1 f=\cdots=\deg_k f=0\right\}.$$

As $\mathrm{Map}_0^*(M,Y)$ is homotopy equivalent to $\mathrm{Map}_0(M,\overline{D}_\infty;Y,Y_a)$,

$$H_*\left(\mathrm{Map}_0^*(M,Y)\right) = H_*\left(\widehat{\mathcal{P}}(M,\overline{D}_\infty)\right) = \lim_{n\to\infty} H_*\left(\mathcal{P}_{J,n}^*(M)\right),$$

for all $J\in\mathcal{C}(M)$.

Fix the complex structure $J\in\mathcal{C}(M)$, let $X$ denote the Riemann surface $M_J$ and put $\mathcal{P}_n^*(X)=\mathcal{P}_{J,n}^*(M)$. If $G$ is a compact Lie group, then

$\mathcal{M}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})$ = the space of based isomorphism classes of holomorphic $G_{\mathbf{C}}$-bundles over $X\times\mathbf{CP}^1$, trivial over $X\vee\mathbf{CP}^1$, based at $(x_\infty,\infty)$ and with characteristic class $n$,

see [**1**].

(7.7) PROPOSITION. *If* $Y=\Omega G$, *then*

$$\mathcal{P}_n^*(X)=\mathcal{M}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}}).$$

PROOF: If $X'=X\setminus\{x_\infty\}$, then a configuration of principal parts in $X$ without a pole at $x_\infty$ can be represented by a holomorphic map $f\colon X'\to\Omega G$, which by Proposition 4.5 is the same as an isomorphism class of a pair $(P',\tau)$, where $P'$ is a holomorphic $G_{\mathbf{C}}$-bundle on $X'\times\mathbf{CP}^1$, and $\tau$ is a trivialization of $P'$ over $X'\times\overline{D}_\infty$. The different choices of $f$ correspond to different trivializations $\tau$, but they all agree on $X'\times\{\infty\}$, i.e. a configuration of principal parts gives a pair $(P',\tau')$, where $\tau'$ is a trivialization of $P'$ over $X'\times\{\infty\}$. We can find a neighbourhood $U$ of $x_\infty$, such that $P'$ is trivial over $(X'\cap U)\times\mathbf{CP}^1$ and $\tau'$ determines the trivialization uniquely. By gluing $P'$ to the trivial bundle over $U\times\mathbf{CP}^1$, we get a bundle $P$ over $X\times\mathbf{CP}^1$, and $\tau'$ extends uniquely to a trivialization over $X\vee\mathbf{CP}^1$. Thus we obtain an element of $\mathcal{M}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})$.

Assume we on the other hand have a bundle $P$ over $X\times\mathbf{CP}^1$, which is trivial over $X\vee\mathbf{CP}^1$. Then the restriction to $X'\times\overline{D}_\infty$ is trivial, and by extending the trivialization over $X'\times\{\infty\}$ to $X'\times\overline{D}_\infty$, the transition functions to sets of the form $U\times\overline{D}$, give us a holomorphic map $f\colon X'\to\Omega G$. Different choices of the trivialization correspond to a multiplication of $f$ with a map $g\colon X'\to L_1^-G_{\mathbf{C}}$, i.e. we get a well-defined configuration of principal parts in $X'$ and hence an element of $\mathcal{P}_n^*(X)$. ∎

From [**1**] we have that $\mathcal{M}_n(\mathbf{CP}^1\times\mathbf{CP}^1, \mathbf{CP}^1\vee\mathbf{CP}^1, G_{\mathbf{C}})$ and $\mathrm{Hol}_n^*(\mathbf{CP}^1,\Omega G)$ are diffeomorphic, and by the remark following lemma 3.1, $\mathcal{P}_n^*(\mathbf{CP}^1)=\mathrm{Hol}_n^*(\mathbf{CP}^1,Y)$, if $Y$ is a generalized flag manifold. All in all we have

