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APPLICATIONS AND MODELLING

in the mathematics curriculum

- State and trends -

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ABSTRACT:

The paper presents an over-all survey of the post-war development and of the present states and trends in the role of applications and modelling in mathematics curricula of all levels. Characteristic phases in the development of elementary and post-elementary curricula respectively are identified. The present situation is described as a period of motion, investigation and experimentation, with the following predominant features: (a) Applications and modelling is being secured a footing in more and more curricula, but a considerable pluralism exists. There are differences between countries with central and countries with decentral curriculum authorities; (b) The distance between "the front" and "the main-stream" of applications and modelling in mathematics curricula is long. The paper finishes by addressing a few key problems: There is a need for a more balanced view of applications and modelling than is usually seen. What are the barriers for applications and modelling to obtain a reasonable position in curricula, and how may these barriers be overcome?

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The paper presents an over-all survey of the post-war development and of the present states and trends in the role of applications and modelling in mathematics curricula of all levels. Characteristic phases in the development of elementary and post-elementary curricula respectively are identified. The present situation is described as a period of motion, investigation and experimentation, with the following predominant features: (a) Applications and modelling is being secured a footing in more and more curricula, but a considerable pluralism exists. There are differences between countries with central and countries with decentral curriculum authorities; (b) The distance between "the front" and "the main-stream" of applications and modelling in mathematics curricula is long. The paper finishes by addressing a few key problems: There is a need for a more balanced view of applications and modelling than is usually seen. What are the barriers for applications and modelling to obtain a reasonable position in curricula, and how may these barriers be overcome?

I. INTRODUCTION

During the last decade the field "Applications and Modelling" has attracted growing attention as an actual and potential part of mathematics curricula at various levels. Attention has grown so strong that "Applications and Modelling" was devoted a section of its own at ICME IV in Berkeley, and again at ICME V in Adelaide. The aim of this paper is to survey the state of affairs and identify present trends in the field. A condition for assessing what is going on and for adopting realistic general strategies and specific measures to influence the development is the presence of a reasonably reliable account of what <u>is</u> going on. To give such an account is not a question of listing a series of facts. It is rather a question of determining which facts are considered important and of interpreting them into a coherent picture. Since selection and interpretation involve values and judgement, tied to national, social and cultural prerequisites, no survey can be indisputable. Therefore, I invite the reader to keep in mind that what I shall be saying in the sequel is filtered through my glasses and marked by my background. This is not an excuse for flaws - indeed, I

The present paper is the slighty revised manuscript of a survey lecture given by the author at ICME V (Adelaide) in a summary session of the Theme Group 6, "Applications and Modelling", for which the author was one of the Chief Organisers.

stand by my exposition - but the reader may very well have different emphases, priorities and views.

II. HISTORICAL RETROSPECT

A. The pre-war period

The notion that school mathematics should take interest in the applicational aspects of mathematics is not at all new, even if the background and shape of its present version is.

During the first two or three decades of this century a utilitarian movement influenced curriculum debates, and to some extent curricula, in most countries in Western Europe and the USA. In opposition to the traditional one-sided emphasis on the formative and aesthetic values of mathematics, the utilitarian movement insisted on the usefulness of mathematics - and of mathematics education - to science and society as its ultimate raison d'etre. The impact of this movement on actual mathematics instruction could be found primarily in the context of arithmetic and basic computational geometry, as those topics were brought into play towards more or less schematized daily-life problems of economy, commerce, trades and crafts, involving counting, computing with the four operations and percents, weighing and measuring etc. We shall call this kind of mathematics and the applications of it "elementary". This term does not refer, in the first place, to a certain organisational segment, labelled "elementary" or "primary" of any particular school system. Elementary mathematics in this sense is usually taught in both primary and secondary school.

It is characteristic that "post-elementary" mathematics was far less influenced by the utilitarian movement. This was due notably to two reasons, the first one being sociological:

Post-elementary mathematics education was given only to a small minority of the youth, the "happy few" that were to form the intellectual élite of society, in particular the scientific and technical parts of it. For the latter group of young people, the post-elementary mathematics they met in school could be considered as a forecourt to the real, advanced mathematical edifice belonging to university level studies. For those bound to become engineers, architects, physicists, actuaries etc., applications of mathematics to areas of relevance would expectedly appear in due time and in due context in course of their studies. So, they had no substantial reason for impatience. In accordance with the general ideology in academic mathematics of that time, the negligible minority bound to become mathematicans presumably did not feel a strong need to be introduced to applications, except perhaps in physics. For the remaining members of the prospective intellectual élite, matematics had in general no applicational perspective. To the extent they stood their mathematics education through, it was because of its internal qualities or because of the common persuasion that mathematics was good for "the thinking muscle".

In any case, the absolute and relative smallness of the élite and its recruiting strata made possible applicational shortcomings of the post-elementary mathematics education a problem of limited importance.

The second reason regards substance matter:
The spectrum of applications of mathematics in other sciences and in practical affairs was much narrower in those days than today. In fact, only areas like physics, astronomy, geodesy, surveying, projection,engineering,insurance, and some economic disciplines took notable advantage of higher level mathematics. Topics related to a few of these areas were included in upper secondary and sometimes also university mathematics curricula, without ever in the period reaching a more than marginal position, though.

Besides being important to the "classical" areas just listed, mathematics became increasingly significant as a tool for frontline developments in biology and agricultural science, in certain social sciences etc., first of all through the then newly emerged discipline mathematical statistics. But it took half a century until material of this kind entered mathematics curricula at a larger scale.

Not much effort was invested by scientists and educationalists in developing a concrete applicational content of the post-elementary mathematics curricula. The dominating point of view was that students' capacity for applying mathematics and handling applications would result pretty directly from a decent, thorough education in mathematics proper.

In contradistinction, applicational aspects of elementary methematics - still in the sense suggested in the beginning of this section - was devoted some attention from educationalists. Firstly, the justification of such aspects was never seriously questioned. Secondly, genuine applications of elementary mathematics abounded, as they do today, in daily life. This made it natural to ask the question 'how should applications be placed in elementary mathematics education?' This is not the occasion to go in details with the different answers to this question given in different countries and quarters at different times - neither do I possess a sufficient background for giving such an exposition. It seems, however, fair to state that applications were taught essentially as in old Mesopotamia: strongly simplified and stereotyped problem situations were described very laconically, only information indispensable to making the embedded arithmetical or geometrical problems well-defined was given. Pupils were then asked to solve the problem. This should be done by selecting <u>the</u> right combination of relatively few standard routines. In many places this "problem solving" process became highly ritualized, with a minimal understanding of problem situations and their mathematical background as a general result. That development led a growing host of mathematicians, educationalist and educators to express - for instance in official reports from governmental or private commissions - their worries about elementary mathematics education. They advocated a greater concern for pupils' understanding of mathematical substance in the school curriculum. However, World War II put a temporary end to these endeavours.

<u>B. From World War II to the "new mathematics" reforms</u>

It is well known that the post-war development of mathematics education in industrialized countries was rooted in the amalgamation of the interests of two different worlds.

The world of government offices, planning agencies and organisations in society engaged itself in providing as rapidly as possible an educational and scientific foundation of accelerated technological and economic growth, and of the increased material and social welfare of the general population that was assumed to result from it. Everybody was convinced that an enhancement and innovation of science education, and above all mathematics education, was a key factor in the achievement of these goals. Mathematics qualifications which can be used flexibly to tackling a variety of known and not yet known problems should be given to an increasing proportion of the labour force and the population in general, including groups that never before were equipped with such qualifications.

The world of mathematicians and mathematics educationalists was even more anxious than before the war to have the mathematics curricula improved and modernized at all levels. Mathematics as a teaching subject should be brought in harmony with and closer to mathematics as a science as it had developed since the last fourth of the 19th century, with an increasing gap between school mathematics and modern higher level mathematics as one consequence.

The political and administrative world called for action and opened doors for innovation. The academic world had ripened conceptions that seemed to meet the societal requirements. So, the academic world was entrusted with the task of renewing mathematics education from school to university.

What came out of it, the "new mathematics", is well-known to everyone reading this paper. Therefore, I shall not go into that matter any further, except as regards one point which has remained largely unnoticed in the numerous critical writings and discussions that have flourished in the wake of the "new math" reform as it rolled across the world. This point is that despite the basically internal orientation of the reform, quite a few of the people involved in it had also implications for the external relations of mathematics in mind. In focussing on the inherent structural components of any situation containing mathematical elements, be it inside or outside the edifice of mathematics as such, they were convinced that mathematics would be a powerful tool for understanding and handling extra-mathematical problems, in that its fundamental structures provide a general conceptual framework that enables people in command of it to attack and analyse unstructured situations.

C. Problems in the post-reform period

No matter what the intentions with the "new mathematics" reform were, it did not take many years before the new curricula came in serious trouble and became intensely questioned from various quarters. It seems that the trouble was particularly manifest in countries were the reform was installed relatively early in the wave, and hence in a relatively "pure" form, whereas countries joining the wave at a later stage were less afflicted, probably because experiences in the "front countries" inspired to adopt modified versions of the reformed curricula.

In short, the trouble arose because it turned out that at large the new curricula did not function satisfactorily. This was the case all the way through, from primary school to university, even if the problems varied, of course, with the level.

In primary school there were two main problems. The first was that pupils' skills in doing simple arithmetic quickly and infallibly were reported widely to have diminished after the introduction of "new mathematics". Although this observation gave rise to "back to basics" movements in a number of countries, the problem was not in itself too serious - it could be remedied by additional training. Really serious was it that primary mathematics instruction did not provide pupils with a capability to utilize mathematics to obtain answers to questions about the world surrounding them. Neither did the old curricula, but it was contrary to intentions and beliefs that the new primary school curricula did not open doors for fruitful links between mathematics and the world around it. In stead, primary school mathematics tended to degenerate in many places to a series of formal, essentially abstract games (despite the extensive use of illustrations, colours, concrete toy-like materials etc.) of sorting, relating and combining objects according to rules which seemed odd to outsiders, especially parents. The games surely did occupy many children, but they remained by and large closed games.

In secondary school the situation was much the same, only translated to a higher level. Even if secondary school mathematics was not supposed or intended to be immediately associable to the world outside mathematics to the extent primary school mathematics was, the situation was problematic. Several interacting reasons were responsible for this. I shall confine myself to mentioning two of them:

(a) The modernization of the educational system, in particular as regards science and mathematics education, was accompanied by a simultaneous expansion of it. As a result, large and new youth groups entered further educational programmes and became exposed to post-elementary mathematics instruction. Only a minority of the young people partaking in this expansion were to proceed to educations and professions involving sophisticated mathematics. Thus, to a majority of students receiving post-elementary mathematics education, that education represented - and represents - a final or almost final stage as far as mathematics is concerned. They are, therefore, not ready to accept that the gains to be won from mathematics education blow in future winds, they want an immediate demonstration that mathematics is an enterprise worthwhile undertaking. This attitude got an amplification from the following factor:

(b) During the sixties and early seventies, a bunch of economic, cultural and political changes manifested themselves in several countries all over the world. One spectacular feature was the so-called student or youth revolt. Although the period was almost a singularity in newer history and is now ever, long-lasting ideological consequences flowed from it. One such consequence is that young people are no longer prepared to blindly rely on authorities or to follow directions, let alone orders, without asking critical questions. They demand arguments and good reasons to be motivated (convinced, stimulated, persuaded) to accept and adopt activities. So, an educational system which does not rely on sanctions cannot get away with using (openly) authoritarian means and styles of teaching.