(7.8) THEOREM. *Let $X$ be Riemann surface and $Y$ a generalized flag manifold or a loop group. Then*

$$H_*\left(\mathrm{Map}_0^*(\mathbf{CP}^1, Y)\right) = \lim_{n\to\infty} H_*\left(\mathrm{Hol}_n^*(\mathbf{CP}^1, Y)\right),$$

*and if $Y = \Omega G$, then*

$$H_*\left(\mathrm{Map}_0^*(X, \Omega G)\right) = \lim_{n\to\infty} H_*\left(\mathcal{M}_n(X \times \mathbf{CP}^1, X \vee \mathbf{CP}^1, G_{\mathbf{C}})\right).$$

The connected components of $\mathrm{Map}^*(\mathbf{CP}^1, Y)$ are the spaces $\mathrm{Map}_{\mathbf{k}}^*(\mathbf{CP}^1, Y)$ of based maps $\mathbf{CP}^1 \to Y$ with multidegree $\mathbf{k} \in \mathbf{Z}^r$. By lemma 6.18 and 6.19, the connected components of $\mathrm{Hol}^*(\mathbf{CP}^1, Y) \cong \mathcal{P}^*(\mathbf{CP}^1)$ are the spaces

$$\mathrm{Hol}_{\mathbf{k}}^*\left(\mathbf{CP}^1, Y\right) = \mathrm{Hol}^*\left(\mathbf{CP}^1, Y\right) \cap \mathrm{Map}_{\mathbf{k}}^*\left(\mathbf{CP}^1, Y\right),$$

with $\mathbf{k} = (k_1, \ldots, k_r)$ and $k_i \geq 0$ for $i = 1, \ldots, r$. Hence we have

(7.9) THEOREM. *If $Y$ is a generalized flag manifold or a loop group, then the inclusion* $\mathrm{Hol}^*(\mathbf{CP}^1, Y) \hookrightarrow \mathrm{Map}^*(\mathbf{CP}^1, Y)$ *induces an injection*

$$\pi_0\left(\mathrm{Hol}^*(\mathbf{CP}^1, Y)\right) \hookrightarrow \pi_0\left(\mathrm{Map}^*(\mathbf{CP}^1, Y)\right).$$

## Appendix

If $B$ is a space, then $F(B) = \{\text{maps: } [0,1] \to B\}$ denotes the space of paths in $B$. There is a standard way of replacing a map $f: X \to B$ with a fibration, see [20]. If $I^f = \{(x, \gamma) \in X \times F(B) \mid f(x) = \gamma(0)\}$ and $p^f(x, \gamma) = \gamma(1)$, then $p^f: I^f \to B$ is a fibration, and the inclusion $X \hookrightarrow I^f$, given by $x \mapsto (X, \text{constant path } f(x))$, is a homotopy equivalence. The fiber of $p^f$ is denoted $T^f$.

If $g: Y \to B$ is another map, then $I_g^f$ denotes the pullback of $p^f: I^f \to B$ over $g$, i.e.

$$\begin{aligned} I_g^f &= \left\{(y, (x, \gamma)) \in Y \times I^f \mid g(y) = p^f(x, \gamma)\right\} \\ &= \left\{(y, x, \gamma) \in Y \times X \times F(B) \mid g(y) = \gamma(1) \text{ and } f(x) = \gamma(0)\right\}. \end{aligned}$$

We see that $I_g^f$ is symmetrical in $f$ and $g$, i.e. $I_g^f = I_f^g$. The space $I_g^f$ is called the *homotopy theoretical fiber product* of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\ g\ } & B. \end{array}$$

A commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{g} & B & \xleftarrow{f} & X \\ {\scriptstyle \phi_Y}\downarrow & & {\scriptstyle \phi_B}\downarrow & & {\scriptstyle \phi_X}\downarrow \\ Y' & \xrightarrow{g'} & B' & \xleftarrow{f'} & X' \end{array}$$

induces a map $\phi\colon I^f_g \to I^{f'}_{g'}$, given by $(y,x,\gamma) \mapsto (\phi_Y(y), \phi_X(x), \phi_B \circ \gamma)$. The homotopy theoretical fiber product is weak homotopy invariant in the following sense :