On this background, receivers of secondary mathematics education began to require, actively or passively, relevance of content and form of their mathematics instruction.

Also in higher educations which use mathematics but do not lead to mathematical professions, students required the mathematical parts of their studies to be visibly relevant to the aim and content of their educational programmes. It is perhaps even more remarkable that many students engaged in mathematical studies proper began to question the sense in studying pure mathematics in isolation from other subjects and sciences and from the world in general. Beside being due to the general ideological current mentioned above, mathematics students' growing concern for the extra-mathematical world was hardly unrelated to the fact that the possibilities of entering university and other research careers in pure mathematics were beginning to dry out rapidly in most countries in the early seventies. Therefore, career perspectives for a growing portion of graduates and PhD's in mathematics were becoming associated with applied mathematics jobs or teaching jobs. And to both categories applications were relevant - as far as teaching is concerned because of the described development in school mathematics.

III. THE CURE: RELEVANCE, APPLICATIONS AND MODELLING

Altogether, at all levels of the educational system the problems in mathematics instruction gave rise to demands of relevance. And right from the beginning relevance was interpreted by students, teachers and educationalists as applicability. The range of applicability should not be taken too narrowly. It was never clearly defined, but ranged from general societal applicability, over applicability to specific sectors and professions in society, and to other (school) subjects, to applicability in students' daily lives.

Whether teachers, educationalists and curriculum planners were sincere, tactical or just pressed, applicational aspects of mathematics entered the curricular scenes in more and more countries. It was far from that this happened in any uniform way.

There was - as there still is - considerable diversity in how applicational aspects were interpreted and introduced in the different curricula. Therefore, in the attempt to identify main lines, ignoring differences implies a lot of simplification. They seem justified, however, if we can dig out trends from the large apparently amorphous collection of local phenomena (local in place, time, boundary conditions and current circumstances).

Our first observation is that the situation in elementary mathematics education - as previously defined - differs from that in the entire range of post-elementary mathematics education in such a way that they should be treated separately, whereas the similarities within the many post-elementary educational levels, as regards the issue of applications, are more predominant than the differences. On this background we shall allow ourselves to deal with the latter levels as a whole.

A. Elementary curricula: Three phases

We shall begin by considering the elementary mathematics curriculum.

As touched upon previously in this paper, elementary mathematics instruction never came to lose every applicational perspective. However, the nature and position of this perspective have been far from constant in time and space. After the "new mathematics" reform the applicational aspects of the elementary curriculum have run through a number of phases. Structural in nature, rather than strictly chronological, are these phases not easy to date. Besides, they occur at different times in different countries and with different strength.

The first phase

formed part of the "new mathematics" reform. From an educational point of view the main endeavour of that reform was to substitute mechanical and ritualized inculcation of stereotyped recipes with understanding of the mathematical concepts, relations and arguments on which procedures and recipes are based. The emphasis on understanding was not limited to internal matters but regarded the application of mathematics as well. What elementary mathematics offered of understanding in connection with extra-mathematical situations was insight in structures, primarily such structures which are formulable in basic, naïve set theory: set operations, relations including orderings, mappings, cardinality etc. In focussing, thus, on classificational matters rather than on understanding and solving specific problems, the application of elementary mathematics tended to concern fairly abstract and academic issues. The growing interest during that phase in adding games, abstract and concrete, to elementary mathematics instruction also can be seen as an expression of the same orientation.

Applications of arithmetic and computational geometry to the spheres of private and societal life were assigned low priority (even if never exterminated), and so was the drill of related computational and technical skills.

The second phase

had its background in the recognition which emerged in many quarters that if pupils are to be capable of activating mathematics to solve problems outside mathematics itself creatively, flexibly and confidently, it is necessary to teach it to them. This presupposes that it can be taught - and learnt. So, problem solving made up the pivot of that phase. In focus was the fostering of pupils' ability to tackle and solve extra-mathematical problems (which in time became increasingly open) by means of mathematics. The providing of students with problem solving attitudes and problem solving strategies attracted much attention. The problem solving process in all its facets were considered more crucial than its content in terms of objects and problem fields. Similarly, the particular product, a solution, of the problem solving process was of secondary importance in comparison to the mere striving to obtain one.

Several examples of carefully constructed, sometimes highly elaborate, problem universes - certainly extra-mathematical but also outside the realm of the "real world" - calling for a variety of problem solving approaches came into being.

As a consequence of the process orientation in the second phase, a general context-free, or at least context-indifferent, notion of problem solving developed.

All this went along with, and was matched by, another general line of development in theory and practice of mathematics education: A humanisation of mathematics instruction. Mathematics should be a friendly, tolerant, open, creative subject, devoted to the personal development of pupils and their social interaction. Mathematics began to be perceived as a framework for activities, not subject to rigorous limitations in method, and not only an edifice of knowledge. Pupils should be stimulated to acquire mathematical insight by experimenting, measuring, guessing, trying out, refuting and much more, and to acquire general attitudes, modes of thinking and of working.