LEMMA 1. *If $\phi_X$, $\phi_Y$ and $\phi_B$ in the diagram above are weak homotopy equivalences, then $\phi$ is a weak homotopy equivalence.*

PROOF: From the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & I^f \\ {\scriptstyle \phi_X}\downarrow\wr & & \downarrow \\ X' & \xrightarrow{\sim} & I^{f'} \end{array}$$

it is seen that the map $I^f \to I^{f'}$ is a weak homotopy equivalence. So from the diagram

$$\begin{array}{ccccc} T^f & \longrightarrow & I^f & \longrightarrow & B \\ \downarrow & & \downarrow\wr & & {\scriptstyle \phi_B}\downarrow\wr \\ T^{f'} & \longrightarrow & I^{f'} & \longrightarrow & B', \end{array}$$

the long exact homotopy sequence and the five lemma, the map $T^f \to T^{f'}$ is a weak homotopy equivalence. Finally, the diagram

$$\begin{array}{ccccc} T^f & \longrightarrow & I^f_g & \longrightarrow & Y \\ \downarrow\wr & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \phi_Y}\downarrow\wr \\ T^{f'} & \longrightarrow & I^{f'}_{g'} & \longrightarrow & Y', \end{array}$$

the long exact homotopy sequence and the five lemma imply that $\phi\colon I^f_g \to I^{f'}_{g'}$ is a weak homotopy equivalence. ∎

The *fiber product* of the diagram

$$\begin{array}{ccc} & & X \\ & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{g} & B \end{array}$$

is the pullback $g^*(X)$ of $X$ over $g$, i.e. $g^*(X) = \{(y,x) \in Y \times X \mid f(x) = g(y)\}$. There is an inclusion $g^*(X) \hookrightarrow I_g^f$ given by $(y,x) \mapsto (y, x, \text{constant path } g(y))$.

Any commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & B \end{array}$$

induces a map $\phi\colon Z \to g^*(X)$ given by $z \mapsto (\tilde{f}(z), \tilde{g}(z))$ and the diagram is called *cartesian*, if the map $\phi$ is a homeomorphism. We can compose $\phi$ with the inclusion $g^*(X) \hookrightarrow I_g^f$ above, and get a map $Z \to I_g^f$. The diagram is called *homotopy cartesian*, if this composition is a homotopy equivalence. It is called *weak homotopy cartesian*, if the map is a weak homotopy equivalence, and it is called *homology cartesian*, if the map is a homology equivalence.

A map $f\colon X \to B$ is called a *quasifibration*, if

$$f_*\colon \pi_*\left(X, f^{-1}(f(x)), x\right) \longrightarrow \pi_*(B, f(x))$$

is an isomorphism for all $x \in X$. Then, just as in the case of fibrations, there is a long exact homotopy sequence

$$\ldots \to \pi_{i+1}(B, f(x)) \to \pi_i(f^{-1}(f(x)), x) \to \pi_i(X, x) \to \pi_i(B, f(x)) \to \ldots,$$

see [**2**]. From this we have

LEMMA 2. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & B \end{array}$$

*be a cartesian diagram. If $f$ and $\tilde{f}$ are quasifibrations, then the diagram is weak homotopy cartesian.*

PROOF: Let $F$ be a fiber of $f\colon X \to B$, and consider the diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \wr} & & \| \\ T^f & \longrightarrow & I^f & \longrightarrow & B. \end{array}$$

The long exact homotopy sequence and the five lemma give that $F \to T^f$ is a weak homotopy equivalence. As the diagram is cartesian, the fibers of $Z \to Y$ and $X \to B$ are homeomorphic, hence we have the diagram

$$\begin{array}{ccccc} F & \longrightarrow & Z & \longrightarrow & Y \\ \downarrow{\scriptstyle \wr} & & \downarrow & & \| \\ T^f & \longrightarrow & I_g^f & \longrightarrow & Y, \end{array}$$

and again the long exact homotopy sequence and the five lemma imply that $Z \to I^f_g$ is a weak homotopy equivalence. ∎