There are, however, pedagogical difficulties in a general context-indifferent problem solving approach. One is that such an approach is likely not to appeal more than do ordinary mathematics to pupils who are unsuccessful in, or just not attracted by, nut-cracking, even if it is extra-mathematical.

To take this difficulty into account was one among several aims of

the third phase.
The main thing should be to detect mathematics involved, or involvable, in the real world surrounding pupils. The interest did not concern structural issues as much as specific problems and answers to them contributed by mathematics.

Since pupils are members of an entire hierarchy of worlds, it is not quite obvious what is meant by "the real world surrounding them". Usually it has been interpreted to mean their immediate, comprehensible daily lives with their families, friends, schools, sports, leisure times, including features from public media (radio, tv, newspapers). Problems involving money play a predominant part in those spheres, hence applications of mathematics to such problems gained momentum. Graphs, tables, elementary descriptive statistics were new topics added to the elementary curriculum.

Also to this phase the humanisation of mathematics instruction was of importance, for instance in that focus was on problems relating to pupils' immediate surroundings. Problems of greater societal significance, also to pupils as future citizens, but farther away from their daily lives and thus containing less subjective relevance, present considerable pedagogical difficulties and challenges. They have been taken into consideration in the curricula only sporadically.

B. Post-elementary curricula: Four phases

Now we shall turn to the post-elementary curricula. Also the applicational aspects of these curricula have run through a number of phases. What was said in the beginning of section A as to the concept of phases adopted there is equally relevant to this section.

The zero'th phase

was a phase of transition. Genuine applications were hardly present in "new mathematics" post-elementary instruction. To the extent that they occurred at all they were based on the allegation, not accompanied by further comments, that from, say, physics its is well-known that... So, by...we can determine ... Thus the applicational component was reduced to a mere wrapping of usual mathematical problems in terminology borrowed from other subjects without really affecting the mathematical treatment. In more fanciful instances no references were included to "serious" parts of other fields, only to mathematical problems disguised in words but not anchored in any sort of reality - whimsical problems as Henry Pollak calls them. The inclusion of probability theory as a new topic in reformed and post-reformed curricula provided an additional source for problems of both kinds.

Another interpretation identified applicational aspects of mathematics with those parts of it which were actually used, or potentially could be used, by somebody outside mathematics itself. A certain line of argument carried that interpretation further as follows: Since history has taught us that virtually any piece of mathematics has found eventually an exterior application, foreseen or unforeseen, we cannot distinguish between pure and applied mathematics. Hence all mathematics is applied or at least applicable mathematics, and so our curricula already do consist of applied/applicable mathematics. It is just up to people (including teachers) in other subjects or fields to go and use it. We, the mathematicians and the mathematics teachers, should have enough respect of the professional requirements involved in the applicational process to not butt into activities beyond our competence. But let no one doubt our sympathy.

Of course this, in the present phrasing somewhat caricatured, position mainly serves defensive purposes, and should not be taken to imply a true inclusion of applicational aspects of mathematics in the curricula. However, what characterized in general the zero'th phase was that educators and educationalists felt sufficiently challenged as to pay attention, were it only preliminarily and vaguely, to the issue of applying mathematics.

During
the first phase
it became clear that the applicational capacity of mathematics had to be demonstrated in the instruction, not just postulated. Since the application of mathematics had become so widespread in a lot of new fields beyond the classical ones, physics and engineering, and since a decreasing proportion of students would come close to such classical applications during their education or in their future jobs, the demonstration had to encompass applications to new areas, in order to be of interest to the remaining students. An additional incentive in this direction was probably also the fact that applications to physics and physics based fields often presupposes reference to, and hence pre-knowledge about the extensive theoretical apparatus

of physics. Applications to new fields having less demanding theoretical foundations have the advantage of being easier to describe and handle - and to "sell" to students.

If a slogan should be coined to characterize the first phase it might well be: "Mathematics can be used to so many things in so many areas; let us show you some of them".

In reality this programme was directed not only to students, and to the world outside mathematics but as much to teachers and educationalists who in general at that time did not possess down-to-the-earth documentation themselves for the frequent assertions of the widespread usefulness of mathematics. Therefore, quite a few seminars and conferences were held, in the seventies primarily , the aims of which were to present and exchange application cases from different fields, and to discuss strategies for their inclusion in curricula. Seminar reports, books and new-founded journals provided a growing bank of examples from which teachers, textbook writers and educationalist could gain inspiration and draw material. Each case exposed a well-arranged application limited in size and degree of difficulty such that it could be treated by methods and content inherent in the existing curricula with only negligible changes of them. By and large they therefore served as elaborate, seriously meant and sometimes realistic illustrations of the actual and potential use of the mathematics that was in the curriculum already.

When taught, applications were treated as any topic in the syllabus: went over in teacher presentations or coined into closed exercises with well-defined questions demanding well-defined answers.

The term "model" was brought into use in this phase. It was never, as far as I know, used in connection with the classical applications related to physics. This was probably due to the fact that only models relying on particularly well-founded parts of physics, such as Newtonian mechanics or thermodynamics, used to be let into the curricula - which, however, did not imply that those theories were presented or examined in the mathematics courses. But in many new areas of application theoretical considerations are capable of pointing out unique models with a firm foundation only rarely. Usually several models can come into question. Therefore, to emphasize this and to prevent confusion between model and reality the term model was introduced and rapidly adopted in common usage.

However, not much attention was actually paid to making analyses of the theoretical background of a given model, quality assessment, confrontation with reality, discussion of alternatives, of significance to the field of application etc., part of the study on a par with the purely mathematical treatment. So, the call for circumspection and caution when dealing with models mainly remained a lip service to general scientific standards.

In addition to including examples of applications and models in "ordinary" mathematics courses, new separate courses with titles like "Applications of mathematics", "Mathematical models (in...)" began to be offered at several institutions round about in the world. A good deal of the material on which such courses were based, stemmed from the many newly published case collections. In return, original examples from such courses served to increase the stock of cases intering new collections.

The second phase

Among the many reasons for bringing applications and models into mathematics instruction, a dominant one was that students should acquire ability to activate mathematics themselves in dealing with problems and situations of other disciplines and from the world surrounding them. They should become able to mathematize extra-mathematical situations, i.e. to build mathematical models related to such situations, investigate their properties and interprete these as properties of objects and relations belonging to the area modelled.

It soon became clear that for such an enterprise to be meaningful, the problem situations at issue had to be open-ended, which means that the questions asked, whether explicite or implicite, are - as also the answers - fuzzy, ill-defined and accessible to students' own influence. This did not imply that the problem situations had to be more extensive, or lead to more complicated or sophisticated models than was the case with the well-groomed examples from the first phase. On the contrary, the models might well be the same, the big difference being that students should not get them served as objects of learning, but arrive at them as a result of their own efforts.

Dealing with this kind of situations demands other forms of instruction and course work, as well as other participants' roles, than the conventional ones. The role of the teacher, for instance, changes from that of a lecturer to that of a guide. Collaboration between students, often working in small groups, permission or even need to use a variety of approaches and a multitude of materials, utilization of longer instructional sequences than single-frozen lessons, etc., were all important tools for mathematization and model-building courses within the mathematics curriculum, or in addition to it. In some places, project organised sequences dealing with problem situations displaying a broad spectrum of complexity, and typically terminated by written reports, have been experimented with, and even put into practise.

The main trend was- and is - that modelling courses remained within the framework of the general mathematics courses. However, quite a few examples of interdisciplinary cooperation involving several disciplines and teachers exist.

Where the first phase saw courses in appl<u>ications</u> and mod<u>els</u> established at many institutions, primarily at university level, courses in app<u>lying</u> mathematics and in mod<u>elling</u>, emphasizing the model-building <u>process</u>, arose in the second phase.

As regards the mathematical substance of models, it remained largely within the respective existing syllabi. The main direction of motion was still <u>from mathematics to applications</u>. The models resulting from the students' mathematization activities were in general known to the teachers, but more and more frequently it happened that the problem situations to be modelled were open not only to students but to their teachers as well.

Despite the relative stability of curriculum content on the various educational levels, there emerged trends to supplement syllabi with elements of new topics under inspiration of the application and modelling wave, topics which made new types of applications possible. Above all, this concerned statistics. Earlier, to the extent that statistics was at all present in mathematics instruction, it was limited to descriptive statistics, and belonged thus partly to the elementary curricula. It was not connected to probability theory, which topic in turn stood pretty isolated in the curriculum. What happended in the second phase was that proper statistical theory and methods - e.g. distributions, estimation, tests - got a footing secured, either as a new component in the mathematics curriculum (this was mainly the case in the schools) or in separate courses added to the mathematical ones (this was mainly the case at college and university level). Since then, mathematical statistics viewed as a teaching subject particularly fit for applications and modelling has become widespread and has expanded considerably, to such an extent that in recent years it has been devoted educational conferences of its own. There is good reason to believe that the application and modelling movement has stimulated greatly this proliferation and expansion. Not only constitute statistical problems a world in themselves, they are also invariably present whenever a mathematical model is being confronted with empirical or experimental data.

But also other new topics found their way to the mathematics courses because of the growing interest in applications and modelling. In particular topics from discrete mathematics (including graph theory, combinatorics, difference equations, algorithms, boolean algebra) have gained momentum. From some quarters, chiefly in the USA, it has been argued that discrete mathematics ought to replace, or at least to be equalized with, college calculus in order to create better feeding-lines to computer science. Other applicationally motivated new topics are optimization and numerical analysis. Interestingly enough, that motivation has played a role in the recent revival of proper geometry in the curricula too. In stead of emphasizing, as used to be the case, either its axiomatic-deductive features or its formative values, geometry is now often being viewed as a particularly rich and interesting source of models of the physical world, and as a unique supplier of illumination of the interplay between mathematical theory and scientific epistemology.

A highly important factor in the development during the second phase was the proliferation of electronic tools for computation and calculation. First the simple hand-held calculator, then programmable versions of it, and recently micro-computers, all having reached prices low enough to make them accessible to "everyman" in the industrialized world, reinforced the application and modelling wave considerably, by making computational treatment of simple models easy and of more complex models possible.

The third phase

is characterized by a reversion of the order of what comes first, mathematics or the applicational situation and its needs. In the preceding phases, a certain mathematical apparatus, in terms of topics and methods, were given, as well as certain boundary conditions. Applications and model-building situations which could be dealt with on that basis were then sought. In the third phase, the field of application and the problems associated with it come first, and only subsequently is mathematics introduced in the context of model-building. The topics and methods that come into play depend on the nature of the field and problems to be dealt with, and on the actual and potential availability, at the educational level at issue, of appropriate mathematical knowledge and skills.

It is not surprising that this approach can be found - but far from as often in practice as one would expect - in disciplines which utilize and hence teach mathematics as a tool, and in which the interest in mathematics goes exactly as far as mathematics has something to contribute to their primary enterprises. What is new in this phase is that also within independent mathematical education not solely aiming at being a subject to serve the interests of other disciplines, experiments based on this approach have been performed. These experiments range from instances where the approach has been used in limited enclaves embedded in more conventional curricula to instances where the entire mathematics programme has been organised on basis of it.