As in [**11**], a map $f\colon X \to B$ is called a *homology equivalence*, if each $b \in B$ has arbitrarily small contractible neighbourhoods $U$, such that the inclusion $f^{-1}(b') \hookrightarrow f^{-1}(U)$ is a homology equivalence for all $b' \in U$.

LEMMA 3. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & B \end{array}$$

*be a cartesian diagram. If $f$ and $\tilde{f}$ are homology fibrations, then the diagram is homology cartesian.*

PROOF: Let $b \in B$ and put $F = f^{-1}(b)$. Choose a contractible neighbourhood $U$ of $b$, such that the inclusion $F \hookrightarrow f^{-1}(U)$ is a homology equivalence. Consider the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\sim} & f^{-1}(U) & \longrightarrow & U \\ \downarrow & & \downarrow\wr & & \| \\ T^f & \xrightarrow{\sim} & p^{f-1}(U) & \longrightarrow & U. \end{array}$$

We see that $F \to T^f$ is a homology equivalence. Now consider the diagram

$$\begin{array}{ccccc} F & \longrightarrow & Z & \longrightarrow & Y \\ \downarrow\wr & & \downarrow & & \| \\ T^f & \longrightarrow & I^f_g & \longrightarrow & Y. \end{array}$$

Just as a fibration, a homology fibration gives a convergent spectral sequence. Thus we have

$$\begin{array}{ccc} E^2_{p,q} = H_p(Y, H_q(F)) & \Longrightarrow & H_*(Z) \\ \downarrow\wr & & \downarrow \\ E'^2_{p,q} = H_p(Y, H_q(T^f)) & \Longrightarrow & H_*(I^f_g), \end{array}$$

and hence $H_*(Z) \to H_*(I^f_g)$ is an isomorphism. ∎

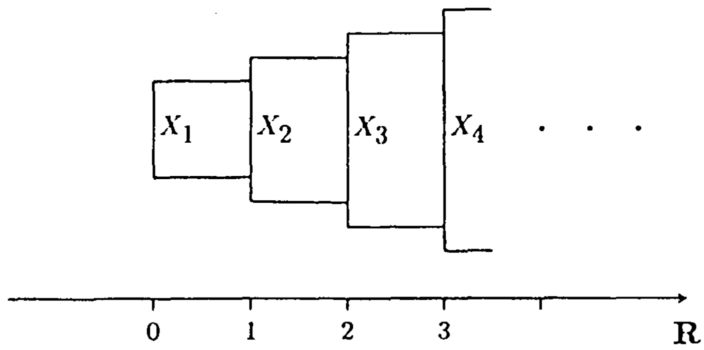
Suppose we have a sequence of maps

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \cdots.$$

The *telescope* of the sequence is the space

$$\hat{X} = \mathrm{Tel}\Big(X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \ldots\Big) = \bigsqcup_{n \in \mathbf{N}} X_n \times [n-1, n] /\!\sim,$$

where $(x, n) \sim (\phi_n(x), n)$. If we regard $X_n$ as a subset of $X_{n+1}$, by the map $\phi_n$, then we have the following picture :




It looks like an infinite telescope, and as it is possible to slide a finite part of the telescope together, we have

LEMMA 4. *The inclusions $i_n\colon X_n \hookrightarrow \widehat{X}\colon x \mapsto (x, n-1)$ induces an isomorphism*

$$i_*\colon \lim_{n\to\infty} H_*(X_n) \xrightarrow{\simeq} H_*(\widehat{X}),$$

*where $\lim_{n\to\infty} H_*(X_n)$ is the direct limit of*

$$H_*(X_1) \xrightarrow{\phi_{1*}} H_*(X_2) \xrightarrow{\phi_{2*}} H_*(X_3) \xrightarrow{\phi_{3*}} \ldots.$$