In addition to making great demands to the enthusiasm and the capacity of the teacher involved, this approach requires much in structuring content and organising teaching. First of all, it requires thorough consideration of how problem-oriented model-building sequences and sequences consisting of work with systematically organised mathematics substance should interact and be proportioned. Another thing is that genuine model-building processes, taking the needs of the underlying applicational situation seriously, tend to be lengthy, often unpredictably so, because you are not finished until the questions which originated the model-building have been answered satisfactorily.

Even if there are, thus, delicate and difficult balances to decide about and look after, students' gains from this approach are considerable when it works - and sometimes it does work. Students obtain first-hand experiences with the crucial stages of the model-building process - with all the mess, complexity and obscurity involved, and all the frustrations coming up, with the countless decisions and compromises to be made, usually on unclear backgrounds - which they cannot obtain by the approaches of the preceding phases.

IV. THE CURRENT SITUATION: STATE AND TRENDS

The meaning of the phases, as they were described in the above sections, should not be over-interpreted. An increase in phase number indicates an increased emphasis on the area of application and issues and demands related to it. The phases are not thought to represent stages of evolution, in the sense that the curriculum of a certain segment of an educational system can only enter a given phase if it has run through the previoues ones or that it will invariably develop into the succeding phase after some time. Neither does this ordering imply that a "later" phase represents a more valuable level of development than an "earlier" one. In addition, a change of stage in real life may well happen to pass from a "later" to an "realier" phase. Finally, it will often be the case that curricula associated with different segments of the educational system of a given country are in different phases.

Although the phases are, thus, defined in structural rather than in chronological terms, as an over-all trend curriculum discussions and curriculum development have followed, in fact, these phases in general outlines also chronologically.

In the current situation of applications and modelling two features seem to be predominant in both elementary and post-elementary curricula:

(1) There is a long distance between "the front" of the debate on and the development of applications and modelling in the curriculum, on the one hand, and the main-stream of mathematics instruction on the other hand. Although much has happened at the front in the past decade, and although this has had an increasing impact on main-stream instruction, we should not be misled to believe that the state at the front - on which our attention is, naturally enough, chiefly focussed - is similar to that in the main-stream. Even in countries where radical educational experiments with applications and modelling take place, much post-elementary mathematics instruction can be found which contains hardly any applicational, let alone modelling, components.

(2) If we do concentrate our attention to the places (in a broad sence, including countries, educational systems, levels, institutions, specific curricula etc.) where applications and models are being worked with, we find a considerable pluralism. Most of what has been mentioned as characteristic to the phases, from modest and singular activities through the most advanced and extensive experiments, is still present in the current situation.

Of course, this pluralism is due, above all, to the large variations that exist in all dimensions between not only educational traditions and systems of different countries, but between different sectors and levels within individual countries as well. It is, however, due also to the fact that the application and modelling enterprise is still in a period of motion , investigation and experimentation. We have not reached a stage of general clarification, where things have been settled up in stable formats. If this will ever happpen is difficult to prophesy.

Anyway, thousand flowers bloom. But certain trends of development may be detected in the pluralism after all. I shall attempt to point at some of them.

Applications, models and model-building have been secured, or are in the process of being secured, a footing in post-elementary mathematics curricula, also in the main-stream, in most countries. There are, indeed, big differences in how far this has been brought.

In countries in which the curriculum authority is decentralized, examples of curricula in which non-trivial application and modelling activities are implemented are most abundant. Such curricula live often side by side with curricula which contain nothing of the kind. In countries of this type innovations strongly depend on, but are also only limited by, the enthusiasm and efforts of outstanding inspirators, groups of teachers and local authorities, be it in schools or universities.

In countries where curriculum planning and implementation are in the hands of central authorities, changing the main-stream of mathematics education requires much more time. This is true for the decision process as well as for the process of carrying out decisions . Huge masses have to be set in motion, which in addition makes the costs considerable. On the other hand, a reform will, when first realised, affect everybody on the educational level concerned and not just a minority. It may well be the case in such countries that prospective reforms are tried out by means of pilot projects in experimental institutions. This may result in fairly advanced experiences.

Generally speaking, in countries with central curriculum authorities the inclusion in post-elementary curricula of proper applications, models and model-building in a large scale, has not been brought very far (although there are exceptions and the situation seems to be changing). This probably has its background in two connected factors. Firstly, the programmes of innovation in mathematics curricula inspired by the "new mathematics" wave were completed in those countries only rather recently, in some countries even very recently, e.g. in the 1980s. Of course, they are not prepared to undertake new extensive reform programmes already. Secondly, the "new mathematics" innovation when it came in these countries never went to extremes in claustrophobic structure inbreeding as was sometimes seen in dedicated "new mathematics" places. So, in many of the late-reform countries applications never disappeared completely, if only present in modest amounts.

By and large, it seems that in countries where the "new mathematics" reforms were implemented early, and hence took its most pure shapes, the inclusion of applications and modelling in the curricula has been brought furthest.

In the <u>third world</u> countries it is a general feature that much emphasis is laid on applications and modelling in the elementary curricula of the primary schools and in relation to daily-life problems and situations in the surroundings of pupils. In principle, this emphasis is found also in the secondary and post-secondary curricula. But in most developing countries education on these levels regards so few students that those who obtain such an education automatically enter, or will enter eventually, the educated élite of their societies, with the professions, tasks and positions resulting thereof. These people will occupy leading roles in the development of their countries, and if their mathematical background contains more or less of applications and modelling is of minor importance compared with the matter of having a mathematical background at all. At least in the moment. For, after having successfully completed the establishing of a sufficient primary education for the majority of their populations, several developing countries have turned their attention to the higher educational levels and have brought applicational and modelling aspects in focus.

There is no reason to doubt that this indicates a path that many developing countries will strive to follow in the future, once they have catered for the more immediate needs of secondary and post-elementary education, such as provision of sufficiently mnay competent teachers, sufficient material resources for instruction, and have solved other basic problems, e.g. those caused by the fact that many Third World countries have a multitude of different languages within their borders.

The over-all trend in applications and modelling in mathematics curricula consists of several components, sub-trends. Every subtrend represents a moving from one position towards another position. A summary of points touched upon previously in this paper, with a few more points added, might be schematized as in the following table, which for the sake of clarity is intentionally over-simplified:

	FROM focus on		TOWARDS focus on
PURPOSE AND GOALS	Applications in the curriculum serve the purpose of defending mathematics against criticism and declining student recruitment.	→	The interest in introducing applications and modelling in the curriculum is an indepentdent one. It goes beyond the scope of mathematics and mathematics education
	The main goals are to provide motivation, or to improve students' acquisition of mathematical concepts, theories and methods.	→	The main goal is to improve the activation of mathematics to situations and problems outside the world of mathematics.
CONTENT	Internal and classical topics	→	Applied/applicable topics, including "new" ones
	Applied/applicable topics	→	Applications
	Topics being applied	→	Problem situations calling for the application of mathematics
	Problem situations aimed at illuminating mathematics	→	Problem situations aimed at illuminating aspects of the extra-mathematical world
	Well-defined problem situations	→	Open-ended problem situations
	Constructed problem situations	→	Real problem situations
	Work inside the model with focus on its mathematical properties	→	Work with the entire model, including its non-mathematical aspects
ORGANISATION	Separate courses in applications and modelling	→	Integration in the general mathematics programme
	Independence of other subjects	→	Interdisciplinary cooperation
	Short applications and modelling sequences	→	Long applications and modelling sequences
METHODS	Initiative is with the teacher, who manages the course of events.	→	Students work independetly, often in groups, under teacher guidance.

V: CONCLUSION: PROBLEMS TO BE FACED

A. Wanted: A balanced view of mathematical models

The accelerating accessibility of computers of all sizes and of related soft-ware packages implies that virtually any kind of mathematical model can be examined by computer calculations, and its behaviour by computer simulation. This makes it possible to treat and investigate a model without going through sophisticated analyses of the mathematics involved and of the model foundation. If we add that the endeavour to include applications and model-building in various curricula have called for much attention and devotion, and for a conviction of the general power of mathematics it is not surprising that balanced conceptions of the modelling capacity of mathematics, and its limitations, are not widespread.

The epistemological status
of a mathematical model of a given segment of reality lies not only with the mathematical form it takes, or with its capability of fitting a set of data nicely. There is much more to it than that. Models which from a purely mathematical point of view are identical may have quite different epistemological foundations. The role of the Newtonian model of gravitation within classical mechanics is crucial to that theory. If the law of gravitation were overruled, so were not only our understanding of gravitation, but of the entire complex of mechanical phenomena. When exactly the same set of mathematical expressions is used to model migration in urban geography, its role is quite different. If, for some region, it does not fit relevant data, it is just not appropiate in that context, which may give rise to further measures, for instance to skipping or modifying the model, or what have you. But our understanding of urban geography is not fatally affected if the model is overruled. The point is not that classical mechanics provide true propositions (we all know that this is a truth only with modifications), whereas urban geography at its present state does not. Instances where an analogue of the gravitational law gives a fine fitting to data of migration phenomena may very well exist. The point is that the conditions for evaluation are completely different in the two cases. Thus, evaluation is not only a matter of central importance, but a highly complex one too. Several other examples illustrating this point could have been put forward.

In some cases a mathematical model "stands alone in the world" and has therefore to be evaluated isolatedly. In other cases it may be closely tied to established theory, or be related to well-founded models of phenomena governed by analogous mechanisms. In some contexts (primarily such that involve natural sciences), the outcome of experiments may go into the evaluation, in others this is not possible, sometimes on grounds of principle. For some models purely mathematical considerations may prove useful to the evaluation, for others not. And so forth. So, it is not a question of establishing an assesment-check-list but of conducting careful analyses involving a variety of criteria for judgement.

I shall take this opportunity to plea for the incorporation of epistemological analyses along these lines - at an appropriate level, of course - whereever applications of mathematics and mathematical model-building is undertaken in the instruction. If it is our main goal to provide students of any level with knowledge about, insight in and sound judgement of the role of mathematics in the world - and to me it is - then mathematics education must encompass such analyses.
If we go too far in preaching the gospel that mathematics is unrestrictedly applicable to any kind of problems whatsoever, we will be overstating our case. Then a backlash is likely to emerge and the dream of a well-founded and sound mathematics education of the general population in our countries will be really bad off.

In the beginning of this section it was pointed out that computer calculation and simulation might tend to rule out mathematical methods of treating models and of examining their properties. Of course, this is not the only effect resulting from the increasing availability of non-expensive electronic calculators and computers. The range and realisticity of problems and models which can be treated in educational contexts is enormously widened by these tools. Also powerful means of graphic illustration and presentation of model results are provided by the computers. The challenge is to utilise the computer for mathematics education but to avoid the inherent danger of unjustified trivialisation of it. This challenge urges for the relations between mathematics, model-building and computers to be put on the agendas of discussion among mathematics educationalists and educators, and of mathematics education itself.