Assume that the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\phi_n} & X_{n+1} \\ {\scriptstyle\phi_n}\downarrow & & \downarrow{\scriptstyle\phi_{n+1}} \\ X_{n+1} & \xrightarrow{\phi_{n+1}} & X_{n+2} \end{array}$$

is homotopy commutative for all $n \in \mathrm{N}$. Then we get a commutative diagram

$$\begin{array}{ccccccc} H_*(X_1) & \xrightarrow{\phi_{1*}} & H_*(X_2) & \xrightarrow{\phi_{2*}} & H_*(X_3) & \xrightarrow{\phi_{3*}} & \ldots \\ \downarrow{\scriptstyle\phi_{1*}} & & \downarrow{\scriptstyle\phi_{2*}} & & \downarrow{\scriptstyle\phi_{3*}} & & \\ H_*(X_2) & \xrightarrow{\phi_{2*}} & H_*(X_3) & \xrightarrow{\phi_{3*}} & H_*(X_4) & \xrightarrow{\phi_{4*}} & \ldots, \end{array}$$

which induces an isomorphism $\phi_*\colon \lim_{n\to\infty} H_*(X_n) \xrightarrow{\simeq} \lim_{n\to\infty} H_*(X_n)$.

LEMMA 5. *Let $\widehat{X}$ be the telescope of a sequence as above and let $\widehat{X}'$ be the telescope of the sequence $X_1' \xrightarrow{\phi_1'} X_2' \xrightarrow{\phi_2'} X_3' \xrightarrow{\phi_3'} \ldots$ . Assume we have homotopy equivalences $\psi_n\colon X_n' \to X_n$, such the diagram*

$$\begin{array}{ccc} X_n' & \xrightarrow{\psi_n} & X_n \\ {\scriptstyle\phi_n'}\downarrow & & \downarrow{\scriptstyle\phi_n} \\ X_{n+1}' & \xrightarrow{\psi_{n+1}} & X_{n+1} \end{array}$$

*commutes for all* $n \in \mathbf{N}$.

*If* $\widehat{\psi}\colon \widehat{X}' \to \widehat{X}$ *is a map, such that* $\widehat{\psi} \circ i'_n$ *and* $i_{n+1} \circ \phi_{n+1} \circ \psi_n\colon X'_n \to \widehat{X}$ *are homotopy equivalent for infinitely many* $n \in \mathbf{N}$, *then* $\widehat{\psi}$ *is a homology equivalence.*

PROOF: There is a commutative diagram

$$\begin{array}{ccccc} H_*(X'_n) & \xrightarrow[\sim]{\psi_{n*}} & H_*(X_n) & \xrightarrow{\phi_{(n+1)*}} & H_*(X_{n+1}) \\ {\scriptstyle i'_{n*}}\downarrow & & & & \downarrow{\scriptstyle i_{(n+1)*}} \\ H_*(\widehat{X}') & & \xrightarrow{\qquad\widehat{\psi}_*\qquad} & & H_*(\widehat{X}) \end{array}$$

for infinitely many $n \in \mathbf{N}$, and by taking the direct limit, we get the commutative diagram

$$\begin{array}{ccccc} \lim\limits_{n\to\infty} H_*(X'_n) & \xrightarrow[\sim]{\psi_*} & \lim\limits_{n\to\infty} H_*(X_n) & \xrightarrow[\sim]{\phi_*} & \lim\limits_{n\to\infty} H_*(X_{n+1}) \\ {\scriptstyle i'_*}\downarrow\wr & & & & {\scriptstyle i_*}\downarrow\wr \\ H_*(\widehat{X}') & & \xrightarrow{\qquad\widehat{\psi}_*\qquad} & & H_*(\widehat{X}), \end{array}$$

and we see that $\widehat{\psi}$ is a homology equivalence. ∎

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