B. Barriers for applications and modelling in mathematics curricula

In conclusion of this paper I shall adress the question "Why is it so difficult to ensure in practice applications and modelling an appropriate position in the mathematics curricula?".

Firstly, the question is hardly urgent for elementary mathematics education. At this level, the main difficulties lie more with giving good answers to the question "how should applications and modelling be taught so as to enable pupils to mathematize problem situations from the world surrounding them with phantasy, flexibility and critical judgement?" than to the questions "should elementary mathematics instruction accept this task?" and " if yes, to what extent?". I dare claim that the general answers to the latter two questions are "yes" and "as much as possible", respectively. So, there really is no barrier to having applications and modelling activities accepted as a substantial part of the elementary curriculum. The "how" question, which is by no means a trivial one, leads, in contrast, to countless different answers and gives rise to much dispute on curricular and pedagogical means, but such matters are not our concern at the moment.

If we turn to the post-elementary mathematics curricula, the situation is quite different, and our opening question is relevant indeed. I shall not attempt at giving a comprehensive treatment of it here but confine myself to advancing a few key points.

(1) If taken seriously, applications and modelling activities are time consuming, and the more so the more realistic the application and model building process is. Since in extensively and delicately balanced educational programmes it is rarely possible to expand the total amount of time allotted to mathematical doings, application and modelling activities have to find their room beside the remaining mathematical activities. This necessarily implies a reduction of the purely mathematical syllabus (in a narrow sense) which can be dealt with in the course.

Many mathematics teachers for post-elementary levels are not ready to accept this consequence. They perceive themselves as mathematicians, or at least as ambassadors of mathematics as a

science, and hence view a reduction of the body of mathematical concepts, theories and methods as a reduction in students' access to insight in and experiences with the realm of mathematics. To such teachers this would represent a decline in quality of the mathematics education we give our students, a decline which is not compensated by the gains resulting from the applications and modelling activities. Not seldom, "purchasers" of pupils, students or graduates receiving them at later stages of their education tend to accept this view. Either because they simply agree, or because they have been accustomed to receiving students or graduates fulfilling certain specifications. They have not accustomed themselves to utilizing the potential of new specifications, and focus in stead on the trouble these cause them, for instance by demanding changes in courses.
We should not, however, underestimate the genuine dilemmas involved in balancing in courses applications and modelling activities with work regarding internal mathematical concepts, theories and methods. The latter kind of work needs attention in its own right. After all, the actual and potential applicability of mathematics to the world outside relies on the properties it possesses.

(2) The second barrier lie with the students. While situations to which applications and models based on elementary mathematics are relevant occur everywhere in our immediate, daily-life surroundings, this is not the case with applications and models based on post-elementary mathematics. Surely, quite a few examples exist. However, many of them are associated with situations and problems which are no less academic and no less alien to students than are purely mathematical problems, and they are often much more difficult to handle because of the multi-dimensional requirements they imply. In other cases, the contribution of mathematics does not regard the core of the problem but only its outskirts, and thus becomes a little far-fetched. Such examples hardly convince students of the relevance and far-reaching powers of mathematics in tackling situations and problems from the world around us. What we need are cases the substance of which are, or may be made, relevant to students and to which mathematics can provide answers that cannot be obtained by other means, or only with difficulty. Rather than insisting on the relevance for students of problems of speciali-

zed academic interest, or on the relevance of mathematics in dealing with problems to which it has only marginal contributions to offer, we should invest a good deal of effort in detecting significant problems to which mathematics may provide crucial assistance. Much more needs to be done in this field even if we shall never be - and should never pretend to be - able to abolish the basic seclusion of students, whether in privacy or in school, from societal activities in which mathematics really matters.

(3) Application and modelling qualifications, especially in handling real problems, are difficult to assess, let alone test, by traditional evaluation tools, in particular in educational systems which make use of centrally constructed written examination papers . So far, this problem has not found satisfactory solutions. As long as this is so, written examination papers will contain only usual mathematical problems. Because of the well-known backwash of examination requirements on the instruction, this will tend to keep application and modelling activities in a marginal position. Taking for granted that the general confidence in examinations in most countries and quarters are not likely to be changed in a foreseeable future, a great task in inventing suitable forms of assessment lies in front of us, if this barrier to applications and modelling is to be overcome.

C. Final remarks

From the exposition in the preceding sections of the present paper it follows that all over the world applications and modelling are in growth and development. This takes place at the front as well as in the main-stream of mathematics education, and regards theoretical as well as practical activities.

There is reason to believe that a stable situation is not close by. As long as the distance between front and main-stream is so long, there will be motion in the field tending to reduce the distance. Much needs to be done in these years in bringing main-stream mathematics instruction in line with the situation at the front. On the other hand, as sections V.A and B indicate the state there is not optimal, a situation which calls for some rethinking and a lot of further development.

In our well-motivated zeal to promote applications and modelling in the mathematics curricula we should always keep in mind that no item of education - including applications and modelling - can serve as its own purpose. So, we should not take to just become applications and modelling lobbyists. Their justification must result from continuing and open-minded analyses of what they are good for and of how their potential qualities can be best unfolded. Applications and modelling have to be viewed under the perspectives of mathematics education as a whole and of education in general.

In societies as they look in the present epoch, it is essential to democracy that every receiver of mathematics education, in primary shool and in PhD-studies, are provided with capability to understand, exercise sound judgement about and act towards the role mathematics in the world (nature and society). To this end are applications and modelling of crucial importance indeed, but it matters very much how they are treated.

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